On Oligopoly with Perfect Complements and Cournot Oligopoly

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Abstract

Previous literature suggests that oligopoly with perfect complements is the dual problem to Cournot oligopoly model. This result crucially relies on the assumption that the firms have zero costs. This paper shows that if the production costs of the firms are different from zero, the nice duality between these two oligopoly settings breaks down. One implication of this break down is that oligopoly with perfect complements can easily be a game of strategic complements while Cournot oligopoly cannot be so in a global sense.

On the other hand, Buchanan and Yoon (2000) argue that the effects on equilibrium profits of having one versus more than one firm are the same in both models, but we show that this need not be the case when costs are included. Using linear demand and linear costs, it is shown that the loss in equilibrium profit relative to the cartel/monopoly is bigger under Cournot oligopoly than under oligopoly with perfect complements.

1 Introduction

Cournot's complementary monopoly theory of 1838 (Cournot, 1960) refers to a market where n firms sell n different products that are useless unless they are used together, i.e., the n goods are perfect complements. Each of the firms produce exactly one good that has no substitutes, which means that they are monopolies in their own product. The property of being a group of monopolies selling goods that are perfect complements explains Cournot's original name for the model, nonetheless, along the lines of modern game theory, this setting is known as oligopoly with perfect complements, which is the way we are going to refer to it for the remainder of this paper.

To enjoy one unit of one of the goods produced by a firm in an oligopoly with perfect complements, the consumer needs to buy one unit of each of the rest of the goods, then, the final price that she has to pay is the sum of the prices of all the products. In consequence, the demand will be given by this cumulative price.

One example of this setting is when different firms produce pieces for electronic devices. To assemble an iPhone, Apple buys components from Samsung, LG, Infineon, Broadcom, ST Microelectronics and Texas Instruments. Theoretically, we can think of the components produced by these firms specifically for the iPhone as totally useless if not bought together by Apple to produce iPhones.

A classical and multi-cited example that Cournot (1960) used to develop his theory is the production of brass: the producer needs both copper and zinc at the same time, otherwise, these elements are useless if only one of them is bought. Another seminal work in oligopolies with perfect complements was developed independently by Ellet in 1839 (Ellet, 1966), the example he used is how two different individuals who own consecutive segments of a canal decide their tolls to shippers.

An important application of this setting in the literature is the anticommons problem. This one arises when multiple agents have the right to exclude people from consuming the good (anticommon good) that they own. This is modeled as each owner choosing the price for the anticommon good that maximizes her profit, the consumers have to pay each one of the owners their price in order to use the anticommon good. In their paper, Buchanan and Yoon (2000) use as an example to illustrate this problem a vacant lot next to a country village that can be used as a parking lot that has a lower capacity than open demand. Thus, the owners establish their prices for the parking lot and the users have to pay all of them.

On the other hand, Sonnenschein (1968) argues that Cournot oligopoly and oligopoly with perfect complements models are formally equivalent; he argues that the only difference is the interpretation placed on symbols. In Cournot oligopoly, prices are determined by the total quantity produced which is the sum of the quantities that the firms produce. In oligopoly with perfect complements, the quantity produced is determined by the total price of the goods, this is, by the sum of the prices of the goods that are perfect complements. Thus, according to the author, if the demand function and the inverse demand function are the same, quantities in Cournot's oligopoly model lead to precisely the same prices in oligopoly with perfect complements model and vice versa. Nevertheless, Sonnenschein's (1968) proof is made under the assumption that the firms' production costs are zero. If we impose a different structure of costs of the firms, the equivalence between both models does not hold in general.

One application of Sonnenschein's (1968) result is discussed by Buchanan and Yoon (2000). They argue that the loss in total equilibrium profit by both the commons and the anticommons problem is equal, even if the effects on quantities are opposite. The commons problem arises when several agents have the usage right over a common good. Here, the common good will be used more than the efficient level which is the level of usage that a single owner would choose. As discussed earlier, the effect in the anticommons setting is the opposite: there will be an under-usage of the good (relative to the usage that a single owner would allow). The authors show that these effects in quantities are opposite but the loss in total profit is the same; again, this result crucially relies on the combination of both the linear demand and costless production.

Although some oligopolies with perfect complements have costless production, we can easily find examples where it is not, for instance, the production of the parts for the iPhone. For the first case, the results presented by Sonnenschein (1968) and Buchanan and Yoon (2000) are useful, but when production is not costless, the results for Cournot oligopoly are not longer applicable to the oligopoly of perfect complements.

The objective of this paper is to show that when the firms are symmetric, the mathematical equivalence between both models breaks down when costs of production are different than zero. The first part of the paper gives existence of symmetric equilibria for an oligopoly with perfect complements with any number of firms and analyzes the effects of entry of new firms (in this case, goods as well) on equilibrium prices, quantities, individual and industry profits, welfare and price-cost margins. The methodolgy used is similar to that of Amir and Lambson (2000), based on lattice programming methods, but the results differ.

To give a idea of why the models are different, let $D(\cdot)$, $P(\cdot)$ and $C(\cdot)$ denote the demand function, the inverse demand function and the common cost function respectively. Under the methodology used, the results for oligopoly with perfect complements rely on the sign of $-D'(z)$, where z stands for the total price for the n goods, which is unambiguously positive when the demand function is strictly decreasing. In contrast, the results for

Cournot oligopoly depend on the sign of $-P'(Q) + C''(q)$, where Q denotes total output and q single-firm output, which can be positive or negative depending on the curvature of the cost function.

The second part of the paper presents one example with linear demand and linear cost that reinforces the fact that Cournot oligopoly and oligopoly with perfect complements are not mathematically equivalent in the presence of production costs (costs different than zero). The example shows that the change in equilibrium total profits when adding firms to the monopoly is different among both models. This result complements the results presented by Buchanan and Yoon (2000). The authors showed that the equilibrium total profits with n firms are the same under both models with costless production. Specifically, in their context, they showed that the equilibrium total profits for n individuals that have rights over the parking lot are the same whether they have usage rights (equivalent to Cournot oligopoly) or exclusion rights (correspondent to oligopoly with perfect complements). As a consequence, the change in equilibrium profits when adding firms to the monopoly is the same among both models. Buchanan and Yoon (2000) used a linear demand and zero costs, the latter drives this particular result which fits in their example but not in applications where production is costly.

The next part of the paper discusses some applications, followed by the model, the results and the conclusions.

1.1 Applications

As mentioned in the Introduction, a classical application of oligopoly with perfect complements is the anticommons problem. This one will be discussed in deeper detail in Section 2.2. In this section, we discuss some other applications that have been relevant in the literature.

The first one under consideration is patents (see Shapiro (2001) and Lerner and Tirole (2004)). This is clearly an application of oligopoly with perfect complements if we think of a firm or consumer that wants to develop a new product but might infringe in a number, say n, of patents owned by different parts in order to do so, then, it has to pay for the usage of all of the patents combined. Then, the patents in this scenario are perfect complements and the consumer needs to buy a license for each of them so that the total price that she ends up paying is the sum of all the n licenses.

It is well-known that Cournot (1960) found that prices are lower if a multi-product monopoly produces all the goods versus having n firm producing the goods, one by each of the firms, also, he noticed that industry profits are higher with a multi-product monopolist, then, an oligopoly with perfect complements hurts both the consumer and the producers. Some authors have focused in finding ways to solve this problem. One possible solution is a patent pool, which is an agreement among patent owners to allow a consumer, who might own or not a patent part of the patent pool, to use a set of their patents, and whose efficiency has been studied by Lerner and Tirole (2004). Although the main focus of the present paper is different than solving this natural inefficiency of the oligopoly with perfect complements discussed by Cournot (1960), it gives insights to differentiate this model from a Cournot oligopoly so that the results for the latter are not mistakenly transferred to the oligopoly with perfect complements. This paper also provides results to make comparative statics in equilibrium prices, profits (individual and total) and welfare when more patents are required in order to develop a product given that the consumer demand remains unchanged.

Another application related to patents but not exclusive to them is to analyze the case where an inventor wants to obtain a patent for its product. In general, permits of different institutions are required. Here, the complementary goods are the permits, not the patents like in the previous example. The model in the present paper allows us to predict what would happen when more agencies are issued the right to give required permits to obtain the patent. If the demand of the inventor does not change, we can predict the direction of change of equilibrium prices of the agencies that issue the permits, number of patents issued and profits of the agencies. This applies in general for situations where permits from different agencies are required for certain activities. Buchanan and Yoon (2000) illustrate this case with an entrepreneur who wanted to invest in a seaside/hunting-preserve resort in Italy, 1999, but had to obtain permits from different agencies such as the tourist board, a hotel-restaurant agency and the wildlife agency.

In a separate topic, Feinberg and Kamien (2001) analyze the hold-up problem in an oligopoly with perfect complements setting, i.e., the problem that rises when a producer of one of the goods is vulnerable to exploitation by another one, especially when the game is sequential. For example, if the government or a firm wants to buy land from different owners in order to make a big construction, let us say a government agency or a mall, one owner can wait until the rest of the owners have agreed to sell their land ir order to get a higher benefit for her part of land given that it is necessary for the project. Clearly, for the government or firm that wants to buy the big piece of land, the small pieces of land owned by different agents are perfect complements. Although hold-up problems are out of the scope of this paper, Feinberg and Kaimen's (2001) example is relevant in this literature and it is the same used by Ellet (1966). They use the metaphor of two different persons owning two sequential segments of a road where there are no alternative routes or exits. Although this has been only an illustrative example for various authors, Gardner, Gaston and Masson (2002) bring this analogy to the real world and apply the oligopoly with perfect complements model to analyze how the Rhine river was tolled in 1254.

Although Cournot oligopoly is a more popular model than oligopoly with perfect complements in the sense that it is more frequently referred to in the literature, it is important to realize that oligopoly with perfect complements is as well a common setting observed in real life and should not be mistakenly considered by a model that is mathematically equivalent to Cournot oligopoly in general. This erroneous notion might lead to wrong results when analyzing it.

2 The Model and the Results

2.1 Equilibrium and Effects of Entry

We consider n symmetric firms in an oligopoly with perfect complements setting proposed by Cournot (1960), i.e., a market situation in which producers sell products that are useless unless they are purchased together in a fixed ratio. In particular, we assume that this ratio is 1:1 given that we can always normalize the quantities. This situation is modeled by each of the n firms choosing the price of their own good; thus, the consumer has to buy one unit of each of the n goods and the total price that she pays is the sum of all the prices set by the firms.

This oligopoly with perfect complements is described by (D, C, K, n), where K is the maximum price than can be charged for any of the goods in the complementary market, n is the number of firms (and therefore goods), $D : [0, nK] \to R^+$ stands for the demand function and $C : [0, \infty) \to R^+$ is the cost function for each firm; R^+ stands for the real numbers that are greater than or equal to zero.

Let x denote the price that the firm under consideration charges, and y, the sum of the prices of the remaining $(n-1)$ firms. Let $z = x + y$, i.e., z is the total price that a consumer has to pay in order to obtain all the complementary goods and get use from them.

The firm under consideration chooses the price x that maximizes its profit given by:

$$\Pi(x, y) = xD(x + y) - C[D(x + y)]. \tag{1}$$

Alternatively, we can think about the same firm choosing z given y, in this case, it will maximize its profit given by:

$$\tilde{\Pi}(z, y) = (z - y)D(z) - C[D(z)]. \tag{2}$$

Let $\Delta(z, y)$ denote the cross-partial derivative of $\tilde{\Pi}$ with respect to z and y, then:

$$\Delta(z, y) = -D'(z). \tag{3}$$

Hereafter, we assume the following standard assumptions,

(A1) $D(\cdot)$ is continuously differentiable and $D'(\cdot) < 0$.

(A2) $C(\cdot)$ is twice continuously differentiable and $C'(\cdot) \geq 0$.

Notice that under assumption (A1), $\Delta(z, y) > 0$ on the lattice

$$\varphi \hat{=} \{(z, y) : 0 \leq y \leq (n-1)K, y \leq z \leq y + K\}.$$

The following theorem guarantees the existence of at least one symmetric equilibrium in the oligopoly with perfect complements; under the standard assumptions, no asymmetric equilibria exist. All the proofs are shown in Section 4.

Theorem 1. *Assume that the standard assumptions (A1) and (A2) hold. Then, for each $n \in N$, the oligopoly with perfect complements has at least one symmetric equilibrium and no asymmetric equilibria.*

The proofs show that differentiability of the demand and cost functions are not necessary for the main results to hold, it is assumed by convenience. The results rely on the fact that the profit function $\tilde{\Pi}$ in equation (2) is supermodular on the lattice φ, which is equivalent to $\Delta > 0$ under smoothness conditions.

The results in Theorem 1 reveal the first difference between Cournot oligopoly and oligopoly with perfect complements. Amir and Lambson (2000) show that asymmetric equilibria in Cournot oligopoly may exist, where $m < n$ firms produce the equilibrium of the oligopoly composed by m firms and the remaining $n - m$ firms do not produce anything. The reason is that the corresponding Δ in Cournot oligopoly[1] may be positive or negative, while in oligopoly with perfect complements it is always positive given that the derivative of the demand function is strictly negative.

When the corresponding Δ in Cournot oligopoly is strictly negative, at least one asymmetric equilibrium exists, where one firm produces the monopolistic output and the rest do not produce anything. Also, we have quasi-anticompetitive results, i.e., prices that rise with the entry of firms. The reason of the ambiguity of the sign of the corresponding Δ in Cournot oligopoly is that in addition to the derivative of the inverse demand function, it also depends on the second derivative of the cost function.

The sign of Δ (corresponding Δ for each of the models) is key when getting existence and comparative statics results, thus, the fact that it takes a different sign whether the model is Cournot oligopoly or oligopoly with perfect complements breaks down the mathematical equivalence between both of them discussed by Sonnenschein (1968).

A natural question is what happens to equilibrium prices, outputs and profits when a new firm enters the market, i.e., when a new product that perfectly complements the goods already in the market is introduced. The following result answers to this question, but first, we introduce the notation.

Given that in general, we do not have uniqueness in the equilibrium of this model, we will denote the set of equilibria for each variable by its corresponding capital letter sub-indexed by the number of firms; i.e., in the presence of n firms, X_n denotes the set of equilibrium prices for a single firm, Y_n is the set of equilibrium cumulative prices for the rest of the $(n-1)$ firms, Z_n is the set of equilibrium total prices, Q_n is the set of equilibrium outputs for a single firm (recall that in this model, every firm produces exactly the same amount of output of different goods that are perfect complements) and Π_n is

[1] In Cournot oligopoly, Δ denotes the cross-partial derivative of a single firm's profit with respect to the total output and the output of the remaining $(n-1)$ firms. This is analogous to oligopoly with perfect complements but the decision variable is quantity instead of price. Specifically, $\Delta = -P'(Q) + C''(q)$, where $P(\cdot)$ denotes the inverse demand function, $C(\cdot)$ the cost function, and Q and q stand for aggregate and single-firm outputs, respectively.

the set of equilibrium profits for a single firm. An element of anyone of the sets will be denoted by the corresponding lower-case letter sub-indexed by the number of firms.

For ease in the presentation of the results, we say that an equilibrium set for a specific variable in the model is increasing or decreasing in n, when the maximal and minimal points of the set are increasing or decreasing in n, respectively. We cannot tell the direction of change of all the points in the equilibrium sets when we increase the number of firms because the comparative statics results in this paper are based on the results by Milgrom and Roberts (1990, 1994), that are valid only for the extremal equilibria of the games.

The proofs in the Appendix specify the role of the maximal and minimal points of the corresponding equilibrium sets. These are represented by the lower-case letter corresponding to the set denoted by an upper and a lower bar respectively.

After specifying the notation, we are ready for our first comparative statics results. Notice that for this to hold, the new product must be produced by a new firm, not by one already in the market. A firm already in the market introducing the new good might change the results. The reason is that in oligopoly with perfect complements, the goods are different and a multiproduct firm producing more than one has to be multiplant; thus, its cost has to be multiplied by the number of goods that it is producing, say $\tilde{n}$, i.e., its cost becomes $\tilde{n}C(\cdot)$ instead of $C(\cdot)$.

Theorem 2. *Assume that the standard assumptions (A1) and (A2) hold. For any number of firms $n \in N$:*

(a) *The equilibrium total price set, Z_n, is increasing in n; consequently, the corresponding equilibrium per-firm output set, Q_n, is decreasing in n.*

(b) *The equilibrium per-firm profit set, Π_n, is decreasing in n.*

In oligopoly with perfect complements model, the addition of one good that perfectly complements the goods that are already in the market increases the equilibrium total price set; this is very intuitive since now, there is an additional good that the consumer has to buy in order to enjoy all of them. Nonetheless, the equilibrium profit of the firms decrease, thus, under the standard assumptions, the firms do not have incentives to let entry into this market structure.

Theorem 2 does not contain information on the direction of change of the equilibrium per-firm price set, X_n, when we increase the number of firms/goods n. It can take any direction of change depending on the slope of the reaction correspondence with respect to the cumulative price of the other $(n-1)$ firms. Theorems 3 and 4 give sufficient conditions for these directions of change.

Theorem 3. *In addition to the standard assumptions (A1) and (A2), assume that* $D(\cdot)$ *is concave and* $C[D(\cdot)]$ *is convex; then, there exists a unique and symmetric equilibrium, with equilibrium per-firm price* x_n *decreasing in the number of firms* n*.*

Theorem 4. *In addition to the standard assumptions (A1) and (A2), assume that* $D(\cdot)$ *is log-convex and* $C[D(\cdot)]$ *concave; then, the equilibrium per-firm price set,* X_n*, is increasing in the number of firms* n*.*

Under the hypotheses of Theorem 3, every selection of the argmax of Π in equation (1) is decreasing in the cumulative price of the remaining $(n-1)$ firms. Moreover, the slopes of these selections are in $[-1, 0]$, then, by Lemma 2.3 in Amir (1996) and Theorem 1 of the present paper, the equilibrium is unique and symmetric. This, plus the result that the equilibrium cumulative price of the rest of the $(n-1)$ firms set is increasing in n, proven in Lemma 1 of the Appendix, imply that the equilibrium per-firm price is decreasing in n. Notice that a necessary condition for $C[D(\cdot)]$ to be convex when the demand is concave, is that the cost function is convex.

On the other hand, if the demand function is log-convex (and under the standard assumptions), a cost function that is concave when composed with the demand function is sufficient to have increasing selections of the reaction correspondence of the firm with respect to the cumulative price of the remainder $(n-1)$ firms. Thus, by Lemma 1 in the Appendix, the equilibrium per-firm price set is increasing in n. Similarly, note that a concave cost function is a necessary condition for $C[D(\cdot)]$ to be concave when the demand function is log-convex.

By Theorem 2 part (a), the equilibrium total price set increases with the number of firms and the equilibrium quantity set goes down, thus, the equilibrium consumer surplus set decreases with more firms in the market. Also, by Theorem 2 part (b), the equilibrium individual profit set decrease with the number of products or firms, nevertheless, the change in equilibrium industry profits is ambiguous. The following result tells us that the equilibrium total profit set also decreases with the number of firms. Combining these results, we conclude that the equilibrium social welfare set is decreasing in n as well.

Proposition 1. *Assume that the standard assumptions (A1) and (A2) hold. Then, for any number of firms $n \in N$:*

(a) The equilibrium consumer surplus set, CS_n, is decreasing in n.

(b) The equilibrium total profit set, $n\Pi_n$, is decreasing in n.

(c) The equilibrium social welfare set, W_n, is decreasing in n.

Finally, we analyze the equilibrium price-cost margin set, M_n, where the price-cost margin in equilibrium is defined by $m_n \triangleq x_n - C'[D(z_n)]$. It turns out that it may increase or decrease with the number of firms depending on whether the demand function is log-convex or log-concave.

Proposition 2. *Suppose that the standard assumptions (A1) and (A2) hold. Then, for any number of firms $n \in N$, the equilibrium price-cost margin set, M_n, is decreasing (increasing) in n if $D(\cdot)$ is log-concave (log-convex).*

The previous results in this paper show that Cournot oligopoly and oligopoly with perfect complements are mathematically different problems when the production is costly. The following section presents one example that reinforces this fact. The example shows that, under linear demand and linear cost functions, the loss in total profit when we change from one to more than one firm is bigger under Cournot oligopoly than under oligopoly with perfect complements. This generalizes the analysis made by Buchanan and Yoon (2000), where they argue that the losses in equilibrium profits are the same for both models. Their result crucially relies on having costless production, which is a reasonable assumption in many models of perfect complements. Nonetheless, if one wants to extend this result for costs of production different than zero, some features in the nature of both models need to be taken into account. For instance, one firm producing n perfect complements should be thought as a multi-product firm with n plants, given that the goods that are different. These technical differences in the models that disappear with costless production, lead us to asymmetric effects in the change of profits when adding firms to the respective monopolist.

2.2 Monopoly vs. Oligopoly

In their paper, Buchanan and Yoon (2000) argue that commons and anticommons problems lead to "symmetric tragedies", a concept that will be explained below. As pointed out in the introduction, the commons problem is an application of Cournot oligopoly model, and the anticommons problem, of oligopoly with perfect complements.

Translated into firms, the commons problem can be seen as a Cournot oligopoly because the owners of the common good decide how much of it to use in order to maximize their profit. A single owner can be seen as a single-product monopolist that decides the quantity that is going to be used/sold.

On the other hand, the anticommons problem fits into an oligopoly with perfect complements model where owners exclude potential users by choosing the price of the permits to use the anticommon good, then, a user needs to buy all the permits in order to enjoy it. A single owner of the anticommon good can be seen as a multi-product monopolist because it gives all the permits required to use it. In this case, the goods produced are the permits.

Buchanan and Yoon (2000) argue that adding firms to either monopoly in either setting reduces equilibrium total profit and that the reduction is the same even if the effects in equilibrium outputs are opposite. In the case of Cournot oligopoly (commons problem), the good is over-used, and in oligopoly with perfect complements (anticommons problem), the good is under-used. By over-use, the authors mean that the Cournot oligopoly produces a higher equilibrium output than a single-product monopolist would; on the other side, by under-use, they mean that the oligopoly with perfect complements produces less than a multi-product monopolist.

Specifically, Buchanan and Yoon (2000) find that the individual profits for the single-product and multi-product monopolies are the same, similarly, the individual profits for a firm in the Cournot oligopoly are the same as those for a firm in an oligopoly with perfects complements. Based on these results, the authors conclude that the effect (loss) on industry profits of adding firms to the single or multi-product monopoly is the same for both problems; nonetheless, their results crucially rely on the specifications of the model, particularly, in the costless production.

The following example, with linear demand and linear cost, shows that individual profits for firms in a Cournot oligopoly or in an oligopoly with perfect complements are in general different when the costs of production are different than zero; similarly for the single-product monopolist versus the multi-product monopolist. Thus, effects in total profits when we increase from one to more than one firm in the market are different for both settings. In particular, for this example, the loss in equilibrium profit when adding firms to the corresponding monopoly is bigger under Cournot Oligopoly than under oligopoly with perfect complements.

Although it is not explicitly illustrated in this paper, it is useful to point out that effects in total profits can even go in opposite directions, i.e., adding firms to a single-product monopolist might result in a gain of equilibrium total profits while adding firms to the multi-product monopolist will always lead to a loss in equilibrium total profits. The reason is a combination of two well known results: adding firms to a single-product monopolist might lead to higher equilibrium profits of the industry under specific conditions, and a multi-product monopolist will always earn at least the same profits as the oligopoly, given that it can always replicate the production decisions of the latter.

Example. Let $P(q) = a - bq$, with $a, b > 0$, be the inverse demand function and $C(q) = cq$, $c \geq 0$, the cost function for each one of the $n \geq 1$ firms in the market. Notice that the demand function is given by $D(z) = \frac{a}{b} - \frac{z}{b}$, and that $c = 0$ represents the costless production which illustrates the setting in Buchanan and Yoon (2000). Finally, let us assume that $a - nc > 0$, this condition ensures the existence of equilibria with positive production in both models (symmetric, when having more than one firm).

If only one homogeneous good is produced, a single-product monopolist chooses the quantity q_1 that maximizes its profit $qP(q) - C(q)$. Thus, it chooses $q_1 = \frac{a-c}{2b}$, which leads to the equilibrium price $z_1 = \frac{a+c}{2}$ and optimal profit $\Pi_1 = \frac{(a-c)^2}{4b}$.

Now suppose that there are $n > 1$ symmetric firms producing the same good in a Cournot oligopoly. By Amir and Lambson (2000), the equilibrium is symmetric and the quantity produced by each one of the firms is given by $q_n = \frac{a-c}{b(n+1)}$, the total quantity in the market is $nq_n = \frac{n(a-c)}{b(n+1)}$, the price is $z_n = \frac{a+nc}{n+1}$, each firm's profit is $\Pi_n = \frac{(a-c)^2}{b(n+1)^2}$ and the total profit, $n\Pi_n = \frac{n(a-c)^2}{b(n+1)^2}$. Notice that having $n > 1$ firms decreases the industry profits in

$$\frac{(a-c)^2(n-1)^2}{4b(n+1)^2}, \tag{4}$$

compared to the single-product monopoly setting.

Now, let us consider $n > 1$ products that perfectly complement each other. If we have a multi-product monopoly producing the n goods and selling them in a bundle, it will choose the price z_1 of the bundle that maximizes its profit $zD(z) - nC[D(z)]$. Because the products are different, the multi-product monopolist needs n different plants to produce them, incurring in n times the cost of producing $D(z_1)$ units of a product. Then, in this example, the optimal output produced is $q_1 = \frac{a-nc}{2b}$, with equilibrium price $z_1 = \frac{a+nc}{2}$ and optimal profit $\Pi_1 = \frac{(a-nc)^2}{4b}$.

If we have an oligopoly with perfect complements, i.e., n firms producing the n perfect complements, each one producing exactly one of them, each of the firms will charge the price $x_n = \frac{a+c}{n+1}$ in equilibrium, the total price that the consumer has to pay is $z_n = \frac{n(a+c)}{n+1}$, the quantity that each firm produces (and that the consumer gets of each product) is $q_n = \frac{a-nc}{b(n+1)}$, individual profit is $\Pi_n = \frac{(a-nc)^2}{b(n+1)^2}$ and total profit, $n\Pi_n = \frac{n(a-nc)^2}{b(n+1)^2}$. Having n firms producing the n goods reduces the multi-product monopoly profits by

$$\frac{(a-nc)^2(n-1)^2}{4b(n+1)^2}. \tag{5}$$

Table 1 summarizes the previous results.

Table 1: Equilibrium total output, total price and industry profit for different settings.

	Total Output	Total Price	Industry Profit
Single-product monopoly	$\frac{a-c}{2b}$	$\frac{a+c}{2}$	$\frac{(a-c)^2}{4b}$
Cournot oligopoly	$\frac{n(a-c)}{b(n+1)}$	$\frac{a+nc}{n+1}$	$\frac{n(a-c)^2}{b(n+1)^2}$
Loss in profit from 1 to n firms			$\frac{(a-c)^2(n-1)^2}{4b(n+1)^2}$
Multi-product monopoly	$\frac{a-nc}{2b}$	$\frac{a+nc}{2}$	$\frac{(a-nc)^2}{4b}$
Oligopoly with perfect complements	$\frac{a-nc}{b(n+1)}$	$\frac{n(a+c)}{n+1}$	$\frac{n(a-nc)^2}{b(n+1)^2}$
Loss in profit from 1 to n firms			$\frac{(a-nc)^2(n-1)^2}{4b(n+1)^2}$

In particular, if $c = 0$, the profits of the industry are the same for both the Cournot oligopoly and the oligopoly with perfect complements and for both the single-product monopoly and the multi-product monopoly, which is the example discussed in Buchanan and Yoon (2000); notice that the monopolies have bigger profits in both settings.

On the other hand, if $c > 0$, increasing from one to $n > 1$ firms leads to different effects on equilibrium total profits. The monopolists get always the biggest profit compared to their respective oligopolies, i.e., in equilibrium, the single-product monopoly earns more profits than the Cournot oligopoly and the multi-product monopoly, more than the oligopoly with perfect complements. Nonetheless, according to equations (4) and (5), the loss in profit of adding firms to either monopoly is bigger under the homogeneous product setting (single-product monopoly vs. Cournot oligopoly) than under the multi-product one (multi-product monopoly vs. oligopoly with perfect complements).

This illustrates that the effects on equilibrium industry profits by adding firms to the single-product or to the multi-product monopoly are, in general, different in the presence of non-zero costs.

The idea that the tragedies of the commons and anticommons are not symmetric is discussed by Vanneste *et. al.* (2006). Using two experiments, a lab experiment versus a scenario experiment, they conclude that the behaviors of the players facing a commons dilemma versus an anticommons dilemma are different, not analogous like a linear demand with slope one predicts. In particular, they find that the players act more aggressively (with higher decision variables) when they face the anticommons dilemma. Although this might be explained through the specification of the demand function, the paper brings up the concern that the commons and anticommons problems are not symmetric in general.

3 Concluding Remarks

This paper shows that including production costs different than zero in the models breaks down the mathematical equivalence (or duality) between Cournot oligopoly and oligopoly with perfect complements first discussed by Sonnenschein (1968). His results are useful for settings were production is costless, they allow us to extend results for Cournot oligopoly to an oligopoly with perfect complements if one keeps track of the interpretation placed on symbols, symplifing the analysis in theses cases. Nonetheless, trying to generalize this duality for any production cost function is, in general, erroneous. For instance, no asymmetric equilibrium can exist in an oligopoly with perfect complement while it may exist in Cournot oligopoly.

The reason of this mathematical difference is that the results presented here and those for Cournot oligopoly in Amir and Lambson (2000), which rely on lattice programming methods, depend on the sign of the corresponding Δ for each of the models. In oligopoly with perfect complements, Δ is always positive, while in Cournot oligopoly, its sign is ambiguous. In other words, the games can be rewritten in such a way that the aggregative game exhibits increasing differences for the oligopoly with perfect complements but not for Cournot oligopoly. In the latter case, whether this reinterpretation of the game exhibits increasing differences or not depends on the curvature of the cost function.

It is important to notice that in oligopoly with perfect complements, the interpretation of the results differs from that of Cournot oligopoly. Here, by entry of a firm, we mean the addition of a new good that perfectly complements the existing ones in the market. In fact, the results in Theorem 2 do not allow us to do comparative statics with respect to the number of firms/products when at least one firm is producing more than one good. These results hold only if one firm is producing exactly one good out of the n goods that are the perfect complements in the market.

An extension of this model is to allow firms to produce more than one good or merging, i.e., firms getting together to produce two or more goods and sell them separately or in a bundle.

In conclusion, this paper shows that although it seems that Cournot oligopoly and oligopoly with perfect complements are equivalent theories with a different interpretation placed on symbols, this is not the case when we include production costs that are different from zero in the models. Previous results in the literature, like those by Sonnenschein (1968) and Buchanan and Yoon (2000) are suitable for applications where production is naturally costless, but if the problem is such that costs are not negligible, a deeper study of the oligopoly with perfect complements has to be done in order to get the corresponding right results.

4 Proofs

Before presenting the proofs, we define the following mapping for every $n \in N$ that can be thought of as a normalized cumulative best-response correspondence. This mapping is analogous to the one used by Amir and Lamson (2000) and it is useful in dealing with symmetric equilibria.

$$B_n : [0, (n-1)K] \longrightarrow 2^{[0,(n-1)K]}$$

where

$$B_n(y) = \frac{n-1}{n}(x' + y). \tag{6}$$

In this mapping, x' represents the firm's best-response, i.e., the price that maximizes its profit in (1) given the cumulative price y for the remaining $(n-1)$ firms. Notice that $x' \in [0, K]$ and $y \in [0, (n-1)K]$ imply that the (set-value) range of B_n is as given. Also, a fixed point of B_n, $\hat{y}$, yields an oligopoly with perfect complements symmetric equilibrium where $\hat{x}' = \hat{y}/(n-1)$, this is, each of the responding firms will set the same price as the other $(n-1)$ firms.

Proof of Theorem 1.

As pointed out earlier, under (A1), the cross partial derivative of the maximand in (2), $\Delta(z, y)$, is strictly positive on the lattice

$$\varphi = \{(z, y) : 0 \leq y \leq (n-1)K, y \leq z \leq y + K\};$$

also, the feasible set $[y, y+K]$ is ascending in y. Then, by Topkis (1978), every selection of the argmax, Z^*, of (2) is increasing in y. As earlier, x' denotes the firm's best-response to y, thus $Z^*(y) = x' + y$. This implies that for every $n \in N$, every selection of B_n as defined by equation (6) is increasing in y. Then, by Tarski's fixed-point theorem, B_n has a fixed-point that implies the existence of a symmetric equilibrium of the oligopoly with perfect complements.

Next, we prove that no asymmetric equilibrium can exist.

Let $\tilde{z}$ be an arbitrary (single-valued) selection of Z^*. By the first part of the proof, we know that $\tilde{z}(y)$ is increasing in y. Let us assume that $\tilde{z}(y_1) = \tilde{z}(y_2)$ for some $y_1 > y_2$. W.l.o.g., take the selections $\tilde{z}(y_1)$ and $\tilde{z}(y_2)$ to be interior, thus, they must satisfy the first-order condition of the maximization of profit in (2):

$$D(\tilde{z}(y_j)) + \{\tilde{z}(y_j) - y_j - C'[D(\tilde{z}(y_j))]\}D'(\tilde{z}(y_j)) = 0, j = 1, 2. \tag{7}$$

Since $\tilde{z}(y_1) = \tilde{z}(y_2)$, equation (7) implies that $y_1 = y_2$, which is a contradiction because we assumed $y_1 > y_2$. Thus, every selection of Z^* is strictly increasing. This means that for each $z' \in Z^*$ corresponds at most one y such that $z' = x' + y$ (z' is the best-response to y); then, for each total equilibrium price z', each firm must charge the same price $x' = z' - y$, with $y = (n-1)x'$, i.e., no asymmetric equilibrium exists. □

Before proceeding with the rest of the proofs, we introduce the following Lemmas that will be useful in doing so.

Lemma 1. *Assume that the standard assumptions (A1) and (A2) hold. Then, for every number of firms $n \in N$, the equilibrium cumulative prices of $(n-1)$ firms set, Y_n, is increasing in n.*

Proof of Lemma 1.

By Topkis's Theorem, the maximal and minimal selections of B_n, denoted by $\overline{B}_n$ and $\underline{B}_n$ respectively, exist. Furthermore, the largest equilibrium cumulative price for $(n-1)$ firms, $\overline{y}_n$, is the largest fixed-point of $\overline{B}_n$. Notice that $\overline{B}_n(y)$ is increasing in n for every fixed y. Hence, by Theorem A.4 in Amir and Lamson (2000), the largest fixed-point of $\overline{B}_n$, $\overline{y}_n$, is also increasing in n. Using an analogous argument with $\underline{B}_n$, shows that the the smallest equilibrium cumulative price for $(n-1)$ firms, $\underline{y}_n$, is increasing in n. □

Lemma 2. *Assume that the standard assumptions (A1) and (A2) hold. Then, for every number of firms $n \in N$, $\overline{\Pi}_n = \Pi(\underline{x}_n, (n-1)\underline{x}_n)$ and $\underline{\Pi}_n = \Pi(\overline{x}_n, (n-1)\overline{x}_n)$.*

Proof of Lemma 2.

We prove that $\overline{\Pi}_n = \Pi(\underline{x}_n, (n-1)\underline{x}_n)$, a similar argument shows that $\underline{\Pi}_n = \Pi(\overline{x}_n, (n-1)\overline{x}_n)$. To this aim, observe that $\tilde{\Pi}(z, y)$ is decreasing in y, then, $\overline{\tilde{\Pi}}_n = \tilde{\Pi}(\underline{z}_n, \frac{(n-1)}{n}\underline{z}_n) = \Pi(\underline{x}_n, (n-1)\underline{x}_n)$. Now, we show that $\overline{\Pi}_n = \Pi(\underline{x}_n, (n-1)\underline{x}_n)$. Suppose not, then it exists $\tilde{x}_n \in X_n$ such that $\Pi(\tilde{x}_n, (n-1)\tilde{x}_n) > \Pi(\underline{x}_n, (n-1)\underline{x}_n)$, then, $\Pi(\tilde{x}_n, (n-1)\tilde{x}_n) = \tilde{\Pi}(\tilde{z}_n, \frac{(n-1)}{n}\tilde{z}_n) > \Pi(\underline{x}_n, (n-1)\underline{x}_n) = \overline{\tilde{\Pi}}_n$, where $\tilde{z}_n = n\tilde{x}_n$, which contradicts the fact that $\overline{\tilde{\Pi}}_n$ is the maximal element in the set $\tilde{\Pi}_n$, thus, $\Pi(\underline{x}_n, (n-1)\underline{x}_n)$ is the maximal per-firm profit equilibrium. □

Proof of Theorem 2.

(a) From the proof of Theorem 1, we know that every selection of the argmax of the profit function given by equation (2) is increasing in y. This plus the fact that $\overline{y}_n$ is increasing in n (Lemma 1) imply that $\overline{z}_n$ is increasing in n. Using an analogous argument, $\underline{z}_n$ is increasing in n.

(b) Consider the following relations and the facts that $(n-1)\underline{x}_n = \underline{y}_n$ and $n\underline{x}_{n+1} = \underline{y}_{n+1}$,

$$\begin{aligned}
\overline{\Pi}_n &= \underline{x}_n D(\underline{x}_n + \underline{y}_n) - C(D(\underline{x}_n + \underline{y}_n)) \\
&\geq (\underline{x}_{n+1} + \underline{y}_{n+1} - \underline{y}_n) D(\underline{x}_{n+1} + \underline{y}_{n+1} - \underline{y}_n + \underline{y}_n) - C(D(\underline{x}_{n+1} + \underline{y}_{n+1} - \underline{y}_n + \underline{y}_n)) \\
&= (\underline{x}_{n+1} + \underline{y}_{n+1} - \underline{y}_n) D(\underline{x}_{n+1} + \underline{y}_{n+1}) - C(D(\underline{x}_{n+1} + \underline{y}_{n+1})) \\
&\geq \underline{x}_{n+1} D(\underline{x}_{n+1} + \underline{y}_{n+1}) - C(D(\underline{x}_{n+1} + \underline{y}_{n+1}) \\
&= \overline{\Pi}_{n+1}.
\end{aligned}$$

The first equality follows by Lemma 2, the first inequality follows by optimality and the second one, by the fact that $\underline{y}_n$ is increasing in n, proved in Lemma 1; thus, $\overline{\Pi}_n$ is decreasing in n.

A similar argument, using $\overline{x}_n$, proves that $\underline{\Pi}_n$ is also decreasing in n. □

Proof of Theorem 3.

First, we show that the profit $\Pi(x,y)$ defined by equation (1) exhibits the dual single-crossing property under the hypotheses of the Theorem, i.e., we show that when $x' > x$, $y' > y$, $\Pi(x,y) \geq \Pi(x',y) \Rightarrow \Pi(x,y') \geq \Pi(x',y')$, which is equivalent to $xD(x+y) - C[D(x+y)] \geq x'D(x'+y) - C[D(x'+y)] \Rightarrow xD(x+y') - C[D(x+y')] \geq x'D(x'+y') - C[D(x'+y')]$.

If the demand function is concave, then it has decreasing differences, which means that $D(x'+y) - D(x'+y') \geq D(x+y) - D(x+y')$, this, plus the assumption that $x' > x$ implies that

$$x'D(x'+y) - x'D(x'+y') \geq xD(x+y) - xD(x+y'), \tag{8}$$

and by assumption,

$$xD(x+y) - C[D(x+y)] \geq x'D(x'+y) - C[D(x'+y)]. \tag{9}$$

Adding equations (8) and (9), we have

$$-x'D(x'+y') - C[D(x+y)] \geq -xD(x+y') - C[D(x'+y)]. \tag{10}$$

But $C[D(\cdot)]$ convex implies that

$$C[D(x+y)] - C[D(x+y')] \geq C[D(x'+y)] - C[D(x'+y')], \tag{11}$$

thus, if we add equations (10) and (11) and rearranging terms we have

$$xD(x+y') - C[D(x+y')] \geq x'D(x'+y') - C[D(x'+y')],$$

which proves that $\Pi(x,y)$ has the dual single-crossing property in (x,y). In consequence, every selection of the argmax of profit in (1), $r(\cdot)$, is decreasing in y. On the other hand, by the proof of Theorem 1, every selection of the best-response of equation (2), $Z^*(\cdot)$, is increasing in y. Since $Z^*(y) = r(y) + y$, these results put together imply that the slope of every selection of $r(\cdot)$ lie in $[-1, 0]$. Thus, the equilibrium is unique by Amir (1996), and symmetric by Theorem 1.

Finally, we know that $x_n = r(y_n)$. Since in equilibrium, y_n is increasing in n (by Lemma 1) and all selections of $r(\cdot)$ are decreasing in y, x_n is decreasing in n. □

Proof of Theorem 4.

We first prove that $\Pi(x, y)$ satisfies the single-crossing property in (x, y). Let $x' > x$, $y' > y$ and $x'D(x'+y) - C[D(x'+y)] \geq xD(x+y) - C[D(x+y)]$. We want to show that $x'D(x'+y') - C[D(x'+y')] \geq xD(x+y') - C[D(x+y')]$.

The assumption that $D(\cdot)$ is log-convex implies that $\frac{D(x'+y')}{D(x+y')} \geq \frac{D(x'+y)}{D(x+y)}$, then,

$$x'D(x'+y) - C[D(x'+y)] \geq xD(x+y) - C[D(x+y)] \geq x\frac{D(x+y')D(x'+y)}{D(x'+y')} - C[D(x+y)].$$

Multiplying the previous inequalities by $\frac{D(x'+y')}{D(x'+y)}$, we have

$$x'D(x'+y') - C[D(x'+y)]\frac{D(x'+y')}{D(x'+y)} \geq xD(x+y') - C[D(x+y)]\frac{D(x'+y')}{D(x'+y)}. \quad (12)$$

Note that $\frac{D(x'+y')}{D(x'+y)} \leq 1$ and $C[D(\cdot)]$ decreasing imply that

$$\left(\frac{D(x'+y')}{D(x'+y)} - 1\right) C[D(x'+y)] \geq \left(\frac{D(x'+y')}{D(x'+y)} - 1\right) C[D(x+y)], \quad (13)$$

thus, adding equations (12) and (13) gives us

$$x'D(x'+y') - C[D(x'+y)] \geq xD(x+y') - C[D(x+y)]. \quad (14)$$

The assumption that $C[D(\cdot)]$ is concave implies

$$C[D(x'+y)] - C[D(x'+y')] \geq C[D(x+y)] - C[D(x+y')], \quad (15)$$

finally, adding equations (14) and (15) we have

$$x'D(x'+y') - C[D(x'+y')] \geq xD(x+y') - C[D(x+y')],$$

which shows that $\Pi(x, y)$ has the single-crossing property in (x, y). Hence, the extremal selections from the argmax of profit in (1), $r(\cdot)$, are increasing in y. Thus $\overline{x}_n = r(\overline{y}_n)$, and given that $\overline{y}_n$ is increasing in n (by Lemma 1), so is $\overline{x}_n$. A similar argument follows for $\underline{x}_n$. □

Proof of Proposition 1:

(a) First notice that the consumer surplus, $CS(\cdot)$, at any total price z and for any $n \in N$ number of firms is given by

$$CS(z) = \int_z^\infty D(t)\ dt,$$

which is decreasing in z.

Then,

$\overline{CS}_n - \overline{CS}_{n+1} = \int_{\underline{z}_n}^{\infty} D(t)\ dt - \int_{\underline{z}_{n+1}}^{\infty} D(t)\ dt$

$= \int_{\underline{z}_n}^{\underline{z}_{n+1}} D(t)\ dt \geq 0.$

The inequality follows by Theorem 2 part (a), $\underline{z}_{n+1} \geq \underline{z}_n$.

A similar argument using $\overline{z}_n$ and $\overline{z}_{n+1}$ proves that $\underline{CS}_n$ is increasing in n.

(b) We prove that $n\overline{\Pi}_n \geq (n+1)\overline{\Pi}_n$. The result that $n\underline{\Pi}_n$ is decreasing in n follows by a similar argument using $\overline{x}_n$. Consider the following relations:

$\overline{\Pi}_n = \underline{x}_n D(n\underline{x}_n) - C[D(n\underline{x}_n)]$

$\geq [(n+1)\underline{x}_{n+1} - (n-1)\underline{x}_n] D[(n+1)\underline{x}_{n+1} - (n-1)\underline{x}_n + (n-1)\underline{x}_n]$

$-C[D[(n+1)\underline{x}_{n+1} - (n-1)\underline{x}_n + (n-1)\underline{x}_n]]$

$\geq \left[(n+1)\underline{x}_{n+1} - \frac{(n-1)(n+1)}{n}\underline{x}_{n+1}\right] D[(n+1)\underline{x}_{n+1}] - C[D[(n+1)\underline{x}_{n+1}]$

$= \frac{(n+1)}{n}\underline{x}_{n+1} D[(n+1)\underline{x}_{n+1}] - C[D[(n+1)\underline{x}_{n+1}]$

$= \frac{(n+1)}{n}\left[\underline{x}_{n+1} D[(n+1)\underline{x}_{n+1}] - \frac{n}{n+1} C[D[(n+1)\underline{x}_{n+1}]]\right]$

$\geq \frac{(n+1)}{n}\left[\underline{x}_{n+1} D[(n+1)\underline{x}_{n+1}] - C[D[(n+1)\underline{x}_{n+1}]]\right]$

$= \frac{(n+1)}{n}\overline{\Pi}_{n+1}.$

The first equality follows by Lemma 2 and the first inequality, by optimality. The second inequality follows by the fact that $(n+1)\underline{x}_{n+1} \geq n\underline{x}_n$ (Theorem 2 part (a)). The last inequality follows because $\frac{n}{n+1} < 1$.

(c) It is clear that $\overline{W}_n = \overline{CS}_n + n\overline{\Pi}_n$. Then

$\overline{W}_n - \overline{W}_{n+1} = [\overline{CS}_n - \overline{CS}_{n+1}] + [n\overline{\Pi}_n - (n+1)\overline{\Pi}_{n+1}] \geq 0.$

The inequality follows because both terms in the right-hand side of the equality are positive by parts (a) and (b). A similar argument proves that $\underline{W}_n$ is decreasing in n. □

Proof of Proposition 2:

Let us consider the maximal point of the equilibrium price-cost margin set, the proof for the minimal point of the set is analogous.

First, notice that $D(\cdot)$ log-concave (log-convex) implies that $\frac{D'(z)}{D(z)} \geq (\leq) \frac{D'(z')}{D(z')}$, for all $z' > z$. Thus,

$$-\frac{D(z')}{D'(z')} + \frac{D(z)}{D'(z)} \leq (\geq) 0. \tag{16}$$

Now, the first-order condition for the oligopoly with perfect complements can be written as

$$D(z_n) + m_n D'(z_n) = 0,$$

which implies that

$$m_n = -\frac{D(z_n)}{D'(z_n)}.$$

If $D(\cdot)$ is log-concave (log-convex), m_n is decreasing (increasing) in z_n, then, $\overline{m}_n = -\frac{D(\underline{z}_n)}{D'(\underline{z}_n)}$ $(\overline{m}_n = -\frac{D(\overline{z}_n)}{D'(\overline{z}_n)})$.

Thus, $\overline{m}_{n+1} - \overline{m}_n = -\frac{D(\underline{z}_{n+1})}{D'(\underline{z}_{n+1})} + \frac{D(\underline{z}_n)}{D'(\underline{z}_n)}$ $\left(\overline{m}_{n+1} - \overline{m}_n = -\frac{D(\overline{z}_{n+1})}{D'(\overline{z}_{n+1})} + \frac{D(\overline{z}_n)}{D'(\overline{z}_n)}\right)$, which is negative (positive) if $D(\cdot)$ is log-concave (log-convex), by equation 16 and Theorem 2 part (a), $\underline{z}_{n+1} \geq \underline{z}_n$ $(\overline{z}_{n+1} \geq \overline{z}_n)$. $\square$

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