Aggregate Preferred Correspondence and the Existence of a Maximin REE

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Abstract

In this paper, a general model of a pure exchange differential information economy is studied. In this economic model, the space of states of nature is a complete probability measure space, the space of agents is a measure space with a finite measure, and the commodity space is the Euclidean space. Under appropriate and standard assumptions on agents' characteristics, results on continuity and measurability of the aggregate preferred correspondence in the sense of Aumann [8] are established. These results together with other techniques are then employed to prove the existence of a maximin rational expectations equilibrium (maximin REE) of the economic model.

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1. Introduction

When traders come to a market with different information about the items to be traded, the resulting market prices may reveal to some traders information originally available only to others. The possibility for such inferences rests upon traders having expectations of how equilibrium prices are related to initial information. This endogenous relationship was considered by Radner in his seminal paper [21], where he introduced the concept of a rational expectations equilibrium by imposing on agents the Bayesian (subjective expected utility) decision doctrine. Under the Bayesian decision making, agents maximize their subjective expected utilities conditioned on their own private information and

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also on the information that the equilibrium prices generate. The resulting equilibrium allocations are measurable with respect to the private information of each individual and also with respect to the information the equilibrium prices generate and clear the market for every state of nature. In papers [1, 2] and [21], conditions on the existence of a Bayesian rational expectations equilibrium (REE) were studied and some generic existence results were proved. However, Kreps [19] provided an example that shows that a Bayesian REE may not exist universally. In addition, a Bayesian REE may fail to be fully Pareto optimal and incentive compatible and may not be implementable as a perfect Bayesian equilibrium of an extensive form game, refer to [13] for more details.

It was pointed out in [18] that the market hypothesis fails if the space of states of nature is of a dimension higher than that of the price simplex. Thus, in generic existence theorems of Allen [1, 2] and Radner [21], the assumption on the space of states of nature being finite or of sufficiently low dimension relative to the dimension of price simplex is essential. However, it was shown in [17] that if the space of states of nature is of a dimension strictly higher than that of the price simplex, then for a residual set of economies there is a rational expectations equilibrium which is given by a two-to-one and almost discontinuous price function. When the dimensions of both spaces coincide, as mentioned in [3], the existence of an equilibrium fails in finite economies. If the space of agents is a unit interval consisting of imperfectly and perfectly informed agents, under the hypothesis of suitably disperse forecasts, it was shown in [3] that for each state of nature the aggregate excess demand is continuous on the price simplex and satisfies Walras's law. This fact allowed Allen to apply a fixed point theorem to obtain the market clearing price vector for each state of nature and obtain the existence of an ε-rational expectations equilibrium for all $\varepsilon > 0$. The convergence as $\varepsilon \to 0$ holds for some cases in which open counterexamples to the existence of rational expectations equilibria are known. In the same year, Allen [4] also considered two types of agents (informed and uninformed) and prices carried only incomplete information, when prices conveyed some information from informed agents to uninformed agents. By applying a fixed point theorem, she obtained a new approximate non-revealing rational expectations equilibrium in the sense that the total discrepancy between demand and supply is small. Allen [5] further showed the existence of a rational expectations equilibrium with (strong) ε-market clearing in the sense that the discrepancy between demand and supply is zero for all but one commodity for which the value can be made arbitrarily small.

In a recent paper [11], de Castro et al. introduced a new notion of REE by a careful examination of Krep's example of the nonexistence of a Bayesian REE. In this formulation, the Bayesian decision making adopted in the papers of [1] and [21] was abandoned and replaced by the maximin expected utility (MEU) (see [12]). In this new setup, agents maximize their MEU conditioned on their own private information and also on the information the equilibrium prices have generated. Contrary to a Bayesian REE, the resulting maximin REE may not be measurable with respect to the private information of each individual or the information that the equilibrium prices generate.

Although Bayesian REE and maximin REE coincide in some special cases (e.g., fully revealing Bayesian REE and maximin REE), these two concepts are in general not equivalent. Nonetheless, the introduction of the MEU into the general equilibrium modeling enables de Castro et al. to prove that a maximin REE exists universally under the standard continuity and concavity assumptions on the utility functions of agents. Furthermore, they showed that a maximin REE is incentive compatible and efficient. Note that in the economic model considered in [11], it is assumed that there are finitely many states of nature and finitely many agents, and the commodity space is finite-dimensional. Thus, one of open questions is whether their existence theorem can be extended to more general cases. The main motivation of this paper is to tackle this question. In this paper, a general economic model of a pure exchange differential information economy is studied. In this model, the space of states of nature is a complete probability measure space, and the space of agents is a measure space with a finite measure. Under appropriate and standard assumptions on agents' characteristics, the existence of a maximin rational expectations equilibrium (maximin REE) in this general economic model is established.

This paper is organized as follows. In Section 2, mathematical preliminaries and key facts used in this paper are presented. The economic model and assumptions are introduced and explained in Section 3. In Section 4, some results on continuity and measurability of agents' aggregate preferred correspondence in the sense of Aumann [8] are established. These results are key techniques employed to prove the existence of a maximin rational expectations equilibrium of the economic model in Section 5. Finally, in Section 6, results and techniques appeared in this paper are compared with relevant results in the literature.

2. Mathematical preliminaries

Let G be a non-empty set. On ℓ-dimensional Euclidean space $\mathbb{R}^\ell$, two different but equivalent norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are used in this paper, where

$$\|x\|_\infty = \max\{|x_i| : 1 \leq i \leq \ell\}, \quad \|x\|_1 = \sum_{1\leq i\leq \ell} |x_i|$$

for each point $x = (x_1, x_2, \cdots, x_\ell) \in \mathbb{R}^\ell$. A *correspondence* $F : G \rightrightarrows \mathbb{R}^\ell$ from G to $\mathbb{R}^\ell$ assigns to each $x \in G$ a subset $F(x)$ of $\mathbb{R}^\ell$. Meanwhile, F can also be viewed as a function $F : G \to 2^{\mathbb{R}^\ell}$, where $2^{\mathbb{R}^\ell}$ denotes the power set of $\mathbb{R}^\ell$. Further, F is called *non-empty valued* (resp. *closed-valued, compact-valued, convex-valued*) if $F(x)$ is a non-empty (resp. closed, compact, convex) subset of $\mathbb{R}^\ell$ for all $x \in G$. The *graph* of F, denoted by Gr_F, is defined by

$$\mathrm{Gr}_F = \{(x, y) \in G \times \mathbb{R}^\ell : y \in F(x) \text{ and } x \in G\}.$$

For $x \in \mathbb{R}^\ell$ and $A \in 2^{\mathbb{R}^\ell} \setminus \{\emptyset\}$, define $\mathrm{dist}(x, A) = \inf\{d(x, y) : y \in A\}$, where d is the Euclidean metric on $\mathbb{R}^\ell$. Let $\mathscr{K}_0\left(\mathbb{R}^\ell\right)$ be the family of non-empty compact subsets of $\mathbb{R}^\ell$. Recall that the *Hausdorff metric* H on $\mathscr{K}_0\left(\mathbb{R}^\ell\right)$ is defined such that for any two $A, B \in \mathscr{K}_0\left(\mathbb{R}^\ell\right)$,

$$H(A,B) = \max\left\{\sup_{a\in A} \operatorname{dist}(a,B),\ \sup_{b\in B} \operatorname{dist}(b,A)\right\}.$$

For equivalent definitions of H, refer to [6]. The *Hausdorff metric topology* $\mathscr{T}_H$ on $\mathscr{K}_0(\mathbb{R}^\ell)$ is the topology generated by H. For a closed subset M of $\mathbb{R}^\ell$, $\mathscr{K}_0(M)$ and the Hausdorff metric H on $\mathscr{K}_0(M)$ can be defined similarly. If G is a topological space, a non-empty compact-valued correspondence $F : G \rightrightarrows \mathbb{R}^\ell$ is called *Hausdorff continuous* if $F : G \to \left(\mathscr{K}_0\left(\mathbb{R}^\ell\right), \mathscr{T}_H\right)$ is continuous. This statement holds when $\mathbb{R}^\ell$ is replaced by a closed subset M of $\mathbb{R}^\ell$.

Let $\{A_n : n \geq 1\}$ be a sequence of non-empty subsets of $\mathbb{R}^\ell$. A point $x \in \mathbb{R}^\ell$ is called a *limit point* of $\{A_n : n \geq 1\}$ if there exist $N \geq 1$ and $x_n \in A_n$ for each $n \geq N$ such that $\{x_n : n \geq N\}$ converges to x. The set of limit points of $\{A_n : n \geq 1\}$ is denoted by $\mathrm{Li}A_n$. Similarly, a point $x \in \mathbb{R}^\ell$ is called a *cluster point* of $\{A_n : n \geq 1\}$ if there exist positive integers $n_1 < n_2 < \cdots$ and for each k an $x_k \in A_{n_k}$ such that $\{x_k : k \geq 1\}$ converges to x. The set of cluster points of $\{A_n : n \geq 1\}$ is denoted by $\mathrm{Ls}A_n$. It is clear that $\mathrm{Li}A_n \subseteq \mathrm{Ls}A_n$, and both $\mathrm{Ls}A_n$ and $\mathrm{Li}A_n$ are closed (possibly empty) sets. If $\mathrm{Ls}A_n \subseteq \mathrm{Li}A_n$, then $\mathrm{Li}A_n = \mathrm{Ls}A_n = A$ is called the *limit* of the sequence $\{A_n : n \geq 1\}$. Note that $\mathrm{Ls}A_n = \mathrm{Ls}\overline{A}_n$ and $\mathrm{Li}A_n = \mathrm{Li}\overline{A}_n$. Hence, if A is the limit of $\{A_n : n \geq 1\}$, then A is also the limit of $\{\overline{A}_n : n \geq 1\}$. If A and all A_n's are closed and contained in a compact subset $M \subseteq \mathbb{R}^\ell$, then it is well known that $\mathrm{Li}A_n = \mathrm{Ls}A_n = A$ if and only if $\{A_n : n \geq 1\}$ converges to A in the Hausdorff metric topology on $\mathscr{K}_0(M)$, refer to [6].

Let (T, Σ, μ) be a measure space and $\{F_n : n \geq 1\}, F : (T, \Sigma, \mu) \rightrightarrows \mathbb{R}^\ell$ be correspondences. Recall that F is said to be *lower measurable* if

$$F^{-1}(V) = \{t \in T : F(t) \cap V \neq \emptyset\} \in \Sigma$$

for every open subset V of $\mathbb{R}^\ell$. It is well known that a non-empty closed-valued correspondence $F : (T, \Sigma, \mu) \rightrightarrows \mathbb{R}^\ell$ is lower measurable if and only if there exists a sequence of measurable selections $\{f_n : n \geq 1\}$ of F such that for all $t \in T$,

$$F(t) = \overline{\{f_n(t) : n \geq 1\}}.$$

If all $F_n's$ are non-empty closed-valued, lower measurable and at least one of $F_n's$ is compact-valued, then $\bigcap_{n\geq 1} F_n$ is lower measurable, refer to [16]. If all $F_n's$ are integrably bounded by the same function, then

$$\mathrm{Ls}\int_T F_n d\mu \subseteq \int_T \mathrm{Ls}F_n d\mu, \quad \text{and} \quad \int_T \mathrm{Li}F_n d\mu \subseteq \mathrm{Li}\int_T F_n d\mu.$$

If $(S, \mathscr{S}, \nu)$ is another measure space and $f : (T, \Sigma, \mu) \times (S, \mathscr{S}, \nu) \to \mathbb{R}^\ell$ is jointly measurable, then it is well known that $\int_T : f(\cdot, \cdot) d\mu : (S, \mathscr{S}, \nu) \to \mathbb{R}^\ell$ is measurable. Let $M \subseteq \mathbb{R}^\ell$ be endowed with the relative Euclidean topology, and (Y, ϱ) be a metric space. A function $f : (T, \Sigma, \mu) \times M \to (Y, \varrho)$ is called *Carathéodory* if $f(\cdot, x)$ is measurable for all $x \in M$, and $f(t, \cdot)$ is continuous for all $t \in T$. It is known that any Carathéodory function is jointly measurable with respect to the Borel structure on M. A *selection* of F is a single-valued function $f : (T, \Sigma, \mu) \to \mathbb{R}^\ell$ such that $f(t) \in F(t)$ for almost all $t \in T$. If a selection f of F is measurable (resp. integrable), then it is called a *measurable* (resp. an *integrable*) *selection*. Let $\mathscr{S}_F$ denote the set of integrable selections of F. The *integration* of F over T in the sense of [7] is a subset of $\mathbb{R}^\ell$, defined as

$$\int_T F d\mu = \left\{ \int_T f d\mu : f \in \mathscr{S}_F \right\}.$$

If F is non-empty closed-valued and integrably bounded, then $\int_T F d\mu$ is compact, refer to [15]. The following two theorems on measurable selections have been employed in this paper.

The following measurable selection theorem can be found in [16, 20].

Theorem 2.1 (Kuratowski-Ryll-Nardzewski Measurable Selection Theorem)**.** *If $F : (T, \Sigma, \mu) \rightrightarrows \mathbb{R}^\ell$ is a closed-valued and lower measurable correspondence, then it has a measurable selection.*

The following measurable selection theorem can be found in [9, 22].

Theorem 2.2 (Aumann-Saint-Beuve Measurable Selection Theorem)**.** *Let B be a Borel subset of $\mathbb{R}^\ell$. If (T, Σ, μ) is a complete finite measure space and $F : T \rightrightarrows B$ has a measurable graph, then there exists a measurable function $f : T \to B$ such that $f(t) \in F(t)$ for all $t \in T$.*

Of course, the above two theorems were stated in more general forms in the literature. But here, they are adapted and presented in particular and simpler forms to fit in this paper.

3. Differential information economies

In this paper, a model of a pure exchange economy $\mathscr{E}$ with differential information is considered. The space of states of nature is a complete probability measure space $(\Omega, \mathscr{F}, \nu)$. The space of agents is a measure space (T, Σ, μ) with a finite measure μ. The commodity space is the ℓ-dimensional Euclidean space $\mathbb{R}^\ell$, and the positive cone $\mathbb{R}^\ell_+$ is the *consumption set* for each agent $t \in T$ in every state of nature $\omega \in \Omega$. Each agent $t \in T$ is associated with her/his *characteristics* $(\mathscr{F}_t, U(t, \cdot, \cdot), a(t, \cdot), q_t)$, where $\mathscr{F}_t$ is the σ-algebra generated by a partition Π_t of Ω representing the *private information*[1] of t; $U(t, \cdot, \cdot) : \Omega \times \mathbb{R}^\ell_+ \to \mathbb{R}$ is a *random utility function* of t; $a(t, \cdot) : \Omega \to \mathbb{R}^\ell_+$ is the *random initial endowment* of t and q_t is a probability measure on Ω giving the *prior belief* of t. The economy extends over two time periods $\tau = 1, 2$. At the ex ante stage ($\tau = 0$), only

[1]For a recent general study of information sets, refer to Hervés-Beloso and Monteiro [14].

the above description of the economy is a common knowledge. At the interim stage $\tau = 1$, agent t only knows that the realized state of nature belongs to the event $\mathscr{F}_t(\omega^*)$, where $\mathscr{F}_t(\omega^*)$ is the unique member of Π_t containing the true state of nature ω^* at $\tau = 2$. At the ex post stage ($\tau = 2$), agents execute the trades according to the contract agreed at period $\tau = 1$, and consumption takes place. The coordinate-wise order on $\mathbb{R}^\ell$ is denoted by $\leq$ and the symbol $x \gg 0$ means that x is an interior point of $\mathbb{R}^\ell_+$, and $x > 0$ means that $x \geq 0$ but $x \neq 0$. Let $L_1\left(\mu, \mathbb{R}^\ell_+\right)$ be the set of equivalent classes of Lebesgue integrable functions from T to $\mathbb{R}^\ell_+$. An *allocation* in $\mathscr{E}$ is a function $f : T \times \Omega \to \mathbb{R}^\ell_+$ such that $f(\cdot, \omega) \in L_1\left(\mu, \mathbb{R}^\ell_+\right)$ for all $\omega \in \Omega$. An allocation f in $\mathscr{E}$ is *feasible* if $\int_T f(\cdot, \omega) d\mu = \int_T a(\cdot, \omega) d\mu$ for all $\omega \in \Omega$.

The following standard assumptions on agents' characteristics shall be used:

($\mathbf{A}_1$) The initial endowment function $a : (T, \Sigma, \mu) \times (\Omega, \mathscr{F}, \nu) \to \mathbb{R}^\ell_+$ is jointly measurable such that $\int_T a(\cdot, \omega) d\mu \gg 0$ for each $\omega \in \Omega$.

($\mathbf{A}_2$) $U(\cdot, \cdot, x) : (T, \Sigma, \mu) \times (\Omega, \mathscr{F}, \nu) \to \mathbb{R}$ is jointly measurable for all $x \in \mathbb{R}^\ell_+$ and $U(t, \omega, \cdot) : \mathbb{R}^\ell_+ \to \mathbb{R}$ is continuous for all $(t, \omega) \in T \times \Omega$.

($\mathbf{A}_3$) For each $(t, \omega) \in T \times \Omega$, $U(t, \omega, \cdot) : \mathbb{R}^\ell_+ \to \mathbb{R}$ is monotone in the sense that if $x, y \in \mathbb{R}^\ell_+$ with $y > 0$, then $U(t, \omega, x + y) > U(t, \omega, x)$.

($\mathbf{A}_4$) For each $(t, \omega) \in T \times \Omega$, $U(t, \omega, \cdot) : \mathbb{R}^\ell_+ \to \mathbb{R}$ is concave.

Let

$$\Delta = \left\{ p \in \mathbb{R}^\ell_+ : p \gg 0 \text{ and } \sum_{h=1}^{\ell} p^h = 1 \right\}$$

be equipped with the relative Euclidean topology and the Borel structure $\mathscr{B}(\Delta)$ generated by this topology. Each element $p \in \Delta$ is viewed as a (normalized) *price system* in the deterministic case. The *budget correspondence* $B : T \times \Omega \times \Delta \rightrightarrows \mathbb{R}^\ell_+$ is defined by

$$B(t, \omega, p) = \left\{ x \in \mathbb{R}^\ell_+ : \langle p, x \rangle \leq \langle p, a(t, \omega) \rangle \right\}$$

for all $(t, \omega, p) \in T \times \Omega \times \Delta$. Obviously, B is non-empty and closed-valued. For each $\omega \in \Omega$, by Theorem 2 in [15, p.151], there are $p(\omega) \in \Delta$ and an allocation $f(\cdot, \omega)$ such that $(f(\cdot, \omega), p(\omega))$ is a Walrasian equilibrium of the deterministic economy $\mathscr{E}(\omega)$, given by

$$\mathscr{E}(\omega) = \left((T, \Sigma, \mu); \mathbb{R}^\ell_+; (U(t, \omega, \cdot), a(t, \omega)) : t \in T \right).$$

Define a function $\delta : \Delta \to \mathbb{R}_+$ by $\delta(p) = \min\left\{ p^h : 1 \leq h \leq \ell \right\}$, where $p = (p^1, ..., p^\ell) \in \Delta$. For any $(t, \omega, p) \in T \times \Omega \times \Delta$, let

$$\gamma(t, \omega, p) = \frac{1}{\delta(p)} \sum_{h=1}^{\ell} a^h(t, \omega), \quad \text{and } b(t, \omega, p) = (\gamma(t, \omega, p), ..., \gamma(t, \omega, p)).$$

Define $X : T \times \Omega \times \Delta \rightrightarrows \mathbb{R}^\ell_+$ by $X(t, \omega, p) = \{x \in \mathbb{R}^\ell_+ : x \leq b(t, \omega, p)\}$ for all $(t, \omega, p) \in T \times \Omega \times \Delta$. Note that X is non-empty, compact- and convex-valued such that $B(t, \omega, p) \subseteq X(t, \omega, p)$ for all $(t, \omega, p) \in T \times \Omega \times \Delta$. It can be readily verified that for every $(t, \omega) \in T \times \Omega$, the correspondence $X(t, \omega, \cdot) : \Delta \rightrightarrows \mathbb{R}^\ell_+$ is Hausdorff continuous. Define correspondences C, $C^X : T \times \Omega \times \Delta \rightrightarrows \mathbb{R}^\ell_+$ by

$$C(t, \omega, p) = \big\{y \in \mathbb{R}^\ell_+ : U(t, \omega, y) \geq U(t, \omega, x) \text{ for all } x \in B(t, \omega, p)\big\}$$

and

$$C^X(t, \omega, p) = C(t, \omega, p) \cap X(t, \omega, p).$$

Obviously,

$$B(t, \omega, p) \cap C(t, \omega, p) = B(t, \omega, p) \cap C^X(t, \omega, p)$$

holds for all $(t, \omega, p) \in T \times \Omega \times \Delta$. Note that under $(\mathbf{A}_2)$, $U(t, \omega, \cdot)$ is continuous on the non-empty compact set $B(t, \omega, p)$. Thus, one has

$$B(t, \omega, p) \cap C(t, \omega, p) \neq \emptyset$$

for all $(t, \omega, p) \in T \times \Omega \times \Delta$.

Proposition 3.1. *Let $(t, \omega, p) \in T \times \Omega \times \Delta$. Under $(\mathbf{A}_3)$, $\langle p, x\rangle \geq \langle p, a(t, \omega)\rangle$ for every point $x \in C^X(t, \omega, p)$.*

Proof. Assume that $\langle p, x_0\rangle < \langle p, a(t, \omega)\rangle$ for some point $x_0 \in C^X(t, \omega, p)$. Then, one can choose some $y \in \mathbb{R}^\ell_+$ such that $y > 0$ and $\langle p, x_0 + y\rangle < \langle p, a(t, \omega)\rangle$. Thus, $x_0 + y \in B(t, \omega, p)$. Since $x_0 \in C^X(t, \omega, p)$, one has $U(t, \omega, x_0) > U(t, \omega, x_0 + y)$. However, $(\mathbf{A}_3)$ implies $U(t, \omega, x_0 + y) > U(t, \omega, x_0)$. This is a contradiction, which completes the proof. □

Following [8], $C^X(t, \omega, p)$ is called the *preferred set* of agent t at the price p and state of nature ω, and $\int_T C^X(\cdot, \omega, p)d\mu$ is called the *aggregate preferred set* at the price p and state of nature ω. Moreover, we call

$$\int_T C^X(\cdot, \cdot, \cdot)d\mu : \Omega \times \Delta \rightrightarrows \mathbb{R}^\ell_+$$

the *aggregate preferred correspondence*. In Section 4, we discuss some descriptive properties of this object. These properties will be used to derive the existence of a maximin rational expectations equilibrium in our economic model.

4. Properties of the aggregate preferred correspondence

In this section, some results on continuity and measurability of the aggregate preferred correspondence are presented.

Proposition 4.1. *Under $(\mathbf{A}_1)$, for every $(\omega, p) \in \Omega \times \Delta$, $B(\cdot, \omega, p) : (T, \Sigma, \mu) \rightrightarrows \mathbb{R}^\ell_+$ and $X(\cdot, \omega, p) : (T, \Sigma, \mu) \rightrightarrows \mathbb{R}^\ell_+$ are lower measurable.*

Proof. Here, only the proof of lower measurability of $B(\cdot,\omega,p)$ is provided. The other case can be done analogously. Fix $(\omega,p) \in \Omega \times \Delta$. Define a function $h : (T,\Sigma,\mu) \times \mathbb{R}^\ell_+ \to \mathbb{R}$ by letting $h(t,x) = \langle p,x\rangle - \langle p,a(t,\omega)\rangle$ for all $(t,x) \in T \times \mathbb{R}^\ell_+$. Then, $h(\cdot,x)$ is measurable for all $x \in \mathbb{R}^\ell_+$. Note that $B(t,\omega,p) = h(t,\cdot)^{-1}((-\infty,0])$. Let $V \subseteq \mathbb{R}^\ell_+$ be a non-empty open subset, and put $V \cap \mathbb{Q}^\ell_+ = \{x_k : k \geq 1\}$. It is worth to point out that if $x \in B(t,\omega,p) \cap V$, then $x_k \in B(t,\omega,p)$ for some $k \geq 1$. Since $h(\cdot,x_k)$ is measurable, $\{t \in T : h(t,x_k) \in (-\infty,0]\} \in \Sigma$ for all $k \geq 1$. Thus,

$$\begin{aligned} B(\cdot,\omega,p)^{-1}(V) &= \bigcup_{k\geq 1} \{t \in T : x_k \in B(t,\omega,p)\} \\ &= \bigcup_{k\geq 1} \{t \in T : h(t,x_k) \in (-\infty,0]\} \end{aligned}$$

belongs to Σ. It follows that $B(\cdot,\omega,p)$ is lower measurable. □

Proposition 4.2. *Under* $(\mathbf{A}_1)$-$(\mathbf{A}_3)$*,* $\int_T C^X(\cdot,\cdot,\cdot)d\mu : \Omega \times \Delta \rightrightarrows \mathbb{R}^\ell_+$ *is non-empty compact-valued.*

Proof. Fix $(\omega,p) \in \Omega\times\Delta$. By $(\mathbf{A}_2)$, $C^X(t,\omega,p)$ is non-empty closed for all $t \in T$. By the lower measurability of $B(\cdot,\omega,p)$, there exists a sequence $\{f_n : n \geq 1\}$ of measurable functions from (T,Σ,μ) to $\mathbb{R}^\ell_+$ such that $B(t,\omega,p) = \overline{\{f_n(t) : n \geq 1\}}$ for all $t \in T$. For each $n \geq 1$, define a correspondence $C_n : T \rightrightarrows \mathbb{R}^\ell_+$ by letting

$$C_n(t) = \left\{x \in \mathbb{R}^\ell_+ : U(t,\omega,x) \geq U(t,\omega,f_n(t))\right\}$$

for all $t \in T$. Obviously, one has $C(t,\omega,p) \subseteq \bigcap_{n\geq 1} C_n(t)$ for all $t \in T$. If $x \in \mathbb{R}^\ell_+ \setminus C(t,\omega,p)$ for some $t \in T$, there exists a point $y \in B(t,\omega,p)$ such that $U(t,\omega,y) > U(t,\omega,x)$. By $(\mathbf{A}_2)$, there exists an $n_0 \geq 1$ such that $U(t,\omega,f_{n_0}(t)) > U(t,\omega,x)$. This implies that $x \notin C_{n_0}(t)$, and thus $C(t,\omega,p) = \bigcap_{n\geq 1} C_n(t)$ for all $t \in T$. Fix $n \geq 1$, and define $h : (T,\Sigma,\mu) \times \mathbb{R}^\ell_+ \to \mathbb{R}$ by

$$h(t,x) = U(t,\omega,f_n(t)) - U(t,\omega,x).$$

Clearly, h is Carathéodory. Similar to Proposition 4.1, one can show that C_n is lower measurable. Since $X(\cdot,\omega,p)$ is compact-valued and

$$C^X(\cdot,\omega,p) = \bigcap_{n\geq 1} C_n(\cdot) \cap X(\cdot,\omega,p),$$

then $C^X(\cdot,\omega,p) : (T,\Sigma,\mu) \rightrightarrows \mathbb{R}^\ell_+$ is lower measurable. By Theorem 2.1, $C^X(\cdot,\omega,p)$ has a measurable selection which is also integrable, as $b(\cdot,\omega,p)$ is so. Since $C^X(\cdot,\omega,p)$ is closed-valued and integrably bounded, $\int_T C^X(\cdot,\omega,p)d\mu$ is compact. □

Next, we establish Hausdorff continuity of the aggregate preferred correspondence with respect to the variable $p \in \Delta$.

Theorem 4.3. *Under* ($\mathbf{A}_1$)-($\mathbf{A}_3$), *for each* $\omega \in \Omega$, $\int_T C^X(\cdot,\omega,\cdot)d\mu : \Delta \rightrightarrows \mathbb{R}^\ell_+$ *is Hausdorff continuous*

Proof. Fix $\omega \in \Omega$. Let $\{p_n : n \geq 1\} \subseteq \Delta$ converge to $p \in \Delta$. Choose $\varepsilon > 0$ and $N \geq 1$ such that $\varepsilon < \delta(p)$ and $\varepsilon < \delta(p_n)$ for all $n \geq N$. Let

$$c = \min\{\delta(p_n), \varepsilon : n = 1, 2, \cdots, N-1\},$$

$$d(t,\omega) = \frac{1}{c}\sum_{h=1}^{\ell} a^h(t,\omega), \quad \text{and} \quad \xi(t,\omega) = (d(t,\omega), ..., d(t,\omega)).$$

Define $M(\omega)$ by

$$M(\omega) = \left\{x \in \mathbb{R}^\ell_+ : x \leq \int_T \xi(\cdot,\omega)d\mu\right\}.$$

Since $X(\cdot,\omega,p_n)$ and $X(\cdot,\omega,p)$ are upper bounded by $\xi(\cdot,\omega)$, then the sets $\int_T C^X(\cdot,\omega,p_n)d\mu$ and $\int_T C^X(\cdot,\omega,p)d\mu$ are contained in the compact subset $M(\omega)$ of $\mathbb{R}^\ell_+$. Thus, one only needs to show that $\left\{\int_T C^X(\cdot,\omega,p_n)d\mu : n \geq 1\right\}$ converges to $\int_T C^X(\cdot,\omega,p)d\mu$ in the Hausdorff metric topology on $\mathscr{K}_0(M(\omega))$, which is equivalent to

$$\text{Li}\int_T C^X(\cdot,\omega,p_n)d\mu = \text{Ls}\int_T C^X(\cdot,\omega,p_n)d\mu = \int_T C^X(\cdot,\omega,p)d\mu.$$

The above equation can be verified in two steps. First, one verifies

$$\text{Ls}\int_T C^X(\cdot,\omega,p_n)d\mu \subseteq \int_T C^X(\cdot,\omega,p)d\mu.$$

To do this, it is enough to verify that $\text{Ls}C^X(t,\omega,p_n) \subseteq C^X(t,\omega,p)$ for any $t \in T$. Pick $t \in T$ and $x \in \text{Ls}C^X(t,\omega,p_n)$. Then, there exist positive integers $n_1 < n_2 < n_3 < \cdots$ and for each k a point $x_k \in C^X(t,\omega,p_{n_k})$ such that $\{x_k : k \geq 1\}$ converges to x. It is obvious that $x \in X(t,\omega,p)$. If $x \notin C^X(t,\omega,p)$, by the continuity of $U(t,\omega,\cdot)$, one can choose some $y \in \mathbb{R}^\ell_+$ such that $\langle p, y\rangle < \langle p, a(t,\omega)\rangle$ and $U(t,\omega,y) > U(t,\omega,x)$. By the Hausdorff continuity of $X(t,\omega,\cdot)$, $\{X(t,\omega,p_{n_k}) : k \geq 1\}$ converges to $X(t,\omega,p)$ in the Hausdorff metric topology. Since $y \in X(t,\omega,p)$, there exists a sequence $\{y_k : k \geq 1\}$ such that $y_k \in X(t,\omega,p_{n_k})$ for all $k \geq 1$ and $\{y_k : k \geq 1\}$ converges to y. It follows that $U(t,\omega,y_k) > U(t,\omega,x_k)$ and $\langle p_{n_k}, y_k\rangle < \langle p_{n_k}, a(t,\omega)\rangle$ for all sufficiently large k, which is a contradiction with $x_k \in C^X\ (t,\omega,p_{n_k})$ for all $k \geq 1$. Therefore, one must have $x \in C^X(t,\omega,p)$. Secondly, one needs to verify

$$\int_T C^X(\cdot,\omega,p)d\mu \subseteq \text{Li}\int_T C^X(\cdot,\omega,p_n)d\mu.$$

It suffices to verify that $C^X(t,\omega,p) \subseteq \text{Li}C^X(t,\omega,p_n)$ for all $t \in T$. Fix $t \in T$ and $d \in C^X(t,\omega,p)$. If $d = b(t,\omega,p)$, then $b(t,\omega,p_n) \in C^X(t,\omega,p_n)$ and $\{b(t,\omega,p_n) : n \geq 1\}$ converges to d. Assume $d < b(t,\omega,p)$. Select $\delta > 0$ such that

$$d + (0,\cdots,\delta,\cdots,0) \leq b(t,\omega,p),$$

and a sequence $\{\delta_i : i \geq 1\}$ in $(0, \delta]$ converging to 0. For each $i \geq 1$, let

$$d^i = d + (0, \cdots, \delta_i, \cdots, 0),$$

and choose a sequence $\{d^i_n : n \geq 1\}$ such that for each n, $d^i_n \in X(t, \omega, p_n)$ and $\{d^i_n : n \geq 1\}$ converges to d^i. It is claimed that for each $i \geq 1$, $d^i_n \in C^X(t, \omega, p_n)$ for sufficiently large n. Otherwise, there must exist an i_0 and a subsequence $\{d^{i_0}_{n_k} : k \geq 1\}$ of $\{d^{i_0}_n : n \geq 1\}$ such that $d^{i_0}_{n_k} \notin C^X(t, \omega, p_{n_k})$. Let $b_k \in B(t, \omega, p_{n_k})$ and $U(t, \omega, b_k) > U(t, \omega, d^{i_0}_{n_k})$ for all $k \geq 1$. Then $\{b_k : k \geq 1\}$ has a subsequence converging to some $b \in B(t, \omega, p)$. $(\mathbf{A}_2)$ and $(\mathbf{A}_3)$ imply

$$U(t, \omega, b) \geq U(t, \omega, d^{i_0}) > U(t, \omega, d),$$

which contradicts with $d \in C^X(t, \omega, p)$. To complete the proof, note that the previous claim implies that for each i, $\{\text{dist}(d^i, C^X(t, \omega, p_n)) : n \geq 1\}$ converges to 0. Since $\{d^i : i \geq 1\}$ converges to d, one concludes that $\{\text{dist}(d, C^X(t, \omega, p_n)) : n \geq 1\}$ converges to 0. This means that $d \in \text{Li}C^X(t, \omega, p_n)$. $\square$

The next result is crucial for the existence theorem in Section 5. In its proof, the following characterization of lower measurability of correspondences in [6] is used: A correspondence $F : (\Omega, \mathscr{F}, \nu) \rightrightarrows \mathbb{R}^\ell_+$ is lower measurable if and only if for all $y \in \mathbb{R}^\ell_+$, $\text{dist}(y, F(\cdot)) : (\Omega, \mathscr{F}, \nu) \to \mathbb{R}_+$ is a measurable function.

Theorem 4.4. *Under* $(\mathbf{A}_1)$*-*$(\mathbf{A}_3)$*, for each* $p \in \Delta$*,* $\int_T C^X(\cdot, \cdot, p) d\mu : (\Omega, \mathscr{F}, \nu) \rightrightarrows \mathbb{R}^\ell_+$ *is lower measurable.*

Proof. Fix $p \in \Delta$. Since a and U are $\Sigma \otimes \mathscr{F}$-measurable and $\Sigma \otimes \mathscr{F} \otimes \mathscr{B}(\mathbb{R}^\ell_+)$-measurable respectively, by the argument of a result in [23, 131], there exist two sequences $\{a_n : n \geq 1\}$ and $\{\psi_n : n \geq 1\}$ of $\Sigma \otimes \mathscr{F}$-measurable and $\Sigma \otimes \mathscr{F} \otimes \mathscr{B}(\mathbb{R}^\ell_+)$-measurable functions respectively such that $\{a_n : n \geq 1\}$ uniformly converges to a on $T \times \Omega$ and $\{\psi_n : n \geq 1\}$ uniformly converges to U on $T \times \Omega \times \mathbb{R}^\ell_+$. For each $n \geq 1$, write a_n and ψ_n as

$$a_n = \sum_{i \geq 1} e_i \chi_{T^n_i \times \Omega^n_i}, \quad \text{and } \psi_n = \sum_{i \geq 1} v_i \chi_{T^n_i \times \Omega^n_i \times B^n_i},$$

where $e_i \in \mathbb{R}^\ell_+$, $v_i \in \mathbb{R}$, and $\{T^n_i \times \Omega^n_i \times B^n_i : i \geq 1\}$ is a partition of $T \times \Omega \times \mathbb{R}^\ell_+$ for all $n \geq 1$. Choose $N \geq 1$ such that $\|a_n - a\|_\infty < 1$ for all $n \geq N$. By the measurability of $a_n(\cdot, \omega)$, $a_n(\cdot, \omega) \in L_1\left(\mu, \mathbb{R}^\ell_+\right)$ for all $\omega \in \Omega$ and all $n \geq 1$ (replacing a_n for all $1 \leq n < N$ by some constant functions, if necessary). Let

$$\gamma_n(t, \omega) = \frac{1}{\delta(p)} \sum_{h=1}^{\ell} a^h_n(t, \omega), \quad \text{and } b_n(t, \omega) = (\gamma_n(t, \omega), \cdots, \gamma_n(t, \omega)).$$

Define $X_n, B_n, C_n : T \times \Omega \rightrightarrows \mathbb{R}^\ell_+$ such that $X_n(t, \omega) = \left\{x \in \mathbb{R}^\ell_+ : x \leq b_n(t, \omega)\right\}$,

$$B_n(t, \omega) = \left\{x \in \mathbb{R}^\ell_+ : \langle p, x \rangle \leq \langle p, a_n(t, \omega) \rangle\right\}$$

and

$$C_n(t,\omega) = \left\{y \in \mathbb{R}^\ell_+ : \psi_n(t,\omega,y) \geq \psi_n(t,\omega,x) \text{ for all } x \in B_n(t,\omega)\right\}.$$

In addition, define $C^X_n : T \times \Omega \rightrightarrows \mathbb{R}^\ell_+$ such that for all $(t,\omega) \in T \times \Omega$,

$$C^X_n(t,\omega) = (C_n(t,\omega) \cup \{b_n(t,\omega)\}) \cap X_n(t,\omega).$$

For every $n \geq 1$, define the correspondence $H_n : (\Omega, \mathscr{F}, \nu) \rightrightarrows L_1\left(\mu, \mathbb{R}^\ell_+\right)$ by letting $H_n(\omega) = \mathscr{S}_{C^X_n(\cdot,\omega)}$. Obviously, $H_n(\omega) \neq \emptyset$ for all $\omega \in \Omega$.

Claim 1. For each $n \geq 1$, H_n is lower measurable. For convenience, let $\Theta : L_1\left(\mu, \mathbb{R}^\ell_+\right) \times \Omega \to \mathbb{R}_+$ be the function such that $\Theta(g,\omega) = \text{dist}(g, H_n(\omega))$ for all $g \in L_1\left(\mu, \mathbb{R}^\ell_+\right)$ and $\omega \in \Omega$. To verify the claim, one needs to verify that for all $g \in L_1\left(\mu, \mathbb{R}^\ell_+\right)$, $\Theta(g,\cdot)$ is measurable. Since $\Theta(\cdot,\omega) : L_1\left(\mu, \mathbb{R}^\ell_+\right) \to \mathbb{R}_+$ is norm-continuous, it suffices to show that $\Theta(g,\cdot) : (\Omega, \mathscr{F}, \nu) \to \mathbb{R}_+$ is measurable for every simple function $g = \sum_{j=1}^r x_j \chi_{T_j}$, where $x_j \in \mathbb{R}^\ell_+$. To this end, consider the function $\Gamma : (T, \Sigma, \mu) \times (\Omega, \mathscr{F}, \nu) \to \mathbb{R}_+$ such that $\Gamma(t,\omega) = \text{dist}\left(g(t), C^X_n(t,\omega)\right)$ for all $(t,\omega) \in T \times \Omega$. Since Γ is constant on each $(T^n_i \cap T_j) \times \Omega^n_i$, it is jointly measurable. Note that $\Gamma(t,\omega) \leq \|g(t) - b_n(t,\omega)\|$ for all $(t,\omega) \in T \times \Omega$. This implies for all $\omega \in \Omega$, $\Gamma(\cdot,\omega)$ is integrable. Thus, $\Theta(g,\cdot)$ is measurable and Claim 1 is verified if one shows for all $\omega \in \Omega$, $\int_T \Gamma(\cdot,\omega) d\mu = \Theta(g,\omega)$. Assume $\int_T \Gamma(\cdot,\omega_0) d\mu < \Theta(g,\omega_0)$ for some $\omega_0 \in \Omega$. Pick $\varepsilon > 0$ such that

$$\int_T \Gamma(\cdot,\omega_0) d\mu + \varepsilon\mu(T) < \Theta(g,\omega_0).$$

Further, pick $t \in T^n_i \cap T_j$ and $y_{(i,j)} \in C^X_n(t,\omega_0)$ so that

$$\|x_j - y_{(i,j)}\| < \Gamma(t,\omega_0) + \varepsilon.$$

Define $\zeta : T \to \mathbb{R}^\ell_+$ by $\zeta(t) = y_{(i,j)}$ for all $t \in T^n_i \cap T_j$. Then, $\zeta \in H_n(\omega_0)$ and

$$\|g - \zeta\|_1 < \int_T \Gamma(\cdot,\omega_0) d\mu + \varepsilon\mu(T),$$

which is a contradiction.

Claim 2. The correspondence $\int_T C^X_n(\cdot,\cdot) d\mu : (\Omega, \mathscr{F}, \nu) \rightrightarrows \mathbb{R}^\ell_+$ is lower measurable. Consider the function $\xi : L_1\left(\mu, \mathbb{R}^\ell_+\right) \to \mathbb{R}^\ell_+$ defined by $\xi(f) = \int_T f d\mu$ for all $f \in L_1\left(\mu, \mathbb{R}^\ell_+\right)$. Let V be an open subset of $\mathbb{R}^\ell_+$. Note that

$$\xi \circ H_n(\omega) = \int_T C^X_n(\cdot,\omega) d\mu$$

for all $\omega \in \Omega$, and

$$(\xi \circ H_n)^{-1}(V) = \{\omega \in \Omega : H_n(\omega) \cap \xi^{-1}(V) \neq \emptyset\}.$$

Since ξ is norm-continuous, by Claim 1, $(\xi \circ H_n)^{-1}(V) \in \mathscr{F}$. This verifies the claim.

Claim 3. For each $\omega \in \Omega$,

$$\mathrm{Li} \int_T C_n^X(\cdot, \omega) d\mu = \mathrm{Ls} \int_T C_n^X(\cdot, \omega) d\mu = \int_T C^X(\cdot, \omega, p) d\mu.$$

To see this, for each $\omega \in \Omega$, put

$$\alpha(\cdot, \omega) = \sup \left\{ b_1(\cdot, \omega), \cdots, b_{N-1}(\cdot, \omega), b(\cdot, \omega, p) + \left(\frac{\ell}{\delta(p)}, \cdots, \frac{\ell}{\delta(p)} \right) \right\}.$$

Then, $C^X(\cdot, \omega, p)$ and all $C_n^X(\cdot, \omega)$ are integrably bounded by $\alpha(\cdot, \omega)$. Now, it suffices to verify that for all $t \in T$,

$$\mathrm{Ls} C_n^X(t, \omega) \subseteq C^X(t, \omega, p), \quad \text{and } C^X(t, \omega, p) \subseteq \mathrm{Li} C_n^X(t, \omega).$$

First, let $x \in \mathrm{Ls} C_n^X(t, \omega)$. If $x = b(t, \omega, p)$, then $\{b_n(t, \omega) : n \geq 1\}$ converges to x and $b_n(t, \omega) \in C_n^X(\cdot, \omega)$ for all $n \geq 1$. Otherwise, there exist positive integers $n_1 < n_2 < n_3 < \cdots$ and for each k a point $x_k \in C_{n_k}^X(t, \omega)$ such that $\{x_k : k \geq 1\}$ converges to x. Obviously, $x_k \neq b_{n_k}(t, \omega)$ for all sufficiently large k, and $x \in X(t, \omega, p)$. If $x \notin C^X(t, \omega, p)$, there exists some $y \in B(t, \omega, p)$ such that $U(t, \omega, y) > U(t, \omega, x)$. By the continuity of $U(t, \omega, \cdot)$, y can be chosen so that $\langle p, y \rangle < \langle p, a(t, \omega) \rangle$. Since $\{X_{n_k}(t, \omega) : k \geq 1\}$ converges to $X(t, \omega, p)$ in the Hausdorff metric topology, there exists a sequence $\{y_k : k \geq 1\}$ such that $y_k \in X_{n_k}(t, \omega)$ for all $k \geq 1$ and $\{y_k : k \geq 1\}$ converges to y. By the inequality

$$\begin{aligned} |U(t, \omega, x) - \psi_{n_k}(t, \omega, x_k)| &< |U(t, \omega, x) - U(t, \omega, x_k)| \\ &+ |U(t, \omega, x_k) - \psi_{n_k}(t, \omega, x_k)|, \end{aligned}$$

the continuity of $U(t, \omega, \cdot)$ and the uniform convergence of $\psi_{n_k}(t, \omega, \cdot)$ to $U(t, \omega, \cdot)$, one concludes that

$$\psi_{n_k}(t, \omega, y_k) > \psi_{n_k}(t, \omega, x_k) \text{ and } \langle p, y_k \rangle < \langle p, a_{n_k}(t, \omega) \rangle$$

for sufficiently large k, which contradicts with the fact that $x_k \in C_{n_k}^X(t, \omega)$ for all $k \geq 1$. Hence, $x \in C^X(t, \omega, p)$. Now, let $d \in C^X(t, \omega, p)$. If $d = b(t, \omega, p)$, there is nothing to verify. Thus, $d \in \mathrm{Li} C_n^X(t, \omega)$. Assume $d < b(t, \omega, p)$. Similar to that in the proof of Theorem 4.3, one can show that $d \in \mathrm{Li} C_n^X(t, \omega)$.

To complete the proof, for each $\omega \in \Omega$, put

$$M(\omega) = \left\{ x \in \mathbb{R}_+^\ell : x \leq \int_T \alpha(\cdot, \omega) d\mu \right\}.$$

Clearly, $\overline{\int_T C_n^X(\cdot, \omega) d\mu}$ and $\int_T C^X(\cdot, \omega, p) d\mu$ are contained in the compact set $M(\omega)$. By Claim 3, $\left\{ \overline{\int_T C_n^X(\cdot, \omega) d\mu} : n \geq 1 \right\}$ converges to $\int_T C^X(\cdot, \omega, p) d\mu$ in $M(\omega)$ in the Hausdorff metric topology. It is well known that a nonempty compact-valued correspondence is lower measurable if and only if it is measurable when viewed as a single-valued function whose range space is the space of nonempty compact sets endowed with the Hausdorff metric topology. By Claim 2, $\int_T C^X(\cdot, \cdot, p) d\mu$ is lower measurable. □

Corollary 4.5. *Under* $(\mathbf{A}_1)$-$(\mathbf{A}_3)$*,* $\int_T C^X(\cdot,\cdot,\cdot)d\mu : (\Omega,\mathscr{F},\nu)\times(\Delta,\mathscr{B}(\Delta)) \to \mathscr{K}_0(\mathbb{R}^\ell_+)$ *is a jointly measurable function, where* $\mathscr{K}_0(\mathbb{R}^\ell_+)$ *is endowed with the Hausdorff metric topology.*

Proof. By Theorem 4.3, for every $\omega \in \Omega$, $\int_T C^X(\cdot,\omega,\cdot)d\mu : \Delta \to \mathscr{K}_0\left(\mathbb{R}^\ell_+\right)$ is continuous. Furthermore, by Theorem 4.4, for every $p \in \Delta$, the correspondence $\int_T C^X(\cdot,\cdot,p)d\mu : (\Omega,\Sigma,\nu) \rightrightarrows \mathbb{R}^\ell_+$ is lower measurable. Hence, for every $p \in \Delta$, $\int_T C^X(\cdot,\cdot,p)d\mu : (\Omega,\Sigma,\nu) \to \mathscr{K}_0(\mathbb{R}^\ell_+)$ is measurable. This means that $\int_T C^X(\cdot,\cdot,\cdot)d\mu : (\Omega,\mathscr{F},\nu)\times(\Delta,\mathscr{B}(\Delta)) \to \mathscr{K}_0(\mathbb{R}^\ell_+)$ is Carathéodory, and therefore is jointly measurable. □

5. The existence of a maximin REE

A *price system* of $\mathscr{E}$ is a measurable function $\pi : (\Omega,\mathscr{F},\nu) \to \Delta$. Let $\sigma(\pi)$ be the smallest sub-algebra of $\mathscr{F}$ such that π is measurable and let $\mathscr{G}_t = \mathscr{F}_t \vee \sigma(\pi)$. For each $\omega \in \Omega$, let $\mathscr{G}_t(\omega)$ denote the smallest element of $\mathscr{G}_t$ containing ω. Given $t \in T$, $\omega \in \Omega$ and a price system π, let $B^{REE}(t,\omega,\pi)$ be defined by

$$B^{REE}(t,\omega,\pi) = \left\{x \in (\mathbb{R}^\ell_+)^\Omega : x(\omega') \in B(t,\omega',\pi(\omega')) \text{ for all } \omega' \in \mathscr{G}_t(\omega)\right\}.$$

The *maximin utility* of each agent $t \in T$ with respect to $\mathscr{G}_t$ at an allocation $f : T\times\Omega \to \mathbb{R}^\ell_+$ in state $\omega \in \Omega$, denoted by $\underline{U}^{REE}(t,\omega,f(t,\cdot))$, is defined by

$$\underline{U}^{REE}(t,\omega,f(t,\cdot)) = \inf_{\omega' \in \mathscr{G}_t(\omega)} U(t,\omega',f(t,\omega')).$$

Comparing with $\underline{U}^{REE}(t,\cdot,\cdot)$, the function $U(t,\cdot,\cdot)$ is sometime called the *ex post* preferences or the *ex post* utility of agent t.

Remark 5.1. The maximin utility formation in the sense of REE was introduced by de Castro et al. in [11], where Ω is finite. In this case, for each $t \in T$, Π_t is a partition of Ω consisting of only finitely many elements and $\sigma(\pi)$ is generated by a partition Π_π also consisting of only finitely many elements. Thus, the σ-algebra $\mathscr{G}_t = \mathscr{F}_t \vee \sigma(\pi)$ is generated by the partition $\Pi_t \vee \Pi_\pi$. For each $\omega \in \Omega$, there exists a unique element $\mathscr{G}_t(\omega)$ in $\Pi_t \vee \Pi_\pi$ containing ω. It is clear that $\mathscr{G}_t(\omega)$ is the smallest element of $\mathscr{G}_t$ containing ω. Moreover, since $\mathscr{G}_t(\omega)$ is a finite set, $\underline{U}^{REE}(t,\omega,f(t,\cdot))$ is well-defined.

In our case, Ω is fairly general, particularly, can be infinite. The structure of $\sigma(\pi)$ can be complicated. If Ω is infinite, $\sigma(\pi)$ may not be generated by a partition. But, for each $\omega \in \Omega$, there always exists a (unique) smallest element in $\sigma(\pi)$ containing ω. This means that there also exists a (unique) smallest element $\mathscr{G}_t(\omega)$ in $\mathscr{G}_t$ containing ω. Since $\mathscr{G}_t(\omega)$ can be infinite, $\underline{U}^{REE}(t,\omega,f(t,\cdot))$ is allowed to take the value $-\infty$ if the above infimum does not exist. This adaptation will not affect the proof of Theorem 5.3 below.

Definition 5.2. Given a feasible allocation f and a price system π, the pair (f,π) is called a *maximin rational expectations equilibrium* (abbreviated as maximin REE) of $\mathscr{E}$ if $f(t,\omega) \in B(t,\omega,\pi(\omega))$ and $f(t,\cdot)$ maximizes $\underline{U}^{REE}(t,\omega,\cdot)$ on

$B^{REE}(t,\omega,\pi)$ for almost all $(t,\omega)\in T\times\Omega$. In this case, f is called a *maximin rational expectations allocation*, and the set of such allocations is denoted by $MREE(\mathscr{E})$.

Definition 5.2 indicates that at a maximin rational expectations allocation, except for some negligible sets of agents and states of nature, each individual maximizes his maximin utility conditioned on his private information and the information generated by the equilibrium prices, subject to the budget constraint. Recently, de Castro et al. [11] showed that $MREE(\mathscr{E})\neq\emptyset$ when Ω and T are finite. Our next theorem extends their result to a more general case.

Theorem 5.3. *Under* $(\mathbf{A}_1)$-$(\mathbf{A}_4)$, $MREE(\mathscr{E})\neq\emptyset$.

Proof. Consider the correspondence $Z:(\Omega,\mathscr{F},\nu)\times(\Delta,\mathscr{B}(\Delta))\rightrightarrows\mathbb{R}^\ell$ defined by

$$Z(\omega,p)=\int_T C^X(\cdot,\omega,p)d\mu-\int_T a(\cdot,\omega)d\mu.$$

By Proposition 4.2, Z is non-empty compact-valued. In addition, by Corollary 4.5 and $(\mathbf{A}_1)$, $Z:(\Omega,\mathscr{F},\nu)\times(\Delta,\mathscr{B}(\Delta))\to\mathscr{K}_0(\mathbb{R}^\ell)$ is jointly measurable. Define another correspondence $F:(\Omega,\mathscr{F},\nu)\rightrightarrows(\Delta,\mathscr{B}(\Delta))$ by

$$F(\omega)=\{p\in\Delta: Z(\omega,p)\cap\{0\}\neq\emptyset\}.$$

By Theorem 2 in [15, p.151], F is non-empty valued. Since

$$\mathrm{Gr}_F=Z^{-1}(\{0\}),\mathrm{Gr}_F\in\mathscr{F}\otimes\mathscr{B}(\Delta),$$

by Theorem 2.2, there exists a measurable function $\hat{\pi}:(\Omega,\mathscr{F},\nu)\to(\Delta,\mathscr{B}(\Delta))$ such that $\hat{\pi}(\omega)\in F(\omega)$ for all $\omega\in\Omega$. By the definition of Z, there exists an allocation f such that $f(t,\omega)\in C^X(t,\omega,\hat{\pi}(\omega))$ and $\int_T f(\cdot,\omega)d\mu=\int_T a(\cdot,\omega)d\mu$ for almost all $t\in T$ and all $\omega\in\Omega$. Proposition 3.1 implies that $\langle\hat{\pi}(\omega),f(t,\omega)\rangle\geq\langle\hat{\pi}(\omega),a(t,\omega)\rangle$ for almost all $t\in T$ and all $\omega\in\Omega$. It follows that $\langle\hat{\pi}(\omega),f(t,\omega)\rangle=\langle\hat{\pi}(\omega),a(t,\omega)\rangle$ for almost all $t\in T$ and all $\omega\in\Omega$. Thus, $f(t,\omega)\in B(t,\omega,\hat{\pi}(\omega))$ for almost all $t\in T$ and all $\omega\in\Omega$. For every $\omega\in\Omega$, define $T_\omega\subseteq T$ by

$$T_\omega=\{t\in T: f(t,\omega)\in B(t,\omega,\hat{\pi}(\omega))\cap C(t,\omega,\hat{\pi}(\omega))\}.$$

Then, $\mu(T_\omega)=\mu(T)$ for all $\omega\in\Omega$. Next, for every $\omega\in\Omega$ and every $t\in T\setminus T_\omega$, as $B(t,\omega,\hat{\pi}(\omega))\cap C(t,\omega,\hat{\pi}(\omega))\neq\emptyset$, one can pick a point

$$h(t,\omega)\in B(t,\omega,\hat{\pi}(\omega))\cap C(t,\omega,\hat{\pi}(\omega)),$$

and then define a function $\hat{f}:T\times\Omega\to\mathbb{R}^\ell_+$ such that

$$\hat{f}(t,\omega)=\begin{cases} f(t,\omega), & \text{if } t\in T_\omega;\\ h(t,\omega), & \text{if } t\in T\setminus T_\omega.\end{cases}$$

It is obvious that $\hat{f}(t,\omega) \in B(t,\omega,\hat{\pi}(\omega)) \cap C(t,\omega,\hat{\pi}(\omega))$ for all $(t,\omega) \in T \times \Omega$. Assume that there are an agent $t_0 \in T$, a state of nature $\omega_{t_0} \in \Omega$ and an element $y(t_0,\cdot) \in B^{REE}(t_0,\omega_{t_0},\hat{\pi})$ such that

$$\underline{U}^{REE}(t_0,\omega_{t_0},y(t_0,\cdot)) > \underline{U}^{REE}(t_0,\omega_{t_0},\hat{f}(t_0,\cdot)).$$

Then, one obtains

$$U(t_0,\omega'_{t_0},y(t_0,\omega'_{t_0})) > U(t_0,\omega'_{t_0},\hat{f}(t_0,\omega'_{t_0}))$$

for some $\omega'_{t_0} \in \mathscr{G}_{t_0}(\omega_{t_0})$, which contradicts with $\hat{f}(t_0,\omega'_{t_0}) \in C(t_0,\omega'_{t_0},\hat{\pi}(\omega'_{t_0}))$. This verifies that $(\hat{f},\hat{\pi})$ is a maximin rational expectations equilibrium of $\mathscr{E}$. □

6. Conclusion

The first application of maximin expected utility functions to the general equilibrium theory with differential information appeared in [10], where an existence theorem for a Walrasian equilibrium in an economy was established. However, their MEU formulation is in the ex ante sense, and REE notion was not considered.

In our paper, an existence theorem on a maximin rational expectations equilibrium (maximin REE) for an exchange differential information economy is proved. Comparing with the existence result on maximin REE in [11], our theorem applies to a more general economic model with an arbitrary finite measure space of agents and an arbitrary complete probability measure space as the space of states of nature, while the later applies only to an economic model which has finitely many agents and finitely many states of nature. Assumptions in our paper are similar to those in [11], except the joint measurability and continuity of utility functions, and the joint measurability of the initial endowment function. The proof techniques in this paper are quite different from those in [11]. Since there are only finitely many agents and states of nature in the model considered in [11], neither measurability nor continuity of utility functions and the initial endowment function plays any role in the proof of the existence of a maximin REE. Instead, the existence of a competitive equilibrium for complete information economies is applied. In contrast, both measurability and continuity of utility functions and the initial endowment function play key roles in this paper. To establish the existence theorem, techniques in [8] are adopted, measurability and continuity of the aggregate preferred correspondence are investigated. However, for special cases, the techniques can be simplified. For instance, if there are finitely many states of nature, one can still apply the approach employed in [11] and obtains an existence theorem. On the other hand, if there are finitely many agents, then one can show that the demand of each agent is $\mathscr{F} \otimes \mathscr{B}(\Delta)$-measurable and so is the aggregate demand. Then, an approach similar to that in the proof of Theorem 5.3 can be applied to establish the existence theorem. Further, since the space of states of nature in our model is an abstract probability space, our existence theorem does not depend on the dimension of the space of states of nature.

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