Bubbly Markov Equilibria*

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March 1, 2013

Preliminary version – comments welcome

Abstract

Bubbly Markov Equilibria (BME) are recursive equilibria on the natural state space which admit a non-trivial bubble. The present paper studies the existence and properties of BME in a stylized overlapping generations (OLG) model with capital accumulation and random production shocks. BME obtain as fixed points of an operator derived from the consumers' Euler equations. Our main result establishes sufficient conditions under which BME exist extending results well-known from deterministic models. We also study the dynamics along the BME and show that the state process converges to a stable set giving rise to a Stationary BME where all equilibrium variables possess invariant probability distributions.

JEL classification: C62, D51, E32

Keywords: Asset Bubbles, Stochastic OLG, Bubbly Markov Equilibria.

*We would like to thank Caren Söhner for many discussions and helpful comments.

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Introduction

A bubble is an intrinsically worthless asset which trades at a positive price such as fiat money, governmental debt, or a bond that never pays any dividends. The theoretical insight that the emergence of bubbles is compatible with fully optimal and rational behavior of investors has triggered a broad literature aiming to identify the conditions under which bubbles emerge and characterize their properties and implications. Despite this interest, however, the vast majority of these studies adopts a deterministic framework. The contribution of the present paper is to study asset bubbles in a stochastic setting with random production shocks which to the best of our knowledge has not been done in the literature.

A natural framework to study asset bubbles also to be adopted in this paper is the Diamond overlapping generations model with production and endogenous capital accumulation. For this class of economies, Tirole (1985) showed that bubbles emerge quite naturally in the presence of dynamic inefficiencies due to an overaccumulation of capital. Moreover, a unique recursive equilibrium where the bubble is a time-independent, nontrivial function on the natural state space exists in his model. In the present paper, this type of equilibrium will be referred to as a Bubbly Markov Equilibrium (BME). In Tirole's deterministic model, a BME corresponds to the saddle-path of his two-dimensional dynamical system.

Tirole's model has been extended in various directions, e.g., to include monetary bubbles as in Weil (1987) or, more recently, in Michel & Wigniolle (2003) and Gali (2013), and to embody financial frictions as in Kunieda (2008), and many others. Common to all these studies is that the production process remains deterministic. Starting with the early contributions of Wang (1993, 1994), however, many studies of overlapping generations economies with production adopt a stochastic setup where production is subjected to exogenous random shocks. A natural question then is how the results on the existence and properties of asset bubbles for deterministic OLG economies carry over to a stochastic setting. Surprisingly, this questions is to the best of our knowledge still unresolved and is therefore the theme of the present paper. Conceptually, we focus on recursive equilibria (RE) where all equilibrium variables are time-invariant functions of capital and the production shock. In Kübler & Polemarchakis (2004), such equilibria are referred to as Markov Equilibria (ME). In the present paper, we extend this notion by referring to ME which admit a bubble as Bubbly Markov Equilibria (BME).

Recent studies of (bubbleless) ME in stochastic overlapping generations models with production may be found in Wang (1993, 1994), Morand & Reffett (2007), McGovern, Morand & Reffett (2012), or Hillebrand (2012b). In these papers, powerful results from functional analysis developed and applied, e.g., in Coleman (1991, 2000) , or Greenwood & Huffman (1995) are used to compute ME as fixed points of an operator derived from the consumers' Euler equations. Fortunately, it turns out that the computation of BME is accessible to these techniques as well. Using this approach, a first objective of this paper is to establish a general existence result of BME. The employed economy is similar to Wang (1993) in that production shocks are i.i.d. and preferences are additive over lifetime consumption. Our existence result requires a set of additional restrictions which reduce to the ones in Tirole (1985) for the special case with degenerate shocks. Based on these findings, a second objective is to characterize the long-run dynamics along the BME and obtain conditions under which a Stationary BME (SBME) corresponding to an invariant probability distribution on the endogenous state space exists. Our second main result establishes conditions under which a SBME exists.

The paper is organized as follows. Section 1 introduces the model. The existence and construction of BME are the theme of Section 2. Section 3 studies the dynamics along the BME and the existence of stable sets. Section 4 concludes, proofs for all results are relegated to the Mathematical Appendix.

1 The Model

1.1 Production sector

A single firm operates a linear homogeneous technology to produce an all-purpose output commodity using capital and labor as inputs. In addition, production in period t is subjected to an exogenous random production shock $\varepsilon_t \in \mathcal{E} \subset \mathbb{R}_{++}$. Per-capita output y_t in period t is produced from capital intensity $k_t \geq 0$ and the shock according to the intensive form technology $f : \mathbb{R}_+ \times \mathcal{E} \longrightarrow \mathbb{R}_+$, $y_t = f(k_t; \varepsilon_t) := \varepsilon_t g(k_t)$. Throughout, the following restrictions on technology are imposed.

Assumption 1
The map $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is C^2 with derivatives $g'' < 0 < g'$ and $\lim_{k\to 0} g'(k) = \infty$.

Denote by $E_\phi(z) := |\frac{z\phi'(z)}{\phi(z)}|$ the elasticity of a differentiable function $\phi \neq 0$. Below we will occasionally impose the following additional restrictions:

$$(T1)\ E_{g'}(k) \leq 1\ \forall k > 0 \qquad (T2)\ E_g(k) + |E_{g'}(k)| \leq 1\ \forall k > 0.$$

Restriction (T1) is equivalent to $k \mapsto k\mathcal{R}(k;\varepsilon)$ being non-decreasing and is often imposed in the literature, cf. Wang (1993) and others. Condition (T2) is equivalent to $k \mapsto E_g(k)$ being non-decreasing. The latter holds for a large class of technologies including Cobb-Douglas ($g(k) = k^\alpha$) and CES production ($g(k) = [1 - a + ak^\varrho]^{\frac{1}{\varrho}}$, $0 < \varrho < 1$ in which case $E_g(k) = \frac{a}{a+(1-a)k^{-\varrho}}$). Evidently, (T2) is stronger and implies (T1).

As in Wang (1993), shocks are i.i.d. over time with (marginal) distribution ν supported on $\mathcal{E}$ where $\{\varepsilon_{\min}, \varepsilon_{\max}\} \subset \mathcal{E} \subset [\varepsilon_{\min}, \varepsilon_{\max}] \subset \mathbb{R}_{++}$. The process $\{\varepsilon_t\}_{t\geq 0}$ induces a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that each ε_t is $\mathcal{F}_t$-measurable. Throughout, the notion of an adapted process $\{\xi_t\}_{t\geq 0}$ refers to this filtration and implies that each ξ_t can depend only on random variables ε_n, $n \leq t$. $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t]$ is the conditional expectations operator.

At equilibrium, labor supply will be constant and normalized to unity. Market clearing and profit maximizing behavior imply that the equilibrium wage w_t and capital return r_t are determined as

$$w_t = \mathcal{W}(k_t;\varepsilon_t) := \varepsilon_t[g(k) - kg'(k)] \tag{1a}$$

$$r_t = \mathcal{R}(k_t;\varepsilon_t) := \varepsilon_t\, g'(k). \tag{1b}$$

1.2 The Bubble

The bubbly asset can be purchased in period t at a price of unity and offers a random return $r^\star_{t+1} > 0$ during the next period. Let $b_t \geq 0$ denote the (value of the) bubble at time $t \geq 0$. As a specific interpretation, one may think of a one-period lived bond in which case b_t is the number of bonds issued at time t. Other interpretations such as real government debt or real money balances apply as well. The return process $\{r^\star_t\}_{t\geq 0}$ determines the risk to which bubbly investments are subjected. The bubble is completely self-financing in the sense that the payment obligation incurred during period t is fully financed by issuing new bubbly assets at time $t+1$. Thus, given a return process $\{r^\star_t\}_{t\geq 0}$ and an initial value $b_0 \geq 0$, the bubble evolves according to the law of motion

$$b_{t+1} = r^\star_{t+1} b_t, \quad t \geq 0. \tag{2}$$

1.3 Consumption Sector

The consumption sector consists of overlapping generations of two-period lived consumers. Young consumers earn income from supplying one unit of labor inelastically to the labor market while old consumers earn capital income. For simplicity, there is no population growth. A young consumer who takes decisions in period t observes her labor income $w_t > 0$ and faces returns r_{t+1} on capital and $r^\star_{t+1}$ on the bubbly asset which enter the decision problem as given random variables. The consumer chooses her investment b in bubbles and capital investment s to maximize expected lifetime utility. Assuming an additive von-Neumann Morgenstern utility function $U(c^y, c^o) = u(c^y) + v(c^o)$ over lifetime consumption, the decision problem reads:

$$\max_{b,s}\Big\{u(w_t - b - s) + \mathbb{E}_t\left[v\big(r^\star_{t+1}\, b + r_{t+1}\, s\big)\right] \Big|\, s \geq 0, b + s \leq w_t\Big\}. \tag{3}$$

Note that no short-selling constraints on bubbles are imposed at the individual level. Throughout, we impose the following restrictions on the utility functions:

Assumption 2
Both u and v are C^2 with derivatives $z'' < 0 < z'$ and $\lim_{c\to 0} z'(c) = \infty$, $z \in \{u, v\}$.

The following additional restrictions will occasionally be used in the sequel:

$$(U1)\; E_{v'}(c) \le 1\; \forall c > 0 \quad (U2)\; E_{v'}(c) \equiv \theta\; \forall c > 0 \quad (U3)\; v(c) \equiv \gamma u(c),\; \gamma > 0.$$

Observe that restriction (U2) of constant relative risk-aversion implies the form

$$v(c) = \begin{cases} \frac{\gamma}{1-\theta}\left(c^{1-\theta} - 1\right) & \text{if } \theta > 0, \theta \neq 1 \\ \gamma \log c & \text{if } \theta = 1 \end{cases}, \gamma > 0.$$

The savings decision s_t derived from (3) determines the per-capita capital stock k_{t+1} of the following period. Exploiting this and (2), one obtains the following Euler equations associated with the consumer's decision problem (3) which must hold at equilibrium:

$$u'(w_t - b_t - k_{t+1}) = \mathbb{E}_t\left[\mathcal{R}(k_{t+1};\cdot)v'\big(b_{t+1} + k_{t+1}\mathcal{R}(k_{t+1};\cdot)\big)\right] \tag{4a}$$
$$u'(w_t - b_t - k_{t+1})b_t = \mathbb{E}_t\left[b_{t+1}v'\big(b_{t+1} + k_{t+1}\mathcal{R}(k_{t+1};\cdot)\big)\right]. \tag{4b}$$

1.4 Equilibrium

The economy is $\mathscr{E} = \langle u, v, f, \nu\rangle$. The most general notion of equilibrium is that of a sequential (or sequence of markets) equilibrium (SE) to be introduced in the following

Definition 1
Given initial values $k_0 > 0$, $\varepsilon_0 \in \mathcal{E}$, and $b_0 \ge 0$, a SE for $\mathscr{E}$ is an adapted stochastic process $\{w_t, r_t, r_t^\star, b_t, k_{t+1}\}_{t\ge 0}$ which satisfies (1a,b), (2), and (4a,b) for all $t \ge 0$.

In this paper, we focus on a particular class of equilibria where all equilibrium variables at time t are determined by time-invariant functions of some state variable x_t with values in the state space $\mathbb{X}$. In the literature, such equilibria are called *Recursive Equilibria (RE)*. We confine ourselves to a class of recursive equilibria where the state variable is $x_t = (k_t, \varepsilon_t)$. The underlying state space is called the natural state space. Note that the factor price mappings $\mathcal{W}$ and $\mathcal{R}$ from (1a,b) already satisfy this property. Following the terminology of Kübler & Polemarchakis (2004), RE on the natural state space are called *Markov equilibria* (ME). Extending this terminology, we will refer to ME which admit a bubble as *Bubbly Markov equilibria* (BME). If the bubble is trivial, i.e., $b_t \equiv 0$, a BME reduces to a ME in the usual sense as studied, e.g., in Wang (1993). Formally, we have the following

Definition 2 (Bubbly Markov Equilibrium (BME))
A SE of $\mathscr{E}$ is called a BME if there exists a Borel set $\mathbb{X} \subset \mathbb{R}_{++} \times \mathcal{E}$ and measurable mappings $\mathcal{K}^E : \mathbb{X} \longrightarrow \mathbb{R}_{++}$ and $\mathcal{B}^E : \mathbb{X} \longrightarrow \mathbb{R}_+$ such that $k_{t+1} = \mathcal{K}^E(k_t, \varepsilon_t)$ and $b_t = \mathcal{B}^E(k_t, \varepsilon_t)$ for all $t \ge 0$.

A BME is called non-trivial if $\mathcal{B}^E > 0$. At any non-trivial BME the bubble return is given by the map $\mathcal{R}^\star : \mathbb{X} \times \mathcal{E} \longrightarrow \mathbb{R}_{++}$,

$$r_{t+1}^\star = \mathcal{R}^\star(k_t, \varepsilon_t, \varepsilon_{t+1}) := \frac{\mathcal{B}^E(\mathcal{K}^E(k_t, \varepsilon_t), \varepsilon_{t+1})}{\mathcal{B}^E(k_t, \varepsilon_t)}, \quad t \geq 0. \tag{5}$$

If the BME is trivial, however, i.e., $\mathcal{B}^E \equiv 0$, then the supporting return process is clearly not uniquely defined, i.e., any process $\{r_{t+1}^\star\}_{t\geq 0}$ which satisfies

$$u'(w_t - k_{t+1}) = \mathbb{E}_\nu[r_{t+1}^\star v'(k_{t+1}\mathcal{R}(k_{t+1}, \cdot))] = 0$$

for all $t \geq 0$ supports the equilibrium. A particular choice would be to set $r_{t+1}^\star = r_{t+1} = \mathcal{R}(k_{t+1}, \varepsilon_{t+1})$ for all $t \geq 0$.

2 Existence of Bubbly Markov Equilibria

2.1 The trivial equilibrium

Under the additional restriction (T1), it is shown in Wang (1993) that $\mathscr{E}$ possesses a unique trivial ME. The latter is associated with the mappings $\mathcal{B}_0^E \equiv 0$ and $\mathcal{K}_0^E = \mathcal{K}_0 \circ \mathcal{W}$ where $\mathcal{K}_0 : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ determines the unique solution $k = \mathcal{K}_0(w)$ to the implicit condition

$$G_0(k, w) := u'(w - k) - \mathbb{E}_\nu[\mathcal{R}(k, \cdot)v'(k\mathcal{R}(k, \cdot))] = 0. \tag{6}$$

Note that the implicit function theorem implies that $\mathcal{K}_0$ is C^1, strictly increasing, and $0 < \mathcal{K}_0(w) < w$ for all $w > 0$. The capital process along the trivial BME evolves as

$$k_{t+1} = \mathcal{K}_0^E(k_t, \varepsilon_t) = \mathcal{K}_0 \circ \mathcal{W}(k_t, \varepsilon_t). \tag{7}$$

Note that $\mathcal{K}_0^E$ is strictly increasing and C^1 in its first argument. Similar to Tirole (1985), the properties of the trivial equilibrium are key to obtain conditions under which non-trivial BME exists. In anticipation of this result, the remainder imposes the following

Assumption 3
The map $\mathcal{K}_0^E(\cdot, \varepsilon_{\max})$ from (7) has a unique non-trivial fixed point $k_{\max}$ which is stable.

Uniqueness and stability of $k_{\max}$ imply $k \leq \mathcal{K}_0(k, \varepsilon_{\max})$ for all $k \in \mathbb{K} :=]0, k_{\max}]$. Thus, $k_t \in \mathbb{K}$ implies $k_{t+1} = \mathcal{K}_0(k_t, \varepsilon_t) \leq \mathcal{K}_0(k_t, \varepsilon_{\max}) \leq k_{\max}$. Assuming $k_0 \in \mathbb{K}$ permits the state space in Definition 2 to be chosen as $\mathbb{X} := \mathbb{K} \times \mathcal{E}$ along the trivial BME. Below we show that this choice extends to the non-trivial case. Thus, a first consequence of Assumption 3 is that it will allow us to obtain a bounded state space. Below we will see that a second property of the fixed point $k_{\max}$ is sufficient for non-trivial BME to exist.

2.2 Constructing an operator

In the sequel, we assume that Assumptions 1, 2, and 3 hold. We seek to obtain conditions under which a non-trivial BME exists. For this purpose, we will use the Euler equations (4a,b) to construct an operator whose non-trivial fixed points give rise to non-trivial BME. Let $k_{\max}$ and $\mathbb{K}$ be defined as in Assumption 3 and the previous subsection. We also define $w_{\max} := \mathcal{W}(k_{\max}, \varepsilon_{\max})$ and $\mathbb{W} :=]0, w_{\max}] = \mathcal{W}(\mathbb{K} \times \mathcal{E})$.

As the current state $x_t = (k_t, \varepsilon_t)$ enters the Euler equations (4a,b) only through the wage $w_t = \mathcal{W}(x_t)$, we conjecture that, similar to the trivial equilibrium, the mappings $\mathcal{K}^E$ and $\mathcal{B}^E$ from Definition 2 can be written as $\mathcal{K}^E = \mathcal{K} \circ \mathcal{W}$ and $\mathcal{B}^E = \mathcal{B} \circ \mathcal{W}$ where $\mathcal{K} : \mathbb{W} \longrightarrow \mathbb{K}$ and $\mathcal{B} : \mathbb{W} \longrightarrow \mathbb{R}_+$. Based on this insight, our goal in this section is to determine a function $\mathcal{B} : \mathbb{W} \longrightarrow \mathbb{R}_{++}$ such that $b_t = \mathcal{B}(w_t)$ for all times $t \geq 0$ consistent with (4a,b). To establish existence of such a solution, we will construct an operator on a suitable function space whose fixed points satisfy this property.

Define the function space

$$\mathscr{G} := \left\{ \psi : \mathbb{W} \longrightarrow \mathbb{R}_+ \;\middle|\; \begin{array}{l} \psi \text{ is continuous }, \psi(w) \leq w \text{ for all } w > 0 \\ w \mapsto \psi(w) \text{ is weakly increasing} \\ w \mapsto w - \psi(w) \text{ is weakly increasing} \end{array} \right\}. \tag{8}$$

The space $\mathscr{G}$ is endowed with the partial order $\psi_1 \geq \psi_2$ ($\psi_1 > \psi_2$) iff $\psi_1(w) \geq \psi_2(w)$ ($\psi_1(w) > \psi_2(w)$) for all $w \in \mathbb{W}$. Our goal is to construct an operator $T : \mathscr{G} \longrightarrow \mathscr{G}$ whose fixed points define BME. The key ingredients to this approach are the Euler equations (4a,b). The idea is as follows: Suppose that the bubble during the following period is determined by some function ψ of next period's wage. Then, for any given value $w \in \mathbb{W}$, the current bubble b and capital investment k must solve the Euler equations (4a,b). Given $\psi \in \mathscr{G}$, let

$$H_1(k, b; w, \psi) \;:=\; u'(w - b - k) - \mathbb{E}_\nu \big[\mathcal{R}(k; \cdot) v'\big(\psi(\mathcal{W}(k; \cdot)) + k\mathcal{R}(k; \cdot)\big)\big] \tag{9a}$$

$$H_2(k, b; w, \psi) \;:=\; u'(w - b - k)b - \mathbb{E}_\nu \big[\psi(\mathcal{W}(k; \cdot)) v'\big(\psi(\mathcal{W}(k; \cdot)) + k\mathcal{R}(k; \cdot)\big)\big] \tag{9b}$$

which are defined for all $w \in \mathbb{W}$, $k \in \mathbb{K}$, and $b \geq 0$ such that $k + b < w$. Existence and uniqueness of this solution is ensured by the next result.

Lemma 1
In addition to Assumptions 1–3, let (U1) and (T1) be satisfied and $\psi \in \mathscr{G}$ be arbitrary. Then, for each $w \in \mathbb{W}$ there exist unique values $\tilde{b} \geq 0$ and $\tilde{k} \in \mathbb{K}$ satisfying $\tilde{k} + \tilde{b} < w$ such that $H_1(\tilde{k}, \tilde{b}; w, \psi) = H_2(\tilde{k}, \tilde{b}; w, \psi) = 0$.

Lemma 1 permits to define functions $\mathcal{K}_\psi : \mathbb{W} \longrightarrow \mathbb{R}_{++}$ and $\mathcal{B}_\psi : \mathbb{W} \longrightarrow \mathbb{R}_+$ which determine the unique solution to $H_1(\tilde{k}, \tilde{b}; w, \psi) = H_2(\tilde{k}, \tilde{b}; w, \psi) = 0$ for each $w \in \mathbb{W}$. This result permits to define an operator T on $\mathscr{G}$ which associates with any function $\psi \in \mathscr{G}$ the new function $T(\psi) := \mathcal{B}_\psi$.

Lemma 2
In addition to Assumptions 1–3, let (U1), (U2), and (T1) hold. Then $T : \mathscr{G} \longrightarrow \mathscr{G}$. *Furthermore,* $\psi > 0$ *implies* $T(\psi) > 0$, $\psi = 0$ *implies* $T(\psi) = 0$, *and* $T(\mathrm{id}_{\mathbb{W}}) < \mathrm{id}_{\mathbb{W}}$. *In addition,* $\mathcal{K}_\psi$ *is continuous and satisfies* $\mathcal{K}_\psi \leq \mathcal{K}_0$ *for all* $\psi \in \mathscr{G}$.

The last result permits the state space to be chosen as $\mathbb{X} = \mathbb{K} \times \mathcal{E}$ along a BME as well.

2.3 Properties of the operator

We seek to establish existence of a non-trivial fixed point $\psi \in \mathscr{G}$ of T, i.e., $\psi > 0$ and $T(\psi) = \psi$. As shown in Lemma 2, the trivial solution $\psi = 0$ is always a fixed point, so an abstract existence result will not help. Instead, we will explicitly construct non-trivial fixed points as pointwise limits of function sequences. This approach also opens up the possibility to compute fixed points numerically. The method follows the one in Greenwood & Huffman (1995), see also the papers by Morand & Reffett (2003, 2007).

Our first task will be to establish additional properties of T such as monotonicity, etc. In this regard, the main obstacle is that the methods from differential calculus including the implicit function theorem are not available for functions in $\mathscr{G}$. To remedy this problem, we will temporarily restrict ourselves (respectively T) to the smaller set

$$\mathscr{G}' := \{\psi \in \mathscr{G} \mid \psi \text{ is } C^1\} \tag{10}$$

of continuously differentiable functions in $\mathscr{G}$. We denote by T' the restriction of T to the smaller set $\mathscr{G}'$. It will turn out that establishing the afore-mentioned properties for T' is sufficient to apply the construction principle below. The following result shows that T' preserves continuous differentiability.

Lemma 3
Under the hypotheses of Lemma 2, $T' : \mathscr{G}' \longrightarrow \mathscr{G}'$.

The following result establishes the desired monotonicity of T' which is key to construct fixed points below.

Lemma 4
In addition to Assumptions 1 – 3, let (U1) and (T1) hold. Then, T' *is monotonically increasing, i.e.,* $\psi_1 \geq \psi_0$ *implies* $T'(\psi_1) \geq T'(\psi_0)$ *and* $\psi_1 > \psi_0$ *implies* $T'(\psi_1) > T'(\psi_0)$.

Observe that $\mathscr{G}'$ still contains the trivial solution $\psi \equiv 0$. Thus, we will need conditions under which the operator T' 'lifts' or increases functions close to zero. For this purpose, the following condition is required which essentially extends the one from Tirole (1985) to the present stochastic setting.

Assumption 4
The largest fixed point from Assumption 3 satisfies $\mathcal{R}(k_{\max}, \varepsilon_{\max}) < 1$.

The merits of Assumption 4 is that we can establish the existence of a lower bound in the following result.

Lemma 5
In addition to Assumptions 1–3, let (U1)–(U3) and (T2) hold. For $\delta \in]0,1[$*, define* $\psi_\delta(w) := \delta w$, $w \in \mathbb{W}$*. If Assumption 4 holds, there exists a* $\underline{\delta} \in]0,1[$ *such that* $T'\psi_{\underline{\delta}} > \psi_{\underline{\delta}}$.

2.4 Constructing non-trivial fixed points

Based on the previous result, we are now in a position to explicitly construct a non-trivial fixed point. In fact, our approach delivers two such solutions which coincide when the fixed point is unique.

As a first step, define a sequence of functions $(\psi_n)_{n\geq 0}$ by setting $\psi_0 := \psi_{\underline{\delta}}$ and $\psi_{n+1} := T'(\psi_n)$. By Lemma 3, this sequence is well-defined, i.e., $\psi_n \in \mathscr{G}'$ for all $n \geq 0$. Further, by virtue of Lemma 5 $\psi_1 > \psi_0$ which implies $\psi_{n+1} > \psi_n$ for all $n \geq 0$ by Lemma 4. For $w \in \mathbb{W}$, let

$$\underline{\mathcal{B}}(w) := \lim_{n\to\infty} \psi_n(w)$$

which is well-defined as the sequence $(\psi_n(w))_{n\geq 1}$ is increasing and bounded by w. We show that the limiting function satisfies $\underline{\mathcal{B}} \in \mathscr{G}$. For each $n \geq 1$, $w \mapsto \psi_n(w)$ and $w \mapsto w - \psi_n(w)$, $w \in \mathbb{W}$ are monotonically increasing. Let $0 < w_1 < w_2 \leq w_{\max}$ be arbitrary. Then, the inequalities $\psi_n(w_1) \leq \psi_n(w_2)$ and $w_1 - \psi_n(w_1) \leq w_2 - \psi_n(w_2)$ being true for all $n \geq 1$ also hold in the limit and imply that $\underline{\mathcal{B}}$ inherits the previous monotonicity properties. Using an argument developed and proved in Morand & Reffett (2003, p.1369), these properties also imply continuity of $\underline{\mathcal{B}}$. Finally, $\underline{\mathcal{B}} > \psi_{\underline{\delta}} > 0$ and, as shown below, $\underline{\mathcal{B}}(w) < w$ for all $w \in \mathbb{W}$. Thus, $\underline{\mathcal{B}} \in \mathscr{G}$.

As a second step, repeat the previous construction by setting $\widetilde{\psi}_0 := \mathrm{id}_{\mathbb{W}}$ and $\widetilde{\psi}_{n+1} := T'(\widetilde{\psi}_n)$. Note that $T'(\widetilde{\psi}_0) < \widetilde{\psi}_0$. Analogous reasonings give rise to the continuous function

$$\overline{\mathcal{B}}(w) := \lim_{n\to\infty} \psi_n(w).$$

Standard arguments imply the following

Lemma 6
Both functions $\underline{\mathcal{B}}$ *and* $\overline{\mathcal{B}}$ *constructed above are fixed points of* T *and satisfy* $0 < \psi_{\underline{\delta}} < \underline{\mathcal{B}} \leq \overline{\mathcal{B}} < \mathrm{id}_{\mathbb{W}}$.

The previous results lead to the following

Lemma 7
Let $\mathcal{B} \in \mathscr{G}$ *be a non-trivial fixed point of* T *and* $\mathcal{K}_{\mathcal{B}}$ *the associated capital function. Then,* $\mathcal{K}^E := \mathcal{K}_{\mathcal{B}} \circ \mathcal{W}$ *and* $\mathcal{B}^E := \mathcal{B} \circ \mathcal{W}$ *define a non-trivial BME for* $\mathscr{E}$ *on* $\mathbb{X} = \mathbb{K} \times \mathcal{E}$.

Combining the construction devised in this section with Lemmata 6 and 7, the main result of this section can be stated in the following

Theorem 1
Under Assumptions 1–4, and the additional restrictions (U1)–(U3) and (T2), the economy $\mathscr{E}$ has at least one non-trivial BME.

3 Dynamics and Stationary BME

3.1 Dynamics along the BME

Let $\mathcal{B} \in \mathscr{G}$ be a non-trivial fixed point of T computed as in the previous section and let $\mathcal{K} := \mathcal{K}_{\mathcal{B}}$ denote the induced capital function. In this section, our goal is to study the equilibrium dynamics along the induced BME. Specifically, we would like to know whether the bubble is persistent in the sense that $\lim_{t\to\infty} b_t > 0$ $\mathbb{P}$–a.s. As $\mathcal{B} > 0$, the bubble is non-persistent if and only if $\lim_{t\to\infty} w_t = 0$ along the BME on a set of shock sequences of positive measure. Our argument will show that the latter is excluded by the equilibrium dynamical system.

Given $w_0 := \mathcal{W}(k_0, \varepsilon_0) > 0$, set $b_0 = \mathcal{B}(w_0)$. The dynamics along the BME read

$$k_{t+1} = \mathcal{K}^E(k_t, \varepsilon_t) := \mathcal{K}(\mathcal{W}(k_t, \varepsilon_t)) \tag{11a}$$
$$b_t = \mathcal{B}^E(k_t, \varepsilon_t) := \mathcal{B}(\mathcal{W}(k_t, \varepsilon_t)). \tag{11b}$$

One observes that the equilibrium dynamics are essentially governed by the capital dynamics (11a) while the equilibrium bubble in (11b) essentially mirrors the induced equilibrium wage process. Note that (11a) is structurally of the same type as the bubbleless case studied in Wang (1993).

3.2 Self-supporting sets

We are interested in characterizing the long-run statistical behavior of (11a) in a fashion similar to Wang (1993). The following result establishes that the capital dynamics are bounded away from zero below and bounded above by the value $k_{\max}$ from Assumption 3 under all shocks.

Lemma 8
Under the hypotheses of Theorem 1, the mappings $\mathcal{K}^E(\cdot, \varepsilon)$ defined in (11a) satisfy the following for each $\varepsilon \in \mathcal{E}$:

(i) $\mathcal{K}^E(k_{\max}, \varepsilon) < k_{\max}$

(ii) $\mathcal{K}^E(k, \varepsilon) > k$ *for k sufficiently small.*

Proof: Let $\varepsilon \in \mathcal{E}$ be arbitrary but fixed. Claim (i) follows from $\mathcal{K}^E < \mathcal{K}_0^E$ and Assumption 3 which implies

$$\mathcal{K}^E(k_{\max}, \varepsilon) \leq \mathcal{K}^E(k_{\max}, \varepsilon_{\max}) < \mathcal{K}_0^E(k_{\max}, \varepsilon_{\max}) = k_{\max}.$$

To prove (ii), suppose first that $w_{\min} := \lim_{k\to 0} \mathcal{W}(k, \varepsilon) > 0$. Then,

$$\lim_{k\to 0}(\mathcal{K}^E(k, \varepsilon) - k) = \mathcal{K}_{\mathcal{B}}(w_{\min}) > 0$$

from which the claim follows. Second, suppose $w_{\min} = 0$. From the Euler equations (4a,b) and (5), we conclude that for each $k > 0$ there exists $\varepsilon' \in \mathcal{E}$ such that

$$\frac{\mathcal{B}^E(\mathcal{K}^E(k, \varepsilon), \varepsilon')}{\mathcal{B}^E(k, \varepsilon)} \geq \mathcal{R}(k, \varepsilon').$$

As $\lim_{k\to 0} \mathcal{R}(k, \varepsilon') = \infty$, we infer that for at least one $\varepsilon' \in \mathcal{E}$, it holds that

$$\lim_{k\to 0} \frac{\mathcal{B}^E(\mathcal{K}^E(k, \varepsilon), \varepsilon')}{\mathcal{B}^E(k, \varepsilon)} = \infty.$$

As shown in the previous section, $\underline{\delta} w < \mathcal{B}(w) < w$ for all $w \in \mathbb{W}$. Thus,

$$\lim_{k\to 0} \frac{\mathcal{W}(\mathcal{K}^E(k, \varepsilon), \varepsilon')}{\underline{\delta}\mathcal{W}(k, \varepsilon)} = \frac{\varepsilon'}{\varepsilon\underline{\delta}} \lim_{k\to 0} \frac{\mathcal{W}(\mathcal{K}^E(k, \varepsilon), \varepsilon)}{\mathcal{W}(k, \varepsilon)} \geq \lim_{k\to 0} \frac{\mathcal{B}^E(\mathcal{K}^E(k, \varepsilon), \varepsilon')}{\mathcal{B}^E(k, \varepsilon)} = \infty$$

from which we conclude that

$$\lim_{k\to 0} \frac{\mathcal{W}(\mathcal{K}^E(k, \varepsilon), \varepsilon)}{\mathcal{W}(k, \varepsilon)} = \infty.$$

Thus, by strict monotonicity of $\mathcal{W}(\cdot, \varepsilon)$, $\mathcal{K}^E(k, \varepsilon) > k$ for k sufficiently small. ■

Using this result, let $\bar{k}_{\min}$ be the smallest fixed point of $\mathcal{K}^E(\cdot, \varepsilon_{\min})$ and $\bar{k}_{\max}$ be the largest fixed point of $\mathcal{K}^E(\cdot, \varepsilon_{\max})$. Then, the interval $\bar{\mathbb{K}} := [\bar{k}_{\min}, \bar{k}_{\max}]$ is self-supporting for the family $(\mathcal{K}^E(\cdot, \varepsilon))_{\varepsilon\in\mathcal{E}}$ in the sense that $k \in \bar{\mathbb{K}}$ implies $\mathcal{K}^E(k, \varepsilon) \in \bar{\mathbb{K}}$ for all $\varepsilon \in \mathcal{E}$. Further, the set $\bar{\mathbb{K}}$ is attracting in the sense that for any $k_0 \in \mathbb{K}$, the process $\{k_t\}_{t\geq 0}$ generated by (11a) converges to $\bar{\mathbb{K}}$. We formally state this insight as

Lemma 9
The set $\bar{\mathbb{K}} = [\bar{k}_{\min}, \bar{k}_{\max}] \subset \mathbb{R}_{++}$ is self-supporting for the family $(\mathcal{K}^E(\cdot, \varepsilon))_{\varepsilon\in\mathcal{E}}$.

At this point, however, note that the set $\bar{\mathbb{K}}$ may not be ergodic for the dynamics (11a). That is, there may exist proper subsets which are self-supporting as well. The latter would be excluded if the family $(\mathcal{K}^E(\cdot, \varepsilon))_{\varepsilon\in\mathcal{E}}$ could be shown to possess a stable fixed-point configuration. In the latter case, the results from Brock & Mirman (1972) can be applied to show that the set $\bar{\mathbb{K}}$ is ergodic and supports a unique invariant probability distribution, see also Wang (1993)).

3.3 Persistence of bubbles

Defining $\bar{w}_{\min} := \mathcal{W}(\bar{k}_{\min}, \varepsilon_{\min})$ and $\bar{w}_{\max} := \mathcal{W}(\bar{k}_{\max}, \varepsilon_{\max})$, the wage process $\{w_t\}_{t\geq 0}$ defined by (11a) will asymptotically converge to the set $\bar{\mathbb{W}} := [\bar{w}_{\min}, \bar{w}_{\max}]$. Likewise, by (11b) the bubble will asymptotically converge to the set $\bar{\mathbb{B}} := \mathcal{B}(\bar{\mathbb{W}}) = [\mathcal{B}(\bar{w}_{\min}), \mathcal{B}(\bar{w}_{\max})]$. Thus, the equilibrium dynamical system (11a,b) converges to a compact set bounded away from zero which is self-supporting under all sequences of shocks. As a consequence, the bubble along the BME is *persistent* in the sense that it remains bounded away from zero and will not converge to zero asymptotically. In the special case where shocks are degenerate, the deterministic finding from Tirole (1985) is recovered where the two-dimensional system (11a,b) is saddle-path stable and converges to the unique golden-rule steady state.

3.4 Stationary BME

In stochastic models, the concept of an invariant probability distribution of the state variables is widely applied to extend the notion of a fixed point in deterministic models. We will follow the literature by calling BME which admit an invariant distribution a *Stationary BME* (SBME). Since all equilibrium variables are measurable functions of the state variables, it follows that all equilibrium variables possess an invariant distribution along a SBME.

To define a SBME formally, let $x_t = (k_t, \varepsilon_t)$, $t \geq 0$ denote the state variable and endow the state space $\mathbb{X} =]0, k_{\max}] \times \mathcal{E}$ with the Borel-σ-algebra $\mathscr{B}(\mathbb{X})$ to become a measurable space. Then, the mapping $\mathcal{K}^E : \mathbb{X} \longrightarrow \mathbb{R}_{++}$ from (11a) and the time-invariant distribution ν of the shock process gives rise to a transition probability $P : \mathbb{X} \times \mathscr{B}(\mathbb{X}) \longrightarrow [0, 1]$ which governs the statistical evolution of the process $\{x_t\}_{t>0}$. The construction of the transition probability is described in detail in Stokey & Lucas (1994, pp.220). For $x \in \mathbb{X}$ and $B \in \mathcal{B}(\mathbb{X})$, the value $P(x, B)$ is the probability that $x_{t+1} \in B$ given that $x_t = x$. In the terminology used by Duffie, Geanakoplos, Mas-Colell & McLennan (1994), P defines a time homogeneous Markov equilibrium (THME) on the state space $\mathbb{X}$.

Suppose that the initial state x_0 is distributed according to some probability measure μ_0 on the measurable space $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Then, P induces a sequence $\{\mu_t\}_{t\geq 0}$ of probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ which is defined recursively as

$$\mu_t(B) = \int_{\mathbb{X}} P(x, B)\, \mu_{t-1}(dx), \quad B \in \mathcal{B}(\mathbb{X}). \tag{12}$$

We are now in a position to define a SBME formally in the following

Definition 3
A SBME is a probability measure μ on the measurable space $(\mathbb{X}, \mathscr{B}(\mathbb{X}))$ that is invariant under the transformation (12).

It is well-known that the existence of SBME is closely connected to the existence of compact self-supporting sets, cf. Brock & Mirman (1972), Wang (1993), or Stokey & Lucas (1994). Based on their results and Lemma 9 which implies that $\bar{\mathbb{X}} := \bar{\mathbb{K}} \times \mathcal{E}$ is a compact self-supporting set of (11a), we have the following

Theorem 2
Under the hypotheses of Theorem 1, there exists a SBME for the economy $\mathscr{E}$.

4 Conclusions

Bubbly Markov equilibria provide a suitable concept to study asset bubbles in OLG models with stochastic production. Using a functional equation approach where BME obtain as fixed points of an operator, the present paper established sufficient conditions under which BME exist extending well-known results from deterministic models. We also showed that bubbles remain persistent along a non-trivial BME and give rise to an invariant probability distribution on the state space. The latter were referred to a Stationary BME.

Several issues are on our research agenda. First, we would like to study the welfare effects of asset bubbles and whether the injection of a bubble is welfare improving. In this regard, we also seek to link our existence conditions to the ones derived in Demange & Laroque (2000). Second, we seek to extend the present setup to include more general preferences, production technologies, and correlated production shocks which follow a Markov process. Existing results from Morand & Reffett (2007), McGovern, Morand & Reffett (2012), and Hillebrand (2012b) suggest that the basic approach in this paper is amendable to all these extensions.

A Mathematical Appendix

A.1 Proof of Lemma 1

(i) Let ψ be given and $w \in \mathbb{W}$ be arbitrary but fixed. For $k \in \mathbb{K}$ and $\varepsilon \in \mathcal{E}$, set $c(k;\varepsilon) := \psi(\mathcal{W}(k;\varepsilon)) + k\mathcal{R}(k;\varepsilon)$ which is a strictly increasing function of k due to monotonicity of ψ and (T1). For $k \in \mathbb{K}$, define the functions

$$\tilde{B}(k) := \frac{\mathbb{E}_\nu\left[\psi(\mathcal{W}(k;\cdot))v'\big(c(k;\cdot)\big)\right]}{\mathbb{E}_\nu\left[\mathcal{R}(k;\cdot)v'\big(c(k;\cdot)\big)\right]} \tag{A.1}$$

and

$$S(k) := k + \tilde{B}(k) = \frac{\mathbb{E}_\nu\left[c(k;\cdot)v'\big(c(k;\cdot)\big)\right]}{\mathbb{E}_\nu\left[\mathcal{R}(k;\cdot)v'\big(c(k;\cdot)\big)\right]} =: \frac{\tilde{N}(k)}{D(k)}. \tag{A.2}$$

Since ψ is continuous, so are the mappings $\tilde{B}$, $\tilde{N}$, D, and S. Observe that $\tilde{N}$ in (A.2) is weakly increasing due to (U1) and monotonicity of $c(\cdot;\varepsilon)$ while D is strictly decreasing which implies that S is strictly increasing. Furthermore, by the boundary conditions imposed in Assumptions 1 and 2

$$\lim_{k\to 0} D(k) = \infty \tag{A.3}$$

which, together with the monotonicity of $\tilde{N}$ implies

$$0 \leq \lim_{k\to 0} \tilde{B}(k) \leq \lim_{k\to 0} S(k) = \lim_{k\to 0} \frac{\tilde{N}(k)}{D(k)} = 0. \tag{A.4}$$

For $k \in \mathbb{K}$, define

$$G(k;w) := u'(w - S(k)) - D(k). \tag{A.5}$$

Then, the desired solution $\tilde{k}$ solves $G(\tilde{k};w) = 0$. Observe that $G(\cdot;w)$ is a strictly increasing function which follows from the monotonicity of S and D and u'. Thus, any zero is necessarily unique. Also observe the boundary behavior $\lim_{k\to 0} G(k;w) = -\infty$ due to (A.3). By continuity, it suffices to find a $k < w$ such that $G(k;w) \geq 0$. Suppose $\psi \equiv 0$. Then the solution is $\tilde{k} = k_0 := \mathcal{K}_0(w)$ defined by (6) and $\tilde{b} = 0$. If $\psi \neq 0$, consider the following two cases. First, $S(k_0) \geq w$. Then, by (A.4) and monotonicity and continuity of S, there exists a unique value $0 < \hat{k} \leq k_0$ such that $S(\hat{k}) = w$ which implies $\lim_{k\nearrow\hat{k}} G(k;w) = \infty$. Second, suppose $S(k_0) < w$. Then, $\lim_{k\nearrow k_0} G(k;w) = u'(w - S(k_0)) - D(k_0) \geq G_0(k_0;w) = 0$ with G_0 defined by (6). Thus, in either case, there exists a solution $0 < \tilde{k} \leq k_0 < w$. Setting $\tilde{b} = \tilde{B}(\tilde{k})$ completes the proof. ■

A.2 Proof of Lemma 2

Let $\psi \in \mathscr{G}$ be arbitrary. As shown in the previous proof, $\mathcal{B}_\psi = \tilde{B}_\psi \circ \mathcal{K}_\psi$ with $\tilde{B}_\psi$ defined in (A.1) and, for each $w \in \mathbb{W}$, $k = \mathcal{K}_\psi(w)$ is the unique solution to $G(k;w) = 0$ with G defined in (A.5). From (A.1), we infer directly that $\mathcal{B}_\psi \geq 0$, $\psi > 0$ implies $\mathcal{B}_\psi > 0$ and $\psi = 0$ implies $\mathcal{B}_\psi = 0$. Furthermore, by (A.5) and the definition of $\mathcal{K}_\psi$, for all $w \in \mathbb{W}$

$$w > S(\mathcal{K}_\psi(w)) = \mathcal{K}_\psi(w) + \tilde{B}_\psi(\mathcal{K}_\psi(w)) > \tilde{B}_\psi(\mathcal{K}_\psi(w)) = \mathcal{B}_\psi(w).$$

We show that $w \longmapsto w - \mathcal{B}_\psi(w)$ is (even strictly) increasing. For this purpose, let $w \in \mathbb{W}$ and $\Delta > 0$ be arbitrary such that $w + \Delta \in \mathbb{W}$. We show that $\mathcal{B}_\psi(w+\Delta) < \mathcal{B}_\psi(w) + \Delta$. By contradiction, suppose $\mathcal{B}_\psi(w+\Delta) \geq \mathcal{B}_\psi(w) + \Delta$. Note that G defined in (A.5) is strictly decreasing in w and strictly increasing in k by strict monotonicity of D and S. These properties imply that $\mathcal{K}_\psi$ is strictly increasing which gives $\mathcal{K}_\psi(w+\Delta) > \mathcal{K}_\psi(w)$. Further, as shown in the previous proof, the function D defined in (A.2) is strictly decreasing which gives $D(\mathcal{K}_\psi(w)) > D(\mathcal{K}_\psi(w+\Delta))$. On the other hand, by (A.5) and our hypothesis

$$\begin{aligned} D(\mathcal{K}_\psi(w+\Delta)) &= u'(w+\Delta-\mathcal{B}_\psi(w+\Delta)-\mathcal{K}_\psi(w+\Delta)) \\ &\geq u'(w-\mathcal{B}_\psi(w)-\mathcal{K}_\psi(w+\Delta)) \\ &> u'(w-\mathcal{B}_\psi(w)-\mathcal{K}_\psi(w)) \\ &= D(\mathcal{K}_\psi(w)) \end{aligned}$$

which is a contradiction and proves the claim.
Next, we show that $\mathcal{B}_\psi$ is increasing. As $\mathcal{B}_\psi = \tilde{B} \circ \mathcal{K}_\psi$ and we have already shown that $\mathcal{K}_\psi$ is strictly increasing, it remains to show that $\tilde{B}$ defined in (A.1) is increasing as well. To avoid trivialities, assume in the remainder that $\psi > 0$. Let $k \in \mathbb{K}$ and $\Delta > 0$ be arbitrary such that $k+\Delta \in \mathbb{K}$. We have to show that $\tilde{B}(k+\Delta) > \tilde{B}(k)$. By (T1), the map $a \longmapsto av'(a+b)$, $a > 0$ is increasing for all $b \geq 0$. Using this in (A.1) and the fact that both $\mathcal{R}(\cdot;\varepsilon)$ and v' are strictly decreasing gives

$$\tilde{B}(k+\Delta) > V(\Delta) := \frac{\mathbb{E}_\nu\left[\psi(\mathcal{W}(k;\cdot))v'\big(\psi(\mathcal{W}(k,\varepsilon)+(k+\Delta)\mathcal{R}(k+\Delta,\cdot))\big]}{\mathbb{E}_\nu\left[\mathcal{R}(k;\cdot)v'\big(\psi(\mathcal{W}(k,\varepsilon)+(k+\Delta)\mathcal{R}(k+\Delta,\cdot))\big]}.$$

As $V(0) = \tilde{B}(k)$, it suffices to show that V is weakly increasing. Under the additional hypothesis (U2) of constant relative risk aversion, the following lemma holds.

Lemma 10
In addition to Assumption 2, let v satisfy (U2). Then, for any bounded random variables $X > 0$ and $Y > 0$ defined on the probability space $(\mathcal{E}, \mathscr{B}(\mathcal{E}), \nu)$, the function

$$H(a) := \frac{\mathbb{E}_\nu[Yv'(Y+aX)]}{\mathbb{E}_\nu[Xv'(Y+aX)]}, a \geq 0$$

is weakly increasing.

Proof of Lemma 10
Define $\tilde{X} := X|v''(Y+aX)|^{\frac{1}{2}}$ and $\tilde{Y} := Y|v''(Y+aX)|^{\frac{1}{2}}$. Under the hypotheses of the lemma, the function H is C^1 and the derivative is positive, if and only if

$$\mathbb{E}_\nu[\tilde{X}^2]\,\mathbb{E}_\nu[Yv'(Y+aX)] \geq \mathbb{E}_\nu[\tilde{X}\tilde{Y}]\,\mathbb{E}_\nu[Xv'(Y+aX)]. \tag{A.6}$$

Under (T2), $v'(Y+aX) = \theta|v''(Y+aX)|(Y+aX)$ permitting (A.6) to be written as

$$\Big(\mathbb{E}_\nu\left[\tilde{X}^2\right]\Big)^{\frac{1}{2}}\Big(\mathbb{E}_\nu\left[\tilde{Y}^2\right]\Big)^{\frac{1}{2}} \geq \mathbb{E}_\nu\left[|\tilde{X}\tilde{Y}|\right]. \tag{A.7}$$

By Hölder's inequality (see Aliprantis & Border (2007, p.463 setting $p = q = 2$ which implies $\frac{1}{p}+\frac{1}{q} = 1$)), (A.7) is indeed satisfied. □

Employing Lemma 10 (setting $Y := \psi(\mathcal{W}(k;\cdot)$, $X := \mathcal{R}(k;\cdot)$, and $a = \frac{(k+\Delta)g'(k+\Delta)}{g'(k)}$ – which is increasing in Δ by (T1) – shows that V is weakly increasing which proves

$$\tilde{B}(k+\Delta) > V(\Delta) \geq V(0) = \tilde{B}(k).$$

Finally, adopting an argument used and proved in Morand & Reffett (2003, p.1360), note that monotonicity of $\mathcal{B}$ and $w \longmapsto w - \mathcal{B}(w)$, $w \in \mathbb{W}$ implies continuity of $\mathcal{B}$. ■

A.3 Proof of Lemma 3

Let $\psi \in \mathscr{G}$ be arbitrary. We only need to show that $T(\psi) = \mathcal{B}_\psi$ is C^1. First note that if ψ is C^1, so are the mappings $\tilde{B}$, S, D, and $\tilde{N}$ defined in (A.1) and (A.2) and the map G defined in (A.5). Recall that for each $w \in \mathbb{W}$, the mapping $\mathcal{K}_\psi$ determines the unique zero of $G(\cdot; w)$. Since $G_1(k, w) > 0$, the map $\mathcal{K}_\psi$ is C^1 by the implicit function theorem. Thus, the composition $\mathcal{B}_\psi = \tilde{B} \circ \mathcal{K}_\psi$ is C^1 as well. ■

A.4 Proof of Lemma 4

We only prove the strict inequalities, as the proof of the weak inequalities is analogous. Let $\psi_1 > \psi_0$ be arbitrary but fixed. For $\lambda \in [0, 1]$, define $\psi_\lambda := \lambda\psi_1 + (1 - \lambda)\psi_0$. Since $\mathscr{G}'$ is convex, $\psi_\lambda \in \mathscr{G}'$ for all λ. Moreover, the map $\lambda \longmapsto \psi_\lambda = \psi_0 + \lambda\Delta$ where $\Delta := \psi_1 - \psi_0 > 0$ is strictly increasing.

Let $w \in \mathbb{W}$ be arbitrary but fixed. By Lemma 1 (and a slight abuse of notation), for each $\lambda \in [0, 1]$ there exists a unique pair (k_λ, b_λ) which solves $H_1(k_\lambda, b_\lambda; w, \lambda) = H_1(k_\lambda, b_\lambda; w, \lambda) = 0$. We will show that the map $\lambda \longmapsto b_\lambda$, $\lambda \in [0, 1]$ is strictly increasing and $\lambda \longmapsto k_\lambda$, $\lambda \in [0, 1]$ is strictly decreasing which implies $b_1 > b_0$ and $k_1 < k_0$. The proof employs the same structure as the one of Lemma 1. Write $c_\lambda(k; \varepsilon) := \psi_\lambda(\mathcal{W}(k; \varepsilon)) + k\mathcal{R}(k; \varepsilon)$. First, the pair (k_λ, b_λ) satisfies $b_\lambda = \tilde{B}(k_\lambda, \lambda)$ where

$$\tilde{B}(k, \lambda) := \frac{\mathbb{E}_\nu\big[\psi_\lambda(\mathcal{W}(k; \cdot))v'\big(c_\lambda(k; \cdot)\big)\big]}{\mathbb{E}_\nu\big[\mathcal{R}(k; \cdot)v'\big(c_\lambda(k; \cdot)\big)\big]} =: \frac{N(k, \lambda)}{D(k, \lambda)}, \quad k \in \mathbb{K}, \lambda \in [0, 1]. \tag{A.8}$$

For later reference, we compute the partial derivatives of D and N. In this regard, note from (1a,b) that $\mathcal{W}_k(k; \varepsilon) = -k\mathcal{R}_k(k; \varepsilon) > 0$ which implies $c_k(k; \varepsilon) = \mathcal{R}(k; \varepsilon) + k\mathcal{R}_k(k; \varepsilon)(1 - \psi'(\mathcal{W}(k; \varepsilon)))$ for $k > 0$ and $\varepsilon \in \mathcal{E}$. Taking the derivative of (A.8) one obtains, exploiting (U1) and omitting some arguments for notational clearness

$$\begin{aligned}
\partial_k N(k, \lambda) &= \mathbb{E}_\nu\big[-k\mathcal{R}_k(k; \cdot)\psi'_\lambda(-)\big(v'(-) + \psi_\lambda(-)v''(-)\big) + \psi_\lambda(-)\mathcal{R}(k; \cdot)v''(-)\big] \\
&> -\mathbb{E}_\nu\big[\psi_\lambda(-)\mathcal{R}(k; \cdot)|v''(-)|\big] > 0 &\text{(A.9)} \\
\partial_k D(k, \lambda) &= \mathbb{E}_\nu\big[\mathcal{R}_k(k; \cdot)\big(v'(-) + k\mathcal{R}(k; \cdot)v''(-)(1 - \psi'_\lambda(-))\big) + \mathcal{R}(k; \cdot)^2 v''(-)\big] \\
&< -\mathbb{E}_\nu\big[\mathcal{R}(k; \cdot)^2|v''(-)|\big] < 0 &\text{(A.10)} \\
\partial_\lambda N(k, \lambda) &= \mathbb{E}_\nu\big[\Delta(k; \cdot)\big(v'(-) - \psi_\lambda(\mathcal{W}(k; \cdot))|v''(-)|\big)\big] \\
&\geq k\mathbb{E}_\nu\big[\Delta(k; \cdot)\mathcal{R}(k; \cdot)|v''\big(-\big)|\big] > 0 &\text{(A.11)} \\
\partial_\lambda D(k, \lambda) &= -\mathbb{E}_\nu\big[\Delta(k; \cdot)\mathcal{R}(k; \cdot)|v''(-)|\big] < 0 &\text{(A.12)}
\end{aligned}$$

where $\Delta(k; \varepsilon) := \psi_1(\mathcal{W}(k; \varepsilon)) - \psi_0(\mathcal{W}(k; \varepsilon)) > 0$.

We show that $\frac{dk_\lambda}{d\lambda} < 0$. As k_λ is the unique solution to $G(k; \lambda) := u'(w - k - \tilde{B}(k, \lambda)) - D(k, \lambda) = 0$, the derivative computes

$$\frac{dk_\lambda}{d\lambda} = -\frac{G_\lambda(k; \lambda)}{G_k(k; \lambda)}\Big|_{k=k_\lambda} = -\frac{|u''(-)|\partial_\lambda\tilde{B}(k_\lambda; \lambda) - \partial_\lambda D(k_\lambda, \lambda)}{|u''(-)|(1 + \partial_k\tilde{B}(k_\lambda; \lambda)) - \partial_k D(k_\lambda, \lambda)}. \tag{A.13}$$

Using (A.11) and (A.12), the partial derivative of (A.8) with respect to λ satisfies

$$\partial_\lambda \tilde{B}(k;\lambda) = \frac{\partial_\lambda N(k,\lambda)D(k,\lambda) - \partial_\lambda D(k,\lambda)N(k,\lambda)}{D(k,\lambda)^2} > 0. \tag{A.14}$$

Also recall that $\partial_k D(k,\lambda) < 0$ by (A.10). Using these results and (A.12) in (A.13) together with $\partial_k \tilde{B}(k;\lambda) \geq 0$ which follows from the monotonicity of $\tilde{B}$ established in the proof of Lemma 2, the claim follows.

Finally, we show that $\frac{db_\lambda}{d\lambda} > 0$. As $b_\lambda = \tilde{B}(k_\lambda, \lambda)$ one obtains

$$\frac{db_\lambda}{d\lambda} = \partial_k \tilde{B}(k_\lambda,\lambda)\frac{dk_\lambda}{d\lambda} + \partial_\lambda \tilde{B}(k_\lambda,\lambda)$$

which, using (A.13) can be rearranged to

$$\frac{db_\lambda}{d\lambda} = \frac{|u''(-)|\partial_\lambda \tilde{B}(k_\lambda;\lambda) + M(k_\lambda;\lambda)}{|u''(-)|(1+\partial_k \tilde{B}(k_\lambda;\lambda)) - \partial_k D(k_\lambda,\lambda)}$$

where $M(k;\lambda) := \partial_k \tilde{B}(k;\lambda)\partial_\lambda D(k;\lambda) - \partial_\lambda \tilde{B}(k;\lambda)\partial_k D(k;\lambda)$. As the denominator and the first term in the numerator are strictly positive, it suffices to show that $M(k_\lambda;\lambda) \geq 0$. Using (A.14) and that

$$\partial_k \tilde{B}(k;\lambda) = \frac{\partial_k N(k,\lambda)D(k,\lambda) - \partial_k D(k,\lambda)N(k,\lambda)}{D(k,\lambda)^2} \tag{A.15}$$

straightforward computations yield that

$$M(k;\lambda) = \frac{\partial_\lambda D(k,\lambda)\partial_k N(k,\lambda) - \partial_k D(k,\lambda)\partial_\lambda N(k,\lambda)}{D(k,\lambda)}$$

From (A.11) and (A.12), observe that $\partial_\lambda N(k,\lambda) \geq -k\partial_\lambda N(k,\lambda)$. It therefore suffices to show that $\partial_k N(k,\lambda) + k\partial_k D(k,\lambda) \leq 0$. Using (A.9) and (A.10) yields the desired result

$$\partial_k N(k,\lambda) + k\partial_k D(k,\lambda) < \mathbb{E}_\nu\left[k\mathcal{R}_k(k;\cdot)v'(-)(1-\psi'_\lambda(-))\right] < 0.$$

■

A.5 Proof of Lemma 5

For all $\delta \in]0,1]$, $\psi_\delta \in \mathscr{G}'$. Using the structure from the proof of Lemma 1, $(T\psi_\delta)(w) = \tilde{B}(\mathcal{K}_\delta(w))$ for all $w \in \mathbb{W}$ where $\tilde{B}$ is defined in (A.1) and $\mathcal{K}_\delta(w) := \mathcal{K}_{\psi_\delta}(w)$ is the unique solution to

$$G(k;\delta,w) := u'\big(w - k - \tilde{B}(k)\big) - \mathbb{E}_\nu\left[\mathcal{R}(k;\cdot)v'\big(\delta\mathcal{W}(k;\cdot) + k\mathcal{R}(k;\cdot)\big)\right] = 0. \tag{A.16}$$

Using (1a) and (1b), the map $\tilde{B}$ can be written as

$$\tilde{B}(k) = \delta \frac{\mathcal{W}(k; \varepsilon_{\max})}{\mathcal{R}(k; \varepsilon_{\max})}.$$

Rearranging terms, it follows that $T\psi_\delta(w) \geq \psi_\delta(w)$ if and only if

$$\frac{\mathcal{W}(\mathcal{K}_\delta(w); \varepsilon_{\max})}{w} - \mathcal{R}(\mathcal{K}_\delta(w); \varepsilon_{\max}) > 0. \tag{A.17}$$

We seek to establish existence of a $\underline{\delta} \in]0,1[$ such that (A.17) holds for all $w \in \mathbb{W}$. As $\mathcal{K}_\delta$ is well-defined for all $\delta \in [0,1]$ and depends continuously on δ, it suffices to show that the l.h.s in (A.17) is bounded away from zero for $\delta = 0$. Thus, we will show that there exists $c > 0$ such that

$$H(w) := \frac{\mathcal{W}(\mathcal{K}_0(w); \varepsilon_{\max})}{w} - \mathcal{R}(\mathcal{K}_0(w); \varepsilon_{\max}) \geq c \tag{A.18}$$

for all $w \in \mathbb{W}$. Here for each $w \in \mathbb{W}$, $\mathcal{K}_0(w)$ is determined by the implicit condition (6). Define $k_{\max} > 0$ as in Assumption 3 and set $w_{\max} := \mathcal{W}(k_{\max}, \varepsilon_{\max})$. By stability and uniqueness of $k_{\max}$, the map $w \longmapsto \mathcal{W}(\mathcal{K}_0(w), \varepsilon_{\max})$ has $w_{\max}$ as its unique fixed point which is stable implying $\mathcal{W}(\mathcal{K}_0(w); \varepsilon_{\max}) > w$ for all $w \in]0, w_{\max}[$.[1] Using Assumption 4 and $k_{\max} = \mathcal{K}_0(w_{\max})$ gives

$$H(w_{\max}) = 1 - \mathcal{R}(k_{\max}; \varepsilon_{\max}) > 0.$$

Furthermore, letting $w_{\min} > 0$ be the unique solution to $\mathcal{R}(\mathcal{K}_0(w); \varepsilon_{\max}) = 1$, it follows that $H(w) > 0$ for all $w \in [w_{\min}, w_{\max}]$. Thus, defining

$$c := \min\Big\{ H(w) | w \in [w_{\min}, w_{\max}] \Big\} > 0$$

the claim in (A.18) will follow if we show that H is strictly decreasing on $]0, w_{\min}[$.

Let $w \in]0, w_{\min}[$ be arbitrary but fixed and set $k := \mathcal{K}_0(w)$. The derivative of H can be written as

$$H'(w) = -\frac{k\mathcal{R}(k; \varepsilon_{\max})}{w^2} \left[\frac{\mathcal{W}(k; \varepsilon_{\max})}{k\mathcal{R}(k; \varepsilon_{\max})} - E_{g'}(k) \frac{\mathcal{K}_0'(w) w}{k} \frac{k + w}{k} \right]. \tag{A.19}$$

As $k = \mathcal{K}_0(w)$ solves $G(k; 0, w) = 0$ with G defined in (A.16), the derivative satisfies

$$\mathcal{K}_0'(w) = \frac{|u''(w-k)|}{|u''(w-k)| + \frac{|g''(k)|}{g'(k)} u'(w-k) + (1 - E_{g'}(k)) \mathbb{E}_\nu [\mathcal{R}(k; \cdot)^2 |v''(-)|]}.$$

[1] To see this, note that for all $w \in \mathbb{W}$ there exists a unique $k \in \mathbb{K}$ such that $w = \mathcal{W}(k; \varepsilon_{\max})$. Assumption 3 and (7) yield $k \leq \mathcal{K}_0^E(k, \varepsilon_{\max}) = \mathcal{K}_0(w)$ which, by strict monotonicity of $\mathcal{W}(\cdot; \varepsilon_{\max})$ gives $w = \mathcal{W}(k; \varepsilon_{\max}) \leq \mathcal{W}(\mathcal{K}_0(w); \varepsilon_{\max})$ where the last inequality is strict if and only if the first one is strict.

Using $\mathbb{E}_\nu[\mathcal{R}(k;\cdot)^2|v''(-)|] = \theta\frac{u'(w-k)}{k}$ by (U2) and (6) and $\frac{u'(w-k)}{|u''(w-k)|} = \frac{w-k}{\theta} \geq w-k$ by (U1)–(U3), it follows that

$$\begin{aligned}
\mathcal{K}_0'(w) &= \frac{1}{1+E_{g'}(k)\frac{w-k}{k\theta}+(1-E_{g'}(k))\frac{w-k}{k}} \\
&\leq \frac{1}{1+E_{g'}(k)\frac{w-k}{k}+(1-E_{g'}(k))\frac{w-k}{k}} = \frac{k}{w}. \qquad \text{(A.20)}
\end{aligned}$$

Using this result and (T2), the bracketed term in (A.19) can be rearranged as

$$\begin{aligned}
\frac{\mathcal{W}(k;\varepsilon_{\max})}{k\mathcal{R}(k;\varepsilon_{\max})} - E_{g'}(k)\frac{\mathcal{K}_0'(w)w}{k}\frac{k+w}{k} &\overset{(1a,b)}{=} \frac{1-E_g(k)}{E_g(k)} - E_{g'}(k)\frac{\mathcal{K}_0'(w)w}{k}\frac{k+w}{k} \\
&\overset{(A.20)}{\geq} \frac{1-E_g(k)}{E_g(k)} - E_{g'}(k)\frac{k+w}{k} \\
&\overset{(T2)}{\geq} (1-E_g(k))\left[\frac{1}{E_g(k)} - \frac{k+w}{k}\right] \\
&= (1-E_g(k))\left[\frac{1-E_g(k)}{E_g(k)} - \frac{w}{k}\right]. \qquad \text{(A.21)}
\end{aligned}$$

The claim will follow if we show that the bracketed term in (A.21) is positive. Now, as argued in Remark **??**, (U2) and (U3) imply that $v'(c) = \gamma u'(c) = \gamma c^{-\theta}$ where $\theta \leq 1$ by (U1). The following auxiliary result shows that Assumption 4 implies further restrictions on these parameters.

Lemma 11

Let $\hat{\gamma} := \gamma\mathbb{E}_\nu[(\mathrm{id}_\mathcal{E}(\cdot)/\varepsilon_{\max})^{1-\theta}]$. *Under (U1)–(U3), Assumption 4 implies that*

$$\frac{\mathcal{W}(k_{\max};\varepsilon_{\max})}{k_{\max}\mathcal{R}(k_{\max};\varepsilon_{\max})} > \frac{1+\hat{\gamma}^{\frac{1}{\theta}}}{\hat{\gamma}^{\frac{1}{\theta}}}.$$

Proof of Lemma 11.

Set $R_{\max} := \mathcal{R}(k_{\max};\varepsilon_{\max})$ and, as before $w_{\max} = \mathcal{W}(k_{\max};\varepsilon_{\max})$. Then, $k_{\max} = \mathcal{K}_0(w_{\max})$ which, by (6) resp. (A.16), is equivalent to $G(k_{\max};0,w_{\max}) = 0$ and can be written as

$$(w_{\max}-k_{\max})^{-\theta} = \gamma\mathbb{E}_\nu\left[\mathcal{R}(k_{\max};\cdot)\big(k_{\max}\mathcal{R}(k_{\max};\cdot)\big)^{-\theta}\right] = \hat{\gamma}R_{\max}(k_{\max}R_{\max})^{-\theta}.$$

Exploiting $R_{\max} < 1$, this can be rewritten to satisfy the following inequalities:

$$k_{\max}R_{\max} = \hat{\gamma}^{\frac{1}{\theta}}R_{\max}^{\frac{1}{\theta}}(w_{\max}-k_{\max}) < \hat{\gamma}^{\frac{1}{\theta}}(w_{\max}-k_{\max}) < \hat{\gamma}^{\frac{1}{\theta}}(w_{\max}-R_{\max}k_{\max}).$$

Rearranging and using $w_{\max} = \mathcal{W}(k_{\max};\varepsilon_{\max})$ and $k_{\max} = \mathcal{K}_0(w_{\max})$ gives the claim. □

Recall that (T2) is equivalent to the map $k \mapsto E_g(k)$ being weakly increasing. Combined with Lemma 11 and (1a,b), $k < k_{\max}$ implies

$$\frac{1-E_g(k)}{E_g(k)} \overset{(1a,b)}{=} \frac{\mathcal{W}(k;\varepsilon_{\max})}{k\mathcal{R}(k;\varepsilon_{\max})} \overset{(T3)}{\geq} \frac{\mathcal{W}(k_{\max};\varepsilon_{\max})}{k_{\max}\mathcal{R}(k_{\max};\varepsilon_{\max})} > \frac{1+\hat{\gamma}^{\frac{1}{\theta}}}{\hat{\gamma}^{\frac{1}{\theta}}}.$$

Thus, positivity of the bracketed term in (A.21) will follow if we show that

$$\frac{1+\hat{\gamma}^{\frac{1}{\theta}}}{\hat{\gamma}^{\frac{1}{\theta}}} \geq \frac{w}{k}. \tag{A.22}$$

Using $\mathcal{R}(k;\varepsilon_{\max}) > 1$ and exploiting (U1)–(U3), it follows from $G(k;0,w) = 0$ that

$$(w-k)^{-\theta} = \gamma \mathbb{E}_\nu \Big[\mathcal{R}(k;\cdot)\big(k\mathcal{R}(k;\cdot)\big)^{-\theta}\Big] \quad = \quad \hat{\gamma}\mathcal{R}(k;\varepsilon_{\max})^{1-\theta}k^{-\theta} \geq \hat{\gamma}k^{-\theta}$$

where $\hat{\gamma}$ is defined as in Lemma 11. Rearranging shows that (A.22) is indeed satisfied. Thus, the bracketed term in (A.21) is positive implying $H'(w) < 0$ for all $w \in]0, w_{\min}[$. This proves (A.18) and the claim. ■

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