Impatience vs. Incentives*

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Abstract

This paper studies the long-run dynamics of Pareto-optimal self-enforcing contracts in a repeated principal-agent framework with differential discounting. Impatience concerns encourage contracts to favor the more patient player in the long run, and incentives concerns encourage contracts to favor the agent in the long run. When the agent is relatively impatient, the impatience and incentives forces are in conflict. If the conflict is strong, we show that optimal contracts oscillate between favoring the principal and favoring the agent as a way to cater to both forces in the long-run. This occurs in the absence of uncertainty or any need to randomize. When the impatience and incentives forces are aligned or one force dominates the other, we show that every optimal contract converges to a steady state in the long run in a well-behaved way. The results of Ray (2002) and Lehrer and Pauzner (1999) can be recovered as limiting cases.

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1 Introduction

In this note, we study optimal contracting in a perfect information repeated principal-agent framework with differential discounting. Differential discounting creates gains from trade across time and tends to push optimal contracts toward favoring the more patient player in the long-run. This impatience force is in addition to the incentives force that tends to push optimal contracts toward favoring the agent in the long-run. In this paper, we analyze how the interplay of impatience and incentives affects the long-run dynamics of Pareto-optimal self-enforcing contracts.

We split the analysis into two cases. First, we assume that there is a strong impatience/incentives conflict. Formally, this means two things: the principal is strictly more patient (so impatience and incentives work against each other) and the steady state optimal contract(s) (we prove existence) is in the interior of the Pareto-frontier. The second assumption reflects the idea that if the steady state was the principal's (agent's) favorite Pareto-optimal self-enforcing contract, then one of the forces must always dominate.

Our first result, Proposition 1, shows in a standard setting with transferable utility, when the impatience/incentives conflict is strong, all optimal contracts except the steady state generically oscillate between favoring the principal and favoring the agent. This allows the contracts to cater to both the impatience force and the incentives force. In many cases, the oscillation exhibits long-run persistence so that convergence to the steady state is not guaranteed.

There has been a growing applied literature looking at the role of differential discounting in dynamic principal-agent relationships.[1] Previous works have shown when the impatience/incentives conflict is strong, the steady state will be distorted in favor of the principal. This distortion is statically inefficient, but ends up being optimal because it facilitates early delivery of utility to the more impatient agent. Our Proposition 1 reveals that oscillation is a second important dimension along which the impatience/incentives conflict can affect the long-run dynamics of optimal contracts. These optimal contracts are randomization proof. Therefore, the oscillation feature we produce is not some "trick feature" that is concealing an underlying advantage to randomization.[2]

The basic intuition for oscillation is contained in Lemma 3 which characterizes "optimal contracts" in the absence of participation constraints. Fix any agent promised value. We show that in the "optimal contract" delivering that promised value, it is efficient for the principal to demand the steady state investment level at all dates and only adjust the monetary transfers, and it is efficient for the incentive-compatibility condition to hold with equality at all dates. These two intuitive requirements suffice to pin down the unique sequence of monetary transfers, which oscillates.

When the oscillation is damped, it is essentially without loss of generality to ignore participation constraints since every oscillating contract converges to the steady state. However, when the sequence of monetary transfers explosively oscillates, even *arbitrarily low* participation constraints will eventually be violated. In order to respect the par-

[1] In particular, see Acemoglu, Golosov, and Tsyvinski (2008), Aguiar, Amador, and Gopinath (2009), and Opp (2012).

[2] Indeed, the setup we use is quite natural, and nests the essential elements of a number of seminal papers. See Examples 1, 2 and 3 in the next section.

ticipation constraints, we show that the optimal contracts now also oscillate investment around the stationary level, helping keep the monetary oscillations in check. In this case, participation constraints can have non-trivial effects on contract dynamics.[3]

In the second part of our analysis, we show that when the impatience/incentives conflict is not strong, every optimal contract converges to a steady state. This result is proved in a more general setting that does not assume transferable utility. Proposition 2 shows that if the unique steady state is the agent's or principal's favorite, then all optimal contracts converge monotonically to the steady state. Proposition 3 shows that if the agent is at least as patient as the principal, then every optimal contract converges to a steady state in a well-behaved way.[4] Thus, steady states capture all long-run dynamics when the impatience/incentives conflict is weak.

Our results help us relate the opposing predictions of Ray (2002) and Lehrer and Pauzner (1999). As Ray (2002) points out, when "the agent is more impatient than the principal... the Lehrer-Pauzner findings and the results of [Ray (2002)] tug in different directions. It may be worth exploring if one of the two factors always dominates." Our results not only show what happens when one factor does dominate (Proposition 2), but also highlights cases when both factors significantly affect all optimal contracts, resulting in oscillation (Proposition 1).

2 Strong Impatience/Incentives Conflict

In this section, we characterize optimal contracts when the impatience/incentives conflict is strong (formally defined in Definition 1) in a model with transferable utility. In the next section, we characterize optimal contracts when the impatience/incentives conflict is weak in a more general setting that does not assume transferable utility.

Consider an infinite horizon, discrete time repeated principal-agent relationship with perfect public information and transferable utility. Time is indexed by $t \in \mathbb{Z}_+$. At each date, an action consists of an investment $e \geq 0$ and a monetary transfer $m \in \mathbb{R}$ from the principal (P) to the agent (A). A negative transfer means the agent pays the principal.[5] The investment parameter can be interpreted a number of ways. In addition to representing effort put in by the agent, it can also capture some labor/capital input from the principal. Utilities are quasi-linear over actions:

$$(u_A, u_P) = (r_A(e) + m, r_P(e) - m) \tag{1}$$

[3]In some cases, the implied comovement of investment and transfers is sufficiently strong, so that the joint sequence will again converge to the steady state. In these cases, participation constraints work in subtle ways: If participation constraints are ignored, any non-steady state "optimal contract" would explosively oscillate and break both constraints permanently. Once participation constraints are respected, optimal contracts adjust so strongly that the constraints now never bind.

[4]Proposition 3 has a similar flavor to Theorem 1 of Acemoglu, Golosov, and Tsyvinski (2008), who study optimal mechanisms in a political economy setting. An important technical contribution of their paper is to also allow for capital, an additional physical state variable.

[5]In some applications, it may be natural to impose a constraint on m so that it must be nonnegative or nonpositive (see Examples 2 and 3). While we do not assume such constraints, in many cases, the optimal contracts we derive satisfy the natural constraints automatically.

Given a sequence of actions $\{e_t, m_t\}$, introduce the time t continuation payoff of the sequence:

$$(U_{A,t}, U_{P,t}) = \left(\sum_{j=t}^{\infty} \delta_A^{j-t} u_A(e_j, m_j), \sum_{j=t}^{\infty} \delta_P^{j-t} u_P(e_j, m_j) \right) \tag{2}$$

A sequence of actions is a *(self-enforcing) contract* if, at each date, continuation is individually rational for both players and the agent cannot profitably deviate. The outside option of player $i = A, P$ is $O_i := o_i/(1-\delta_i)$, $i = A, P$. Since the setting is perfect public information, we assume a deviation at date t triggers exit at date $t+1$ with both parties receiving their outside options.[6] At date t, the deviating agent is able to hold onto a constant fraction $\theta \in [0, 1]$ of the planned transfer m_t as well as a portion of the investment by shirking or stealing capital.[7] Define the net deviation utility for the agent at date t to be $D(e_t, m_t) := d(e_t) - (1-\theta)m_t$. The incentive-compatibility constraint is:

$$\delta_A U_{A,t+1} \geq D(e_t, m_t) + \delta_A O_A \tag{3}$$

We assume r_A, r_P, and d are differentiable and $r_A' + d'/(1-\theta) > 0$. These conditions assure differentiability of the Pareto-frontier. We assume $r_A + r_P$ is strictly concave with interior maximum e^*. Lastly, we assume the model is sufficiently "convex": Fix actions (e_1, m_1) and (e_2, m_2) and any $\lambda \in [0, 1]$. There exists an action (e_λ, m_λ) such that $u_i(e_\lambda, m_\lambda) \geq \lambda u_i(e_1, m_1) + (1-\lambda)u_i(e_2, m_2)$ for $i = A, P$, with at least one of them strict if $e_1 \neq e_2$, and $D(e_\lambda, m_\lambda) \leq \lambda D(e_1, m_1) + (1-\lambda)D(e_2, m_2)$.

Example 1. *In Thomas and Worrall (1994), the government* (A) *allows a multinational firm (P) to invest* e *in the country and generate returns* $Y(e)$*. In exchange, the multinational firm pays the government taxes* τ*. The government can deviate by expropriating the return* $Y(e)$ *but then forfeits tax income. In this case,* $r_A(e) = 0$*,* $r_P(e) = Y(e) - e$*,* $m = -\tau$*,* $\theta = 0$ *and* $D(e) = Y(e) - \tau$*.*[8]

Example 2. *In Ray (2002) and a simplified version of Thomas and Worrall (1988), a worker* (A) *exerts costly effort* $c(e)$ *to produce* e *for the owner* (P)*. In return, the worker receives an upfront wage* w*. The worker can deviate by shirking and stealing the wage. In this case,* $r_A(e) = -c(e)$*,* $r_P(e) = e$*,* $m = w$*,* $\theta = 1$ *and* $D(e) = c(e)$*.*

Example 3. *In Ray (2002) and a simplified version of Albuquerque and Hopenhayn (2004), an entrepreneur* (A) *borrows money* l *from a lender* (P) *and invests in projects that generate* $F(l)$*, where* r *is the riskless rate. In return, the lender receives a loan repayment* R*. The entrepreneur can deviate by not repaying the loan. Then* $r_A(l) = F(l)$*,* $r_P(l) = -(1+r)l$*,* $m = -R$*,* $\theta = 0$ *and* $D(l) = R$*.*

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[6] The model can be easily adapted to include renegotiation. In this case, the outside option O_i can be endogenously determined based on the optimal value function (see, for example, van Damme, 1991).

[7] The bound on θ reflects an assumption of weakly inefficient deviation. Assumption (A.3) in the related paper by Ray (2002) is in the same spirit.

[8] Acemoglu, Golosov, and Tsyvinski (2008), Aguiar, Amador, and Gopinath (2009), and Opp (2012) consider similar setups.

In these examples, θ is either 0 or 1, suggesting that the monetary transfer either occurs fully before or after the principal observes the agent's action choice. But in practice, a government like in Example 1 may be able to demand a portion of the tax be paid up front; a worker like in Example 2 may only be able to steal wage inefficiently (such as the banker in Calomiris and Kahn, 1991); and a lender as in Example 3 may require collateral for the loan. In these cases, the fraction θ recouped by the agent need no longer be all or nothing. Our analysis, which also considers $\theta \in (0,1)$, will have something to say about these situations.

The payoff set generated by all contracts is convex and compact. Therefore, a well-defined Pareto-frontier V exists. Viewed as a function over U_A, $V(U_A)$ is a compact, weakly concave curve. The points of V are precisely the payoffs of the Pareto-optimal contracts - call them V-contracts. Moreover, since the continuation contract of a V-contract is a V-contract, the sequence of continuation payoffs for any V-contract lies on V. These facts fall out of the convexity assumption and will be proved in greater generality in the next section. Our goal is to characterize the law of motion of V-contracts.

Lemma 1. *Fix a V-contract with nontrivial initial investment $e_0 > 0$ and a payoff point $(U_A, V(U_A))$ in the interior of V. Then V is differentiable and $V'(U_A) = [r'_P(e_0) - d'(e_0)/(1-\theta)]/[r'_A(e_0) + d'(e_0)/(1-\theta)]$.*[9]

The expression for V' comes from considering small perturbations of the initial action while holding the net deviation utility fixed. These perturbations alter the agent's and principal's payoffs at rates equal to the denominator and numerator of the expression for V'. By construction, the perturbations are still contracts and they create a differentiable payoff set through $(U_A, V(U_A))$. Since V must lie weakly above this payoff set but remain weakly concave, it must also be differentiable with the same slope.

By assuming that $(r_A + r_P)'(0)$ is sufficiently high or $d'(0)$ is sufficiently close to zero or something similar, it is easy to ensure that Lemma 1's mild assumption that optimal investment is nontrivial holds. From now on, we will assume differentiability of V.[10]

Lemma 2. *There exists a unique stationary V-contract $\{(e^s, m^s)\}$ - the steady state.*

Existence is implied by next section's more general Lemma 6 and uniqueness falls out of Lemma 4.

Definition 1. *When $\delta_A \geq \delta_P$ or the unique steady state is at one of the corners of V, we say the impatience/incentives conflict is weak. Otherwise, we say the impatience/incentives conflict is strong.*

This section focuses on the strong impatience/incentives conflict case. The next section will contain results in a more general setting characterizing V-contracts when the conflict is weak.

[9] When $\theta = 1$, the slope is -1.

[10] All of the differentiability assumptions we have made so far are purely for simplicity and are not necessary for an explicit characterization of V-contracts. For example, in the general theory of the next section, we make no such assumptions.

The previous literature has shown that the steady state investment level is generically below the static optimum: $e^s < e^*$. See Corollary 1. This holds even if e^* can be indefinitely sustained in a contract. Loosely speaking, distorting steady state investment level downwards shifts the steady state to the left, causing V-contracts to become more frontloaded/less backloaded. This is desirable as it exploits dynamic trading gains due to the relative impatience of the agent. Moreover, in the specific models considered in the previous literature, all non-steady state V-contracts converge to the steady state over time. In the next section, when we look at the weak conflict case, we also get results (Proposition 2 and Proposition 3) that show convergence to a steady state. Thus,

Incumbent Intuition. *Steady state investment distortions capture all long-run effects of the impatience/incentives conflict.*

In the analysis below, we show that there is a second way in which the impatience/incentives conflict can affect V-contracts: Oscillation.

Definition 2. *A sequence $\{x_t\}$ oscillates around x^* if for all t, $x_t \neq x^*$, $x_t < x^* \Rightarrow x_{t+1} > x^*$, and $x_t > x^* \Rightarrow x_{t+1} < x^*$.*

We are now ready to state the main result of this section:

Proposition 1. *Assume strong impatience/incentives conflict. If $\theta = 0$ or 1, every non-steady state V-contract converges monotonically to the steady state. If $\theta \in (0, 1)$, every non-steady state V-contract oscillates around the steady state. If $\theta \in [\frac{\delta_P}{\delta_P+\delta_A}, \frac{1}{1+\delta_A}]$, all oscillating V-contracts eventually perpetually oscillate between two distinct points. If $\theta \in (0, \frac{\delta_P}{\delta_P+\delta_A}) \cup (\frac{1}{1+\delta_A}, 1)$, all oscillating V-contracts converge to the steady state.*

Proposition 1 demonstrates the importance of oscillation. For any non-corner θ, every non-steady state V-contract oscillates.[11] For a precise description of the oscillation movement, see the proof in the Appendix. To understand the intuition behind oscillation, consider the relaxed version of the model without participation constraints. The following lemma is an important step in the proof of Proposition 1

Lemma 3. *Let $\theta \in (0, 1)$ and $\rho := \frac{1}{\delta_A}\frac{1-\theta}{\theta}$. Let $\{(m^s, e^s)\}$ be the steady state with payoff (U_A^s, U_P^s). For every $m \in \mathbb{R}$, the sequence $\{(m^s + (-\rho)^t m, e^s)\}_{t=0}^{\infty}$ satisfies the IC-constraint at every date and has payoff $(U_A^s + \frac{m}{1+\delta_A\rho}, U_P^s - \frac{m}{1+\delta_P\rho})$. These sequences generate a linear payoff set through (U_A^s, U_P^s) and are therefore Pareto-optimal disregarding participation constraints. Call them the optimal incentive-contracts.*

Lemma 3 reveals that all V-contracts would oscillate around the steady state in a simple way were it not for participation constraints.

What is it about Lemma 3's optimal incentive-contracts that make them oscillate? In the absence of participation constraints, there is no reason for optimal incentive contracts to feature lax incentive constraints or distortions away from the constrained-efficient investment level e^s of the steady state. Indeed, it can be shown that the Lemma 3

[11] Part of the reason this behavior seems to have been missed by the literature is that previous papers have focused on $\theta = 0$ and 1, or $\delta_A = \delta_P$. Recall, strong impatience/incentives conflict requires $\delta_A < \delta_P$.

optimal incentive-contracts are *uniquely* characterized as the set of action sequences with the following two properties: the steady state investment level e^s is sustained and IC-constraints hold with equality at each date. These two properties are responsible for generating the oscillation dynamic.

To see why, suppose the principal wants to write a contract delivering $U_{A,0}$ to the agent where $U_{A,0} > U_A^s$. Since investment has to stay at e^s, pay must increase. But a pay increase affects incentives by lowering the net deviation utility $D(e, m)$ when $\theta < 1$. To keep the IC-constraint holding with equality, the continuation value cannot simply go back to U_A^s. Instead, it must decrease further to some $U_{A,1} < U_A^s$ to compensate for the smaller $D(e, m)$. This is beneficial since it heightens backloading to the more patient principal: $V(U_{A,1}) > V(U_A^s)$.

Now, to deliver $U_{A,1}$ to the agent while keeping investment at e^s requires a pay decrease. Compared to a pay increase, a pay decrease has the opposite effect on incentives, increasing the net deviation utility. This forces $U_{A,2} > U_A^s$. Reusing the previous arguments, we find that $U_{A,3} < U_A^s$, $U_{A,4} > U_A^s \ldots$ and oscillation emerges.

Participation constraints will, in general, cause V-contracts to depart from the simple blueprint of Lemma 3. The degree to which participation constraints affect the structure of V-contracts can be parameterized by the size of θ.

Suppose $\theta > 1/(1+\delta_A)$. Then $\rho < 1$ and the transfer distortions $-(\rho)^t m$ in Lemma 3 diminish in magnitude over time. Thus all Lemma 3 incentive-contracts eventually satisfy both participation constraints permanently. More specifically, if a Lemma 3 incentive-contract satisfies participation constraints for the first two dates, it will satisfy participation constraints for all future dates. This fact is used to show in the proof of Proposition 1 that every V-contract becomes a Lemma 3 incentive-contract starting from date $t = 2$ at the latest when $\theta > 1/(1 + \delta_A)$. Thus, distortions driven by participation constraints are transitory, disappearing after two dates.

When $\theta < \frac{1}{1+\delta_A}$ or equivalently $\rho > 1$, the transfer distortions $-(\rho)^t m$ in Lemma 3 are explosive. Every non-steady state Lemma 3 incentive-contract will eventually break both participation constraints permanently. As a result, V-contracts will have to depart from Lemma 3 incentive-contracts in significant and persistent ways. V-contracts will exhibit persistent distortions in both transfers and investments, and the Pareto-frontier will no longer be linear (see, for example, Figure 1). However, we can exploit the changing slope of V to provide an implicit characterization of both transfer and investment dynamics.

Lemma 4. *For any V-contract with payoff $(U_{A,t}, U_{P,t} = V(U_{A,t}))$ and continuation payoff $(U_{A,t+1}, U_{P,t+1} = V(U_{A,t+1}))$, the following differential conditions must be satisfied:*

$$V'(U_{A,t+1}) = -\frac{\delta_A}{\delta_P}\frac{1}{1-\theta} - \frac{1}{\rho\delta_P}V'(U_{A,t}) \tag{4}$$

The unique fixed point of the equation is $V'^s := -\delta_A/[(1-\theta)\delta_P + \theta\delta_A]$.

Corollary 1. *Suppose the impatience/incentives conflict is strong. Then steady state investment is strictly below e^* if $\theta < 1$.*[12]

[12]If $\theta = 1$, then $r'_A + r'_P = d'\frac{\delta_P - \delta_A}{\delta_A}$ at the steady state investment level. Thus, so long as $d'(e^*) > 0$, investment distortions also occur for $\theta = 1$.

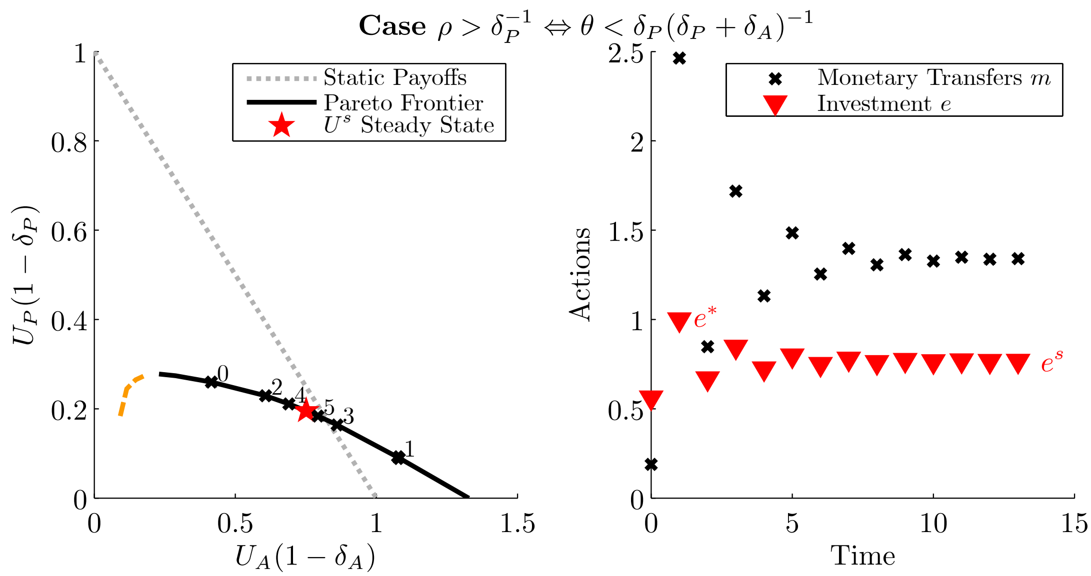

Figure 1: **Pareto Frontier and Dynamics.** The graphs plot a version of Example 2 when $\theta < \frac{\delta_P}{\delta_P+\delta_A}$. The left panel plots the Pareto Frontier in the normalized payoff space $(1-\delta_i)U_i$ with the initial time-0 allocation favoring the principal. The right panel plots the oscillating dynamics of investment and transfers.

Lemma 4 shows that even if V-contracts have to depart from Lemma 3 incentive-contracts in significant and persistent ways due to participation constraints, the oscillation property survives. Equation (4) implies that the slope of the value function oscillates around the steady state value of V'^s according to a first-order difference equation. By Lemma 1, this law of motion also governs the dynamics of investment. Hence, Lemma 4 can be used to characterize non-stationary V-contracts for $\theta < 1/(1+\delta_A)$ much in the same way Lemma 3 characterizes V-contracts for $\theta > 1/(1+\delta_A)$. When $\theta \in (\frac{\delta_P}{\delta_P+\delta_A}, \frac{1}{1+\delta_A})$, non-steady-state V-contracts explosively oscillate until reaching a unique maximal oscillating contract. This maximal oscillating contract oscillates between two points of V, at least one of which is a corner point. This results means that when $\theta \in (\frac{\delta_P}{\delta_P+\delta_A}, \frac{1}{1+\delta_A})$ and V-contracts cannot explosively oscillate forever like the Lemma 3 incentive-contracts, they do next best thing: They oscillate explosively until the participation constraints prevent them from increasing their amplitude any further.

Lastly, when $\theta \in (0, \frac{\delta_P}{\delta_P+\delta_A})$, all non-steady-state V-contracts damped oscillate, converging to the steady state (see Equation (4)). In the previous case, Lemma 3 incentive-contracts explosively oscillate and the V-contracts copy them as best they can. In this case, Lemma 3 incentive-contracts also explosively oscillate, but it is no longer optimal for V-contracts to try and mimic the behavior. This divergence in behavior implies that while participation constraints are not reached in the long run when $\theta \in (0, \frac{\delta_P}{\delta_P+\delta_A})$, their mere existence still strongly affects the dynamics of V-contracts and the shape of the Pareto frontier. This surprising result is possibly due to the fact that ρ is increasing as θ decreases, so that when $\theta \in (0, \frac{\delta_P}{\delta_P+\delta_A})$, the explosiveness of the Lemma 3 oscillation becomes quite severe and not worth mimicking. Indeed, when $\theta = 0$, $\rho = \infty$ and

V-contracts completely lose their oscillation and exhibit monotone convergence instead.

Lemma 4 also implies that when V-contracts become distorted by participation constraints, investment begins to oscillate as well. This is because Lemma 1 ties the investment level to the slope of V. Figure 1, which plots a version of Example 2 with $\theta \in (0, \frac{\delta_P}{\delta_P+\delta_A})$ demonstrates how this typically works. As a V-contract moves between agent payoffs where the slope of V is steeper and shallower than V'^s, the investment level moves between levels above and below the steady state level, mirroring the oscillation in transfers. The shadow oscillation of investment partially compensates for the V-contracts' inability, due to participation constraints, to oscillate its transfers as much as Lemma 3 requires. As the oscillations of the V-contract's continuation payoff dampen toward the steady state, so do the transfer and investment oscillations.

3 Weak Impatience/Incentives Conflict

The previous section considered a contracting model with transferable utility and showed how oscillation is a key optimal arrangement when the impatience/incentives conflict is strong. This section proves every V-payoff can be achieved by a V-contract that converges in a stable way to a steady state when the impatience/incentives conflict is weak. This result, comprising Propositions 2 and 3, is derived in a general setting that does not require transferable utility.

In this general setting, let $(\mathscr{A}, \geq)$ denote the compact and convex action space over which the continuous utility functions u_A, u_P are defined.[13] There is a continuous net deviation function $D : \mathscr{A} \longrightarrow \mathbb{R}$ characterizing the best possible one period net deviation as a function of the current period action. A (self-enforcing) *contract* is a sequence of actions $\{a_t\}$ such that for each date t, the participation constraint $(U_{A,t}, U_{P,t}) \geq (O_A, O_P)$ and the incentive-compatibility constraint $\delta_A U_{A,t+1} \geq D(a_t) + \delta_A O_A$ are both satisfied.

Definition 3. *D is semi-convex over $(\mathscr{A}, u_A, u_P)$ if for any $a_1, a_2 \in \mathscr{A}$ and $\lambda \in [0,1]$ there exists an $a_\lambda \in A$ such that $u_i(a_\lambda) \geq \lambda u_i(a_1) + (1-\lambda)u_i(a_2)$ for $i = A, P$, and $D(a_\lambda) \leq \lambda D(a_1) + (1-\lambda)D(a_2)$. D is strictly semi-convex if for at least one player i, $u_i(a_\lambda) > \lambda u_i(a_1) + (1-\lambda)u_i(a_2)$.*

Throughout the rest of the paper, we will assume semi-convexity of D. Let C denote the set of contract payoffs. It is easy to show that C is compact and convex.[14] So, let V denote the set of all Pareto-optimal payoffs of C. As an abuse of notation, we will also think of $V = V(U_A)$ as a function over agent payoffs. Therefore, V-contracts are randomization-proof and as a function,

Lemma 5. *$V(U_A)$ is weakly concave with compact domain.*

The following fundamental result holds for all pairs of discount factors:

[13] The action space of the transferable utility model was not compact. However, given the participation constraints, the relevant action set could effectively be made compact.

[14] Convexity is an immediate consequence of the semi-convexity of D. Compactness means bounded and closed. Boundedness follows from the participation constraints. Closedness follows from an application of the diagonalization trick.

Lemma 6. *There exists at least one steady state. It is unique if D is strictly semi-convex.*

We are now ready to state the first result characterizing V-contracts when the impatience/incentives conflict is weak.

Proposition 2. *If the only steady state is V_R, the rightmost payoff of V, then every V-contract converges monotonically to the steady state. A symmetric result holds with V_R replaced with V_L, the leftmost payoff of V.*

While the condition in Proposition 2 is not on fundamentals, it is still of practical value. Oftentimes, it is straightforward to impose conditions on fundamentals guaranteeing that impatience dominates incentives or vice versa, causing the unique steady state to be at one of the corners of V. For instance, Ray (2002) considers a version of Example 2 in our model with equal discounting and imposes an upper bound on the common discount factor. This sufficiently strengthens the incentives force and effectively guarantees that the principal's participation constraint must bind in the steady state, implying that the steady state is V_R. Thus, the Ray (2002) result that all self-enforcing contracts converge to the agent's favorite can be seen as a special case of Proposition 2 when incentives dominates impatience.[15]

Proposition 2 is also useful when players become arbitrarily patient. Lehrer and Pauzner (1999) shows in a game-theoretic setting that the continuation payoff sequence of an efficient equilibrium will move in the direction of the more patient player as discount factors tend to one. The intuition is that as players become arbitrarily patient, one shot deviation gains become trivial relative to the surplus generated by the relationship. Moreover, the best way to generate surplus is to exploit trading gains across time, backloading rewards to the more patient player. This intuition is broadly applicable in our setting and implies that the location of the steady state tends to move toward the more patient player's corner. Once the discount factors are sufficiently high, the steady state will reach either V_R or V_L depending on who's more patient. Then, Proposition 2 characterizes the dynamics of all V-contracts. The following result is an application of this intuition to the transferable utility model. In particular, it shows that when impatience dominates incentives, oscillation is no longer optimal.

Corollary 2. *In the transferable utility model with $\delta_A < \delta_P$, let the discounts vary while holding $\log \delta_A / \log \delta_P$ constant. Then for all sufficiently high δ_A, δ_P, the steady state is V_L and all V-contracts converge monotonically to the steady state.*

In the transferable utility model, if the discounts were reversed so that $\delta_A > \delta_P$, then the steady state would be V_R and once again we would have monotone convergence to the steady state. In the general setting without transferable utility, $\delta_A > \delta_P$ is not enough to imply that the steady state must be V_R or is even unique. Nevertheless, when $\delta_A \geq \delta_P$, the impatience force and the incentives force are (weakly) consonant. That is, they both promote backloading of rewards to the agent.

[15] Technically, the general model of Ray (2002) does not fit within our setting. However, both of the motivating examples he cites do.

In the last result of this paper, we show that weak consonance (i.e. $\delta_A \geq \delta_P$) is sufficient to prove stable convergence to a steady state.[16]

Definition 4. *A sequence* $\{x_t\}$ *converges to* x^* *in a stable way if* $\lim x_t = x^*$, *and for all* t, i, $x_t \leq x^* \Rightarrow x_{t+i} \geq x_t$ *and* $x_t \geq x^* \Rightarrow x_{t+i} \leq x_t$.

Proposition 3. *If* $\delta_A \geq \delta_P$, *then every* V*-payoff can be achieved by a contract that converges to a steady state in a stable way.*

Proposition 3 shows that only when the agent is strictly more impatient can there be any sustained nontrivial dynamics in the long-run. In the previous section, we showed that with transferable utility, oscillation is the unique nontrivial dynamic that can be sustained in the long run. No doubt more exotic behaviors could possibly occur in the general setting. However, the following lemma, which is important in the proof of Proposition 3, shows that perpetual oscillation between two states is, in some sense, the fundamental alternative to stable convergence to a steady state.

Lemma 7. *For any pair of discount factors* δ_A, δ_P, *either every* V*-contract converges to a steady state in a stable way or there exists at least one* V*-contract perpetually oscillating between two states.*

Lastly, Proposition 3 provides a useful generalization of the Ray (2002) result, albeit in a slightly different setting. Recall, in that paper, it is shown that if $\delta_A = \delta_P$ and the unique steady state is V_R, then all V-contracts converge to it. Proposition 3 relaxes the assumptions on discount factors, and the uniqueness and location(s) of the steady state(s), while showing that convergence still occurs and can be assumed to be stable. In contrast, it is worth highlighting the potential discontinuity when moving from equal discounting to relative impatience of the agent. Even arbitrarily small differences in discount factors might then imply explosive oscillation instead of stable convergence - compare Propositions 1 and 3.

4 Conclusion

In this paper, we explore the implications for self-enforcing contracting when the principal and agent have potentially different discount factors. When the agent is at least as patient as the principal, we showed that every Pareto-optimal self-enforcing contract converges to a steady state.

However, when the agent is strictly more impatient, there is a conflict between the impatience force and the incentive force. We find that if the conflict is strong, then a robust compromise between these two forces is oscillation. By always switching between principal-preferred and agent-preferred states, such arrangements are able to continue (partially) catering to the more patient principal in the long-run, while finding an interesting solution to the incentive-compatibility problem: The agent has too much to gain

[16] Weak consonance does not rule out damped oscillation nor does it rule out multiplicity of steady states. However, it does rule out explosive oscillation, even if it is temporary. Furthermore, recall that if D is strictly semi-convex then there is a unique steady state.

tomorrow from staying when the current state is bad, and has too much to lose today from deviating when the current state is good.

The strict optimality of oscillation implies that simple stationary contracts likes those in Levin (2003) can be insufficient to characterize optimal allocations even in seemingly well-behaved, non-stochastic models when there are arbitrarily small differences in discount factors. We hope our findings serve as a catalyst for further research on the implications of impatience in agency relationships.

A Proofs

Proof of Lemma 1. Fix a V-contract with initial action $(e_0 > 0, w_0)$ and interior payoff point $(U_{A,0}, V(U_{A,0}))$. Let Δ denote an infinitesimal quantity. Consider the action sequence that is identical to our V-contract starting at date 1 but has initial action $(e_0 + \Delta, w_0 + d'(e_0)\Delta/(1-\theta))$. This action sequence is incentive-compatible, satisfies the participation constraints, and is therefore a contract. Its payoff point is $(U_{A,0} + r'_A(e_0)\Delta + d'(e_0)\Delta/(1-\theta), V(U_{A,0}) + r'_P(e_0)\Delta - d'(e_0)\Delta/(1-\theta))$. Since this contract is weakly Pareto-dominated by a V-contract, it must be that $d^-V(U_{A,0}) \leq [r'_P(e_0) - d'(e_0)/(1-\theta)]/[r'_A(e_0) + d'(e_0)/(1-\theta)] \leq d^+V(U_{A,0})$. But since V is weakly concave, $d^-V \geq d^+V$ everywhere, and the result is proved. □

Proof of Lemma 2. Let $\mathcal{S}$ denote the set of payoffs (U_A, U_P) in V satisfying $V'(U_A) = \frac{-\delta_A}{\theta\delta_A + (1-\theta)\delta_P}$. Lemma 4 implies that any steady state lies on $\mathcal{S}$. Suppose there are two distinct steady states with actions (m^s, e^s) and (m', e'). Then $\mathcal{S}$ is a line segment with slope $\frac{-\delta_A}{\theta\delta_A + (1-\theta)\delta_P}$. Suppose $e^s \neq e'$. Given the assumption that the model is "strictly convex" along investment (see discussion after Equation (3)), any point on $\mathcal{S}$ strictly between the two stationary payoffs is in the interior of C. Contradiction. This implies that $m^s \neq m'$. But then, the slope of $\mathcal{S}$ is $-\frac{1-\delta_A}{1-\delta_P} \neq \frac{-\delta_A}{\theta\delta_A + (1-\theta)\delta_P}$. Contradiction. Thus, there is a unique steady state. □

Proof of Lemma 3. It suffices to show that every optimal incentive contract satisfies its date 0 IC-constraint. Fix an $m \in \mathbb{R}$. The corresponding optimal incentive contract's date 1 continuation payoff is $U_{A,1} = U_A^s - (\rho m)/(1 + \delta_A\rho)$. We then have the following sequence of relations:

$$\delta_A U_{A,1} = \delta_A \left[U_A^s - (\rho m)/(1 + \delta_A\rho)\right] = \delta_A U_A^s - (1-\theta)m \geq$$

$$D(e^s, m^s) - (1-\theta)m + \delta_A O_A = D(e^s, m^s + m) + \delta_A O_A$$

And the result is proved. □

Proof of Lemma 4. Let (e_0, m_0) be the initial action. Incentive-compatibility requires $\delta_A U_{A,1} \leq d(e_0) - (1-\theta)m_0 + \delta_A O_A$. For every $\epsilon \neq 0$, consider the perturbed contract C_ϵ with initial action $(e_0, m_0 + \epsilon)$ followed by a continuation V-contract with agent payoff $U_{A,1} - \frac{(1-\theta)\epsilon}{\delta_A}$. Contract C_ϵ's agent payoff is $U_{A,0}^\epsilon = U_{A,0} + \theta\epsilon$. Its principal payoff is

$$U_{P,0}^\epsilon = r_P(e) - (m + \epsilon) + \delta_P V\left(U_{A,1} - \frac{(1-\theta)\epsilon}{\delta_A}\right) =$$

$$V(U_{A,0}) - \epsilon + \delta_P \left(V \left(U_{A,1} - \frac{(1-\theta)\epsilon}{\delta_A} \right) - V(U_{A,1}) \right)$$

The efficiency of C requires $V(U_{A,0} + \theta\epsilon) \geq U^{\epsilon}_{P,0}$, which implies

$$V \left(U_{A,1} - \frac{(1-\theta)\epsilon}{\delta_A} \right) - V(U_{A,1}) \leq \frac{\epsilon + V(U_{A,0} + \theta\epsilon) - V(U_{A,0})}{\delta_P}$$

Dividing both sides by ϵ and taking the limit as ϵ goes to zero, we have the condition. □

Proof of Corollary 1. Lemma 1 implies steady state investment satisfies $[r'_P(e^s) - d'(e^s)/(1-\theta)]/[r'_A(e^s) + d'(e^s)/(1-\theta)] = V'^s > -1 = [r'_P(e^*) - d'(e^*)/(1-\theta)]/[r'_A(e^*) + d'(e^*)/(1-\theta)]$. □

Proof of Proposition 1.
Case 1: $\theta \geq \frac{1}{1+\delta_A}$
Let W denote the Lemma 3 payoff points. We aim to prove the following characterization of V-contracts: $W \cap V$ is an interval containing V_L or V_R and, of course, the steady state. Therefore, any V-contract with payoff in $V \cap W$ is a Lemma 3 contract, and $V - W$ will be an interval (possibly empty) around V_R or V_L. If $V - W$ is around V_R, then any V-contract with payoff in $V - W$ has V_L as its date 1 continuation payoff. A symmetric result holds if $V - W$ is around V_L.

Let $\mathcal{S}$ denote the set of payoffs (U_A, U_P) in V satisfying $V'(U_A) = \frac{-\delta_A}{\theta\delta_A + (1-\theta)\delta_P}$. Let (e^s, m^s) and (U^s_A, U^s_P) be the action and payoff point of the steady state. Consider an action sequence of the type considered in Lemma 3: $C(e^s, m^s, m) := \{(e^s, m^s + (-\rho)^t m)\}_{t=0}^{\infty}$ where $\rho := \frac{1}{\delta_A}\frac{1-\theta}{\theta}$. Since $\theta \geq \frac{1}{1+\delta_A}$, $\rho < 1$. This implies that if $C(e^s, m^s, m)$'s payoff $(U^s_A + \frac{m}{1+\delta_A\rho}, U^s_P - \frac{m}{1+\delta_P\rho})$ and continuation payoff $(U^s_A - \frac{\rho m}{1+\delta_A\rho}, U^s_P + \frac{\rho m}{1+\delta_P\rho})$ satisfy the participation constraints, $C(e^s, m^s, m)$ is a contract.

Since the steady state is assumed to be in the interior of V, its participation constraints do not bind. Therefore m can be picked to be nonzero. Let $\overline{m} > 0$ be the largest m such that $C(e^s, m^s, m)$ is a contract. Then $U^s_P - \frac{\overline{m}}{1+\delta_P\rho} = O_P$ or $U^s_A - \frac{\rho\overline{m}}{1+\delta_A\rho} = O_A$. Otherwise, $\overline{m}$ could be increased and the resulting action sequence would still be a contract, contradicting the maximality of $\overline{m}$. Similarly, let $\underline{m} < 0$ be the smallest m such that $C(e^s, m^s, m)$ is a contract. Then $U^s_A + \frac{\underline{m}}{1+\delta_A\rho} = O_A$ or $U^s_P + \frac{\rho\underline{m}}{1+\delta_P\rho} = O_P$.

The locus of contract payoffs $\mathcal{W} := \{(U^s_A + \frac{m}{1+\delta_A\rho}, U^s_P - \frac{m}{1+\delta_P\rho})\}_{m \in [\underline{m}, \overline{m}]}$ is linear, has slope $\frac{-\delta_A}{\theta\delta_A + (1-\theta)\delta_P}$, and clearly contains either V_L or V_R. Thus $\mathcal{W} \subset \mathcal{S} \subset V$. If $\theta = 1$ then $\mathcal{W} = \mathcal{S} = V$ and we're done.

So fix $\theta \in [\frac{1}{1+\delta_A}, 1)$. Suppose there exists a point $p \in \mathcal{S}$ not in $\mathcal{W}$. WLOG, p is to the right of $\mathcal{W}$. This implies that $V_L \in \mathcal{W}$. Now, if it is not the case that every continuation payoff of every contract with payoff in $\mathcal{S} - \mathcal{W}$ is in $\mathcal{W}$, then there is a steady state in $\mathcal{S} - \mathcal{W}$. This follows from Kakutani's fixed point theorem and is a contradiction.

So pick a contract with payoff p and continuation payoff $p_1 \in \mathcal{W}$. Let (e', m') denote this contract's initial action and let p_1 be the payoff of $C(e^s, m^s, m)$ for some m. Since V is linear between p and p_1 and the model is strictly convex along investment, it must be that $e' = e^s$. But this forces $m' = m/(-\rho)$ and $p \in \mathcal{W}$. Contradiction and so $\mathcal{W} = \mathcal{S} = W \cap V$.

Finally, let p be a payoff of V not in $\mathcal{S}$. WLOG, p is to the right of $\mathcal{S}$. Then $V_L \in \mathcal{S}$. Lemma 4 then implies that any contract with payoff p must have continuation payoff V_L. This simultaneously determines the initial action and all subsequent actions and the contract is completely characterized.

Case 2: $\theta < \frac{1}{1+\delta_A}$
Again, introduce the set $\mathcal{S}$. Like before, any $\mathcal{S}$-payoff is uniquely supported by an action sequence of the form $C(e^s, m^s, m)$. However, since $\theta < \frac{1}{1+\delta_A}$, any non-stationary $\theta < \frac{1}{1+\delta_A}$ will eventually break the participation constraint and is not feasible. Therefore, $\mathcal{S} = (U_A^s, U_P^s)$.

Suppose $\theta = \frac{\delta_P}{\delta_P+\delta_A}$. Consider the contract that supports the payoff V_L. Denote its time t continuation payoff by $(U_{L,A,t}, U_{L,P,t})$. Suppose $(U_{L,A,1}, U_{L,P,1}) \neq V_R$, then Lemma 4 implies that $(U_{L,A,2}, U_{L,P,2}) = V_L$ and the V_L contract perpetually oscillates between the two states. In this case, Lemma 4 implies that any non-steady state V-contract with payoff between V_L and $(U_{L,A,1}, U_{L,P,1})$ perpetually oscillates between two states. Finally, any payoff of V to the right of $(U_{L,A,1}, U_{L,P,1})$ is supported by a contract whose continuation contract is the V_L-contract. A symmetric result holds with L and R reversed. The only remaining case is when the date 1 continuation payoff of the V_L-contract is V_R and vice versa. In this case, every non-steady state V-contract perpetually oscillates between two states.

Now suppose $\theta \in (\frac{\delta_P}{\delta_P+\delta_A}, \frac{1}{1+\delta_A})$. Using the same argument as before, we can show that either the V_L- or the V_R-contract perpetually oscillates between two states. WLOG, assume it's the V_L-contract. Again, just like before, any payoff of V to the right of $(U_{L,A,1}, U_{L,P,1})$ is supported by a contract whose continuation contract is the V_L-contract. Moreover, Lemma 4 implies that any V-contract with payoff between V_L and $(U_{L,A,1}, U_{L,P,1})$ explosively oscillates until it eventually becomes the V_L-contract.

Lastly, suppose $\theta < \frac{\delta_P}{\delta_P+\delta_A}$. Then Lemma 4 implies that every V-contract exhibits damped oscillation and converges to the steady state. □

Proof of Lemma 6. For each payoff $v \in V$, define the set $\kappa(v) \subset V$ to consist of all continuation payoffs of contracts with payoff v. κ is a convex valued correspondence with closed graph. The result then follows from Kakutani's Fixed Point Theorem. □

Proof of Proposition 2. Recall κ from the proof of Lemma 6. For each set $\kappa(v)$, let $\kappa(v)_L$ and $\kappa(v)_R$ denote the leftmost and rightmost points of $\kappa(v)$. Suppose for all $v \in V - V_R$, $\kappa(v)_L$ is strictly to the right of v. Then the lemma holds. Otherwise, let v' be a point such that $\kappa(v')_L$ is to the left of v'. If $\kappa(v')_R$ is to the right of v', then v' is a fixed point. Contradiction. So suppose $\kappa(v')_R$ is also to the left of v'. Consider the subset of V: $V|_{[V_L,v']}$ and the correspondence $\kappa' := \eta \circ \kappa$ where $\eta : V \to V$ is the continuous map that is the identity on $[V_L, v']$ and maps $(v', V_R]$ to v'. By Kakutani's Fixed Point Theorem, κ' has a fixed point $v^* \in V|_{[V_L,v']}$. Since $\kappa(v')_R$ is to the left of v', $v^* \neq v'$. But then that means v^* must also be a fixed point of κ. Contradiction. □

Lemma 8. *Let F be a continuous function from $[0,1]$ to itself and consider the set of all iteration sequences generated by F. Either every iteration sequence converges stably or at least one of them oscillates perpetually between two points.*

Proof. Suppose there exists a sequence $\{F^n(x)\}$ which is not stable convergent. If $\{F^n(x)\}$ is stable, then $F(\limsup\{F^n(x)\}) = \liminf\{F^n(x)\}$ and $F(\liminf\{F^n(x)\}) = \limsup\{F^n(x)\}$ and we have a perpetual oscillation between two points. Otherwise, there exists some $y = F^N(x)$ such that $F^2(y) < y < F(y)$ or $F(y) < y < F^2(y)$. WLOG assume $F(y) < y < F^2(y)$. This implies the existence of a fixed point z^* of F in the interval $(F(y), y)$. Since $F^2(y) - y > 0$ and $F^2(1) \leq 1$, there exists a value $z^{**} \in (y, 1]$ such that $F^2(z^{**}) = z^{**}$. If $F < z^*$ on $(y, 1]$ then $\{F(z^{**}), z^{**}\}$ comprise a perpetual oscillation between two points. Otherwise, let z be the smallest value in $(y, 1]$ such that $F(z) = z^*$. Then $F^2(z) - z < 0$. Therefore, z^{**} may be chosen to be in (y, z). Again, $\{F(z^{**}), z^{**}\}$ comprise a perpetual oscillation between two points. □

Proof of Lemma 7. Assume D is strictly semi-convex over $\mathscr{A}$.[17] Then the κ from the proof of Lemma 6 is a continuous function. The result follows immediately from Lemma 8. □

Proof of Proposition 3. Suppose the proposition is false. Then Lemma 8 implies there exists an oscillating V-contract $a_1 \circlearrowleft a_2$. Let $u_1 := (u_A(a_1), u_P(a_1))$, $u_2 := (u_A(a_2), u_P(a_2))$ and $U_{12} = (U_{12,A}, U_{12,P}) :=$ payoff of $a_1 \circlearrowleft a_2$. Of course $a_2 \circlearrowleft a_1$ is also an V-contract with payoff $U_{21} = (U_{21,A}, U_{21,P})$.

Semi-convexity implies that there exists an action a_h with utility $u_h := (u_A(a_h), u_P(a_h))$ such that $u_h \geq \frac{u_1+u_2}{2}$ and $D(a_h) + \delta_A O_A \leq \frac{D(a_1)+D(a_2)}{2} + \delta_A O_A \leq \frac{\delta_A(U_{12,A}+U_{21,A})}{2} = \frac{\delta_A(u_A(a_1)+u_A(a_2))}{2}$. Therefore $u_h \in F$. Also, $u_2 \in F$. Thus there exists a λ such that $\lambda u_h + (1-\lambda)u_2 \geq U_{21}$. Semi-convexity implies there exists an $\tilde{a}$ with utility $\tilde{u} := (u_A(\tilde{a}), u_P(\tilde{a}))$ such that $\tilde{u} \geq \lambda u_h + (1-\lambda)u_2$ and $D(\tilde{a}) + \delta_A O_A \leq \delta_A(\lambda u_A(a_h) + (1-\lambda)u_A(a_2))$. Therefore $\tilde{u} \in F$ and either $\tilde{u} > U_{21}$ or $\tilde{u} = U_{21} \in F$. Both are contradictions.

□

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