STRICTLY COHERENT PREFERENCES
NO HOLDS BARRED

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I shall explain a general result asserting that any additive, homogeneous (i.e., independent) weak order $(\mathscr{L}, \succsim)$ respecting the principle of weak dominance (admissibility) can be represented by a normalized, strictly positive linear function $\mathbb{E}$ on $\mathscr{L}$—that is, an *expectation function*—taking values in a (possibly non-Archimedean) ordered field extension of the system of real numbers in the sense that for all gambles f and g in $\mathscr{L}$, $f \succsim g$ if and only if $\mathbb{E}(f) \geqslant \mathbb{E}(g)$. In addition to foregoing all regularity conditions on the function space of gambles $\mathscr{L}$ (e.g., boundedness and measurability requirements), a distinctive feature of my approach is that rather than employing an ultraproduct construction to obtain the ordered field extension, my proof uses a version of the *Hahn Embedding Theorem* ultimately to construct the desired ordered field in which the expectation function is to take values.

Roughly speaking, I shall demonstrate that $\mathbb{E}$ takes values in an ordered field in which every number can be written as *formal power series* in a *single* infinitesimal ϵ:

$$\sum_{a \in \mathbb{G}} r_a \epsilon^a$$

where $r_a \in \mathbb{R}$ and the nonzero r_a form a well-ordered subset of a given ordered Abelian group $\mathbb{G}$. In fact, the image of $(\mathscr{L}, \succsim)$ under $\mathbb{E}$, $\mathbb{E}(\mathscr{L})$, forms an ordered vector space such that the ordered subcollection $\{\epsilon^a \in \mathbb{E}(\mathscr{L}) : a \in \mathbb{G}\}$ of infinitesimals from $\mathbb{E}(\mathscr{L})$ is order-isomorphic to the ordered collection of *Archimedean equivalence classes* of gambles from $(\mathscr{L}, \succsim)$. In other words, any difference in $\{\epsilon^a \in \mathbb{E}(\mathscr{L}) : a \in \mathbb{G}\}$ corresponds to a difference in $\mathscr{L}$, so if f, g in $\mathscr{L}$ are such that $f, g \succ \mathbf{0}$, then if f is infinitely smaller than g, thus belonging to distinct Archimedean equivalence classes, then there are unique elements a_f and a_g in $\mathbb{G}$ such that a_f is greater than a_g and so ϵ^{a_f} is less than ϵ^{a_g}. We can therefore *trace* infinitely large differences in $\mathbb{E}(\mathscr{L})$ to infinitely large differences in $\mathscr{L}$, providing a tangible idea of where the numbers come from. Hence, the route by way of the Hahn Embedding Theorem usefully locates the origins of numbers of interest.

In connection with the foregoing developments, I shall introduce a *qualitative criterion of coherence* for *qualitative comparisons* of gambles. This criterion is reminiscent of de Finetti's quantitative criterion of coherence for betting, yet it does not impose an Archimedean condition on qualitative comparisons. Moreover, unlike de Finetti's criterion of coherence, the qualitative criterion respects the principle of weak dominance, aligning with a condition of rational decision making. Just as de Finetti formulated his criterion of coherence for an arbitrary collection of bounded gambles, absent of any measurability or closure conditions, I too shall formulate the qualitative criterion of coherence for an arbitrary collection

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of gambles $\mathscr{X}$, in fact not requiring that the gambles be *bounded* and thereby allowing the collection of gambles to be *any* collection of gambles of interest. Furthermore, unlike many other developments of comparative probability, I shall not require that the binary relation $\gtrapprox$ on $\mathscr{X}$ reflecting an agent's judgments of comparative expectations to satisfy *reflexivity, completeness* or even *transitivity*.

Despite these weak structural assumptions, I shall explain a *qualitative analogue* of de Finetti's Fundamental Theorem of Prevision ([de Finetti, 1974a, pp. 133-114], [de Finetti, 1974b, pp. 336-337]). Specifically, I shall describe a theorem asserting that any coherent system of comparative expectations can be extended to a weakly ordered coherent system of comparative expectations over any collection of gambles containing the initial collection of gambles of interest. Thus, if a system of comparative expectations $(\mathscr{X}, \gtrapprox)$ satisfies the qualitative criterion of coherence, and $\mathscr{Y}$ is a collection of gambles containing $\mathscr{X}$, then $\gtrapprox$ can be extended to a weak order $\gtrsim$ on $\mathscr{Y}$ in such a way that $(\mathscr{Y}, \gtrsim)$ is system of comparative expectations satisfying the qualitative criterion of coherence. In addition to the qualitative version of the principle of weak dominance, the extended weakly ordered coherent system of comparative expectations satisfies familiar additivity and homogeneity properties (i.e., the independence postulate) when the extended collection forms a linear space.

Among other things, the technical developments have bearing on interesting results due to Dubins [1977] concerning the extendability of real-valued finite order-preserving linear functionals. Dubins [1977] showed that very few finite real-valued order-preserving linear functionals can be defined for all real-valued functions, while nonetheless every finite order-preserving linear functional can be extended to all real-valued functions if infinity is also admitted as a value for the functional. The foregoing technical developments show that if an ordered field extension of the real numbers is admitted as a carrier for an order-preserving linear function, then not only can every order-preserving linear function be extended to all real-valued functions, but also the linear function can countenance a richer system of comparisons of real-valued functions, comparisons which are impossible even in the extended framework that Dubins employs.

Finally, the foregoing technical developments not only contain key ingredients of an unconditionally *full* account of non-Archimedean expected utility in the style of Anscombe and Aumann [1963], but also form the basis of an account of imprecise probabilities and previsions in the style of, for example, Walley [1991] and Williams [1975] (see also [Smith, 1961] and [Levi, 1974]).

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