Effort-Maximizing Contingent Prize Allocation Rule in Three-Battle Contests

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Abstract

This paper studies the effort-maximizing prize allocation rule in three-battle contests. The organizer has a fixed prize budget, and rewards the players contingent on the number of battles they win. The battles can be simultaneously or sequentially played between two same individuals, or each played by different paired players from two opponent teams. A full spectrum of contest technologies in the Tullock family is accommodated. We find winner-take-all is optimal unless the contest is sequentially played between two same individuals and the discriminatory power of the contest technology is in the intermediate range. With sequential battles between two same contestants, when the discriminatory power falls in the intermediate range, the reward strictly increases with the count of wins, and the reward to a single win strictly increases with the discriminatory power but never goes beyond one-third of the total prize. For the same contest, interestingly, when the discriminatory power falls in the high range, a wide span of allocation rules ranging from winner-take-all to proportional division induces the maximal total expected effort. Therefore, the optimal prize allocation rule should in principle give additional award to the grand winner of the whole contest. Our findings rationalize the commonly observed winner-take-all prize structure as well as intermediate prizes in multi-battle contests.

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1 Introduction

Many economic and social competitions, including patent races, electoral campaigns and sports, can be viewed as contests in which players compete in multiple battles by expending non-refundable costly effort and their rewards are determined by the outcomes of all battles.

Winner-take-all has been a typical prize allocation rule in multi-battle contests; in many occasions, intermediate prizes awarded to winner of component battles are also widely observed. One interesting

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question thus arises: How does the contest organizer’s choice of the prize allocation rule depend on the contest structure? In particular, at which situations should the allocation rule must rely on the overall performance aggregating over all the battles, and at which situations the winners of the battles can be awarded separately in each battle at the optimum without loss of generality? In this paper, we study these issues and target to rationalize the well adopted prize allocation rules from a perspective of effort elicitation by the contest organizer. We fully characterize the effort-maximizing prize allocation rule contingent on the battles outcomes in three-battle contest environments where a whole spectrum of contest technology in the Tullock family is accommodated, and the battles can be sequentially or simultaneously played between two same individual players or each battle is played by paired players from two opponent teams. Our study reveals that the temporal structure, the discriminatory power of the contest technology and the player structure play crucial roles in determining the optimal contingent prize allocation rule.

Equilibrium analysis of multi-battle contests have been well established in the literature. Borel (1921) first introduces the Colonel Blotto game, in which two players must simultaneously each allocate a fixed amount of a resource (use-it-or-lose-it) to a finite number of battles that are modelled as all-pay auctions. Players’ payoffs depend linearly on the count of wins across all battles. The Colonel Blotto game has caught much attention in the literature since its formulation, and many variations and extensions were examined for addressing important issues in a wide range of environments. Friedman (1958) and Robson (2005) extend the analysis by assuming Tullock contest technology in component battles. Roberson (2006) extends the equilibrium analysis to asymmetric budgets across players. Kvasov (2007) and Roberson and Kvasov (2008) study the cases with budget-constrained linear costs.

The extension of the original Colonel Blotto game to the majoritarian objective is examined by Borel and Ville (1938). Szentes and Rosenthal (2003a) and (2003b) further carry out the equilibrium analysis with majoritarian objective and linear costs. The extensions of the original Colonel Blotto game to the majoritarian objective with linear costs and Tullock technology are carried out by Snyder (1989). Clark and Konrad (2007) and Kovenock and Roberson (2009) further examine environments where one player has a best-shot objective and the other has a weakest-link objective. Kovenock and Roberson (2012) provide an excellent and comprehensive survey on this literature on simultaneous multi-battle contests between the same two players.

Environments where the battles are contested sequentially have been analyzed by Harris and Vickers (1987), Ferall and Smith (1997), Klumpp and Polborn (2006), Konrad and Kovenock (2009), McFall, Knoeber and Thurman (2010), Malueg and Yates (2010) and Sela (2011) among others. Harris and Vickers (1987) study a multi-battle patent race. Klumpp and Polborn (2006) model U.S. presidential primaries as a multi-battle dynamic contest between two candidates. Malueg and Yates (2010) study the players’ strategic effort supply in best-of-three contests and test the theoretical prediction empirically using tennis data. All these studies identify the so-called strategic momentum effect in dynamic multi-battle contests between the same two players. Konrad and Kovenock (2009) provide a complete characterization of the unique subgame perfect equilibrium in multi-battle contests with intermediate prizes, in which the component contests are modelled as all-pay auctions. They find that even a large
lead by one player might not fully discourage the other when a component battle awards a positive intermediate prize. Sela (2011) compares the best-of-three all-pay auction to the standard one-stage all-pay auction.¹

Fu, Lu and Pan (2015) study multi-battle team contests, which accommodates in general homogeneity-of-degree-zero contest technologies in each battle and different information structures for contestants. In this environment, two teams with an equal number of players compete in a contest. Players from rival teams form pairwise matches to fight in multiple component battles that can be carried out simultaneously or sequentially. A team wins if and only if its players secure a majority number of victories. Each player benefits from his team’s win, while he can also receive a private reward for winning his own battle. They find that the strategic momentum effect typically identified in dynamic multi-battle contests between the same two players is completely nullified in this team contest setting. Häfner (2012) studies multi-battle team contests in a tug-of-war setting with sequential battles and an all-pay auction technology in each battle.

In this paper, we first study the optimal contingent prize allocation rule that elicits the highest total expected total effort in a sequential-play multi-battle contest between the same two risk neutral players with unit marginal effort cost. In every component battle, both players observe the outcomes of previous battles and exert their effort simultaneously. We allow a full spectrum of contest technology within the Tullock family in component battles, which are indexed by the discriminatory power \( r \) of the corresponding success function. Specifically, the contest organizer has a fixed budget (normalized as 1) that can be used to fund nonnegative prizes to competing parties in a three-battle contest.² She has the flexibility of fully allocating this budget contingent on the outcomes of the battles, i.e. the sum of wins of each party, subject to a monotonicity condition which requires a party with more wins being rewarded a higher prize.³ An interesting but intricate issue is whether a positive prize should be granted to a player with a single win, as will be illustrated in our analysis.

We find that the optimal prize allocation rule crucially depends on the discriminatory power \( r \) of the contest technology adopted in component battles. The discriminatory power in a Tullock contest measures the importance of a player’s effort in determining his winning probability. A higher discriminatory power \( r \) means that the winning chances are more determined by players’ effort rather than other random factors that also affect the contestants’ performances.

We fully characterize the optimal contingent prize allocation for every discriminatory power \( r(>0) \). For each \( r \), we first characterize the subgame perfect equilibrium by backward induction for each eligible contingent prize allocation rule. We then compare across all eligible contingent prize allocation rules to identify the optimal rule. The procedure requires lengthy computations and multi-step comparisons which must be carried out very carefully with great patience and persistence. In particular, computing players’ total expected effort for a given prize structure requires aggregating the players’ effort across

¹Irfanoglua et al. (2010) and Mago and Sheremeta (2012) test the implications of these theoretical studies by experiments.

²If the organizer’s budget is indivisible, she can equivalently use winning probabilities as design instruments.

³We assume the prize allocation does not depend on the identities of the competing parties.
every possible path. Because of the difficulties generated by potential ex-post asymmetry due to the sequential nature of the contest and its effect on players’ strategies, we have to consider separately multiple overlapping subsets of eligible prize structures, and obtain the optimal prize allocation rule within each subset. A comparison across all these restricted optimums yields the globally optimal prize allocation rule.

The main findings are as follows. For convenience, we use \( v(n) \) to denote the prize awarded to a player winning \( n \in \{0, 1, 2, 3\} \) battles. First, when the discriminatory power is small enough such that \( r \leq r \) with \( 1 < r \leq \tilde{r} < 1.2 \), a non-contingent rule of winner-take-all (i.e. the party wins two or more battles is rewarded with the whole budget, and the other party gets nothing) uniquely elicits the highest expected aggregate effort. In this case, \( v(0) = v(1) = 0 \) and \( v(2) = v(3) = 1 \). Second, when \( r \in [\tilde{r}, 2] \), we have a unique prize allocation rule with \( v(0) = 0 \leq v(1) \leq \frac{1}{3} \) and \( \frac{2}{3} \leq v(2) \leq v(3) = 1 \). More specifically, when \( r \) increases from \( \tilde{r} \) to 2, prize \( v(1) \) increases from 0 to \( \frac{1}{3} \) while prize \( v(2) \) decreases from 1 to \( \frac{2}{3} \). In particular, when \( r = 2 \), proportional prize allocation is optimal. Third, when \( r > 2 \), the optimal prize allocation rule is not unique. All allocation rules with \( v(0) = 0 \), \( v(3) = 1 \), \( v(1) \in [0, \frac{1}{3}] \) and \( v(2) = 1 - v(1) \) would elicit the highest total expected effort from players. In other word, a wide range of prize allocation rules spanning from proportional allocation to winner-take-all can be optimal, as long as the losing player (who wins less than one battle) is rewarded no more than proportionally according to his count of wins.

The economics and intuitions behind these characterizations can be illustrated as follows. It is natural that \( v(0) = 0 \) (and thus \( v(3) = 1 \)) is necessary to elicit maximal effort from players as rewarding a player without a single win definitely dampens the players’ incentive. The more interesting and intricate trade-off lies in the balance between \( v(1) \) and \( v(2) \), the prizes for a single win and two wins. In other words, should a positive prize be granted to a player with a single win? A change of \( v(1) \) from 0 to \( \varepsilon(> 0) \) generates different marginal impacts on the incentive of players through changes in effective prize spreads upon different contingencies of early battle outcomes. When one player wins both of the first two battles, a change in \( v(1) \) would increase the effective prize spread by \( \varepsilon \) in the third battle, which leads to higher effort supply. When each of the two players wins one battle of the first two, such a change in \( v(1) \) reduces the effective prize spread by \( 2\varepsilon \) in the third battle, which leads to lower effort supply. Both effects increase with the discriminatory power \( r \) since the third battle always has a symmetric prize spread though it depends on the outcomes of the first two battles. Therefore, fix the winning probabilities in the second battle, the positive effect would dominate the negative effect if and only if the probability of the event that one player wins the first two battles is twice higher than the probability that each player wins one of the first two battles.

However, the marginal impact of change in \( v(1) \) on the second battle winning probabilities also plays a role when comparing its two effects on the third battle effort as different efforts are induced in the subsequent subgames. For low \( v(1) \), the third battle induces more effort if each player has won one early battle as the prize spread is higher. This means that the change in second battle wining probabilities due to the \( \varepsilon \) increase in \( v(1) \) would increase the effort supply in the third battle if and only if the momentum effect is reduced by the change in \( v(1) \).
How does the above \( \varepsilon \) increase in \( v(1) \) affect the prize spreads for both players in the second battle and their effort supply in this battle? We first consider the winner of the first battle. This change is determined by the changes in a player’s payoffs when he wins both the first two battles and when he wins only one of the first two battles. On one hand, the expected payoff of a player conditional his winning the first two battles must drop since the expected prize decrease but effort cost increase due to increased prize spread in the third battle.\(^4\) On the other hand, the expected payoff of a player conditional on his winning only one of first two battles must increase since the expected prize does not change but effort cost decrease due to decreased prize spread in the third battle.\(^5\) Therefore, an \( \varepsilon \) increase in \( v(1) \) decreases the prize spread in the second battle for the winner of the first battle.

We now turn to the loser of the first battle. On one hand, we have explained that the expected payoff of a player conditional on his winning only one of first two battles must increase. On the other hand, the expected payoff of a player conditional his losing the first two battles must also increase since the expected prize turns positive from zero through effort cost also increases due to increased prize spread in the third battle.\(^6\) Therefore, the impact of an \( \varepsilon \) increase in \( v(1) \) on the second-battle prize spread of the loser of the first battle can be ambiguous. For \( r \geq 2 \), we have a prize spread \( v(1) - v(0) \), which increases with \( v(1) \); For \( r = 0 \), we have a prize spread of \( \frac{v(2) - v(0)}{2} \), which decreases with \( v(1) \). Due to continuity of the prize spread as a function of \( r \), we reasonably expect that the second battle prize spread of a loser of the first battle decreases with \( v(1) \) when \( r \) is low, but increases with \( v(1) \) when \( r \) is high.

The above illustrated changes in players’ second battle prize spread mean that higher \( r \) would definitely render higher effort supply; and with lower \( r \), the both players’ second-battle prize spreads decrease, which means that most likely the induced effort supply is lower. When \( r \) is high, the smaller gap between players’ second battle prize spreads reduces the momentum effect, which enlarges the positive effect on expected effort supply in the third battle. When \( r \) is small, both players’ second battle prize spreads drop with \( v(1) \), the change in their difference is less clear. It is possible that the momentum effect can even be enhanced by an increase in \( v(1) \), which contributes to the negative effect on expected effort supply in the third battle.

We now turn to the first battle. How does the above \( \varepsilon \) increase in \( v(1) \) affect the common prize spread in the first battle? We first look at how the expected payoffs of the winner and the loser of the first battle are affected. When \( r \) is high, due to the reduced momentum effect, the winner of the first battle would win less expected prize; and the loser of the first battle would win more expected prize. On the other hand, both of their second stage and third stage expected effort supply is higher. Therefore, the expected payoff of the winner of the first battle is lower. While the change in the expected payoff of the loser of the first battle cannot be immediately pinned down. It is reasonable to expect that an increase in \( v(1) \) likely eventually benefits him. Therefore, with high \( r \), the first battle prize spread should be lower, which renders lower first battle effort supply.

\(^4\) At the unique symmetric equilibrium, the player concerned player wins \( v(3) \) and \( v(2) \) with same probability of \( \frac{1}{2} \).

\(^5\) At the unique symmetric equilibrium, each player wins \( v(1) \) and \( v(2) \) with same probability of \( \frac{1}{2} \). Note \( v(1) + v(2) = 1 \).

\(^6\) At the unique symmetric equilibrium, the player concerned player wins \( v(0) \) and \( v(1) \) with same probability of \( \frac{1}{2} \).
When \( r \) is low, whether the momentum effect is reduced or enhanced is not clear, which makes it hard to pin down the signs of changes in the expected prizes and expected effort supply (both second stage and third stage) for both the winner and loser of the first battle. Nevertheless, intuitively an increase in \( v(1) \) would most likely hurt more the winner and benefit more the loser of the first battle. Therefore, one can expect a lower first battle prize spread results, which means lower first battle effort supply.

Based the above discussions, the first battle most likely generates less total effort with an \( \varepsilon \) increase in \( v(1) \) regardless of the magnitude of \( r \). However, a high \( r \) would generate higher effort from the second and third battle while the impact of a lower \( r \) on the effort generated from from the second and third battle can be ambiguous. These observations rationalize how the optimal allocation should reply on \( r \). When \( r \) is small (i.e. when \( r \leq r^* \)), the ambiguous effect of an \( \varepsilon \) increase in \( v(1) \) on the second and third battle is dominated by its negative effect on the first battle effort supply, which leads to the optimality of a zero \( v(1) \). When \( r \) is big (i.e. when \( r > r^* \)), the positive effect of an \( \varepsilon \) increase in \( v(1) \) on the second and third battle at least offsets its negative effect on the first battle effort supply, which leads to the optimality of a positive \( v(1) \). When \( r \in (r^*, 2) \), the direct positive impact on the second and third battle is more likely to dominate the more indirect negative impact on the first battle effort supply, we thus have \( v(1) \) increases with \( r \) in this range. When \( r \geq 2 \), we have that each battle reaches its maximal efficiency in eliciting effort supply. In particular, the rents are fully dissipated in the first battle in which the two players' prize spreads are symmetric. Higher \( v(1) \) increases effort supply in the second and third battle, however, the change in the effort supply is fully offset by the effort decrease in the first battle due to the full surplus extraction in the first stage, as long as the strategic momentum for the winner of the first battle still persists.

For comparison purpose, we also analyze three alternative settings, i.e. a simultaneous-play multi-battle contest between the same two players, as well as simultaneous-play and sequential-play multi-battle team contests between two teams with the same number of players. For team contest, we assume that the prize awarded to a team is a public good to every members within the team. We find winner-take-all is optimal in all these alternative environments. A common key driving force that facilitates this result is that for all three alternatives, the players in a particular battle can assume that the other two battles are simply a fair draw in equilibrium.

Our paper primarily belongs to the well established literature on multi-battle contests. To our best knowledge, this is the first study on optimal contingent prize allocation in the environment of multi-battle contests with simultaneous or sequential play between the same two players, or simultaneous or sequential pairwise play between two teams with same number of players. A winner-take-all prize structure is commonly adopted in practice in multi-battle contests, and typically assumed in the literature. Our paper rationalizes the optimality of this popular prize allocation rule in a wide range of environments in multi-battle contest context. This finding extends the validity of the winner-take-all principle which has been established in many other settings, including Clark and Riis (1998), Krishna and Morgan (1998), Moldovanu and Sela (2001), Lai and Matros (2005), Fu and Lu (2012) and Schweinzer, Paul and Segev (2012) among others.
We find that a prize allocation rule that rewards every positive number of wins (i.e. split awards contingent on overall battle outcomes) generates the maximal expected total effort when the battles proceed sequentially between the same two players and the discriminatory power \( r \) is high \((r > \bar{r})\). In particular, the reward for winning one battle increases and the reward for winning two battles decreases with \( r \in [\bar{r}, \hat{r}] \). These findings suggest intermediate prizes can facilitate effort elicitation at many occasions in dynamic multi-battle contests between two players.\(^7\) Intermediate prizes are frequently observed in multi-battle races in reality. For instance, in Formula I car races, each Grand Prix generates some benefits to the winner, besides the grand championship that is awarded on an annual basis. Similarly, the PGA tour awards winners of all component tournaments, and a grand prize is awarded to the overall best performer at the end of the tour. Our paper provides a rationale for such practice.

The rest of the paper proceeds as follow. In Section 2, we set up the model in the framework of sequentially played three-battle contests between two same players, and introduce some notations and useful existing results in two-player single stage simultaneous-play contests of Tullock family allowing for the full span of discriminatory powers. In Section 3, we derive the subgame perfect equilibrium for every eligible contest technology by backward induction, and identify the optimal contingent prize allocation rule. In Section 4, we study the same contest but assuming the battles are simultaneously played between the two contestants. In Section 5, we study team contests with multiple pairwise battles. Section 6 provides some concluding remarks. The appendix collects some technical proofs.

2 The model setup

Two players \( A \) and \( B \) compete in a dynamic three-battle contest. Both of them are risk neutral and have unit marginal effort cost. They fight the three battles sequentially, and observe the past outcome (i.e. the state of the contest) before making effort in the current battle.

The contest organizer has a prize budget \( V \), which is normalized as 1. The organizer’s prize allocation rule is contingent on the contest outcome, i.e. the number of battles the each player wins. Let \( v(n) \), \( n \in \{0,1,2,3\} \) denote the prize that a player wins if he wins \( n \) battles. Alternatively, \( v(n) \) can be interpreted as the winning probability of a single indivisible grand prize with value 1. We thus have the following feasibility restrictions on the prize allocations:

\[
\begin{align*}
v(n) + v(n') &= 1, \forall n, n' \in \{0,1,2,3\}, n + n' = 3; \\
v(n) &\geq v(n'), \forall n \geq n'; \\
v(n) &\geq 0, \forall n.
\end{align*}
\]

The first constraint says that the sum of prizes to the two players cannot go beyond the total prize budget, i.e. it exhausts the whole budget. The second constraint says that the player with

\(^7\)Mago, Sheremeta and Yates (2013) find that rewarding intermediate battle prizes could boost effort supply in dynamic contests even with small \( r \). However, in their study, the extra battle prizes are funded by additional budget.
higher number of wins is awarded a higher prize. The third constraint means the prizes cannot be negative, which is natural when \( v(n) \) is interpreted as winning probabilities or the players are subject to limited liability.

A generalized Tullock contest technology is adopted for each component battle, in which both players exert their effort simultaneously. Let \( e_A \) and \( e_B \) denote the players’ effort in a battle. The player \( i \)'s probability of winning the battle is specified by

\[
p_i = \frac{x_i^r}{x_i^r + x_j^r}
\]

where \( i, j \in \{A, B\} \) and \( r \in (0, \infty) \) denotes the discriminatory power of the Tullock contest.

In this paper, we study the optimal prize allocation rule that elicits the highest expected total effort in the contest.

### 2.1 Preliminaries: equilibrium strategies in Tullock contests

We first present some existing results on equilibrium analysis in a two-player Tullock contest with asymmetric values and an arbitrary discriminatory power \( r \). These results pave the foundation of our analysis. Consider two players \( i \) and \( j \) competing in a generalized Tullock contest with discriminatory power \( r \). The value of player \( i \) is \( v_i \); and the value of player \( j \) is \( v_j \). Without loss of generality assuming \( v_i \geq v_j > 0 \). Player \( i \)'s winning probability is given by

\[
p_i = \frac{x_i^r}{x_i^r + x_j^r}
\]

where \( x_i \) and \( x_j \) are players’ effort, and \( r \in (0, \infty) \) denotes the discriminatory power of the contest. We use \( x_i(v_i, v_j; r) \) and \( x_j(v_i, v_j; r) \) denote the players’ equilibrium strategy, which can be in pure or mixed strategy.

**Definition 1** Assume \( 0 < z \leq 1 \). A cutoff \( \hat{r}(z) \in (1, 2] \) is defined as the unique solution to

\[
r = 1 + z^r.
\]

Nti (1999) establishes that a pure-strategy equilibrium exists if and only if \( r \) is bounded from above by a cutoff \( \hat{r}(\frac{v_i}{v_j}) \leq 2 \) and provides a complete characterization of the equilibrium strategy. Wang (2010) analyzes the case of \( r \in (\hat{r}(\frac{v_i}{v_j}), 2] \) and obtains a closed-form solution to the equilibrium strategy.\(^9\) The mixed-strategy equilibrium in an all-pay auction has been analyzed extensively in the literature. Alcalde and Dahm (2010) analyze the case of \( r > 2 \). They identify an “all-pay-auction” equilibrium in mixed strategies, although a closed form solution of the equilibrium strategies remains less than explicit. These characterizations are summarized as follows:

**Lemma 1** Assuming \( v_i \geq v_j > 0 \). The equilibrium bidding strategies \( x_i(v_i, v_j; r) \) and \( x_j(v_i, v_j; r) \) are:

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\(^8\)Otherwise, we can relabel the two players.

\(^9\)The cutoff \( \hat{r}(\frac{x}{y}) \) converges to 2 when \( x \) approaches \( y \), i.e., when the two players are symmetric. In that case, the particular case analyzed by Wang (2010) vanishes.
(i) If \( r \leq \hat{r}(\frac{v_i}{v_j}) \),

\[
\begin{align*}
  x_i(v_i, v_j; r) &= \frac{r v_i^{r+1} v_j^r}{(v_i^r + v_j^r)^2}, \\
  x_j(v_i, v_j; r) &= \frac{r v_j^{r+1} v_i^r}{(v_i^r + v_j^r)^2}.
\end{align*}
\]

(ii) If \( r \in (\hat{r}(\frac{v_i}{v_j}), 2] \),

\[
\begin{align*}
  x_i(v_i, v_j; r) &= \left( \frac{1}{r-1} \right)^{\frac{r}{2}} (1 - \frac{1}{r}) v_j, \\
  x_j(v_i, v_j; r) &= \begin{cases} 
    (1 - \frac{1}{r}) v_j, & \text{with probability } q = \frac{v_j}{v_i} \left( \frac{1}{r-1} \right)^{\frac{r}{2}}, \\
    0, & \text{with probability } 1 - q.
  \end{cases}
\end{align*}
\]

(iii) If \( r > 2 \),

\[
\begin{align*}
  x_i(v_i, v_j; r) &= \mu^*, \\
  x_j(v_i, v_j; r) &= \begin{cases} 
    \mu^*, & \text{with probability } q = \frac{v_j}{v_i}, \\
    0, & \text{with probability } 1 - q.
  \end{cases}
\end{align*}
\]

where \( \mu^* \) is the (symmetric) equilibrium mixed strategy identified by Baye, Kovenock, and de Vries (1994) in a two-player Tullock contest with \( r > 2 \), and fully dissipates the rent in the symmetric game when both have valuation \( v_j \).

It should be noted that a closed-form expression of the mixed strategy \( \mu^* \) is yet to be identified in the literature. Baye, Kovenock, and de Vries (1994) verify its existence and full rent dissipation in the associated equilibrium of a two-player symmetric contest. We use this result to obtain the equilibrium effort outlays in contests with stochastic entry.

In Case (i), the proposed pure-strategy equilibrium is unique. The uniqueness of the equilibrium in Case (ii) and Case (iii), however, has not been established in the literature. In all three cases, \( x_i(v_i, v_j; r) \) is a more aggressive strategy than \( x_j(v_i, v_j; r) \) as long as \( v_i \geq v_j \), i.e., contestant with value \( v_i \) tends to exert more effort because of his higher valuation.

### 2.2 Other notations

For convenience, we first introduce some notations that are used in our subsequent analysis.

**Definition 2** We define \( \tau \in (1, 1.2) \) as the unique solution of \( r = 1 + (\frac{1-\frac{r}{2}}{2 + \frac{r}{4}})^r \).

Clearly, a solution \( \tau \) must fall in \((1, 2)\). The uniqueness of \( \tau \) follows from the facts that when \( r \in (1, 2) \), the left hand side of the equation increases with \( r \) and the right hand side decreases with \( r \). Moreover, we have \( \tau < 1.2 \) as \((\frac{1-\frac{r}{2}}{2 + \frac{r}{4}})^r < 0.2 \) when \( r = 1.2 \). One can obtain the numerical solution of \( \tau \approx 1.1935 \).
Recall \( v(0) + v(3) = 1 \) and \( v(1) + v(2) = 1 \). It is thus sufficient for us to focus on \((v(0), v(2))\) to fully describe the prize structure. The following definitions are useful for deriving the optimal prize allocation rule when \( r \in (0, 2] \).

For convenience, we introduce the following definitions.

**Definition 3** Let \( w_A = \left(\frac{1}{2} - \frac{r}{4}\right)[v(1) - v(0)] + \left(\frac{1}{2} + \frac{r}{4}\right)[v(2) - v(1)] \), \( w_B = \left(\frac{1}{2} + \frac{r}{4}\right)[v(1) - v(0)] + \left(\frac{1}{2} - \frac{r}{4}\right)[v(2) - v(1)] \) and \( \eta = \frac{w_B}{w_A} \).

Using the budget constraints, \( w_A \) and \( w_B \) can be alternatively written as \( w_A = \left(\frac{1}{2} - \frac{r}{4}\right)[1 - v(2) - v(0)] + \left(\frac{1}{2} + \frac{r}{4}\right)[2v(2) - 1] \) and \( w_B = \left(\frac{1}{2} + \frac{r}{4}\right)[1 - v(2) - v(0)] + \left(\frac{1}{2} - \frac{r}{4}\right)[2v(2) - 1] \). Thus \( w_A, w_B \) and \( \eta \) only depend on \( r, v(0) \) and \( v(2) \). The economic meaning of these two terms will become clearer as the analysis proceeds. The proof of Lemma 3 will reveal that they can be interpreted as the players’ prize spreads in the second battle. Specifically, \( w_A \) stands for the prize spread of the winner of the first battle, and \( w_B \) stands for the prize spread of the loser of the first battle.

**Definition 4** (i) \( \forall r, \) we define \( \mathcal{V} = \{(v(0), v(2)) : 0 \leq v(0) \leq \frac{1}{2} \leq v(2) \leq 1 \) and \( v(2) + v(0) \leq 1 \}; \)

(ii) \( \forall r \in (0, 2], \) we define \( \mathcal{V}_0 = \mathcal{V} \cap \{(v(0), v(2)) : r \leq 1 + \left(\frac{w_B}{w_A}\right)^r \text{ and } w_A \leq w_B\} \), \( \mathcal{V}_1 = \mathcal{V} \cap \{(v(0), v(2)) : r \leq 1 + \left(\frac{w_B}{w_A}\right)^r \leq 2 \text{ and } w_A \leq w_B\} \), \( \mathcal{V}_2 = \mathcal{V} \cap \{(v(0), v(2)) : 1 + \left(\frac{w_B}{w_A}\right)^r < r \leq 2 \text{ and } w_A \geq w_B\} \), \( \mathcal{V}_3 = \mathcal{V} \cap \{(v(0), v(2)) : 1 + \left(\frac{w_B}{w_A}\right)^r < r \leq 2 \text{ and } w_A \leq w_B\} \), \( \mathcal{V}_4 = \mathcal{V} \cap \{(v(0), v(2)) : v(2) - v(1) \geq v(1) - v(0)\} \) and \( \mathcal{V}_5 = \mathcal{V} \cap \{(v(0), v(2)) : v(2) - v(1) \leq v(1) - v(0)\} \).

The restrictions \( 0 \leq v(0) \leq \frac{1}{2} \leq v(2) \leq 1 \) and \( v(2) + v(0) \leq 1 \) in the definition of \( \mathcal{V} \) are equivalent to the non-negativity and monotonicity of \( v(n) \) under budget constraints \( v(0)+v(3) = 1 \) and \( v(1)+v(2) = 1 \). Note that \( \mathcal{V}_2 \) and \( \mathcal{V}_3 \) can be empty for some \( r \) as will be illustrated.

**Property 1** When \( r \in (0, 2] \), we have \( \mathcal{V} = \cup_{i=0}^{3} \mathcal{V}_i \); and when \( r > 2 \), \( \mathcal{V} = \cup_{i=4}^{5} \mathcal{V}_i \).

## 3 Optimal prize allocation

We consider the optimal prize allocation rule for three ranges of the discriminatory power \( r \). In case 1, \( r \in (0, \bar{r}] \), in case 2, \( r \in (\bar{r}, 2] \), and in case 3, \( r \in (2, +\infty) \). For each case, the results of Lemma 1 can be utilized to solve for the subgame perfect equilibrium and compute the expected total effort from all three battles.

### 3.1 Case 1: \( r \in (0, \bar{r}] \)

The following lemma shows that for the current case, we only need to consider the prize structure covered by \( \mathcal{V}_0 \) and \( \mathcal{V}_1 \).

**Lemma 2** \( \forall r \in (0, \bar{r}], \) we have \( \mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1; \forall r \in (\bar{r}, 2], \) we have \( \mathcal{V}_2 \neq \emptyset \).
**Proof.** For the first part, it suffices to show that \( w_A \) and \( w_B \) satisfy either \( r \leq 1 + \left( \frac{w_B}{w_A} \right)^r \) when \( w_A \leq w_B \) or \( r \leq 1 + \left( \frac{w_B}{w_A} \right)^r \) when \( w_A \geq w_B \) for any given \( r \in (0, \tau] \). By the definitions of \( w_A \) and \( w_B \), we can easily verify that \( \left( \frac{1}{2} + \frac{r}{4} \right)^r \leq \left( \frac{w_A}{w_B} \right)^r \) when \( w_A \leq w_B \); and \( \left( \frac{2 - 1}{2} \right)^r \leq \left( \frac{w_B}{w_A} \right)^r \) when \( w_A \geq w_B \) for any \( r > 0 \). By the definition of \( \tau \) and the fact \( r \in (0, \tau] \), we have \( r \leq 1 + \left( \frac{2 - 1}{2 + 4} \right)^r \). Therefore \( r \leq 1 + \left( \frac{2 - 1}{2 + 4} \right)^r \leq 1 + \left( \frac{w_A}{w_B} \right)^r \) when \( w_A \leq w_B \); and \( r \leq 1 + \left( \frac{1}{2} \right)^r \leq 1 + \left( \frac{w_B}{w_A} \right)^r \) when \( w_A \geq w_B \), \( \forall r \in (0, \tau] \).

For the second part, we can verify that prize structure with \( v(0) = v(1) = 0, v(2) = v(3) = 1 \) must belong to \( \mathcal{V}_2, \forall r \in (\tau, 2] \). For this prize structure, we have \( w_A = \frac{1}{2} + \frac{r}{4} > w_B = \frac{1}{2} - \frac{r}{4} \), it is clear that \( 1 + \left( \frac{w_B}{w_A} \right)^r = 1 + \left( \frac{1}{2} \right)^r < r \) when \( r \in (\tau, 2] \). So \( \mathcal{V}_2 \neq \emptyset \).

Based on Lemma 1, we focus on the prize structures in \( \mathcal{V}_0 \cup \mathcal{V}_1 \), and pin down the total effort elicited at the subgame perfect equilibrium. The equilibrium is characterized in the proof of Lemma 3 following standard backward induction procedure. Appropriate parts of Lemma 1 are utilized in different battles depending on the relation between the players’ prize spreads and the discriminatory power in each battle.

**Lemma 3** \( \forall r \in (0, \tau] \), for prize structures in \( \mathcal{V}_0 \cup \mathcal{V}_1 \), the total expected effort elicited at equilibrium is

\[
TE_1 = \frac{w_A^r w_B^r}{[w_A^r + w_B^r]^2} \left[ (1 - \frac{r}{2}) w_A + (1 + \frac{r}{2}) w_B \right] + \frac{r}{2} [2v(2) - 1] + \frac{w_A^r r [1 - v(2) - v(0)]}{w_A^r + w_B^r}.
\]

**Proof.** We solve the subgame perfect equilibrium by backward induction. We use \( (n_A, n_B) \) to denote the number of wins that players have secured.

We first look at the third battle. When \( (n_A, n_B) = (2, 0) \), by the budget constraints, the two players have a common effective prize spread

\[
v_A(2, 0) = v_B(2, 0) = v(3) - v(2) = v(1) - v(0) \geq 0.
\]

By Lemma 1(i), we have the following equilibrium effort supply

\[
x_A(2, 0) = x_B(2, 0) = \frac{rv_A(2, 0)}{4} = \frac{r}{4} (v(1) - v(0)).
\]

Each player’s winning probability for this battle is

\[
p_A(2, 0) = p_B(2, 0) = \frac{1}{2}.
\]

The same results hold when \( (n_A, n_B) = (0, 2) \).

When \( (n_A, n_B) = (1, 1) \), the two players’ common effective prize spread is

\[
v_A(1, 1) = v_B(1, 1) = v(2) - v(1) \geq 0.
\]
By Lemma 1(i), we have the following equilibrium effort supply

\[ x_A(1, 1) = x_B(1, 1) = \frac{rv_A(2, 0)}{4} = \frac{r}{4}(v(2) - v(1)). \]

Each player’s winning probability for this battle is

\[ p_A(1, 1) = p_B(1, 1) = \frac{1}{2}. \]

We now turn to the second battle. When \((n_A, n_B) = (1, 0)\), the effective prize spread for player \(A\) is

\[ v_A(1, 0) = [p_A(2, 0)v(3) + p_B(2, 0)v(2) - x_A(2, 0)] - [p_A(1, 1)v(2) + p_B(1, 1)v(1) - x_A(1, 1)] \]
\[ = \left[ \frac{1}{2}v(3) + \frac{1}{2}v(2) - \frac{r}{4}(v(1) - v(0)) \right] - \left[ \frac{1}{2}v(2) + \frac{1}{2}v(1) - \frac{r}{4}(v(2) - v(1)) \right] \]
\[ = w_A. \]

Similarly, the effective prize spread for player \(B\) is \(v_B(1, 0) = w_B\).

Note that the prize structures are restricted to \(V_0 \cup V_1\). Thus Lemma 1(i) applies to the second battle. We have the following equilibrium effort

\[ x_A(1, 0) = \frac{rv_A(1, 0)}{[v_A(1, 0) + v_B(1, 0)]^2}; \quad x_B(1, 0) = \frac{rv_B(1, 0)}{[v_A(1, 0) + v_B(1, 0)]^2}. \]

The players’ winning probabilities are

\[ p_A(1, 0) = \frac{x_A(1, 0)}{v_A(1, 0) + v_B(1, 0)} = \frac{v_A(1, 0)}{v_A(1, 0) + v_B(1, 0)}; \]
\[ p_B(1, 0) = \frac{x_B(1, 0)}{v_A(1, 0) + v_B(1, 0)} = \frac{v_B(1, 0)}{v_A(1, 0) + v_B(1, 0)}. \]

When \((n_A, n_B) = (0, 1)\), similarly we have

\[ x_A(0, 1) = x_B(1, 0) = \frac{rv_A(1, 0)}{[v_A(1, 0) + v_B(1, 0)]^2}; \quad x_B(0, 1) = x_A(1, 0) = \frac{rv_B(1, 0)}{[v_A(1, 0) + v_B(1, 0)]^2}. \]

Then players’ winning probabilities are

\[ p_A(0, 1) = p_B(1, 0) = \frac{v_B(1, 0)}{v_A(1, 0) + v_B(1, 0)}; \quad p_B(0, 1) = p_A(1, 0) = \frac{v_A(1, 0)}{v_A(1, 0) + v_B(1, 0)}. \]
Now we consider the first battle. The effective prize spreads are symmetric across the two players:
\[
v_A(0, 0) = v_B(0, 0)
\]
\[
= \{p_A(1, 0)[p_A(2, 0)v(3) + p_B(2, 0)v(2) - x_A(2, 0)]
+ p_B(1, 0)[p_A(1, 1)v(2) + p_B(1, 1)v(1) - x_A(1, 1)] - x_A(1, 0)\}
- \{p_A(0, 1)[p_A(1, 1)v(2) + p_B(1, 1)v(1) - x_A(1, 1)]
+ p_B(0, 1)[p_A(0, 2)v(1) + p_B(0, 2)v(0) - x_A(0, 2)] - x_A(0, 1)\}
= \frac{rv_A'(0, 0)v_B'(1, 0)}{v_A'(1, 0) + v_B'(1, 0)}[v_B(1, 0) - v_A(1, 0)] + \frac{v_A'(1, 0)}{v_A'(1, 0) + v_B'(1, 0)}[v(2) - v(0)].
\]

Applying Lemma 1(i), we have the following equilibrium effort
\[
x_A(0, 0) = x_B(0, 0) = \frac{r}{4}v_A(0, 0),
\]
and players’ winning probabilities are
\[
p_A(0, 0) = p_B(0, 0) = \frac{1}{2}.
\]

The total effort thus can be calculated as follows by aggregating over all three battles:
\[
TE_1 = 2x_A(0, 0) + [x_A(1, 0) + x_B(1, 0)] + p_A(1, 0)[x_A(2, 0) + x_B(2, 0)] + p_B(1, 0)[x_A(1, 1) + x_B(1, 1)]
= \frac{rw_A'^r w_B'^r}{[w_A'^r + w_B'^r]^2}[(1 - \frac{r}{2})w_A + (1 + \frac{r}{2})w_B] + \frac{r}{2}(v(2) - v(1)) + r(v(1) - v(0)) \frac{w_A'^r}{w_A'^r + w_B'^r},
\]
which gives the desired result by incorporating the budget constraints. ■

Note that \(TE_1\) solely depend on prizes \(v(0)\) and \(v(2)\), with \(0 \leq v(0) \leq \frac{1}{2} \leq v(2) \leq 1\) and \(v(2) + v(0) \leq 1\). The next lemma pins down how \(TE_1\) changes with \(v(0)\) for given \(v(2) \in [\frac{1}{2}, 1]\).

\(\text{Lemma 4} \ \forall r \in (0, \bar{r}], \text{ we have } \frac{dT E_1}{dv(0)} < 0, \forall v(0) \in [0, 1]. \text{ Therefore, we have } v(0) = 0 \text{ and } v(3) = 1 \text{ at the optimum.} \)

\(\text{Proof.} \ \text{See Appendix.} \ ■ \)

Lemma 4 shows that \(TE_1\) always decreases with \(v(0)\) in the eligible range determined by \(v(2)\). We thus conclude that the optimal \(v(0) = 0\) and \(v(3) = 1\). This result is quite intuitive, at optimum, no reward should be awarded to a player with zero win to boost players’ incentive for exerting high effort. Doing so would save budget for rewarding better performance and increase effective prize spreads for players.

We next pin down prizes \(v(1)\) and \(v(2)\).

\(\text{Definition 5} \ \text{Define } \bar{r} \text{ to be the unique solution of } D(\frac{1 - \bar{r}}{2}, \bar{r}) = 0 \text{ on } (0, \bar{r}], \text{ where } D(\eta, r) \equiv [(3r - 1) + (\bar{r} + 1)\eta][1 - (\bar{r}) - 1 + (r - 1)\eta^r] + (1 + \bar{r})(-r + 1 + (r + 1)\eta^r)\eta + 2\eta(1 + \eta^r)(2 - \frac{3}{4}\bar{r} + \bar{r}^3). \)
The existence and uniqueness of $r$ is revealed by the following property.

**Property 2** $D(\frac{1-r}{2+\frac{r}{4}}, r)$ strictly decreases with $r \in (0, \bar{r}]$ and has a unique root of $r \approx 1.0884$ in this range.

$$D(\frac{1-r}{2+\frac{r}{4}}, r) \text{ can be simplified as } \frac{1}{2} \left(\frac{2-\bar{r}}{2+\bar{r}}\right)^2 r + (5r^2 + 12) \left(\frac{2-\bar{r}}{2+\bar{r}}\right)^{r} - 11r^2 + 8.$$ Property 2 can be verified by using standard tool such as Mathematica. Let $\xi(r) = 4 \left(\frac{2-\bar{r}}{2+\bar{r}}\right)^2 r + (5r^2 + 12) \left(\frac{2-\bar{r}}{2+\bar{r}}\right)^{r} - 11r^2 + 8$. It can be verified that $\xi'(r) \leq 0$ on $[0, 2]$, which means $D(\frac{1-r}{2+\frac{r}{4}}, r)$ has a unique root $r$ on $[0, 2]$. This generalized property will be further used in Section 3.2 when we talk about the case of $r \in (\bar{r}, 2]$.

**Property 3** $\forall r \in (0, \bar{r}], D(\eta, r)$ increases with $\eta$ when $\eta \geq \frac{1-r}{\frac{2+r}{4}}$.

**Proof.** See Appendix. ■

We now are ready to fully pin down the optimal prize structure. Recall $\eta(v(0) = 0, v(2), r) = \frac{w_B}{w_A} = \frac{\left(\frac{1}{2} \frac{r}{4}\right)(1-v(2)) + \left(\frac{1}{2} \frac{r}{4}\right)(2v(2)-1)}{\left(\frac{1}{2} \frac{r}{4}\right)(1-v(2)) + \left(\frac{1}{2} \frac{r}{4}\right)(2v(2)-1)}$. Note $\eta(v(0) = 0, v(2), r)$ decreases with $v(2) \in \left[\frac{1}{2}, 1\right]$ when $r \leq 2$. Thus $\frac{2+r}{4} \geq \eta(v(0) = 0, v(2), r) \geq \frac{1-r}{\frac{2+r}{4}}$. Note $D(\eta(v(0) = 0, v(2), r), r) > 0, \forall v(2) \in \left[\frac{1}{2}, 1\right], r \in (0, \bar{r}]$. By the Property 2 of $D(\frac{1-r}{2+\frac{r}{4}}, r)$, we know $D(\eta(v(0) = 0, v(2) = 1, r), r) = D(\frac{1-r}{2+\frac{r}{4}}, r) < 0, \forall r \in (\bar{r}, \bar{r}]$. Moreover, $D(\eta(v(0) = 0, v(2) = \frac{3}{2}, r), r) = 3r^2 + 12 > 0$. By Property 3, let $v^*(2) \in \left(\frac{3}{2}, 1\right)$ to be the unique solution of $D(\eta(v(0) = 0, v(2), r), r) = 0, \forall r \in (\bar{r}, \bar{r}]$. By simulation, we can verify that $v^*(2)$ decreases with $r \in (\bar{r}, \bar{r}]$.

**Theorem 1** Consider a sequentially played three-battle contest between two same individual contestants.

(i) $\forall r \in (0, \bar{r}]$, we have that prize structure $v(0) = v(1) = 0, v(2) = v(3) = 1$ uniquely induces the highest expected total effort;

(ii) $\forall r \in (\bar{r}, \bar{r}]$, we have that prize structure $v(0) = 0, v(1) = 1 - v^*(2), v(2) = v^*(2) \in \left(\frac{3}{2}, 1\right)$, and $v(3) = 1$ uniquely induces the highest expected total effort.

**Proof.** Based on Lemma 4, to fully identify the optimal prize allocation for a given $r \in (0, \bar{r}]$, we need to maximize $TE_1(v(0) = 0, v(2))$ by choosing the optimal $v(2) \in \left[\frac{1}{2}, 1\right]$. For this purpose, we look at the first order derivative of $TE_1(v(0) = 0, v(2))$ with respect to $v(2)$, which eventually generates the deserved results as stated in the Theorem. The details are provided in the Appendix. ■

### 3.2 Case 2: $r \in (\bar{r}, 2]$

We now turn to the case where $r \in (\bar{r}, 2]$. Recall in this case, we have we have $\mathcal{V} = \cup_{i=0}^{3} \mathcal{V}_i$ by Property 1. We show that all prize profiles not in $\mathcal{V}_1$ can never be optimal in a few steps. We then focus on $\mathcal{V}_1$ to fully characterize the optimal profile when $r \in (\bar{r}, 2]$.

We first establish the following lemma which says that we can ignore prize structures in $\mathcal{V}_0$ for the optimal prize structure in the sense that these are dominated by those in $\mathcal{V}_1$. Note that $\mathcal{V}_1$ covers the prize structures such that $w_A \geq w_B$, i.e. the winner of the first battle has a higher prize spread.
Lemma 5 \( \forall r \in (\pi, 2] \), any prize profile in \( V_0 \) is strictly dominated by a prize profile in \( V_1 \), unless it is the profile with \( v(0) = 0, v(2) = \frac{2}{3} \), which belongs to \( V_0 \cap V_1 \).

Proof. For a prize profile in \( V_0 \) or \( V_1 \), we have that the effort is given by \( TE_1 \) in Lemma 3. Take any prize profile in \( V_0 \) and let it differ from \( (v(0) = 0, v(2) = \frac{2}{3}) \). We prove this lemma by finding a prize structure in \( V_1 \) that induces a strictly higher level of effort than the given prize structure in \( V_0 \). The detailed proof is in the appendix. ■

Lemma 5 thus requires that the optimal prize structure should render a higher prize spread for the winner of the first battle, which is very intuitive if the contest organizer wants to provide better incentive for the contestants to exert higher effort.

The following lemma provides two lower bounds for the maximal effort inducible in \( V_1 \) by the effort levels generated by two prize profiles, one in \( V_1 \) and one in \( V_0 \). \( \forall r \in (\pi, 2] \), we have that prize profiles with \( \eta = \frac{w_B}{w_A} \in [(r - 1)^{\frac{1}{2}}, 1] \) belongs to \( V_1 = V \setminus \{(v(0), v(2)) : r \leq 1 + \left( \frac{w_B}{w_A} \right)^r \text{ and } w_A \geq w_B \} \), and prize profiles with \( \eta = \frac{w_B}{w_A} \in [1, (r - 1)^{-\frac{1}{2}}] \) belong to \( V_0 = V \setminus \{(v(0), v(2)) : r \leq 1 + \left( \frac{w_B}{w_A} \right)^r \text{ and } w_A \leq w_B \} \). In particular, \( (v(0) = 0, v(2) = \frac{2r[(r - 1)^{\frac{1}{2}} + 1]}{3r + 3r(r - 1)^{\frac{1}{2}} - 2(r - 1)^{\frac{1}{2}} + 2}) \) belongs to \( V_1 \), with \( \eta = (r - 1)^{\frac{1}{2}} \); and \( (v(0) = 0, v(2) = \frac{2r[(r - 1)^{\frac{1}{2}} + 1]}{3r + 3r(r - 1)^{\frac{1}{2}} - 2(r - 1)^{\frac{1}{2}} + 2}) \) belongs to \( V_0 \), with \( \eta = (r - 1)^{-\frac{1}{2}} \). Note that \( v(2) = \frac{2r[(r - 1)^{\frac{1}{2}} + 1]}{3r + 3r(r - 1)^{\frac{1}{2}} + 2(r - 1)^{\frac{1}{2}} + 2} \geq \frac{2}{3} \) and \( v(2) = \frac{2r[(r - 1)^{\frac{1}{2}} + 1]}{3r + 3r(r - 1)^{\frac{1}{2}} - 2(r - 1)^{\frac{1}{2}} + 2} \leq \frac{2}{3} \) when \( r \in [\pi, 2] \). Moreover, \( (v(0) = 0, v(2) = \frac{2r[(r - 1)^{\frac{1}{2}} + 1]}{3r + 3r(r - 1)^{\frac{1}{2}} - 2(r - 1)^{\frac{1}{2}} + 2}) \) is on the boundary separating \( V_1 \) and \( V_2 \); and \( (v(0) = 0, v(2) = \frac{2r[(r - 1)^{\frac{1}{2}} + 1]}{3r + 3r(r - 1)^{\frac{1}{2}} - 2(r - 1)^{\frac{1}{2}} + 2}) \) is on the boundary separating \( V_0 \) and \( V_3 \).

Lemma 6 \( \forall r \in (\pi, 2] \), we have \( \max_{V_1} TE_1 \geq \max \left\{ \pi + \frac{(r - 1)(2 - r)}{1 + \frac{r}{2} + (1 + \frac{r}{2})(r - 1)^{\frac{1}{2}}} \left( \frac{5r}{2} + \frac{5r}{2} - 2 + \frac{(\frac{r}{2} + 1)(r - 1)^{\frac{1}{2}}}{2} \right) \right\} \).

Proof. See appendix. ■

The following lemma pins down the expected total effort induced by a prize profile in \( V_2 \).

Lemma 7 \( \forall r \in (\pi, 2] \), a prize profile in \( V_2 \) induces the total effort

\[
TE_2 = \frac{r}{2}(1 - 2v(0)) + (1 - \frac{1}{r})\left( \frac{1}{r - 1} \right)^{\frac{1}{2}}(2 - r)\left[ \frac{r}{2} - (\frac{1}{2} + \frac{r}{4})v(0) + (\frac{1}{2} - \frac{3r}{4})v(2) \right].
\]

Proof. The game is solved by backward induction. Recall \( (n_A, n_B) \) denotes the history of the game. We first consider the third battle, which can be solved in the same way as Lemma 3.

When \( (n_A, n_B) = (2, 0) \) or \( (0, 2) \)

\[
x_A(2, 0) = x_B(2, 0) = x_A(0, 2) = x_B(0, 2) = \frac{rv_A(2, 0)}{4} = \frac{r}{4}(v(1) - v(0)).
\]

The winning probabilities are

\[
p_A(2, 0) = p_B(2, 0) = p_A(0, 2) = p_B(0, 2) = \frac{1}{2}.
\]
When \((n_A, n_B) = (1, 1)\), we have

\[ x_A(1, 1) = x_B(1, 1) = \frac{r v_A(1, 1)}{4} = \frac{r}{4}(v(2) - v(1)). \]

The winning probabilities are

\[ p_A(2, 0) = p_B(2, 0) = \frac{1}{2}. \]

Next, we look at the second battle. When \((n_A, n_B) = (1, 0)\), recall from the proof of Lemma 3 that the effective prize spreads are as follows:

\[ v_A(1, 0) = \frac{1}{2}(1 - v(1) - v(0)) + \frac{r}{4}(1 - 3v(1) + v(0)) = w_A, \]

\[ v_B(1, 0) = \frac{1}{2}(1 - v(1) - v(0)) + \frac{r}{4}(-1 + 3v(1) - v(0)) = w_B. \]

\(v_A(1, 0) \geq v_B(1, 0)\) is equivalent to \(w_A \geq w_B\). As the prize profiles are in \(V_2\), applying Lemma 1(ii) gives the equilibrium effort:

\[ \tilde{x}_A(1, 0) = (\frac{1}{r-1})^{\frac{1}{r}}(1 - \frac{1}{r})v_B(1, 0), \]

\[ \tilde{x}_B(1, 0) = \begin{cases} (1 - \frac{1}{r})v_B(1, 0) & \text{with probability } q = \frac{v_B(1, 0)}{v_A(1, 0)}(\frac{1}{r-1})^{\frac{1}{r}}, \\ 0 & \text{with probability } 1 - q. \end{cases} \]

The winning probabilities are

\[ p_A(1, 0) = 1 - (1 - \frac{1}{r})q; \quad p_B(1, 0) = (1 - \frac{1}{r})q. \]

Similarly, when \((n_A, n_B) = (0, 1)\), we have

\[ \tilde{x}_A(0, 1) = \tilde{x}_B(1, 0) = \begin{cases} (1 - \frac{1}{r})v_B(1, 0) & \text{with probability } q = \frac{v_B(1, 0)}{v_A(1, 0)}(\frac{1}{r-1})^{\frac{1}{r}}, \\ 0 & \text{with probability } 1 - q. \end{cases} \]

\[ \tilde{x}_B(0, 1) = \tilde{x}_A(1, 0) = (\frac{1}{r-1})^{\frac{1}{r}}(1 - \frac{1}{r})v_B(1, 0). \]

The winning probabilities are

\[ p_A(0, 1) = (1 - \frac{1}{r})q; \quad p_B(0, 1) = 1 - (1 - \frac{1}{r})q. \]

We have

\[ E[\tilde{x}_A(0, 1)] = E[\tilde{x}_B(1, 0)] = q(1 - \frac{1}{r})v_B(1, 0) = (1 - \frac{1}{r})(\frac{1}{r-1})^{\frac{1}{r}}\frac{v_B(1, 0)}{v_A(1, 0)}. \]
Now we come to the first battle. The common prize spread is

\[ v_A(0, 0) = v_B(0, 0) \]
\[ = \{p_A(1, 0)[p_A(2, 0)v(3) + p_B(2, 0)v(2) - x_A(2, 0)] + p_B(1, 0)[p_A(1, 1)v(2) + p_B(1, 1)v(1) - x_A(1, 1)] - \bar{x}_A(1, 0)\} \]
\[ - \{p_A(0, 1)[p_A(1, 1)v(2) + p_B(1, 1)v(1) - x_A(1, 1)] + p_B(0, 1)[p_A(0, 2)v(1) + p_B(0, 2)v(0) - x_A(0, 2)] - E[\bar{x}_A(0, 1)]\} \]
\[ = [1 - (1 - \frac{1}{r})q]\{v(2) - v(0)\} + (1 - \frac{1}{r})v_B(1, 0)\bar{q} + \frac{(r - 1)}{r - 1}\bar{q}. \]

Thus the effort supply is

\[ x_A(0, 0) = x_B(0, 0) = \frac{r}{4} v_A(0, 0), \]

and winning probabilities are

\[ p_A(0, 0) = p_B(0, 0) = \frac{1}{2}. \]

Aggregating over the three battle, we have the total effort:

\[ TE_2 = 2x_A(0, 0) + x_A(1, 0) + E[\bar{x}_B(1, 0)] \]
\[ + p_A(1, 0)(x_A(2, 0) + x_A(2, 0)) + p_B(1, 0)(x_A(1, 1) + x_B(1, 1)) \]
\[ = \frac{r}{2} (1 - 2v(0)) + (1 - \frac{1}{r})(\frac{1}{r - 1})\hat{f}(2 - r)(\frac{r}{2} - (\frac{1}{2} + \frac{r}{4})v(0) + (\frac{1}{2} - \frac{3r}{4})v(2)). \]

In the next two propositions, we show that we can ignore the prize profiles in \( \mathcal{V}_2 \) and \( \mathcal{V}_3 \) when searching for the optimal prize structure.

**Proposition 1** \( \forall r \in (\bar{r}, 2], \) we have \( \sup_{\mathcal{V}_2} TE_2 = TE_2^* := \frac{r}{2} + \frac{(r-1)(2-r)}{[(r-1)^{\frac{1}{2}}(1+\frac{r}{2})+\frac{r}{2}-1]} \). However, there exists no prize structure in \( \mathcal{V}_2 \) that can induce \( \sup TE_2 \).

**Proof.** We want to maximize effort \( TE_2 \) subject to prize profiles are in \( \mathcal{V}_2 \), i.e. \( w_A \geq w_B \) and \( 1 + (\frac{w_B}{w_A})^r < r \leq 2 \), as well as the non-negativity and monotonicity of the prizes. Since \( r \in (\bar{r}, 2] \), constraints \( w_A \geq w_B \) and \( 1 + (\frac{w_B}{w_A})^\bar{r} < r \leq 2 \) can be written as \( (r - 1)^{\frac{1}{2}} w_A > w_B \), i.e. \( v(2) > \frac{(r-1)^{\frac{1}{2}} - \frac{1}{2}r(r-1)^{\frac{1}{2}} - \frac{1}{2}r}{\frac{1}{2}r(r-1)^{\frac{1}{2}} + \frac{1}{2}r(r-1)^{\frac{1}{2}} + \frac{1}{2}r} v(0) + \frac{1}{2}r + \frac{1}{2}r(r-1)^{\frac{1}{2}} \). Clearly, \( TE_2 \) is decreasing in both \( v(0) \) and \( v(2) \) when \( r \in (\bar{r}, 2] \). We thus can set \( v(0) = 0 \) and consider simplified constraint \( v(2) > \frac{v(1)(r-1)^{\frac{1}{2}}}{(r-1)^{\frac{1}{2}}(1+\frac{r}{2})+\frac{r}{2}-1} \), which is less than 1 when \( r > \bar{r} \). We thus have for any prize profile in \( \mathcal{V}_2 \), the effort induced is converging to but
smaller than \( TE_2(v(0) = 0, v(2) = \frac{r(1 + (r - 1)^{\frac{1}{2}})}{r(1 + \frac{3}{2}r + \frac{3r}{2} - 1)} \), which is the following.

\[
TE_2(v(0) = 0, v(2) = \frac{r(1 + (r - 1)^{\frac{1}{2}})}{[(r - 1)^{\frac{1}{2}}(1 + \frac{3}{2}r + \frac{3r}{2} - 1)]} = \frac{r}{2} + (1 - \frac{1}{r})(\frac{1}{r - 1})^{\frac{1}{2}}(2 - r)\frac{r}{2} + (\frac{1}{2} - \frac{3r}{4})\frac{r(1 + (r - 1)^{\frac{1}{2}})}{[(r - 1)^{\frac{1}{2}}(1 + \frac{3}{2}r + \frac{3r}{2} - 1)]}
\]

\[
= \frac{r}{2} + (1 - \frac{1}{r})(\frac{1}{r - 1})^{\frac{1}{2}}(2 - r)\frac{1}{[(r - 1)^{\frac{1}{2}}(1 + \frac{3}{2}r + \frac{3r}{2} - 1)]}\frac{r}{2}(r - 1)^{\frac{1}{2}}
\]

\[
= \frac{r}{2} + \frac{(r - 1)(2 - r)}{[(r - 1)^{\frac{1}{2}}(1 + \frac{3}{2}r + \frac{3r}{2} - 1)]}.
\]

Similar to Lemma 7 and Proposition 1, we provide the total effort induced by prize profiles in \( V_3 \), and establish an upper bound which is not attainable in \( V_3 \).

**Proposition 2** \( \forall r \in (\tau, 2] \), a prize profile in \( V_3 \) induces the total effort

\[
TE_3 = \frac{r}{2} (2v(2) - 1) + (1 - \frac{1}{r})(\frac{1}{r - 1})^{\frac{1}{2}}(2 + r)(v(0) - \frac{r}{2} - (\frac{1}{2} - \frac{r}{4})v(0) + (\frac{1}{2} + \frac{3}{4}r)v(2)).
\]

We have \( \sup_{V_3} TE_3 = TE_3 := \frac{(r^2 + \frac{7}{4}(r - 1)^{\frac{1}{2}} + \frac{7}{2}r^2 + \frac{7}{2} - 2)}{[1 + \frac{3}{2}r + (\frac{3r}{2} - 1)r^1]}. \) Moreover, there exists no prize structure in \( V_3 \) that can induce \( \sup TE_3 \).

**Proof.** See appendix. ■

According to Lemma 6, Propositions 1 and 2, we have \( \max_{V_1} TE_1 \geq \max\{\sup_{V_2} TE_2, \sup_{V_3} TE_3\}. \) Together with Lemma 5, we have the following result.

**Proposition 3** \( \forall r \in (\tau, 2] \), the optimal prize profile cannot be in \( V_1 \setminus V \) and it must be in \( V_1 \) if it exists.

The next Theorem establishes the existence and uniqueness of optimal prize profile, which is explicitly characterized. Note that the discussion following Property 2 has revealed that \( D(\frac{\tau^2}{\tau^2 + \frac{7}{2}}, r) \) has a unique root \( \tau \) on \( [0, 2] \).

**Theorem 2** \( \forall r \in (\tau, 2] \), the optimal prize profile exists and is unique. Thus it is in \( V_1 \). Let \( r^* \in (\tau, 2) \) such that \( D(v = (r - 1)^{\frac{1}{2}}, r)|_{v=r^*} = 0 \).

(i) When \( r \in (\tau, r^*] \), the prize profile with \( v(0) = 0, v(1) = 1 - v^*(2), v(2) = v^*(2) \) and \( v(3) = 1 \) is optimal, where \( v^*(2) \in [\frac{3}{2}, 1] \) is determined by \( D(\eta (v(0) = 0, v^*(2), r), r) = 0 \);

(ii) When \( r^* \leq r \leq 2 \), the prize profile with \( v(0) = 0, v(1) = 0 \) is optimal.
(ii) When \( r \in (r^*, 2] \), the prize profile with \( v(0) = 0 \), \( v(1) = \frac{\frac{1}{2} + \frac{r}{4} + \frac{r}{4}(r-1)^{\frac{1}{3}}}{\frac{1}{2} + \frac{r}{4} + \frac{r}{4}(r-1)^{\frac{1}{3}}}, v(2) = 1 - v(1) \) and \( v(3) = 1 \) is optimal.

**Proof.** Similar to Lemma 4 and Theorem 1, we can obtain that \( v^*(0) = 0 \) and \( \frac{d}{dv(2)}(TE_1(v(0)=0,v(2))) \) sgn \( D(\eta, r) \), where \( D(\eta, r) \) is increasing in \( \eta \) for each \( r \in [\tau, 2] \).

For \( \mathcal{V}_1 \), \( r \leq 1 + (\frac{w_B}{w_A})^r \) and \( w_A \geq w_B \) leads to \( \frac{-\frac{1}{2} + \frac{r}{4} + \frac{r}{4}(r-1)^{\frac{1}{3}}}{\frac{1}{2} + \frac{r}{4} + \frac{r}{4}(r-1)^{\frac{1}{3}}} \leq v(1) \) and \( v(1) \leq \frac{1}{3} \) respectively, as a result, \( v(2) \in [\frac{2}{3}, 1 - \frac{-\frac{1}{2} + \frac{r}{4} + \frac{r}{4}(r-1)^{\frac{1}{3}}}{\frac{1}{2} + \frac{r}{4} + \frac{r}{4}(r-1)^{\frac{1}{3}}}]. \) Note that \( r \geq \tau \) implies that \( \eta := \frac{w_B}{w_A} \in [(r-1)^{\frac{1}{3}}, 1] \subset [\frac{1}{2} + \frac{r}{4}, 1] \). Direct calculation yields that \( \eta(v(0) = 0, v(2), r) \) is decreasing in \( v(2) \) and thus \( D(\eta(v(0) = 0, v(2), r), v(2), r) \) is decreasing in \( v(2) \).

By the similar arguments in Theorem 1, we have \( D(\eta = (r-1)^{\frac{1}{3}}, r) < 0 \) for \( r \in (\tau, r^*) \) and \( D(v(r-1)^{\frac{1}{3}}) \) \( D(v(r) > 0 \) for \( r \in (r^*, 2] \), where \( r^* \) satisfies \( D(v(r-1)^{\frac{1}{3}}, r) = 0 \). Therefore, when \( r \in (\tau, r^*) \), the optimal \( v^*(2) \) is determined by \( D(\eta(v(0) = 0, v^*(2)), r) = 0 \) because \( D(\eta(v(0) = 0, v^*(2), r)^v < v^*(2), r) < 0 \) and \( D(\eta(v(0) = 0, v^*(2), r), r) < 0 \), while \( v^*(2) = 1 - \frac{-\frac{1}{2} + \frac{r}{4} + \frac{r}{4}(r-1)^{\frac{1}{3}}}{\frac{1}{2} + \frac{r}{4} + \frac{r}{4}(r-1)^{\frac{1}{3}}} \) is optimal when \( r \in (r^*, 2] \).

The first part holds because of Proposition 3. The details are included in the appendix. ■

By Theorem 1 and 2, we have \( \frac{d}{dv(2)}(TE_1(v(0)=0,v(2))) \) sgn \( D(\eta, r) \), where

\[
\eta \equiv \frac{w_B}{w_A} = \frac{\left(\frac{1}{2} + \frac{r}{4}\right)(1 - v(2)) + \left(\frac{1}{2} - \frac{r}{4}\right)(2v(2) - 1)}{\left(\frac{1}{2} - \frac{r}{4}\right)(1 - v(2)) + \left(\frac{1}{2} + \frac{r}{4}\right)(2v(2) - 1)}.
\]

Note that \( \eta \geq \frac{1 - \frac{r}{2}}{2 + \frac{r}{4}} \), and \( \eta = \frac{1 - \frac{r}{2}}{2 + \frac{r}{4}} \) can be achieved only by \( v(2) = 1 \).

When \( r \leq \tau \), if \( D(\eta = \frac{1 - \frac{r}{2}}{2 + \frac{r}{4}}, r) > 0 \), then \( D(\eta, r) > 0 \) for all feasible \( \eta \) because \( D(\eta, r) \) increases with \( \eta \) (Property 3), as a result, \( v^*(2) = 1 \); if \( D(\eta = \frac{1 - \frac{r}{2}}{2 + \frac{r}{4}}, r) < 0 \), \( D(\eta = \frac{1 - \frac{r}{2}}{2 + \frac{r}{4}}, r) < 0 \), so reducing \( v(2) \) from 1 would elicit higher \( TE_1 \) until \( D(\eta = \frac{1 - \frac{r}{2}}{2 + \frac{r}{4}}, r) = 0 \).

When \( r > \tau \), \( \frac{1 - \frac{r}{2}}{2 + \frac{r}{4}} \) is still a lower bound for \( \eta \), while there is another lower bound \( \eta \geq (r-1)^{\frac{1}{3}} \) within \( \mathcal{V}_1 = \mathcal{V} \cap \{(v(0), v(2)) \text{ such that } r \leq 1 + \eta^r \text{ and } w_A \geq w_B \} \). Note that \( \eta = (r-1)^{\frac{1}{3}} \) can be reached by a \( (v(0), v(2)) \) with \( r = 1 + \eta^r \) and \( w_A \geq w_B \). \( (r-1)^{\frac{1}{3}} \) is a better lower bound as \( (r-1)^{\frac{1}{3}} \geq \frac{1 - \frac{r}{2}}{2 + \frac{r}{4}} \) when \( r > \tau \), so we consider the sign of \( D(\eta = (r-1)^{\frac{1}{3}}, r) \) here. The rest arguments are the same as the case of \( r \leq \tau \).

In Figure 1, we plot \( D(\eta = \text{lower bound}, r) = D(\eta = \max\left\{\frac{1 - \frac{r}{2}}{2 + \frac{r}{4}}, (r-1)^{\frac{1}{3}}\right\}, r) \) to summarize the arguments above.

In the graphs, \( r \approx 1.09 \) is the horizontal ordinate of the intersection of the black line and the blue line; \( r^* \approx 1.31 \) is the horizontal ordinate of the intersection of the red line and the black line. Figure 2 plots the optimal prize \( v(1) \) as a function of \( r \in [1, 2] \). Note that by Figure 2, \( v(1) \) strictly increases with \( r \in [\tau, 2] \) and is bounded above by \( \frac{1}{3} \).
3.3 Case 3: $r > 2$

We now consider the remaining case of $r > 2$, in which all prize profiles are covered by sets $V_4$ and $V_5$. In Lemmas 7 and 8, we will provide the total effort for prize profiles in $V_4$ and $V_5$ respectively. In Propositions 4 and 5, we will characterize the optimal prizes in $V_4$ and $V_5$ respectively. Combining both propositions fully pins down the optima prize structures when $r > 2$.

**Lemma 8** When $r > 2$, the expected aggregate effort $T E_4 = 1 - 2v(0)$ for all prize profiles in $V_4 = \mathcal{V} \cap \{(v(0), v(2)) : v(2) - v(1) \geq v(1) - v(0)\}$.

**Proof.** Note that when $r > 2$, we have Lemma 1(iii) applies. As usual, we solve the game by backward induction.

First, consider battle 3. at history $(n_A, n_B) = (2, 0)$: For history $(2, 0)$, the common effective prize spread is:

$$v_A(2, 0) = v_B(2, 0) = v(1) - v(0) \geq 0.$$ 

Thus effort supply (in the mixed-strategy equilibrium) is given by

$$\widetilde{x}_A(2, 0) \sim G_A^{(2,0)}(x) = \frac{x}{v(1) - v(0)} \text{ in } [0, v(1) - v(0)],$$

$$\widetilde{x}_B(2, 0) \sim G_B^{(2,0)}(x) = \frac{x}{v(1) - v(0)} \text{ in } [0, v(1) - v(0)],$$

where $G_{(n_A,n_B)}^i(\cdot), i = A, B$ denotes the cumulative distribution function of player $i$.

The winning probabilities are

$$p_A(2, 0) = p_B(2, 0) = \frac{1}{2}.$$ 

History $(0,2)$ is symmetric. The effective prize spread is

$$v_A(0, 2) = v_B(0, 2) = v(1) - v(0) \geq 0.$$ 

The effort supply is

$$G_A^{(0,2)}(x) = G_B^{(0,2)}(x) = \frac{x}{v(1) - v(0)} \text{ in } [0, v(1) - v(0)].$$

The winning probabilities are

$$p_A(0, 2) = p_B(0, 2) = \frac{1}{2}.$$ 

When $(n_A, n_B) = (1, 1)$, the common effective prize spread is

$$v_A(1, 1) = v_B(1, 1) = v(2) - v(1) \geq 0.$$
Thus effort supply is

\[ G^A_{(1,1)}(x) = G^B_{(1,1)}(x) = \frac{x}{v(2) - v(1)} \text{ in } [0, v(2) - v(1)]. \]

The probabilities are

\[ p_A(1, 1) = p_B(1, 1) = \frac{1}{2}. \]

We now consider the second battle. When \((n_A, n_B) = (1, 0)\), the effective prize spreads are as follows. For player 1,

\[
\begin{align*}
\varv_A(1, 0) &= [p_A(2, 0)v(3) + p_B(2, 0)v(2) - E\tilde{x}_A(2, 0)] - [p_A(1, 1)v(2) + p_B(1, 1)v(1) - E\tilde{x}_A(1, 1)] \\
&= \frac{1}{2}(v(3) - v(1)) + E\tilde{x}_A(1, 1) - E\tilde{x}_A(2, 0) \\
&= v(2) - v(1).
\end{align*}
\]

Similarly,

\[ \varv_B(1, 0) = v(1) - v(0). \]

We have \(\varv_A(1, 0) \geq \varv_B(1, 0)\) if and only if \(v(2) - v(1) \geq v(1) - v(0)\), which is the case considered in this lemma, i.e. prize profiles in \(V_4\).

Thus the effort supply is

\[
\begin{align*}
G^A_{(1,0)}(x) &= \frac{x}{v(1) - v(0)} \text{ in } [0, (v(1) - v(0))], \\
G^B_{(1,0)}(x) &= \frac{(v(2) - v(1)) - (v(1) - v(0)) + x}{(v(2) - v(1))} \text{ in } [0, v(1) - v(0)].
\end{align*}
\]

The winning probabilities are

\[ p_A(1, 0) = 1 - \frac{1}{2}q, \quad p_B(1, 0) = \frac{q}{2}, \]

where \(q = \frac{v(1) - v(0)}{v(2) - v(1)}\).

Players’ expected effort is

\[
\begin{align*}
E[\tilde{x}_A(1, 0)] &= \frac{1}{2}(v(1) - v(0)), \quad E[\tilde{x}_B(1, 0)] = \frac{1}{2}(v(1) - v(0))^2 \\
&+ \frac{1}{2}(v(2) - v(1)).
\end{align*}
\]

History \((0,1)\) is symmetric. We have effort supply

\[
G^A_{(0,1)}(x) = G^B_{(1,0)}(x), \quad G^B_{(0,1)}(x) = G^A_{(1,0)}(x).
\]

The winning probabilities are

\[ p_A(0, 1) = \frac{1}{2}q, \quad p_B(0, 1) = 1 - \frac{1}{2}q. \]
Now we come to the first battle. The common effective prize spread is
\[
v_A(0, 0) = v_B(0, 0)
\]
\[
= \{p_A(1, 0) [p_A(2, 0)v(3) + p_B(2, 0)v(2) - E[\bar{x}_A(2, 0)]]
+ p_B(1, 0) [p_A(1, 1)v(2) + p_B(1, 1)v(1) - E[\bar{x}_A(1, 1)]] - E[\bar{x}_A(1, 0)]\}
- \{p_A(0, 1) [p_A(1, 1)v(2) + p_B(1, 1)v(1) - E[\bar{x}_A(1, 1)]]
+ p_B(0, 1) [p_A(0, 2)v(1) + p_B(0, 2)v(0) - E[\bar{x}_A(0, 2)] - x_A(0, 1)]\}
= (1 - \frac{1}{2}q)(v(2) - v(0)) + (1 - \frac{1}{2}q)(v(1) - v(0)).
\]

Thus the effort supply is
\[
G^A_{(0, 0)} = \frac{x}{v_A(0, 0)} \quad \text{in} \quad [0, v_A(0, 0)],
\]
and the winning probabilities are
\[
p_A(0, 0) = p_B(0, 0) = \frac{1}{2}.
\]

Total effort thus is as follows:
\[
TE_4 = 2E[\bar{x}_A(0, 0)] + E[\bar{x}_A(1, 0)] + E[\bar{x}_B(1, 0)]
+ p_A(1, 0) (E[\bar{x}_A(2, 0)] + E[\bar{x}_A(2, 0)]) + p_B(1, 0) (E[\bar{x}_A(1, 1)] + E[\bar{x}_A(1, 1)])
= v_A(0, 0) + \frac{1}{2}(1 + q)(v(1) - v(0)) + (1 - \frac{q}{2})(v(1) - v(0)) + \frac{q}{2}(v(2) - v(1))
= 2(v(1) - v(0)) + (v(2) - v(1))
= v(2) + v(1) - 2v(0)
= 1 - 2v(0).
\]

\[
\Box
\]

**Proposition 4** When \( r > 2 \), any prize profile in \( \mathcal{V}_4 \) with \( v^*(0) = 0 \), \( v^*(2) \in [\frac{2}{3}, 1] \), \( v^*(1) = 1 - v^*(2) \), and \( v^*(3) = 1 \) is optimal, which induces the highest possible total effort, which equals 1.

**Proof.** By Lemma 8, maximizing the total effort \( TE_4 \) among prize structures in \( \mathcal{V}_4 \) yields the optimal allocations \( v(0) = 0, v(2) \in [\frac{2}{3}, 1], v(1) = 1 - v(2), \) and \( v(3) = 1 \). As a result, \( TE_4^* = 1 \), i.e. the rent is fully dissipated. \( \Box \)

Following similar procedure, we obtain the effort supply as in the following lemma when prize profiles are in \( \mathcal{V}_5 \). To save space, the proof is relegated to the appendix.

**Lemma 9** When \( r > 2 \), the expected aggregate effort is \( TE_5 = 3(v(2) - v(1)) \) for all prize structures in \( \mathcal{V}_5 = \mathcal{V} \cap \{(v(0), v(2)) : v(2) - v(1) \leq v(1) - v(0)\} \).
**Proof.** See appendix.

**Proposition 5** When $r > 2$, prize profile $v(0) = 0, v(1) = \frac{1}{3}, v(2) = \frac{2}{3}, v(3) = 1$ is the unique prize structure in $V_5$ that induces the highest possible total effort, which equals 1.

**Proof.** $TE_5 = 3(v(2) - v(1))$ by Lemma 9. Therefore the designer’s problem is

\[
\begin{align*}
\max & \quad TE_5 \\
\text{s.t.} & \quad (v(1) - v(0)) \geq (v(2) - v(1)), \\
& \quad 0 \leq v(0) \leq \frac{1}{2} \leq v(2) \leq 1, \\
& \quad v(2) + v(0) \leq 1,
\end{align*}
\]

which yields $v(0) = 0, v(1) = \frac{1}{3}, v(2) = \frac{2}{3}$ and $v(3) = 1$. For this prize profile, we have $TE_5^* = 1$. ■

Note that $V = V_4 \cup V_5$ when $r > 2$. Combining Propositions 4 and 5 immediately gives the following theorem, which fully characterizes the optimal prize profile when $r > 2$.

**Theorem 3** When $r > 2$, the family of $v^*(0) = 0, v^*(2) \in \left[\frac{2}{3}, 1\right], v^*(1) = 1 - v^*(2),$ and $v^*(3) = 1$ provides all the optimal effort-maximizing prize profiles, which yields a total effort $TE^* = 1$.

Based on Theorems 1 to 3, we have the following observation regarding the optimal prize allocation rule, which says that proportional division rule is in general not optimal. In other words, in multi-battle contests, the optimal prize allocation rule should in principle give additional award to the grand winner of the whole contest.

**Corollary 1** $\forall r > 0$, we always have the optimal $v(1) \in [0, \frac{1}{3}]$. Moreover, $v(1) < \frac{1}{3}$ unless $r \geq 2$.

## 4 Simultaneous play

In the analysis of Section 3, we have been focusing on the setting with sequential battles. In this section, we study how the temporal structure of the battles affects the optimal prize allocation rule. In particular, we now look at the case of simultaneous play.

**Theorem 4** In simultaneous-play contest, winner-take-all (i.e. $v(3) = v(2) = 1, v(1) = v(0) = 0$) is the unique optimal effort-maximizing prize allocation rule for every $r > 0$.

**Proof.** We focus on equilibrium that the players adopt the same bidding strategy in each battle. Therefore, the other battles are fair random draws when players make decisions in a particular battle. Without losing the generality, we consider batter 1. The common effective prize spread in the battle is as follow:

\[
\begin{align*}
\Delta v = & \quad \frac{1}{4}[v(3) - v(2)] + \frac{1}{2}[v(2) - v(1)] + \frac{1}{4}[v(1) - v(0)] \\
= & \quad \frac{1}{4}[v(3) - v(0)] + \frac{1}{4}[v(2) - v(1)].
\end{align*}
\]
In view of the restrictions on \( v(n) \), whenever the strategy a player uses satisfies the property that a higher prize leads to more effort, it is clear that the optimal prize structure should be as follows

\[
v(3) = v(2) = 1, v(1) = v(0) = 0.
\]

Theorem 5 reveals that the temporal structure of the multi-battle contest plays an important role when the organizer decides on the effort-maximizing prize allocation rule. With simultaneous play, a winner-take-all rule is uniquely optimal; while with sequential play, a winner-take-all is uniquely optimal if and only if the discriminatory power of the contest technology is low.

5 Team contests with pairwise battles

In Sections 3 and 4, we have analyzed multi-battle contests between the same two players. In this section, we analyze the multi-battle contest between two teams with equal number of players. The team contests can be simultaneously or sequentially played with multiple pairwise battles. Each battle is played by different paired players, each from a different team. This environment has been first studied in Fu, Lu and Pan (2015) for equilibrium analysis purpose for a large class of contest technology of homogeneity-of-degree-zero while adopting the winner-take-all prize allocation rule. Häfner (2012) studies equilibrium analysis in a tug-of-war variant of this environment while assuming all-pay-auction technology and winner-take-all prize allocation rule.

We assume the same prize allocation framework as specified in Section 2: \( v(n), n \in \{0, 1, 2, 3\} \) denotes the prize allocated to the winning team. All three players in a team evaluate the prize at \( v(n) \), i.e. the prize is a public good within a team. Alternatively, the players within a same team equally split the prize their team wins. This alternative optimal prize sharing rule does not be affect the optimal prize allocation.

We first note that with simultaneous play, the optimal prize allocation is no different from that for multi-battle contests between two same players. For players in a particular battle, each other battle can be viewed as an independent random draw. It follows that the analysis of Section 4 remains valid: the winner-take-all rule is uniquely optimal.

We thus next focus on the case of sequentially played three-battle team contest with pairwise battles.

5.1 Sequentially play

Based on the insight of Fu, Lu and Pan (2015), each battle has a symmetric prize spread for both players involved. However, depending the magnitude of discriminatory power \( r \), the equilibrium bidding strategy takes different forms. We thus consider two cases. In case 1, \( r \leq 2 \), and in case 2, \( r > 2 \). In case 1, we have in each battle that the equilibrium is in pure strategy that is given in Lemma 1(i); and in case 2, we have in each battle that the equilibrium is in mixed strategy that is given in Lemma 1(iii).
The procedure for deriving the subgame perfect equilibrium is standard, and thus is relegated to the appendix to save space.

5.1.1 Case 1: \( r \leq 2 \)

**Lemma 10** When \( r \leq 2 \), the expected total effort is \( T E_2 = \frac{3r}{4} (v(2) - v(0)) \) for team contest.

**Proof.** See appendix. ■

**Theorem 5** When \( r \leq 2 \), winner-take-all (i.e. \( v(3) = v(2) = 1, v(0) = v(1) = 0 \)) uniquely maximizes the total effort supply.

**Proof.** Maximizing the total effort \( T E_2 = \frac{3r}{4} (v(2) - v(0)) \) given by Lemma 10 under required constraints for \( v(n) \) yields \( v(2) = v(3) = 1 \) and \( v(0) = v(1) = 0 \) in this case. ■

5.1.2 Case 2: \( r > 2 \)

**Lemma 11** When \( r > 2 \), the expected total effort is \( T E_2 = \frac{3}{2} (v(2) - v(0)) \) for team contest.

**Proof.** See appendix. ■

**Theorem 6** When \( r > 2 \), winner-take-all (i.e. \( v(3) = v(2) = 1, v(0) = v(1) = 0 \)) uniquely maximizes the total effort supply.

**Proof.** Maximizing the total effort \( T E_2 = \frac{3}{2} (v(2) - v(0)) \) given by Lemma 11 under required constraints for \( v(n) \) yields \( v(2) = v(3) = 1 \) and \( v(0) = v(1) = 0 \) in this case. ■

We thus see that optimal prize allocation rule can be different across contests played between the same two players and two teams of multiple players. When the discriminatory power \( r \) is low (i.e. \( r \leq \bar{r} \)), both two forms of contests require winner-take-all as the optimal prize allocation rule. When discriminatory power \( r \) is in the middle range (i.e. \( r \in (\bar{r}, 2) \)), team contest still requires a winner-take-all prize allocation for effort maximization while the contest between two same individual players requires awarding positive prizes to a player winning even a single battle. When discriminatory power \( r \) is in the high range (i.e. \( r \geq 2 \)), a winner-take-all prize allocation rule still uniquely maximizes the total effort supply in team contest while a wide span of prize allocation rules ranging from winner-take-all to proportional division rule is optimal in a contest between two same individual players.

6 Conclusion

In this paper, we completely characterize the optimal contingent prize allocation rule in simultaneous-play or sequential-play three-battle contests between two same players, or team contests with three pairwise battles between two teams each with three players. The prize a player wins depends on the
count of his wins. The full spectrum of contest technologies of Tolluck family are accommodated in our analysis.

We find that a winner-take-all rule (a party wins all prize money if he wins more than two battles) induces the maximal total expected effort as long as the contest is between teams or it is simultaneously played. When the battles are sequential and between two same individual players, the discriminatory power of the contest technology plays a crucial role in determining the optimal prize allocation rule. Specifically, when discriminatory power is in the low range, a winner-take-all allocation remains optimal. When the discriminatory power is in the intermediate range, the reward strictly increase with the count of wins. Moreover, the reward to a single strictly increase with the discriminatory power but never goes beyond one-third of the total prize. When the discriminatory power falls in the high range, a wide range of allocation rules in between winner-take-all and proportional division induce the maximal total expected effort. These findings illustrate the differences in strategic environments of multi-battle contests between two individual and two teams, and between simultaneous play and sequential play.

Our findings provide rationals from a perspective of effort elicitation for the commonly adopted winner-take-all prize allocation in multi-battle contests, as well as the practice of setting intermediate prizes in many occasions. Nevertheless, our analysis sets a upper bound on the maximal prize for the player with a single win. Its optimal level should never go beyond one-third of the total prize budget. In other words, the optimal prize allocation rule should in principle give additional award to the grand winner of the whole contest.

We would like to emphasize that our findings apply no matter the prize budget is divisible. When it is indivisible, the contest organizer can adopt allocation probabilities as the design instrument, which delivers an identical set of results. Moreover, with indivisible prize budget, intermediate prize directly linked to each battle is infeasible, which makes the current framework of prize allocation rules that rely on the overall battle outcomes more appealing. Nevertheless, with divisible prize budget, an alternative framework of prize allocation rule can formulated as follows. The designer assigns each battle an intermediate prize \( v \), and a grand prize \( V \) to the winner of the whole contest, subject to the usual prize budget constraint. It would be interesting to investigate the optimal prize allocation rule in this framework. We expect similar if not equivalent insights would prevail.\(^{10}\)

Another interesting extension is to allow the prize allocation fully contingent on the ordered outcome vector of the battles. In this analytical framework, there will be four choice variables for a sequential-play three-battle contest instead of two in the current analysis, which greatly increases the burden of computation.

In team contest environments, an interesting issue on internal prize allocation arises if the coaches try to maximize their teams' winning chances by choosing internal prize allocation rule that splits a winner-take-all prize the team wins. In particular, one can investigate whether equally sharing rule among all players or among winners of component battles can be an equilibrium.

\(^{10}\)One can verify that this alternative prize structure is covered by our analytical framework by setting \( v(0) = 0, v(1) = v, v(2) = 2v + V \) and \( v(3) = 3v + V \).
Appendix

Proof of Lemma 4

**Proof.** Note that $v(1) - v(0) = \frac{w_A + w_B}{2} - \frac{w_A - w_B}{r}$, $v(2) - v(1) = \frac{w_A + w_B}{2} + \frac{w_A - w_B}{r}$ and $\frac{d}{dv(0)} \frac{w_B}{w_A} = \frac{w_B dv(0)}{(w_A)^2}$. Thus

$$\frac{d}{dv(0)} \left( \frac{TE_1}{r} \right) = (1 - \frac{r^2}{4})(v(2) - v(1)) \frac{d}{dv(0)} \left( \frac{w_A}{w_B} \right) \left[ (1 + \frac{r^2}{4})(v(1) - v(0)) + \\
(1 - \frac{r^2}{4})(v(2) - v(1)) \right] + \frac{w_A}{w_B + w_A} \frac{w_B}{w_B + w_A} \frac{(1 + \frac{r^2}{4})(v(1) - v(0))}{(1 + \frac{r^2}{4})(v(2) - v(1))} \left( \frac{w_B}{w_B + w_A} \right) \left( \frac{w_B}{w_B + w_A} \right)
$$

where $\eta = \frac{w_B}{w_A}$. Note that $\frac{1 - \frac{r}{2}}{\frac{1}{3} + \frac{r}{4}} \leq \eta \leq \frac{1}{2} + \frac{r}{4}$.

We will prove $\frac{d}{dv(0)} \frac{TE_1}{r} < 0$ in the following two cases.

**Case 1:** $w_A \geq w_B$. In this case, we see $1 \geq \eta$ and $\eta \geq \frac{1 - \frac{r}{2}}{\frac{1}{3} + \frac{r}{4}} \geq \frac{1 - \frac{r}{2}}{\frac{1}{3} + \frac{r}{4}} \geq \frac{3 - 1}{\frac{1}{3} + \frac{r}{4}} \geq \frac{3 - 1}{\frac{1}{3} + \frac{r}{4}}$.

Notice that $\xi = [(1 - \frac{r}{2})(-r - 1 + (r + 1)\eta^r) + (1 + \frac{r}{2})(-r + 1 + (r + 1)\eta^r)]^r$ increases in $\eta \in [\frac{1}{4}, 1]$ for $r \leq \frac{1}{2} \in (1, 1.2)$. \(\frac{d\xi}{d\eta} = (1 - \frac{r}{2})(r - 1)\eta^r - 1 + (1 + \frac{r}{2})(-r + 1 + (r + 1)\eta^r) \eta^r + (1 + \frac{r}{2})\eta(r + 1)\eta^r - 1 = (1 + \frac{1}{2})r(1 - \eta^r) + (1 + \frac{1}{2})(1 + r)\eta(r + 1)

When $r \leq 1$, we have $(1 - \frac{r}{2})(r - 1)\eta^r - 1 = (1 - \frac{1}{2})(1 - \eta^r) \eta(r + 1) \geq \frac{1}{2}$ and $(1 + \frac{1}{2})(1 + r)\eta(r + 1) \geq \eta \geq \frac{1}{2}$. Thus $\eta \geq 0$. Clearly, $\eta \geq 0$ when $r \leq 1$. Therefore $\frac{d\xi}{d\eta} \geq 0$.

When $r \in [\frac{1}{2}, \frac{1}{2}]$, we have $\xi \geq -1.6 \cdot 0.2 = -0.32$. $\eta \geq (\frac{1}{4})^{0.23} \geq (\frac{1}{4})^{0.53} = \frac{3}{4} > 0.32$. Thus $\frac{d\xi}{d\eta} \geq 0$.

Therefore, we have

$$\frac{(1 + \eta^r)\frac{d}{dv(0)} \frac{TE_1}{r}}{r} \leq \frac{1}{4}\eta^r - 1[(r + 2) + (r - 2)\eta][(1 - \frac{r}{2})(-2) + (1 + \frac{r}{2})(2)] - (1 + \eta^r)[1 + (2 + \frac{r^2}{4})\eta^r]$$

It suffices to show that $\frac{1}{2} < \frac{(1 + \eta^r)[1 + (2 + \frac{r^2}{4})\eta^r]}{\eta^r - 1[(r + 2) + (r - 2)\eta]} = \frac{\eta^r - 1 + \eta^r}{\eta^r - 1[(r + 2) + (r - 2)\eta]}$ for all $\frac{1}{4} \leq \eta \leq 1$.

Note that $(\eta^{1 - r} + \eta)[1 + (2 + \frac{r^2}{4})\eta^r]$ is increasing in $\eta$ for $\frac{1}{4} \leq \eta \leq 1$ and $0 \leq r \leq \frac{1}{2}$. The
monotonicity of $(\eta^{1-r} + \eta)[1 + (2 + \frac{r^2}{4})\eta^r]$ is clear when $r \leq 1$. When $r \in [1, \tau]$, the first order derivative of $(\eta^{1-r} + \eta)[1 + (2 + \frac{r^2}{4})\eta^r]$ is $(1 - r)\eta^{-r} + 1 + (2 + \frac{r^2}{4})(1 + (r + 1)\eta^r) = 1 + (2 + \frac{r^2}{4}) - \frac{1 - r}{\eta^r} + (2 + \frac{r^2}{4})(r + 1)\eta^r \geq 3\frac{1}{4} - \frac{0.2}{(\frac{2}{3})^{2.4}} + (2 + \frac{1}{4})(2 + 1)(\frac{1}{4})^{1+2}> 0$. Moreover, $\frac{5}{2} < ((\frac{1}{4})^{1-r} + \frac{1}{4})\frac{[1+(2+\frac{r^2}{4})(\frac{1}{4})^r]}{[r+(2)+(r-2)(\frac{1}{4})]}$ holds for $0 \leq r \leq \tau$. $\frac{5}{2} < ((\frac{1}{4})^{1-r} + \frac{1}{4})\frac{[1+(2+\frac{r^2}{4})(\frac{1}{4})^r]}{[r+(2)+(r-2)(\frac{1}{4})]}$ is equivalent to $\frac{5}{2} < \frac{(1+4^{r})(1+(2+\frac{r^2}{4})(\frac{1}{4})^r)}{[5r+6]}$, which is equivalent to $1 + 4r + \frac{r^2}{4} + \frac{r^2}{4} + 1 - \frac{5}{2}r^2 - 3r > 0$. Let $\zeta = 1 + 4r + \frac{r^2}{4} + \frac{r^2}{4} + 1 - \frac{5}{2}r^2 - 3r$. We want to show $\zeta(r) > 0$ for $0 \leq r \leq \tau$. $\zeta(r) = 4^r\ln 4 - \frac{2}{(4^r)^2}\ln 4 + \frac{r}{2^r} - \frac{r^2}{2^r}\ln 4 - 5r - 3$. Consider the three components $4^r\ln 4 - 5r - 2$, $\frac{r}{2^r} - 1$ and $-\frac{r^2}{2^r}\ln 4 - \frac{r^2}{2^r}\ln 4$. Clearly the last two components are negative. The first component is maximized when $r = \frac{\ln 5 - 2\ln(\ln 4)}{\ln 4} \approx 0.3$. Thus the first component is also negative for all relevant $r \leq 1.2$. We thus know that $\zeta$ is minimized when $r = 1.2$. Note $\zeta(1.2) = 1 + 5.28 + \frac{2}{5.28} + \frac{1.44}{5.28} + 1 - \frac{5}{2} \cdot 1.44 - 3 \cdot 1.2 > 0$. Therefore, $\zeta(r) > 0$ for $0 \leq r \leq \tau$.

We thus have $\frac{d}{d\theta(0)}TE_1 < 0$ in this case.

**Case 2:** $w_A \leq w_B$. In this case, we have $1 \leq \eta$ and $\eta \leq \frac{\frac{\eta + \frac{r^2}{4}}{\eta + \frac{r^2}{4}}}{\frac{\eta + \frac{r^2}{4}}{\eta + \frac{r^2}{4}}} \leq \frac{\frac{\eta + \frac{r^2}{4}}{\eta + \frac{r^2}{4}}}{\frac{\eta + \frac{r^2}{4}}{\eta + \frac{r^2}{4}}} \leq 4$.

Note that $[(1 - \frac{5}{2})(-r - 1 + (r - 1)\eta^r) + (1 + \frac{5}{2})\eta(-r + 1 + (r + 1)\eta^r)] \geq [(1 - \frac{5}{2})-r - 1 + (r - 1)\eta^r] + (1 + \frac{5}{2})(-r + 1 + (r + 1)\eta^r)] = r(3\eta^r - 1) > 0$. When $[(r + 2) - (r - 2)\eta] \leq 0$, it is clear $\frac{d}{d\theta(0)}TE_1 < 0$ holds. If $[(r + 2) - (r - 2)\eta] > 0$, it suffices to show that $[(1 - \frac{5}{2})(-r - 1 + (r - 1)\eta^r) + (1 + \frac{5}{2})\eta(-r + 1 + (r + 1)\eta^r)] < \frac{2}{r}(\eta^{-r} + \eta)(1 + (2 + \frac{r^2}{4})\eta^r)$ for all $1 \leq \eta \leq 4$ and $0 \leq r \leq \tau$, that is, $(1 + \frac{5}{2})\eta(-r + 1 + (r + 1)\eta^r) < \frac{2}{\eta}(\eta^{-r} + \eta)(1 + (2 + \frac{r^2}{4})\eta^r) + (1 - \frac{5}{2})(r + 1 + (1 - r)\eta^r)$.

Note $(1 - \frac{5}{2})(r + 1 + (1 - r)\eta^r) \geq 0$ for all $1 \leq \eta \leq 4$ and $r \leq 1.2$. This is clear when $r \leq 1$. When $r \in [1,1.2]$, we have $r + 1 + (1 - r)\eta^r \geq 2 - 0.2 \cdot 4^{1.2} > 0$. Thus, we just need to show that $(1 + \frac{5}{2})\eta(-r + 1 + (r + 1)\eta^r) \leq \frac{2}{r}(\eta^{-r} + \eta)(1 + (2 + \frac{r^2}{4})\eta^r)$, which is equivalent to $(1 + \frac{5}{2})(-r + 1 + (r + 1)\eta^r) < \frac{2}{r}(\eta^{-r} + \eta)(1 + (2 + \frac{r^2}{4})\eta^r)$, i.e. $1 - \frac{5}{2} - \frac{r^2}{4} + (1 + \frac{3\eta}{2} + \frac{r^2}{2})\eta^r < \frac{2}{r}[(2 + \frac{r^2}{4})\eta^r + 3 + \frac{r^2}{4}] + \frac{5}{2}\eta^{-r}$. This holds because $\frac{5}{2}r + r + \frac{r^2}{2} - 1 + (\frac{4}{5} - r - \frac{r^2}{4})\eta^r > 0$, which is implied by $(\frac{4}{5} - r - \frac{r^2}{4}) = \frac{1}{5}(4 - r^2 - \frac{r^2}{4} - r) > 0$ and $\frac{5}{2}r + r + \frac{r^2}{2} - 1 = \frac{1}{r}(6 + r^2 + \frac{r^2}{2} - r) > 0$ when $1 \leq \eta \leq 4$ and $0 \leq r \leq \tau \leq 1.2$.

Hence, $\frac{d}{d\theta(0)}TE_1 < 0$ holds for $0 \leq r \leq \tau \leq 1.2$ in this case.

Combining the two cases, we conclude that $\frac{d}{d\theta(0)}TE_1 < 0$. ■
Proof of Property 3

We prove that \( D(\eta, r) \) is increasing in \( \eta \) when \( \eta \geq \frac{1 - \frac{r}{2}}{\frac{3r}{2} + \frac{1}{4}} \) for each \( r \in (0, \pi] \subset (0, 1.2] \). Note that we \( r \in (0, \pi] \subset (0, 1.2], \) and \( \eta \geq \frac{1 - \frac{r}{2}}{\frac{3r}{2} + \frac{1}{4}} > \frac{2 - \frac{r}{2}}{\frac{3r}{2} + \frac{1}{4}} \mid_{r=1.2} = \frac{1}{4} \). The derivative of \( D(\eta, r) \) wrt. \( \eta \) is given by

\[
\begin{align*}
(\frac{3r}{2} + 1)[(1 - \frac{r}{2})(-r - 1 + (r - 1)(r + 1)\eta^r) + (1 + \frac{r}{2})(-2r + 2 + (r + 1)(r + 2)\eta^r)\eta] \\
+ (\frac{3r}{2} - 1)[(1 - \frac{r}{2})r\eta^{r-1} + (1 + \frac{r}{2})(-r + 1 + (r + 1)^2\eta^r)] + (2\eta(1 + \eta^r)(2 - \frac{3r^2}{4} + \eta^r))' \\
\geq (\frac{3r}{2} + 1)[(1 - \frac{r}{2})(-r - 1 + (r - 1)(r + 1)\eta^r) + (1 - \frac{r}{2})(-2r + 2 + (r + 1)(r + 2)\eta^r)] \\
+ (\frac{3r}{2} - 1)[(1 - \frac{r}{2})(-r - 1)(r - 1)\eta^{r-1} + (1 + \frac{r}{2})(-r + 1 + (r + 1)^2\eta^r)] + (2\eta(1 + \eta^r)(2 - \frac{3r^2}{4} + \eta^r))' \\
= [-4r^2 + \frac{3}{2}r^3 + (r - 3r^2 + \frac{11}{4}r^3 - \frac{3}{4}r^4)\eta^{r-1} + (3r + 6r^2 + \frac{9}{4}r^3 - \frac{3}{4}r^4)\eta^r] \\
+ [4 - \frac{3r^2}{2} + (6 + 6r - \frac{3r^2}{2} - \frac{3r^3}{2})\eta^r + (2 + 4r)\eta^{2r}] \\
= (4 - \frac{11r^2}{2} + \frac{3r^3}{2}) + (r - 3r^2 + \frac{11}{4}r^3 - \frac{3}{4}r^4)\eta^{r-1} + (6 + 9r + \frac{9}{2}r^2 + \frac{3}{4}r^3 - \frac{3}{4}r^4)\eta^r + (2 + 4r)\eta^{2r} \\
> (4 - \frac{11r^2}{2} + \frac{3r^3}{2}) + (r - 3r^2 + \frac{11}{4}r^3 - \frac{3}{4}r^4)\eta^{r-1} + (6 + 9r + \frac{9}{2}r^2 + \frac{3}{4}r^3 - \frac{3}{4}r^4)(\frac{1}{4}) \eta^{-1} \\
= (4 - \frac{11r^2}{2} + \frac{3r^3}{2}) + (3 + \frac{13}{4}r - \frac{15}{16}r^2 + \frac{47}{16}r^3 - \frac{15}{16}r^4) \eta^{-1} \\
\geq (4 - \frac{11r^2}{2} + \frac{3r^3}{2}) + (\frac{3}{2} + \frac{13}{4}r - \frac{15}{8}r^2 + \frac{47}{16}r^3 - \frac{15}{16}r^4) \frac{3}{4} \\
= \frac{41}{16} + \frac{39}{16}r - \frac{221}{32}r^2 + \frac{237}{64}r^3 - \frac{45}{64}r^4 \\
\geq 0.
\end{align*}
\]  

Proof of Theorem 1

**Proof.** By the Lemmas 2 and 4, to find the optimal prize allocation for a given \( r \in (0, \pi] \), we only need to identify the optimal \( v(2) \in [\frac{1}{2}, 1] \) by maximizing the \( TE_1(v(0) = 0, v(2)) \). Differentiating \( (TE_1(v(0)=0,v(2)) \) with respect to \( v(2) \) gives

\[
\begin{align*}
\frac{d}{dv(2)}(TE_1(v(0)=0,v(2))) \\
= (1 - 2\frac{w_A^r}{w_A^r + w_B^r})\frac{d}{dv(2)}\left(\frac{w_A^r}{w_A^r + w_B^r}\right)\left([1 + \frac{r^2}{4})v(1) + (1 - \frac{r^2}{4})(v(2) - v(1))\right] \\
+ \frac{w_A^r}{w_A^r + w_B^r}\frac{w_B^r}{w_A^r + w_B^r}\left([1 + \frac{r^2}{4})(-1) + (1 - \frac{r^2}{4})2\right) + 1 + \frac{d}{dv(2)}\left(\frac{w_A^r}{w_A^r + w_B^r}\right)v(1) + \frac{w_A^r}{w_A^r + w_B^r}(-1)
\end{align*}
\]

\[30\]
and because I

\[ w_A + w_B (1 - 3r^2 / 4) + 1 + \frac{d}{dv(2)}(w_A + w_B) - \frac{d}{dv(2)}(w_A + w_B)^2 \]

Therefore, we have

\[ w_A(0) = 0 \]

Then, we have

\[ I = \frac{w_A^r - w_B^r}{w_A + w_B} (1 + (r^2 / 4)(v(1) + (1 - r^2 / 4)(v(2) - v(1)))\frac{d}{dv(2)}(\frac{w_A^r}{w_A + w_B}) \geq 0 \]

because \[ \frac{d}{dv(2)}(\frac{w_A^r}{w_A + w_B}) \geq 0 \], II \equiv (w_A + w_B)(1 - 3r^2 / 4) \geq -1 / 2, III \equiv 1, IV \equiv \frac{d}{dv(2)}(w_A + w_B)(v(1) \geq 0 \), and V \equiv \frac{w_A^r}{w_A + w_B} \leq 1 / 2 because \[ w_B > w_A \]. Thus, \[ \frac{d}{dv(2)}(\frac{TE_1(v(0))}{v(1)}) > 0 \] when \[ v(2) \in [0, \frac{2}{3}] \]. As a result, \[ TE_1(v(0) = 0, v(2) = \frac{2}{3}) \]

We only need to consider \[ \frac{d}{dv(2)}(\frac{TE_1(v(0) = 0, v(2))}{v(2)}) \] for any \( v(2) \in [0, \frac{2}{3}] \).

Similar to \[ \frac{d}{dv(2)}(\frac{w_A^r}{w_A + w_B}) \] as in the the proof of Lemma 4, one can verify that \[ \frac{d}{dv(2)}(\frac{w_A^r}{w_A + w_B}) = -r \frac{(w_A^r - w_B^r)}{[1 + (w_A + w_B)^2] \frac{1}{w_A^r} \frac{(w_A + w_B)}{2} - \frac{3}{4}(w_A + w_B)] \]. Recall \[ v(1) - v(0) = w_A + w_B - \frac{w_A - w_B}{v(2)} \] and \[ v(2) - v(1) = \frac{w_A - w_B}{v(2)} \]. Thus \[ \frac{(1 + r^2 / 4)(v(1) - v(0)) + (1 - r^2 / 4)(v(2) - v(1)))}{w_A^r} \]

Using notation \( \eta \), we have

\[ \eta = \frac{r^2 - 1}{1 + \eta^2}[\frac{3}{2}r - 1 + (3r + 1)\eta][(1 - \frac{r}{2}) + (1 + \frac{r}{2})\eta] \]

and II + III + V = \[ \frac{\eta^r - 1}{\eta^2 [2 - \gamma^2 + \eta^r]} \]

After substitution and rearrangement, we have

\[ \frac{d}{dv(2)}(\frac{TE_1(v(0) = 0, v(2))}{v(2)}) \]

Therefore, \[ sign(\frac{d}{dv(2)}(\frac{TE_1(v(0) = 0, v(2))}{v(2)}) = sign(D(\eta, r)) \]

when \[ r \leq 2 \]. Thus \[ \frac{1 + r^2 / 4}{2} \geq \eta(0) = 0, v(2), r \] \[ \frac{1 - 2}{2} \].
Note $D(\eta(v(0) = 0, v(2), r), r) > 0, \forall v(2) \in [\frac{1}{2}, 1], r \in (0, \bar{r}]$ by Properties 2 and 3. We thus have the optimal $v(2) = 1$ when $r \in (0, \bar{r}]$.

By the Property 2 of $D(\frac{1}{2} - \frac{r}{2}, r)$, we know $D(\eta(v(0) = 0, v(2) = 1, r), r) = D(\frac{1}{2} - \frac{r}{2}, r) < 0, \forall r \in (\bar{r}, \bar{r}]$. Moreover, $D(\eta(v(0) = 0, v(2) = \frac{2}{3}, r), r) = 3r^2 + 12 > 0$. By Property 3, let $v^*(2) \in (\frac{2}{3}, 1)$ to be the unique solution of $D(\eta(v(0) = 0, v(2), r), r) = 0, \forall r \in (\bar{r}, \bar{r}]$. We thus have the optimal $v(2) = v^*(2)$ when $r \in (\bar{r}, \bar{r}]$. 

**Proof of Lemma 5**

**Proof.** By Lemma 3, we have

\[
TE_1 = \frac{rw_Aw_B}{w_A + w_B^2}[(1 + \frac{r^2}{4})(v(1) - v(0)) + (1 - \frac{r^2}{4})(v(2) - v(1))] + \frac{r}{2}(v(2) - v(1)) + r(v(1) - v(0))\frac{w_A}{w_A + w_B^2},
\]

for prize structures in $\mathcal{V}_0 \cup \mathcal{V}_1$ when $r \in (\bar{r}, 2]$.

Recall $\bar{r} \in (1, 1.2)$. We first look at the case where $r \in (1.3, 2)$. For any given prize structure \{v(0), v(1), v(2), v(3)\} $\in \mathcal{V}_0$, recall that $w_B \geq w_A$, which is equivalent to $v(1) - v(0) \geq v(2) - v(1)$. This inequality means $v(2) \in [\frac{1}{2}, \frac{2-v(0)}{3}]$. Construct a new prize structure \{\(\tilde{v}(0), \tilde{v}(1), \tilde{v}(2), \tilde{v}(3)\}\} $\in \mathcal{V}_1$ such that $\tilde{v}(2) - \tilde{v}(1) = v(1) - v(0)$ and $\tilde{v}(1) - \tilde{v}(0) = v(2) - v(1)$. Specifically, let $\tilde{v}(2) = \frac{v(1) - v(0) + 1}{2}, \tilde{v}(1) = 1 - \tilde{v}(2), \tilde{v}(0) = \tilde{v}(1) - (v(2) - v(1))$ and $\tilde{v}(3) = 1 - \tilde{v}(0)$. One can verify the new structure is eligible, and we have $\tilde{w}_A = (\frac{1}{2} - \frac{r}{4})(\tilde{v}(1) - \tilde{v}(0)) + (\frac{1}{2} + \frac{r}{4})(\tilde{v}(2) - \tilde{v}(1)) = (\frac{1}{2} - \frac{r}{4})(v(2) - v(1)) + (\frac{1}{2} + \frac{r}{4})(v(1) - v(0)) = w_B$ and $\tilde{w}_B = w_A$. The corresponding total effort under the new prize structure is

\[
\bar{TE}_1 = \frac{rw_Aw_B}{w_A + w_B^2}[(1 + \frac{r^2}{4})(\tilde{v}(1) - \tilde{v}(0)) + (1 - \frac{r^2}{4})(\tilde{v}(2) - \tilde{v}(1))] + \frac{r}{2}(\tilde{v}(2) - \tilde{v}(1)) + r(\tilde{v}(1) - \tilde{v}(0))\frac{w_A}{w_A + w_B^2}
\]

\[
= \frac{rw_Aw_B}{w_A + w_B^2}[(1 + \frac{r^2}{4})(v(2) - v(1)) + (1 - \frac{r^2}{4})(v(1) - v(0))] + \frac{r}{2}(v(1) - v(0)) + r(v(2) - v(1))\frac{w_B}{w_A + w_B^2}.
\]

Recall $v(1) - v(0) = \frac{w_A + w_B}{2} - \frac{w_A - w_B}{r}v(2) - v(1) = \frac{w_A + w_B}{2} + \frac{w_A - w_B}{r}$. We have that for $w_B > w_A$.
and \( r \in [1.25, 2] \),

\[
\widetilde{TE}_1 - TE_1 = \frac{r w_A w_B}{[w_A + w_B]^2} \left[ (-\frac{r^2}{2})(v(1) - v(0)) + \frac{r^2}{2}(v(2) - v(1)) + \frac{r}{2}[(v(1) - v(0)) - (v(2) - v(1))] \right.
\]

\[
+ r [(v(2) - v(1)) \frac{w_B}{w_A + w_B} - (v(1) - v(0)) \frac{w_A}{w_A + w_B}] \]

\[
\geq \frac{r}{2} \left[ \frac{w_B^2}{w_A + w_B} \left( (1 + \frac{2}{r})w_A + (1 - \frac{2}{r})w_B \right) \right. - \left. \frac{w_A^2}{2} \left( (1 - \frac{2}{r})w_A + (1 + \frac{2}{r})w_B \right) \right]
\]

\[
\geq \frac{r}{2} \left[ \left( (1 + \frac{2}{r})w_A w_B (w^{-1} - w_{-1}) \right) + (1 - \frac{2}{r})w_{-1} - w_{A+1} \right]
\]

\[
= \frac{r}{2} \frac{w_A + w_B}{w_A + w_B} [(2 + r)(\frac{w_B}{w_A})^r - 4 \frac{w_B}{w_A} + (2 - r)]
\]

\[
> 0.
\]

When \( \frac{w_B}{w_A} \geq 1 \), \( (2 + r)(\frac{w_B}{w_A})^r - 4 \frac{w_B}{w_A} + (2 - r) \) strictly increases with \( \frac{w_B}{w_A} \) when \( r \geq 1.25 \). When \( \frac{w_B}{w_A} = 1 \), \( (2 + r)(\frac{w_B}{w_A})^r - 4 \frac{w_B}{w_A} + (2 - r) = 0 \).

We now look at the case where \( r \in (\pi, 1.25] \). For this case, we first show \( TE_1 \) decreases with \( v(0) \) when \( \frac{w_B}{w_A} \geq 1 \); second we show that \( TE_1(v(0) = 0, v(2)) \) where \( v(2) \in [\frac{1}{2}, \frac{3}{2}] \) is strictly dominated by \( TE_1(v(0) = 0, v(2) = \frac{3}{2}) \).

In a first step, we show \( TE_1 \) decreases with \( v(0) \) when \( \frac{w_B}{w_A} \geq 1 \) and \( r \in (\pi, 1.25] \). This can be achieved by following the same procedure as in the case 1 in the proof of Lemma 4. To save space, please refer to the proof of Lemma 4. Note we have \( \eta = \frac{w_B}{w_A} \in [1, \frac{3}{2}] \). Recall in the proof of Lemma 4 \( \varphi_2 = (1 + \frac{\pi}{2})(1 - r) \) and \( \varphi_1 = \eta^{-1}[(1 - \frac{\pi}{2})(r - 1) + (1 + \frac{\pi}{2})(1 + r)] \). When \( r \in [\pi, 1.25] \), we have \( \varphi_2 \geq -1.625 \cdot 0.25 = -0.40625 \). \( \varphi_1 \geq \frac{3}{13} \cdot 0.25[(1 - \frac{1.25}{2})(1.2 - 1)1.2 + (1 + \frac{1.25}{2})(1.2 + 1)] \frac{3}{13} 2.2 = 1.3204 > 0.40625. \) Thus \( \frac{d}{d\eta} \geq 0 \).

Therefore, we have

\[
(1 + \eta)(3^2) d \frac{TE_1}{r} \leq \frac{r}{2} \eta^{-1}[(r + 2) - (r + 2) \eta - (1 + \eta)[1 + (2 + \frac{r^2}{4}) \eta]]. \tag{6}
\]

It suffices to show that \( \frac{r}{2} \leq \frac{1}{\eta^{-1}[(r + 2) - (r + 2) \eta]} \leq \frac{1}{[r + 2] + (r - 2) \eta]} \) for all \( \frac{3}{13} \leq \eta \leq 1 \) and \( 1.25 \geq r \geq \pi \). We show that \( (1 - r) \eta^{-1} + \eta[(1 - \frac{r}{2}) + (1 + \frac{r}{2}) \eta] \) is increasing in \( \eta \) for \( \frac{3}{13} \leq \eta \leq 1 \) and \( 1.25 \geq r \geq \pi \).

The first order derivative of \( (1 - r) \eta^{-1} + \eta[(1 + \frac{r}{2}) + (1 + \frac{r}{2}) \eta] \) is \( (1 - r)\eta^{-2} + 1 + (2 + \frac{r^2}{4})(1 + (r + 1) \eta) \) \( = 1 + (2 + \frac{r^2}{4}) - 1 + (2 + \frac{r^2}{4})(r + 1) \)

\[
\geq 3 \frac{1}{2} + 0.25 \pi + (2 + \frac{1}{4})(2 + 1)(\frac{3}{13})^{1.25} = 2.766 > 0. \]

Moreover, \( \frac{r}{2} < ((\frac{1}{4})^{1-r} + \frac{1}{4})^{1 + (2 + \frac{r^2}{4})} \) holds for \( 1.25 \geq r \geq \pi \). \( \frac{r}{2} < (\frac{1}{4})^{1-r} + \frac{1}{4} \) is equivalent to \( \frac{r}{2} \leq \frac{(1 + \eta)(1 + (2 + \frac{r^2}{4}) \eta)}{[r + 2] + (r - 2) \eta} \), which
is equivalent to $1 + 4r + \frac{2}{4r^2} + 1 - \frac{5}{2} r^2 - 3r > 0$. Let $\zeta = 1 + 4r + \frac{2}{4r^2} + 1 - \frac{5}{2} r^2 - 3r$. We want to show $\zeta(r) > 0$ for $1.25 \geq r \geq \overline{r}$. $\zeta'(r) = 4r \ln 4 - \frac{2}{4r^2} \ln 4 + \frac{r}{4r^2} - r^2 \ln 4 - 5r - 3$. Consider the three components $4r \ln 4 - 5r - 2, \frac{1}{4r^2} - 1$ and $- \frac{1}{4r^2} \ln 4 - \frac{r^2}{4r^2} \ln 4$. Clearly the last two components are negative. The first component is maximized when $r = \frac{\ln 5 - 2\ln(\ln 4)}{\ln 4} \approx 0.3$. Thus the first component is also negative for all relevant $r \leq 1.25$. We thus know that $\zeta$ is minimized when $r = 1.25$. Note $\zeta(1.25) = 1 + 4^{1.25} + \frac{2}{4^{1.25}} + 1 - \frac{5}{2} \cdot 1.25^2 - 3 \cdot 1.25 = 0.4178 > 0$. Therefore, $\zeta(r) > 0$ for $1.25 \geq r \geq \overline{r}$. We thus have $\frac{d}{dv(0)} TE_1 < 0$ in this case, which means that we can focus on price structures with $v(0) = 0$ to search for the optimum in $\mathcal{V}_0$.

In a second step, we show that $TE_1(v(0) = 0, v(2))$ where $v(2) \in [\frac{1}{2}, \frac{2}{3}]$ (i.e. prizes are in $\mathcal{V}_0$) is strictly dominated by $TE_1(v(0) = 0, v(2) = \frac{2}{3})$. The proof is identical to the part of the proof of Theorem 1 to show $\frac{d}{dv(0)}(TE_1(v(0)=0,v(2))) > 0$ when $w_B > w_A$. Note that price structure with $v(0) = 0, v(2) = \frac{2}{3}$ is a common element of $\mathcal{V}_0 \cap \mathcal{V}_1$. ■

Proof of Lemma 6

Proof. In $\mathcal{V}_1$, we have $w_A \geq w_B$ and $r \leq 1 + (\frac{w_B}{w_A})^r$, which imply that $v(1) \leq \frac{1+v(0)}{3}$, $v(1) \geq -\frac{1}{2} + \frac{3}{4} + \frac{(1 + \frac{3}{4}) (r-1)^{\frac{1}{r}}} {2 + \frac{3}{4} + \frac{(1 + \frac{3}{4}) (r-1)^{\frac{1}{r}}}}$, and $\eta := \frac{w_B}{w_A} \in [(r-1)^{\frac{1}{r}}, 1]$. Note that $r > \overline{r}$ gives $-\frac{1}{2} + \frac{3}{4} + \frac{(1 + \frac{3}{4}) (r-1)^{\frac{1}{r}}} {2 + \frac{3}{4} + \frac{(1 + \frac{3}{4}) (r-1)^{\frac{1}{r}}}} > 0$ because of the definition of $\overline{r}$, and thus $-\frac{1}{2} + \frac{3}{4} + \frac{(1 + \frac{3}{4}) (r-1)^{\frac{1}{r}}} {2 + \frac{3}{4} + \frac{(1 + \frac{3}{4}) (r-1)^{\frac{1}{r}}}} > 0$. One can easily verify that prize profile with $v(0) = 0, v(1) = -\frac{1}{2} + \frac{3}{4} + \frac{(1 + \frac{3}{4}) (r-1)^{\frac{1}{r}}} {2 + \frac{3}{4} + \frac{(1 + \frac{3}{4}) (r-1)^{\frac{1}{r}}}$, and $\eta = (r-1)^{\frac{1}{r}}$ belongs to $\mathcal{V}_1$.

To facilitate computation, we first come up a different way of writing the $TE_1$.

$$TE_1 = \frac{r w_A^r w_B^r}{[w_A + w_B^r]^2} \left[ (1 + \frac{r^2}{4})(v(1) - v(0)) + (1 - \frac{r^2}{4})(v(2) - v(1)) \right]$$

$$+ \frac{r}{2} (v(2) - v(1)) + r(v(1) - v(0)) - \frac{w_A^r}{w_A + w_B^r}$$

$$= \frac{rw_A^r w_B^r}{[w_A + w_B^r]^2} \left[ (1 - \frac{r}{2}) w_A + (1 + \frac{r}{2}) w_B \right] + \frac{r}{2} \left[ (1 + \frac{2}{r}) w_A + (1 - \frac{2}{r}) w_B \right]$$

$$+ \frac{r}{2} \left[ (1 - \frac{2}{r}) w_A + (1 + \frac{2}{r}) w_B \right] - \frac{w_A^r}{w_A + w_B^r}$$

$$= \frac{rw_A^r}{1 + v^r} \left[ (1 - \frac{r}{2}) + (1 + \frac{r}{2}) v \right] w_A + \frac{r}{4} \left[ (1 + \frac{2}{r}) + (1 - \frac{2}{r}) \right] w_A$$

$$+ \frac{r}{2} \left[ (1 - \frac{2}{r}) + (1 + \frac{2}{r}) v \right] w_A + \frac{1}{1 + v^r}$$

$$= \left[ 1 - \frac{2}{r} \right] w_A \left[ \frac{r}{r-1} v - \frac{1}{r} + \frac{r}{2} \right] + \frac{r}{2} \left[ (1 + \frac{r}{2}) \frac{v^r + 1}{(v^r + 1)^2} + \frac{1}{2} \frac{r}{2} \right] w_B$$

$$+ \frac{r}{2} \left[ (1 + \frac{2}{r}) \frac{v^r + 1}{(v^r + 1)^2} + \frac{1}{2} \frac{r}{2} \right] w_A$$
Denote \( A := [(1 + \frac{r}{2})(r+1)v^{(r+1)} + \frac{1}{2}(\frac{r}{2} - 1)] \) and \( B := [(1 - \frac{r}{2})(r-1)v^{(r-1)} + \frac{1}{2}(\frac{r}{2} + 1)] \). Therefore,

\[
TE_1 = Bw_A + Aw_B
\]

\[
= [(\frac{1}{2} - \frac{r}{4})B + (\frac{1}{2} + \frac{r}{4})A][v(1) - v(0)] + [(\frac{1}{2} + \frac{r}{4})B + (\frac{1}{2} - \frac{r}{4})A][1 - 2v(1)]
\]

\[
= [-\frac{1}{2}(A + B) + \frac{3r}{4}(A - B)]v(1) - \frac{1}{2}(A + B) + \frac{r}{4}(A - B)v(0) + \frac{1}{2}(A + B) + \frac{r}{4}(B - A)
\]

where \( A + B = r[\frac{3v^r + 1}{(v^r + 1)^2} + \frac{1}{2}] \) and \( A - B = \frac{1}{(v^r + 1)^2}[(2 + r^2)v^r + 2] - 1 \).

When \( v(0) = 0 \) and \( v(1) = \frac{-\frac{1}{2} + \frac{r}{4} + (\frac{1}{2} + \frac{r}{4})(r - 1)^\frac{1}{2}}{-\frac{1}{2} + \frac{3r}{4} + (\frac{1}{2} + \frac{3r}{4})(r - 1)^\frac{1}{2}} \), the total effort induced is given by the following:

\[
TE_1(v(0) = 0, v(1)) = \frac{1}{2}(A + B)\frac{\frac{r}{2} + \frac{r}{2}(r - 1)^\frac{1}{2}}{-\frac{1}{2} + \frac{3r}{4} + (\frac{1}{2} + \frac{3r}{4})(r - 1)^\frac{1}{2}}
\]

\[
+ \frac{r}{4}(A - B)\frac{-\frac{3}{2} + \frac{3r}{4} + (\frac{3}{2} + \frac{3r}{4})(r - 1)^\frac{1}{2}}{-\frac{1}{2} + \frac{3r}{4} + (\frac{1}{2} + \frac{3r}{4})(r - 1)^\frac{1}{2}} - 1
\]

\[
= \frac{r}{4}(A + B)\frac{-1 + (r - 1)^\frac{1}{2}}{-\frac{1}{2} + \frac{3r}{4} + (\frac{1}{2} + \frac{3r}{4})(r - 1)^\frac{1}{2}}
\]

\[
+ \frac{r}{4}(A - B)\frac{-1 + (r - 1)^\frac{1}{2}}{-\frac{1}{2} + \frac{3r}{4} + (\frac{1}{2} + \frac{3r}{4})(r - 1)^\frac{1}{2}}
\]

\[
= \frac{rB}{-1 + \frac{3r}{2} + (1 + \frac{3r}{2})(r - 1)^\frac{1}{2}} + \frac{rB}{-1 + \frac{3r}{2} + (1 + \frac{3r}{2})(r - 1)^\frac{1}{2}}
\]

\[
= \frac{r}{-1 + \frac{3r}{2} + (1 + \frac{3r}{2})(r - 1)^\frac{1}{2}}[A(r - 1)^\frac{1}{2} + B]
\]

\[
= \frac{r}{-1 + \frac{3r}{2} + (1 + \frac{3r}{2})(r - 1)^\frac{1}{2}}[\frac{3r}{4} + \frac{1}{2}(r - 1)^\frac{1}{2} + \frac{5}{2} + \frac{r}{4} - \frac{2}{r}]
\]

\[
= \frac{r}{2} + \frac{r}{-1 + \frac{3r}{2} + (1 + \frac{3r}{2})(r - 1)^\frac{1}{2}}
\]

Similarly, prize profile with \( v(0) = 0, v(1) = \frac{(\frac{r}{2} - 1)(r+1)v^{(r+1)} + \frac{1}{2}(\frac{r}{2} + 1)}{(3r/2 - 1)(r+1)v^{(r+1)} + \frac{1}{2}(3r/2 + 1)} \) and \( \eta = \frac{1}{(r-1)^\frac{1}{2}} \) belongs to \( \mathcal{V}_0 \). The
total effort induced is

\[
TE_1(v(0) = 0, v(1) = \frac{(r/2 - 1)(r - 1)^{\frac{1}{r}} + r/2 + 1}{(3r/2 - 1)(r - 1)^{\frac{1}{r}} + 3r/2 + 1})
\]

\[
= \frac{1}{2}(A + B) \frac{r(1 + (r - 1)^{\frac{1}{r}})}{(3r/2 - 1)(r - 1)^{\frac{1}{r}} + 3r/2 + 1}
\]

\[
+ \frac{r}{4}(A - B)\frac{-2(r - 1)^{\frac{1}{r}} + 2}{(3r/2 - 1)(r - 1)^{\frac{1}{r}} + 3r/2 + 1}
\]

\[
= 1\frac{1}{2}(A + (r - 1)^{\frac{1}{r}}B)
\]

\[
= \frac{1}{2}(\frac{3r}{2} - 1)(r - 1)^{\frac{1}{r}} + 3r/2 + 1
\]

\[
= \frac{5r^2}{2} + 2\frac{r}{2} - 2 + \frac{(r^2 + r)}{2}(r - 1)^{\frac{1}{r}}
\]

\[
= \frac{(r^2 - 1)(r - 1)^{\frac{1}{r}} + 3r/2 + 1}{(3r/2 - 1)(r - 1)^{\frac{1}{r}} + 3r/2 + 1}.
\]

\[\blacksquare\]

**Proof of Proposition 2**

**Proof.** The third battle can be analyzed identically as in Lemma 7. We now look at the second battle. Recall \((n_A, n_B)\) denotes the history of the contest. When \((n_A, n_B) = (1, 0)\) the effective prize spreads are respectively

\[v_A(1, 0) = \frac{1}{2}(1 - v(1) - v(0)) + \frac{r}{4}(1 - 3v(1) + v(0)) = w_A,\]

\[v_B(1, 0) = \frac{1}{2}(1 - v(1) - v(0)) + \frac{r}{4}(-1 + 3v(1) - v(0)) = w_B.\]

\[v_A(1, 0) \leq v_B(1, 0)\] is equivalent to \(w_A \leq w_B\). As the prize profiles are in \(\mathcal{V}_3\), applying Lemma 1(ii) gives the equilibrium effort:

\[
\tilde{x}_A(1, 0) = \begin{cases} 
(1 - \frac{1}{r})v_A(1, 0) & \text{with probability } q = \frac{v_A(1, 0)}{v_B(1, 0)}(\frac{1}{r - 1})^{\frac{1}{r}}, \\
0 & \text{with probability } 1 - q.
\end{cases}
\]

\[
\tilde{x}_B(1, 0) = (\frac{1}{r - 1})^{\frac{1}{r}}(1 - \frac{1}{r})v_A(1, 0)
\]

The winning probability are

\[p_A(1, 0) = (1 - \frac{1}{r})q, \ p_B(1, 0) = 1 - (1 - \frac{1}{r})q.\]

Similarly, when \((n_A, n_B) = (0, 1)\), we have

\[
\tilde{x}_A(0, 1) = \tilde{x}_B(1, 0) = (\frac{1}{r - 1})^{\frac{1}{r}}(1 - \frac{1}{r})v_A(1, 0).
\]
$$\tilde{x}_B(0, 1) = \tilde{x}_A(1, 0) = \begin{cases} (1 - \frac{1}{r})v_A(1, 0) & \text{with probability } q = \frac{v_A(1, 0)}{v_B(1, 0)} \left(\frac{1}{r-1}\right)^\frac{1}{2}, \\ 0 & \text{with probability } 1 - q. \end{cases}$$

The winning probability are

$$p_A(0, 1) = 1 - (1 - \frac{1}{r})q, \quad p_B(0, 1) = (1 - \frac{1}{r})q.$$ 

We have

$$E[\tilde{x}_B(0, 1)] = E[\tilde{x}_A(1, 0)] = q(1 - \frac{1}{r})v_A(1, 0) = (1 - \frac{1}{r}) \left(\frac{1}{r-1}\right)^\frac{1}{2} \frac{v_A^2(1, 0)}{v_B(1, 0)}.$$

Now we come to the first battle. We pin down the common effective prize spread:

$$v_A(0, 0) = v_B(0, 0)$$

$$= \{p_A(1, 0)[p_A(2, 0)v(3) + p_B(2, 0)v(2) - x_A(2, 0)]$$

$$+ p_B(1, 0)[p_A(1, 1)v(2) + p_B(1, 1)v(1) - x_A(1, 1)] - E[\tilde{x}_A(1, 0)]\}$$

$$- \{p_A(0, 1)[p_A(1, 1)v(2) + p_B(1, 1)v(1) - x_A(1, 1)]$$

$$+ p_B(0, 1)[p_A(0, 2)v(1) + p_B(0, 2)v(0) - x_A(0, 2)] - \tilde{x}_A(0, 1)\}$$

$$= (1 - \frac{1}{r})q(v(2) - v(0)) + (1 - \frac{1}{r})v_A(1, 0)\left(\frac{1}{r-1}\right)^\frac{1}{2} - q].$$

Thus we have the effort supply

$$x_A(0, 0) = x_B(0, 0) = \frac{r}{4} v_A(0, 0),$$

and the winning probabilities

$$p_A(0, 0) = p_B(0, 0) = \frac{1}{2}.$$
Total effort thus is as follow:

\[
TE_3 = 2x_A(0,0) + E[\bar{x}_A(1,0)] + \bar{x}_B(1,0) \\
+ p_A(1,0)(x_A(2,0) + x_B(2,0)) + p_B(1,0)(x_A(1,1) + x_B(1,1)) \\
= \frac{r}{2}(v(2) - v(1)) + r(1 - \frac{1}{r})q(v(1) - v(0)) \\
+ (1 + \frac{r}{2})(1 - \frac{1}{r})(\frac{1}{r-1})^\frac{1}{r}w_A + (1 - \frac{r}{2})(1 - \frac{1}{r})qw_A \\
= \frac{r}{2}(v(2) - v(1)) + r(1 - \frac{1}{r})(\frac{1}{r-1})^\frac{1}{r}w_A \frac{1}{w_B} [(1 - \frac{2}{r})w_A + (1 + \frac{2}{r})w_B] \\
+ (1 + \frac{r}{2})(1 - \frac{1}{r})(\frac{1}{r-1})^\frac{1}{r}w_A + (1 - \frac{r}{2})(1 - \frac{1}{r})(\frac{1}{r-1})^\frac{1}{r}w_A \frac{w_A}{w_B} \\
= \frac{r}{2}(2v(2) - 1) + (1 - \frac{1}{r})(\frac{1}{r-1})^\frac{1}{r}(2 + r)w_A \\
= \frac{r}{2}(2v(2) - 1) + (1 - \frac{1}{r})(\frac{1}{r-1})^\frac{1}{r}(2 + r)[\frac{1}{2} + \frac{r}{2})(2v(2) - 1) + (1 - \frac{2}{r})w_A/((1 - \frac{2}{r})w_A) \\
= \frac{r}{2}(2v(2) - 1) + (1 - \frac{1}{r})(\frac{1}{r-1})^\frac{1}{r}(2 + r)[-\frac{2}{2} - (1 - \frac{2}{r})w_A + (1 + \frac{3}{4} r)w_A].
\]

Hence, \( TE_3 \) is increasing in \( v(2) \) and decreasing in \( v(0) \).

We want to maximize effort \( TE_3 \) subject to prize profiles are in \( \mathcal{V}_3 \), i.e. \( w_A \leq w_B \) and \( 1 + \frac{w_A}{w_B} < r \leq 2 \), as well as the non-negativity and monotonicity of the prizes. Since \( r \in (\tau, 2] \), constraints \( w_A \leq w_B \) and \( 1 + \frac{w_A}{w_B} < r \leq 2 \) can be written as \( (r - 1)^{\frac{1}{r}}w_B > w_A \), i.e. \( v(2) < \frac{r(1+(r-1)^{\frac{1}{r}})}{(r-1)^{\frac{1}{r}}(\frac{3r}{2} - 1) + (\frac{3r}{2} + 1)} \). We thus can set \( v(0) = 0 \) and consider simplified constraint \( v(2) < \frac{r(1+(r-1)^{\frac{1}{r}})}{(r-1)^{\frac{1}{r}}(\frac{3r}{2} - 1) + (\frac{3r}{2} + 1)} \), which is less than 1 when \( r > \tau \). We thus have for any prize profile in \( \mathcal{V}_3 \), the effort induced is converging to but smaller than \( TE_3(v(0) = 0, v(2)) = \frac{r(1+(r-1)^{\frac{1}{r}})}{(r-1)^{\frac{1}{r}}(\frac{3r}{2} - 1) + (\frac{3r}{2} + 1)} \), which is the following.

\[
TE_3(v(0)) = 0, v(2) = \frac{r(1+(r-1)^{\frac{1}{r}})}{(r-1)^{\frac{1}{r}}(\frac{3r}{2} - 1) + (\frac{3r}{2} + 1)} \\
= r \frac{r(1+(r-1)^{\frac{1}{r}})}{(r-1)^{\frac{1}{r}}(\frac{3r}{2} - 1) + (\frac{3r}{2} + 1)} - \frac{r}{2} \\
+ (1 - \frac{r}{2})(1 - \frac{1}{r})^\frac{1}{r}(2 + r)[\frac{1}{2} + \frac{3}{4} r] - \frac{r}{2} \frac{r(1+(r-1)^{\frac{1}{r}})}{(r-1)^{\frac{1}{r}}(\frac{3r}{2} - 1) + (\frac{3r}{2} + 1)}.
\]
where, to show that

Note that, of Lemma 4, we calculate the derivative of

\[ \frac{d}{d\eta}(TE_1) = \frac{1}{(1+\eta)^3} \left\{ \frac{1}{4} \eta^{r-1}[(r+2)+(r-2)\eta][(1-\frac{r}{2})(-r-1+(r-1)\eta)] + \right. \]

\[ \left. (1+\frac{r}{2})(-r+1+(r+1)\eta)\eta - (1+\eta^r)[1+(2+\frac{r^2}{4})\eta^r] \right\} \]  

\[ \left( \frac{2}{4} + \frac{2}{3}(r-1)^{\frac{3}{2}} + \frac{2}{2} \right) \]

\[ 1 + \frac{3r}{2} + (\frac{3r}{2} - 1)(r-1)^{\frac{3}{2}} \].

\[ \text{Step 1: we prove } \frac{d}{d\eta}(TE_1) < 0 \text{ when } r \in (\tau, 2] \text{ and } \eta \in [(r-1)^{\frac{3}{2}}, 1] \text{ as follows. From the proof of Lemma 4, we know that} \]

\[ \frac{d}{d\eta}(TE_1) = \frac{1}{(1+\eta)^3} \left\{ \frac{1}{4} \eta^{r-1}[(r+2)+(r-2)\eta][(1-\frac{r}{2})(-r-1+(r-1)\eta)] + \right. \]

\[ \left. (1+\frac{r}{2})(-r+1+(r+1)\eta)\eta - (1+\eta^r)[1+(2+\frac{r^2}{4})\eta^r] \right\} \]  

\[ \left( \frac{2}{4} + \frac{2}{3}(r-1)^{\frac{3}{2}} + \frac{2}{2} \right) \]

\[ 1 + \frac{3r}{2} + (\frac{3r}{2} - 1)(r-1)^{\frac{3}{2}} \].

\[ \text{Proof of Theorem 2} \]

\[ \text{Step 2: From the proof of Theorem 1, we obtain that } sign\left(\frac{d}{d(\eta)}(TE_1(\eta(0)=0,\eta(2)))\right) = sign(D(\eta, r)), \]

where \( D(\eta, r) \equiv [(\frac{3r}{2} - 1) + (\frac{3r}{2} + 1)\eta][(1-\frac{r}{2})(-r-1+(r-1)\eta^r) + (1+\frac{r}{2})(-r+1+(r+1)\eta^r)] + 2\eta(1+\eta^r)(-2 - \frac{3r^2}{4} + \eta^r). \]

In this step, we show that \( D(\eta, r) \) is an increasing function in \( \eta \) when \( \eta \in [(r-1)^{\frac{3}{2}}, 1] \subset [\frac{1}{2}, 1] \) for each \( r \in (\tau, 2] \).

Recall that we calculate the derivative of \( D(\eta, r) \) with respect to \( \eta \) in the proof of Theorem 1, which
is greater than
\[
(4 - \frac{11r^2}{2} + \frac{3r^3}{2}) + (r - 3r^2 + \frac{11}{4}r^3 - \frac{3}{4}r^4)\eta^{-1} + (6 + 9r + \frac{9}{2}r^2 + \frac{3}{4}r^3 - \frac{3}{4}r^4)\eta + (2 + 4r)\eta^2r
\]
\[
> (4 - \frac{11r^2}{2} + \frac{3r^3}{2}) + (6 + 9r + \frac{9}{2}r^2 + \frac{3}{4}r^3 - \frac{3}{4}r^4)(r - 1)
\]
\[
> 0. \tag{10}
\]

The first inequality holds as \((r - 3r^2 + \frac{11}{4}r^3 - \frac{3}{4}r^4) > 0\) when \(r \in (\tau, 2]\) and \(\eta \in [(r - 1)^\frac{3}{2}, 1]\).

The rest proof is same as the one in the Theorem 1, except that we use \((r - 1)^\frac{3}{2}\) as the lower bound of \(\eta\) when \(r \in (\tau, 2]\), rather than \(\frac{1}{2}\) as before. \(\blacksquare\)

**Proof of Lemma 9**

**Proof.** We use backward induction to solve the game. Note Lemma 1(iii) applies to all three battles. The third-battle results for history (2,0), (1,1) and (0,2) remain same as in the proof of Lemma 8. Next, we look at the second battle. The expressions for the prize spreads remain same as in the proof of Lemma 8.

History (1,0): \(\bar{v}_A(1, 0) \leq \bar{v}_B(1, 0)\) if and only if \(v(2) - v(1) \leq v(1) - v(0)\), and we are considering \(\mathcal{V}_5\).

Thus the effort supply is
\[
G^A_{(1,0)}(x) = \frac{(v(1) - v(0)) - (v(2) - v(1)) + x}{(v(1) - v(0))} \text{ in } [0, v(2) - v(1)],
\]
\[
G^B_{(1,0)}(x) = \frac{x}{v(2) - v(1)} \text{ in } [0, v(2) - v(1)].
\]

The winning probabilities are
\[
p_A(1, 0) = \frac{1}{2} q, \quad p_B(1, 0) = 1 - \frac{q}{2},
\]
where \(q = \frac{v(2) - v(1)}{v(1) - v(0)}\). We thus have
\[
E[\bar{x}_A(1, 0)] = \frac{1}{2} (v(1) - v(0)), \quad E[\bar{x}_B(1, 0)] = \frac{1}{2} (v(2) - v(1))^2.
\]

History (0,1) is the dual case of history (1,0).
Now we come to the first battle. We first pin down the common effective prize spread as follows:

\[ v_A(0, 0) = v_B(0, 0) \]
\[ = \{p_A(1, 0)[p_A(2, 0)v(3) + p_B(2, 0)v(2) - E[\hat{x}_A(2, 0)]] + p_B(1, 0)[p_A(1, 1)v(2) + p_B(1, 1)v(1) - E[\hat{x}_A(1, 1)]] - E[\hat{x}_A(1, 0)]\} \]
\[ - \{p_A(0, 1)[p_A(1, 1)v(2) + p_B(1, 1)v(1) - E[\hat{x}_A(1, 1)]] + p_B(0, 1)[p_A(0, 2)v(1) + p_B(0, 2)v(0) - E[\hat{x}_A(0, 2)] - x_A(0, 1)\} \]
\[ = \frac{1}{2}q(v(2) - v(0)) + \left( \frac{1}{2} - \frac{q}{2} \right)(v(2) - v(1)). \]

Thus the effort supply is

\[ G_A^{(0, 0)}(x) = G_B^{(0, 0)}(x) = \frac{x}{v_A(0, 0)} \text{ in } [0, v_A(0, 0)]. \]

The winning probabilities are

\[ p_A(0, 0) = p_B(0, 0) = \frac{1}{2}. \]

Total effort thus is as follow:

\[ TE_5 = 2E[\hat{x}_A(0, 0)] + E[\hat{x}_A(1, 0)] + E[\hat{x}_B(1, 0)] \]
\[ + p_A(1, 0)(E[\hat{x}_A(2, 0)] + E[\hat{x}_A(2, 0)]) + p_B(1, 0)(E[\hat{x}_A(1, 1)] + E[\hat{x}_A(1, 1)]) \]
\[ = v_A(0, 0) + \left( \frac{1}{2} + q \right)(v(2) - v(1)) - \frac{q}{2}(v(1) - v(0)) + \left( 1 - \frac{q}{2} \right)(v(2) - v(1)) \]
\[ = 3(v(2) - v(1)). \]

\[ \square \]

**Proof of Lemma 10**

**Proof.** Note that Lemma 1(i) applies to all three battles when \( r \leq 2 \). Before a battle is fought, the history of past battles, or the state of the contest, is observed by players involved. The history of the contest is denoted \( (n_A, n_B) \) where \( n_A \) is the number of wins secured by team \( i = A, B \). We solve the game by backward induction. We first look at the third battle.

History \( (2, 0) \): the effective prize spreads are

\[ v_A(2, 0) = v(3) - v(2) \geq 0, \]
\[ v_B(2, 0) = v(1) - v(0) \geq 0. \]

The budget constraint \( v(3) + v(0) = v(2) + v(1) \) implies \( v(3) - v(2) = v(1) - v(0) \), so \( v_A(2, 0) = v_B(2, 0) \)

By Lemma 1(i), we have effort supply
The winning probabilities are
\[ p_A(2, 0) = \frac{x_A(2, 0)}{x_A(2, 0) + x_B(2, 0)} = \frac{1}{2}, \quad p_B(2, 0) = \frac{x_B(2, 0)}{x_A(2, 0) + x_B(2, 0)} = \frac{1}{2} \]

History (0, 2) is similar. We now look at history (1, 1). For history (1, 1), the common prize spread is
\[ v_A(1, 1) = v_B(1, 1) = v(2) - v(1) \]

Thus effort supply is
\[ x_A(1, 1) = x_B(1, 1) = \frac{r}{4}(v(2) - v(1)) \]

The winning probabilities are
\[ p_A(1, 1) = p_B(1, 1) = \frac{1}{2} \]

We now look at the second battle. The history can be (1, 0) or (0, 1).

History (1, 0): we first pin down the effective prize spreads:
\[ v_A(1, 0) = \frac{1}{2} (v(3) - v(1)) = \frac{1}{2} (v(2) - v(0)) \]
\[ v_B(1, 0) = \frac{1}{2} (v(2) - v(0)) \]

Thus effort supply is
\[ x_A(1, 0) = x_B(1, 0) = \frac{r}{4} v_A(1, 0) = \frac{r}{8} (v(2) - v(0)) \]

The winning probabilities are
\[ p_A(1, 0) = p_B(1, 0) = \frac{1}{2} \]

History (0, 1): This is dual case of history (1, 0). We have prize spread
\[ v_A(0, 1) = v_B(0, 1) = \frac{1}{2} (v(2) - v(0)) \]

and effort supply
\[ x_A(0, 1) = x_B(0, 1) = \frac{r}{8} (v(2) - v(0)) \]

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The winning probabilities are
\[ p_A(0, 1) = p_B(0, 1) = \frac{1}{2}. \]

Now we come to the first battle. We pin down the common effective prize spread:

\[
v_A(0, 0) = v_B(0, 0) = \{p_A(1, 0)[p_A(2, 0)v(3) + p_B(2, 0)v(2)] + p_B(1, 0)[p_A(1, 1)v(2) + p_B(1, 1)v(1)]\] 
- \{p_A(0, 1)[p_A(1, 1)v(2) + p_B(1, 1)v(1)] + p_B(0, 1)[p_A(0, 2)v(1) + p_B(0, 2)v(0)]\} 
= \frac{1}{2}(v(2) - v(0)).
\]

Thus effort supply is
\[ x_A(0, 0) = x_B(0, 0) = \frac{r}{4}v_A(0, 0) = \frac{r}{8}(v(2) - v(0)). \]

The winning probabilities are
\[ p_A(0, 1) = p_B(0, 1) = \frac{1}{2}. \]

Thus, total effort can be calculated as follow:
\[
TE^{1}_{S} = 2x_A(0, 0) + [x_A(1, 0) + x_B(1, 0)] 
+ p_A(1, 0)[x_A(2, 0) + x_B(2, 0)] + p_B(1, 0)[x_A(1, 1) + x_B(1, 1)] 
= 2 \times \left[ \frac{r}{8}(v(2) - v(0)) \right] + 2 \times \left[ \frac{r}{8}(v(2) - v(0)) \right] 
+ \frac{1}{2} \times 2 \times \frac{r}{4}(v(1) - v(0)) + \frac{1}{2} \times 2 \times \frac{r}{4}(v(2) - v(1)) 
= \frac{3r}{4}(v(2) - v(0)).
\]

**Proof of Lemma 11**

**Proof.** Note Lemma 1(iii) applies to each battle. Moreover, Fu, Lu and Pan (2015) reveals that the prize spread is common for the two players in each battle. We solve the game by backward induction. We first look at the third battle.

History (2,0): For history (2,0), we first describe the two players’ effective prize spreads:

\[ v_A(2, 0) = v(3) - v(2) \geq 0, v_B(2, 0) = v(1) - v(0) \geq 0. \]

We have \( v_A(2, 0) = v_B(2, 0) \) follows from the budget constraints.

Thus effort supply is
\[ \tilde{x}_A(2, 0) \sim G^A_{(2,0)}(x) = \frac{x}{v(1) - v(0)} \text{ in } [0, v(1) - v(0)]. \]
\[ \tilde{x}_B(2, 0) \sim G^B(2, 0) \left( x \right) = \frac{x}{v(1) - v(0)} \text{ in } [0, v(1) - v(0)], \]

where \( G^i_{\{n,A,n,B\}}(.) \) denotes the cumulative distribution function of the player \( i \)'s mixed strategy in equilibrium. The winning probabilities are

\[ p_A(2, 0) = p_B(2, 0) = \frac{1}{2}. \]

History (0,2) is similar. We now look at history (1,1). For history (1,1), the common effective prize spread is

\[ v_A(1, 1) = v_B(1, 1) = v(2) - v(1) \geq 0. \]

Thus effort supply is given by

\[ \frac{G_A(1, 1)}{G^A(1, 1)}(x) = \frac{G^B(1, 1)}{G_B(1, 1)}(x) = \frac{x}{v(2) - v(1)} \text{ in } [0, v(2) - v(1)]. \]

The winning probabilities are

\[ p_A(1, 1) = p_B(1, 1) = \frac{1}{2}. \]

History (0,2): For history (0,2), we first describe the two players’ effective prize spreads.

\[ v_A(0, 2) = v_B(0, 2) = v(1) - v(0) \geq 0 \]

Thus

\[ \frac{G_A(1, 1)}{G^A(1, 1)}(x) = \frac{G^B(1, 1)}{G_B(1, 1)}(x) = \frac{x}{v(1) - v(0)} \text{ in } [0, v(1) - v(0)]. \]

Then teams’ winning probability are

\[ p_A(1, 1) = p_B(1, 1) = \frac{1}{2}. \]

History (1,0): we first pin down the common effective prize spread:

\[ v_B(1, 0) = v_A(1, 0) = [p_A(2, 0)v(3) + p_B(2, 0)v(2)] - [p_A(1, 1)v(2) + p_B(1, 1)v(1)] \]

\[ = \frac{1}{2}(v(3) - v(1)). \]

Thus effort supply is

\[ \frac{G_A(1, 0)}{G^A(1, 0)}(x) = \frac{G^B(1, 0)}{G_B(1, 0)}(x) = \frac{2x}{v(2) - v(0)} \text{ in } [0, \frac{1}{2}(v(2) - v(0))]. \]

The winning probabilities are

\[ p_A(1, 0) = p_B(1, 0) = \frac{1}{2}. \]

History (0,1) is symmetric.
Total effort thus is as follow:

\[
TE^2_B = 2E[\tilde{x}_A(0,0)] + (E[\tilde{x}_A(1,0)] + E[\tilde{x}_B(1,0)])
\]

\[
+ p_A(1,0)(E[\tilde{x}_A(2,0)] + E[\tilde{x}_B(2,0)])
\]

\[
+ p_B(1,0)(E[\tilde{x}_A(1,1)] + E[\tilde{x}_B(1,1)])
\]

\[
= 2E[\tilde{x}_A(0,0)] + 2E[\tilde{x}_A(1,0)] + E[\tilde{x}_A(2,0)] + E[\tilde{x}_A(1,1)]
\]

\[
= \frac{3}{2}(v(2) - v(0)),
\]

Where

\[
E[\tilde{x}_A(0,0)] = \int \tilde{x}_A(0,0)dG^A_{(0,0)}(x) = \frac{1}{4}(v(2) - v(0)),
\]

\[
E[\tilde{x}_A(1,0)] = \frac{1}{4}(v(2) - v(0)),
\]

\[
E[\tilde{x}_A(2,0)] = \frac{1}{4}(v(1) - v(0)),
\]

\[
E[\tilde{x}_A(1,1)] = \frac{1}{4}(v(2) - v(1)).
\]

References


