On Pure-Strategy Equilibria in Games with Correlated Information

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Abstract: Aumann (1974) introduced the notions of secret and objective events in a setting with correlated information and subjective beliefs, but a decisive and celebrated example of Radner-Rosenthal (1982) questioned the hypotheses of the result that the set of independent objective pure-strategy equilibrium payoffs of a suitably-formulated incomplete-information game coincides with the set of mixed-strategy equilibrium payoffs of the original complete information game of Nash (1950, 1951). We present a two-player game with correlated information modeled as a subset in the product of the extended Lebesgue interval, as proposed in Khan-Zhang (2012), and show that the sub-σ-algebra of a player’s secret events is rich enough to adequately respond to this criticism, and to rescue Aumann’s original motivation towards a descriptive theory of pure-strategy equilibrium in games with correlated information. We also show that a saturated information structure, as emphasized by Keisler-Sun (2009), is necessary to guarantee the existence of pure-strategy equilibrium in a simple subclass of such games. Our results generalize beyond the toy-model, but we emphasize the simplest setting with many illustrative examples.

(169 words)

Key Words: Correlated information games, Kakutani Lemma, Lebesgue extension, secret event, regular information structure, subjective (objective) pure-strategy equilibrium, saturated probability spaces.

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1 Introduction

The assumption of information available to a player that is independent of that available to his or her others in a finite game of incomplete information is violated in innumerable applied settings in economic and game theory. The exclusion of correlated information and subjective beliefs regarding the other players’ information or types is surely unjustified and unavoidable if a model is to have relevance and meaningfully address substantive issues. However, Radner and Rosenthal (1982) present a matching-penny game with correlated information, where each player’s type is modeled by the usual Lebesgue unit interval, and the set of the states of the world (the set of the pair of types), is the lower half-triangle of the Lebesgue square. This example of a two-player game with a common two-action set, objective beliefs and the simplest of correlated information, is decisive in that it shows that the set of pure-strategy Nash equilibria is empty. Even though results for various specific and concrete settings are available in the literature, the RR example is perhaps directly responsible for the lack of general theorems on the existence of a pure-strategy Nash equilibrium in finite games with correlated information or types. Thus, the question of a compelling assumption of some generality under which the RR example can be ruled out has remained an open question for the last three decades. In this paper, we address this question.

However, there is another perspective from which to frame the results reported here. In a pioneering paper, Aumann (1974) launched a substantive investigation of the “concepts of communication, correlation, commitment, and contract,” seeing them as the very basis for the distinction between cooperative and non-cooperative game theory. In particular, in what can be seen in hindsight as a conceptual triangulation of subjectivity, inter-subjectivity and objectivity in any game, Aumann explored issues concerning secrecy, subjectivity and objectivity, and their underpinning by the statistical notions of correlation and independence. To elaborate, an event is a secret of player $i$ in the subjective register of his or her information if it is independent of all player $j$’s events with respect to $j$’s prior. “Thus $i$ is informed regarding each $i$-secret event, but players other than $i$ can get no hint of it, even by pooling their knowledge.” And objective events are simply those on which

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1It is of interest in this connection that in [34], greater importance is given to this and other examples than to the existence theorems themselves. Throughout the sequel, we shall refer to this example as the RR example, but remind the reader here that there are two other examples in [34] that are important for issues not considered by us here.

2The importance of this example, and its re-direction of the research towards the existence of approximate pure-strategy Nash equilibria, is attested to in the subsequent literature; see [3, 35, 31, 9] and subsequent work that relies on ordered structures on action sets.

3With this emphasis on a triangulation in the formulation of a game, we connect Aumann and game theory to the work of Davidson (2001), a theme that we hope to pursue elsewhere.
the individual beliefs (inter-subjectivities) coincide.\textsuperscript{4} With this vocabulary in place, Aumann then considers an individualized (subjective) sub-$\sigma$-algebra of secret events for each player on which the individualized belief (subjective prior) of any other player is atomless, and with this formalization of secrecy, shows that the set of independent objective pure-strategy equilibrium payoffs\textsuperscript{5} of the game with incomplete information coincides with the set of mixed-strategy Nash equilibrium payoffs of the same finite game but with complete information, as in the classical setting of Nash (1950, 1951).\textsuperscript{6} As an easy corollary, he established the existence of objective pure-strategy equilibria for finite games with incomplete information in games with finite actions and type-independent payoff functions.\textsuperscript{7}

One has to be clear that an articulation and analytical implementation of Aumann’s ideas hinges crucially on the individualized sub-$\sigma$-algebras of secret events being \textit{diffused} and \textit{disparate}: being rich and plentiful for the \textit{atomlessness} assumption to make sense, and yet be independent in keeping with the non-cooperative requirements of the conclusion of his theorem.\textsuperscript{8} It is straightforward to check that the main assumption in [2] does not hold in the correlated information structure in the RR example.\textsuperscript{9} This can now be seen as a total stall on this direction if additional assumptions on the information space are not to be invoked. To underscore the point, there are no such independent and atomless sub-$\sigma$-algebras of secret events if information is modeled as the Lebesgue square! This lack of an abundant-enough measure-theoretic structure on a player’s set of secret events gives a certain vacuity to Aumann’s result, and it is presumably because of this apparent dead-end

\textsuperscript{4}Given Aumann’s setting of type-independent payoffs, a natural formulation of \textit{objectivity} pertains to a situation where inter-subjectivities coincide; see [2, Section 9(d)] where the distinction of this notion as compared to the conventional one is emphasized. Also see the reference in Footnote 3 above in this connection.

\textsuperscript{5}See the Main Assumption in Section 2 below. We remind the reader that it is precisely to establish this independence and objectivity that the Lyapunov theorem is invoked; see the proofs of Lemma 7.1 and 4.1 in [2]. To our knowledge, this is the first direct application of Lyapunov’s theorem to non-cooperative n-person game theory.

\textsuperscript{6}To re-emphasize, two different games are being connected here. Of course, the classical game of complete information is a trivial game of incomplete information in which private information is represented by the trivial $\sigma$-algebra and only constant functions are permitted as strategies.

\textsuperscript{7}Throughout this paper, a function from the state of the world to the player’s action set is termed a pure-strategy of a player if it is measurable with respect to the player’s information. In Aumann’s 1974 language, it is a randomized strategy for this player. Aumann also makes a distinction between a \textit{randomized} strategy and what he terms a \textit{mixed} strategy, the latter being a randomized strategy which is measurable with respect to the $\sigma$-algebra of a player’s secret events; see the definition and the discussion after Assumption II in Section 3 of [2]. Given the way the subsequent literature on games with incomplete information has developed, we entirely avoid this terminology for such games.

\textsuperscript{8}The adjectives \textit{diffused} and \textit{disparate} are those that Radner-Rosenthal use for games with private information as per their formulation; we expropriate them for our own usage here pertaining to secret events.

\textsuperscript{9}We remind the reader that this is Assumption II in Section 3 of [2]. We repeat the basic point, and reinterpret the RR example another way. Since Aumann’s results are mathematically correct, and there is no pure-strategy equilibrium, it means as a matter of necessity that Aumann’s Assumption II does not hold. And this is to assert that the sub-$\sigma$-algebra of secret events must be empty. It is this that we want to signal through our use of the word “vacuous.”
that ideas of secrecy are jettisoned in the literature, and the assumption of independence directly invoked on the individualized σ-algebras. And in this direct setting, one freed from all nuance of secrecy and objectivity, even type-dependent payoffs can be easily accommodated. The literature then takes an alternative tack, and shifts attention from the underpinning of a classical game of complete information by one of incomplete information – a substantive equivalence theorem – to the technical question of the existence of pure strategies and the purification of mixed strategies in a given game of private information.

But it surely stands to reason that Aumann’s ideas on secrecy and subjectivity are not necessarily to be pegged to any arbitrary abstract measure space of information, much less to a Lebesgue space. The relevant issue is how a sub-σ-algebra of the secret events of a particular player in a multi-player setting sits with it being atomless with respect to others’ subjective beliefs. Is there a formulation of a space of information that is rich enough that it admits sub-σ-algebras that are atomless, one that is hospitable to the invocation and use of the Lyapunov theorem, and yet allows independence and objectivity? This is the alternative framing of the question addressed here.

Towards a preliminary response, and to fix ideas, we can pose the question in the context of a toy model of two-person game in which the correlated information is given an explicit pictorial representation. Inspired by the RR example, the type space of each player is modeled by the usual Lebesgue unit interval and Ω, the set of the states of the world, is modeled by a subset of the Lebesgue square. This set is the source of correlation in the sense that, for each type $t_i$ of player $i$, conditional on $t_i$, the player believes that the types of the other player can not lay outside of the $t_i$-section of Ω provided this $t_i$-section is nonempty. For example, if Ω is a set that lies below a non-zero constant function in the square, and both players’ prior or subjective belief is the uniform distribution on Ω, this yields an independent information structure. With this geometric representation, one can ask the preliminary question as to the extent to which correlated information can be admitted in Aumann’s theorem if one sticks with Lebesgue measure. We can show that Aumann’s assumption is rendered vacuous for many examples of correlated information structure. This preliminary result is a benchmark to what follows. The question is how to extend the Lebesgue square to yield a non-vacuous version of Aumann’s result for more generally correlated information while still remaining in the toy model.

It is the answer to this question that delivers the first important finding of this paper. It is that for a general class of two-player games with correlated information, Aumann’s assumption

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10 The reader can look at Figure 1 below if he or she so wishes.
11 The reader is referred to Examples 2-4 below in this connection.
is fully executable and implementable provided that each player’s type space is modeled by an extended Lebesgue interval as in [22]; see Theorem 1. In [22], the Lebesgue $\sigma$-algebra is augmented or extended by adding a number of Lebesgue nonmeasurable subsets to the Lebesgue algebra such that there exists a sub-$\sigma$-algebra of the extended algebra such that it is independent of the Lebesgue $\sigma$-algebra, and the extended Lebesgue measure is atomless on this sub-algebra.\textsuperscript{12} This is exactly what we are after. The basic idea behind the finding is that if each player’s type space is modeled by the extended Lebesgue interval, which is to say that each player has a sufficiently rich information structure, then it becomes fully possible for players to have an abundant secret-event structure as well. As a consequence, Aumann’s equivalence theorem can be fully rescued to yield an independent objective pure-strategy equilibrium for a general class of two-person games with correlated information that yields any given mixed strategy Nash equilibrium for the classical reduced setting with complete information.

Our framework also facilitates the analysis of the structure of pure-strategy equilibria in this class of games. It is shown in [2] that in a two person game, under Aumann’s main assumption, if a mild condition is satisfied, namely, it is not the case that one player assigns zero subjective probability to an event to which the other player assigns positive subjective probability, then the set of subjective independent pure-strategy equilibrium payoffs coincides with the set of objective independent pure-strategy equilibrium payoffs. We offer illustrative examples of both zero-sum and non-zero sum games in which the importance of the above mild condition can be investigated, and the relationship between the set of subjective independent pure-strategy equilibrium payoffs and all the objective independent equilibrium payoffs clearly seen; see Examples 7 and 8 below.

Finally, we indicate how we go beyond the Lebesgue interval. Once the basic conceptualizations are understood, it is easy to see that the results in Theorem 1 can be generalized to a situation where each player’s type space is modeled by an abstract probability space which admits an atomless independent supplement for a certain sub-$\sigma$-algebra, and the state of the world is modeled by what we term a \textit{regular} subset in the product of these two sub-$\sigma$-algebras.\textsuperscript{13} One straightforward but important corollary, Corollary 2 below, is that if player 1’s type space is modeled by a saturated probability space,\textsuperscript{14} player 2’s type space by the Lebesgue extension as in [22], then for any two-player finite game with correlated information modeled by the (regular) hypograph of a measurable function from player 1’s type space to the unit Lebesgue interval, then there exists a pure-strategy equilibrium.

\textsuperscript{12}This property is called that the extended Lebesgue algebra admits an \textit{atomless independent supplement} with respect to the Lebesgue algebra in [14]; see Definition 2 below. See also the conditions in Theorem 1 of [10].

\textsuperscript{13}See Definition 1 below.

\textsuperscript{14}This concept is by now well-understood and used in several recent papers; see Definition 3 below.
equilibrium in such a correlated information game. Given this result, the relevant question is whether we are imposing too strong a condition on the information structure of player 1 to guarantee the existence of pure-strategy equilibrium for the games with the type of correlated information that we investigate. In terms of the results of Corollary 2 mentioned above, we find that if player 1’s type space is modeled by an abstract atomless probability space, and if the matching-penny game associated with a very small class of correlated information structure each of which has a pure-strategy equilibrium, then the underlying modeling of player 1’s type space must be a saturated space. That is a testimony to the fact that the requirement of player 1’s type space being modeled by a saturated space is a rather weak requirement.

The paper is organized as follows. Section 2 presents the toy model of games with correlated information, 3 reworks the RR example and provides another one that illustrates how Aumann’s assumption has no bite when the information is correlated in another simple way. This is to say that the result is true but vacuous in the sense that it main hypothesis does not hold. The construction of Lebesgue extension is reviewed in Section 4 in a language that is hopefully more user-friendly, and the main result pertaining to it is presented in Section 5. The relationship between subjective and objective pure-strategy equilibria is illustrated by examples in Section 6. Finally, a necessity result is introduced in Section 7, followed by discussion of related work in Section 8. All proofs of the results are provided in Appendix.

2 The Model

We just follow [2] to define a 2-person game. A game consists of a finite set of players \( N = \{1, 2\} \), each of whom, say player \( i \), can take actions from a finite set \( A_i \). Let \( A \) be the Cartesian product of all \( A_i \). The states of the world are represented by a subset \( \Omega \) in the Lebesgue square \([0, 1] \times [0, 1]\), together with a \( \sigma \)-algebra \( B \) of subsets of \( \Omega \). For each player \( i \), a sub-\( \sigma \)-algebra \( J_i \) of \( B \) consists of those events regarding which \( i \) is informed, moreover, for each player \( i \), a probability measure \( p_i \) defined on \( B \) (not only on \( J_i \)) is the subjective prior of \( i \).

For player \( i \), a pure strategy is a \( J_i \)-measurable map from \( \Omega \) to \( A_i \). Accordingly, the expected payoff for player \( i \) under a pure strategy profile \((s_1, s_2)\) is

\[
\mathbb{E}_{p_i} u_i(s_1, s_2) = \sum_{a_1 \in A_1, a_2 \in A_2} u_i(a_1, a_2) p_i(\{s_1 = a_1, s_2 = a_2\}).
\]

A pure strategy profile \((s_1, s_2)\) is called an equilibrium if for \( i = 1, 2 \), \( \mathbb{E}_{p_i}(s_i, s_j) \geq \mathbb{E}_{p_i}(s'_i, s_j) \), for all pure strategy \( s'_i \) of player \( i \).
For player $i$, an event $B \in \mathcal{J}_i$ is called an $i$-secret event if $B$ is independent with $\mathcal{J}_j$, i.e., for all $j \neq i$, and $C \in \mathcal{J}_j$, $p_j(B \cap C) = p_j(B) \times p_j(C)$. An event $B$ is $i$-secret means that the other player $j$ can get no hint about this event, even by pooling all the events $j$ has been informed. A pure strategy profile is called objective if $p_1(\{s_i = a_i\}) = p_2(\{s_i = a_i\})$ for all $a_i \in A_i$; otherwise it is a subjective pure strategy profile. A strategy profile $(s_1, s_2)$ is called independent if for any player $i$, the event $\{s_i = a_i\}$ is an $i$-secret event for all $a_i \in A_i$. In the sequel, we will concern more on the subjective independent pure-strategy equilibria and objective independent pure-strategy equilibria for the game model mentioned above.\(^\text{15}\)

A probability measure is atomless or nonatomic on a $\sigma$-algebra means that for any event in the $\sigma$-algebra which occurs with a positive probability, part of this event also belongs to the $\sigma$-algebra and it occurs with a strictly smaller positive probability.

We are now ready to present Assumption II in Aumann [2],

**Main Assumption** For each player $i$, there is a sub-$\sigma$-algebra of $i$-secret events such that it is independent with $\mathcal{J}_j$ and $p_j$ is atomless on it for $j \neq i$.

Roughly speaking, the above assumption simply means that there are as many $i$-secret events of as many different sizes with respect to player $j$’s subjective prior $p_j$ as one requires. With this assumption at hand, Aumann showed that the set of all objective independent pure-strategy equilibrium payoff vectors coincides with the set of all Nash mixed-strategy equilibrium payoff vectors for the original finite game with complete information; see Proposition 4.3 therein. As a result, for games with such a structure of information, there certainly exists an objective independent pure-strategy Nash equilibrium, since a mixed-strategy Nash equilibrium always exists for the original game with complete information.

Throughout the paper, we consider a special class of games with correlated information in which the states of the world are modeled by a subset of the square, $[0, 1] \times [0, 1]$, and on which a particular information structure is imposed. Given any state, now a two-dimensional vector in the square, player 1 only observes the first component of the state and player 2 the second one. Though it is a particular restrictive game-theoretic model, we shall see by examples that even in this small class of games, the Main Assumption can not hold for a general correlated information structure. As brought out in the introduction, the whole purpose of this paper is to present a theory sophisticated enough to establish the existence of an independent pure-strategy equilibrium, either subjective or objective, for this class of games.

\(^{15}\)Independent pure-strategy equilibria was termed as “mixed-strategy equilibrium” in the language of [2]; see also Footnote 7 above.
The following example serves as an illustration of the basic vernacular of the model.

**Example 1.** Consider the following Figure 1.

![Figure 1: Ω is hypograph of a nonzero constant function](image)

In this example, the states of the world Ω is modeled by the lower half of the Lebesgue square \([0, 1] \times [0, 1]\), and the associated σ-algebra \(\mathcal{B}\) is obtained by restricting the Lebesgue product σ-algebra to Ω. For any vector \(\omega \in \Omega\), player 1 is informed the first component of \(\omega\), and player 2 the second component. As a result, player 1’s information \(\mathcal{J}_1\) is generated by the sets of form \(B_1 \times [0, 1/2]\), where \(B_1\) is a Lebesgue subset in \([0, 1]\); similarly, player 2’s information \(\mathcal{J}_2\) is generated by the sets of form \([0, 1] \times B_2\), where \(B_2\) is a Lebesgue subset in \([0, 1/2]\). Let \(p = p_1 = p_2\) be the uniform distribution on Ω, i.e., the corresponding density function is 2 on Ω with respect to the Lebesgue measure on the square. It is clear that any event in \(\mathcal{J}_i\) is an \(i\)-secret event and \(p\) is atomless on these both σ-algebras. As a result, the Main Assumption is automatically satisfied and the all the results therein can be delivered.

Actually, the information structure above is independent in the following sense. Player 1’s type space is modeled by the Lebesgue unit interval \([0, 1]\) and Player 2’s type space is modeled by the half Lebesgue interval \([0, 1/2]\). For any type \(t_1 \in [0, 1]\) of player 1, he believes that player 2’s types are uniformly distributed on \([0, 1/2]\); and for any type \(t_2 \in [0, 1/2]\) of player 2, he believes that player 1’s types are uniformly distributed on \([0, 1]\).

**Example 2.** Consider the following Figure 2.

Divide the Lebesgue square evenly into four smaller squares along the two bisectors \(t_1 = 1/2\) and \(t_2 = 1/2\). The states of the world Ω is represented by the Lebesgue square by removing the quarter sub-square in the northwest, and the associated σ-algebra \(\mathcal{B}\) is obtained by restricting the Lebesgue product σ-algebra to Ω. As in Example 1, player 1 (or player 2) is informed of the first (or second) component of any vector in Ω. As a result, \(\mathcal{J}_1\) is generated by the sets of form \((B_1 \times [0, 1]) \cap \Omega\),
where $B_1$ is a Lebesgue subset in $[0, 1]$; similarly, player 2’s information $J_2$ is generated by the sets of form $([0, 1] \times B_2) \cap \Omega$, where $B_2$ is a Lebesgue subset in $[0, 1]$. Let $p_1 = p_2 = p$ be the uniform distribution on $\Omega$. That is, the corresponding density function for $p$ is the constant $4/3$ at all points in $\Omega$, with respect to the Lebesgue measure on the square.

Let $\mathcal{R}_1$ be a $\sigma$-algebra generated by all subsets of the form $E_s = ([0, s] \cup [1/2, 1/2 + s]) \times [0, 1]$ for all $s \in [0, 1/2]$. It is clear that for any such $s$, $E_s$ is independent of any event in $J_2$. Moreover, $p$ is atomless on $\mathcal{R}_1$. Similarly, there is a sub-algebra of $J_2$ which is independent of $J_1$ and when restricted to which $p$ is atomless. Therefore, the Main Assumption holds.

Similarly, it is worthwhile to point out that if the states of the world are modeled by the hypograph of any non-zero valued simple function $h$ from $[0, 1]$ to $(0, 1]$, i.e., $\{(t_1, t_2) \in [0, 1] \times [0, 1] | t_2 \leq h(t_1)\}$, the Main Assumption holds. Though such a setting is not one of independent information, (e.g., in Example 2), it can nevertheless be viewed as one of independence conditional on a finite set of public states in the sense of [30]. As a result of Theorem 5 therein, for any two player finite action game associated with this information structure, there still exists a pure-strategy Bayesian-Nash equilibrium.

### 3 Examples with Correlated Information

Next, we provide several examples to illustrate that the two requirements of the Main Assumption, independence and atomlessness, are too strong to hold for models even with a rather simple-form correlated information. This was after all the substantive thrust of the RR example and the results in [34].

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$^{16}$For example, let $F = [0, 1] \times [0, t], t < 1/2$, $p(E_s) = 2s, p(F) = \frac{4t}{3}$, while $p(E \cap F) = \frac{8st}{3}$; as a result $E_s$ is independent with $F$. 

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Example 3. (Example 1 in [34])

In this example, the states of the world $\Omega$ are represented by half of the Lebesgue square which lies below the 45 degree line, and the associated $\sigma$-algebra is the product of two copies of Lebesgue $\sigma$-algebra restricted on this triangle. As above, players 1 and 2 are informed of the first and second component respectively of any vector of the states of the world, and $\mathcal{J}_1$ is generated by the sets of form $(B_1 \times [0, 1]) \cap \Omega$, where $B_1$ is a Lebesgue subset in $[0, 1]$, and $\mathcal{J}_2$ is generated by the sets of form $([0, 1] \times B_2) \cap \Omega$, where $B_2$ is a Lebesgue subset in $[0, 1]$. Let $p_1 = p_2 = p$ be the uniform distribution on $\Omega$. That is, with respect to the Lebesgue measure on the square, the density function for $p$ is the constant 2 at all points in $\Omega$.

It is clear that there is no nontrivial $i$-secret event at all in this model, for $i = 1, 2$, it then results in a failure of the Main Assumption. For example, for any set of the form $E = ([0, 1] \times (a, b)) \cap \Omega$, which is in $\mathcal{J}_2$, there exists two different subsets $F_1 = ((c_1, d_1) \times [0, 1]) \cap \Omega$ and $F_2 = ((c_2, d_2) \times [0, 1]) \cap \Omega$ in $\mathcal{J}_1$ such that $d_1 - c_1 = d_2 - c_2$, and either (1) $p(E \cap F_1) = p(E \cap F_2)$, or (2) $p(E^c \cap F_1) = p(E^c \cap F_2)$, where $E^c$ is the complement of $E$ with respect to $\Omega$. This implies that $E$ can not be independent with $F_1$ and $F_2$ simultaneously, as a result, $E$ can never be an 2-secret event.

The information structure in this example is correlated in the following sense. Both players’ type spaces are modeled by the Lebesgue unit interval $[0, 1]$, for any type $t_1 \in [0, 1]$, player 1 thinks that player 2’s types are uniformly distributed in $[0, t_1]$; similarly, and for any type $t_2 \in [0, 1]$, player 2 believes that player 1’s types are uniformly distributed in $[t_2, 1]$.

In Example 1 of [34], Radner and Rosenthal consider the matching penny game in Table 1.

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\[17\] The information structure we present here is a slightly different from Example 1 of [34], where the states of the world is modeled by the upper triangle of the Lebesgue square above the 45 degree diagonal.

\[18\] This implies that $p(F_1) = d_1^2 - c_1^2 \neq d_2^2 - c_2^2 = p(F_2)$. 

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below associated with the above correlated information structure, where \( H \) represents Head, \( T \) means Tail. They characterize the equilibrium in this game must satisfy the condition that “for any player and given any value of his information variable, the conditional probabilities of the two moves of the other player should be equal,” then the nonexistence of a pure-strategy equilibrium in this simple game is resulted as a failure to exist a Lebesgue measurable subset in \([0, 1]\) which is independent with all open intervals. However, a formal proof of this equilibrium characterization result is lacked in [34]. We provide a formal proof in this paper, which is a special case of Lemma 2 in the Appendix.

\[
\begin{array}{c|cc}
\text{Player 2} & H & T \\
\hline
\text{Player 1} & H & 1, -1 & 0, 0 \\
& T & 0, 0 & 1, -1
\end{array}
\]

Table 1: Payoffs in a matching-penny game

**Example 4.** Unlike Examples 1-3, we next present an information structure with subjective priors. The information structure is in Figure 4. Divide the Lebesgue square into four quarters by the two diagonal lines. The states of the world \( \Omega \) contains two quarters below the 45 degree line, the lower quarter \( E_2 \) and the right one \( E_1 \). As before, player 1 is informed the first component of a state and 2 the second component for any vector in the states of the world. Let \( p_i \) be the uniform distribution on \( E_i \).

\[
\begin{align*}
E_2 &= \frac{1}{2} \\
E_1 &= \frac{1}{2} \\
\Omega &= E_1 \cup E_2, \ p_i^\varepsilon = (1 - \varepsilon)p_i + \varepsilon p_j \quad \text{where} \quad p_i \text{ is uniform on } E_i, i = 1, 2
\end{align*}
\]

Figure 4: \( \Omega = E_1 \cup E_2, \ p_i^\varepsilon = (1 - \varepsilon)p_i + \varepsilon p_j \) where \( p_i \) is uniform on \( E_i, i = 1, 2 \)

Let \( \varepsilon \in (0, 1/2] \). Let Player \( i \)'s subjective probability measure \( p_i^\varepsilon = (1 - \varepsilon)p_i + \varepsilon p_j \) for \( i = 1, 2 \). That means, player \( i \) is more likely to think that the states of the world is in the part \( E_i \) not the

\[\text{See the last paragraph in page 406 in [34].}\]
other part $E_j$. In particular, when $\varepsilon = 1/2$, this information structure is exactly the information structure in the previous example. When $\varepsilon > 0$, the same arguments in the previous example can be applied here to show that there is no non-trivial secret event for both players, as a result, the Main Assumption fails to hold.

However, in the extreme case that $\varepsilon = 0$, $p^0_i = p_i$ for both players. That is, player $i$ thinks that the states of the worlds occur exactly in $E_i$, while he thinks that it can not happen in $E_j$. In this correlated information structure, for $t \in [0, 1/2)$, let $F_2(t) = ([0, 1] \times [[0,t] \cup [1/2, 1-t]]) \cap \Omega$. It is clear that $p_2(F_2(t)) = 4t(1-t)$, $p_1(F_2(t)) = 1/2$, as a result, if $t = \frac{2-\sqrt{2}}{4}$, $F_2(t)$ is an objective event with probability $\frac{1}{2}$. It is an easy exercise to check that $F_2(t)$ is a nontrivial 2-secret event.

Furthermore, a nontrivial 2-secret event should be of size $1/2$ with respect to $p_1$ and of the form as $[0, 1] \times [B_1 \cup ([1/2, 1] \setminus (1/2 + B_1))]$, where $B_1$ is a Lebesgue subset in $[0, 1/2]$ and $1/2 + B_1 = \{1/2 + x : x \in B_1\}$. As a result, the largest possible sub-$\sigma$-algebra of 2-secret events which is independent with $J_1$ is $\{\emptyset, \Omega, F, F^c\}$, where $F$ is of the above form. As a result, $p^0_i = p_1$ cannot be atomless on these $\sigma$-algebras of 2-secret events, the atomlessness condition in the Main Assumption does not hold.

Now consider the matching penny game in Table 1 associated with this information structure. When $\varepsilon > 0$, there does not exist a subjective or objective independent pure-strategy equilibrium, since there is no nontrivial secret events for both players. However, when the subjective prior in the extreme case that $\varepsilon = 0$, there do exist independent pure-strategy equilibrium for this game. Consider the following subjective independent pure-strategy profile: both players take $H$ if their own signal is less than one half, $T$ otherwise. It is easy to check that it is a subjective independent pure-strategy equilibrium, and the resulted equilibrium payoff vector is $(0.5, -0.5)$ where $0.5$ is the value of the matching penny game. Moreover, player 2 taking the same strategy as above, and player 1 choosing $T$ for any signal is another subjective independent pure-strategy equilibrium; and the resulted subjective independent equilibrium payoff vector is $(0.5, 0)$. Similarly, there is another subjective independent equilibrium payoff vector $(1, -0.5)$. It is worthwhile to note that $(0.5, -0.5)$ is also an independent objective independent equilibrium payoff.

We now conclude this section. Examples 3–4 above illustrate the difficulty to obtain nontrivial secret events for very simple correlated information structures. They attest to the fact that the requirements in the Main Assumption are too strong to hold even for such simple information structures. As a result, even for such particular information structures, it is impossible to implement Aumann’s Proposition 4.3, to obtain a pure-strategy equilibrium in rather simple two-person games.
4 A Lebesgue Extension

The principal message of this paper is that these examples can be resolved, and Aumann’s Main Assumption vindicated and implemented, if each player’s space of information is upgraded from the Lebesgue interval to the extended Lebesgue interval as formulated in [22]. In this section, we attempt to lay out for the general reader the basic intuitions underlying the construction of this extension rather than simply using it as a black-box that furnishes a pure-strategy equilibrium in a class of games that do not possess such an equilibrium. To put the point another way, the principles underlying the extension go beyond the technical to the substantive considerations that Aumann emphasis in the formulation of secrecy and the atomlessness of beliefs on the class of secret events.

We first start from the Lebesgue (unit) interval \((L = [0, 1], \mathcal{L}, \lambda)\). The Lebesgue \(\sigma\)-algebra \(\mathcal{L}\) is generated by all the open subsets in \([0, 1]\), and all the sets with length 0; and the Lebesgue measure of any open interval is exactly the length of this interval. We next review the construction of an extension, denoted by \((I = [0, 1], \mathcal{L}^e, \lambda^e)\), of the Lebesgue interval in the sense that \(\mathcal{L}\) is a sub-\(\sigma\)-algebra in \(\mathcal{L}^e\), and when restricted to any Lebesgue measurable subset \(E\), the measure under \(\lambda^e\) is the Lebesgue measure of \(E\).

The following result of Kakutani [17] is crucial for the next stage of the construction.

**Kakutani’s Lemma:** There exists a partition of uncountable cardinality of \(L = [0, 1]\), denoted by \(\{C_k : k \in K = [0, 1]\}\), such that the Lebesgue outer-measure\(^{20}\) of \(C_k\) is one for all \(k \in K = [0, 1]\).

Here each subset \(C_k\) has Lebesgue outer measure one is crucial, roughly it means that the points in this set is rather dispersed in the Lebesgue interval. Since the index set \(K = [0, 1]\), then it can be also endowed with a Lebesgue structure as described above. We can now lift the Kakutani partition and consider the set

\[
C = \bigcup_{k \in [0, 1]} C_k \times \{k\} \subseteq L \times K.
\]

Note that \(C\) is a subset of the square.

Note that \(C\) is a subset in the Lebesgue square, the product of two Lebesgue intervals. Since each horizontal section of \(C\) has Lebesgue outer measure one, it implies that \(C\) itself has outer measure one in the sense of Lebesgue product measure on the square. One can appeal to a standard

\(^{20}\)Given a measure space \((T, \mathcal{T}, \mu)\), the associated outer measure, denoted by \(\mu^*\), is defined as follows: for any subset \(E \subseteq T\), \(\mu^*(E) = \inf \{\sum_n \mu(E_n) : E_n \in \mathcal{T}, E \subseteq \bigcup_n E_n\}\), it bears emphasis that the infimum is taken over all countable covers of \(E\).
procedure in measure theory to obtain an extension of the Lebesgue square such that in which $C$ is measurable and of measure one; then restricted this extension of the Lebesgue square to $C$, one can obtain a probability structure $(C, T, \gamma)$.\(^{21}\) Moreover, for any set of the form $((l, l') \times (k, k')) \cap C$, its measure with respect to $\gamma$ inherits that of the half-open interval $(l, l') \times (k, k')$ in the product. In particular, for all $0 \leq l < l' \leq 1$ and $0 \leq k < k' \leq 1$, 

$$\gamma \left[ ((l, l') \times (k, k')) \cap C \right] = (l' - l)(k' - k).$$

All that remains is to take this extended probability structure in the square to the unit interval we began with.

It is a consequence of the Kakutani partition that the projection $p$ from $C$ to $I$ is a one-to-one mapping, and can therefore be used to induce a probability structure on $I$ from the probability structure on $C$. Denote the new probability structure on $[0, 1]$ by $([0, 1], \mathcal{L}^e, \lambda^e)$, and this is the extension of the Lebesgue unit interval that we seek. It is now worthwhile to summarize the procedure. Each type has a double identity: an explicit identity or trait (say, e.g., the location) indexed by elements of $L$ and another implicit identity or trait (say, e.g., the wealth level) indexed by elements of $K$, and the two traits co-exist in single-dimensional set $I$.

The point of consequence is that these two traits are governed by two independent $\sigma$-algebras, and the extended Lebesgue measure is atomless on both. Next, we turn to this. For all $0 \leq k < k' \leq 1$, let $D_{kk'} = \bigcup_{k < k'' < k'} C_{k''}$, which is the set of all implicit traits lying between $k$ and $k'$. Notice that $p^{-1}(D_{kk'}) = ([0, 1] \times [k, k')) \cap C$, and by virtue of the way that the extended $\sigma$-algebra $\mathcal{L}^e$ was obtained on $I$, $D_{kk'} \in \mathcal{L}^e$. Furthermore, by virtue of the way that the extended Lebesgue measure was obtained on $\mathcal{L}^e$, we have 

$$\lambda^e(D_{kk'}) = \gamma \left[ p^{-1}(D_{kk'}) \right] = \gamma \left[ ([0, 1] \times (k, k')) \cap C \right] = k' - k.$$

That is, the probability of a type whose implicit trait lies between $k$ and $k'$ is exactly $k' - k$. Let $\mathcal{R}$ be the $\sigma$-algebra generated by all $D_{kk'}$ for all $0 \leq k < k' \leq 1$. It is clear that $\mathcal{R}$ is characterized by all the Borel sets in $K$. Moreover, note that $\lambda^e$ is atomless on $\mathcal{R}$.

Next, we claim that $\mathcal{R}$ and $\mathcal{L}$ are independent. Fix $0 \leq k < k' \leq 1$ and $0 \leq l < l' \leq 1$, consider the probability of types whose implicit trait lies between $k$ and $k'$ and whose explicit trait between $l$ and $l'$. Independence of the two traits simply means that the probability of the types that lie in the intersection of the two sets, which is to say that it simultaneously lies between the two levels

\(^{21}\)For more details, see Step 3 in Section 2.2 of [22].
be the product of the probability that the type lies in each of the sets. But this clear on account of the fact that 

\[ p^{-1}(D_{kk'} \cap (l,l')) = ((l,l') \times (k,k')) \cap C, \]

and thus

\[ \lambda^c(D_{kk'} \cap (l,l')) = \gamma[p^{-1}(D_{kk'} \cap (l,l'))] = \gamma[((l,l') \times (k,k')) \cap C] = (l' - l)(k' - k). \tag{1} \]

This means that the probability of a type with an implicit trait between \( k \) and \( k' \) in any neighborhood of the form \((l,l')\) is always \((l' - l)(k' - k)\). Notice that the Borel \( \sigma \)-algebra \( B \) is generated by all neighborhoods of the form \((l,l')\), \( 0 \leq l < l' \leq 1 \), as a result, \( D_{kk'} \) is independent of the the Borel \( \sigma \)-algebra, hence the Lebesgue \( \sigma \)-algebra \( L \). Since \( k, k' \) are also arbitrary, it follows that \( R \) and \( L \) are independent \( \sigma \)-algebras.

We now summarize the above as below,

**Lemma 1.** There is a sub \( \sigma \)-algebra \( R \) in \( L^c \) such that it is independent of the Lebesgue \( \sigma \)-algebra \( L \) and on which the probability measure \( \lambda^c \) is atomless.

We now invite the reader to return to Aumann’s Main Assumption and read it in conjunction with the above result. It furnishes precisely what Aumann wants to assume, and thereby implements his program. Where the single identity of the Lebesgue interval fails, the double identity of the extended Lebesgue interval succeeds. It is this success that we turn to next.

## 5 A Result based on Lebesgue Extension

In this section, we use the Lebesgue extension to model a player’s information, and relying on the fact that it is copious enough to accommodate secrecy and atomlessness, present a result that implements the Main Assumption in the setting of what we call regular information. This condition is new to the best of our knowledge, and we turn to it.

**Definition 1.** Take as given two probability spaces \((T_i, T_i, \tau_i), i = 1, 2\) and their usual product space \((T_1 \times T_2, T_1 \otimes T_2, \tau_1 \otimes \tau_2)\). A nonnegligible subset \( E \in T_1 \otimes T_2 \) is called regular if for \( \tau_i \)-almost all \( t_i \in T_i \), the \( t_i \)-section of \( E \) is either the empty set or of positive \( \tau_j \)-measure.

In Examples 1–4, the states of the world are all modeled by regular subsets in the Lebesgue square. It is clear that the hypograph of a non-zero constant function \( h \equiv c \) for \( 1 \geq c > 0 \), i.e., \( \{(x,y) : 0 \leq x, y \leq 1, y \leq c\} \), is a regular subset in the Lebesgue square. More generally, the hypograph of a continuous function \( h : [0, 1] \to [0, 1] \) is regular in the Lebesgue square if and only if that the set of roots of \( h \), \( \{h = 0\} \) is a Lebesgue null set. For example, the lower triangle below the 45 degree line in Example 3 is regular. Though a nonnegligible measurable subset in the Lebesgue square
square may not be regular, it can become regular after some “physical surgery” by removing a Lebesgue null-set from the subset itself, see the right picture in Figure 5.

![Figure 5](image_url)

Figure 5: The left figure is regular in the Lebesgue square, while the right is not. And the right figure could become regular after a “physical surgery” by cutting off the line segment from the origin (0, 0) to (2/3, 0).

We are now ready to present our first main result.

**Theorem 1.** Let $\Omega$ be a regular Lebesgue measurable subset in the extended Lebesgue square $([0,1] \times [0,1], \mathcal{L}^e \otimes \mathcal{L}^e, \lambda^e \otimes \lambda^e)$, and $p_1 = p_2 = p$ the restricted probability measure $\lambda^e \otimes \lambda^e$ on $\Omega$. Then for every two-person game with this information structure, the set of independent pure-strategy equilibrium payoffs coincides with the set of mixed-strategy Nash equilibrium payoffs in the original two-person game with complete information.

A straightforward corollary of Theorem 1 is that there exists a pure-strategy equilibrium in such a correlated information game since there exists a mixed-strategy Nash equilibrium in the original finite game with complete information. The statement in this theorem is exactly that in Proposition 4.3 of [2], and the main idea of this result is to show that the Main Assumption can still hold if the states of the world is modeled by any Lebesgue regular subset in the extended Lebesgue square, even though the information structure is allowed to be correlated. This provides a sharp contrast with the phenomena in Examples 3–4 where the states of the world is modeled by the same regular subsets in the Lebesgue square, and it might be even difficult to obtain a nontrivial secret event. In this regard, the Lebesgue extension $([0,1], \mathcal{L}^e, \lambda^e)$ provides an analytical framework to study problems on pure-strategy equilibria in such games.

**Remark 1.** Though there is no subjective priors in Theorem 1, this result can be generalized if certain subjective priors are assumed. For example, if $p_2$ can be dis-integrated in a way such that (1)
the marginal distribution on player 2’s types is absolutely continuous w.r.t. \( \lambda^e \), and (2) conditional on any \( t_2 \) which is meaningful for player 2, i.e., \( t_2 \) is in the projection of \( \Omega \) to player 2’s types, player 2 believes that player 1’s type obeys a distribution which is absolutely continuous w.r.t. the restricted distribution of \( \lambda^e \) on the \( t_2 \)-section of \( \Omega \), denote it by \( \Omega_{t_2} = \{ t_1 : (t_2, t_1) \in \Omega \} \), and the corresponding Radon-Nikodym derivative function is Lebesgue measurable. Similar requirements can be imposed on \( p_1 \). In this case, it is clear that \( p_1 \) is not necessarily to be identical to \( p_2 \). A consequence, under this subjective information structure, the statements of Theorem 1 still hold.

For example, if the states of the world is the same as that in Example 4, and it is instead put in the extended Lebesgue square, this subjective correlated information structure satisfies the conditions mentioned above.

The following result is straightforward from Theorem 1.

**Corollary 1.** Let \( h : [0, 1] \to [0, 1] \) be a Lebesgue measurable function such that the hypograph of \( h \), \( \Gamma_h = \{ (x, y) : 0 \leq x \leq 1, 0 \leq y \leq h(x) \} \), is a regular set in \( \mathcal{L} \otimes \mathcal{L} \). For any two person game where the correlated information is modeled by \( \Gamma_h \) in the extended Lebesgue square, the conclusion in Theorem 1 holds.

We next revisit Example 3, and see how the negative result in this example can be resolved by working with the extended Lebesgue square.

**Example 5** (Example 3 revisited, in the extended Lebesgue square). For the matching penny game in Example 3, we assume now that the states of the world is now in a new world, the extended Lebesgue square \( [0, 1] \times [0, 1], \mathcal{L}^e \otimes \mathcal{L}^e, \lambda^e \otimes \lambda^e \). According to Lemma 1, let \( R = D_{0,t_2} \), which is an event in \( \mathcal{R} \) with measure \( \lambda^e(R) = \frac{1}{2} \), and it is independent with the Lebesgue \( \sigma \)-algebra \( \mathcal{L} \). Now consider the following strategy profile: for player 2, if \( t_1 \in R \), he plays \( H \), \( T \) otherwise; for player 1, at \( t_1 = 0 \), he takes the same action as player 1’s action when \( t_1 = 0 \), if \( t_2 \neq 0 \), he takes \( H \) if \( t_2 \in R \), otherwise \( T \).

It is easy to check that this strategy profile is a pure-strategy Bayesian-Nash equilibrium for this game. In fact, given player 1’s strategy, for each type \( t_2 \in [0, 1) \) of player 2, his expectation for player 1 playing \( H \) is \( \lambda^e(R \cap [t_2, 1]) / \lambda^e([t_2, 1]) = \lambda^e(R) = 1/2 \) where the first equation follows from the independence between \( R \) and \( [t_1, 1] \), similarly, player 2’s expectation for his opponent playing \( T \) is also one half; thus, both \( H \) and \( T \) are player 2’s best response at \( t_2 \). Analogously, one can argue that given player 2’s strategy, for any \( t_1 > 0 \), player both \( H \) and \( T \) are best responses since \( \lambda^e([0, t_1] \cap R) / \lambda^e([0, t_1]) = \frac{1}{2} \) for all \( t_1 > 0 \). At \( t_1 = 0 \), player 1 knows the type of player 2 is \( t_2 = 0 \).
for sure, as a result, his best response is to take the same action of player 2’s at \( t_2 = 0 \). Hence player 1’s strategy is optimal given player 2’s.

In the underlying Bayesian game, each player’s type space is modeled now by the measurable space \( ([0,1], \mathcal{L}^e) \). For each type \( t_i \in [0,1] \), \( t_i \) has two identities, one is the location \( t_i \), and the other is the corresponding wealth level of this location. Notice that in the above equilibrium profile, for every player at his own type, the action only depends on whether this type is in \( R \) or not, in other words, whether the associated wealth level of this type satisfies certain condition or not. Put differently, the location identity of the type itself is totally ignored in such an equilibrium strategy profile. Notice that the correlation of the types only has bites on the location identities for the type profiles, as a result, it is no wonder that the difficulty stemmed from the correlation on locations can be bypassed by taking actions pegged on the wealth identity, which is independent with the location.

**Remark 2.** In the pure-strategy Bayesian Nash equilibrium in Example 5, almost surely, both players take \( H \) when their type is in \( R \) and \( T \) if their type is not in \( R \). However, by the construction of the Lebesgue extension in Section 4, the set \( R = D_0 \) is a set such that it contains no subinterval but it has non-empty intersection with any interval. This means that both players take actions \( H \) and \( T \) alternatively in a very frenzy way. Indeed, this is exactly an idealization of the approximated pure-strategy Bayesian-Nash equilibrium obtained Section 3 in [3], where \([0,1]\) is equally divided into a large even number of subintervals, and each player takes \( H \) and \( T \) in each sub-interval alternatively.

**Example 6** (Example 3 revisited again, in the one-sided extended Lebesgue square). One may curious that whether the negative result in Example 3 can be resolved by modeling the states of the world in a “smaller world” than the extended Lebesgue square as in Example 5. In particular, how about if only one player’s type space is modeled by the extended Lebesgue interval and the other’s remains the Lebesgue interval. We provide a negative answer for this question.

Assume that player 1’s type space is the Lebesgue unit interval \( ([0,1], \mathcal{L}, \lambda) \), and player 2’s is the extended Lebesgue interval \( ([0,1], \mathcal{L}^e, \lambda^e) \), and the states of the world is still the lower triangle below the 45 degree line. Suppose there exists a pure-strategy Bayesian-Nash equilibrium in this underlying matching penny Bayesian game, we claim that this equilibrium must be also a pure-strategy Bayesian-Nash equilibrium in the matching penny game in Example 3. Thus a contradiction can be obtained. In particular, we can show that:
Claim 1. Given a pure-strategy of player 1, which is a Lebesgue measurable function from \([0, 1]\) to \(\{H, T\}\), then the best response for player 2, against player 1’s strategy, which is one to one for \(\lambda^e\)-almost all \(t_2 \in [0, 1]\), is also Lebesgue measurable.

Remark 3 (On the regularity condition). One may ask whether the “regularity” condition can be dispensed in the statement of Theorem 1. This is clearly not possible. Consider the matching penny game in Table 1 associated with the states of the world being modeled by the right picture in Figure 5 which is irregular. It is clear that there exists no pure-strategy Bayesian-Nash equilibrium in this correlated information game because at any type \(t_1 \in [0, 2/3]\), player 1’s best response is to chose the same action as the action of player 2 at \(t_2 = 0\), however, at \(t_2 = 0\), player 2 will avoid choosing the same action as player 1 at \(t_1 \in [0, 2/3]\).

6 Subjectivity versus Objectivity

One advantage to model the correlated information structure as a regular subset in the extended Lebesgue interval is that we could calculate all the possible subjective or objective independent pure-strategy Nash equilibrium for a given correlated information game. Aumann’s Proposition 5.1 shows that in 2-person games, unnecessarily to be zero-sum, under certain mild condition on players’ subjective priors, the set of subjective independent equilibrium payoffs coincides with the set of objective independent equilibrium payoffs, and as a result of Proposition 4.3 therein, coincides with the set of mixed-strategy equilibrium payoffs of the underlying game with complete information. The mild condition, i.e., (5.2) therein, require that it is not the case that one player assigns zero probability to an event on which another player assigns positive probability. For example, in the subjective prior structure in Example 4, this condition is fulfilled but it is not in the extreme case that \(\epsilon = 0\). In this section, we provide two examples, one is a zero-sum game and the other is a non-zero sum, to illustrate the relationship between subjective and objective independent equilibrium payoffs if this condition is dispensed.

Example 7. Consider the same matching penny game in Example 4, the subjective priors are in an extreme case that \(\epsilon = 0\), but the states of the world is instead in the extended Lebesgue square. The Lebesgue extension ensures us the existence of secret event of any size, not merely of size \(1/2\), such that each player’s subjective measure is atomless when restricted on a \(\sigma\)-algebra of secret events of the other player. In particular, the Main Assumption is valid in this information structure. It follows from Theorem 1 and Remark 1 that there exist an objective independent pure-strategy equilibrium and all such objective independent equilibria results in the same expected payoff vector.
(0.5, −0.5), where 0.5 is the value of this zero-sum game, since the matching penny game with complete information has a unique mixed-strategy equilibrium.

The following claim characterizes all the pure-strategy equilibrium payoffs in this game if the states of the world is modeled in the extended Lebesgue square.

**Claim 2.** In this example, all the subjective independent equilibrium payoffs are of the following form, \{(0.5, −a), (1 − b, −0.5) : a, b ∈ [0, 1/2]\}; see the following Figure 6.

![Figure 6: Independent pure-strategy equilibrium (expected) payoffs of the matching penny game if the states of the world is modeled in the extended square.](image)

There is a sharp difference between the set of independent pure-strategy equilibrium payoffs in Example 4 where the states of the world is a subset in the Lebesgue square and this example. In Example 4, (0.5, 0), (1, −0.5) and (0.5, −0.5) are all the possible subjective independent equilibrium payoff vectors, and (0.5, −0.5) is also an objective independent equilibrium payoff. In this example, (0.5, 0.5) is still the unique objective pure-strategy equilibrium payoff, and much more subjective independent pure-strategy equilibrium payoffs are yielded other than \{(0.5, 0), (1, −0.5), (0.5, −0.5)\}. Particularly, every point in the set \{(0.5, −a), (1 − b, −0.5) : a, b ∈ [0, 1/2]\} can be yielded by a subjective pure-strategy equilibrium in this example.

**Example 8.** Next, we see the impact of the lack of the above mild condition in the context of non-zero sum games. First, consider the following 2 × 2 game as in Table 2. The information structure is the same as in the previous example.

In this game, there is a unique equilibrium in which player 1 takes \textit{H} with probability 2/3, \textit{T} with 1/3 and player 2 takes \textit{H} with probability 1/3, and \textit{T} with 2/3. As a result, the unique Nash equilibrium payoff vector is \((4/3, −2/3)\).

Second, let the above 2 × 2 game be associated with the information structure as in Example 4, the subset in the Lebesgue square. There is no independent objective pure-strategy equilibrium
Table 2: Payoffs in a non zero-sum game.

<table>
<thead>
<tr>
<th></th>
<th>Player 1</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>H</td>
<td>T</td>
</tr>
<tr>
<td>H</td>
<td>4, -1</td>
<td>0, 0</td>
</tr>
<tr>
<td>T</td>
<td>0, 0</td>
<td>2, -2</td>
</tr>
</tbody>
</table>

because there is no objective secret event with measure 1/3 for each individual player. However, there are three subjective independent pure-strategy equilibria, as follows,

<table>
<thead>
<tr>
<th>1’s strategy</th>
<th>2’s strategy</th>
<th>Exp. equi. payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>= T for ( t_1 &lt; 1/2 ), = H otherwise</td>
<td>= T for ( t_2 &gt; 1/2 ), = H otherwise</td>
<td>((2, -1/2))</td>
</tr>
<tr>
<td>= H for all ( t_1 )</td>
<td>= H for ( t_2 &gt; 1/2 ), = T otherwise</td>
<td>((2, 0))</td>
</tr>
<tr>
<td>= T for ( t_1 &lt; 1/2 ), = H otherwise</td>
<td>= H for all ( t_2 )</td>
<td>((4, -1/2))</td>
</tr>
</tbody>
</table>

Table 3: Subjective equilibria and the equilibrium payoff when the information is in Figure 4

Finally, we consider the subjective independent pure-strategy equilibria when this game is associated with the extended Lebesgue square as in Example 7. Here the Main Assumption is valid now, and it follows from Theorem 1 that there is an objective independent pure-strategy Nash equilibrium, furthermore, every independent objective pure-strategy equilibrium yields the same equilibrium payoff vector \((4/3, -2/3)\).

**Claim 3.** All the independent subjective equilibrium payoff vectors in this example are given in Figure 7.

Figure 7: \((2, 0), (2, -0.5), (4, -0.5)\) are all the independent subjective pure-strategy equilibrium payoff vectors when the information is in the Lebesgue square. The independent objective equilibrium payoff vector \((\frac{4}{3}, \frac{-2}{3})\) and all other points in the line or in the shaded area are subjective independent equilibrium payoffs resulted from the extended square.
It may be worthwhile to summarize the findings in the above two examples. When the mild condition on subjective priors is relaxed, there is a sharp contrast between independent subjective and objective equilibrium payoff structure. In particular, independent subjective equilibria could deliver many more equilibrium payoffs than objective equilibria. Meanwhile, the richer the states of the world being modeled in the square, the more independent pure-strategy equilibrium points could be achieved. This is because the richer the subjective information structure, the more space it allows to accommodate secrecy and secret events for an individual players.

Finally, since the Lebesgue extension is helpful to obtain the set of all independent subjective (objective) equilibrium payoff vectors, given any fixed correlated information structure. It is of special interest to check whether this equilibrium payoff structure possesses certain upper semi-continuity in terms of the change of subjective priors. Examples provide a negative answer to this question. When $\epsilon > 0$, as a result of Propositions 4.3 and 5.2 of [2], in both examples, the set of subjective independent pure-strategy equilibrium payoffs is a singleton set, i.e., $\{(1/2, -1/2)\}$ and $\{(-2/3, 4/3)\}$ respectively, however, in the extreme case $\epsilon = 0$, the set of all subjective independent pure-strategy equilibrium payoffs has higher dimension, i.e., of dimension 1 and 2 respectively. Thus as $\epsilon \to 0$, the correspondence of subjective independent pure-strategy equilibrium payoffs is lack of upper semi-continuity.

7 Generalizations beyond the Toy-Model

7.1 A Richer Information Structure

We first introduce the idea of atomless independent supplement, this idea can be traced back to [26] and [15], and summarized as the current name in [14].

**Definition 2.** Given an atomless probability space $(\mathcal{T}, \mathcal{T}, \mu)$ and a sub-$\sigma$-algebra $\mathcal{H}$ of $\mathcal{T}$, $\mathcal{H}$ admits an atomless independent supplement in $\mathcal{T}$ if there exists a sub-$\sigma$-algebra $\mathcal{S}$ of $\mathcal{T}$ such that (1) the probability measure $\mu$ is atomless on the sub-$\sigma$-algebra $\mathcal{S}$, and (2) the two sub-$\sigma$-algebras $\mathcal{H}$ and $\mathcal{S}$ are independent.

Lemma 1 exactly illustrates the idea of atomless independent supplement. In particular, the Lebesgue $\sigma$-algebra $\mathcal{L}$ which is generated by the location or the identity map on the interval, admits an atomless independent supplement in the Lebesgue extension $([0,1], \mathcal{L}^\epsilon, \lambda^\epsilon)$. In particular, the sub-$\sigma$-algebra $\mathcal{R}$ generated by the wealth mapping is an atomless independent supplement of $\mathcal{L}$.

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22See Lemma 4.4 (iv) in [15], and see more equivalent properties in [14].
Assume that player 2’s type space is the Lebesgue extension \( T \); suppose that \( T \) admits an atomless independent supplement in \( T \). Let \( \Omega \subseteq T \times T_2 \) be a regular subset in \( T \times T_2 \). Let \( p_1 = p_2 = p \) be the restricted probability measure \( \mu_1 \otimes \mu_2 \) on \( \Omega \). Then the results of Theorem 1 hold.

**Definition 3.** Let \( X \) and \( Y \) be two complete metric spaces, \( \nu \) is a Borel probability measure on \( X \times Y \). A probability space \( (T, \mathcal{T}, \mu) \) has the saturation property for \( \nu \) if for any random variable \( f : T \to X \) which induces the marginal distribution of \( \nu \) on \( X \), there is a random variable \( g : T \to Y \) such that \( (f, g) \) induces \( \nu \). A saturated probability space is then defined to be a space with the saturation property for any Borel probability measure on the product of any two complete metric spaces.

**Corollary 2.** Assume that player 2’s type space is the Lebesgue extension \([0, 1], \mathcal{L}^e, \lambda^e\) and player 1’s is a saturated probability space \((T, \mathcal{T}, \mu)\). Assume that \( h \) is a measurable function from \((T, \mathcal{T}, \mu)\) to the Lebesgue unit interval, and \( \Gamma_h \subseteq T \times [0, 1] \) a regular hypograph in \((T \times [0, 1], \mathcal{T} \otimes \mathcal{L}, \mu \otimes \lambda)\). Let \( \Gamma_h \) be the model of the states of the world, and \( p_1 = p_2 = p \) the restricted probability measure \( \mu \otimes \lambda^e \) on \( \Gamma_h \). Then the results of Theorem 1 hold.

Given an atomless probability space \((T, \mathcal{T}, \mu)\). Let \( h \) be a \( \mathcal{T} \)-measurable function from \( T \) to the Lebesgue interval \([0, 1], \mathcal{L}, \lambda\). This function \( h \) is said to be measure-preserving if the induced distribution of \( h \) on the interval \([0, 1]\) is the Lebesgue measure, i.e., for any Lebesgue subset \( B \in \mathcal{B} \), \( \mu(h^{-1}(B)) = \lambda(B) \). It is clear that for any atomless probability space \((T, \mathcal{T}, \mu)\), there exists a measure-preserving function \( h : T \to [0, 1] \). Moreover, the hypograph of \( h \), \( \Gamma_h = \{(t, s) \in T \times [0, 1] : 0 \leq s \leq h(s)\} \), is a regular subset in \( T \times [0, 1] \). In fact, since \( h \) is measure-preserving, \( \mu(h^{-1}(\{0\})) = \lambda(\{0\}) = 0 \). This means that for \( \mu \)-almost all \( t \in T \), the \( t \)-section of \( \Gamma_h \) is exactly the interval \([0, h(t)]\), which is of positive measure \( h(t) \). Moreover, for any \( \ell \in [0, 1] \), the \( \ell \)-section of the hypograph is \( \{t \in T : h(t) \geq \ell\} = h^{-1}([\ell, 1]) \). It follows from the measure-preserving property that \( \mu(h^{-1}([\ell, 1])) = \lambda([\ell, 1]) = 1 - \ell \).

It follows from Corollary 2 that for any matching penny game as in Table 1 associated with the states of the world \( \Gamma_h \), where \( h \) is a \( \mathcal{T} \)-measurable, measure-preserving function from \( T \) to the Lebesgue interval \([0, 1], \mathcal{L}, \lambda\), there exists an independent objective pure-strategy Nash equilibrium in this game. We now have the following necessity result for the type space being modeled by a saturated probability space.

**Theorem 2.** Assume that player 2’s type space is the Lebesgue extension \([0, 1], \mathcal{L}^e, \lambda^e\) and player 1’s is an atomless probability space \((T, \mathcal{T}, \mu)\). If the matching penny game (see Table 1) associated with
the information structure $\Gamma_h$ has a Bayesian-Nash equilibrium for every measure-preserving map $h$ from $(T, T, \mu)$ to the Lebesgue unit interval, then $(T, T, \mu)$ must be a saturated probability space.

This result illustrates that the matching penny game as in Table 1 associated with the states of the world being modeled by the hypographs of all $T$-measurable, measure-preserving function from $T$ to the Lebesgue interval $([0, 1], \mathcal{L}, \lambda)$, which is a smaller class of correlated information games studied in this paper, serve as a diagnostic tool for the existence of pure-strategy Nash equilibrium (PSNE) in more bigger class of Bayesian games, two-person games with any finite actions associated with a regular hypograph of any $T$-measurable function from $T$ to the Lebesgue interval $([0, 1], \mathcal{L}, \lambda)$. In particular, if every game in the smaller class has a pure-strategy Nash equilibrium, then player 1’s type space must be modeled by a saturated space, according to Corollary 2, every correlated information game in the bigger class also has a pure-strategy Nash equilibrium. Similar results has been achieved in Bayesian games with independent types but with uncountable infinite actions spaces in [23].

7.2 The $n$-Player Case

The main results, Theorem 1 and Proposition 1 can be extended to a setting with more than two players. First, the idea of regular subsets in Definition 1 can be generalized as

**Definition 1’.** Given $n$ probability spaces $(T_i, T_i, \tau_i), i = 1, \cdots, n$ and their product space $(T, T, \tau)$, where $T = T_1 \times \cdots \times T_n$, $T = T_1 \otimes \cdots \otimes T_n$, $\tau = \tau_1 \otimes \cdots \otimes \tau_n$. A nonnegligible subset $E \in T$ is called regular if for $\tau_i$-almost all $t_i \in T_i$, the $t_i$-section of $E$ is either the empty set or of positive $\tau_{-i}$-measure.

**Proposition 2.** For $i = 1, 2, \cdots, n$, let $(T_i, T_i, \mu_i)$ be an atomless probability space and $\mathcal{H}_i$ a sub-$\sigma$-algebra of $T_i$; suppose that $\mathcal{H}_i$ admits an atomless independent probability measure in $T_i$. Let $\Omega \subseteq T$ be a regular subset in $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. Let $p_1 = p_2 = \cdots = p_n = p$ be the restricted probability measure $\mu_1 \otimes \cdots \otimes \mu_n$ on $\Omega$. Then the results of Theorem 1 hold.

8 Discussion

(a) **On purification.** In the literature on the existence of pure-strategy equilibria for Bayesian games with independent information, the existence of pure-strategy equilibria has been established by purifying a (measure valued) mixed-strategy equilibrium. For finite-action Bayesian games with private information with certain independence on the distribution of types, the existence of pure
equilibria was obtained in [34] and [30]. As pointed out in [19], all these existence results can be delivered by applying a general purification principle established by [5, 6], which is a generalization of the well-known Lyapunov theorem on the range of the vector measures. However, for such private independent information games with uncountable actions, if the information is modeled by the Lebesgue square, there may not exist a pure-strategy Nash equilibrium; see [18] for a counterexample. This negative result is resolved by modeling the type spaces by atomless Loeb probability spaces in [18], then by general saturated probability spaces in [23]. As far as the counterexample in [18] is concerned, [23] also find that the negative results can be subdued by modeling the information space by the extended Lebesgue square. In [13], the authors show that there exists a pure-strategy equilibrium in games with independent private information and with uncountable action spaces, provided that a condition is fulfilled that the prior on the private information is “relatively diffused” compared with the information relevant to players’ payoffs. Actually, the property of “atomless independent supplement” has been proved to be equivalent to the “setwise coarser” condition to establish the “relative diffuseness” in [13], see Proposition 1 in [14].

The independence assumption on the distribution of types in the work mentioned above is crucial to implement the purification principal or to apply the Lyapunov theorem, and the RR example illustrates that such an independence assumption is indispensable for this approach. However, Aumann’s results (especially Proposition 4.3 in [2]) shed some light on applying the purification principle to establish the existence of pure strategy equilibrium for games with correlated information, provided an analogue of the independence assumption, the Main Assumption, is fulfilled, though this assumption can not be satisfied in many examples as illustrated by the RR example, see also other examples in Section 3. The heart of this Main Assumption is the co-existence of secret private information among players and the subjective priors are atomless on such secret private information. We in this paper find that the idea of “atomless independent supplement” is an ideal mathematical instrument to make the main assumption still work for a general class of correlated information games, see Theorem 1 and Proposition 1.

In [10], the authors also realize the possible application of the purification principle to games with correlated information, uncountable action spaces, and payoff relevant types, their model itself is a generalization of Aumann’s model in [2]. They show that under the assumption that each player has private secret information, which may not be independent of other parts of players’ private information, and the given common prior is saturated over this private information, the

\[\text{In [10], the authors use “superatomless” spaces instead of saturated spaces, these two kind of spaces are equivalent;\]
purification principle can still hold and as a result, there exists a pure-strategy equilibrium for such a Bayesian game with possible correlated information.

Rather to study a sophisticated model of games with uncountable action spaces, type relevant types as in [10], we in this paper consider a very simple model of games (finite action, type irrelevant, etc) with correlated information modeled by subsets in a product of type spaces. Since our model is far more simpler, we do not require that the prior to be saturated over players’ private secret information, while a far more simpler Lebesgue extension works well to model the type space to guarantee the existence of pure-strategy equilibrium. Moreover, since our model of correlated information games is simpler, we are also at ease to fully characterize the structure of all subjective or objective independent pure strategy equilibria. Finally, this paper also answers why saturated probability spaces must be used to model players private information. We show that the underlying type spaces must be saturated, if matching penny games with a very small class of simple correlated information structures each has a pure-strategy equilibrium. This result is new in the literature of correlated information games and serves as a complement to [10] on the usage of saturated spaces.

(b) On affiliation. This approach is new in the literature on Bayesian games with correlated information, compared to the existing literature in which certain order structure is imposed on the strategy spaces or particular conditions are imposed on the payoff structures; see, among others, [1], [27, 28], [36].

In the literature, the idea of affiliation is widely used to study correlated information structure. Here affiliation “means that a high value of one bidder’s estimate makes high values of the others’ estimates more likely” (Milgrom and Weber [29], p. 1096). Formally and simply, for \( i = 1, 2 \), assume that player \( i \)'s valuation for an object is represented by \( t_i \in [0, 1] \). Let \( f(t_1, t_2) \) be the probability density function of the valuation of both players. The information structure is affiliated if for all \( t_1 \leq t_1', t_2 \leq t_2' \), we have \( f(t_1, t_2)f(t_1', t_2) \geq f(t_1, t_2')f(t_1', t_2) \).

Actually, the information structure in Example 1 of [34] (see Example 3) itself is affiliated, in which the density function taking value 2 in the triangle below the 45 degree line, and value 0 otherwise. Moreover, if the states of the world is represented by a regular hypograph of a function in the square \([0, 1] \times [0, 1]\), and the prior probability measure for both players are the restricted probability measure of the Lebesgue product measure, the density function takes a constant positive number on this subset and 0 otherwise. In this case, the correlated information structure is affiliated if and only the function itself is an increasing function. From this point of view, affiliated information structure is rather restrictive in all the correlated information structures, even for the

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see Section 5 of [10] etc.
toy model we studied in this paper.

Notice that there is no way to impose an order structure in the strategies to ensure the payoffs in the matching penny game in Table 1 to be supermodular. Radner-Rosenthal’s example illustrates that even for games with affiliated information structure, the supermodularity in the payoff structure is indispensable. In this regard, one advantage of the approach introduced in this paper (Theorem 1 and Proposition 1) is that if we model the information space by a richer information spaces, it is also possible to establish existence of pure-strategy Bayesian-Nash equilibria even for Bayesian games with general correlated information structure (not necessarily to be affiliated) or with general payoff structures (not necessarily to be supermodular or satisfy other additional properties).

(c) On payoff-relevant types. In most parts of this paper, we discussed the existence of pure-strategy Bayesian-Nash equilibrium for two person games with payoff-irrelevant types. The purpose of this paper is to use these simplest games to illustrate how particular models of correlated information can affect such an existence result.

Consider a game with payoffs as in Table 4. Compared to the payoff of the matching penny game in Table 1, each player’s payoff also depends on his own type, i.e., player 1’s payoff is a scaled to \( a(t_1) \), and player 2’s to \( b(t_2) \). It is clear to see that the underlying two-person game is not necessarily to be zero-sum.

<table>
<thead>
<tr>
<th></th>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( H )</td>
<td>( T )</td>
</tr>
<tr>
<td>( H )</td>
<td>( a(t_1), -b(t_2) )</td>
<td>( 0, 0 )</td>
</tr>
<tr>
<td>( T )</td>
<td>( 0, 0 )</td>
<td>( a(t_1), -b(t_2) )</td>
</tr>
</tbody>
</table>

Table 4: A game with payoff-relevant payoffs, where \( a, b \) are positive-valued functions on \([0, 1]\)

Consider this two-person game associated with the states of the world is modeled by \( \Gamma_h \) in the Lebesgue square where \( h \) is a Lebesgue measurable function from the Lebesgue interval to \([0, 1]\). It is not hard to check that if \((\sigma_1, \sigma_2)\) is a pure-strategy Nash equilibrium of the above game, then it is also a pure-strategy Nash equilibrium for the matching penny game in Table 1 associated with the same correlated information structure. In fact, e.g., given player 2’s strategy \( \sigma_2 \), for any type of \( t_1 \), notice that \( \sigma_1(t_1) \) is a best response to his expected payoff, which is \( a_1(t_1) \) times the expected payoff of player 1, so it is also an optimal to the expected payoff in the matching penny game.

Indeed, the main results of this paper, Theorem 1 and Proposition 1, can be generalized to finite-action Bayesian games with general private information setting, i.e., the payoff of each player
only depends on his own type, not on the other player. Moreover, the results on the existence of pure-strategy equilibrium in [34] and [30] can be generalized to a setting where certain correlated information is allowed but the states of the world is modeled by a subset in a product of two measure-theoretically richer probability spaces, since one can directly apply the purification principle on the atomless independent supplements, or the atomless $\sigma$-algebras of secret events.

(d) On Savage games. In [9], a model of incomplete information games is proposed in Savage’s framework of purely subjective uncertainty, and the existence of pure-strategy equilibrium in such games is also obtained. The RR example, as well as many other games considered in this paper, does not satisfy an assumption of their model about the regularity of best responses, as a result, such correlated information games are excluded in their Savage games; see Assumption A.7 and Example 3.10 of [9]. However, in our model of correlated information games, we show that for a class of Bayesian games with correlated information including RR example, there exists a pure-strategy equilibrium in such games. Our measure-theoretic approach has also been proved useful to handle counterexample on the nonexistence of pure-strategy equilibrium in games with independent private information, for instance the KRS example in [18], see [24]. However, the KRS example is also excluded consideration in [9], see Footnote 4 therein. As a result, our measure-theoretic approach is very different from that considered in [9].

9 Appendix: Proofs of Results

9.1 Proofs: Section 5

Proof of Theorem 1. We will show that Assumption II in [2] can still be fulfilled if the states of the world $\Omega$ is represented by a regular Lebesgue measurable subset in the extended Lebesgue square, $([0,1] \times [0,1], \mathcal{L}^e \otimes \mathcal{L}^e, \lambda^e \otimes \lambda^e)$. As a consequence, the assertion in this theorem is straightforward from Proposition 4.3 therein.

First, the $\sigma$-algebra on $\Omega$ is obtained by restricting $\mathcal{L}^e \otimes \mathcal{L}^e$ to $\Omega$. Since players 1 and 2 are only informed the first and second component of a state respectively, $\mathcal{J}_1$ is the $\sigma$-algebra generated by all subsets of the form $(S_1 \times [0,1]) \cap \Omega$, while $\mathcal{J}_2$ is the $\sigma$-algebra generated by all $([0,1] \times S_2) \cap \Omega$, where $S_1$ and $S_2$ are both $\mathcal{L}^e$-measurable subsets. Now consider the following sub-$\sigma$-algebra of $\mathcal{J}_i$ for $i = 1, 2$. $\mathcal{J}_i'$ is the $\sigma$-algebra generated by $(R_1 \times [0,1]) \cap \Omega$, while $\mathcal{J}_2'$ is the $\sigma$-algebra generated by $([0,1] \times R_2) \cap \Omega$, where both $R_1$ and $R_2$ are $\mathcal{R}$-measurable subsets. It is clear that $\mathcal{J}_i'$ is a sub-$\sigma$-algebra of $\mathcal{J}_i$ since $\mathcal{R}$ is a sub-$\sigma$-algebra of $\mathcal{L}^e$ for $i = 1, 2$.

We next claim that: For $i = 1, 2$, $\mathcal{J}_i'$ is a $\sigma$-algebra generated by $i$-secret events. In other words,
the sub-σ-algebra $\mathcal{J}'$ is independent with $\mathcal{J}_j$.

Before proving this claim, we first make some preparations. Let $\Omega_1$ and $\Omega_2$ be the projection of $\Omega$ to the vertical and horizontal part respectively. Since $\Omega$ is a regular Lebesgue measurable subset in the square, w.l.o.g., assume that for any $t_i \in \Omega_i$, the $t_i$-section of $\Omega$, denoted by $\Omega_{t_i} = \{t_j : (t_2, t_1) \in \Omega\}$, is a Lebesgue measurable subset and $\lambda(\Omega_{t_i}) > 0$. Since $([0, 1], \mathcal{L}^e, \lambda^e)$ is a Lebesgue extension, we have $\Omega_{t_i} \in \mathcal{L}^e$ and $\lambda^e(\Omega_{t_i}) = \lambda(\Omega_{t_i}) > 0$ for each $t_i \in \Omega_i$. Let $q_i$ be the marginal of the common prior $p$ on player $i$’s type space, it is clear that $q_i$ is absolutely continuous w.r.t. the extended Lebesgue measure $\lambda^e$. Let the corresponding Radon-Nikodym derivative be $f_i : [0, 1] \to \mathbb{R}^+$, which is an $\mathcal{L}^e$-measurable function. Because $p$ is the restricted probability of $\lambda^e \otimes \lambda^e$ on $\Omega$, then for any $E \in \mathcal{L}^e \otimes \mathcal{L}^e$,

$$p(E \cap \Omega) = \int_{\Omega_1 \cap E_1} \frac{\lambda^e(E_{t_1} \cap \Omega_{t_1})}{\lambda^e(\Omega_{t_1})} f_1(t_1) \lambda^e(dt_1) = \int_{\Omega_2 \cap E_2} \frac{\lambda^e(E_{t_2} \cap \Omega_{t_2})}{\lambda^e(\Omega_{t_2})} f_2(t_2) \lambda^e(dt_2)$$

(2)

where $E_i$ is the projection of $E$ to player $i$’s type space, and $E_{t_i}$ is the $t_i$-section of $E$ for every player $i$’s type $t_i$.

We are finally ready to prove the above claim. Fix $R \in \mathcal{R}$ and $S \in \mathcal{L}^e$, and let $E = (\mathcal{J}_2 \times R) \cap \Omega$ and $F = (S \times [0, 1]) \cap \Omega$. It is clear that $E \in \mathcal{J}'_2$ and $F \in \mathcal{J}_1$. It is easy to calculate $p(E)$ as follows,

$$p(E) = \int_{\Omega_2 \cap [0, 1]} \frac{\lambda^e(R \cap \Omega_{t_2})}{\lambda^e(\Omega_{t_2})} f_2(t_2) \lambda^e(dt_2)$$

$$= \int_{\Omega_2} \frac{\lambda^e(R) \cdot \lambda^e(\Omega_{t_2})}{\lambda^e(\Omega_{t_2})} f_2(t_2) \lambda^e(dt_2)$$

$$= \lambda^e(R) \int_{\Omega_2} f_2(t_2) \lambda^e(dt_2)$$

$$= \lambda^e(R),$$

(3)

where the first equation follows from Equation (2), the second from the property that $R$ is independent with the Lebesgue measurable subset $\Omega_{t_2}$, and the last from the definition of Radon-Nikodym derivative. Similarly, we have

$$p(E \cap F) = p((S \times R) \cap \Omega)$$

$$= \int_{\Omega_2 \cap S} \frac{\lambda^e(R \cap \Omega_{t_2})}{\lambda^e(\Omega_{t_2})} f_2(t_2) \lambda^e(dt_2)$$

$$= \lambda^e(R) \int_{\Omega_2 \cap S} f_2(t_2) \lambda^e(dt_2)$$

$$= \lambda^e(R) \cdot p(F)$$

$$= p(E) \cdot p(F),$$

29
where the second and fourth equations follows from Equation (2), the third from that $R$ is independent with $\Omega_2$, the last from Equation (3). We thus show that $E$ and $F$ are independent events. Since the choice of $R$ and $S$ are arbitrary, hence the sub-$\sigma$-algebra $\mathcal{F}_2^t$ is independent with $\mathcal{F}_1$. That is, $\mathcal{F}_2^t$ is a $\sigma$-algebra of 2-secret events. It is clear from Equation (3) that $p_2 = p$ is atomless on $\mathcal{F}_1^t$. Similarly, we can show that $\mathcal{F}_1^t$ is a $\sigma$-algebra of 1-secret events, and $p_1 = p$ is atomless on $\mathcal{F}_1^t$. We thus complete the proof of the claim.

Proof of Claim 1. Since the type space of player 1 is modeled by the Lebesgue unit interval $([0, 1], \mathcal{L}, \lambda)$, a pure-strategy for player 1 is a $\{H, T\}$-valued Lebesgue measurable map. Let $E$ be a Lebesgue measurable subset such that player 1 takes $H$ if and only if $t_1 \in E$. We now consider the best response for player 2 at his type $t_2 \in [0, 1]$. At $t_2$, player 2 thinks $t_1$ is distributed as the restricted probability measure of $\lambda^e$ on $[t_2, 1]$. Therefore, given player 1’s strategy as above, player 2’s best response depends on the value of $\frac{\lambda^e(E \cap [t_2, 1])}{\lambda^e([t_2, 1])}$. Notice that both $E$ and $[t_2, 1]$ are Lebesgue measurable subset, $\frac{\lambda^e(E \cap [t_2, 1])}{\lambda^e([t_2, 1])} = \frac{\lambda(E \cap [t_2, 1])}{1-t_2}$. In particular,

$$B_2(t_2) = \begin{cases} H, & \text{if } \frac{\lambda(E \cap [t_2, 1])}{1-t_2} < \frac{1}{2}; \\ T, & \text{if } \frac{\lambda(E \cap [t_2, 1])}{1-t_2} > \frac{1}{2}; \\ H \text{ or } T, & \text{if } \frac{\lambda(E \cap [t_2, 1])}{1-t_2} = \frac{1}{2}. \end{cases}$$

Let $g(t_2) = \lambda(E \cap [t_2, 1]) - \frac{1-t_2}{2}$, for all $t_2 \in [0, 1]$. It is clear that $g : [0, 1] \to [0, 1]$ is a continuous function in $t_2$. As a result, both $\{t_2 : g(t_2) > 0\}$ and $\{t_2 : g(t_2) < 0\}$ are open subsets in $[0, 1]$. As a result, $\{t_2 : g(t_2) = 0\}$, denoted it by $\{g = 0\}$, is also a Lebesgue measurable subset. We next show by contradiction that $\lambda(\{g = 0\}) = 0$. Suppose not, $\lambda(\{g = 0\}) > 0$, then it contains an closed interval, say it $[c, d]$. As a result, for any $t_2 \in [c, d]$, $\lambda(E \cap [t_2, 1]) = \frac{1-t_2}{2}$. Thus for any $t_2, t'_2 \in [c, d]$, we have

$$\lambda(E \cap [t_2, t'_2]) = \lambda(E \cap [t_2, 1]) - \lambda(E \cap [t'_2, 1]) = \frac{1-t_2}{2} - \frac{1-t'_2}{2} = \frac{t'_2 - t_2}{2}.$$ 

This contradicts the fact that for the nonnegligible Lebesgue subset $E \cap [c, d]$, there exists an interval such that the intersection is of measure strictly greater than $3/4$; see [11, p. 68, Theorem A].

Therefore, the $\lambda$-almost all $t_2$, the best response for player 2 is single-valued except for a Lebesgue null subset. As a result, the best response for player 2, against player 1’s strategy, is a Lebesgue measurable function. 

\footnote{Actually, if $\{g = 0\}$ contains a dense set in the interval $[c, d]$, it follows from the continuity of $g$ that every point of $[c, d]$ is a root of $g$.}
9.2 Proofs: Section 6

Proof of Claim 2. Let \( \mathcal{R}' \) be the \( \sigma \)-algebra generated by \( \mathcal{R} \) and \( \{[0,1/2), [1/2,1]\} \), where the former is the atomless independent supplement of the Lebesgue \( \sigma \)-algebra in \( \mathcal{L}^e \). It is clear that \( \mathcal{R}_1 = \mathcal{R}' \otimes \mathcal{L}^e \) is a \( \sigma \)-algebra of 1-secret events since and \( p_2 \) is atomless on \( \mathcal{R}_1 \). Similarly, \( \mathcal{R}_2 = \mathcal{L}^e \otimes \mathcal{R}' \) is a \( \sigma \)-algebra of 2-secret events since and \( p_1 \) is atomless on \( \mathcal{R}_2 \). Thus, the Main Assumption holds. Let \( F_1, F_2 \in \mathcal{R}' \), it is clear that a subjective independent pure-strategy equilibrium can be represented by \( (F_1, F_2) \), such that player \( i \) takes action \( H \) if and only if \( t_i \in F_i \) for \( i = 1, 2 \). Assume that \( \lambda^e(F_1) = x, \lambda^e(F_2) = y \). We next consider all the possible equilibrium payoffs for subjective independent pure-strategy equilibria of this form.

(Case A) \((1/2, 1] \cap F_1 \neq \emptyset \) and \((1/2, 1] \cap F_1^c \neq \emptyset \). In this case, the best response for player 1 is \( H \) for some \( t_1 \in F_1 \cap (1/2, 1] \) and \( T \) for other \( t_1 \in F_1^c \cap (1/2, 1] \). As a result, we have \( y = 1 - y \), that is, \( y = 1/2 \). Consider the best responses of player 2, there are the following three possible cases,

(Case A1) \([0,1/2] \cap F_2 \neq \emptyset \) and \([0,1/2] \cap F_2^c \neq \emptyset \). In this case, the best response for player 2 is \( H \) for some \( t_2 \in [0,1/2] \cap F_2 \) and \( T \) for some other \( t_2 \in [0,1/2] \cap F_2^c \). Hence, we have \( -x = -(1 - x) \), i.e., \( x = 1/2 \). Thus, the resulted subjective equilibrium expected payoff is \((1/2, -1/2)\).

(Case A2) \([0,1/2] \subseteq F_2^c \). In this case, the best response for player 2 for all \( t_2 \in [0,1/2] \) is \( T \). As a result, \( -x \leq -(1 - x) \), i.e., \( x \geq 1/2 \). Therefore, the subjective expected equilibrium payoff vector is of the form \((1/2, -(1 - x))\) for all \( 1 \geq x \geq 1/2 \). In this case, all the subjective expected equilibrium payoff vectors consist of the interval

\[
\left\{ \frac{1}{2} \right\} \times \left( -\frac{1}{2}, 0 \right).
\]

(Case A3) \([0,1/2] \subseteq F_2 \). It is clear that \( y = \lambda^e(F_2) \geq 1/2 \). In this case, the best response for player 2 for all \( t_2 \in [0,1/2] \) is \( H \). As a result, \( -x \geq -(1 - x) \), i.e., \( x \leq 1/2 \). Therefore, the subjective expected equilibrium payoff vector is of the form \((1/2, -x)\) for all \( 0 \leq x \leq 1/2 \). In this case, all the subjective expected equilibrium payoff vectors consist of the interval

\[
\left\{ \frac{1}{2} \right\} \times \left( -\frac{1}{2}, 0 \right).
\]

(Case B) \([1/2,1] \subseteq F_1^c \). It is clear that \( x = \lambda^e(F_1) \leq 1/2 \). Moreover, in this case, the best response for player 1 at any type \( t_1 \in [1/2,1] \) is \( T \), as a result, \( y \leq 1 - y \), i.e., \( y \leq 1/2 \). Consider the best responses of player 2, there are possibly the following three cases,

(Case B1) \([0,1/2] \cap F_2 \neq \emptyset \) and \([0,1/2] \cap F_2^c \neq \emptyset \). As in A1, it follows that \( x = 1/2 \). Therefore, the subjective expected equilibrium payoff vector in this case is of the form \((1 - y, -1/2)\), for all

...
$y \in [0,1/2]$. That is, in this case, all the subjective expected equilibrium payoff vectors consist of the interval
\[
\left( \frac{1}{2},1 \right) \times \left\{ -\frac{1}{2} \right\}.
\]

(Case B2) $[0,1/2] \subseteq F_2^c$. As in A2, $x \geq 1/2$. According to $x \leq 1/2$, $x = 1/2$. Therefore, the subjective expected equilibrium payoff vector in this case is of the form $(1-y,-1/2)$, for all $y \in [0,1/2]$. That is, in this case, all the subjective expected equilibrium payoff vectors consist of the interval
\[
\left( \frac{1}{2},1 \right) \times \left\{ -\frac{1}{2} \right\}.
\]

(Case B3) $[0,1/2] \subseteq F_2$. It is clear that $y = \lambda^e(F_1) \geq 1/2$. It follows from $y \leq 1/2$ that $y = 1/2$. As in A3, $x \leq 1/2$. Therefore, the subjective expected equilibrium payoff vector in this case is of the form $(1/2,-x)$, for all $x \in [0,1/2]$. That is, in this case, all the subjective expected equilibrium payoff vectors consist of the interval
\[
\left\{ \frac{1}{2} \right\} \times \left( -\frac{1}{2},0 \right).
\]

(Case C) $[1/2,1] \subseteq F_1$. It is clear that $x = \lambda^e(F_1) \geq 1/2$. Moreover, in this case, the best response for player 1 at any type $t_1 \in [1/2,1]$ is $H$, as a result, $y \geq 1-y$, i.e., $y \geq 1/2$. Consider the best responses of player 2, there are the following three possible cases.

(Case C1) $[0,1/2] \cap F_2 \neq \emptyset$ and $[0,1/2] \cap F_2^c \neq \emptyset$. As in A1, $x = 1/2$. As a result, the resulted subjective equilibrium expected payoff is $(y,-1/2)$ for $y \in [1/2,1]$. That is, all the subjective expected equilibrium payoff vector is the interval
\[
\left\{ \frac{1}{2} \right\} \times \left( -\frac{1}{2},0 \right).
\]

(Case C2) $[0,1/2] \subseteq F_2^c$. It is clear that $y = \lambda^e(F_2) \leq 1/2$. According to $y \geq 1/2$, $y = 1/2$. Also as in A2, one can obtain $x \geq 1/2$. A subjective expected equilibrium payoff vector in this case is of the form $(1/2,-(1-x))$ for all $x \in [1/2,1]$. Hence all the the subjective expected equilibrium payoff vectors are the following interval
\[
\left\{ \frac{1}{2} \right\} \times \left( -\frac{1}{2},0 \right).
\]
(Case C3) $[0, 1/2] \subseteq F_2$. In this case, as in A3, $x \leq 1/2$. Notice also $x \geq 1/2$, as a result $x = 1/2$. Therefore, the subjective expected equilibrium payoff vector is of the form $(y, -1/2)$, for all $y \in [1/2, 1]$. That is, in this case, all the subjective expected equilibrium payoff vectors consist of the interval

$$\left(\frac{1}{2}, 1\right) \times \left\{ -\frac{1}{2} \right\}.$$

Combine all the above subjective equilibrium vectors together, one gets the Picture 6.

Proof of Claim 3. The proof of this claim is very similar to that of Claim 2. Assume that any subjective (pure-strategy) equilibrium can be represented by $(F_1, F_2)$, for some required $F_1, F_2$ with $x^e(F_1) = x, x^e(F_2) = y$ as therein. We next consider all the possible equilibrium payoffs for subjective independent pure-strategy equilibria of this form.

(Case A) $(1/2, 1] \cap F_1 \neq \emptyset$ and $(1/2, 1] \cap F_1^c \neq \emptyset$. In this case, the best response for player 1 is $H$ for some $t_1 \in F_1 \cap (1/2, 1]$ and $T$ for other $t_1 \in F_1^c \cap (1/2, 1]$. As a result, we have $4y = 2(1 - y)$, that is, $y = 1/3$. Consider the best responses of player 2, there are the following two possible cases,

(Case A1) $[0, 1/2) \cap F_2 \neq \emptyset$ and $[0, 1/2) \cap F_2^c \neq \emptyset$. In this case, the best response for player 2 is $H$ for some $t_2 \in [0, 1/2) \cap F_2$ and $T$ for other $t_2 \in [0, 1/2) \cap F_2$. Hence, we have $-x = -2(1 - x)$, i.e., $x = 2/3$. Thus, the resulted subjective equilibrium expected payoff is $(\frac{4}{3}, -\frac{2}{3})$.

(Case A2) $[0, 1/2) \subseteq F_2^c$. In this case, the best response for player 2 for all $t_2 \in [0, 1/2)$ is $T$. As a result, $-x \leq -2(1 - x)$, i.e., $x \geq 2/3$. Therefore, the subjective expected equilibrium payoff vector is of the form $(4/3, -2(1 - x))$ for all $1 \geq x \geq 2/3$. In this case, all the subjective expected equilibrium payoff vectors consist of the interval

$$\left\{ \frac{4}{3} \right\} \times \left( -\frac{2}{3}, 0 \right).$$

(Case A3) $[0, 1/2) \subseteq F_2^c$. It is clear that $y = x^e(F_2) \geq 1/2$. This contradicts $y = \frac{4}{3}$. Thus this case can not take place.

(Case B) $(1/2, 1] \subseteq F_1^c$. It is clear that $x = x^e(F_1) \leq 1/2$. Moreover, in this case, the best response for player 1 at any type $t_1 \in (1/2, 1]$ is $T$. As a result, $4y \leq 2(1 - y)$, i.e., $y \leq 1/3$. Consider the best responses of player 2, there are possibly the following two cases,

(Case B1) $[0, 1/2) \cap F_2 \neq \emptyset$ and $[0, 1/2) \cap F_2^c \neq \emptyset$. As in A1, it follows that $x = 2/3$. This contradicts $x \leq 1/2$, as a result, this case can not take place.

(Case B2) $[0, 1/2) \subseteq F_2^c$. As in A2, $x \geq 2/3$. This case can not happen either because it contradicts $x \leq 1/2$.

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(Case B3) \([0, 1/2) \subseteq F_2\). It is clear that \(y \geq 1/2\). This contradicts with \(y \leq 1/3\). This case can not happen either.

(Case C) \((1/2, 1] \subseteq F_1\). It is clear that \(x = \lambda^c(F_1) \geq 1/2\). Moreover, in this case, the best response for player 1 at any type \(t_1 \in (1/2, 1]\) is \(H\), as a result, \(4y \geq 2(1 - y)\), i.e., \(y \geq 1/3\). Consider the best responses of player 2, there are the following three possible cases.

(Case C1) \([0, 1/2) \cap F_2 \neq \emptyset\) and \([0, 1/2) \cap F_3^c \neq \emptyset\). As in A1, \(x = 2/3\). As a result, the resulted subjective equilibrium expected payoff is \((4y, -2/3)\) for \(y \in [1/3, 1]\). That is, all the subjective expected equilibrium payoff vector is the interval

\[\left[\frac{4}{3}, 4\right] \times \left\{-\frac{2}{3}\right\}.

(Case C2) \([0, 1/2) \subseteq F_2^c\). It is clear that \(y = \lambda^c(F_2) \leq 1/2\). As in A2, one can obtain \(x \geq 2/3\). A subjective expected equilibrium payoff vector is of the form \((4y, -2(1 - x))\) for all \(y \in [1/3, 1/2]\), \(x \in [2/3, 1]\). Hence all the the subjective expected equilibrium payoff vectors are the following rectangle

\[\left[\frac{4}{3}, 2\right] \times \left[-\frac{2}{3}, 0\right].

(Case C3) \([0, 1/2) \subseteq F_2\). It is clear that \(y = \lambda^c(F_2) \geq 1/2\). Moreover, in this case, the best response for player 2 at any for \(t_2 \in [0, 1/2)\) is \(H\); as a result, \(-x \geq 2(1 - x)\), i.e., \(x \leq 2/3\). Therefore, the subjective expected equilibrium payoff vector is \((4y, -x)\), for all \(y \in [1/2, 1]\), \(x \in [1/2, 2/3]\). That is, all the the subjective expected equilibrium payoff vectors in this case consist of the following rectangle

\[\left[2, 4\right] \times \left[-\frac{2}{3}, -\frac{1}{2}\right].

Combine all the above subjective equilibrium vectors together, one gets the Picture 7. ■

9.3 Proofs: Section 7

Proof of Proposition 1. For \(i = 1, 2\), let \(S_i\) be an atomless independent supplement of \(H_i\) in \(\mathcal{T}_i\). Let \(R_1 = S_1 \otimes T_2\) and \(R_2 = T_1 \otimes S_2\). The approaches in the proof of Theorem 1 can be applied here to show that \(R_i\) is a \(\sigma\)-algebra of \(i\)-secret events in the new information structure, and moreover, \(p_j\) is atomless on \(R_i\). As a result, Assumption II in [2] is fulfilled.

Proof of Corollary 2. Let \(\mathcal{H}\) be a sub-\(\sigma\)-algebra of \(\mathcal{T}\) generated by the \(\mathcal{T}\)-measurable function \(h\), i.e., \(\mathcal{H} = \sigma\{h^{-1}[s, 1] : s \in [0, 1]\}\). Because the hypograph of \(h\), \(\Gamma_h\), is a regular subset in \(T \times [0, 1]\), \(\Gamma_h \in \mathcal{H} \otimes \mathcal{L}\). We next show that \(\mathcal{H}\) admits an atomless independent supplement in \(\mathcal{T}\) and then this corollary follows straightforward from Lemma 1 and Proposition 1.
Towards this end, first let $\nu = \nu' \otimes \lambda$ be a product probability measure on the square $[0, 1] \times [0, 1]$, where $\lambda$ is the Lebesgue measure and $\nu'$ the induced Borel probability measure induced by $h$, i.e., $\nu' = \mu \circ h^{-1}$. By applying the saturation property on this $\nu$, there exists a $T$-measurable function $g : T \to [0, 1]$ such that $(h, g)$ induces the product probability measure $\nu = \nu' \otimes \lambda$ on the square. It is clear that $g$ and $h$ are two independent random variables, as a result, the sub-$\sigma$-algebra generated by $g$, denoted it by $S \subseteq T$, is independent with the sub-$\sigma$-algebra induced by $h$, $\mathcal{H}$. Finally, it is clear that $\mu$ is atomless on $S$ because $g$ induces the Lebesgue measure on the interval. We thus complete the proof. 

Before proving Theorem 2, we first consider the following Lemma.

**Lemma 2.** For the matching penny game associated with information structure $\Gamma_h$ for a measure-preserving map $h$ from $(T_1, T_1, \mu)$ to the Lebesgue unit interval, if there exists a pure-strategy Bayesian-Nash equilibrium, it must be the case that given player $i$’s type $t_i$, the conditional probability of player $j$ playing $H$ and $T$ are equal. In particular, let the pure-strategy equilibrium is represented by $(E_1, E_2)$, where $E_2 \in \mathcal{L}^c, E_1 \in T_1$ such that player $i$ takes $H$ if and only if $t_i \in E_i$, then

$$\lambda^c (E_2 \cap [0, h(t_1)]) = \frac{1}{2} h(t_1), \quad \forall \ t_1 \in T_1; \tag{4}$$

$$\mu (E_1 \cap h^{-1} [t_2, 1]) = \frac{1}{2} \mu (h^{-1} [t_2, 1]), \quad \forall \ t_2 \in [0, 1]. \tag{5}$$

If both players’ type spaces are modeled by the Lebesgue unit interval and the states of the world is the lower triangle of the Lebesgue square under the 45 degree line, the matching penny game associated with this information structure is exactly Example 1 of [34]; see also Example 3 in Section 3. In this game, the statement in Lemma 2 is exactly the following claim therein but a formal proof is not provided, “for each player $i$ and almost every value $z_i$ of his information variable $Z_i$, the conditional probabilities of the two moves of the other player (given that $Z_i = z_i$) should be equal”.

**Proof of Lemma 2.** Let the pure-strategy equilibrium is represented by $(E_1, E_2)$, where $E_2 \in \mathcal{L}^c, E_1 \in T$. Given $E_2, T_2 = [0, 1]$ has a partition $\{T_2^>, T_2^\leq, T_2^=\}$, where

$$T_2^\leq = \left\{ t \in [0, 1] : \lambda^c ([0, t] \cap E_2) < \frac{t}{2} \right\},$$

and $T_2^<, T_2^=$ are defined similarly. Since player 2’s type space $([0, 1], \mathcal{L}^e, \lambda^e)$ is a Lebesgue extension, it is clear that $\lambda^e ([0, t_2] \cap E_2)$ is a continuous function in $t_2$, as a result, $T_2^<, T_2^>$ are both open sets
in the Lebesgue interval. Since $h$ is a measure preserving map, it is clear that $1 = \sup h(T_2)$. We now prove this result in three steps.

**Steps 1**. We first show by contradiction that $1 \in T^<_2$. Suppose that $1 \in T^<_2$. By the continuity of $\lambda^e([0,t_2] \cap E_2)$ in the parameter $t_2$, there is an interval containing 1 which lies in $T^<_2$. Let $a = \inf \{ t_2 \in [0,1] : (t_2,1] \subseteq T^<_2 \}$. That is, for any $t_2 \in (a,1]$ we have,

$$
\lambda^e([0,t_2] \cap E_2) < \frac{t_2}{2}. \tag{6}
$$

We prove by contradiction that $a > 0$. Suppose $a = 0$, that means $(0,1] \subseteq T^<_2$. As a result, for any $t_1 \in T_1$ with $h(t_1) > 0$, $\lambda^e([0,h(t_1) \cap E_2]) < \frac{h(t_1)}{2}$, that is, the best response for player 1 at $t_1$ is $T$. Note that $\{ t_1 : h(t_1) = 0 \}$ is a $\mu$-null set; as a result, for $\mu$-almost all $t_1 \in T_1$, player 1 takes $T$. Therefore, the best response for almost all $t_2$ is to take $H$; i.e., $E_2 = [0,1]$. That is, $(0,1] \subseteq T^<_2$, a contradiction.

Since $1 > a > 0$, it follows from the continuity of $\lambda^e([0,t_2] \cap E_2)$ in $t_2$ that

$$
\lambda^e([0,a] \cap E_2) = \frac{a}{2}. \tag{7}
$$

It is clear from the measure-preserving property that $h^{-1}(a,1]$ is of $\mu$-measure $1 - a > 0$. For every $t_1 \in h^{-1}(a,1]$, according to Eq. (6), the only best response for player 1 at this type $t_1$ is $T$, i.e., $h^{-1}(a,1] \subseteq E_1^*$. As a result, for any $t_2 \in (a,1]$,

$$
\mu(h^{-1}[t_2,1] \cap E_1) = 0 < \frac{1}{2} \mu(h^{-1}[t_2,1]) = \frac{1 - t_2}{2},
$$

where the last equation follows from the measure-preserving property. Hence player 2 will take $H$ for sure at every $t_2 \in (a,1]$, i.e., $(a,1] \subseteq E_2$. As a result, for any $t_2 \in (a,1]$

$$
\lambda^e([0,t_2] \cap E_2) = \lambda^e([0,a] \cap E_2) + (t_2 - a) = t_2 - \frac{a}{2} > \frac{t_2}{2},
$$

a contradiction to Eq. (6). Similarly, one can show that it is not the case that $1 \in T^>_2$.

**Steps 2**. Actually, the proof of Step 1 can still be applied to show that there can not a subset of the form $(a,1)$ or $(a,1]$ such that this subset belongs to $T^>_2$ or $T^<_2$. As a result, it must be the case that there is $c < 1$, such that $[c,1] \subseteq T^<_2$, i.e., for any $t_2 \in [c,1]$

$$
\lambda^e([0,t_2] \cap E_2) = \frac{t_2}{2}. \tag{8}
$$

It is clear that for every $t_2 \in [c,1]$, $\mu(h^{-1}[t_2,1]) = 1 - t_2 > 0$, we next show that

$$
\mu(h^{-1}[t_2,1] \cap E_1) = \frac{1}{2} \mu(h^{-1}[t_2,1]) = \frac{1 - t_2}{2}. \tag{9}
$$
We first prove Eq. (9) for any \( t_2 \in (c, 1) \). Suppose not, let \( t_0 \in (c, 1) \) be a point such that 
\[
\mu(h^{-1}[t_0, 1] \cap E_1) > \frac{1}{2} \mu(h^{-1}[t_0, 1]) = \frac{1-t_0}{2} > 0.
\]
Notice that as functions, both \( \mu(h^{-1}[t, 1]) \) and \( \mu(h^{-1}[c, 1] \cap E_2) \) are continuous at \( t_0 \), as a result, there exists \( t_0' \in (c, t_0) \) such that for any \( t_2 \in (t_0', t_0) \),
\[
\mu(h^{-1}[t_2, 1] \cap E_1) > \frac{1}{2} \mu(h^{-1}[t_2, 1]) = \frac{1-t_2}{2}.
\]
As a result, the best response for player 2 at every \( t_2 \in (t_0', t_0) \) is \( T \), i.e., \( (t_0', t_0) \subseteq E_2^c \). Hence,
\[
\lambda^e([0, t_0'] \cap E_1) = \lambda^e([0, t_0] \cap E_1),
\]
it follows from Eq. (8) that \( \frac{t_0'}{2} = \frac{t_0}{2} \), then \( t_0 = t_0' \), a contradiction. Similarly, one can show that it is not the case that \( \mu(h^{-1}[c, 1] \cap E_1) > \frac{1}{2} \mu(h^{-1}[c, 1]) \) for any point in \( (c, 1) \).

By the continuity of both \( \mu(h^{-1}[c, 1]) \) and \( \mu(h^{-1}[c, 1] \cap E_1) \) at \( c \), we have that
\[
\mu(h^{-1}[c, 1] \cap E_1) = \frac{1}{2} \mu(h^{-1}[c, 1]) = \frac{1-c}{2}.
\]
(Step 3). Let \( c = \inf\{t_2 \in [0, 1] : [t_2, 1] \subseteq T_2^=\} \). Step 1 shows that the set \( \{t_2 \in [0, 1] : [t_2, 1] \subseteq T_2^=\} \) is nonempty. Given that \( [c, 1] \subseteq T_2^= \) for some \( 0 < c < 1 \), one can repeat the arguments of in Step 1 to show that there can not be an open interval of the form \((d, c)\) such that \((d, c) \subseteq T_2^>\) or \( T_2^< \). As a result \( c = 0 \), that is, \([0, 1] \in T_2^=\) and Eq. (4) holds. Then one can repeat the argument in Step 2 to show that Eq. (5) also holds. We thus complete that proof. ■

We are now ready to prove Theorem 2.

**Proof of Theorem 2.** We proceed by proving the following equivalent result: if \((T, \mathcal{T}, \mu)\) is not a saturated probability space, there exists a measure-preserving map \( h \) from \( T \) to the Lebesgue interval such that the underlying matching-penny game associated with the information structure \( \Gamma_h \) does not have a pure-strategy Bayesian Nash equilibrium.

Suppose that \((T, \mathcal{T}, \mu)\) is not a saturated probability space, it follows from Theorem 3B.7 of [7, page 47] that there exists a non-negligible \( \mathcal{T} \)-measurable subset \( S \subseteq T \), assume that \( \mu(S) = s > 0 \), such that the restricted probability space \((S, \mathcal{T}^S, \mu^S)\) has countable Maharam type, where \( \mathcal{T}^S = \{S \cap E : E \in \mathcal{T}\} \). By Mahram’s theorem, the measure algebra of \((S, \mathcal{T}^S, \mu^S)\), the quotient algebra of \( \mathcal{T}^S \) over all the \( \mu^S \)-null subsets, is isomorphic to the measure algebra of the Lebesgue unit interval. It is a standard result\(^{25}\) that there exists a \( \mathcal{T} \)-measurable map \( h_S : S \rightarrow [0, s] \) such that it is measure-preserving and the measure algebra isomorphism can be realized by this map,

\(^{25}\)See, e.g., Theorem 4.12 in [8].
in particular, for any $E \in T^S$, there exists a Lebesgue measurable subset $E' \subseteq [0,s]$ such that $\mu\left(h_S^{-1}(E') \Delta E\right) = 0$, where $\Delta$ is the symmetric difference in $T^S$.

If $s < 1$, that is, the complement of $S$ in $T$, $T \setminus S$, is also a non-negligible subset in $T$, it is clear that $\mu$ is still atomless on the restricted measurable space on $T \setminus S$. Moreover, it is straightforward to get a measure-preserving map from $T \setminus S$ to the Lebesgue interval $[s,1]$. Therefore, there is a measure-preserving map $g$ from $(T, \mathcal{T}, \mu)$ to the Lebesgue unit interval such that $g|_S = h_S$.

Finally, we complete the proof by showing that there does not exist a pure-strategy Bayesian-Nash equilibrium in the matching penny game associated with the information structure $\Gamma_g$. Suppose not, according to Lemma 2, there exists a $\mathcal{T}$-measurable set $E$ such that, for any $t \in [0,1]$, $\mu\left(E \cap g^{-1}[t,1]\right) = \frac{1}{2} \mu\left(g^{-1}[t,1]\right) = \frac{1-t}{2}$. In particular, $\mu(E) = \frac{1}{2}$. Moreover, for any $t \leq s$, $\mu\left(E \cap g^{-1}(0,t]\right) = \mu(E) - \mu\left(E \cap g^{-1}[t,1]\right) = \frac{1}{2}$. Actually, for any $t \leq s$, because $g^{-1}(0,t] = h_S^{-1}[0,t] \subseteq S$ for any $t \in [0,s)$,

$$\mu\left(E \cap g^{-1}(0,t]\right) = \mu\left((E \cap S) \cap h_S^{-1}[0,t]\right).$$

Therefore, for any $t \in [0,s]$,

$$\mu\left((E \cap S) \cap h_S^{-1}[0,t]\right) = \frac{t}{2}.$$

By the definition of $h_S$, there is a Lebesgue measurable subset $F \subseteq [0,s]$ such that $h_S^{-1}(F)$ and $E \cap S$ differ up to a $\mu$-null set. As a result,

$$\mu\left((E \cap S) \cap h_S^{-1}[0,t]\right) = \mu\left(h_S^{-1}(F) \cap h_S^{-1}[0,t]\right) = \mu\left(h_S^{-1}(F \cap [0,t])\right) = \lambda(F \cap [0,t]), \quad (11)$$

where the last equation is due to the induced distribution of $h_S$ is the Lebesgue measure on $[0,s]$. As a result, there is a Lebesgue measurable subset $F \subseteq [0,s]$ such that $\lambda(F \cap [0,t]) = \frac{t}{2}$ for all $t \in (0,s]$. A contradiction.

References


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