Nash Equilibrium in Games with Quasi-Monotonic Best-Responses

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Abstract

This paper develops a new existence result for pure-strategy Nash equilibrium. In succinct form, for a two-player game with scalar action sets, existence entails that one reaction curve be increasing and continuous and the other quasi-increasing (i.e., not have any downward jumps). The latter property amounts to strategic pseudo-complementarities. We also prove some extensions to n-player games, at the cost of some further plausible assumptions. Along the way, the paper provides a number of ancillary results of independent interest, including sufficient conditions for a quasi-increasing argmax, comparative statics of equilibria, and new sufficient conditions for uniqueness of fixed points. For maximal accessibility of the results, in addition to a general lattice-theoretic treatment, the main results are presented in a Euclidean setting. We argue that all these results have broad and elementary applicability by providing simple illustrations with four commonly used models from applied microeconomic fields.

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Key words and phrases: Existence of Nash equilibrium, quasi-monotone functions, non-monotone comparative statics, supermodularity.

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1 Introduction

In the course of developing new game-theoretic models to describe economic behavior in various situations, the existence of Nash equilibrium often emerges as the first critical test to discriminate between alternative candidate models. In most economic settings, a long-standing preference for pure-strategy Nash equilibrium (henceforth, abbreviated PSNE) still constitutes the dominant norm. The primary requirement of existence applies to any type of investigation, independently of whether the analysis is meant to proceed along specific or general functional forms. In the latter case, existence of Nash equilibrium is virtually always obtained via the application of a fixed point theorem, following a long-standing practice going back to Nash (1951).

While Nash considered mixed-strategy equilibrium for finite games, Rosen (1965) extended his basic insight to the case of pure-strategy equilibrium and Euclidean action spaces. In this traditional approach, existence follows from Brouwer’s (or Kakutani’s) fixed point theorem, and is therefore predicated on the best response functions (or correspondences) being continuous on compact and convex action spaces. Stepping back to the payoff functions, the relevant properties are joint continuity in the strategies and quasi-concavity in own action. For obvious reasons, the underlying method has come to be known as the topological approach.

In more recent times, a new approach to the existence of pure-strategy Nash equilibrium, which relies on the best response mapping (and thus the “reaction curves”) being increasing functions (or selections) and the action spaces being complete lattices, was proposed by Topkis (1978, 1979). Based instead on Tarski’s well known fixed point theorem for monotone functions (Tarski, 1955), this approach of an order-theoretic or algebraic nature has given rise to the recently much studied class of supermodular games. In addition to opening a new realm for addressing the fundamental issue of existence, this approach has also proven useful for the characterization of equilibrium properties, in particular with respect to comparative statics conclusions (Topkis, 1979, Vives, 1990, and Milgrom and Roberts, 1990).

The purpose of the present paper is to develop a new class of games that possess pure-strategy Nash equilibrium, which is not covered by either of the two aforementioned existence paradigms. In its most general form, the result pertains to two-player games with scalar action sets. This new result imposes different requirements on the two players’ reaction curves. For one player, this curve must be both continuous and increasing, while for the other player all that is needed is that his reaction curve not possess any downward jump discontinuities. Fig. 1 below illustrates our basic fixed point result.

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1These requirements have since been partially relaxed in the recent literature dealing with discontinuous games.
We follow a lesser known part of Tarski's (1955) classical paper, and call “quasi-increasing” functions that cannot have downward jumps.\(^2\) The principal aim of this part of Tarski’s paper is to prove an intersection point theorem between a quasi-increasing function and a quasi-decreasing function whenever these have the same domains and ranges (both complete chains) and the former starts above the latter and ends below it.\(^3\) An important special case of this Theorem obtains when one takes the quasi-decreasing function to be the identity function, in which case the result boils down to a fixed point theorem for quasi-increasing functions. Interestingly, variants of this fixed point theorem have been rediscovered in the economics literature and applied a number of times to establish existence of symmetric PSNE in symmetric oligopoly settings. As a result, the existence of a symmetric PSNE then follows at once from the existence of a fixed point for the common reaction curve to all the players.\(^4\)

While the economic applications of this result have so far all shared the critical property that the underlying game is symmetric, the starting point for the present paper is the idea that the underlying logic may be used to to establish existence of PSNE for asymmetric two-player games. To this end, it suffices to apply this fixed point theorem to the composition of the two reaction curves, one of which is a quasi-increasing function and the other an increasing and continuous function, upon making the key observation that the composition of two such functions is itself necessarily quasi-increasing. While this structure makes it clear that the underlying class of games overlaps with the two existing classes of games that are known to possess PSNEs, as described above, it is easy to see that it is not nested with either of

\(^2\)Tarski’s definition of quasi-increasing functions, reproduced in Section 5, is given in purely lattice-theoretic terms for functions mapping chains into chains. The simpler version for real numbers is formalized in Section 2.

\(^3\)A quasi-decreasing function is defined by the dual property of not possessing any upward jump discontinuities. See Section 2 for a more detailed explanation and illustration of Tarski’s intersection point theorem.

them.

It is instructive to provide some basic insight into the restrictiveness of the needed assumptions for
this new existence result by comparing them to those underlying supermodular games as a benchmark.
The comparison yields a mixed outcome. On the one hand, for one player, the present framework requires
continuity in addition to upward monotonicity of the reaction curve. On the other hand, for the other
player, the requirement of upward monotonicity has been relaxed to that of just quasi-monotonicity.
Amending the standard terms to refer to these properties in the economics literature in a suitable way,
one may designate such games as being characterized by continuous strategic complementarity for one
player and limited strategic substitutability for the other player, or what we may call *strategic quasi-
complementarity*. While the former property may be seen as combining the requirements from each of
two known approaches to existence of PSNE so far, i.e., continuity and monotonicity, no such connection
holds for the property of quasi-increasing reaction curves. Recall that the latter rules out only downward
jump discontinuities.

Despite the partial link with the topological approach just alluded to, it is important to stress at the
outset that the present approach is predominantly lattice-theoretic, as will become quite clear in the last
section of this paper. One consequence of this fact is that the result admits an order dual for games
where the reaction function of one player is continuous and decreasing and that of the other player is
quasi-decreasing. While the two dual results are mathematically equivalent, they tend to apply to quite
different classes of economic models (we shall have more to say on this point below).

In delineating the proper scope for the new result at hand, it is important to point out two basic
shortcomings. The first is that the players’ strategy spaces must be a chain or a totally ordered set, a
limitation that stems directly from the use of Tarski’s intersection point theorem. Indeed, the latter result
does not generalize to partially ordered sets. The second drawback is that this basic existence result does
not extend at the same level of generality to games with more than two players. On the other hand,
we do propose multi-player extensions, but at the expense of some further assumptions. In one version,
players 3 to \( n \) are added into the basic two-player setting in such a way that the new players’ payoffs
satisfy the usual requirements for a supermodular game and the payoffs of the two “original players”
satisfy suitable complementarity conditions that make their reaction curves shift up with the actions of
the new players, in addition to the new assumptions discussed above. Thus by combining the logic of
the two-player existence result with some basic comparative statics results (that we also prove as part
of equilibrium characterization in the two-player case), we are able to carry the basic existence result
to \( n \)-player games. However, this extension imposes quite some restrictions on the composition of the
players, such as partial symmetry.

While the existence conclusions described thus far constitute the main goal of the paper, the underlying analysis requires a number of ancillary results that are of independent interest. Some of these are crucially needed as building blocks to construct the basic machinery for the existence result, while others are useful supplementary results. The first of the building blocks is to derive some simple lemmas that capture the essential features of quasi-monotonic functions, provide a basic calculus for useful operations involving them, as well as sufficient conditions that isolate a useful subclass of quasi-monotonic functions (the so-called upper or lower Lipschitz functions). The second block consists in the elaboration of sufficient conditions on a parametric optimization problem to yield a quasi-monotonic argmax correspondence. To this end, we follow some existing work (in particular Granot and Veinott, 1985 and Curtat, 1996) and introduce parameter-dependent changes of variable that allow the desired conclusion to obtain in much the same way as the usual conclusion of monotonicity of argmax’s.

In line with the theory of supermodular games, the basic existence results under consideration do not address uniqueness of PSNE in any way. Instead, the equilibrium set is shown to constitute a chain under great generality (this is akin to the result that the PSNE set for supermodular games is a complete lattice). Nonetheless, some key supplementary results provide novel sufficient conditions for the uniqueness of PSNE. While the underlying conditions are all related in one form or another to a contraction property in the reaction curves, they require this basic property to hold only in a local sense. As such, the results are more reminiscent of the uniqueness results in equilibrium theory that are based on degree theory (see e.g., Dierker, 1972). In addition to applying to some of settings covered by the existence results given here, these uniqueness results could also apply more broadly to other classes of games. In particular, the uniqueness results apply to symmetric PSNE of symmetric games, for which we also state the basic underlying existence result that forms the general version of those that have appeared in specific oligopoly contexts (e.g., Roberts and Sonnenschein, 1976).

Last but certainly not least, as with any advance in abstract theory, one needs to address a crucial test: How broad is the scope of applicability of the novel results? A related subquestion is, how accessible is the overall tool kit developed to facilitate the use of the new results here? In order to make a compelling case that both questions can be answered along very positive lines, we provide several detailed examples of well known economic models for which new existence results are obtained via the direct use of the examples of the paper. Furthermore, as we provide all the concomitant details of the various steps needed to apply the results for the different models, the reader can easily appreciate the practical value of the basic results of this paper. While a large variety of applications may be given, it suffices to develop the
following selection of well established economic models: two standard model of Cournot and Bertrand competition with differentiated products, a model of non-cooperative care provision from the law and economics literature, and a widely used model of non-cooperative pollution abatement. In addition, the Bertrand model is also used as a vehicle to illustrate some of the ancillary results of the paper.

The organization of the paper reflects an unusual choice in the order of exposition. We begin in Section 2 with a full exposition of the definitions of the new notions and a derivation of the basic results in the special case of (scalar) Euclidean action sets. In Section 3, the new results are stated in the form of existence results for PSNE in games, along with the associated uniqueness results. Section 4 contains a detailed discussion of the economic applications of the new theory. It is thus only in Section 5 that the full lattice-theoretic treatment of the new existence theory are presented, along with all the proofs, including the ones omitted in previous sections (to avoid repetition). We felt that this type of exposition facilitates access to the new results by potential users who may not be sufficiently interested in the basic theory to invest in the basics of order theory. This will come at the expense of having to refer readers forward for some proofs that are only given in full generality in the last section of the paper.

2 Quasi-monotone functions on $\mathbb{R}$

In the framework of real parameter and decision spaces, this section lays out all the fundamental notions and basic results that are needed as preliminaries for our new approach to the existence of pure-strategy Nash equilibrium (henceforth PSNE) for games with scalar real action spaces. The present theory is based on the properties of quasi-increasing and quasi-decreasing functions, introduced by Tarski (1955) for general completely ordered lattices (or chains). A more general, order-theoretic treatment is covered in section 5.\footnote{The motivation behind this separation is accessibility. As our results are readily applicable to many basic models of micro-economic theory, we wish to present the basic results in a setting that most economists are familiar with.} We begin in Section 2.1 with the basic definitions and properties of quasi-monotone functions in (one-dimensional) Euclidean space along with some basic practical tests for this property. We provide practical sufficient conditions for quasi-increasingness in section 2.2, before moving to the analysis of parametric optimization problems that yield quasi-monotonic functions as optimal solutions in Section 2.3. Then Section 2.4 describes our fixed point results, which will be used in section 3 for equilibrium existence in games. Proofs of some of these results are provided only in the (more general) order-theoretic treatment in section 5.
2.1 Definition and basic properties

In the same article that contains his well known fixed point theorem for increasing maps on a complete lattice, Tarski (1955) also proved an intersection point theorem (his Theorem 3). This subsection presents this latter, much less known, result for the special case of real-valued functions on a real domain. The main new concepts needed are the following.

**Definition 1** Let $X, Y \subset \mathbb{R}$. A function $f : X \to Y$ is quasi-increasing if for every $x \in X$,

\[
\limsup_{y \uparrow x} f(y) \leq f(x) \leq \liminf_{y \downarrow x} f(y).
\]

(1)

$f$ is quasi-decreasing if if for every $x \in X$,

\[
\liminf_{y \uparrow x} f(y) \geq f(x) \geq \limsup_{y \downarrow x} f(y).
\]

(2)

A function is quasi-monotone if it is either quasi-increasing or quasi-decreasing.

Fig. 2 below illustrates the concepts of quasi-increasing and quasi-decreasing functions.

![Image of functions](image)

Fig. 2: $f$ is quasi-increasing; $g$ is quasi-decreasing.

From the definition, the following facts, formally stated and proved in section 5, should be intuitive:

1. If a function is increasing, then it is quasi-increasing. Analogously, if a function is decreasing, then it is quasi-decreasing.\(^6\)

2. A function is continuous if and only if it is quasi-increasing and quasi-decreasing.

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\(^6\)Throughout this paper, we call "increasing" any function that is weakly increasing (i.e., nondecreasing). We use "strictly increasing" for the strict version of this concepts. The same applies to decreasing functions.
3. Both compositions $f \circ g$ and $g \circ f$ are quasi-increasing if $f$ is quasi-increasing and $g$ is continuous and increasing. This result may fail if $g$ is just increasing. Analogously, $f \circ g$ and $g \circ f$ are both quasi-decreasing if $f$ is quasi-decreasing and $g$ is continuous and decreasing.

Tarski (1955) proved a theorem whose real-valued version reads as follows.

**Theorem 2 (Tarski’s Intersection Point Theorem)** If $f : [a, b] \to \mathbb{R}$ is quasi-increasing, $g : [a, b] \to \mathbb{R}$ is quasi-decreasing, $f(a) \geq g(a)$ and $f(b) \leq g(b)$, then the set $\{ x \in [a, b] : f(x) = g(x) \}$ is non-empty, and has as largest element $\vee \{ x \in [a, b] : f(x) \geq g(x) \}$ and as smallest element $\wedge \{ x \in [a, b] : f(x) \leq g(x) \}$.

Theorem 2 is graphically illustrated in Fig. 3 below. The conditions $f(a) \geq g(a)$ and $f(b) \leq g(b)$ are indispensable. Fig. 2 above illustrates a case where these conditions fail and there is no intersection point.

![Fig. 3: $f$ is quasi-increasing; $g$ is quasi-decreasing and they intersect in a nonempty set.](image)

Since Theorem 2 pertains to two functions having the same domains and the same ranges, it is more aptly called an intersection point theorem (between two curves). We shall refer to it as such, motivated also by the need to distinguish it from the well known fixed point theorem by Tarski (1955).

Unaware of Tarski’s intersection point theorem, Milgrom and Roberts (1994) use (1) to define quasi-increasing functions, which they referred to as “continuous but for upward jumps”. Indeed, this terminology is quite revealing since one crucial implication of (1) is that jump discontinuities in quasi-increasing functions must be upward (likewise, jumps in quasi-decreasing functions must be downward).

Milgrom and Roberts (1994) proved a fixed-point result for quasi-increasing self maps on a compact interval, which can be obtained as a special case of Theorem 2 by taking $f$ to be the mapping of interest

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7We shall prove in section 5 that for functions on the reals, (1) is equivalent to the more general definition used by Tarski on abstract ordered spaces.
and $g$ to be the $45^\circ$ line, in which case the extra conditions $f(a) \geq g(a)$ and $f(b) \leq g(b)$ are automatically satisfied (see Corollary 10). In fact, the latter result has a remarkable history in the economics literature in that special cases was independently discovered by McManus (1964) and Roberts and Sonnenschein (1976). These two studies used this fact as an intermediate result with the principal aim of establishing existence of a symmetric equilibrium for symmetric Cournot oligopoly. In their result, the key property of the underlying reaction curve (of any one firm) in symmetric Cournot oligopoly with convex costs is that all its slopes are bounded below by $-1$, which is a sufficient condition for a function to be quasi-increasing. For a generalization to the case of symmetric firms with non-convex costs, using Topkis’s monotonicity result for the first time in this literature, see Amir and Lambson (2000) and Amir (1996).

We shall derive useful results for both quasi-increasing and quasi-decreasing functions. However, since properties related to quasi-decreasing functions can be obtained directly from analogous ones for quasi-increasing functions using standard duality arguments (about which we give more details in section 5), we shall limit most of our discussion to quasi-increasing functions.

While the property of quasi-monotonicity might at first sight appear quite esoteric as far as its relevance to economic modeling is concerned, we shall derive functional and convenient sufficient conditions for quasi-increasingness that arise in quite natural ways in economics. The next subsection discusses some of these conditions.

### 2.2 On some subclasses of quasi-monotone functions

With the exception of the key implication of ruling out downward jumps, the general definition of quasi-increasing imposes hardly any useful structure on functions that would make them amenable to practical analysis from the perspective of economic applications. For instance, since every continuous function is both quasi-increasing and quasi-decreasing, quasi-monotonic functions may fail to possess left and right limits at points of their domains, or to possess any smoothness properties. In this section, we derive sufficient conditions for quasi-monotonicity that impart crucial structure to the associated subclass of functions, akin to that enjoyed by monotonic functions. Importantly, these sufficient conditions correspond precisely to properties that are naturally satisfied when quasi-monotonic functions arise as argmax’s of parametric optimization problems whose objective functions satisfy some quasi-complementarity conditions to be identified below.

An important sub-class of quasi-increasing functions that arise naturally in economics is the class of lower-Liptschitz functions, defined as follows. A function $f : X \to \mathbb{R}$ is $K$-lower-Liptschitz if for some
$K \in \mathbb{R}$, we have $f(x) - f(y) \geq K(x - y)$ for all $x, y \in X$ such that $x \geq y$.\(^8\) A function $f : X \to \mathbb{R}$ is $K$-upper-Liptschitz if $-f$ is $K$-lower-Liptschitz.

**Lemma 3** Let $X \subset \mathbb{R}$ and assume that a function $f : X \to \mathbb{R}$ is $K$-lower-Liptschitz (resp. $K$-upper-Liptschitz). Then

(a) $f$ is quasi-increasing (resp. quasi-decreasing), and

(b) $f$ is a function of bounded variation.

**Proof.** (a) The property of lower-Liptschitz can be rewritten as $x \geq y \implies f(x) - Kx \geq f(y) - Ky$, that is, the function $\tilde{f}(x) = f(x) - Kx$ is increasing. Therefore, it is quasi-increasing by Lemma 29 (in section 5). If we add to it the function $x \mapsto Kx$, which is continuous and therefore also quasi-increasing, the sum $x \mapsto f(x)$ is quasi-increasing by Lemma 32.

(b) From part (a), we have $f(x) = \tilde{f}(x) + Kx$. Hence, when $K \geq 0$, $f$ is increasing and when $K < 0$, $f$ is the difference between two increasing functions. Thus in either case, $f$ is a function of bounded variation.

The proof for the $K$-upper-Liptschitz case is analogous.\(^9\)

A useful consequence of this result is that lower-Liptschitz functions inherit all the useful properties of functions of bounded variation, such as the existence of a left and right limit at every point of their domain and differentiability almost everywhere.

The following lemma characterizes another subclass of quasi-monotone functions that will also prove useful for subsequent results.

**Lemma 4** Let $X, Y \subset \mathbb{R}$, $r : Y \to X$ and $\beta : X \times Y \to \mathbb{R}$ be two functions, and define $f : Y \to \mathbb{R}$ as $f(y) \equiv \beta(r(y), y)$.

(a) If $r$ is increasing and $\beta$ is increasing in its first argument and jointly continuous, then $f$ is quasi-increasing.

(b) If in addition to the assumptions of (a), $\beta$ is $K$-lower-Liptschitz in $y$ for each $x$, then $f$ is $K$-lower-Liptschitz.

(c) If in addition to the assumptions of (a), $X, Y$ are compact intervals and $\beta$ is continuously differentiable in $y$ for each $x$, then $f$ is $K$-lower-Liptschitz.

\(^8\)Recall that $f$ is $K$-Liptschitz if for some $K \geq 0$, we have $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in X$.

\(^9\)Note that if $K \geq 0$, one can directly observe that $f$ will be increasing and, therefore, trivially quasi-increasing. This lemma is useful, therefore, when $K < 0$. 

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Proof. (a) Let \( \{y_n\}_{n \in \mathbb{N}} \) be such that \( y_n \uparrow x \) and \( f(y_n) \to \limsup_{y \uparrow x} f(y) \). Since \( y_n \uparrow x \) and \( r \) is increasing, 
\[ r(y_1) \leq r(y_n) \leq r(x). \]
Since \( r \) is increasing and \( \{y_n\}_{n \in \mathbb{N}} \) is an increasing sequence, \( \{r(y_n)\}_{n \in \mathbb{N}} \) converges, so that \( r(y_n) \to \bar{r} \leq r(x) \). Since \( \beta \) is continuous, 
\[ f(y_n) = \beta(r(y_n), y_n) \to \beta(\bar{r}, x) \]
and \( \beta \) is increasing in its first coordinate, 
\[ \beta(\bar{r}, x) \leq \beta(r(x), x) = f(x), \]
that is, \( \limsup_{y \uparrow x} f(y) \leq f(x) \).

Taking \( \{y_n\}_{n \in \mathbb{N}} \) to be such that \( y_n \downarrow x \) and \( f(y_n) \to \liminf_{y \downarrow x} f(y) \) and repeating the same arguments, we conclude that \( \liminf_{y \downarrow x} f(y) \geq f(x) \). Thus, \( f \) is quasi-increasing.

(b) For any \( y' > y \), since \( \beta \) is increasing in its first argument and \( r \) is increasing, we have
\[
\frac{f(y') - f(y)}{y' - y} = \frac{\beta(r(y'), y') - \beta(r(y), y)}{y' - y} \geq \frac{\beta(r(y), y') - \beta(r(y), y)}{y' - y} \geq K.
\]
Hence \( f \) is \( K \)-lower-Lipschitz.

(c) Since \( \beta \) is continuously differentiable in \( s \) for each \( x \), and \( Y \) and \( X \) are compact, \( \partial \beta / \partial y \) is uniformly bounded. Hence \( \beta \) is lower-Lipschitz in \( y \) for each \( x \). The result follows from part (b).

One noteworthy aspect of the above result is that the function \( \beta \) is not required to be monotonic in its second argument \( y \). In general, \( f \) as defined in this Lemma may well fail to be increasing.

We are now ready for an investigation of the emergence of the aforementioned subclasses of quasi-monotone functions as solutions to parametric optimization problems.

### 2.3 Sufficient conditions for quasi-increasing argmax

In what follows, we shall be primarily interested in quasi-monotone functions that arise as selections of players’ reaction correspondences in a game. To that end, we must therefore investigate how such functions arise as solutions to parametric optimization problems. Thus we shall consider a selection

\[ r : Y \to X \]

of the correspondence \( R : Y \rightrightarrows X \), with \( X, Y \subset \mathbb{R} \) defined by:

\[ r(y) \in R(y) \equiv \arg\max_{x \in X} M(x, y), \tag{3} \]

for some objective function \( M : X \times Y \to \mathbb{R} \). We wish to provide conditions on \( M \) such that \( r \) is quasi-increasing in the parameter \( y \).

It is useful first to recall that (at least one of the selections) \( r \) will be increasing if \( M \) satisfies a single-crossing condition with respect to \( (x; y) \), i.e., if for any \( (x'; y') \succeq (x; y) \), we have\(^{10}\)

\[ M(x', y') \succeq (>)M(x, y) \implies M(x', y') \succeq (>)M(x, y'). \tag{4} \]

\(^{10}\)As it is well known, this condition does not imply that all selections are increasing, but that at least the maximal and minimal selections are. Throughout the paper, while we use the notions of increasing differences and single-crossing as sufficient conditions on an objective function to guarantee an increasing argmax, it is clear that we could as well use the interval dominance order (Quah and Strulovici, 2009) for the same purpose. (In fact, the latter is the most general condition...
For an argmax to be quasi-increasing in the parameter \( y \), we need instead the notion of *shifted single-crossing*, which we introduce next via a judicious (non-separable) change of decision variable.

Let there be given a continuous function \( \alpha : X \times Y \to Z \) that is strictly increasing in \( x \) for fixed \( s \) and increasing in \( y \) for fixed \( x \). Here, \( Z \) is the range of \( \alpha \) and it is given by \( Z \equiv [\alpha(x, y^0), \alpha(x, y^s)] \), where \( x = \inf X, \alpha = \sup X, y = \inf Y \), and \( y = \sup Y \).

If one defines a new variable \( z = \alpha(x, y) \), then since \( \alpha \) is continuous and strictly increasing in \( x \), there exists a (parametrized inverse) function \( \beta : Z \times Y \to X \) such that \( x = \beta(z, y) \). In other words, \( \alpha \) has an inverse in its first argument, that is, there exists a function \( \beta \) satisfying \( \alpha(\beta(z, y), y) = z \) and \( \beta(\alpha(x, y), y) = x \). It is not difficult to see that \( \beta \) must be increasing in its first argument. Further, assume that \( \beta \) is continuous.

In view of the monotonicity properties of \( \alpha \), and of the fact that \( X = [\underline{x}, \bar{x}] \) is independent of \( y \), the set of feasible values of \( z \) for a fixed value of \( y \) is \( Z(y) \equiv [\alpha(\underline{x}, y), \alpha(\bar{x}, y)] \), which is clearly ascending in \( y \) (since \( \alpha \) is increasing in \( y \)). We have the following:

**Definition 5** Let \( \alpha : X \times Y \to Z \) be continuous, strictly increasing in \( x \) and increasing in \( y \), and \( \beta(z, y) \) be the continuous inverse of \( \alpha \) with respect to the first variable. A function \( M : X \times Y \to \mathbb{R} \) satisfies a \( \beta \)-shifted single-crossing property with respect to \( (x; y) \) if \( \tilde{M}(z, y) \equiv M(\beta(z, y), y) \) has the single-crossing property (4) with respect to \( (z; y) \in Z \times Y \), that is, for any \((z'; y') \geq (z; y)\) we have

\[
\tilde{M}(z', y) \geq (>\tilde{M}(z, y) \implies \tilde{M}(z', y') \geq (>\tilde{M}(z, y').
\]

(5)

Moreover, \( M \) satisfies a \( \beta \)-shifted strict single crossing property with respect to \( (x; y) \) if \( \tilde{M}(z, y) \equiv M(\beta(z, y), y) \) has the strict single-crossing property with respect to \( (z; y) \in Z \times Y \), that is, for any \((z'; y') \geq (z; y), (z'; y') \neq (z; y)\), we have

\[
\tilde{M}(z', y) \geq \tilde{M}(z, y) \implies \tilde{M}(z', y') > \tilde{M}(z, y').
\]

(6)

Naturally, a sufficient (but in general non-necessary) condition for the \( \beta \)-shifted single-crossing property is what one would naturally call \( \beta \)-shifted increasing differences, defined by \( \tilde{M}(z, y) \equiv M(\beta(z, y), y) \) having increasing differences in \( (z, y) \), when \( \beta \) is continuous in \( (z, y) \) and strictly increasing in \( z \). If both of the three listed here when the action and parameter are real variables). In addition, since increasing differences is the only one that survives addition without any restrictions (see Quah and Strulovici, 2012 for more on this point), we shall make extensive use of this property in the applications section.
$M$ and $\beta$ are $C^2$, this is equivalent to\textsuperscript{11,12}

$$\tilde{M}_{12}(z, y) = [M_{11}(\beta(z, y), y)\beta(z, y) + M_{12}(\beta(z, y), y)]\beta_1(z, y) + M_1(\beta(z, y), y)\beta_{21}(z, y) \geq 0$$

Defining the set-valued functions $R : Y \rightrightarrows X$ and $Z^* : Y \rightrightarrows Z$ by

$$R(y) \equiv \arg\max_{x \in X} M(x, y) \quad (7)$$

and

$$Z^*(y) \equiv \arg\max_{z \in Z(y)} M(\beta(z, y), y) \quad (8)$$

we have a one-to-one mapping between selections $r$ of $R$ and selections $z^*$ of $Z^*$, that is, given $z^*(\cdot) \in Z^*(\cdot)$, we have $r(y) = \beta(z^*(y), y) \in R(y)$ and, conversely, given $r(\cdot) \in R(\cdot)$, we have $z^*(y) = \alpha(r(y), y) \in Z^*(y)$.

**Proposition 6** Assume that $M : X \times Y \to \mathbb{R}$ satisfies a $\beta$-shifted single-crossing property with respect to $(x; y)$, with $\beta$ jointly continuous and strictly increasing in its first argument. Assume that $R(y) \neq \emptyset$ for all $y \in Y$. Then, the following holds:

(a) The maximal and minimal selections of $R$, $\overline{r}$ and $\underline{r}$, are both quasi-increasing in $y$.

(b) If, in addition, $\beta$ is continuously differentiable, then $\overline{r}$ and $\underline{r}$ are both $K$-lower Lipschitz in $y$ for some $K$.

(c) If $M$ satisfies the $\beta$-shifted strict single-crossing property, then all selections of $R$ are quasi-increasing in $y$.

**Proof.** (a) Since $M(\beta(z, y), y)$ has the single-crossing property with respect to $(z; y)$, and the feasible set $Z(y) \equiv [\alpha(\overline{x}, y), \alpha(\underline{x}, y)]$ is ascending in $y$ (since $\alpha$ is increasing in $y$), we know from the Topkis-Milgrom-Shannon monotonicity theorem that the extremal selections of $S(y)$, $\overline{s}^*$ and $\underline{s}^*$ are increasing functions of $y$. Since $\beta$ is increasing in its first coordinate and continuous by assumption, the assumptions of Lemma 4(a) are satisfied by $\beta$ and $\overline{s}^*$ and $\underline{s}^*$ and we conclude that $\overline{r}(y) = \beta(\overline{s}^*(y), y)$ and $\underline{r}(y) = \beta(\underline{s}^*(y), y)$ are both quasi-increasing in $y$.

(b) This follows directly from Lemma 4(c).

(c) It is well known that the strict single-crossing property implies that all selections are monotonic. Thus, the proof of (c) is similar to the proof of (a) and therefore omitted. ■

\textsuperscript{11}In this paper, we use subscripts in functions for partial derivatives.

\textsuperscript{12}This equivalence is a well-known result about increasing differences.
The following economic application illustrates the change of variable used above in the context of a familiar model. In particular, this provides some guidance as to how a suitable choice of the function $\alpha$ is arrived at. Several more relevant examples are given in the applications section.

**Example 7** Consider a Bertrand duopoly with differentiated substitute products wherein firms 1 and 2 choose prices $x, y$ (in some given price set $[0, \bar{p}]$) and face a demand system $(D, \hat{D})$ for their products, respectively. Assume linear cost functions with marginal costs $c$ and $\hat{c}$. In what follows, we focus only on firm 1 (say). Its profit function is

$$F(x, y) = (x - c)D(x, y)$$  \hspace{1cm} (9)

Its demand $D$ is continuously differentiable and satisfies $D_1 < 0$ (the law of demand) and $D_2 > 0$ (products are substitutes in demand).

Our aim here is to show that Firm 1’s reaction correspondence $f(y) = \arg\max_{x \in [0, \bar{p}]} (x - c)D(x, y)$ is quasi-decreasing in $y$, under the assumption that

$$D_2D_1^2 - D[D_1D_{12} - D_2D_{11}] > 0 \text{ for all } (x, y).$$  \hspace{1cm} (10)

To see this, let firm 1 respond by choosing its own output $z$ instead of its price $x$, i.e. let $z = D(x, y)$. Since $D_1 < 0$, parametric inversion will yield a function $h$ such that

$$x = h(z, y) \iff z = D(x, y).$$

As is easy to check, the partials of $h$ and $D$ are then related by

$$h_1 = \frac{1}{D_1}, h_2 = -\frac{D_2}{D_1}, \text{ and } h_{12} = \frac{1}{D_1^3}(D_2D_{11} - D_1D_{12}).$$  \hspace{1cm} (11)

Given $y$, the best response problem of firm 1 may be equivalently viewed as

$$\max \left\{ \tilde{F}(z, y) \triangleq z[h(z, y) - c] : z \in [D(\bar{p}, y), D(0, y)] \right\}. $$  \hspace{1cm} (12)

The first step is to derive conditions under which the argmax $z^*(y)$ is increasing in $y$. Since the feasible set $[D(\bar{p}, y), D(0, y)]$ is ascending in $y$ (as $D_2 > 0$), by Topkis’s Theorem, all the selections of $z^*(y)$ are increasing in $y$ if $\tilde{F}$ has strictly increasing differences in $(z, y)$. For this, it suffices that $\tilde{F}_{12}(z, y) = h_2(z, y) + zh_{12}(z, y) > 0$, for all $(z, y)$. Using (11), the latter is equivalent to

$$-\frac{D_2}{D_1} + D_1 \frac{1}{D_1^3}(D_1D_{12} - D_2D_{11}) > 0,$$

which is the same as (10).
Since the argmax’s of (9) and (12) are related by \( z^*(y) = D(f(y), y) \) or \( f(y) = h(z^*(y), y) \), and \( h \) is decreasing in its first argument, \( f(y) \) is quasi-decreasing in \( y \) by Lemma 4.

The interpretation is that when firm 2 raises its price \( y \), firm 1 may optimally react by increasing or decreasing its price \( x \), but, in the latter case, never by so much that firm 1’s output would end up increasing. In other words, while strategic complementarity in pricing decisions is allowed to any extent, a limited form of strategic substitutability can also be accommodated by condition (10). More precisely, condition (10) accommodates what we named strategic quasi-complementarity.

We shall return to this particular economic application below to illustrate other results from the present paper.

The following remark introduces a particular change of variable with a separable structure that will prove useful in some of the economic applications presented in section 4.

**Remark 8** For many problems, a simple change of variable is as follows. Let \( z = \alpha(x, y) = x + k(y) \) for some strictly increasing function \( k \). Then for \( Z^*(\cdot) \) to be increasing in \( y \), it is sufficient that \( \tilde{M}(z, y) \equiv M(z - k(y), y) \) has the single-crossing property with respect to \((z; y)\). A sufficient condition that is easy to check is that \( \tilde{M}(z, y) \) has increasing differences with respect to \((z, y)\). When \( \tilde{M} \) and \( k \) are both twice continuously differentiable, this is equivalent to

\[
\tilde{M}_{12}(z, y) = -M_{11}(z - k(y), y)k'(y) + M_{12}(z - k(y), y) \geq 0.
\] (13)

The idea of a change of variable to perform comparative statics of a non-monotonic sort has already appeared repeatedly in the literature on supermodular optimization and games. In a setting with scalar decision and parameter variables, Granot and Veinott (1985) define the notion of doubly-increasing differences for an objective function as being the conjunction of the properties of increasing differences and of \( \beta \)-shifted increasing differences with an additively separable function (as in the previous remark). Curtat (1996) extends their result to multi-dimensional decision and parameter sets. Similar ideas have also appeared in the context of oligopoly applications with quasi-increasing reaction curves, e.g., in Amir (1996) and Amir and Lambson (2000), where the relevant change of variable is \( z = x + y \).

The modern theory of monotone comparative statics is often qualified as being of a qualitative nature. Indeed, it aims to predict the directions of change of endogenous variables in response to changes in exogenous parameters, but usually not the associated magnitudes of these changes. In contrast, the conclusion that an argmax is quasi-decreasing in a parameter can be viewed as a comparative statics result of a non-monotonic and quantitative sort. As an illustration, consider the Bertrand duopoly
example above. The derived conclusion may be re-stated as \( f'(y) \leq -\frac{D_2(f(y),y)}{D_1(f(y),y)} \), for almost all \( y \) (w.r.t. Lebesgue measure), which provides a lower bound on the rate of decrease of \( f(y) \) as \( y \) changes. If one adds the reasonable further assumption on demand that \( \frac{D_2(f(y),y)}{|D_1(f(y),y)|} < 1 \) for all \( y \), then one can conclude the firm 1 never lowers its price by as much as the increase in its rival’s price, a conclusion of a clearly quantitative nature.

Observing that a similar (dual) argument can handle the derivation of upper bounds on the rate of the change of argmax’s, as will be illustrated in the last section below, this method can easily be used to provide sufficient conditions on the players’ reaction curves in a game to constitute contraction mappings, thus ensuring uniqueness of PSNE. For instance, Amir (1996) uses such arguments to establish uniqueness of Cournot equilibrium.

### 2.4 Our fixed point results

This subsection states the simplest form of our basic fixed point result in the Euclidean case, and discusses its direct connection to Tarski’s intersection point Theorem. To fix ideas, the functions may be thought of as selections from players’ best response correspondences in a strategic game. In the next section, we provide sufficient conditions directly on the payoff functions of a game that yield the following properties on players’ best responses.

**Theorem 9 (Fixed Point Theorem)** Let \( f : [a,b] \to [c,d] \) be quasi-increasing, \( g : [c,d] \to [a,b] \) be continuous and increasing, and define \( h(x,y) = (g(y),f(x)) \). Then there exists \((\bar{x},\bar{y}) \in [a,b] \times [c,d]\) such that \( h(\bar{x},\bar{y}) = (\bar{x},\bar{y}) \).

A more elaborate version of Theorem 9, including monotone comparative statics results, is stated and proved in section 5—see Theorem 35. The idea of the proof is, however, straightforward. We consider the function \( g \circ f : [a,b] \to [a,b] \). Since \( g \) is continuous and increasing and \( f \) is quasi-increasing, \( g \circ f \) is quasi-increasing—see fact 3 in the list of observations in section 2.1. Since the identity function is continuous and hence quasi-decreasing, by Tarski’s Intersection Point Theorem 2, the set \( \{ x \in [a,b] : g \circ f(x) = x \} \) is non-empty. If \( \bar{x} \in \{ x \in [a,b] : g \circ f(x) = x \} \) and we define \( \bar{y} = f(\bar{x}) \), then \((\bar{x},\bar{y})\) is a fixed point of \( h \).

This proof sketch makes it apparent that the key idea here is to translate Tarski’s result from an intersection point result to a fixed point theorem for an important subclass of bivariate maps, namely those that are formed by the conjunction of two one-dimensional functions, as is the best response mapping.

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This is justified since \( f \) is a function of bounded variation (Lemma 3).
for a two-player game. Put differently, the idea is to translate an intersection point result between two functions with the same domains and the same ranges to an intersection point result between two functions with interchanged domains and ranges, in line with the usual graphical depiction of intersecting reaction curves in economics as a simple way of representing PSNE.

A well-known interesting and immediate corollary is the following.

**Corollary 10**

(a) Let \( f : [a, b] \rightarrow [a, b] \) be quasi-increasing. Then \( f \) has a fixed-point.

(b) Let \( f : [a, b] \rightarrow [a, b] \) be such that \( \frac{f(x') - f(x)}{x' - x} \geq -k \), for some \( k \geq 0 \) and any \( x', x \in [a, b], x \neq x' \). Then \( f \) has a fixed-point.

**Proof.** (a) Simply apply Theorem 9 to \( f \) and \( g(x) = x \) (the identity).

(b) The given slope condition means that \( f \) is \( k \)-lower-Liptschitz, and hence quasi-increasing (Lemma 3). Then use part (a).

As noted earlier, the result in part (b) with \( k = 1 \) was proved and used by MacManus (1964) and Roberts and Sonnenschein (1976) to establish existence of symmetric Cournot equilibrium in symmetric Cournot oligopoly with convex costs.\(^\text{14}\) The latter property alone ensures that each firm’s reaction curve has all its slopes above \(-1\) (though it may be discontinuous), so that each firm always reacts to rivals’ output in a way that increases total output. Existence then follows from this property alone, even though the game is neither of strategic substitutes nor of strategic complements. Amir and Lambson (2000) extends this result to oligopolies with some level of non-convex costs. Milgrom and Roberts (1994) prove part (a) independently and use it to conduct comparative statics of equilibrium points.

Besides the stronger version of Theorem 9 (Theorem 35), the reader will find in section 5 other fixed point theorems, that we summarize below. Theorem 37 is the dual of Theorem 35, which is the more complete version of Theorem 9. It implies that if \( f \) is quasi-decreasing and \( g \) is continuous and decreasing, then \( h(x, y) = (g(y), f(x)) \) has a fixed point.

We nonetheless state the Euclidean version now. While this is simply the order-dual of Theorem 9, the economic models to which it might apply can be substantially different from those associated with Theorem 9, as will be confirmed by some of the applications later on.

**Theorem 11** Let \( f : [a, b] \rightarrow [c, d] \) be quasi-decreasing, \( g : [c, d] \rightarrow [a, b] \) be continuous and decreasing, and define \( h(x, y) = (g(y), f(x)) \). Then there exists \((\bar{x}, \bar{y}) \in [a, b] \times [c, d]\) such that \( h(\bar{x}, \bar{y}) = (\bar{x}, \bar{y}) \).

\(^\text{14}\) The proof of this result by Mac Manus is not fully rigorous.
The fixed point theorems presented in this section will be translated into equilibrium existence results for games in section 3. The next subsection deals with the issue of uniqueness of fixed points of quasi-monotone functions and will be useful to establish uniqueness of PSNE.

2.5 Uniqueness of fixed points

In some applications, beyond the issue of existence, the uniqueness of fixed points is often a highly desirable property. In fact, most studies in applied microeconomics postulate game-theoretic models with a unique PSNE, independently of whether specific functional forms are adopted or not. The standard methods used to establish uniqueness of fixed points or PSNEs typically rely on dominant diagonal conditions on payoff functions or, equivalently, on contraction arguments for best response mappings (Rosen, 1965; Milgrom and Roberts, 1990). Both of these conditions are generally postulated to hold in a global sense. In this section, we present two results that establish the uniqueness of fixed points in the present setting without requiring global contraction arguments. Our first result of this form allows the function to be quasi-decreasing (instead of continuous, or often even smooth) and satisfy a local contraction property along the diagonal, that is, only at candidate fixed points.\footnote{To underscore the novelty of this result, an application to Bertrand competition is presented in Section 4.}

Proposition 12 Let \( f : [a, b] \to [a, b] \) be quasi-decreasing and satisfy the following property: for every \( x \in [a, b] \) such that \( x = f(x) \), we have

\[
\lim_{y \to x} \frac{f(y) - f(x)}{y - x} < 1. \tag{14}
\]

Then \( f \) has at most one fixed point.

Proof. Suppose that \( \bar{x}, \bar{y} \in [a, b] \) are both fixed points of \( f \), with (say) \( \bar{x} < \bar{y} \). Because of (14), there are neighborhoods \( U_1 \) of \( \bar{x} \) and \( U_2 \) of \( \bar{y} \) such that \( y \in U_1 \cap ([a, b] \setminus \{\bar{x}\}) \implies \frac{f(y) - \bar{x}}{y - \bar{x}} < 1 \) and 
\( y \in U_2 \cap ([a, b] \setminus \{\bar{y}\}) \implies \frac{f(y) - \bar{y}}{y - \bar{y}} < 1 \). Then we can pick \( y_1 \in U_1 \cap (\bar{x}, \bar{y}) \) and \( y_2 \in U_2 \cap (\bar{x}, \bar{y}) \), \( y_1 < y_2 \) such that \( \frac{f(y_1) - \bar{x}}{y_1 - \bar{x}} < 1 \) and \( \frac{f(y_2) - \bar{y}}{y_2 - \bar{y}} < 1 \). Hence \( f(y_1) < y_1 \) and \( f(y_2) > y_2 \).

Define the function \( g(y) = f(y) - y \) on \([y_1, y_2]\). Since \( f \) is quasi-decreasing, \( g \) is quasi-decreasing. (See Corollary 31 and Lemma 32.) Moreover, \( g(y_1) < 0 < g(y_2) \). We can apply Tarski’s intersection point theorem (Theorem 34) to the function \( g \) thus defined and the constant function \( c(y) = 0, \forall y \), which is continuous and thus quasi-increasing. Theorem 34 states that the supremum of \( \{ y \in [y_1, y_2] : 0 \geq g(y) \} \) is contained in, and is the supremum of, \( \{ y \in [y_1, y_2] : 0 = g(y) \} \). Denote by \( \hat{y} \in [y_1, y_2] \) this supremum. It
is clear that $\hat{y} < y_2$, since $g(y_2) > 0$ and $\hat{y}$ is the highest fixed point of $f$ in $[y_1, y_2]$. Using again property (14) for $\hat{y}$, we can find $\tilde{y} \in (\hat{y}, y_2)$ such that $f(\tilde{y}) < \hat{y}$. Defining as before $\hat{g} : [\tilde{y}, y_2] \to \mathbb{R}$ by $\hat{g}(y) = f(y) - y$, we see that it is a quasi-decreasing function satisfying the assumptions of Theorem 34. Therefore, there exists $\hat{\tilde{y}} \in (\hat{y}, y_2)$, such that $\hat{g}(\hat{\tilde{y}}) = f(\hat{\tilde{y}}) - \hat{\tilde{y}}$, that is, $\hat{\tilde{y}} \in [y_1, y_2]$ is a fixed point of $f$ and $\hat{\tilde{y}} > \hat{y} > \hat{\tilde{y}}$, which contradicts the fact that $\hat{y}$ is the highest fixed point of $f$ in $[y_1, y_2]$. This contradiction establishes the property.}

**Remark 13** While this Proposition might a priori appear to be directly suitable for use as a uniqueness argument only for symmetric PSNE of symmetric games, one can also use it for two-player asymmetric games by applying it to the composition of the two players’ reaction functions, which maps (say) player 1’s action space to itself. Indeed, it is well known that the set of fixed points of such a composition coincides with the set of PSNEs of the game (Vives, 1990). In this form, all that is needed for a unique PSNE is that the mentioned composition is 1-upper Lipschitz in a neighborhood of any fixed point (as captured by (14)), and not necessarily a global contraction (i.e., a globally 1-upper and 1-lower Lipschitz function).

Relying on first order conditions under smoothness assumptions, another convenient test for the uniqueness of a fixed point that is non-global in character can be given in the form of the following sufficient condition.

**Proposition 14** Let $X = [a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$. Assume that $M : X^n \to \mathbb{R}$ is differentiable in its first coordinate and satisfies the following:

$$x, x' \in X, x' > x \text{ and } M_1(x, x, ..., x) \leq 0 \implies M_1(x', x', ..., x') < 0.$$  \hspace{1cm} (15)

Then a function $r : X \to X$ satisfying $r(x) \in \arg \max_{y \in X} M(y, x, ..., x)$ has at most one fixed point.

**Proof.** The proof rules out multiple interior fixed points, and then multiple corner fixed points. Since $r(x) \in \arg \max_{y \in X} M(y, x, ..., x)$ and $M$ is differentiable in its first variable, we must have $M_1(r(x), x, ..., x) = 0$ if $x$ is an interior point of $X$. Assume that $x$ and $x'$ are interior fixed points of $r$, with $x' > x$. Then, $M_1(x, x, ..., x) = 0$ and $M_1(x', x', ..., x') = 0$, but this contradicts (15). Assume now that the endpoint $a$ is a fixed point of $r$. In this case, we must have $M_1(a, ..., a) \leq 0$. By (15), $M_1(x', ..., x') < 0$ for all $x' > a$, which shows that there is no other fixed point of $r$ above $a$. Similarly, if the endpoint $b$ is a fixed point of $r$, then $M_1(b, ..., b) \geq 0$ and (15) implies that $M_1(x, ..., x) > 0$ for all $x < b$. Therefore, there are no other fixed point of $r$ below $b$. This concludes the proof. \hfill \blacksquare
The above proposition will be used to establish uniqueness of symmetric equilibria in symmetric games later on. A slight adaptation of the results above can yield analogous uniqueness results for asymmetric games.

Interestingly, (15) may be seen as a strict dual single-crossing condition for the partial $M_1(x, ..., x)$, viewed as a function of one variable. As such, it becomes transparent that a sufficient condition for (15) is that $M_1(a, a, ..., a)$ is strictly decreasing in $a$, which (if $M$ is twice continuously differentiable) is in turn implied by

$$M_{11}(a, ..., a) + \sum_{j \neq 1} M_{1j}(a, a, ..., a) < 0.$$ 

The latter condition is of the dominant diagonal type; it says that any row sum of the Hessian matrix of $M$ is negative. Nevertheless, this condition is in general significantly less restrictive than the typical related conditions in the literature, in that it is not required to hold globally, but rather only along the diagonal of the domain $X^n$.

3 Pure-strategy Nash equilibrium in games

This section contains our results about the existence and uniqueness of pure-strategy Nash equilibrium (henceforth PSNE) in games. Its main objective is to translate our fixed points results into conclusions about the existence of PSNE. In the process, the relevant sufficient conditions shall be placed on the primitives of the game.

We begin by describing results for two players games in subsection 3.1. Then $n$-player symmetric games are the object of subsection 3.2, while subsection 3.3 considers $n$-player asymmetric games.

3.1 Two-player games

Consider a two-player strategic game with action spaces $X$ and $Y$ and payoff functions $F, G : X \times Y \to \mathbb{R}$. The following result translates the assumptions of Theorem 9 (or its more general version, Theorem 35 presented in section 5 below) onto sufficient conditions on the primitives of the game.

**Theorem 15** Assume that $X$ and $Y$ are compact intervals in $\mathbb{R}$, $F$ and $G$ are upper semi-continuous in own action, and that:

(a) $F$ satisfies a $\beta$-shifted single-crossing property with respect to $(x; y)$ for some $\beta$ that is continuous and increasing, and
(b) $G$ is strictly quasi-concave in $y$ for each fixed $x$, and satisfies the single-crossing property with respect to $(y; x)$.

Then the set $E$ of PSNE is non-empty.$^{16}$

**Proof.** Due to the upper semi-continuity assumption, the best-reply correspondences have non-empty values. By Proposition 6, the maximal and minimal selections of the best-reply correspondence for the first player, $\bar{x}(\cdot)$ and $\underline{x}(\cdot)$, are quasi-increasing.

From the assumption that $G$ is strictly quasi-concave and upper semi-continuous in $y$, we know that the best-reply correspondence is a single-valued continuous function denoted $\bar{y}(\cdot)$. From the single-crossing property, $\bar{y}(\cdot)$ is increasing in $x$. By Theorem 9, there exists $(x^*, y^*)$ such that $\bar{x}(y^*) = x^*$ and $\bar{y}(x^*) = y^*$, that is, $(x^*, y^*)$ is a PSNE of the game and $E$ is non-empty.$^{17}$

It is instructive to contrast this result with its counterpart from the theory of supermodular games for the present setting. While the latter relies on Tarski’s well known fixed point theorem for increasing maps, the present result is based on a reinterpretation of Tarski’s intersection point theorem as a fixed point theorem for bivariate maps that arise as best-response maps of two-player games. In terms of scope, the two approaches are not nested. On the one hand, for player 1, the present result imposes less structure since his reaction curve is only quasi-decreasing and not necessarily increasing. On the other hand, for player 2, the present result requires continuity of the reaction curve in addition to monotonicity while the latter property is all that is needed for supermodular games. Put differently, the present result relaxes strategic complementarity to strategic quasi-complementarity (i.e., allows a limited form of strategic substitutability) for one player, but imposes continuity as an extra condition on the reaction curve of the other player.

Since the result relies on a mix of continuity and generalized monotonicity conditions, it may be regarded as a synthesis of the two existing methodologies for establishing existence of PSNE in general games: the classical (topological) approach via Brouwer’s or Kakutani’s fixed point theorem (Nash, 1951 and Rosen, 1965) and the algebraic (supermodularity) approach via Tarski’s fixed point theorem (Topkis, 1979).

Theorem 15 admits an order-dual, which is as follows.

**Theorem 16** Assume that $X$ and $Y$ are compact intervals in $\mathbb{R}$, and $F$ and $G$ are upper semi-continuous

$^{16}$It is actually a non-empty complete chain.

$^{17}$The strong claim that $E$ is a non-empty complete chain comes from Theorem 35 in section 5.
in own action. Let $\alpha$ be a continuous function on $X \times Y$ that is strictly decreasing in $x$ and decreasing in $y$, and $\beta(\cdot, y)$ be the inverse of $\alpha$ with respect to the first variable. If

(a) $\bar{F}(z, y) \equiv F(\beta(z, y), y)$ satisfies the dual single-crossing property with respect to $(x; y)$, and
(b) $G$ is strictly quasi-concave in $y$ for each fixed $x$, and satisfies the dual single-crossing property with respect to $(y; x)$.

Then the set of PSNE of this game is non-empty.

**Proof.** If the order on (say) player 2’s action set is reversed, the assumptions of this theorem turn into those of Theorem 15. Hence, the conclusion follows from the latter result. ■

While equivalent to Theorem 15 from a mathematical point of view, in terms of economics applications, this order-dual will apply to models that are quite distinct from those of Theorem 15. In fact, this version will be directly invoked in some of the applications later on.

One can extend Theorem 16 to also deliver uniqueness of PSNE by using the novel insight from Proposition 12.

**Proposition 17** In addition to the assumptions of Theorem 16, assume that the composition of the two reaction curves $f \circ g$ satisfies (14) at all of its fixed points. Then there exists a unique PSNE.

**Proof.** Since the composition $f \circ g$ is quasi-decreasing, the uniqueness conclusion follows directly from Proposition 12, when one takes into account that the set of PSNE coincides with the set of fixed points of the composition $f \circ g$. ■

At the level of generality at which they are stated, Theorem 15 and its order-dual (Theorem 16) are a priori valid only for two-player games. Nevertheless, these results can be extended to $n$-player games upon the inclusion of some additional assumptions on the primitives. This can be done in a number of different ways. The first and more natural extension is to work with symmetric players. This is considered in subsection 3.2 below. Then, subsection 3.3 presents results for asymmetric games.

### 3.2 N-player symmetric games

Consider a $n$-player game, where all players have the same action space $X$, assumed to be a compact interval in $\mathbb{R}$, and the same payoff function $F : X \times X^{n-1} \rightarrow \mathbb{R}$, where the first entry is the player’s own action. With the usual abuse of notation, we can write a joint action vector $x \in X^n$ as $(x_i, x_{-i})$ for any $i \in N$.  

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We now state a basic existence result for symmetric games, special versions of which have surfaced in the economics literature a number of times in specific contexts (e.g., Amir and Lambsomboon, 2000 and Milgrom and Roberts, 1994). This result provides conditions on the payoff function that lead to the (common) reaction correspondence satisfying Corollary 10(a).

**Theorem 18** Assume that

(a) $F$ is upper semi-continuous in own action (or first entry);

(b) for each $a \in X$, $\bar{F}(x,a) \equiv F(x,a,\ldots,a)$ satisfies a $\beta$-shifted single-crossing property with respect to $(x;a)$ for some $\beta : X \times X \to X$ that is continuous and increasing.

Then the set of symmetric PSNE is non-empty.

**Proof.** The assumption that $F$ is upper semi-continuous in its first entry guarantees that the best-reply correspondence is non-empty. By Proposition 6, the maximal best-reply restricted to symmetric actions $\bar{x} : X \to X$ is quasi-increasing. By Corollary 10, the set of fixed points of $\bar{x}$ is a non-empty chain. Since a fixed point of $\bar{x}$ is an equilibrium, the set of equilibria is non-empty.

For the next uniqueness result, existence of PSNE may be guaranteed either by the continuity of the (common) reaction curve or by the fact that it is quasi-increasing.

**Theorem 19** Assume that $F$ is $C^2$ and

(a) for each $a \in X$, $\bar{F}(x,a) \equiv F(x,a,\ldots,a)$ is either strictly quasi-concave in $x$ or satisfies a $\beta$-shifted single-crossing property with respect to $(x;a)$ for some $\beta : X \times X \to X$ that is continuous and increasing.

(b) for each $a \in X$, $\bar{F}$ satisfies $\bar{F}_{11}(a,a) + \bar{F}_{12}(a,a) < 0$.

Then, there exists a unique symmetric PSNE of this game.

**Proof.** The existence of a symmetric PSNE follows from either of the two assumptions in (a), the quasi-concavity in $x$ or the $\beta$-shifted single-crossing property, see Theorem 18.

Since $F$ is $C^2$, we know from (b) that for each $\bar{a}$, $\bar{F}_{11}(x,\bar{a}) + \bar{F}_{12}(x,\bar{a}) < 0$ for all $(x,a)$ in some neighborhood of $(\bar{x},\bar{a})$. Hence, with the change of variable $z = x + a$, $\bar{F}(z-a,a)$ has strongly increasing differences in $(z,a)$, i.e., $\partial \bar{F} / \partial a$ is strictly increasing in $z$; see Amir (1996b) or [Topkis, 1998, p. 79]. Therefore, all the selections of $z^*(a) \equiv \bar{F}(z-a,a)$ are strictly increasing in $a$. In other words, $x^*(a)$ has all its slopes strictly less than 1 in a neighborhood of $\bar{a}$. The uniqueness of symmetric PSNE then follows from Proposition 12.

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18 As before, Corollary 39 allows to conclude that this set is actually a non-empty complete chain.
Again, the main novelty in the underlying argument is that the assumption in part (b) generates a local contraction property for the reaction curve along the diagonal, which is not required to hold in a global sense. Due to the latter point, multiple asymmetric PSNEs are not ruled out.

Before considering an extension to a class of n-player games, we provide a new uniqueness result for symmetric pure-strategy Nash equilibrium for symmetric normal-form games, which is of independent interest for many potential economic applications.

**Theorem 20** Assume that:

(a) $F$ is differentiable in its first variable (own action) and
(b) For any $x', x \in X$ with $x' > x$, $F_1(x, x, ..., x) \leq 0 \implies F_1(x', x', ..., x') < 0$.

Then, there exists at most one symmetric equilibrium of this game.

**Proof.** This follows directly from Proposition 14. ■

Observe here that the given assumptions do not preclude the existence of other PSNEs as long as they are asymmetric.

A sufficient condition for the assumption in Theorem 20(b) is that $\partial_1 F_i(x, ..., x)$ is strictly decreasing in $x$ (see comments after Proposition 14).

### 3.3 N-player asymmetric games

In this section, we explore the extent to which the present PSNE existence result can be extended in a natural way to general $n$-player games. We offer two such results, each of which combines the two-player existence theorem with some basic comparative statics results, and also makes use of well known results on supermodular games.

We begin by laying out a suitable setting and the related notation. Consider a normal-form, $n$-player game ($n \geq 3$), such that player 1 has action set $X$ and utility function $F : X \times Y \times \prod_{j=3}^{n} Z_j \to \mathbb{R}$, player 2 has action set $Y$ and utility function $G : X \times Y \times \prod_{j=3}^{n} Z_j \to \mathbb{R}$ and player $j$ (for $j = 3, ..., n$) has action space $Z_j$ and utility function $H^j : X \times Y \times \prod_{i=3}^{n} Z_i \to \mathbb{R}$. The sets $X, Y, Z_j$, for $j = 3, ..., n$, are compact intervals of real numbers. The following result is a natural extension of Theorem 15 to an $n$-player game.$^{19}$

---

$^{19}$Here the assumption of continuous payoff functions can be relaxed using well known results from the theory of discontinuous games (continuity is useful here in the usual sense when using the maximum theorem). We refrain from doing so to keep the presentation focused on salient features from the new approach at hand.
Theorem 21 Assume that:

(a) $F$, $G$ and $H^j$ for $j = 3, \ldots, n$ are continuous in own action,
(b) for every $y \in Y$ and $z \in \prod_{j=3}^{n} Z_j$, $F(x, y; z)$ satisfies a $\beta$-shifted single-crossing property with respect to $(x; y)$ for some $\beta$ that is continuous and increasing.

Moreover, for each fixed $y$, $F$ satisfies the single-crossing property in $(x; z)$,
(c) $G(x, y; z)$ is strictly quasi-concave in $y$ for each fixed $x$ and $z \in \prod_{j=3}^{n} Z_j$, and satisfies the single-crossing property with respect to $(y; (x, z))$,
(d) for each $j = 3, \ldots, n$, $H^j(x, y; z_j, z_{-j})$ satisfies the single-crossing property with respect to $(z_j; (x, y, z_{-j}))$.

where $z_{-j} \in \prod_{i=3, i \neq j} Z_i$.

Then, there exists a PSNE of this game.

Proof. By (a), the best-reply correspondences are non-empty and upper semi-continuous. By (b) and Proposition 6, the maximal best-reply of player 1 is quasi-increasing on $Y$ and increasing on $\prod_{j=3}^{n} Z_j$.

By (c), the best-reply correspondence of player 2 is single-valued and, therefore, a continuous function of $x$. Moreover, it is increasing in $(x, z)$ by Milgrom-Shannon’s theorem. By (d) and Milgrom-Shannon’s theorem, the maximal best-reply function of player $j$ is increasing in $(x, y, z_{-j})$. The assumptions of Theorem 40 in section 5 are thus satisfied, and we conclude that there exists a PSNE of the game.

This theorem (and its companion, Theorem 40 in section 5) can be interpreted as pointing out a trade-off between quasi-increasingness and continuity. That is, if we instead of having an increasing function, we only have a quasi-increasing function, this causes no problem to the existence of a fixed point (or equilibrium), if we impose the stronger requirement of continuity in the reaction function of other player.

This theorem can be used to guarantee equilibrium in games where there are two special players, with well-defined roles, among a set of other players whose actions are strategic complements.

For the next result, we envision a different setting. Namely, we consider a game with a total of $n_1 + n_2$ players, with $n_1 \geq 2$ and $n_2 \geq 1$, such that players $i \in N_1 \equiv \{1, \ldots, n_1\}$ are all symmetric, while players $j \in N_2 \equiv \{n_1 + 1, \ldots, n_1 + n_2\}$ can be asymmetric. More specifically, players $i \in N_1$ have same action space $X$, which is a compact interval of real numbers, and player $j \in N_2$ has action space $Y_j$, which is a complete lattice for each $j$. For simplicity of notation, we denote $\prod_{j \in N_2} Y_j$ by $Y$. Player $i \in N_1$ has payoff $F(x_i, x_{-i}, y)$. Player $j \in N_2$ has utility function $G^j : X^{n_1} \times Y \to \mathbb{R}$. The following result generalizes Theorem 18.

Theorem 22 Assume that:
(a) $F$ and $G^j$ (for $j \in N_2$) are continuous in own action,
(b) for every $y \in Y = \prod_{j \in N_2} Y_j$ and $b \in X$, $F(x, b, ..., b; y)$ satisfies a $\beta$-shifted single-crossing property with respect to $(x; b)$ for some $\beta : Y \rightarrow X$ that is continuous and increasing. Moreover, for each $b \in X$ fixed, $F(x, b, ..., b; y)$ satisfies the single-crossing property on $(x; y)$,
(c) $G^j(x; y)$ satisfies the single-crossing property with respect to $(y_j; (x, y_{-j}))$.

Then, there exists a PSNE for this game in which all players $i = 1, ..., n_1$ play exactly the same action.

Proof. By (a), all best-reply correspondences are non-empty. By (b) and Proposition 6, the maximal selection of player $i \in N_1$ best-reply correspondence is quasi-increasing in $b \in X$, when all players $j \in N_1, j \neq i$ play $b$. Moreover, it is also increasing in $y$. By (c), and Milgrom-Shannon’s theorem, the maximal selection of player $j \in N_2$’s best-reply correspondence is increasing in $(x, y_{-j})$. Thus, the assumptions of Theorem 42 in section 5 are satisfied and we have a fixed point theorem or PSNE with the stated property. ■

4 Some Selected Applications

This section presents a selection of well known models in applied microeconomics for which the results of the present paper apply quite naturally and in a straightforward manner to yield novel results. Despite the fact that some of these models have extensive literature dealing specifically with the existence of PSNEs, the results proposed below constitute either significant generalizations of their counterparts in the literature or new versions that are not nested with existing ones.

In the presentations below, we always assume that that the primitive functions of each model are twice continuously differentiable. This is only for convenience in establishing the relevant complementarities via Topkis’s cross partial test. Since the results presented below are actually new to the separate literatures dealing with each model, we present the results in the form of formal propositions with concise proofs.

The first application below serves to illustrate the results on existence of symmetric PSNE for symmetric games, and on the uniqueness of PSNE. The other applications deal with the existence of PSNE for asymmetric games.

4.1 Symmetric Bertrand Competition

Consider a symmetric Bertrand oligopoly with differentiated substitute products wherein $n$ identical firms, labeled $1, 2, ..., n$, choose prices $x, y, ..., x_n$ (in a given price set $[0, \overline{p}]$) and face a symmetric demand
system \((D^1, ..., D^n)\) for their products, respectively. Production costs are assumed to be linear with unit cost \(c\) for each firm. Then (say) firm 1’s profit function is

\[
F(x_1, x_2, ..., x_n) = (x_1 - c)D^1(x_1, x_2, ..., x_n)
\]

Each \(D^j\) satisfies \(D^1_1 < 0\) (the standard law of demand) and \(D^j_{ij} > 0\) for all \(i \neq j\), for products to be substitutes in demand.\(^{20}\) It will be useful to introduce the notation \(D(x, y) \equiv D^1(x, y, ..., y)\).

Assuming firms 2, ..., \(n\) use the same price \(y\), we can write firm 1’s profit (when using price \(x\)) as

\[
F(x, y, ..., y) = (x - c)D^1(x, y, ..., y) = (x - c)D(x, y)
\]

and its corresponding reaction curve as \(f(y) = \arg \max_{x \in [0, p]} (x - c)D(x, y)\). Invoking symmetry, we can restrict attention to the latter problem.

We shall show that under the following assumptions on demand \(D\), this game fits Theorem 19.

**Assumptions**

(A0). \(\frac{1}{D(x, y)}\) is strictly convex in \(x\) for each \(y\), i.e., \(D(x, y)D_11(x, y) - 2D^2_2(x, y) < 0\) for all \((x, y)\).

(A1). \(D(D_{12} - kD_{11}) + D_1(kD_1 - D_2) > 0\) for some \(k > 0\) and all \((x, y)\).

(A2). \(D_2D^2_1 - D[1D_{12} - 2D_{11}] > 0\) for all \((x, y)\).

(A3). \(D_2(x, x) < |D_1(x, x)|\) for all \(x\).

Assumption (A0) is not very restrictive and is known to yield a profit function that is quasi-concave in own price. It can be seen by inspection that both Assumptions (A1) and (A2) impose very minor restrictions on a demand curve. Indeed, every term in each condition is either positive or has a strong tendency to be positive. It is easy to verify that most demand functions used for price competition satisfy these properties (see e.g., Vives, 1999).\(^{21}\)

As to Assumption (A3), it pertains only to the Chamberlinian demand \(D^1(x, x, ..., x)\) and therefore is very broadly satisfied. It is also quite intuitive in that its economic content is that at equal prices, own price effects on a firm’s demand outweigh rivals’ price effects. Going back to the original demand function, say \(D^1\), Assumption (A3) becomes

\[
\sum_{j=2}^n D^j_1(x, x, ..., x) < |D^1_1(x, x, ..., x)| \text{ for all } x.
\]

\(^{20}\)Recall that subscripts denote partial differentiation with respect to the indicated variable(s).

\(^{21}\)To shed further light on Assumption A1, rewrite it in the alternative form \((DD_{12} - D_1D_2) + k(D^2_1 - DD_{11}) > 0\), and observe that the first parenthesis is positive if \(D\) is log-supermodular in the two prices and the second is positive if \(D\) is log-concave in own price. By way of comparison, recall that the usual sufficient condition for \(f\) to be increasing is the log-supermodularity of \(D^2\) or \(D^2D^2_{12} - D^2_1D^2_2 \geq 0\) (Milgrom and Roberts, 1990).
When imposed on all firms, this condition is clearly a dominant-diagonal condition on the Jacobian matrix of the demand system, and has the same economic interpretation as Assumption (A3) in a qualitative sense (but considering rivals’ aggregate price effects).

The main result for this model is as follows.

**Proposition 23** For a symmetric Bertrand duopoly with substitute products, the following hold:

(a) Under either assumption (A0) or (A1), a symmetric PSNE exists.

(b) If in addition to assumption (A0) or (A1), assumptions (A2) and (A3) hold, then the game possesses a unique symmetric PSNE.

**Proof.** (a) If Assumption (A0) holds, i.e., $\frac{1}{D_1(x,y)}$ is strictly convex in $x$ for each $y$, then $F(x,y)$ is strictly quasi-concave in $x$ for each $y$, and thus the reaction correspondence $f(y)$ of firm 2 is a continuous function (Caplin and Nalebuff, 1991). Hence, by Brouwer’s fixed point theorem, $f$ has a fixed point, $x^*$, which is easily seen to be a symmetric PSNE ($x^*, x^*, ..., x^*$).

If instead of (A0), Assumption (A1) holds, we now show that $f(y)$ is quasi-increasing. Consider the change of variables $z = x + ky, k > 0$, and the associated objective

$$
\max \left\{ \tilde{F}(z,y) \triangleq (z - ky - c)D(z - ky, y) : z \in [ky, \infty) \right\},
$$

or taking logs,

$$
\max \left\{ \log \tilde{F}(z,y) \triangleq \log(z - ky - c) + \log D(z - ky, y) : z \in [ky, \infty) \right\}.
$$

Since $[ky, \infty)$ is ascending in $y$, the argmax $z^*(y)$ will be increasing, or equivalently $\frac{f'(y') - f'(y)}{y' - y} \geq -k$, if log $\tilde{F}(z,y)$ is supermodular in $(z,y)$ or $D(D_{12} - kD_{11}) + D_1(kD_1 - D_2) > 0$, which is just Assumption A1. It follows that $f$ is quasi-increasing and maps $[0, \bar{p}]$ into itself. Hence, by Corollary 10, it has a fixed point, $x^*$, which is easily seen to be a symmetric PSNE ($x^*, x^*, ..., x^*$).

(b) To show uniqueness of symmetric PSNE, we first need to show that $f(y)$ is quasi-decreasing in $y$, due to Assumption (A2). This step has already been proved in the Example 7 given in Section 2, and is thus omitted here.

Next, we observe that $f(y)$ is actually a continuous function. Under Assumption (A0), this fact was noted above. Under (A1), since $f(y)$ is both quasi-decreasing and quasi-increasing in $y$, it is continuous in $y$. A standard (implicit function) argument then shows that $f(y)$ is differentiable. Since $z^*(y) = D(x^*(y), y)$ is increasing in $y$, we have $D_1(f(y), y)f'(y) + D_2(f(y), y) \geq 0$, or $f'(y) \leq -\frac{D_2(f(y), y)}{D_1(f(y), y)}$.  

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In particular, at \( x^* \), \( f'(x^*) \leq \frac{D_2(x^*, x^*)}{D_1(x^*, x^*)} < 1 \) by Assumption (A3). Since this holds at all candidate symmetric PSNE, there is a unique symmetric PSNE by Proposition 12. ■

A key aspect of the uniqueness result here is that the reaction curve is required to be contractive only locally (around fixed points); it need not be a global contraction. For the latter property, one would need to strengthen Assumption (A3) to hold at all pairs \((x, y)\). This would be a much more restrictive assumption, which is not satisfied by many known (non-linear) demand systems.

4.2 Cournot duopoly without strategic substitutes

Consider the standard model of quantity competition with differentiated products with firms 1 and 2 producing output levels \( x \) and \( y \), and facing inverse demands \( P^1(x, y) \) and \( P^2(x, y) \), and linear cost functions \( c_1x \) and \( c_2y \) (for simplicity). Assume \( P^1_2(x, y) = P^2_1(x, y) < 0 \), i.e., that the two products are substitutes.

The profit function of firms 1 and 2 are then

\[
F(x, y) = xP^1(x, y) - c_1x \quad \text{and} \quad G(x, y) = yP^2(x, y) - c_2y.
\]

We make assumptions on the primitives of this game so as to make it exactly fit Theorem 16.

**Assumptions**

(A1). (a) \( \frac{1}{P(x, y)} \) is strictly convex in \( y \) for each \( x \), or \( P^1 P^2_{12} - 2(P^1_2)^2 < 0 \)

(b) \( P \) is log-submodular, or \( P^1 P^1_{12} - P^2_1 P^1_1 < 0 \)

(A2). \( P^2_1 + 2P^2_2 + y(P^2_{12} + P^2_{22}) < 0 \) for all \( x, y \)

For substitute products, (A2) is seen by inspection to impose very few restrictions on a demand curve. Indeed, every term in (A2) is either negative or has a high tendency to be negative. It is easy to verify that most demand functions used for quantity competition satisfy (A2) (e.g., Vives, 1999).

**Proposition 24** Every Cournot duopoly with substitute products satisfying assumptions (A1)-(A2) possesses a PSNE.

**Proof.** For firm 1, (A1)(a) ensures that \( F(x, y) \) is strictly quasi-concave in \( x \) for each \( y \), and thus that the reaction correspondence \( f(y) \) is a continuous function (Caplin and Nalebuff, 1991). In addition, (A1)(b) implies that \( F(x, y) \) is log-submodular, so that \( f(y) \) is a decreasing function.
For firm 2, we now show that its reaction correspondence, \( g \), is such that 
\[
\frac{g(x') - g(x)}{x' - x} \leq 1 \text{ for all } x', x.
\]

Let \( z = y - x \) and consider 
\[
\max_{z \geq -x} \tilde{G}(x, z) = (z + x)P^2(x, z + x) - c_2(z + x)
\]

Since the feasible set \([-x, \infty)\) is descending in \( x \), and 
\[
\frac{\partial^2 \tilde{G}(x, z)}{\partial z \partial x} = P_1^2 + 2P_2^2 + (z + x)(P_{12}^2 + P_{22}^2) < 0 \text{ by (A2)},
\]
\( z^*(x) \) is decreasing in \( x \), so that the argmax \( y^*(x) = g(x) = z^*(x) + x \) is such that 
\[
\frac{g(x') - g(x)}{x' - x} \leq 1.
\]
Hence \( g \) is quasi-decreasing.

The existence of a PSNE then follows from Theorem 16.

In contrast, the two strands of existing results in the literature postulate assumptions that make the Cournot duopoly either a game of strategic substitutes (Vives, 1999), or a game with continuous reaction curves, thus yielding existence as a general property of these respective classes of games. Relative to the standard results, the above Proposition allows a relaxation of the usual requirement of strategic substitutability for one firm in accommodating a limited level of strategic complementarity, but at the same time imposes quasi-concavity in own action for the other player’s payoff.

4.3 A game of accident liability

Consider a standard and simple model of accident liability (Brown, 1973). Two risk-neutral agents are engaged in an activity that may result in an accident, the probability of which depends negatively on their individual levels of care \( x, y \in [0, 1] \), aimed at avoiding accidents. The expected costs to the two agents are respectively given by \( L^1(x, y) \) and \( L^2(x, y) \). The agents’ private costs corresponding to the levels of care \( (x, y) \) are given by \( C_1(x) \) and \( C_2(y) \). The (expected) payoffs are then the negative of total (expected) costs for each agent, i.e.,
\[
F(x, y) = -L^1(x, y) - C_1(x) \quad \text{and} \quad G(x, y) = -L^2(x, y) - C_2(y)
\]

We shall show that under the following assumptions, this game fits Theorem 16.

**Assumptions**

(A0). \( L^1_i(x, y) < 0, L^2_i(x, y) < 0 \) and \( C^i_i(x) > 0, i = 1, 2 \).

(A1). (a) \( L^1 \) has increasing differences in \( (x, y) \)
(b) $L_{11}^1(x,y) + C''_1(x) > 0$ for all $x,y$

(A2). $L_2^2(x,y)L_1^2(x,y) - L_1^1(x,y)L_{12}^2(x,y) < 0$ for all $x,y$.

Assumptions (A0) and (A1) are standard in the related literature. (A0) holds that each effort by either agent lowers the probability of an accident, and (A1)(a) says that the two effort levels are strategic substitutes for agent 1 (so free riding on the part of agent 1 is expected to arise). (A1)(b) is a standard concavity or diminishing returns condition. The novel condition here is (A2), which states that $L_1^2(x,y)/L_2^2(x,y)$ is decreasing in $x$, or that $x$ and $y$ are weak complements for agent 2. Clearly, this condition is significantly more general than assuming that $L^2(x,y)$ is supermodular.

**Proposition 25** Every game of accident liability satisfying (A0)-(A2) possesses a PSNE.

**Proof.** The two parts of (A1) imply respectively that $F$ has decreasing differences in $(x,y)$ and is strictly concave in $x$. Hence the reaction correspondence of country 1, $f$, is a continuous and decreasing function.

As to country 2, we show that its reaction correspondence, $g$, is quasi-decreasing. Consider the change of variables $z = L^2(x,y)$. Taking a parametric inverse, one has $y = h(x,z)$. Consider player 2’s best response problem in the equivalent form

$$\max_z \bar{G}(x,z) = -z - C_2[h(x,z)]$$

Since the feasible set for $z$, $[L^2(x,1), L^2(x,0)]$, is descending in $y$, and (using the inverse relations for derivatives in (11)),

$$\frac{\partial^2 \bar{G}(x,z)}{\partial z \partial y} = -C''_2[h(z,y)]h_1(z,y)h_2(z,y) - C'_2[h(z,y)]h_{12}(z,y)$$

$$= C''_2(y)\frac{L_2^2}{(L_1^2)^2} - C'_2(y)\frac{1}{(L_1^2)^3}(L_2^2L_{11}^2 - L_1^2L_{12}^2)$$

$$< 0 \text{ by (A0) and (A2)},$$

we conclude that $z^*(x)$ is decreasing in $x$, so that the argmax $y^*(x) = g(x) = h[x, z^*(x)]$ is quasi-decreasing in $x$.

Existence of a PSNE then follows directly from Theorem 16. ■

The extensive literature on this topic typically assumes that both players payoffs satisfy Assumption (A1) (in addition to (A0)). In other words, the players’ reaction curves are then both continuous and decreasing functions. In contrast, the present approach dispenses with the need for any concavity assumption for player 2 (i.e., (A1)(b)) and relaxes the increasing differences assumption (A1)(a) to the
more general condition (A2). From an intuitive viewpoint, the latter assumption may be visualized as follows. In response to an increase in player 1’s level of care, player 2 need not decrease his care level (as would be dictated by the prevalent assumption of strategic substitutes), but instead he would never increase it so much that his expected liability $L^2$ would end up increasing overall.

4.4 A game of global pollution

Consider a standard simple model of inter-country emissions choice where beneficial production requires a global pollutant as input (see e.g., Carraro and Siniscalco, 1993; Chander and Tulkens, 1997; and Diamantoudi and Sartzetakis, 2006). The idea is that each country must trade off the private benefits from productive activity that generates environmental emissions against the public damage caused by aggregate emissions from a global pollutant. For countries 1 and 2, let $x$ and $y$ denote their emissions levels, $B_1(x)$ and $B_2(y)$ their private benefit functions, and $D_1(x+y)$ and $D_2(x+y)$ their environmental damage functions. The payoff functions of the two countries are:

$$F(x,y) = B_1(x) - D_1(x+y) \quad \text{and} \quad G(x,y) = B_2(y) - D_2(x+y)$$

We make assumptions on the primitives of this game to make it exactly fit Theorem 16.

**Assumptions**

(A0). $B_i'(\cdot) > 0$ and $D_i'(\cdot) > 0$.

(A1). (a) $D_1''(\cdot) > 0$.

(b) $B_1''(x) - D_1''(x+y) < 0$ for all $x, y$.

(A2). $B_2''(y) - 2D_2''(x+y) \leq 0$ for all $x, y$.

The content of these assumptions is quite transparent and thus left to the reader.

**Proposition 26** Every game of global pollution satisfying assumptions (A0)-(A2) possesses a PSNE.

**Proof.** The two parts of (A1) imply respectively that $F$ has decreasing differences in $(x,y)$ and is strictly concave in $x$. Hence the reaction correspondence of country 1, $f$, is a continuous and decreasing function. As to country 2, to show that its reaction correspondence, $g$, is quasi-decreasing, consider the change of variables $z = y - x$. Then

$$\tilde{G}(x,z) = C(z + x) - E(z + 2x)$$

22The fact that damage depends on total emissions reflects the global nature of the pollutant at hand. Nevertheless, the two damage functions may differ as a result of different atmospheric conditions.
and
\[ \frac{\partial^2 \tilde{G}(x, y)}{\partial z \partial x} = C''(z + y) - 2E''(z + 2y) \leq 0. \]

Hence, by Topkis’s Theorem, the argmax \( z^*(x) \) is decreasing in \( x \), so that the argmax \( y^*(x) = g(x) = z^*(x) + x \) is such that \( \frac{g(x') - g(x)}{x' - x} \leq 1 \). Hence \( g \) is quasi-decreasing.

Existence of a PSNE then follows directly from Theorem 16. ■

In contrast, the existing results in the literature postulate a strictly concave benefit function and a convex damage function, ensuring the game has payoffs that are strictly concave in own action, and thus continuous reaction curves, and applying Brouwer’s fixed point theorem (see e.g., Chander and Tulkens, 1997). Therefore, our assumptions on country 2 are more general than those made in the literature. Nevertheless, although our approach to existence is fundamentally different, the assumptions on country 1 are actually the same as in the existing literature.

5 An order-theoretic treatment

This section provides a full mathematical treatment of the material covered so far in Euclidean spaces. All definitions and results in this part are presented at the level of generality of Tarski’s original paper, relating to his Intersection Point Theorem. We begin in subsection 5.1 with a brief review of the relevant order-theoretic preliminaries (for a more detailed coverage, the reader can consult Tarski, 1955 or Topkis, 1998). We then define and establish some general properties of quasi-monotone functions in subsection 5.2, which sets the stage for some useful and practical calculus using this class of functions.

5.1 Definitions and well known results

A binary relation \( \geq \subset X \times X \) is a partial order if it is transitive, reflexive and anti-symmetric.\(^{23}\) If \( \geq \) is a partial order on \( X \), we say that the pair \( (X, \geq) \) or, simply \( X \), is a partially ordered set, or poset. An upper (lower) bound of a set \( A \subset X \) is \( z \in X \) such that \( z \geq a (a \geq z) \) for any \( a \in A \). If \( z \in X \) is an upper bound of \( A \subset X \) and \( w \geq z \) for any \( w \in X \) that is also an upper bound of \( A \), we say that \( z \) is the lowest upper bound or supremum of \( A \) and we denote it by \( \lor A \). If \( A = \{x, y\} \), we can also denote it by \( x \lor y \). Since \( \geq \) is anti-symmetric, there is at most one supremum for any set. The greatest lower bound or infimum of a set \( A \) is defined analogously and denoted by \( \land A \).\(^{24}\) If \( A = \{x, y\} \), we denote the infimum

\(^{23}\)Recall that \( \geq \) is anti-symmetric if \( x \geq y \) and \( y \geq x \) implies \( x = y \).

\(^{24}\)For completeness: \( z \) is the greatest lower bound of \( A \) if it is a lower bound of \( A \) and \( z \geq w \) for any \( w \) that is also a lower bound of \( A \).
by $x \land y$. A poset $(X, \geq)$ is a lattice if $x \lor y$ and $x \land y$ exist in $X$ for any $x, y \in X$. A lattice $(X, \geq)$, or simply $X$ for short, is complete if $\lor X'$ and $\land X'$ exist in $X$ for any subset $X'$ of $X$.

For any poset $(X, \geq)$, if $x \geq y$ and $y \neq x$, we write $x > y$. Occasionally, we may write $y \leq x$ and $y < x$ if $x \geq y$ and $x > y$, respectively. If $(X, \geq)$ and $(Y, \geq)$ are posets, a function $f : X \to Y$ is increasing if $x' \geq x$ implies $f(x') \geq f(x)$. A function $f : X \to Y$ is strictly increasing if $x' > x$ implies $f(x') > f(x)$.

Tarski’s classical fixed-point theorem is now recalled (for a proof, see Tarski, 1955):

**Theorem 27 (Tarski Fixed Point Theorem)** Let $X$ be a complete lattice and $f : X \to X$ be an increasing function. Then the set of fixed-points of $f$, $\{x : f(x) = x\}$, is a nonempty complete lattice. Moreover:

$$\lor \{x : f(x) = x\} = \lor \{x : f(x) \geq x\} \quad \text{and} \quad \land \{x : f(x) = x\} = \land \{x : f(x) \leq x\}.$$ 

A partial order $\geq$ is total if $x \geq y$ or $y \geq x$ holds for any $x, y \in X$. Obviously, if $\geq$ is total, then $(X, \geq)$ is a lattice. A poset with a total partial order is called a chain. A chain $X$ is dense if for any $x, y \in X$, such that $x > y$, there exists $z \in X$ such that $x > z > y$.

If $X$ is a chain, its order topology is generated by the sub base of open rays $\{x \in X : x > a\}$ for $a \in X$. If $X_i$ are partially ordered sets for $i = 1, ..., n$, the product space $X = \prod_{i=1}^{n} X_i$ is naturally endowed with the product partial order defined by: $x = (x_1, ..., x_n) \geq y = (y_1, ..., y_n)$ iff $x_i \geq y_i$ for all $i = 1, ..., n$.

### 5.2 Quasi-increasing functions: definition and properties

The order-theoretic definitions of quasi-increasing and quasi-decreasing functions are as follows.

**Definition 28 (Quasi-increasing and Quasi-decreasing)** Let $X$ and $Y$ be complete lattices, and $f : X \to Y$ be a function. Then:

1. $f$ is quasi-increasing if for every nonempty subset $E \subset X$,

$$f(\lor E) \geq \land f(E) \quad \text{and} \quad f(\land E) \leq \lor f(E).$$  \hfill (16)

2. $f$ is quasi-decreasing if for every nonempty subset $E \subset A$,

$$f(\lor E) \leq \lor f(E) \quad \text{and} \quad f(\land E) \geq \land f(E).$$  \hfill (17)
Clearly, the two concepts are duals, in the sense that $f$ is quasi-increasing if and only if $-f$ is quasi-decreasing. Indeed,

$$f(\lor E) \geq \land f(E) \iff -f(\lor E) \leq -\land f(E) \iff (-f)(\lor E) \leq \lor (-f)(E),$$

and a similar equivalence holds for the other inequalities. Therefore, instead of proving results for both quasi-increasing and quasi-decreasing functions, we can prove just for one of them and appeal to the duality.

In the remainder of this subsection, we assume that $X$ and $Y$ are complete lattices, unless otherwise stated. The following observations are almost immediate:

**Lemma 29** If $f : X \rightarrow Y$ is increasing, then it is quasi-increasing. If $f : X \rightarrow Y$ is decreasing, then it is quasi-decreasing.

**Proof.** If $f$ is increasing and $e \in E$ then $\lor E \geq e$ and then $f(\lor E) \geq f(e) \geq \land f(E)$. Similarly, $\land E \leq e$ and $f(\land E) \leq f(e) \leq \lor f(E)$. The other result follows by duality. □

The next fact provides an important link between the order-theoretic treatment and the Euclidean case. It is also a very useful and practical test when working in Euclidean settings.

**Lemma 30** Let $X \subset \mathbb{R}$. A function $f : X \rightarrow \mathbb{R}$ is quasi-increasing if and only if

$$\limsup_{y \uparrow x} f(y) \leq f(x) \leq \liminf_{y \downarrow x} f(y). \quad (18)$$

**Proof.** *Necessity.* Let $y_n \uparrow x$ be such that $\lim_{n \rightarrow \infty} f(y_n) = \liminf_{y \uparrow x} f(y)$ and let $E_n = \{ y_j : j \geq n \}$. Then, $\lor E_n = x$ and the condition $f(\lor E_n) \geq \land f(E_n)$ means $f(x) \geq \inf_{j \geq n} f(y_j)$. Then, $f(x) \geq \liminf_{y \uparrow x} f(y)$. The other inequality is obtained analogously.

* Sufficiency. Let $E \subset X$ and suppose that $f(\lor E) < \land f(E)$. Let $\bar{x} \equiv \lor E$. Then $E \subset \{ y \in X : y \leq x \}$. Therefore, $\land f(E) = \inf_{x \in E} f(x) \leq \limsup_{y \uparrow \bar{x}} f(y)$. Then (18) implies that

$$f(x) < \land f(E) = \inf_{x \in E} f(x) \leq \limsup_{y \uparrow \bar{x}} f(y) \leq f(\bar{x}),$$

which is a contradiction. The proof for the other condition is analogous. □

By duality, it follows that $f : X \rightarrow \mathbb{R}$ is quasi-decreasing if and only if:

$$\liminf_{y \downarrow x} f(y) \geq f(x) \geq \limsup_{y \uparrow x} f(y).$$

As a consequence, we also obtain the following:
Corollary 31 A function $f : X \subset \mathbb{R} \to \mathbb{R}$ is continuous if and only if it is both quasi-increasing and quasi-decreasing.

The following lemma can also be useful:

**Lemma 32** Let $\lambda \geq 0$ and $X \subset \mathbb{R}$. If $f, g : X \to \mathbb{R}$ are quasi-increasing functions, then $\lambda f + g$ is quasi-increasing. Similarly, if $f, g$ are quasi-decreasing functions, then $\lambda f + g$ is quasi-decreasing.

**Proof.** Let $f$ and $g$ be quasi-increasing functions. Using the fact that (a) $\liminf_{y \uparrow x} \lambda f(x) = \lambda \liminf_{y \uparrow x} f(x)$ and that a similar property holds for $\limsup$, and (b) the subadditivity of the $\limsup$ operation and the superadditivity of the $\liminf$ operation, we have:

$$\limsup_{y \uparrow x} (\lambda f(y) + g(y)) \leq \lambda \limsup_{y \uparrow x} f(y) + \limsup_{y \uparrow x} g(y) \leq \lambda f(x) + g(x),$$

and

$$\liminf_{y \downarrow x} (\lambda f(y) + g(y)) \geq \lambda \liminf_{y \downarrow x} f(y) + \liminf_{y \downarrow x} g(y) \geq \lambda f(x) + g(x).$$

This establishes the result for quasi-increasing functions. The proof of the other claim is analogous. ■

**Lemma 33** (i) If $f$ is quasi-increasing and $g$ is continuous and increasing, then $g \circ f$ and $f \circ g$ are also quasi-increasing.

(ii) If $f$ is quasi-increasing and $g$ is continuous and decreasing, then $g \circ f$ and $f \circ g$ are quasi-decreasing.

Moreover, the condition that $g$ is continuous cannot be dispensed with.

**Proof.** We give the proof of part (i), as that of part (ii) is similar.

Let $E$ be nonempty. We know that $f(\vee E) \geq \land f(E)$. Since $g$ is increasing, $g(f(\vee E)) \geq g(\land f(E))$. Since $g$ is quasi-decreasing (because it is continuous), $g(\land f(E)) \geq \land g(f(E))$. This shows that $g(f(\vee E)) \geq \land g(f(E))$. Using the same arguments, we also have:

$$g(f(\land E)) \leq g(\lor f(E)) \leq \lor g(f(E)).$$

Thus, $g \circ f$ is quasi-increasing. If $f$ is quasi-increasing and $g$ is continuous and decreasing, then

$$g(f(\vee E)) \leq g(\land f(E)) \leq \lor g(f(E))$$

and analogously for the other inequality in (17). The proofs for $f \circ g$ are analogous.
For a counterexample for when $g$ is just increasing, let $f, g : [0, 1] \to [0, 1]$ be defined by $f(x) = 1 - x$ and

$$g(x) = \begin{cases} \frac{x}{2} & \text{if } x < \frac{1}{2} \\ \frac{1}{2} & \text{if } x = \frac{1}{2} \\ \frac{1+x}{2} & \text{if } x > \frac{1}{2} \end{cases}$$

It is easy to see that $f$ is continuous and therefore quasi-increasing, while $g$ is increasing but not continuous.

We have:

$$\limsup_{y \uparrow \frac{1}{2}} (g \circ f)(y) = \frac{3}{4} > \frac{1}{2} = (g \circ f)(\frac{1}{2}) = \liminf_{y \downarrow \frac{1}{2}} (g \circ f)(y),$$

contradicting (18). □

Tarski’s intersection point theorem (between quasi-increasing and quasi-decreasing functions) is as follows (see Tarski, 1955, Theorem 3).

**Theorem 34 (Tarski’s Intersection Theorem)** Let $X, Y$ be complete and dense chains and $f, g : X \to Y$ be two functions satisfying: $f(\land X) \geq g(\land X)$ and $f(\lor X) \leq g(\lor X)$, $f$ is quasi-increasing and $g$ is quasi-decreasing. Then $\{x : f(x) = g(x)\}$ is a nonempty complete chain. Moreover:

$$\lor\{x \in X : f(x) = g(x)\} = \lor\{x \in X : f(x) \geq g(x)\}$$

and

$$\land\{x \in X : f(x) = g(x)\} = \land\{x \in X : f(x) \leq g(x)\}.$$

### 5.3 Fixed Point Theorems

We are ready for the statement of our main fixed-point theorem in full generality (i.e., in ordered spaces). We integrate the companion comparative statics result, which predicts the direction of change of the extremal fixed points as an underlying parameter increases. All sets below are assumed to be non-empty, unless otherwise observed.

**Theorem 35** Let $X$ be a complete and dense chain, $Y$ a chain and $\Theta$ a partially ordered set. Endow $X$ and $Y$ with their order topologies. Consider two functions $f : Y \times \Theta \to X$ and $g : X \times \Theta \to Y$ and assume that

1. $f$ is continuous and increasing in $y$ for fixed $\theta \in \Theta$ and increasing in $\theta$ for fixed $y$;

2. $g$ is quasi-increasing in $x$ for fixed $\theta \in \Theta$ and increasing in $\theta$ for fixed $x$. 

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Then, for each $\theta \in \Theta$, the set $E(\theta) = \{(x,y) \in X \times Y : f(x,\theta) = y \text{ and } g(y,\theta) = x\}$ is a non-empty complete chain. Moreover, $e(\theta) = \wedge E(\theta)$ and $e(\theta) = \vee E(\theta)$ are both increasing in $\theta$.

**Proof.** For simplicity, we will omit all reference to $\theta$ in the first part of the proof; the reader should think of $\theta$ as fixed in this first part. By Lemma 33, the composition $f \circ g : X \to X$ is quasi-increasing. Let $\iota : X \to X$ be the identity. Since $\iota$ is continuous, Lemma 31 implies it is quasi-decreasing. If $\inf X = \underline{x}$ and $\sup X = \overline{x}$, we have $f \circ g(\underline{x}) \geq \underline{x} = \iota(\underline{x})$ and $f \circ g(\overline{x}) \leq \overline{x} = \iota(\overline{x})$.

Therefore, the assumptions of Tarski’s Intersection Point Theorem 34 are satisfied for $f \circ g$ and $\iota$ and we conclude that the set $\overline{X}_1 \equiv \{x \in X : f(g(x)) = x\}$ is a nonempty complete chain. Define the set $E \equiv \{(x,y) \in \overline{X}_1 \times Y : y = g(x)\}$. If we denote by $E_2$ the projection of $E$ on $Y$, it is easy to see that $\overline{X}_1 = f(E_2)$. Since $f$ is continuous and increasing, $y \leq y' \iff f(y) \leq f(y')$, which implies that $E$ is also a complete chain.

Now, let us consider explicitly the dependence on $\theta$ to establish the last claims. By Tarki’s Intersection Theorem 34, the supremum and infimum of $\overline{X}_1(\theta)$ are respectively $\overline{x}_1(\theta) \equiv \vee X^+(\theta)$ and $\underline{x}_1(\theta) \equiv \wedge X^-(\theta)$, where $X^+(\theta) \equiv \{x \in X : f(g(x,\theta),\theta) \geq x\}$ and $X^-(\theta) \equiv \{x \in X : f(g(x,\theta),\theta) \leq x\}$. Since $e(\theta) = (\overline{x}_1(\theta), g(\overline{x}_1(\theta)))$ and $e(\theta) = (\underline{x}_1(\theta), g(\underline{x}_1(\theta)))$ and $g$ is increasing, the result then follows if we show that $\theta < \theta'$ implies that $X^+(\theta) \subset X^+(\theta')$ and $X^-(\theta) \supset X^-(\theta')$.

Let $\theta < \theta'$ and let $x \in X^+(\theta)$. Since $g$ is increasing in $\theta$, $g(x,\theta) \leq g(x,\theta')$. Since $f$ is increasing in both arguments and $x \in X^+(\theta)$, we have

$$x \leq f(g(x,\theta),\theta) \leq f(g(x,\theta'),\theta) \leq f(g(x,\theta'),\theta').$$

(19)

Therefore, $x \in X^+(\theta')$. Similarly, if $x \in X^-(\theta')$, that is, $f(g(x,\theta'),\theta') \leq x$, (19) shows that $x \in X^-(\theta)$.

**Remark 36** This theorem is useful for games, where the function $f$ is the reaction function of a player 1 and $g$ is the reaction function of a player 2. In this case the set $E(\theta)$ is clearly contained in the set of equilibria of the game when the parameter is $\theta$, if the reaction functions are just a selection of the correspondence of best-replies. If there are no other selections, $E(\theta)$ will actually be the set of all equilibria.

We have shown the monotonicity of the fixed points using only monotonicity of the functions with respect to the parameter but not with respect to $x$ ($g$ is only quasi-increasing, and not necessarily
increasing in $x$). This comparative statics result may fail if $g$ is not quasi-increasing (counter-examples can be easily found).

Theorem 35 admits an order-dual, which we state without proof below.

**Theorem 37** Let $X$ be a complete and dense chain, $Y$ is a chain and $\Theta$ is a partial order set. Moreover, $X$ and $Y$ are endowed with their order topology. Consider two functions $f : Y \times \Theta \to X$ and $g : X \times \Theta \to Y$ and assume the following:

1. $f$ is continuous and decreasing in $y$ for each fixed $\theta \in \Theta$ and increasing in $\theta$ for each $y$;
2. $g$ is quasi-decreasing in $x$ for each $\theta \in \Theta$ and increasing in $\theta$ for each $x$.

Then, for each $\theta \in \Theta$, the set $E(\theta) = \{(x, y) \in X \times Y : f(x, \theta) = y$ and $g(y, \theta) = x\}$ is a non-empty complete chain. If we define $r : X \times Y \times \Theta \to X \times Y$ by $r(x, y, \theta) = (f(y, \theta), g(x, \theta))$, then $E(\theta)$ is the set of fixed points of $r(\cdot, \theta)$.

Moreover, $\underline{e}(\theta) = \wedge E(\theta)$ and $\bar{e}(\theta) = \vee E(\theta)$ are both increasing in $\theta$.

From the proof of Theorem 35, we can see that the following is true:

**Corollary 38** Let $X$ be a complete and dense chain and $\Theta$ a partially ordered set. Consider a function $r : X \times \Theta \to X$ that is quasi-increasing in $X$ and increasing in $\Theta$. Then, for each $\theta \in \Theta$, the set of fixed points of $r E(\theta) = \{x \in X : r(x, \theta) = x\}$ is a non-empty complete chain. Moreover, $\underline{e}(\theta) \equiv \wedge E(\theta)$ and $\bar{e}(\theta) \equiv \vee E(\theta)$ are both increasing in $\theta$.

This leads to the following result, which is useful for existence of PSNE in symmetric games.

**Corollary 39** Let $X$ be a complete and dense chain and $\Theta$ a partially ordered set. Consider a function $r : X^n \times \Theta \to X^n$ given by $r(x, \theta) = (r^i(x_{-i}, \theta))_{i=1}^n$ where all functions $r^i : X^{n-1} \times \Theta \to X$ are identical and $x \in X \mapsto r^i(x, \ldots, x, \theta) \in X$ is quasi-increasing on $X$. Then for each $\theta \in \Theta$ there exists a point $\bar{e}(\theta) \in X$ such that $(\bar{e}(\theta), \ldots, \bar{e}(\theta)) \in X^n$ is a fixed point of $r(\cdot, \theta)$ and $\theta \mapsto \bar{e}(\theta)$ is increasing. The set of such fixed points is also a non-empty complete chain for each $\theta$.

The above fixed point theorems will be used to obtain results in two-player and $n$-player symmetric games, since they deal with the case of one or two functions. For $n$-player asymmetric games, we need to extend them to include many functions, as done next.

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25It is enough to consider $r = f \circ g$ in the proof. All the arguments work without problems. There is a just a little adaptation in (19).
Theorem 40 Let $X$ be a complete and dense chain, $Y$ be a chain, and $Z_j$ be complete lattices, for $j = 3, ..., n$. Sets $X$ and $Y$ are endowed with their order topologies. Let $Z \equiv \prod_{j=3}^{n} Z_j$ and $Z_{-j} \equiv \prod_{i=3, ..., n, i \neq j} Z_i$. Consider functions $r^j : X \times Y \times Z_{-j} \to Z_j$, for $i = 1, ..., n$, and assume that

1. the function $f : Y \times Z \to X$ is continuous in $y$ for each fixed $z \in Z$ and increasing on $Y \times Z$;
2. the function $g : X \times Z \to Y$ is quasi-increasing in $x$ and increasing on $Z$ for each fixed $x \in X$;
3. the functions $r^j$ are increasing, for $j = 3, ..., n$.

Define $r : X \times Y \times Z \to X \times Y \times Z$ by

$$r(x, y, z) = (f(y, z), g(y, z), (r^j(x, y, z_{-j}))_{j=3}^{n}).$$

Then the set of fixed points of $r$ is nonempty.

Proof. Let us consider $z = (z_3, ..., z_n) \in Z$ as a fixed parameter $\theta \in \Theta = Z$ and define the function $r^\star : X \times Y \times Z \to X \times Y$ by $r^\star(x, y, z) = (f(y, z), g(x, z))$. The assumptions of Theorem 35 are thus satisfied and we conclude that for each $z \in Z$, the function $r^\star(\cdot, z)$ has a fixed point $\bar{e}(z)$ which is increasing in $z$. If we define the function $r^Z : Z \to Z$ by $r^Z(z) = (r^3(\bar{e}(z), z_{-3}), ..., r^n(\bar{e}(z), z_{-n}))$, then $r^Z$ is increasing. By Tarski Fixed Point Theorem 27, it has a fixed point $\bar{z} \in Z$. It is not difficult to see that $(\bar{e}(\bar{z}), \bar{z})$ is a fixed point of $r$. \]

Remark 41 This theorem can be interpreted as pointing out a trade-off between quasi-increasingness and continuity. That is, if instead of having an increasing function, we only have a quasi-increasing function, this causes no problem to the existence of a fixed point, if the other function is also continuous (in addition to being increasing).

A similar result also works if instead of a quasi-increasing and a continuous function, we have $n$ symmetric quasi-increasing functions. Formally the result is as follows.

Theorem 42 Let $n_1, n_2 \in \mathbb{N}$, $n = n_1 + n_2$ and $N_1 = \{1, ..., n_1\}$, $N_2 = \{n_1 + 1, ..., n\}$, $n_1 \geq 2$. Let $X$ be a complete and dense chain and $Z_j$ be complete lattices, for $j \in N_2$. Let $Z \equiv \prod_{j \in N_2} Z_j$, $Z_{-j} \equiv \prod_{i \in N_2, i \neq j} Z_i$ and let $z (z_{-j})$ denote a typical element of $Z (Z_{-j})$. Consider functions $r^i : X^{n_1-1} \times Z \to X$, for $i \in N_1$ and $r^j : X^{n_1} \times Z_{-j} \to Z_j$ for $j \in N_2$. Assume the following:
1. For \( i \in N_1 \), the functions \( r^i : X^{n_1-1} \times Z \rightarrow X \) are equal to each other and satisfy the following property: the function \( \tilde{r}_1 : X \times Z \rightarrow X \) given by \( \tilde{r}_1(x, z) = r^i(x, ..., x, z) \) is quasi-increasing in \( X \) and increasing in \( z \in Z \).

2. For all \( j \in N_2 \), the functions \( r^j \) are increasing.

Define \( r : X^{n_1} \times Z \rightarrow X^{n_1} \times Z \) in the obvious way by combining the functions \( r^i \) and \( r^j \) for \( i \in N_1 \) and \( j \in N_2 \). Then the set of fixed points of \( r \) is nonempty.

**Proof.** By (a), the assumptions of Corollary 39 are satisfied with \( \Theta = Z \). Thus, there exist an increasing function \( \tilde{e} : Z \rightarrow X \) such that \( \tilde{e}(z) \) satisfies \( r^i(\tilde{e}(z), ..., \tilde{e}(z), z) = \tilde{e}(z) \) for all \( i \in N_1 \) and all \( z \in Z \).

Define the function \( \tilde{r} : Z \rightarrow Z \) by its coordinates \( \tilde{r}_j : Z \rightarrow Z_j \), for \( j \in N_2 \) as follows: \( \tilde{r}_j(z) = r_j((\tilde{e}(z), ..., \tilde{e}(z), z-j)) \). Then the function \( \tilde{r} \) is increasing and defined on the complete lattice \( Z \). By Tarski’s Fixed Point Theorem 27, it has a fixed point \( \tilde{z} \). It is easy to see that \((\tilde{e}(z), ..., \tilde{e}(z), \tilde{z})\) is a fixed point of \( r \). \( \blacksquare \)

### 6 Conclusion

By building on an intersection point theorem due to Tarski (1955), the main result of this paper demonstrates that a pure-strategy Nash equilibrium exists in two-player games when one reaction curve is continuous and increasing and the other has no downward jumps. We elaborate in some detail on this kind of functions, called quasi-increasing in Tarski (1955), by deriving a number of results on natural operations involving such functions. These results include sufficient conditions for quasi-increasing functions to arise as argmax’s of parametric optimization problems. Some novel uniqueness results are also proved, which rely on a local (instead of the commonly used global) contraction property. The latter results may also be useful for other classes of games beyond those considered in this paper. Our approach also yields monotone comparative statics results for PSNE. Combining the latter results with the two-player existence theorem, different extensions of the existence result are developed for \( n \)-player games, which require some plausible additional complementarity and monotonicity structure. The special case of symmetric \( n \)-player games is also covered, thus unifying some existing results dealing mostly with Cournot oligopoly.

In an important part of the paper, we argue that the new results here have a promising scope of application for a wide variety of economic models, including standard oligopoly models (of price and quantity competition), pollution abatement games, and care provision games in law and economics. We illustrate in elementary ways all the various steps needed to actually apply some of the results of this paper.
for each of these models, tacitly establishing that strategic quasi-complementarity (or a quasi-increasing reaction curve) forms a convenient relaxation of strategic complementarity, and arises naturally in well-known economic models.

References


