

The natural projection approach to smooth production economies*

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Abstract

Smooth economies with private ownership of production of the kind previously considered in the literature are shown to be formally identical to exchange economies with demand functions that are properly adjusted for production. Smooth production economies therefore satisfy the same properties as smooth exchange economies. Some of these properties had resisted extension to production because of their reliance on properties of the no-trade equilibria, a concept specific to the exchange model. The transitivity of consumers' preferences is not even necessary for those properties to hold true.

1. Introduction

The study of smooth exchange economies has started with Debreu's proof of the generic finiteness and generic continuity of equilibrium allocations and price equilibria [8]. Properties about the nature and structure of singular and regular equilibria and economies, the existence of an index number with applications to the number of equilibria and, more generally, properties of the equilibrium manifold and its projection map into the parameter space have followed. See, for example, [1, 2, 6, 9, 11, 18, 19].

Nevertheless, only three properties of the exchange model have been extended to smooth production economies so far. The generic finiteness and generic continuity of equilibria is proved by Fuchs and Smale [10, 20]. The extension of Dierker's index number is achieved by T. Kehoe [15]. The diffeomorphism of the equilibrium manifold with a Euclidean space is partially extended by Jouini as the pathconnectedness and simple connectedness of the equilibrium manifold for production economies [13].

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A possible explanation for this relatively small number of extensions to production in general, and to smooth production in particular, is that several important properties of the exchange model depend on several remarkable properties that are satisfied at the no-trade equilibria as, for example, the regularity and the negative (semi-)definiteness of the Jacobian matrix of aggregate excess demand and also on the fact that the preimage of the set of Pareto optima by the natural projection is the set of no-trade equilibria [2]. The lack of obvious equivalent for production economies of the no-trade equilibria of the exchange model is therefore a major obstacle to the extension of these properties to production. The main contribution of this paper is to show that smooth economies with private ownership of production satisfy essentially the same properties as exchange economies. All that is required is that production satisfies smoothness and regularity properties that are usual in this kind of studies. See [10, 13, 15] and [20].

Our approach to the study of smooth production economies is to adjust consumers' demand functions for production. The exchange model defined by these production adjusted demand functions is then equivalent to the original production model. The production adjusted demand functions are very close to satisfy the properties considered in [6] that guarantee that the main properties of the exchange model are satisfied. Many proofs of [6] readily apply to the setup defined by these production adjusted demand functions. Only a small number have to be adapted to the new setup. As a result, the production model features exactly the same properties as the standard exchange model. The mathematical prerequisites for reading this paper can be found in the first chapters of [12] and [17] and are the same as those required by Debreu's paper [8].

This paper is organized as follows. Section 2 is essentially devoted to definitions and assumptions and to setting the notation. Section 3 defines the smooth production model and shows its equivalence with an exchange model defined by demand functions that have been production adjusted. Section 4 shows that smoothness and Walras law for consumers' demand functions suffice to imply smoothness and Walras law for the production adjusted demand function from which follows the global structure of the equilibrium manifold for the smooth production model. Section 5 deals with the properness of the natural projection and its implications for the smooth production model. Section 6 goes one step further by assuming that individual demand functions satisfy a property of their Slutsky matrices that is roughly equivalent to the convexity of (not necessarily transitive) individual preferences. This property then implies the uniqueness of equilibrium at an equilibrium allocation and the regularity of all equilibrium allocations in the exchange model associated with the production adjusted demand functions. These properties are essential in getting a complete picture of the relations between the equilibrium manifold and the endowment or parameter set. Concluding comments end this paper

with Section 7.

2. Assumptions, definitions and notation

2.1. Prices and goods

There is a finite number ℓ of goods. All prices are strictly positive. Depending on the context, prices are normalized by the numeraire convention $p_\ell = 1$ (the default normalization), by the simplex convention $p_1 + \dots + p_\ell = 1$ (convenient to deal with prices tending to zero), or simply not normalized in the few cases where we want to compute derivatives with respect to the numeraire price p_ℓ . The set of numeraire normalized prices is denoted by $S = \mathbb{R}_{++}^{\ell-1} \times \{1\}$ while S_Σ denotes the strictly positive simplex and \overline{S}_Σ the closed positive simplex of \mathbb{R}^ℓ .

2.2. Consumers and their demand functions

There is a finite number m of consumers. Consumer i is characterized by a demand function $f_i : S \times \mathbb{R} \rightarrow \mathbb{R}^\ell$ where $f_i(p, w_i)$ represents consumer i 's demand given the (numeraire normalized) price vector $p \in S$ and the consumer's wealth $w_i \in \mathbb{R}$. The same notation is used with non normalized or simplex normalized prices. With non-normalized prices, demand functions are homogenous of degree zero. Consumption faces no sign restrictions as in [4, 7, 11] and [13] for example.

We will state in a moment a few properties of these functions viewed as defined on $S \times \mathbb{R}$ and taking values in \mathbb{R}^ℓ . Before that, we recall the definition of the Slutsky matrix of some arbitrary smooth function $f_i : S \times \mathbb{R} \rightarrow \mathbb{R}^\ell$.

Slutsky matrix

The $\ell \times \ell$ Slutsky matrix $Sf_i(p, w_i) = (s_{jk}(p, w_i))$ associated with the map $f_i : S \times \mathbb{R} \rightarrow \mathbb{R}^\ell$ is defined for non-normalized prices by

$$s_{jk}(p, w_i) = \frac{\partial f_i^j}{\partial p_k}(p, w_i) + \frac{\partial f_i^j}{\partial w_i}(p, w_i) f_i^k(p, w_i).$$

Note $p^T Sf_i(p, w_i) = Sf_i(p, w_i) p = 0$.

Properties of consumers' demand functions

(S) (Smoothness) f_i is smooth.

(W) (Walras law) $p \cdot f_i(p, w_i) = w_i$ for all $(p, w_i) \in S \times \mathbb{R}$.

- (B)** (Boundedness from below) For K_i compact subset of \mathbb{R}^ℓ , there exists $B_i \in \mathbb{R}^\ell$ such that $B_i \leq f_i(p, p \cdot \omega_i)$ for $\omega_i \in K_i$ and $p \in S$.
- (A)** (Desirability) For any sequence (p^q, w_i^q) of *simplex* normalized price vectors and income converging to some (p^0, w_i^0) where $p^0 \in \overline{S_\Sigma} \setminus S_\Sigma$ (i.e., some coordinates of p^0 are equal to zero), then $\lim \|f_i(p^q, w_i^q)\| = +\infty$.
- (NQD)** (Negative quasi-definiteness of the Slutsky matrix) Assuming (S), the restriction of the quadratic form $z \in \mathbb{R}^\ell \rightarrow z^T S f_i(p, w_i) z$ to the hyperplane $H(p) = \{z \in \mathbb{R}^\ell \mid p^T z = 0\}$ perpendicular to the price vector p is negative definite for every $(p, w_i) \in S \times \mathbb{R}$.
- (NQSD)** (Negative quasi-semidefiniteness of the Slutsky matrix) Assuming (S), $z^T S f_i(p, w_i) z \leq 0$ for any $z \in \mathbb{R}^\ell$ and $(p, w_i) \in S \times \mathbb{R}$.

Comments

The above properties are satisfied by the demand functions that are derived from the budget constrained maximization of utility functions satisfying standard assumptions [4].

Differentiability (S) could be weakened to second order differentiability at almost no cost. Walras law (W) means that the value of consumer i 's demand is equal to the consumer's wealth. It is satisfied whenever the budget constraint $p \cdot x_i \leq w_i$ (where $x_i \in \mathbb{R}^\ell$ is the consumer's demand) is binding. **From now on, the default assumption will be that all consumers' demand functions satisfy (S) and (W).**

Property (NQD) is equivalent to the inequality $z^T S f_i(p, w_i) z < 0$, with $z \neq 0$ not collinear with the price vector p . Note that (NQD) is nothing more than the smooth but slightly stronger version of the strict quasi-concavity of the utility functions that represent preferences. However, the definition of (NQD) itself does not require the symmetry of the Slutsky matrix $S f_i(p, w_i)$. Some authors then prefer to talk of quasi-definiteness instead of definiteness to avoid any involuntary inference of symmetry.

Desirability (A) was introduced for individual demand functions by Debreu [8]. This property can be weakened without impairing the main properties of the exchange model. We will do that when we deal with production.

Boundedness from below (B) excludes the possibility for the demand of some goods to tend to $-\infty$ when endowments are bounded from below. We need this property because consumption is not restricted to be positive nor even bounded from below in our setup. The following lemma will be useful:

Lemma 1. Let $f_i : S \times \mathbb{R} \rightarrow \mathbb{R}^\ell$ be a demand function satisfying (B). For K_i compact subset of \mathbb{R}^ℓ and L compact interval of $[0, +\infty)$, there exists $B'_i \in \mathbb{R}^\ell$ such that $B'_i \leq f_i(p, p \cdot \omega_i + w_i)$ for $\omega_i \in K_i$, $w_i \in L$ and $p \in S$.

Proof. There exist x_i and $x'_i \in \mathbb{R}^\ell$ such that $x_i \leq \omega_i \leq x'_i$ for all $\omega_i \in K_i$. Let $\alpha_i > 0$ such that $0 \leq \lambda \leq \alpha_i$ for $\lambda \in L$. Let $e_\ell = (0, 0, \dots, 0, 1)$ be the unit vector parallel to the ℓ -th coordinate axis. Let $x''_i = x'_i + \alpha_i e_\ell$. The union of the two line segments $K'_i = [x_i, x'_i] \cup [x'_i, x''_i]$ is compact. For any $p \in S$, the intersection $K'_i \cap \{z_i \in \mathbb{R}^\ell \mid p \cdot z_i = p \cdot \omega_i + w_i\}$ is reduced to a point ω'_i and $f_i(p, p \cdot \omega_i + w_i) = f_i(p, p \cdot \omega'_i)$. It then suffices to apply (B) to the compact set K'_i . \square

2.3. The exchange model

A “consumer” is characterized by a “demand function” $f_i : S \times \mathbb{R} \rightarrow \mathbb{R}^\ell$ and an endowment vector $\omega_i \in \mathbb{R}^\ell$. The exchange model consists of m consumers whose demand functions are kept fixed while the endowment vector $\omega = (\omega_1, \dots, \omega_m)$ is varied in the parameter or endowment set $\Omega = (\mathbb{R}^\ell)^m$. An equilibrium of the exchange model is defined as:

Definition 1. *The pair $(p, \omega) \in S \times \Omega$ is an equilibrium of the exchange model defined by the m “demand functions” f_i , with $i = 1, \dots, m$, if:*

$$\sum_i f_i(p, p \cdot \omega_i) = \sum_i \omega_i. \quad (1)$$

The set E of $(p, \omega) \in S \times \Omega$ satisfying Equation (1) is known as the equilibrium manifold of the exchange model defined by the m demand functions (f_i).

The left hand-side of Equation (1) can be interpreted as the aggregate demand and the right hand-side as the aggregate supply in the exchange economy defined by the endowment vector ω and the demand functions (f_i).

We also recall the definition of a no-trade equilibrium.

Definition 2. *The pair $(p, \omega) \in S \times \Omega$ is a no-trade equilibrium of the exchange model defined by the m demand functions (f_i) if $\omega_i = f_i(p, p \cdot \omega_i)$ for $i = 1, \dots, m$. The set of no-trade equilibria is denoted by T .*

Recall our default assumption that all consumers' demand functions satisfy (S) and (W). The properties of the exchange model and of its equilibrium equation are studied in [6] for demand functions (f_i) satisfying some of the assumptions listed in Section 2.2.

2.4. Firms and their supply functions

There is a finite number n of firms. The activity of firm j is represented by a vector $y_j \in \mathbb{R}^\ell$. Outputs and inputs are represented by the positive and negative coordinates respectively of the activity vector y_j .

The activity of firm j is assumed to be a function $g_j(p)$ of the price vector $p \in S$. That function is known as the firm's supply function. The same

notation is used when prices are not normalized or simplex normalized. In the case of non-normalized prices, the supply function $g_j(p)$ is homogenous of degree zero, i.e., $g_j(\lambda p) = g_j(p)$ for any $\lambda > 0$.

The characterization of firms by their supply functions leads us to consider more generally the set of continuous functions $\gamma : S \rightarrow \mathbb{R}^\ell$. In the case where γ is differentiable, we denote by $D\gamma(p)$ the $\ell \times \ell$ Jacobian matrix of the map γ for the non-normalized price vector $p = (p_1, \dots, p_\ell)$. We consider in particular the following properties:

(PS) (Production smoothness) γ is smooth.

(PM) (Profit maximization) $p \cdot \gamma(p) > p \cdot \gamma(p')$ for $p' \neq p \in S$.

(PP) (Profit positivity) $p \cdot \gamma(p) \geq 0$ for any $p \in S$.

(PC) (Profit convexity) Assuming (PS), the restriction of the quadratic form $z \rightarrow z^T D\gamma(p) z$ to the hyperplane $H(p) = \{z \in \mathbb{R}^\ell \mid p^T z = 0\}$ is positive definite for every p .

Comments

All four properties are satisfied by the supply function g_j of a firm j whose production set Y_j is convex, contains the zero activity vector $0 \in \mathbb{R}^\ell$, satisfies the free disposal property and such that the efficient boundary $\partial Y_j^{\text{eff}}$ (i.e., the set of efficient activity vectors) is a smooth hypersurface with non-zero Gaussian curvature at its interior points. See [4] and [10].

More specifically, property (PS) results from the strict convexity of the efficient boundary Y_j^{eff} combined with non-zero Gaussian curvature. Property (PM) is nothing more than profit maximization. Property (PP) results from the possibility of inactivity. Property (PC) expresses the smooth strict convexity of the (numeraire normalized) “profit function” $p \rightarrow p \cdot \gamma(p)$, convexity that results from the production set Y_j being strictly convex. See in this respect Proposition 3 and its corollary in the next section. For more details, see [4]. From now on, **the default assumption for the firms’ supply functions is that (PS) and (PM) are satisfied.**

Note that (PS) could be weakened to second order differentiability at no cost.

2.5. Some properties of firms’ supply functions

The following properties are given here for their contribution to the global understanding of supply functions. Let $\gamma : S \rightarrow \mathbb{R}^\ell$ be an arbitrary supply function, i.e., a map that satisfies (PS) and (PM).

Proposition 1. We have

$$p \cdot \frac{\partial \gamma}{\partial p_k}(p) = 0$$

for $1 \leq k \leq \ell$ (non-normalized price vector).

Proof. Follows from the necessary first order conditions applied to the function $p' \rightarrow p \cdot g_j(p')$ that reaches its maximum at $p' = p$ by (PM). \square

Let $s(p) = p \cdot \gamma(p)$ be the “profit function” associated with the supply function $\gamma : S \rightarrow \mathbb{R}^\ell$. Let $Ds = \left(\frac{\partial s}{\partial p_1}, \dots, \frac{\partial s}{\partial p_\ell} \right)$ denote the gradient vector of s with respect to the non-normalized price vector $p = (p_1, \dots, p_\ell)$.

Proposition 2. We have $\gamma = Ds$.

Proof. The derivative of $s(p) = p \cdot \gamma(p)$ with respect to p_k (non-normalized price vector) is equal to

$$\frac{\partial s}{\partial p_k} = \gamma^k(p) + p \cdot \frac{\partial \gamma}{\partial p_k}(p) = \gamma^k(p)$$

by Proposition 1. \square

Restrict the “profit function” $s(p) = p \cdot \gamma(p)$ to *numeraire* normalized prices $p \in S$. Let $(D^2s)_{\ell\ell}$ be the associated $(\ell - 1) \times (\ell - 1)$ Hessian matrix of second order derivatives:

Proposition 3. Property (PC) is equivalent to matrix $(D^2s)_{\ell\ell}$ being positive definite.

Proof. From Proposition 2 comes $\gamma = Ds$, which implies $D\gamma = D^2s$ computed with non-normalized prices. Let $(D\gamma)_{\ell\ell}$ denote the $(\ell - 1) \times (\ell - 1)$ matrix obtained from the $(\ell \times \ell)$ matrix $D\gamma$ by deleting the ℓ -th row and column. It follows from the homogeneity of degree zero of γ with respect to non-normalized prices that (PC) is equivalent to $(D\gamma)_{\ell\ell}$ being positive definite and, therefore, to $(D^2s)_{\ell\ell}$ also positive definite. \square

Corollary 1. Assume (PC). The “profit function” $p \rightarrow s(p) = p \cdot \gamma(p)$ is a strictly convex function of the numeraire normalized prices $p \in S$.

Remark 1. The utility of profit positivity (PP) will show up later.

3. The private production model

3.1. Definition

Recall that there are finite numbers m and n of consumers and firms. Consumer i is endowed with the resources represented by the vector $\omega_i \in \mathbb{R}^\ell$ and the ownership of the fraction θ_{ij} of firm j , with $0 \leq \theta_{ij} \leq 1$. Let $\Theta = \{\theta = (\theta_{ij}) \in (\mathbb{R}_+^n)^m \mid \sum_j \theta_{ij} = 1 \text{ for } 1 \leq i \leq m\}$. An economy with private ownership of production is therefore represented by the pair $(\omega, \theta) \in \Omega \times \Theta$ where $\omega = (\omega_i) \in \Omega$ and $\theta = (\theta_{ij}) \in \Theta$.

3.2. Equilibrium equation

Let $p \in S$ be some price vector. Consumer i 's wealth is, for the price vector $p \in S$, equal to $w_i = p \cdot \omega_i + \sum_j \theta_{ij} p \cdot g_j(p)$. Consumer i 's demand is equal to $f_i(p, p \cdot \omega_i + \sum_j \theta_{ij} p \cdot g_j(p))$. Aggregate demand is the sum of the m consumers' demands.

Aggregate supply consists of two terms: 1) The sum of the individual endowments $\sum_i \omega_i$; 2) The aggregate supply of the m firms, namely the vector $\sum_j g_j(p)$.

Definition 3. *The price vector $p \in S$ is an equilibrium price vector of the economy with private ownership of production (ω, θ) if there is equality of aggregate demand and supply:*

$$\sum_i f_i(p, p \cdot \omega_i + \sum_j \theta_{ij} p \cdot g_j(p)) = \sum_i \omega_i + \sum_j g_j(p) \quad (2)$$

The triple $(p, (\omega, \theta))$ is then called an equilibrium of the private production model defined by the demand functions f_i and supply function g_j for $1 \leq i \leq m$ and $1 \leq j \leq n$.

The equilibrium analysis of the private production model consists in the study of the properties of the equilibria $(p, (\omega, \theta))$ as a function of the parameter $(\omega, \theta) \in \Omega \times \Theta$.

We take the ownership structure of production as fixed, which is equivalent to $\theta \in \Theta$ constant. Only the endowment parameter $\omega = (\omega_i) \in \Omega$ can vary. An equilibrium of the private production model for a given ownership structure $\theta \in \Theta$ is then a pair $(p, \omega) \in S \times \Omega$ that satisfies the equilibrium equation (2) for $\theta \in \Theta$ given. The set of all these equilibria is known as the "equilibrium manifold" though, at this stage, it is by no means obvious that this set is indeed a smooth manifold. Establishing this smooth manifold structure will be one of the properties we prove for the production model.

3.3. Equivalence between the private production model with fixed ownership structure and the exchange model

Given the price vector $p \in S$, define $\gamma_i(p) = \sum_j \theta_{ij} g_j(p)$. Consumer i receives from the firms he owns a wealth amount equal to $\sum_j \theta_{ij} p \cdot g_j(p)$, expression that is equal to $p \cdot \gamma_i(p)$. This leads us to define:

Definition 4. *The production adjusted demand function of consumer i associated with the fixed ownership structure of production $\theta \in \Theta$ is the map $h_i : S \times \mathbb{R} \rightarrow \mathbb{R}^\ell$ where*

$$h_i(p, w_i) = f_i(p, w_i + p \cdot \gamma_i(p)) - \gamma_i(p).$$

We associate with the m production adjusted demand functions $h_i : S \times \mathbb{R} \rightarrow \mathbb{R}^\ell$ an “exchange model” where the m consumers would each be characterized by a demand function equal to h_i and an endowment vector equal to ω_i , with $i = 1, 2, \dots, m$. We then have:

Proposition 4. *The pair $(p, \omega) \in S \times \Omega$ is an equilibrium of the private production model (with $\theta \in \Theta$ given) if and only if it is an equilibrium of the “exchange model” defined by the m (production adjusted) demand functions $h_i : S \times \mathbb{R} \rightarrow \mathbb{R}^\ell$.*

Proof. Let us write the equilibrium equation (2) as

$$\begin{aligned} \sum_i f_i(p, p \cdot (\omega_i + \gamma_i(p))) &= \sum_i \omega_i + \sum_j g_j(p), \\ \sum_i f_i(p, p \cdot (\omega_i + \gamma_i(p))) &= \sum_i \omega_i + \sum_i \left(\sum_j \theta_{ij} g_j(p) \right), \\ \sum_i f_i(p, p \cdot (\omega_i + \gamma_i(p))) &= \sum_i \omega_i + \sum_i \gamma_i(p), \end{aligned}$$

from which follows

$$\begin{aligned} \sum_i \left(f_i(p, p \cdot (\omega_i + \gamma_i(p))) - \gamma_i(p) \right) &= \sum_i \omega_i, \\ \sum_i h_i(p, p \cdot \omega_i) &= \sum_i \omega_i, \end{aligned}$$

the equilibrium equation of the exchange model defined by the m demand functions $h_i : S \times \mathbb{R} \rightarrow \mathbb{R}^\ell$. \square

The properties of the “exchange model” for the (production adjusted) demand functions (h_i), with $i = 1, \dots, m$, obviously depend on the properties of these “demand functions.” We will see shortly that these properties are very close (when they are not identical) to those required in [6] for the main properties of the exchange model to be satisfied.

4. Global structure of the equilibrium manifold of the production model

In this section, we show that (S) smoothness and (W) Walras law are satisfied by the production adjusted demand functions h_i . We then describe the implications of these two (default) properties for the global structure of the equilibrium manifold of the production model.

4.1. An auxiliary result

We start with:

Lemma 2. *Property (PS) (resp. (PM), (PP), and (PC)) is satisfied by any linear non-negative combination of functions from S into \mathbb{R}^ℓ satisfying (PS) (resp. (PM), (PP), and (PC)).*

Proof. Obvious. □

Proposition 5. *Assume that the m supply functions $g_j : S \rightarrow \mathbb{R}^\ell$ satisfy (PS) (resp. (PM), (PP) and (PC)). For θ given, the function $\gamma_i : S \rightarrow \mathbb{R}^\ell$ satisfies (PS) (resp. (PM), (PP) and (PC)) for $i = 1, \dots, m$.*

Proof. Follows readily from Lemma 2. □

4.2. Smoothness (S) and Walras law (W) for production adjusted demand functions

Recall the default assumption that all demand functions f_i satisfy (S) and (W) and all supply function g_j (PS) and (PM).

Proposition 6. *The production adjusted demand function h_i satisfies (S) and (W) for $i = 1, \dots, m$.*

Proof. Smoothness (S) is obvious. Walras law (W) for h_i follows readily from Walras law (W) for f_i by

$$\begin{aligned} p \cdot h_i(p, w_i) &= p \cdot (f_i(p, w_i + p \cdot \gamma_i(p)) - \gamma_i(p)), \\ p \cdot h_i(p, w_i) &= w_i + p \cdot \gamma_i(p) - p \cdot \gamma_i(p), \\ p \cdot h_i(p, w_i) &= w_i. \end{aligned}$$

□

Note that (S) and (W) are satisfied by h_i even if the default assumptions (PM) is (resp. (PS) and (PM) are) not satisfied.

4.3. Applications to the global structure of the equilibrium manifold

The main global properties of the equilibrium manifold in the exchange model do not require more than the default assumption that (S) and (W) are satisfied by all demand functions [4, 6]. This readily implies the following properties for the production model with fixed ownership structure $\theta \in \Theta$ and more particularly for its equilibrium manifold E and its subset of no-trade equilibria T associated with the production adjusted demand functions h_i .

Theorem 1.

- P1.** *The equilibrium manifold E is a smooth submanifold of $S \times \Omega$ of dimension ℓm .*
- P2.** *The set of no-trade equilibria T is diffeomorphic to $S \times \mathbb{R}^m$ and, therefore, to $\mathbb{R}^{\ell+m-1}$.*
- P3.** *The equilibrium manifold E is diffeomorphic to $T \times \mathbb{R}^{(\ell-1)(m-1)}$.*

Recall that it is not readily obvious from the definition of the equilibrium manifold by equation (2) that this set is indeed a smooth manifold and, even more, a smooth submanifold of $S \times \Omega$. This property follows from (P1).

Proof. It follows from Proposition 6 that all production adjusted demand functions h_i satisfy (S) and (W). It then suffices to apply the relevant global structure theorems of [4]. □

Remark 2. Note that the combination of (P2) and (P3) implies that the equilibrium manifold E is diffeomorphic to $\mathbb{R}^{\ell m}$. Property (P3) also gives us a global coordinate system that is very convenient when it comes to characterizing sets of equilibria that satisfy properties like being regular or stable for example. This diffeomorphism is also particularly useful when it comes to analyze properties that are satisfied at the no-trade equilibria but that fail to be satisfied for large trade vectors.

The only global property known previously for the equilibrium manifold of smooth production economies was its pathconnectedness and simple connectedness proved by Jouini [13].

5. Properness of the natural projection for the production model

We have not established that the production adjusted demand functions h_i satisfy desirability (A) or boundedness from below (B) because this does not

happen under realistic assumptions for the firms' supply functions g_j . These two properties would require stronger assumptions than those that are generally accepted. For example, Property (B) would be satisfied if every production set were bounded from above, which is a rather strong assumption.

The main role of (A) for at least one demand function f_i and (B) for all demand functions (f_i) is to imply the properness of the natural projection $\pi : E \rightarrow \Omega$ in the exchange model associated with those demand functions (f_i). See [6]. A solution is therefore to substitute a property that would make more economic sense than (A) and (B) in the production environment. This leads us to consider the following widely accepted property of the aggregate supply function.

5.1. The aggregate supply function

The aggregate supply function $g : S \rightarrow \mathbb{R}^\ell$ is the sum $\sum_j g_j$ of all net supply functions. We then have:

Proposition 7. *Let the m supply functions $g_j : S \rightarrow \mathbb{R}^\ell$ satisfy (PS) (resp. (PM), (PP) and (PC)). The aggregate supply function $g = \sum_j g_j : S \rightarrow \mathbb{R}^\ell$ satisfies (PS) (resp. (PM), (PP) and (PC)).*

Proof. Follows again from Lemma 2. □

Let us denote by $g^+(p)$ and $g^-(p)$ the positive and the opposite of the negative components of $g(p)$. This definition implies $g(p) = g^+(p) - g^-(p)$. The vector $g^-(p)$ represents the quantities of inputs (measured positively) while $g^+(p)$ represents the quantities of outputs. We then define the following property:

(FIFO) (Finite inputs finite outputs) For any $A \in \mathbb{R}_{++}^\ell$, there exists $B \in \mathbb{R}_{++}^\ell$ such that the inequality $g^-(p) \leq A$ implies the inequality $g^+(p) \leq B$.

Property (FIFO) simply states the impossibility of producing infinite quantities of some goods with only finite quantities of inputs. This property is often interpreted as expressing an idea of irreversibility in production. For alternative formulations of the same idea, see for example [10].

5.2. Properness of the natural projection

Theorem 2. *In addition to the default assumptions, all consumers' demand functions f_i satisfy (B) (boundedness from below), one consumer's demand function f_i at least satisfies (A) (desirability), all supply functions g_j satisfy (PP) (profit positivity) and the aggregate supply function $g = \sum_j g_j$ satisfies (FIFO) (finite inputs finite outputs).*

P4. *The natural projection $\pi : E \rightarrow \Omega$ is a proper map.*

P5. The modulo 2 degrees of the natural projection is defined and equal to one. The topological degree is also defined and equal to one for suitable orientations of the equilibrium manifold E and parameter space Ω .

Proof of P4. Let K be a compact subset of Ω . Let us show that the preimage $\pi^{-1}(K)$ is a compact subset of the equilibrium manifold E . First, the set $\pi^{-1}(K)$ is closed in E as the preimage of a closed set by a continuous map.

The compactness of $\pi^{-1}(K)$ will be proved by showing that every infinite sequence (p^q, ω^q) in $\pi^{-1}(K)$ has a convergent subsequence. It follows from the compactness of K that the sequence ω^q has a subsequence that converges to $\omega^0 \in K$. There is no loss in generality in considering directly this subsequence (p^q, ω^q) .

Let \tilde{p}^q be the simplex normalized price vector corresponding to the numeraire normalized price vector p^q . It follows from the compactness of the closed price simplex $\overline{S_\Sigma}$ that there is a subsequence of the sequence \tilde{p}^q that converges to $\tilde{p}^0 \in \overline{S_\Sigma}$.

If \tilde{p}^0 belongs to the interior S_Σ , then p^0 , the numeraire normalized price vector corresponding to \tilde{p}^0 belongs to S and is the limit of the (sub)sequence p^q . There is nothing more to prove.

Let us show therefore that the limit \tilde{p}^0 cannot belong to the boundary $\partial S_\Sigma = \overline{S_\Sigma} \setminus S_\Sigma$.

The proof proceeds by contradiction. Assume the contrary. The equilibrium equation satisfied by (p^q, ω^q) is

$$\sum_i f_i(p^q, p^q \cdot \omega_i^q + p^q \cdot \gamma_i(p^q)) = \sum_i \omega_i^q + g(p^q)$$

for any $q \geq 0$. It follows from $p^q \cdot \gamma_i(p^q) \geq 0$ by (PP) combined with Lemma 1 that $f_i(p^q, p^q \cdot \omega_i^q + p^q \cdot \gamma_i(p^q))$ is bounded from below for every i .

There exists therefore $B_i \in \mathbb{R}^\ell$ such that

$$B_i \leq f_i(p^q, p^q \cdot (\omega_i^q + \gamma_i(p^q))), \quad (3)$$

for $1 \leq i \leq m$ and any integer $q \geq 0$. This implies the inequality

$$\sum_i B_i \leq \sum_i f_i(p^q, p^q \cdot (\omega_i^q + \gamma_i(p^q))) \quad (4)$$

for any $q \geq 0$.

The image of the compact set K by the map $\omega \rightarrow \sum \omega_i$ is a compact set and, therefore, is bounded from above by some $A \in \mathbb{R}^\ell$: $\sum_i \omega_i \leq A$ for any $\omega \in K$.

The inequality

$$\sum_i B_i \leq A + g(p^q).$$

then follows from inequality (4) combined with the fact that (p^q, ω^q) is an equilibrium. The sequence $g(p^q)$ is therefore bounded from below. This implies that it is also bounded from above by (FIFO).

The boundedness of the sequence $g(p^q)$ has two consequences: 1) Each sequence $f_i(p^q, p^q \cdot (\omega_i^q + \gamma_i(p^q)))$ is bounded from below and from above, hence bounded; 2) We can find a subsequence of p^q such that the corresponding subsequence $g(p^q)$ converges to some limit $g^0 \in \mathbb{R}^\ell$.

It follows from the second consequence that the inner product $p^q \cdot g(p^q)$ tends to the finite limit $p^0 \cdot g^0$ and is finite. It follows from the positivity of profit (PP) satisfied by γ_i that the inequality $0 \leq p^q \cdot \gamma_i(p^q) \leq p^q \cdot g(p^q)$ implies that the sequence $p^q \cdot \gamma_i(p^q)$ belongs to a compact interval of the set of real numbers and, by considering a subsequence, converges to some non-negative real number w_i^0 .

There is no loss in generality in assuming that it is the demand function f_1 of consumer 1 that satisfies (A). We have seen that the sequence $f_1(p^q, p^q \cdot (\omega_1^q + \gamma_1(p^q)))$ is bounded. We have also seen that $p^q \cdot (\omega_1^q + \gamma_1(p^q))$ tends to a finite limit. These two conditions are incompatible with (A) if the limit p^0 belongs to the boundary $\partial S_\Sigma = \overline{S_\Sigma} \setminus S_\Sigma$, hence a contradiction.

Proof of P5. Smoothness (S) and (P4) (properness) imply that it is possible to define two concepts of degrees for the natural projection $\pi : E \rightarrow \Omega$: 1) the modulo 2 degree that is the remainder of the division by two of the number of equilibria associated with a regular value of the map π ; 2) the topological or Brouwer degree that counts the number of "oriented" equilibria above a regular value, a positively oriented equilibrium counting for +1 and a negatively oriented one for -1.

In order to compute the degrees of the natural projection, it then suffices to use the invariance of both degrees by proper homotopy combined with the global coordinate system for the equilibrium manifold given by (P3). The degree of the natural projection for arbitrary demand functions (h_i) is then the same as for analytically simpler demand functions. Simple computations then show that it is equal to +1 (for suitable orientations of the equilibrium manifold E and endowment set Ω in the case of the topological degree) and to one for the modulo two degree. See [6] for details.

□

Comments

It is a textbook property that a smooth proper map defines a finite open covering of its set of regular values. The set of singular values Σ of the map $\pi : E \rightarrow \Omega$ is therefore closed by the properness of π and this set has measure zero in Ω by Sard's theorem. Its complement, the set of regular values $\mathcal{R} = \Omega \setminus \Sigma$, is open with full measure. In addition, the number of equilibria associated with $\omega \in \mathcal{R}$ is then finite and locally constant. An equilibrium price selection map is

also defined and is smooth in some open neighborhood of every $\omega \in \mathcal{R}$. These are essentially the properties established for exchange economies by Debreu in [8], properties that were extended to smooth production economies by Fuchs [10]. The degree of a regular exchange economy was defined by Dierker [9] and generalized to smooth production economies by T. Kehoe [15].

6. Regularity of no-trade equilibria and number of equilibria

6.1. Property (NQD) for production adjusted demand functions

Before stating and proving (NQD) for the production adjusted demand functions h_i , we begin by a useful lemma with the default assumptions:

Lemma 3. *The Jacobian matrix at $p' = p$ of the map*

$$p' \rightarrow f_i(p', p' \cdot (\omega_i + \gamma_i(p'))) - f_i(p', p' \cdot (\omega_i + \gamma_i(p)))$$

is equal to 0.

Proof. The only term of row j and column k that is not obviously equal to zero in the computation of this Jacobian matrix is

$$\frac{\partial f_i^j}{\partial w_i}(p, p \cdot (\omega_i + \gamma_i(p))) \cdot p \cdot \frac{\partial \gamma_i}{\partial p_k}(p).$$

This term is equal to zero because $p \cdot \frac{\partial \gamma_i}{\partial p_k}(p)$ is equal to zero by Proposition 1.

□

We now address (NQD).

Proposition 8. *Assume that all supply functions $g_j : S \rightarrow \mathbb{R}^\ell$ satisfy (PP) and (PC). The production adjusted demand function $h_i : S \times \mathbb{R} \rightarrow \mathbb{R}^\ell$ satisfies (NQD) if the demand function $f_i : S \times \mathbb{R} \rightarrow \mathbb{R}^\ell$ satisfies (NQSD).*

Proof. A no-trade equilibrium in the exchange model with production adjusted demand function (h_i) is a pair $(p, \omega) \in S \times \Omega$ such that $\omega_i = h_i(p, p \cdot \omega_i)$ for $i = 1$ to m . We then have :

$$\omega_i + \gamma_i(p) = f_i(p, p \cdot (\omega_i + \gamma_i(p))).$$

Pick some ω_i that satisfies that relation and let $w_i = p \cdot \omega_i$.

The Slutsky matrix $Sh_i(p, w_i)$ of the (production adjusted) demand function h_i at (p, w_i) is the Jacobian matrix at $p' = p$ of the map

$$p' \rightarrow h_i(p', p' \cdot w_i) = f_i(p', p' \cdot (\omega_i + \gamma_i(p'))) - \gamma_i(p').$$

By writing

$$f_i(p', p' \cdot (\omega_i + \gamma_i(p'))) - \gamma_i(p') = f_i(p', p' \cdot (\omega_i + \gamma_i(p))) - \gamma_i(p') + (f_i(p', p' \cdot (\omega_i + \gamma_i(p'))) - f_i(p', p' \cdot (\omega_i + \gamma_i(p))))$$

this Jacobian matrix is the sum of the Jacobian matrices of the maps $p' \rightarrow f_i(p', p' \cdot (\omega_i + \gamma_i(p))) - \gamma_i(p')$ and $p' \rightarrow f_i(p', p' \cdot (\omega_i + \gamma_i(p'))) - f_i(p', p' \cdot (\omega_i + \gamma_i(p)))$.

The Jacobian matrix of the second map at $p' = p$ is equal to zero by Lemma 3. The Jacobian matrix of the first map is itself the sum of two matrices, the Jacobian matrix of the map $p' \rightarrow f_i(p', p' \cdot (\omega_i + \gamma_i(p)))$ and of the map $p' \rightarrow -\gamma_i(p')$. The restriction to the hyperplane $H(p)$ of \mathbb{R}^ℓ perpendicular to $p \in S$ of the quadratic form defined by the second matrix is negative definite by (PC). The first matrix is a standard Slutsky matrix since $\omega_i + \gamma_i(p) = f_i(p, p \cdot (\omega_i + \gamma_i(p)))$. The restriction of the quadratic form it defines to the hyperplane $H(p)$ is again negative quasi-semi-definite by (NQSD). The sum of a semidefinite negative form and a negative definite quadratic form is negative definite. □

6.2. Relations between T and $\pi(T)$

For the following theorem, let $J(p, \omega)$ be the $\ell \times \ell$ Jacobian matrix $J(p, \omega)$ of the aggregate excess demand map $p \rightarrow \sum_i h_i(p, p \cdot \omega_i) - \sum_i \omega_i$ for non-normalized prices $p \in \mathbb{R}_{++}^\ell$. This aggregate excess demand map is homogeneous of degree zero. The matrix $J(p, \omega)$ has a determinant equal to zero and its rank is less than or equal to $\ell - 1$. The equilibrium $(p, \omega) \in E$ is regular if and only if the rank of $J(p, \omega)$ is equal to $\ell - 1$. See [6]. This characterization is important because this Jacobian matrix coincides with the sum of individual Slutsky matrices $\sum_i Sh_i(p, p \cdot \omega_i)$ at the no-trade equilibrium $(p, \omega) \in T$.

Theorem 3. *Let all supply functions $g_j : S \rightarrow \mathbb{R}^\ell$ satisfy (PP) and (PC) and all demand functions $f_i : S \times \mathbb{R} \rightarrow \mathbb{R}^\ell$ satisfy (NQSD).*

P6. *We have $\pi^{-1}(\pi(T)) = T$.*

P7. *The quadratic form $z \in \mathbb{R}^\ell \rightarrow z^T J(p, \omega) z$ restricted to the hyperplane $H(p) = \{z \in \mathbb{R}^\ell \mid p^T z = 0\}$ is definite negative at every no-trade equilibrium $(p, \omega) \in T$.*

Proof. It suffices to reproduce the proofs given in [4]. □

Comments

Theorem 3 tells us that there is a unique (regular) equilibrium associated with the endowment vector $\omega = (\omega_i) \in \Omega$ whenever $\omega_i + \gamma_i(p) = f_i(p, \omega_i + p \cdot \gamma_i(p))$

for $i = 1, \dots, m$ and $(p, w_1, \dots, w_m) \in S \times \mathbb{R}^m$. The set of these endowments is diffeomorphic to $S \times \mathbb{R}^m$. At variance with the exchange model, these endowment vectors do not coincide with the Pareto optima of the production model when consumers' preferences are defined by utility functions.

Properties (P6) and (P7) give us a very simple way of computing the degrees of the natural projection under the stronger assumptions of (NQSD) for all demand functions f_i : any equilibrium allocation $\omega \in \pi(T)$ is then a regular value of the natural projection and the preimage of an equilibrium allocation being unique, the modulo 2 degree is obviously equal to 1 and the topological degree is also equal to 1 for a suitable orientation of the equilibrium manifold E and of Ω . See [2].

Property (P6) is considerably stronger than the two theorems of welfare economics. The latter are only equivalent to the diffeomorphism between the set of no-trade equilibria T and its image $\pi(T)$ (itself identical to the set of Pareto optima when preferences are representable by utility functions). Having a diffeomorphism does not exclude the possibility for other points of the equilibrium manifold to project in $\pi(T)$. That possibility is excluded by (P6) [2].

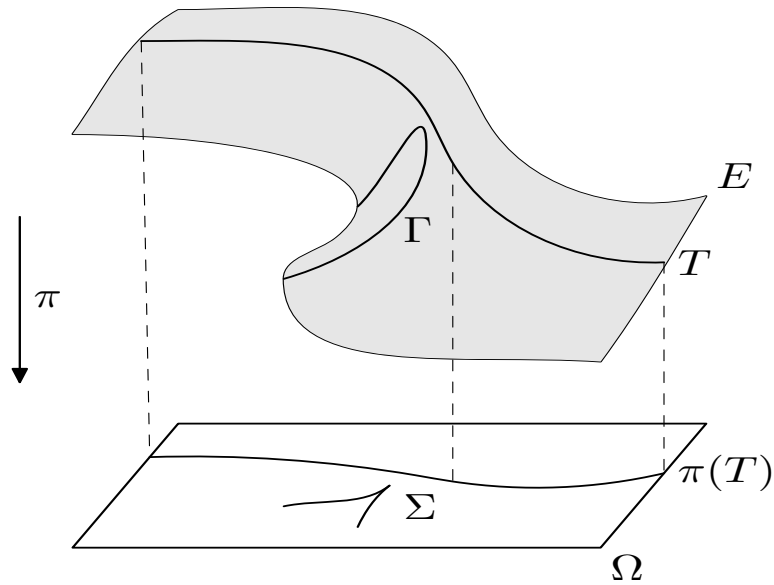


Figure 1: The equilibrium manifold and the natural projection for the smooth production model

Property (P7) implies that the no-trade equilibria are regular points of the natural projection. Combined with (P6), this implies that the set $\pi(T)$ is contained in the set of regular values \mathcal{R} . In addition, since T is connected by (P4), the set $\pi(T)$ is contained in one connected component of \mathcal{R} [2].

Another consequence of (P7) is that the set of critical points of the natural projection (i.e., the critical equilibria) is a closed subset with measure zero of the equilibrium manifold [5]. That property implies in turn that the set of singular economies is also closed with measure zero. Note that Sard's theorem enabled us to prove that property without requiring (NQSD) for all demand functions (f_i).

Property (P7) also implies that the no-trade equilibria are tatonnement stable [2], which in turn suffices to imply the pathconnectedness of the set of tatonnement stable equilibria [3].

The relations between the equilibrium manifold E and some of its subsets and the endowment set Ω and some of its subsets are illustrated by the picture.

7. Concluding comments

The main result of this paper is the extension of the equilibrium manifold and natural projection approach to the setup of smooth production economies. This implies far more than the generic finiteness and continuity of equilibria or the homeomorphism of the equilibrium manifold with a Euclidean space. For example, this approach underlines the remarkable role played by the amount of trade in creating discontinuities of equilibrium price selections in situations of multiple equilibria, a phenomenon that is therefore not limited to exchange economies. The identification of that phenomenon for smooth production economies is new.

A goal for future research will be to extend the results of this paper to production economies that feature convex polyhedral or conical production sets. The extension of the generic finiteness and continuity of equilibrium selections by Mas-Colell [16] and of the topological degree by T. Kehoe [14] is a good omen for the extension of the equilibrium manifold and natural projection approach to that setup.

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