An Influence-Based Theory of Strategic Voting in Large Elections

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Abstract

We consider a voting game where players' preferences depend both on the identity of the winning candidate and on the fraction of votes each candidate receives. Players' preferences are stochastic but, when voting, each player knows the entire profile of preferences. In this voting game, we show that the probability of having, in some equilibria, at least one player voting strategically (i.e, not voting for his favorite candidate) converges to zero as the number of players converges to infinity.

We then consider a version of the same voting game where each player can choose to influence other players. Influence is costly and its consequence is that some candidates are eliminated from the choice set of the influenced players. In the voting game with influence, we show that the probability of having, in every equilibria, at least one player voting strategically converges to one as the number of players converges to infinity.

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1 Introduction

During election times, it is not uncommon to hear voters arguing that, although they prefer candidate a, they will vote for candidate b to prevent candidate c from winning. This voting behavior may, in fact, be expected. As Duverger (1955) has argued:

> In cases where there are three parties operating under the simple-majority single-ballot system the electors soon realize that their votes are wasted if they continue to give them to a third party: whence their natural tendency to transfer their vote to the less evil of its two adversaries in order to prevent the success of the greater evil.

The above example illustrates the occurrence of strategic voting. By definition, voting is sincere if each voter votes for his favorite candidate and is strategic when it is not sincere. The above reasoning for the prevalence of strategic voting led Duverger to formulate what is now called the Duverger's Law, according to which, only two candidates should be expected to receive a positive fraction of votes under the simple-majority single-ballot system. In particular, the Duverger's Law predicts a significant fraction of voters (those who favor candidates without a real change of winning) to vote strategically.

A difficulty with the Duverger's Law is that, when the number of voters is large, voters should be essentially indifferent between all candidates. This is certainly the case when the vote of a single candidate has no effect on the winning candidate, or when the probability of affecting the winning candidate is small. In line with this intuition, we present a complete information model of a simple-majority single-ballot system where each voter attaches a (possible small but) positive weight to the fraction of votes each candidate receives, with his favorite candidate's fraction of votes being weighted more. Although voters vote knowing the entire profile of preferences, these profiles are unknown ex-ante. Specifically, before the game is played, players' preferences are selected independently and identically according to a given probability distribution. In contrast with the Duverger's Law, we show that, under some assumptions, the probability of having a profile of preferences such that all Nash equilibria of the resulting game feature sincere voting converges to one as the number of players goes to infinity.

Intuitively, the above result holds because only rarely will a player, simultaneously, be able to influence the winner and find it optimal to do so by voting strategically. Whenever this is the case, an example of which being presented by the model outlined above, the rationale for strategic voting must go beyond the nature of the simple-majority single-ballot system.

In order to rationalize strategic voting, we introduce to the model described above the possibility that voters influence others before the election takes place. We formalize influence abstractly as a costly activity that each voter can undertake and whose potential consequence is that some candidates are eliminated from the choice sets of other voters. Our formalization of influence allows us to rationalize strategic voting since, when influence is permitted, we obtain that the probability of having a profile of preferences such that all subgame perfect equilibria of the resulting game feature strategic voting converges to one as the number of players goes to infinity.

This result is in sharp contrast with the one obtained when influence is not present. The reason for the latter is that, when the number of players is large, each player's payoffs depend essentially only on his own choice. This is not the case when influence is allowed. Because influence is assumed to be costly, a player will chose to influence only if this choice has a sufficiently large effect on the outcome of the election, in particular, by having some voters voting strategically. For this to be the case, there can be at most a relatively small number of voters choosing to influence. Thus, the possibility of influence brings the strategic nature present in elections with a small number of voters to elections featuring a large number of voters.

2 The Model

There are three candidates and n players. The set of candidates is $C = \{a, b, c\}$ and the set of players is $I_n = \{1, \ldots, n\}$. There are two periods. In the first period, each player chooses whether to influence others or not and, in case he decides to influence, how to influence. The effect of influence is that it may change the vote of those who are being influenced. This is modeled by restricting the action space of an influenced player in period 1. In the second period, there is an election and players vote for a candidate (abstention is not allowed).

Each player's preferences depend on the outcome of the election, on the player's ranking of the candidates and on whether or not he chooses to influence. For each player i, i's action set in the first period is $A_i^1 = C \cup \{e\}$ with the following interpretation: a choice of $a_i^1 = k \in C$ means that player i influences others to vote for candidate k, whereas a choice of $a_i^1 = e$ means that player i is not influencing. Influencing costs $0 < c < 1$.

Player i's ranking of the candidates is formalized in the following way: Let Θ be the set of $\theta = (\theta_k)_{k \in C}$ such that for all $x \in \{0, 1, 2\}$ there exists a unique $k \in C$ such that $\theta_k = x$. For example, $\theta = (2, 0, 1)$ belongs to Θ and means that a is the favorite candidate and c is the second favorite candidate.

Each player's preferences depends also on the fraction of votes each candidate receives. Let $\Delta(C)$ be the set of probability distributions over C. For all $k, k' \in C$, let $W_k = \{\pi \in \Delta(C) : \pi_k > \pi_{\hat{k}} \text{ for all } \hat{k} \neq k\}$ be the set of distributions in which candidate k wins, $T_{k,k'} = \{\pi \in \Delta(C) : \pi_k = \pi_{k'} > \pi_{\hat{k}} \text{ where } \hat{k} \in C \setminus \{k, k'\}\}$ be the set of distributions in which candidates k and k' tie and $T = \{\pi \in \Delta(C) : \pi_a = \pi_b = \pi_c\}$ the set of distribution where all candidates tie. The payoff of player i depends on the fraction of votes each candidate k receives, weighted by the corresponding weight θ_k, and on the identity of the winner. The fraction of votes each candidate receives is weighted less than the identity of the winner. Specifically, for all $n \in \mathbb{N}$, the weight of the latter is 1, while that of the former is $\lambda_n \in (0, 1)$ with $\lim_n \lambda_n < c/2$. Furthermore, when there is a tie, each candidate that ties has an equal probability of winning. Thus, for all $i \in I_n$, the preference $v_i(a_i^1, \pi, \theta)$ of player i is defined by

$$v_i(e, \pi, \theta_i) = \lambda_n \theta_i \cdot \pi + \sum_{k \in C} \left(\theta_{i,k} 1_{W_k}(\pi) + \sum_{k' \in C \setminus \{k\}} \frac{\theta_{i,k} + \theta_{i,k'}}{2} 1_{T_{k,k'}}(\pi) + \frac{\theta_{i,k}}{3} 1_T(\pi) \right) \text{ and}$$

$v_i(a_i^1, \pi, \theta_i) = v_i(e, \pi, \theta) - c$ if $a_i^1 \neq e$.

The following notation will be used. For all $\theta \in \Theta$, let $f(\theta)$ denote the favorite candidate and $f_2(\theta)$ denote the second favorite candidate for the preferences determined by θ. Thus, for all $k \in C$, $f(\theta) = k$ if and only if $\theta_k = 2$ and $f_2(\theta) = k$ if and only if $\theta_k = 1$. It is clear that θ provides a strict ordering of the candidates. While the particular cardinal representation is not important, it facilitates the analysis. Furthermore, note that $f(\theta)$ and $f_2(\theta)$ fully determine θ and so sometimes we abuse notation and write $\theta = (f(\theta), f_2(\theta))$.

For all $i \in I_n$, player i's preferences are determined by a distribution μ on Θ. For all $k, k' \in C$ such that $k \neq k'$, let $\mu_{(k,k')} > 0$ denote the probability of $(f(\theta), f_2(\theta)) = (k, k')$. Let $\mu_i^n = \mu$ for all $i \in I_n$ and assume that $\{\mu_i^n\}_{i=1}^n$ are independent. Let μ_n denote the corresponding product measure on Θ^n. For all $k \in C$, let $\mu_k = \sum_{k' \neq k} \mu_{(k,k')}$. We assume that $\mu_b < \mu_a$ and $\mu_c + \mu_{(k,c)} < \mu_{k'}$ for all $k, k' \in C$ such that $k \neq k'$. Thus, candidate a has the highest probability of being each player's favorite, although this probability can be only slightly higher than that of candidate b. In contrast, the probability of candidate c being the favorite is smaller and bounded away (by $\min_{k \neq c} \mu_{(c,k)}$) from that of candidates b and a.

Our goal is to analyze the equilibria of the following game as the number n of players converges to infinity. For all $n \in \mathbb{N}$, the set of players is I_n. Players' ranking of candidates $\theta^n = (\theta_i^n)_{i \in I_n}$ is drawn according to μ_n. Players observe θ^n and chose, in period 1, $a^1 = (a_1^1, \ldots, a_n^1)$, where $a_i^1 \in A_i^1 = C \cup \{e\}$ for all $i \in I_n$. Let $A^1 = A_1^1 \times \cdots \times A_n^1 = (C \cup \{e\})^n$. As a result of the choice of a^1, we obtain action spaces $A_i^2(\theta^n, a^1)$ for player i in period 2 with $A_i^2(\theta^n, a^1) \subseteq C$ for all $i \in I_n$. See Section 3 for the assumptions we place on how influence works. In period 2, players observe the choice of a^1 and chose $a^2 = (a_1^2, \ldots, a_n^2)$ such that $a_i^2 \in A_i^2(\theta^n, a^1)$ for all $i \in I_n$. We denote such game by $G_n(\theta^n)$.

We next define the notion of strategy. We first consider the case of pure strategies, the case of mixed strategies being considered in Section 4. For all $n \in \mathbb{N}$ and $\theta^n \in \Theta^n$, a strategy profile (a strategy, for short) in $G_n(\theta^n)$ is $s = (s_i^1, s_i^2)_{i \in I_n}$ such that $s_i^1 \in A_i^1$ and s_i^2 maps A^1 into C with $s_i^2(a^1) \in A_i^2(\theta^n, a^1)$ for all $i \in I_n$ and $a^1 \in A^1$.

We turn to the definition of players' payoff functions. For all action profiles a^2, the distribution over candidates induced by a^2 is $\pi(a^2)$ defined by $\pi_k(a^2) = |\{i \in I_n : a_i^2 =$

$k\}|/n$ for all $k \in C$. This notion is used to define, for all strategies s, the distribution $\pi(s)$ over candidates induced by the strategy s in the following way: $\pi(s) = \pi(s^2(s^1))$. Thus, for all $k \in C$, $\pi_k(s)$ is the fraction of players who vote for candidate k. As mentioned above, $\pi(s)$ influences the payoff of each player, which depends also on the player's ranking of candidates and on whether or not he has chosen to influence. For all $i \in I_n$, player i's payoff in $G_n(\theta^n)$ is

$$u_i(s) = v_i(s_i^1, \pi(s), \theta_i^n)$$

for all strategies s.

We use subgame perfect equilibrium as our equilibrium concept. For all $n \in \mathbb{N}$, $\theta^n \in \Theta^n$ and $a^1 \in A^1$, let $G_n(\theta^n, a^1)$ denote the subgame of $G_n(\theta^n)$ starting at period 2 and after a^1 (and θ^n) has been observed. A strategy s is a subgame perfect equilibrium (an equilibrium, for short) of $G_n(\theta^n)$ if s is a Nash equilibrium of $G_n(\theta^n)$ and $s^2(a^1)$ is a Nash equilibrium of $G_n(\theta^n, a^1)$ for all $a^1 \in A^1$. Let $E(\theta^n, a^1)$ denote the set of Nash equilibria of $G_n(\theta^n, a^1)$ and $E(\theta^n)$ the set of subgame perfect equilibria of $G_n(\theta^n)$.

Finally, we present the definition of sincere and of strategic voting. For all $n \in \mathbb{N}$ and $\theta^n \in \Theta^n$, a second period action profile a^2 is sincere relative to θ^n if $a_i^2 = f(\theta_i^n)$ for all $i \in I_n$ and is strategic relative to θ^n if it is not sincere relative to θ^n. A strategy s is sincere (resp. strategic) relative to θ^n if $s^2(s^1)$ is sincere (resp. strategic) relative to θ^n.

3 Influence and Strategic Voting

In this section, we discuss the assumptions we place on how influence operates in our model. With these assumptions in place, we show (Theorem 2 below) that the probability of having, in every equilibria, at least one player voting strategically converges to one as the number of players converges to infinity. This is in contrast with the case where players are not allowed to influence. In this case, we show (Theorem 1 below) that the probability of having, in some equilibria, at least one player voting strategically converges to zero as the number of players converges to infinity.

Our formalization of influence focus only on its consequences, and not on the way influence operates. In our model, the consequence of influence is that some players may be convinced to vote for a candidate (formally, their action set in the second period is a singleton) or not to vote for a candidate (formally, that candidate will not belong to their action set in the second period). Since the impact of influence is on players' second period action sets, our assumptions impose conditions on those sets.

In the case where no one influences, there should be no consequence on the players' second period action sets. In other words, there should be no restriction on players' choices and so each player should be able to vote for any candidate. Formally, letting $\bar{e}$ denote the first period action profile a^1 satisfying $a_i^1 = e$ for all $i \in I_n$, we assume that $A_i^2(\theta^n, \bar{e}) = C$ for all $i \in I_n$.

As a consequence of this assumption, the subgame $G_n(\theta^n, \bar{e})$ corresponds, for all $n \in \mathbb{N}$ and $\theta^n \in \Theta^n$, to a voting game where players can vote for all candidates and have preferences that depend both on who is elected and on the representation that each candidate has. The latter aspect implies that there is a cost of voting for a least preferred candidate because that has a (small, but positive) impact on the fraction of votes casted for the player's favorite candidate. Hence, unless the election results in a tie, one expects players to vote for their favorite candidate, i.e., that they vote sincerely. Furthermore, one expects ties to be unlikely when there is a large number of players. Theorem 1 shows that this intuition is correct by establishing that, with a probability converging to 1, all equilibria of $G_n(\theta^n, \bar{e})$ are sincere.

Theorem 1 *The following holds:*

$$\lim_n \mu_n(\{\theta^n \in \Theta^n : s^2 \textit{ is sincere relative to } \theta^n \textit{ for all } s^2 \in E(\theta^n, \bar{e})\}) = 1.$$

Theorem 1 implies that some modification of the (standard, no influence) voting game is needed in order to allow for strategic voting in equilibrium. Thus, we seek conditions on influence that imply strategic voting in equilibrium.

Although the assumptions we make are on the consequences of the influence process, they are meant to reflect properties that a reasonably specified influence process would have. In particular, we view influence are a costly process requiring players to process information and to organize their thoughts. This view is already reflected in the assumption that players that influence have their payoff reduced (by $c > 0$). The following two assumptions present what may regarded as consequences of the costly nature of influence.

Our first assumption says that if a player influences for some candidate k, then his action space in the second period is the singleton set $\{k\}$. This means that players who chose to influence have convinced themselves to vote for (and support) a candidate, and therefore they will not consider the possibility of voting for a different candidate.

Assumption 1 *For all $n \in \mathbb{N}$, $\theta^n \in \Theta^n$, $a^1 \in A^1$ and $i \in I_n$, if $a_i^1 = k \in C$, then $A_i^2(\theta^n, a^1) = \{k\}$.*

The following assumption limits the ability of influence by ruling out the possibility that a player be convinced to vote for his least favorite candidate. Moreover, it requires that if a player bothers to eliminate a candidate, then that player will also eliminate his least preferred candidate. This assumption is weakened by imposing this condition only on first period action profiles a^1 that are sincere relative to θ^n in the following sense: for all $n \in \mathbb{N}$ and $\theta^n \in \Theta^n$, a first period action profiles $a^1 \in A^1$ is sincere relative to θ^n if, for all $i \in I_n$, $a_i^1 = k \in C$ implies $k = f(\theta_i^n)$.

Assumption 2 *For all $\theta^n \in \Theta^n$, $i \in I_n$, $k \in C$ and $a^1 \in A^1$ that are sincere relative to θ^n, if $A_i^2(\theta^n, a^1) \neq C$ and $\theta_{i,k}^n = 0$, then $k \notin A_i^2(\theta^n, a^1)$.*

Assumption 2 implies that, when influence is sincere, either $A_i^2(\theta^n, a^1) = C$, $A_i^2(\theta^n, a^1) = \{f(\theta_i^n), f_2(\theta_i^n)\}$, $A_i^2(\theta^n, a^1) = \{f(\theta_i^n)\}$ or $A_i^2(\theta^n, a^1) = \{f_2(\theta_i^n)\}$.

Although we model influence as affecting players' second period action spaces, we can think (and it will be convenient to do so) that the effect of influence is to change players' preferences. For example, if a player has a singleton second period action set $\{k\}$, we can think that player's ranking of candidates, after influence, is such that k is his favorite candidate. With this view in mind, influence would be defined by a function that maps the set of individuals' rankings into itself. Although this way of representing influence entails some loss of information (in the example above, we can think that k is the favorite candidate, but more is true: the player can only vote for k), it is convenient because it allows for a comparison between the case where no influence is allowed and the case where some influence is being exerted (this shows up in Lemma 1 and Assumption 3 below). This is done by comparing the corresponding vectors of players' rankings of candidates: θ^n when there is no influence and $g(\theta^n, a^1)$ when there is influence (which shows as part of a^1). For all $\theta^n \in \Theta^n$ and $a^1 \in A^1$, define $g(\theta^n, a^1) \in \Theta^n$ by

$$g_i(\theta^n, a^1) = \begin{cases} \theta_i^n & \text{if } A_i^2(\theta^n, a^1) = C, \\ (k_1, k_2) & \text{if } A_i^2(\theta^n, a^1) = \{k_1, k_2\} \text{ and } \theta_{i,k_1}^n > \theta_{i,k_2}^n, \\ (k_1, k_2) & \text{if } A_i^2(\theta^n, a^1) = \{k_1\} \text{ and } \theta_{i,k_2}^n > \theta_{i,k_3}^n, k_3 \neq k_1. \end{cases}$$

Before we discuss the definition of $g_i(\theta^n, a^1)$, note that under Assumption 2, if a^1 is sincere relative to θ^n and $|A_i^2(\theta^n, a^1)| = 2$, then $A_i^2(\theta^n, a^1) = \{f(\theta_i^n), f_2(\theta_i^n)\}$ and so $g_i(\theta^n, a^1) = \theta_i^n$. Thus, for those a^1 that are sincere relative to θ^n, we can equivalently define $g_i(\theta^n, a^1)$ by

$$g_i(\theta^n, a^1) = \begin{cases} (k_1, k_2) & \text{if } A_i^2(\theta^n, a^1) = \{k_1\} \text{ and } \theta_{i,k_2}^n > \theta_{i,k_3}^n, k_3 \neq k_1, \\ \theta_i^n & \text{otherwise.} \end{cases}$$

In words, the first condition says that if the second period action set contains only one candidate, then this candidate is the favorite candidate under $g_i(\theta^n, a^1)$ and that the two remaining candidates are ranked as they were under θ_i^n. The second condition says that there is no change in the preferences of players that are not influenced and on those who eliminate only their least favorite candidate. For convenience, for all $a^1 \in A^1$, let $g_{a^1} : \Theta^n \to \Theta^n$ denote the function $g(\cdot, a_1)$.

The usefulness of defining a new vector $g_{a^1}(\theta^n)$ of rankings can be seen in Assumption 3 below. Intuitively, voting is sincere when influence is not allowed (in the sense of Theorem 1) because strategic voting occurs only if there are ties and and ties are infrequent. Thus, one easy way to obtain strategic voting through influence would be to assume that influence makes ties became frequent. However, intuitively, this case seems to be non-generic and Assumption 3 rules it out. To formally write this assumption, we first use $g_{a^1}(\theta^n)$ to characterize the strategic equilibria of $G_n(\theta^n, a^1)$ in Lemma 1 below, and then use the characterization of this lemma to impose conditions on g_{a^1} that rule out the non-generic case described above.

The following concepts are used in Lemma 1. For all $\theta^n \in \Theta^n$ and $a^1 \in A^1$, let the distribution $\pi(g_{a^1}(\theta^n))$ over candidates induced by players' ranking be defined by

$$\pi_k(g_{a^1}(\theta^n)) = \frac{|\{i \in I_n : f(g_i(\theta^n, a^1)) = k \text{ and } k \in A_i^2(\theta^n, a^1)\}|}{n} \text{ and}$$
$$\pi_{(k,k')}(g_{a^1}(\theta^n)) = \frac{|\{i \in I_n : (f, f_2)(g_i(\theta^n, a^1)) = (k, k') \text{ and } \{k, k'\} \subseteq A_i^2(\theta^n, a^1)\}|}{n}$$

for all $k, k' \in C$ such that $k \neq k'$. Thus, $\pi_{(k,k')}(g_{a^1}(\theta^n))$ is the fraction of players who can vote for k and k' and for whom these are the two favorite candidates, with k being preferred to k'. A particular case of interest corresponds to $a^1 = \bar{e}$ and we let $\pi(\theta^n) = \pi(g_{\bar{e}}(\theta^n))$, i.e., $\pi_k(\theta^n) = |\{i \in I_n : f(\theta_i^n) = k\}|/n$ for all $k \in C$. The following definition extends the notion of a sincere action profile. An action profile a^2 is sincere relative to $g_{a^1}(\theta^n)$ if $a_i^2 = f(g_i(\theta^n, a^1))$ for all $i \in I_n$ and is strategic relative to $g_{a^1}(\theta^n)$ if it is not sincere relative to $g_{a^1}(\theta^n)$.

Lemma 1 *If Assumption 2 holds, $n > 2$, $\theta^n \in \Theta^n$, $a^1 \in A^1$ is sincere relative to θ^n, $a^2 \in E(\theta^n, a^1)$ and a^2 is strategic relative to $g_{a^1}(\theta^n)$, then one of the following conditions hold:*

1. *There exist $k \in C$ and $k' \in C$, $k' \neq k$, such that $\pi_{(k,k')}(g_{a^1}(\theta^n)) = 0$.*

2. *For some permutation (k_1, k_2, k_3) of C,*

$$\pi_{k_1}(g_{a^1}(\theta^n)) + \pi_{(k_3,k_1)}(g_{a^1}(\theta^n)) = \pi_{k_2}(g_{a^1}(\theta^n)) + \pi_{(k_3,k_2)}(g_{a^1}(\theta^n)).$$

Assumption 3 requires that if the necessary condition for strategic equilibria obtained in Lemma 1 is not satisfied when the ranking of candidates is described by $\theta^n = g_{\bar{e}}(\theta^n)$, then it is also not satisfied when the ranking of candidates is described by $g_{a^1}(\theta^n)$. This requirement is weakened by imposing it only when influence is sincere and for players' rankings in a sequence of sets with probability converging to one.

Assumption 3 *For all* $n \in \mathbb{N}$*, there exist* $X_n \subseteq \Omega^n$ *such that* $\lim_n \mu_n(X_n) = 1$ *and the following conditions hold for all* $\theta^n \in X_n$ *and* $a^1 \in A^1$ *such that* a^1 *is sincere relative to* θ^n*:*

1. *For all* $k, k' \in C$ *such that* $k \neq k'$*, if* $\pi_{(k,k')}(\theta^n) > 0$*, then* $\pi_{(k,k')}(g_{a^1}(\theta^n)) > 0$*.*

2. *For all permutations* (k_1, k_2, k_3) *of* C*, if* $\pi_{k_1}(\theta^n) + \pi_{(k_3,k_1)}(\theta^n) \neq \pi_{k_2}(\theta^n) + \pi_{(k_3,k_2)}(\theta^n)$*, then* $\pi_{k_1}(g_{a^1}(\theta^n)) + \pi_{(k_3,k_1)}(g_{a^1}(\theta^n)) \neq \pi_{k_2}(g_{a^1}(\theta^n)) + \pi_{(k_3,k_2)}(g_{a^1}(\theta^n))$*.*

As a consequence of Assumption 2, the form of influence that will necessarily lead to strategic voting is when a player's second period action set becomes the singleton set consisting of his second favorite candidate. The following assumption imposes a lower bound on the fraction of players that are influenced in this sense. It states that, for sincere first period action profiles, if all players that influence are influencing for the same candidate, then the fraction of players that will have that candidate has their single possible choice is sufficiently high (at least equal to $\mu_a - \mu_b$). Recall that the probability of candidate a being the favorite is higher than that of candidate b and that the difference between the two is therefore $\mu_a - \mu_b > 0$. Thus, by the law of large numbers, the same is true for the fraction of players favoring these two candidates. Thus, the requirement for the above lower bound is necessary for Theorem 2 since otherwise there would be an equilibrium where all players vote sincerely and chose not to influence, and where candidate a wins by a positive margin.

We can also interpret Assumption 4 as placing a standard Inada-type condition on the marginal impact of influence. In fact, Assumption 4 would follows from the assumption that the marginal impact of influence, at the point where no one is influencing, is larger than $\mu_a - \mu_b$, and that further increases on the number of players influencing for a given candidate are nonnegative.

Assumption 4 *For all* $n \in \mathbb{N}$*, there exist* $Y_n \subseteq \Omega^n$ *and* $\delta_n \in (0, 1)$ *such that* $\lim_n \mu_n(Y_n) = 1$*,* $\lim_n \delta_n = \delta \in (\mu_a - \mu_b, \mu_a)$ *and the following condition holds for all* $\theta^n \in Y_n$ *and* $a^1 \in A^1$ *such that* a^1 *is sincere relative to* θ^n*:*

If there is $k \in C$ *such that* $|\{j \in I_n : f(\theta^n_j) \neq k\}|/n \geq \delta_n$ *and* $a^1_i = k$ *for all* $i \in I_n$ *with* $a^1_i \in C$*, then* $|\{j \in I_n : f(\theta^n_j) \neq k \text{ and } A^2_j(\theta^n, a^1) = \{k\}\}|/n \geq \delta_n$*.*

Our final assumption imposes monotonicity conditions on the influence process. The first part considers a decline in the number of players influencing for a particular candidate. It states that if a player has not eliminated some of the other candidates to start with, then the player will not eliminate that candidate after the decline occurs. The second part states that if no player influences for a given candidate, then it cannot be the case that that candidate is the only one considered by some player.

Assumption 5 *The following conditions hold for all $n \in \mathbb{N}$, $\theta^n \in \Theta^n$ and $a^1 \in A^1$ such that a^1 is sincere relative to θ^n:*

1. *If, for some $i \in I_n$ and $k \in C$, $a_i^1 = k$ and $\bar{a}^1 = (e, a_{-i}^1)$, then, for all $j \in I_n$ and all $k' \in C \setminus \{k\}$, $k' \in A_j^2(\theta^n, a^1)$ implies $k' \in A_j^2(\theta^n, \bar{a}^1)$.*
2. *If there exists $k \in C$ such that $a_i^1 \neq k$ for all $i \in I_n$, then $A_i^2(\theta^n, a^1) \neq \{k\}$ for all $i \in I_n$.*

Under Assumptions 1–5, it follows that the probability that *all* equilibria are strategic converges to *one*.

Theorem 2 *If Assumptions 1–5 hold, then*

$$\lim_n \mu_n(\{\theta^n \in \Theta^n : s^2(s^1) \text{ is strategic relative to } \theta^n \text{ for all } s \in E(\theta^n)\}) = 1.$$

Theorem 2 is in sharp contrast with the case where no influence is allowed, since there, the probability of having *at least one* equilibrium with strategic voting converges to *zero*. This provides a sense in which influence explains strategic voting.

4 Mixed Strategies and Existence of Equilibrium

Theorems 1 and 2 characterize the pure strategy equilibria of voting game when influence is allowed and when it is not. Thus, whenever an equilibrium exists in those games, it will have the property described in those results.

The question of existence of equilibrium arises naturally for two reasons. First, Theorems 1 and 2 are trivially true when the equilibrium set is empty. Second, it is conceivable that our assumptions be incompatible with the existence of a pure strategy equilibrium.

Theorem 3 below shows that, with a probability converging to one, pure strategy equilibria exist in the voting game where influence is not allowed. Since, with a probability converging to one, all pure strategy equilibria are sincere in that game, Theorem 3 will be a consequence of the following lemma that provides a sufficient condition for sincere voting to be an equilibrium.

Lemma 2 *Let $n \in \mathbb{N}$, $\theta^n \in \Theta^n$, $a^1 \in A^1$ and a^2 be sincere relative to $g_{a^1}(\theta^n)$. If there exist $k \in C$ such that $\pi_k(g_{a^1}(\theta^n)) > \pi_{k'}(g_{a^1}(\theta^n)) + 1/n$ for all $k' \neq k$, then $a^2 \in E(\theta^n, a^1)$.*

In light of Lemma 2, we establish existence of pure strategy, sincere, equilibrium in $G_n(\theta^n, \bar{e})$ by showing that the probability of the set of candidate rankings θ^n such that $\pi_a(\theta^n) > \pi_k(\theta^n) + 1/n$ for all $k \neq a$ converges to one.

Theorem 3 *The following holds:*

$$\lim_n \mu_n(\{\theta^n \in \Theta^n : E(\theta^n, \bar{e}) \neq \emptyset \text{ and } s^2 \text{ is sincere relative to } \theta^n \text{ for all } s^2 \in E(\theta^n, \bar{e})\}) = 1.$$

We turn to the question of existence of equilibria when influence is allowed. This question can be decomposed into two parts: existence of equilibrium in the first period and in the second period.

As argued above, a pure strategy equilibrium exists with high probability when influence is not allowed. In order to avoid our formalization of influence to be regarded as special, or non-generic, it may be desirable to have a similar property holding when influence is allowed. Assumption 6 presents such requirement, weakened by imposing it only to sincere first period action profiles.

Assumption 6 *For all $n \in \mathbb{N}$, there exist $Z_n \subseteq \Omega^n$ such that $\lim_n \mu_n(Z_n) = 1$ and the following condition holds for all $\theta^n \in Z_n$ and $a^1 \in A^1$ such that a^1 is sincere relative to θ^n: There exists $k \in C$ such that $\pi_k(g_{a^1}(\theta^n)) > \pi_{k'}(g_{a^1}(\theta^n)) + 1/n$ for all $k' \neq k$.*

When in place, Assumption 6 implies that sincere voting relative to $g_{a^1}(\theta^n)$ is a Nash equilibrium of the second period subgame $G_n(\theta^n, a^1)$ whenever $\theta^n \in Z_n$.

It is unlikely to obtain, for all the games that satisfy the assumptions of Section 3, the probability of having at least one pure strategy equilibrium of $G_n(\theta^n)$ converging to one as the number of players converges to infinity. For example, we can construct a game where the ability to influence others is large when there are one player influencing and small otherwise. In this case, one player can change the election result if he is the only influencing, but other player can make the original winner to win by influencing, and thus making all the impacts small. But then the first player has no longer an incentive to influence. In other words, such game has equilibrium where a pure strategy is played in the first period.

Nevertheless, Theorem 2 is still relevant whenever an equilibrium exists, and an example of this will be presented in Section 5.

Furthermore, we establish a result analogous to Theorem 2 for mixed strategies, which, due to the fact that equilibria always exist in mixed strategies, avoids the triviality problem discussed in this section.

For all finite sets F, $\Delta(F)$ denotes the set of probability distributions over F. For all $\sigma \in \Delta(F)$, let $\text{supp}(\sigma)$ denote the support of σ, i.e., the set of $x \in F$ such that $\sigma(x) > 0$.

For all $n \in \mathbb{N}$ and $\theta^n \in \Theta^n$, a mixed strategy profile (a mixed strategy, for short) in $G_n(\theta^n)$ is $\sigma = (\sigma_i^1, \sigma_i^2)_{i \in I_n}$ such that $\sigma_i^1 \in \Delta(A_i^1)$ and σ_i^2 maps A^1 into $\Delta(C)$ with $\sigma_i^2(a^1) \in \Delta(A_i^2(\theta^n, a^1))$ for all $i \in I_n$ and $a^1 \in A^1$ (strictly speaking, σ is a behavioral strategy). Let $\tilde{\sigma_n}$ denote the product probability distribution on A^1 defined by $\sigma_1^1, \ldots, \sigma_n^1$. Let $\tilde{E}(\theta^n, a^1)$ denote the set of mixed Nash equilibria of $G_n(\theta^n, a^1)$ and $\tilde{E}(\theta^n)$ the set of mixed subgame perfect equilibria of $G_n(\theta^n)$.

Recall that, under Assumption 6, a pure strategy equilibrium exists in $G_n(\theta^n, a^1)$ for all $n \in \mathbb{N}$, $\theta^n \in Z_n$ and $a^1 \in A^1$ that are sincere relative to θ^n. Due to this, we focus on mixed strategy equilibrium of $G_n(\theta^n)$ with the property that $\sigma^2(a^1)$ is pure for all a^1 that are sincere relative to θ^n. Formally, for all $n \in \mathbb{N}$ and $\theta^n \in \Theta^n$, let $\Sigma(\theta^n)$ denote the set of mixed strategies σ such that $\sigma^2(a^1)$ is pure for all a^1 that are sincere relative to θ^n and let $\bar{E}(\theta^n) = \tilde{E}(\theta^n) \cap \Sigma(\theta^n)$ be the set of mixed strategy equilibria of $G_n(\theta^n)$ that belong to $\Sigma(\theta^n)$.

We first observe that Theorem 2 holds also for those mixed strategy equilibria σ in $\bar{E}(\theta^n)$ such that σ^1 is a pure strategy. In fact, such result follows from the argument used to establish Theorem 2 and, in this sense, it is a corollary of its proof. In order to state it formally, let $\hat{E}(\theta^n)$ denote the set of $\sigma \in \bar{E}(\theta^n)$ such that σ^1 is pure. When $\sigma \in \hat{E}(\theta^n)$, we abuse notation and write $\sigma^2(\sigma^1)$ for $\sigma^2(a^1)$ where $\text{supp}(\sigma^1) = \{a^1\}$.

Corollary 1 *If Assumptions 1–5 hold, then*

$$\lim_n \mu_n(\{\theta^n \in \Theta^n : \sigma^2(\sigma^1) \text{ is strategic relative to } \theta^n \text{ for all } \sigma \in \hat{E}(\theta^n)\}) = 1.$$

We turn to the mixed strategy version of Theorem 2. It requires the following terminology. For all $n \in \mathbb{N}$ and $\theta^n \in \Theta^n$, a strategy $\sigma \in \Sigma(\theta^n)$ is sincere relative to θ^n if $\tilde{\sigma}_n(\{a^1 \in A^1 : \sigma^2(a^1) \text{ is sincere relative to } \theta^n\}) = 1$. Finally, a strategy σ is strategic relative to θ^n if it is not sincere relative to θ^n, i.e., if $\tilde{\sigma}_n(\{a^1 \in A^1 : \sigma^2(a^1) \text{ is strategic relative to } \theta^n\}) > 0$.

Having mixed strategies does not change the conclusion reached in Theorem 2 according to which, when influence is allowed and with a probability converging to one, all equilibria have some players voting strategically. This follows from Lemma 3 which shows that all non-pure equilibria must be strategic. The intuition for the result is as follows. If the probability of strategic voting is zero, then $\sigma^2(a^1)$ is sincere for all $a^1 \in \text{supp}(\sigma^1)$. Since $a_i^1 = k \in C$ implies $A_i^2(\theta^n, a^1) = \{k\}$, then it follows that $\text{supp}(\sigma_i^1) \subseteq \{e, f(\theta_i^n)\}$ for all $i \in I_n$, since, otherwise, there would be a strictly positive probability of player i voting for $k \neq f(\theta_i^n)$. If σ^1 is not pure, there exists $i \in I_n$ such that $\text{supp}(\sigma_i^1) = \{e, f(\theta_i^n)\}$. A zero probability of strategic voting implies that the continuation is the same independently of the choice of e or $f(\theta_i^n)$. But then player i strictly prefers e since this avoids the cost of influencing.

Lemma 3 *Under Assumption 1, if $\sigma \in \tilde{E}(\theta^n)$ is sincere, then σ^1 is a pure strategy.*

As a consequence of Lemma 3, we obtain that the probability of having, in every equilibria (pure and mixed), at least one player voting strategically with a strictly positive probability converges to one as the number of players converges to infinity.

Corollary 2 *If Assumptions 1–5 hold, then*

$$\lim_n \mu_n(\{\theta^n \in \Theta^n : \sigma \textit{ is strategic relative to } \theta^n \textit{ for all } \sigma \in \bar{E}(\theta^n)\}) = 1.$$

Furthermore, if, in addition, Assumption 6 holds, then

$$\lim_n \mu_n(\{\theta^n \in \Theta^n : \bar{E}(\theta^n) \neq \emptyset \textit{ and } \sigma \textit{ is strategic relative to } \theta^n \textit{ for all } \sigma \in \bar{E}(\theta^n)\}) = 1.$$

The above result is easy to understand but has the drawback that the probability of observing strategic voting might be rather small. This would be the case, allowed in Corollary 2, when $\tilde{\sigma}_n(\{a^1 \in A^1 : \sigma^2(a^1)$ is strategic relative to $\theta^n\})$ converges to zero. Note that this is in contrast with the conclusion of Theorem 2, since there $\tilde{\sigma}_n(\{a^1 \in A^1 : \sigma^2(a^1)$ is strategic relative to $\theta^n\}) = 1$. Thus, we seek a stronger version of Corollary 2 in which the probability of first period action profiles that induce strategic voting in bounded away from zero.

Such strong version of Corollary 2 is obtained in Theorem 4 below. It requires the following terminology. For all $n \in \mathbb{N}$, $\varepsilon > 0$ and $\theta^n \in \Theta^n$, a strategy $\sigma \in \Sigma(\theta^n)$ is ε-strategic relative to θ^n if $\tilde{\sigma}_n(\{a^1 \in A^1 : \sigma^2(a^1)$ is strategic relative to $\theta^n\}) > \varepsilon$.

Theorem 4 *If Assumptions 1–5 hold, then there is $\varepsilon > 0$ such that*

$$\lim_n \mu_n(\{\theta^n \in \Theta^n : \sigma \textit{ is } \varepsilon\textit{-strategic relative to } \theta^n \textit{ for all } \sigma \in \bar{E}(\theta^n)\}) = 1.$$

Furthermore, if, in addition, Assumption 6 holds, then

$$\lim_n \mu_n(\{\theta^n \in \Theta^n : \bar{E}(\theta^n) \neq \emptyset \textit{ and } \sigma \textit{ is } \varepsilon\textit{-strategic relative to } \theta^n \textit{ for all } \sigma \in \bar{E}(\theta^n)\}) = 1.$$

5 Examples

We present several examples to illustrate our results.

5.1 Monopolistic Influence

In this example, there is a single player (player 1) that can have a substantial impact on the choice of the other players. Formally, assume that the following holds for all

$n \in \mathbb{N}$, $\theta^n \in \Theta^n$, $a^1 \in A^1$ and $i \in I_n$ (where δ_n and β_n are number that will be defined below):

1. If $a_i^1 = k \in C$, then $A_i^2(\theta^n, a^1) = \{k\}$;
2. If $a_1^1 = a_i^1 = e$, then $A_i^2(\theta^n, a^1) = C$;
3. If $a_1^1 = k \in C$, $a_i^1 = e$, $k' \neq k$ and $|\{j < i : a_j^1 = e, f(\theta_j^n) = k'$ and $f_2(\theta_j^n) = k\}| < n\delta_n/2$, then $A_i^2(\theta^n, a^1) = \{k\}$;
4. If $a_1^1 = k \in C$, $a_i^1 = e$ and $|\{j < i : a_j^1 = e$ and $f(\theta_j^n) = k\}| < n\delta_n$, then $A_i^2(\theta^n, a^1) = \{k\}$;
5. Otherwise, $A_i^2(\theta^n, a^1) = C$.

Condition 1 simply repeats Assumption 1: if a player influences for a candidate k, then the player can only chose to vote for k. When a player is not influencing, the remaining conditions say that the second period action set of each player depends only on the action of player 1. If player 1 influences for candidate k, then the first $n\beta_n$ players (ordered according the their index) for whom k is their favorite candidate will have $A_i^2(\theta^n, a^1) = \{k\}$. Furthermore, $n\delta_n$ players for whom k is their second favorite candidate will have $A_i^2(\theta^n, a^1) = \{k\}$ and (if possible) half of these will favor each of the remaining candidates. Finally, for the remaining players, $A_i^2(\theta^n, a^1) = C$.

In order to check easily that Assumption 3.2 is satisfied, we assume that the above conditions are modified as follows if a^1 is sincere relative to θ^n, at least one player $i \neq 1$ chooses $a_i^1 = e$ and $\pi_a(g_{a^1}(\theta^n)) + \pi_{(c,a)}(g_{a^1}(\theta^n)) = \pi_b(g_{a^1}(\theta^n)) + \pi_{(c,b)}(g_{a^1}(\theta^n))$. In this case, relabel players $\{2, \ldots, n\}$ so that $a_n^1 = e$, apply the above rules to all players except player n and set $A_n^2(\theta^n, a^1) = \{a\}$ if $f(g_i(\theta^n, a^1)) \neq a$ and $A_n^2(\theta^n, a^1) = \{b\}$ if $f(g_i(\theta^n, a^1)) = a$.

Assume further that $\mu_{(k,k')} = \mu_k/2$ for all $k, k' \in C$, $k \neq k'$ and that there is $\zeta > 0$ such that $\mu_c > 8\zeta$, $\mu_c + 9\zeta < \mu_b$ and $\mu_b + \zeta = \mu_a$. Furthermore, assume that $\zeta < \delta < 2\zeta$. For all $n \in \mathbb{N}$, define $\hat{\delta}_n$ to be the greatest integer less than or equal to $n\delta$ and $\delta_n = \hat{\delta}_n/n$.

In this example, we can show (see Section A.10) the existence of strategic voting in the following sense: for all $n \in \mathbb{N}$, there is $C_n \subseteq \Omega^n$ such that, for all $\theta^n \in C_n$, $G_n(\theta^n)$ has an equilibrium (possibly involving mixed strategies outside the equilibrium path) that equilibria satisfy: $s^2(s^1)$ is sincere relative to $g_{s^1}(\theta^n)$, $s_i^1 = e$ for all $i \neq 1$, $s_1^1 = e$ if $f(\theta_1^n) = a$ or $\theta_1^n = (c, a)$ and $s_1^1 = b$ if $f(\theta_1^n) = b$ or $\theta_1^n = (c, b)$. Thus, no player, with the possible exception of player 1, influences and player 1 influences when needed to change the outcome of the election in the second period. In fact, as a result of our assumptions, c will never win and so the only possible winner are a and b, but b will win only if player 1 influences for b. Thus, player 1 influences if and only if he prefers b to a.

Furthermore, note that player 1 can favor candidate c but influence for candidate b (this happens with a probability close to $\mu_{(c,b)} > 0$ when n is large). In this way, this example allows for strategic influencing.

6 Concluding Remarks

To be added.

A Appendix

A.1 Lemmas

Lemma 4 *The following hold:*

$$\lim_k \frac{(2k)!}{2^{2k}k!k!} = 0.$$

Proof. For all $k \in \mathbb{N}$, let $x_k = \frac{(2k)!}{2^{2k}k!k!}$ and note that

$$x_{k+1} = \frac{(2k+2)(2k+1)}{4(k+1)(k+1)} x_k = \left(1 - \frac{1}{2(k+1)}\right) x_k.$$

Hence, $x_k = x_1 \left(\prod_{j=2}^k \left(1 - \frac{1}{2j}\right)\right)$ and so $-\ln x_k = -\ln x_1 + \sum_{j=2}^k \left(-\ln\left(1 - \frac{1}{2j}\right)\right)$. Since

$$-\ln\left(1 - \frac{1}{2j}\right) = \ln 1 - \ln\left(1 - \frac{1}{2j}\right) \geq \frac{d\ln(1 - 1/2j)}{dx}\frac{1}{2j} = \frac{1}{2j-1},$$

then, $\lim_k \sum_{j=2}^{k} \left(-\ln\left(1-\frac{1}{2j}\right)\right) \geq \lim_k \sum_{j=2}^{k} \frac{1}{2j-1} = \infty$. Thus, $\lim_k(-\ln x_k) = \infty$ and so $\lim_k x_k = 0$. ■

Let $\gamma > 0$ and, for all $n \in \mathbb{N}$, A_n^γ be the set of $\theta^n \in \Theta^n$ such that

$$|\mu_{(k,k')} - \pi_{(k,k')}(\theta^n)| \leq \gamma$$

for all $k, k' \in C$ such that $k \neq k'$.

Lemma 5 *For all* $\gamma > 0$, $\lim_n \mu_n(A_n^\gamma) = 1$.

Proof. Let $k, k' \in C$ be such that $k \neq k'$ and let, for all $n \in \mathbb{N}$, $A_n^\gamma(k,k')$ be the set of $\theta^n \in \Theta^n$ such that $|\mu_{(k,k')} - \pi_{(k,k')}(\theta^n)| \leq \gamma$. Also, let $B_n^\gamma(k,k')$ be the complement of $A_n^\gamma(k,k')$.

Fix $n \in \mathbb{N}$ and $k, k' \in C$ such that $k \neq k'$. For all $i \in I_n$, let $X_i : \Theta^n \to \{0,1\}$ be defined by $X_i(\theta^n) = 1$ if $\theta_i^n = (k,k')$ and $X_i = 0$ otherwise. Let $\mu_i = \mu_n(\{\theta^n : X_i(\theta^n) = 1\}) = \mu_{(k,k')}$ for all $i \in I_n$. Then, $\sum_{i \in I_n} X_i(\theta^n)/n = \pi_{(k,k')}(\theta^n)$ and $\sum_{i \in I_n} \mu_i/n = \mu_{(k,k')}$. Thus, by Kalai (2004, Lemma 2, p. 1654), it follows that $\mu_n(B_n^\gamma(k,k')) \leq 2e^{-2n\gamma^2}$.

Letting $K = \{(k,k') \in C^2 : k \neq k'\}$, we have that $|K| = 6$ and that

$$\mu_n(\cup_{(k,k') \in K} B_n^\gamma(k,k')) \leq 12e^{-2n\gamma^2} \to 0$$

since $n\gamma^2 \to \infty$. Hence,

$$\mu_n(A_n^\gamma) = \mu_n(\cap_{(k,k') \in K} A_n^\gamma(k,k')) = 1 - \mu_n(\cup_{(k,k') \in K} B_n^\gamma(k,k')) \to 1,$$

as claimed. ■

For all $n \in \mathbb{N}$, let $B_n(a,b) = \{\theta^n \in \Theta^n : \pi_a(\theta^n) > \pi_b(\theta^n)\}$. For all $n \in \mathbb{N}$ and all permutations (k_1,k_2,k_3) of C, let $B_n(k_1,k_2,k_3) = \{\theta^n \in \Theta^n : \pi_{k_1}(\theta^n) + \pi_{(k_3,k_1)}(\theta^n) \neq \pi_{k_2}(\theta^n) + \pi_{(k_3,k_2)}(\theta^n)\}$. Let K be the set of all permutations of C.

Lemma 6 *The following holds:*

$$\lim_{n\to\infty} \mu_n((\cap_{(k_1,k_2,k_3) \in K} B_n(k_1,k_2,k_3)) \cap B_n(a,b)) = 1.$$

Proof. Let $\gamma = (\mu_a - \mu_b)/2 > 0$. By Lemma 5, we have that $\lim_n \mu_n(A_n^\gamma) = 1$. Hence, it suffices to show that $\lim_n \mu_n(A_n^\gamma \cap (\cap_{(k_1,k_2,k_3)\in K} B_n(k_1,k_2,k_3)) \cap B_n(a,b)) = 1$. Thus, it suffices to show that $\lim_n \mu_n(A_n^\gamma \cap B_n^c(a,b)) = 0$ and that $\lim_n \mu_n(A_n^\gamma \cap B_n^c(k_1,k_2,k_3)) = 0$ for all permutations (k_1,k_2,k_3) of C.

Let $n \in \mathbb{N}$. Since $\theta^n \in A_n^\gamma$ implies that $\pi_a^n(\theta^n) > \alpha_a - \gamma = \alpha_b + \gamma > \pi_b^n(\theta^n)$, then $A_n^\gamma \cap B_n^c(a,b) = \emptyset$. Thus, $\lim_n \mu_n(A_n^\gamma \cap B_n^c(a,b)) = 0$ as desired.

Fix $(k_1,k_2,k_3) \in K$. Note that, for all $n \in \mathbb{N}$ and $\theta^n \in \Theta^n$, we have that $[\pi_{k_1}(\theta^n) + \pi_{(k_3,k_1)}(\theta^n)] + [\pi_{k_2}(\theta^n) + \pi_{(k_3,k_2)}(\theta^n)] = 1$. Hence, if $\theta^n \in B_n^c(k_1,k_2,k_3))$, then $\pi_{k_1}(\theta^n) + \pi_{(k_3,k_1)}(\theta^n) = 1/2$ and so $|\{i \in I_n$: either $f(\theta_i^n) = k_1$ or $f(\theta_i^n) = k_3$ and $f_2(\theta_i^n) = k_1\}| = n/2$. This implies that if n is odd, then $B_n^c(k_1,k_2,k_3)) = \emptyset$. Hence, it suffices to show that $\lim_{n,n \text{ even}} \mu_n(B_n^c(k_1,k_2,k_3)) = 0$.

Let n be even. For all $l = 1,2$, let $D_l = \{\theta \in \Theta : f(\theta) = k_1$ or $[f(\theta) = k_3$ and $f_2(\theta) = k_l]\}$ and D be the set of $\theta^n \in \Theta^n$ such that $|\{i \in I_n : \theta_i^n \in D_l\}| = n/2$ for all $l = 1,2$. Thus, if $\theta^n \in B_n^c(k_1,k_2,k_3))$, then $\theta^n \in D$. Let $k = n/2$ and $\tilde{\mu}_l = \mu_{k_l} + \mu_{(k_3,k_l)}$ for all $l = 1,2$. Note that $\tilde{\mu}_1^k \tilde{\mu}_2^k = \tilde{\mu}_1^k (1-\tilde{\mu}_1)^k \leq (1/2)^k (1/2)^k = 2^{-2k}$. Hence, it follows that

$$\mu_n(B_n^c(k_1,k_2,k_3)) \leq \mu_n(D) = \frac{n!}{\left(\frac{n}{2}\right)! \left(\frac{n}{2}\right)!} \tilde{\mu}_1^{n/2} \tilde{\mu}_2^{n/2} \leq \frac{(2k)!}{k!k!} \frac{1}{2^{2k}}.$$

It then follows by Lemma 4 that $\lim_n \tilde{\mu}_n(B_n^c(k_1,k_2,k_3)) = 0$. ■

A.2 Proof of Lemma 1

Let $n > 2$, $\theta^n \in \Theta^n$, $a^1 \in A^1$, $a^2 \in E(\theta^n, a^1)$ and suppose that a^2 is strategic relative to $g_{a^1}(\theta^n)$. Suppose that $\pi_{(k,k')}(g_{a^1}(\theta^n)) > 0$ for all $k, k' \in C$ such that $k \neq k'$ (i.e., that condition 1 in the statement does not hold). We will show that for some permutation (k_1,k_2,k_3) of C, we have that

$$\pi_{k_1}(g_{a^1}(\theta^n)) + \pi_{(k_3,k_1)}(g_{a^1}(\theta^n)) = \pi_{k_2}(g_{a^1}(\theta^n)) + \pi_{(k_3,k_2)}(g_{a^1}(\theta^n)).$$

(i.e., that condition 2 in the statement holds).

In order to simplify notation, we write f_i (respectively, f_i^2) for $f(g_i(\theta^n, a^1))$ (resp., $f_2(g_i(\theta^n, a^1)))$. Note first that the definition of $g_i(\theta^n, a^1)$ implies that if $a_i^2 \neq f_i$, then $A_i^2(\theta^n, a^1)$ is not a singleton. Thus, either $A_i^2(\theta^n, a^1) = C$ or $A_i^2(\theta^n, a^1) = \{f(\theta_i^n), f_2(\theta_i^n)\}$. It then follows that $f_i = f(\theta_i^n)$, $f_i^2 = f_2(\theta_i^n)$ and $a_i^2 = f_2(\theta_i^n)$ (in equilibrium, no player votes for his least preferred candidate).

We consider five possible cases regarding a^2.

Case 1: There is $k_1 \in C$ such that $\pi_{k_1}(a^2) > \pi_k(a^2) + 1/n$ for all $k \neq k_1$. In this case, we claim that all voters vote sincerely relative to $g_{a_1}(\theta^n)$. In order to show this claim, suppose that there is $i \in I_n$ such that $a_i^2 \neq f_i$. Then, $u_i(a^2) = \lambda_n \theta_i^n \cdot \pi(a^2) + \theta_{i,k_1}^n$, $\pi_{k_1}(f_i, a_{-i}^2) \geq \pi_k(f_i, a_{-i}^2)$ for all $k \neq k_1$ and $\pi_{k_1}(f_i, a_{-i}^2) = \pi_k(f_i, a_{-i}^2)$ only if $k = f_i$ and $\pi_{k_1}(a^2) = \pi_k(a^2) + 2/n$. Thus, $u_i(f_i, a_{-i}^2) \geq \lambda_n \theta_i^n \cdot \pi(a^2) + \lambda_n(\theta_{i,f_i}^n - \theta_{i,f_i^2}^n)/n + \theta_{i,k_1}^n > u_i(a^2)$, a contradiction.

Case 2: There is $k_1 \in C$ such that $\pi_{k_1}(a^2) = \pi_k(a^2) + 1/n$ for all $k \neq k_1$. In this case, we claim that all voters vote sincerely relative to $g_{a_1}(\theta^n)$. In order to show this claim, suppose that there is $i \in I_n$ such that $a_i^2 \neq f_i$. Then, $u_i(a^2) = \lambda_n \theta_i^n \cdot \pi(a^2) + \theta_{i,k_1}^n$. If $f_i = k_1$, then $u_i(f_i, a_{-i}^2) = \lambda_n \theta_i^n \cdot \pi(a^2) + \lambda_n(\theta_{i,f_i}^n - \theta_{i,f_2}^n)/n + \theta_{i,k_1}^n > u_i(a^2)$, a contradiction. If $f_i \neq k_1$ and $f_i^2 = k_1$, then $u_i(f_i, a_{-i}^2) = \lambda_n \theta_i^n \cdot \pi(a^2) + \lambda_n(\theta_{i,f_i}^n - \theta_{i,f_i^2}^n)/n + \theta_{i,f_i}^n > \lambda_n \theta_i^n \cdot \pi(a^2) + \theta_{i,f_i^2}^n = u_i(a^2)$, a contradiction. Finally, if $f_i \neq k_1$ and $f_i^2 = k' \neq k_1$, then $(\theta_{i,f_i}^n, \theta_{i,f_i^2}^n, \theta_{i,k_1}^n) = (2, 1, 0)$, $u_i(f_i, a_{-i}^2) = \lambda_n \theta_i^n \cdot \pi(a^2) + \lambda_n(2 - 1)/n + (2+0)/2 > \lambda_n \theta_i^n \cdot \pi(a^2) = u_i(a^2)$, a contradiction.

Case 3: There is a permutation (k_1, k_2, k_3) of C such that $\pi_{k_1}(a^2) = \pi_{k_2}(a^2) + 1/n > \pi_{k_3}(a^2) + 1/n$. By assumption, there is $i \in I_n$ such that $(f_i, f_i^2) = (k_3, k_2)$ and $k_2, k_3 \in A_i^2(\theta^n, a^1)$. We claim that $a_i^2 = k_3$, i.e., i votes sincerely relative to $g_{a_1}(\theta^n)$. If not, then $a_i^2 = k_2$, $u_i(a^2) = \lambda_n \theta_i^n \cdot \pi(a^2)$ and $u_i(f_i, a_{-i}^2) = \lambda_n \theta_i^n \cdot \pi(a^2) + \lambda_n(\theta_{i,f_i}^n - \theta_{i,f_i^2}^n)/n > u_i(a^2)$, a contradiction.

Since player i votes sincerely, then $a_i^2 = k_3$, $u_i(a^2) = \lambda_n \theta_i^n \cdot \pi(a^2)$ and $u_i(k_2, a_{-i}^2) = \lambda_n \theta_i^n \cdot \pi(a^2) - \lambda_n(\theta_{i,f_i}^n - \theta_{i,f_i^2}^n)/n + 1/2 \geq \lambda_n \theta_i^n \cdot \pi(a^2) - 1/n + 1/2 > u_i(a^2)$ since $n > 2$. This is a contradiction.

Case 4: $\pi_a(a^2) = \pi_b(a^2) = \pi_c(a^2)$. In this case, all voters vote sincerely relative to $g_{a_1}(\theta^n)$. Suppose that there is $i \in I_n$ such that $a_i^2 \neq f_i$. Then, $u_i(a^2) = \lambda_n \theta_i^n \cdot \pi(a^2) + (2+1+0)/3$ and $u_i(f_i, a_{-i}^2) = \lambda_n \theta_i^n \cdot \pi(a^2) + \lambda_n(2-1)/n + 2 > u_i(a^2)$, a contradiction.

Case 5: There is a permutation (k_1, k_2, k_3) of C such that $\pi_{k_1}(a^2) = \pi_{k_2}(a^2) >$

$\pi_{k_3}(a^2)$. We first show that if $i \in I_n$ is such that $f_i = k_j$ for some $j \in \{1, 2\}$, then $a_i^2 = f_i$. Suppose not; then, $u_i(a^2) = \lambda_n \theta_i^n \cdot \pi(a^2) + (2 + \theta_{i,k'}^n)/2$, where $k' \in \{k_1, k_2\} \setminus \{f_i\}$, and $u_i(f_i, a_{-i}^2) = \lambda_n \theta_i^n \cdot \pi(a^2) + \lambda_n(2 - 1)/n + 2 > u_i(a^2)$, a contradiction.

Second, we show that if $i \in I_n$ is such that $f_i = k_3$ and $A_i^2(\theta^n, a^1) \neq \{k_3\}$, then $a_i^2 = f_i^2$. Since $A_i^2(\theta^n, a^1) \neq \{k_3\}$, then either $A_i^2(\theta^n, a^1) = C$ or $A_i^2(\theta^n, a^1) = \{f_i, f_i^2\}$. In both cases, $k_3 = f(\theta_i^n)$ and $f_i^2 = f_2(\theta_i^n)$. Suppose, in order to reach a contradiction, that $a_i^2 \neq f_i^2$. Then, $a_2^i = k_3$, $u_i(a^2) = \lambda_n \theta_i^n \cdot \pi(a^2) + (1 + 0)/2$, $u_i(f_i^2, a_{-i}^2) = \lambda_n \theta_i^n \cdot \pi(a^2) - \lambda_n(2 - 1)/n + 1 \geq \lambda_n \theta_i^n \cdot \pi(a^2) - 1/n + 1 > u_i(a^2)$ since $n > 2$, a contradiction.

Hence, it follows that

$$\pi_{k_1}(g_{a_1}(\theta^n)) + \pi_{(k_3,k_1)}(g_{a_1}(\theta^n)) = \pi_{k_1}(a^2) = \pi_{k_2}(a^2) = \pi_{k_2}(g_{a_1}(\theta^n)) + \pi_{(k_3,k_2)}(g_{a_1}(\theta^n)).$$

This completes the proof.

A.3 Proof of Lemma 2

Let $n \in \mathbb{N}$, $\theta^n \in \Theta^n$, $a^1 \in A^1$, a^2 be sincere relative to $g_{a^1}(\theta^n)$ and $k \in C$ be such that $\pi_k(g_{a^1}(\theta^n)) > \pi_{k'}(g_{a^1}(\theta^n)) + 1/n$ for all $k' \neq k$.

Since a^2 is sincere relative to $g_{a^1}(\theta^n)$, then $\pi(a^2) = \pi(g_{a^1}(\theta^n))$. This means that candidate k wins and so, for all $i \in I_n$, $u_i(a^2) = \lambda_n \theta_i^n \cdot \pi(a^2) + \theta_{i,k}^n$.

Let $i \in I_n$ be such that $A_i^2(\theta^n, a^1) \neq \{f(g_i(\theta^n, a^1))\}$ and $\bar{a}_i^2 = f_2(g_i(\theta^n, a^1))$. If $f(g_i(\theta^n, a^1)) = k$, then $\pi_k(\bar{a}_i^2, a_{-i}^2) = \pi_k(a^2) - 1/n \geq \pi_{k'}(a^2) + 1/n \geq \pi_{k'}(\bar{a}_i^2, a_{-i}^2)$. Hence, candidate k either ties or wins when $(\bar{a}_i^2, a_{-i}^2)$ is played. Since $f(g_i(\theta^n, a^1)) = k$, the case where k wins provides an upper bound for player i's payoff when $(\bar{a}_i^2, a_{-i}^2)$ is played. Therefore, $u_i(\bar{a}_i^2, a_{-i}^2) \leq \lambda_n \theta_i^n \cdot \pi(a^2) - \lambda_n(\theta_{i,k}^n - \theta_{i,\bar{a}_i^2}^n)/n + \theta_{i,k}^n < u_i(a^2)$.

If $f(g_i(\theta^n, a^1)) = k' \neq k$, then $\pi_k(\bar{a}_i^2, a_{-i}^2) \geq \pi_k(a^2) > \pi_{k'}(a^2) + 1/n \geq \pi_{k'}(\bar{a}_i^2, a_{-i}^2)$. Hence, candidate k wins. Thus, $u_i(\bar{a}_i^2, a_{-i}^2) = \lambda_n \theta_i^n \cdot \pi(a^2) - \lambda_n(\theta_{i,k'}^n - \theta_{i,\bar{a}_i^2}^n)/n + \theta_{i,k}^n < u_i(a^2)$.

Thus, it follows that $a^2 \in E(\theta^n, a^1)$.

A.4 Proof of Lemma 3

Let $\sigma \in \tilde{E}(\theta^n)$ be sincere and suppose, in order to reach a contradiction, that σ^1 is not pure.

Let $i \in I_n$ and $k \in C \cap \text{supp}(\sigma_i^1)$. Let $a^1 \in A^1$ be such that $a_i^1 = k$ and $a_j^1 \in \text{supp}(\sigma_j^1)$ for all $j \neq i$. Then, $\tilde{\sigma}(a^1) > 0$, $A_i^2(\theta^n, a^1) = \{k\}$ and $\text{supp}(\sigma_i^2(a^1)) = \{k\}$. Since σ is sincere, it follows that $k = f(\theta_i^n)$. Hence, $\text{supp}(\sigma_i^1) \subseteq \{e, f(\theta_i^n)\}$.

Since σ^1 is not pure, there exist $i \in I_n$ such that $\text{supp}(\sigma_i^1) = \{e, f(\theta_i^n)\}$. Since σ is sincere, then $\sigma^2(a^1)$ is sincere for all $a^1 \in \text{supp}(\sigma^1)$. In particular, for all $a_{-i}^1 \in \text{supp}(\sigma_{-i}^1)$, $\sigma^2(e, a_{-i}^1)$ and $\sigma^2(f(\theta_i^n), a_{-i}^1)$ are sincere. For convenience, let

$$h(\theta^n) = \sum_{k \in C} \left(\theta_{i,k} 1_{W_k}(\pi(\theta^n)) + \sum_{k' \in C \setminus \{k\}} \frac{\theta_{i,k} + \theta_{i,k'}}{2} 1_{T_{k,k'}}(\pi(\theta^n)) + \frac{\theta_{i,k}}{3} 1_T(\pi(\theta^n)) \right).$$

Then,

$$u_i((e, \sigma_i^2), \sigma_{-i}) = \sum_{a_{-i}^1 \in \text{supp}(\sigma_{-i}^1)} \sigma_{-i}^1(a_{-i}^1)[\lambda_n \theta_i^n \cdot \pi(\theta^n) + h(\theta^n)] = \lambda_n \theta_i^n \cdot \pi(\theta^n) + h(\theta^n) \text{ and}$$
$$u_i((f(\theta_i^n), \sigma_i^2), \sigma_{-i}) = -c + \lambda_n \theta_i^n \cdot \pi(\theta^n) + h(\theta^n).$$

Thus, $u_i((e, \sigma_i^2), \sigma_{-i}) > u_i((f(\theta_i^n), \sigma_i^2), \sigma_{-i})$. Since $\sigma \in \tilde{E}(\theta^n)$, this is a contradiction.

A.5 Proof of Theorem 1

For all $n \in \mathbb{N}$, let $S_n = \{\theta^n \in \Theta^n : s^2 \text{ is sincere relative to } \theta^n \text{ for all } s^2 \in E(\theta^n, \bar{e})\}$.

Let $\alpha = \min_{k,k' \in C, k \neq k'} \mu_{(k,k')}$. Let $\gamma = \alpha/2$ and A_n be the set of $\theta^n \in \Theta^n$ such that $|\mu_{(k,k')} - \pi_{(k,k')}(\theta^n)| \leq \gamma$. Let B_n be the set of $\theta^n \in \Theta^n$ such that $\pi_{k_1}(\theta^n) + \pi_{(k_3,k_1)}(\theta^n) \neq \pi_{k_2}(\theta^n) + \pi_{(k_3,k_2)}(\theta^n)$ for all permutations (k_1, k_2, k_3) of C.

We have that $\lim_n \mu_n(A_n) = \lim_n \mu_n(B_n) = 1$ by Lemmas 5 and 6. Therefore, $\lim_n (A_n \cap B_n) = 1$. We claim that $A_n \cap B_n \subseteq S_n$ for all $n > 2$, which will imply that $\lim_n (S_n) = 1$.

Let $n > 2$ and $\theta^n \in A_n \cap B_n$. Since $\theta^n \in A_n$, then $\pi_{(k,k')}(\theta^n) \geq \mu_{(k,k')} - \min_{\tilde{k},\hat{k} \in C, k \neq k'} \mu_{(\tilde{k},\hat{k})}/2 > 0$ for all $k, k' \in C$ with $k \neq k'$. Since $\theta^n \in B_n$, then $\pi_{k_1}(\theta^n) + \pi_{(k_3,k_1)}(\theta^n) \neq \pi_{k_2}(\theta^n) + \pi_{(k_3,k_2)}(\theta^n)$ for all permutations (k_1, k_2, k_3) of C.

Thus, Lemma 1 then implies that if $s^2 \in E(\theta^n, \bar{e})$, then s^2 is sincere relative to θ^n. Thus, $A_n \cap B_n \subseteq S_n$.

A.6 Proof of Theorem 2

For all $n \in \mathbb{N}$, let $S_n = \{\theta^n \in \Theta^n : s^2(s^1)$ is strategic relative to θ^n for all $s \in E(\theta^n)\}$.

Let $\varepsilon > 0$ be such that $\delta > \mu_a - \mu_b + \varepsilon$ and $\delta < \mu_a - \varepsilon$ (recall Assumption 4) and $\alpha = \min_{k,k' \in C, k \neq k'} \mu_{(k,k')}$. Let $\gamma = \min\{\varepsilon/4, \alpha/8\}$ and A_n be the set of $\theta^n \in \Theta^n$ such that $|\mu_{(k,k')} - \pi_{(k,k')}(\theta^n)| \leq \gamma$. Let B_n be the set of $\theta^n \in \Theta^n$ such that $\pi_a(\theta^n) > \pi_b(\theta^n)$ and $\pi_{k_1}(\theta^n) + \pi_{(k_3,k_1)}(\theta^n) \neq \pi_{k_2}(\theta^n) + \pi_{(k_3,k_2)}(\theta^n)$ for all permutations (k_1, k_2, k_3) of C. Finally, let X_n and Y_n be as in Assumptions 3 and 4, and define $C_n = A_n \cap B_n \cap X_n \cap Y_n$.

We have that $\lim_n \mu_n(X_n) = \lim_n \mu_n(Y_n) = 1$ by definition and $\lim_n \mu_n(A_n) = \lim_n \mu_n(B_n) = 1$ by Lemmas 5 and 6. Therefore, $\lim_n \mu_n(C_n) = 1$. We claim that $C_n \subseteq S_n$ for all n sufficiently large, which will imply that $\lim_n \mu_n(S_n) = 1$.

Let $N \in \mathbb{N}$ be such that $\mu_a - \mu_b + \varepsilon < \delta_n < \mu_a - \varepsilon$ and $\lambda_n < c/2$ (recall that $\lim_n \lambda_n < c/2$). Let $n \geq N$, $\theta^n \in C_n$ and suppose, in order to reach a contradiction, that $\theta^n \notin S_n$.

Since $\theta^n \notin S_n$, let $s_n \in E(\theta^n)$ be such that $s_n^2(s_n^1)$ is sincere for all $n \in N$. Note that s_n^1 is sincere relative to θ^n since, otherwise, Assumption 1 would imply that $s_n^2(s_n^1)$ is not sincere relative to θ^n.

Since $\theta^n \in A_n$, then $\pi_{(k,k')}(\theta^n) \geq \mu_{(k,k')} - \min_{\tilde{k},\hat{k} \in C, k \neq k'} \mu_{(\tilde{k},\hat{k})}/8 > 0$ for all $k, k' \in C$ with $k \neq k'$. Thus, by Assumption 3.1, $\pi_{(k,k')}(g_{a^1}(\theta^n)) > 0$ for all a^1 that are sincere relative to θ^n.

Since $\theta^n \in B_n$, then $\pi_{k_1}(\theta^n) + \pi_{(k_3,k_1)}(\theta^n) \neq \pi_{k_2}(\theta^n) + \pi_{(k_3,k_2)}(\theta^n)$ for all permutations (k_1, k_2, k_3) of C. Thus, by Assumption 3.2, it follows that $\pi_{k_1}(g_{a^1}(\theta^n)) + \pi_{(k_3,k_1)}(g_{a^1}(\theta^n)) \neq \pi_{k_2}(g_{a^1}(\theta^n)) + \pi_{(k_3,k_2)}(g_{a^1}(\theta^n))$ for all permutations (k_1, k_2, k_3) of C and all a^1 that are sincere relative to θ^n.

Lemma 1 then implies that if a^1 is sincere relative to θ^n, then $s_n^2(a^1)$ is sincere relative to $g_{a^1}(\theta^n)$.

Since $s_n^2(s_n^1)$ is sincere relative to θ^n, then $\pi(s_n) = \pi(\theta^n)$. Since $\theta^n \in A_n \cap B_n$, then $\pi_a(\theta^n) > \pi_b(\theta^n)$ and $\pi_b(\theta^n) > \pi_c(\theta^n)$ (in fact, $\pi_c(\theta^n) < \mu_c + \alpha/4 \leq \mu_b - \mu_{(b,c)} + \alpha/4 \leq$

$\mu_b - 3\alpha/4 < \pi_b(\theta^n) - 2\alpha/4 < \pi_b(\theta^n))$. Due to this, a wins.

For convenience, let $a^1 = s_n^1$ and $a^2 = s_n^2(s_n^1)$. Since a wins when s_n is played, we have that

$$u_i(s_n) = \begin{cases} -c + \lambda_n \theta_i^n \cdot \pi(a^2) + \theta_{i,a}^n & \text{if } a_i^1 \in C, \\ \lambda_n \theta_i^n \cdot \pi(a^2) + \theta_{i,a}^n & \text{if } a_i^1 = e. \end{cases}$$

Claim 1 shows that no player influences for b.

Claim 1 *For all $i \in I_n$, $s_{n,i}^1 \neq b$.*

Proof of Claim 1. Suppose, in order to reach a contradiction, that there is $i \in I_n$ such that $s_{n,i}^1 = b$. Consider $\bar{a}^1 = (e, a_{-i}^1)$, and let $\bar{a}^2 = s_n^2(\bar{a}^1)$. Note that $\bar{a}^1$ is sincere relative to θ^n, which implies that $\bar{a}^2$ is sincere relative to $g_{\bar{a}_1}(\theta^n)$.

Fix $j \in I_n$. Since a^2 is sincere relative to θ^n, then $a_j^2 = f(\theta_j^n)$. In particular, if $k \in \{a, c\}$, then $a_j^2 = k$ implies that $k \in A_j^2(\theta^n, a^1)$. Assumption 5.1 implies that $k \in A_j^2(\theta^n, \bar{a}^1)$. Thus, $k = f(\theta_j^n) \in A_j^2(\theta^n, \bar{a}^1)$ implies $k = f(g_j(\theta^n, \bar{a}^1))$ and, since $\bar{a}^2$ is sincere relative to $g_{\bar{a}_1}(\theta^n)$, it follows that $\bar{a}_j^2 = k$.

The above argument establishes that $\pi_a(\bar{a}^2) \geq \pi_a(a^2)$ and $\pi_c(\bar{a}^2) \geq \pi_c(a^2)$. Since $\sum_{k \in C} \pi_k(a^2) = \sum_{k \in C} \pi_k(\bar{a}^2) = 1$, then $\pi_b(\bar{a}^2) \leq \pi_b(a^2)$.

We next show that if $a_j^2 = b$ and $f_2(\theta_j^n) = a$, then $\bar{a}_j^2 \neq c$. We consider two cases. First, if $a \notin A_j^2(\theta^n, \bar{a}^1)$, then, by Assumption 2, $c \notin A_j^2(\theta^n, \bar{a}^1)$ and so $\bar{a}_j^2 = b$. Second, if $a \in A_j^2(\theta^n, \bar{a}^1)$, then either $b = f(g_j(\theta^n, \bar{a}^1))$ (which happens if and only if $b \in A_j^2(\theta^n, \bar{a}^1)$) or $a = f(g_j(\theta^n, \bar{a}^1))$. Since $\bar{a}^2$ is sincere relative to $g_{\bar{a}_1}(\theta^n)$, then either $\bar{a}_j^2 = b$ or $\bar{a}_j^2 = a$.

Since $a_j^2 = b$ holds if and only if $f(\theta_j^n) = b$, then the above argument implies that $\pi_c(\bar{a}^2) \leq \pi_c(a^2) + \pi_{(b,c)}(\theta^n)$. Since $\theta^n \in A_n$, then $\pi_c(a^2) + \pi_{(b,c)}(\theta^n) = \pi_c(\theta^n) + \pi_{(b,c)}(\theta^n) < \mu_c + \alpha/4 + \mu_{(b,c)} + \alpha/8 < \mu_a - 5\alpha/8 < \pi_a(\theta^n) = \pi_a(a^2) \leq \pi_a(\bar{a}^2)$. This implies that a wins in $\bar{a}^2$.

Hence,

$$\begin{aligned} u_i(\bar{a}^1, \bar{a}^2) &= \lambda_n \theta_i^n \cdot \pi(a^2) + \lambda_n \sum_{k \in C} \theta_{i,k}^n [\pi_k(\bar{a}^2) - \pi_k(a^2)] + \theta_{i,a}^n \\ &\geq \lambda_n \theta_i^n \cdot \pi(a^2) - 2\lambda_n + \theta_{i,a}^n \\ &= u_i(s_n) - 2\lambda_n + c > u_i(s_n). \end{aligned}$$

Since $s_n \in E(\theta_n)$, this is a contradiction. ■

Claim 2 *For all $i \in I_n$, $s^1_{n,i} \neq c$.*

Proof of Claim 2. Suppose, in order to reach a contradiction, that there is $i \in I_n$ such that $s^1_{n,i} = c$. Consider $\bar{a}^1 = (e, a^1_{-i})$, and let $\bar{a}^2 = s^2_n(\bar{a}^1)$. Note that $\bar{a}^1$ is sincere relative to θ^n, which implies that $\bar{a}^2$ is sincere relative to $g_{\bar{a}_1}(\theta^n)$.

Fix $j \in I_n$. As in the proof of Claim 1, it follows that, for all $k \in \{a, b\}$, if $a^2_j = k$ then $\bar{a}^2_j = k$. Thus, $\pi_k(\bar{a}^2) \geq \pi_k(a^2)$ for all $k \in \{a, b\}$. Also as in Claim 1, we have that if $a^2_j = c$ and $f(\theta^n_j) = a$, then $\bar{a}^2_j \neq b$.

We next show that if $a^2_j = c$ and $f_2(\theta^n_j) = b$, then $\bar{a}^2_j = c$. By Claim 1, we have that $a^1_l \neq b$ for all $l \in I_n$ and so $\bar{a}^1_l \neq b$ for all $l \in I_n$. Assumption 2 implies that $A^2_j(\theta^n, \bar{a}^1)$ is equal to either C, $\{b, c\}$, $\{b\}$ or $\{c\}$. Assumption 5.2 implies that $A^2_j(\theta^n, \bar{a}^1) = \{b\}$ is not possible. It follows that $c = f(\theta^n_j) \in A^2_j(\theta^n, \bar{a}^1)$ and so $f(g_j(\theta^n, \bar{a}^1)) = c$. Since $\bar{a}^2$ is sincere relative to $g_{\bar{a}_1}(\theta^n)$, it follows that $\bar{a}^2_j = c$.

Thus, it follows that if $a^2_j \in \{a, c\}$, then $\bar{a}^2_j \neq b$. Thus, $\pi_b(\bar{a}^2) \leq \pi_b(a^2)$. Together with $\pi_b(\bar{a}^2) \geq \pi_b(a^2)$, this implies that $\pi_b(\bar{a}^2) = \pi_b(a^2)$. Since $\pi_a(\bar{a}^2) \geq \pi_a(a^2)$, then $\pi_c(\bar{a}^2) \leq \pi_c(a^2)$. This implies that a wins when $\bar{a}^2$ is played. Thus, as in the proof of Claim 1, we obtain that $u_i(\bar{a}^1, \bar{a}^2) \geq u_i(s_n) - 2\lambda_n + c > u_i(s_n)$. Since $s_n \in E(\theta_n)$, this is a contradiction. ■

Claim 3 *For all $i \in I_n$, $s^1_{n,i} \neq a$.*

Proof of Claim 3. Suppose, in order to reach a contradiction, that there is $i \in I_n$ such that $s^1_{n,i} = a$. Consider $\bar{a}^1 = (e, a^1_{-i})$, and let $\bar{a}^2 = s^2_n(\bar{a}^1)$. Note that $\bar{a}^1$ is sincere relative to θ^n, which implies that $\bar{a}^2$ is sincere relative to $g_{\bar{a}_1}(\theta^n)$.

Fix $j \in I_n$. By Assumption 2, either $A^2_j(\theta^n, \bar{a}^1) = C$, $A^2_j(\theta^n, \bar{a}^1) = \{f(\theta^n_j), f_2(\theta^n_j)\}$, $A^2_j(\theta^n, \bar{a}^1) = \{f(\theta^n_j)\}$ or $A^2_j(\theta^n, \bar{a}^1) = \{f_2(\theta^n_j)\}$. Claims 1 and 2 imply that $a^1_l \notin \{b, c\}$ for all $l \in I_n$ and so $\bar{a}^1_l \notin \{b, c\}$ for all $l \in I_n$. Thus, Assumption 5.2 implies that either $A^2_j(\theta^n, \bar{a}^1) = C$, $A^2_j(\theta^n, \bar{a}^1) = \{f(\theta^n_j), f_2(\theta^n_j)\}$ or $A^2_j(\theta^n, \bar{a}^1) = \{a\}$. Hence, because $\bar{a}^2$ is sincere relative to $g_{\bar{a}_1}(\theta^n)$, then either $\bar{a}^2_j = f(\theta^n_j)$ or $\bar{a}^2_j = a$.

Therefore, it follows that $\pi_a(\bar{a}^2) \geq \pi_a(a^2)$ and $\pi_k(\bar{a}^2) \leq \pi_k(a^2)$ for all $k \neq a$. This implies that a wins when $\bar{a}^2$ is played. Since a^1 is sincere relative to θ^n, then $\theta^n_{i,a} = 2$. Since $\pi_a(\bar{a}^2) - \pi_a(a^2) = \pi_b(a^2) - \pi_b(\bar{a}^2) + \pi_c(a^2) - \pi_c(\bar{a}^2)$, it follows that

$$\begin{aligned} u_i(\bar{a}^1, \bar{a}^2) &= \lambda_n \theta^n_i \cdot \pi(a^2) + \lambda_n \sum_{k \in C} \theta^n_{i,k}[\pi_k(\bar{a}^2) - \pi_k(a^2)] + \theta^n_{i,a} \\ &\geq \lambda_n \theta^n_i \cdot \pi(a^2) + \theta^n_{i,a} \\ &= u_i(s_n) + c > u_i(s_n). \end{aligned}$$

Since $s_n \in E(\theta_n)$, this is a contradiction. ■

It follows from Claims 1, 2 and 3 that $s^1_{n,i} = e$ for all $i \in I_n$. We will obtain a contradiction by showing that any player $i \in I_n$ such that $f(\theta^n_i) = b$ gains by deviating from e to b.

Let $i \in I_n$ be such that $f(\theta^n_i) = b$ and $f_2(\theta^n_i) = c$ (since $\theta^n \in A_n$, such i exists). For convenience, let $a^1 = s^1_n$ and $a^2 = s^2_n(s^1_n)$. Since a wins when s_n is played, we have that $u_i(s_n) = \lambda_n \theta^n_i \cdot \pi(a^2) + \theta^n_{i,a} = \lambda_n \theta^n_i \cdot \pi(a^2)$.

Consider $\bar{a}^1 = (b, a^1_{-i})$ and let $\bar{a}^2 = s^2_n(\bar{a}^1)$. Note that $\bar{a}^1$ is sincere relative to θ^n, which implies that $\bar{a}^2$ is sincere relative to $g_{\bar{a}_1}(\theta^n)$.

Fix $j \in I_n$. By Assumption 2, either $A^2_j(\theta^n, \bar{a}^1) = C$, $A^2_j(\theta^n, \bar{a}^1) = \{f(\theta^n_j), f_2(\theta^n_j)\}$, $A^2_j(\theta^n, \bar{a}^1) = \{f(\theta^n_j)\}$ or $A^2_j(\theta^n, \bar{a}^1) = \{f_2(\theta^n_j)\}$. Assumption 5.2 implies that either $A^2_j(\theta^n, \bar{a}^1) = C$, $A^2_j(\theta^n, \bar{a}^1) = \{f(\theta^n_j), f_2(\theta^n_j)\}$ or $A^2_j(\theta^n, \bar{a}^1) = \{b\}$. Hence, because $\bar{a}^2$ is sincere relative to $g_{\bar{a}_1}(\theta^n)$, then either $\bar{a}^2_j = f(\theta^n_j)$ or $\bar{a}^2_j = b$ and $A^2_j(\theta^n, \bar{a}^1) = \{b\}$.

Therefore, it follows that, for all $k \neq b$,

$$\begin{aligned} \pi_k(\bar{a}^2) &= \pi_k(a^2) - \frac{|\{l \in I_n : f(\theta^n_l) = k \text{ and } A^2_l(\theta^n, \bar{a}^1) = \{b\}\}|}{n} \text{ and} \\ \pi_b(\bar{a}^2) &= \pi_b(a^2) + \frac{|\{l \in I_n : f(\theta^n_l) \neq b \text{ and } A^2_l(\theta^n, \bar{a}^1) = \{b\}\}|}{n} \end{aligned}$$

Since $\theta^n \in A_n$, it follows that $|\{j \in I_n : f(\theta^n_j) \neq b\}|/n = \pi_a(\theta^n) + \pi_c(\theta^n) > \mu_a + \mu_c - \varepsilon > \mu_a - \varepsilon > \delta_n$. We then obtain $\pi_k(\bar{a}^2) \leq \pi_k(a^2)$ for all $k \neq b$ and, by Assumption 4, $\pi_b(\bar{a}^2) \geq \pi_b(a^2) + \delta_n$. Furthermore, since $\theta^n \in A_n$,

$$\begin{aligned} \pi_a(\bar{a}^2) \leq \pi_a(\theta^n) < \mu_a + \frac{\varepsilon}{2} &= \mu_b + \mu_a - \mu_b + \frac{\varepsilon}{2} \\ < \pi_b(\theta^n) + \mu_a - \mu_b + \varepsilon &< \pi_b(\theta^n) + \delta_n \leq \pi_b(\bar{a}^2). \end{aligned}$$

This implies that b wins when $\bar{a}^2$ is played. Since $c < 1$, it follows that

$$\begin{aligned} u_i(\bar{a}^1, \bar{a}^2) &= \lambda_n \theta_i^n \cdot \pi(a^2) + \lambda_n \sum_{k \in C} \theta_{i,k}^n [\pi_k(\bar{a}^2) - \pi_k(a^2)] + 2 - c \\ &\geq \lambda_n \theta_i^n \cdot \pi(a^2) + 2 - c > u_i(s_n). \end{aligned}$$

Since $s_n \in E(\theta_n)$, this is a contradiction.

A.7 Proof of Theorem 3

For all $n \in \mathbb{N}$, let $S_n = \{\theta^n \in \Theta^n : s^2$ is sincere relative to θ^n for all $s^2 \in E(\theta^n, \bar{e})\}$ and $T_n = \{\theta^n \in \Theta^n : E(\theta^n, \bar{e}) \neq \emptyset\}$. We want to show that $\lim_n \mu_n(S_n \cap T_n) = 1$. Since $\lim_n \mu_n(S_n) = 1$ by Theorem 1, it suffices to show that $\lim_n(T_n) = 1$.

Let $\varepsilon > 0$ be such that $\mu_a > \mu_b + \varepsilon$ and $\mu_b > \mu_c + \varepsilon$. Let $\gamma = \varepsilon/8$ and A_n be the set of $\theta^n \in \Theta^n$ such that $|\mu_{(k,k')} - \pi_{(k,k')}(\theta^n)| \leq \gamma$.

We have that $\lim_n \mu_n(A_n) = 1$ by Lemma 5. Let $N \in \mathbb{N}$ be such that $1/n < \varepsilon/2$ for all $n \geq N$. We claim that $A_n \subseteq T_n$ for all $n \geq N$, which implies $\lim_n(T_n) = 1$.

Let $n \geq N$ and $\theta^n \in A_n$. Then $\pi_a(\theta^n) \geq \mu_a - \varepsilon/4 > \mu_b + 3\varepsilon/4 \geq \pi_b(\theta^n) + \varepsilon/2 > \pi_b(\theta^n) + 1/n$. Similarly, $\pi_b(\theta^n) > \pi_c(\theta^n) + 1/n$. Letting a^2 be sincere relative to θ^n, it follows from Lemma 2 that $a^2 \in E(\theta^n, \bar{e})$. Hence, $A_n \subseteq T_n$.

A.8 Proof of Corollary 2

For all $n \in \mathbb{N}$, let $S_n = \{\theta^n \in \Theta^n : \sigma$ is strategic relative to θ^n for all $\sigma \in \bar{E}(\theta^n)\}$ and $T_n = \{\theta^n \in \Theta^n : \bar{E}(\theta^n) \neq \emptyset\}$.

We first show that, under Assumptions 1–5, $\lim_n \mu_n(S_n) = 1$. To this end, for all $n \in \mathbb{N}$, let $W_n = \{\theta^n \in \Theta^n : \sigma$ is strategic relative to θ^n for all $\sigma \in \hat{E}(\theta^n)\}$ and note that $S_n^c \subseteq W_n^c$. In fact, if $\theta^n \in S_n^c$, then there exist $\sigma \in \bar{E}(\theta^n)$ such that σ is sincere. Lemma 3 implies that σ^1 is pure and so $\sigma \in \hat{E}(\theta^n)$. Since σ is sincere, then $\theta^n \in W_n^c$.

Corollary 1 implies that $\lim_n \mu(W_n) = 1$. Since $S_n^c \subseteq W_n^c$, it follows that $W_n \subseteq S_n$ and so $\lim_n \mu_n(S_n) = 1$.

If Assumption 6 holds, and Z_n is as in that assumption, then $Z_n \subseteq T_n$ by Lemma 2. Thus, $\lim_n \mu_n(T_n) = 1$. Since $\lim_n \mu_n(S_n) = 1$, it follows that $\lim_n \mu_n(S_n \cap T_n) = 1$.

A.9 Proof of Theorem 4

For all $n \in \mathbb{N}$ and $\varepsilon > 0$, let $S_n^\varepsilon = \{\theta^n \in \Theta^n : \sigma$ is ε-strategic relative to θ^n for all $s \in \bar{E}(\theta^n)\}$ and $T_n = \{\theta^n \in \Theta^n : \sigma^2(\sigma^1)$ is strategic relative to θ^n for all $\sigma \in \hat{E}(\theta^n)\}$ (recall that $\hat{E}(\theta^n)$ denotes the set of $\sigma \in \bar{E}(\theta^n)$ such that σ^1 is pure).

Let $\zeta > 0$ be such that $\delta > \mu_a - \mu_b + \zeta$ and $\delta < \mu_a - \zeta$ (recall Assumption 4) and $\alpha = \min_{k,k' \in C, k \neq k'} \mu_{(k,k')}$. Let $\gamma = \min\{\zeta/4, \alpha/8\}$ and A_n be the set of $\theta^n \in \Theta^n$ such that $|\mu_{(k,k')} - \pi_{(k,k')}(\theta^n)| \leq \gamma$. Let B_n be the set of $\theta^n \in \Theta^n$ such that $\pi_a(\theta^n) > \pi_b(\theta^n)$ and $\pi_{k_1}(\theta^n) + \pi_{(k_3,k_1)}(\theta^n) \neq \pi_{k_2}(\theta^n) + \pi_{(k_3,k_2)}(\theta^n)$ for all permutations (k_1, k_2, k_3) of C. Finally, let X_n and Y_n be as in Assumptions 3 and 4, and define $C_n = A_n \cap B_n \cap X_n \cap Y_n \cap T_n$.

We have that $\lim_n \mu_n(X_n) = \lim_n \mu_n(Y_n) = 1$ by definition, $\lim_n \mu_n(T_n) = 1$ by 1 and $\lim_n \mu_n(A_n) = \lim_n \mu_n(B_n) = 1$ by Lemmas 5 and 6. Therefore, $\lim_n \mu_n(C_n) = 1$. We claim that there exist $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $D_n \subseteq S_n^\varepsilon$ for all $n \geq N$.

We turn to the proof of the above claim. Suppose, in order to reach a contradiction, that the claim does not hold. Hence, for all $m \in \mathbb{N}$, there exists $n_m \geq m$ such that $C_{n_m} \cap (S_{n_m}^{1/m})^c \neq \emptyset$. Thus, for all $m \in \mathbb{N}$, there is $\theta^{n_m} \in \Theta^{n_m}$ and $\sigma_m \in \bar{E}(\theta^{n_m})$ such that $\tilde{\sigma}_m(\{a^1 \in A^1 : \sigma_m^2(a^1)$ is strategic relative to $\theta^{n_m}\}) < 1/m$.

Fix $m \in \mathbb{N}$ such that $\mu_a - \mu_b + \zeta < \delta_{n_m} < \mu_a - \zeta$ and $2\lambda_{n_m}(1 - 1/m) + 4/m < c$. Define $\bar{A}^1 = \{a^1 \in A^1 : \sigma_m^2(a^1)$ is sincere relative to $\theta^{n_m}\}$. For all $i \in I_{n_m}$ and $a_i^1 \in A_i^1$, let $\bar{A}_{-i}^1(a_i^1) = \{a_{-i}^1 \in A_{-i}^1 : (a^1, a_{-i}^1) \in \bar{A}^1\}$. Note that $\bar{A}_1^c = \cup_{a_i^1 \in A_i^1} \left(\{a_i^1\} \times \bar{A}_{-i}^1(a_i^1)^c\right)$ and so $\sigma_{m,-i}^1(\bar{A}_{-i}^1(a_i^1)^c) < 1/m$ for all $i \in I_{n_m}$ and $a_i^1 \in A_i^1$.

Assumption 1 implies that $\bar{A}^1 \subseteq \{a^1 \in A^1 : a^1$ is sincere relative to $\theta^{n_m}\}$ and so $\tilde{\sigma}_m(\{a^1 \in A^1 : a^1$ is sincere relative to $\theta^{n_m}\}) > 1 - 1/m$.

Recall that $\sigma_m^2(a^1)$ is pure for all a^1 that are sincere relative to θ^{n_m}. Thus, using the definition of A_{n_m} and B_{n_m} as in the proof of Theorem 2, Lemma 1 implies that if a^1 is sincere relative to θ^{n_m}, then $\sigma_m^2(a^1)$ is sincere relative to $g_{a^1}(\theta^{n_m})$.

If $a^1 \in \bar{A}^1$, then $\pi_k(\sigma_m^2(a^1)) = \pi_k(\theta^{n_m})$ for all $k \in C$. The definition of A_{n_m} and B_{n_m} implies, as in the proof of Theorem 2, that $\bar{A}_1 \subseteq \{a^1 \in A^1 : \pi_a(\sigma_m^2(a^1)) > \pi_k(\sigma_m^2(a^1))$ for all $k \neq a\}$. Thus, $\tilde{\sigma}_m(\{a^1 \in A^1 : \pi_a(\sigma_m^2(a^1)) > \pi_k(\sigma_m^2(a^1))$ for all $k \neq$

$a\}) > 1 - 1/m$.

Furthermore, for all $i \in I_{n_m}$ and $a^1 \in \bar{A}^1$, we have that $u_i(\sigma_m^2(a^1)) = \lambda_{n_m}\theta_i^{n_m} \cdot \pi(\sigma_m^1(a^1)) + \theta_{i,a}^{n_m}$. Note that $u_i(a^2) \leq 4$ for all second-period action profiles a^2. Let, for all $a^1 \in A^1$, $\tilde{\sigma}_m^2(a^1)$ be the product probability measure defined by $\sigma_{m,1}^2(a^1), \ldots, \sigma_{m,n_m}^2(a^1)$. Therefore, it follows that, for all $a_i^1 \in \operatorname{supp}(\sigma_{m,i}^1)$,

$$\begin{aligned} u_i((a_i^1, \sigma_{m,i}^2), \sigma_{m,-i}) &= -c1_C(a_i^1) + \sum_{a_{-i}^1 \in A_{-i}^1} \sigma_{m,-i}^1(\{a_{-i}^1\}) \left[\sum_{a^2} \tilde{\sigma}_m^2(a_i^1, a_{-i}^1)(\{a^2\}) u_i(a^2) \right] \\ &\leq -c1_C(a_i^1) + \sum_{a_{-i}^1 \in \bar{A}_{-i}^1(a_i^1)} \sigma_{m,-i}^1(\{a_{-i}^1\}) u_i(\sigma_m^2(a_i^1, a_{-i}^1)) + 4/m. \end{aligned}$$

Claim 4 *For all $i \in I_{n_m}$, $\sigma_{m,i}^1(\{b\}) = 0$.*

Proof of Claim 4. Suppose, in order to reach a contradiction, that there is $i \in I_{n_m}$ such that $\sigma_{m,i}^1(\{b\}) > 0$.

We have that if $a_{-i}^1 \in \bar{A}_{-i}^1(b)$, then $u_i(\sigma_m^2(e, a_{-i}^1)) \geq u_i(\sigma_m^2(b, a_{-i}^1)) - 2\lambda_{n_m}$. This inequality follows from the arguments used in Claim 1 since both (b, a_{-i}^1) and (e, a_{-i}^1) are sincere relative to θ^{n_m}.

Therefore,

$$\begin{aligned} u_i((e, \sigma_{m,i}^2), \sigma_{m,-i}) - u_i((b, \sigma_{m,i}^2), \sigma_{m,-i}) &\geq c - 2\lambda_{n_m}\sigma_{m,-i}^1(\bar{A}_{-i}(b)) - 4\sigma_{m,-i}^1(\bar{A}_{-i}(b)^c) \\ &> c - 2\lambda_{n_m}(1 - 1/m) - 4/m > 0. \end{aligned}$$

Since $\sigma_m \in \tilde{E}(\theta^{n_m})$, this is a contradiction. ■

Claim 5 *For all $i \in I_{n_m}$, $\sigma_{m,i}^1(\{c\}) = 0$.*

Proof of Claim 5. Suppose, in order to reach a contradiction, that there is $i \in I_{n_m}$ such that $\sigma_{m,i}^1(\{c\}) > 0$.

We will show that if $a_{-i}^1 \in \bar{A}_{-i}^1(c)$, then $u_i(\sigma_m^2(e, a_{-i}^1)) \geq u_i(\sigma_m^2(c, a_{-i}^1)) - 2\lambda_{n_m}$. This inequality follows from the arguments used in Claim 2 (using Claim 4, instead of Claim 1) since both (b, a_{-i}^1) and (e, a_{-i}^1) are sincere relative to θ^{n_m}.

Therefore, as in Claim 4, $u_i((e, \sigma_{m,i}^2), \sigma_{m,-i}) - u_i((c, \sigma_{m,i}^2), \sigma_{m,-i}) > c - 2\lambda_{n_m}(1 - 1/m) - 4/m > 0$. Since $\sigma_m \in \tilde{E}(\theta^{n_m})$, this is a contradiction. ■

Claim 6 *For all $i \in I_{n_m}$, $\sigma^1_{m,i}(\{a\}) = 0$.*

Proof of Claim 6. Suppose, in order to reach a contradiction, that there is $i \in I_{n_m}$ such that $\sigma^1_{m,i}(\{a\}) > 0$.

We will show that if $a^1_{-i} \in \bar{A}^1_{-i}(a)$, then $u_i(\sigma^2_m(e, a^1_{-i})) \geq u_i(\sigma^2_m(a, a^1_{-i}))$. This inequality follows from the arguments used in Claim 3 (using Claims 4 and 5 instead of Claims 1 and 2) since both (a, a^1_{-i}) and (e, a^1_{-i}) are sincere relative to θ^{n_m}.

Therefore, $u_i((e, \sigma^2_{m,i}), \sigma_{m,-i}) - u_i((a, \sigma^2_{m,i}), \sigma_{m,-i}) > c - 4/m > 0$. Since $\sigma_m \in \tilde{E}(\theta^{n_m})$, this is a contradiction. ■

It follows from Claims 1, 2 and 3 that $\sigma^1_{m,i} = e$ for all $i \in I_n$. Hence, σ^1_m is pure and $\sigma^2_m(\sigma^1_m)$ is sincere relative to θ^{n_m}. This implies that $\theta^{n_m} \in T^c_{n_m}$. But this contradicts $\theta^{n_m} \in C_{n_m} \subseteq T_{n_m}$.

A.10 Proof of the Claims in Example 5.1

We first show that all the assumptions of Theorem 2 hold. It is clear that Assumptions 1 and 4 hold. Assumption 2 holds because if a^1 is sincere that $A^2_i(\theta^n, a^1)$ equals either C or $\{f(\theta^n_i)\}$.

Let $\gamma = \zeta/8$ and define $X_n = Y_n = A^\gamma_n$, where A^γ_n is as in Lemma 5. Note that, for all $\theta^n \in \Theta^n$ and $i \in I_n$, if $a^1 \in A^1$ is sincere relative to θ^n, then either $g_i(\theta^n, a^1) = \theta^n_i$ or $g_i(\theta^n, a^1) = (f_2(\theta^n_i), f(\theta^n_i))$ and $|\{j \in I_n : g_i(\theta^n, a^1) = (f_2(\theta^n_i), f(\theta^n_i))\}|/n \leq \delta_n$. Also, for all $\theta^n \in X_n$,

$$\pi_{(k,k')}(\theta^n) \geq \mu_{(k,k')} - \gamma \geq \frac{\mu_c}{2} - \gamma > \frac{\delta}{2}.$$

Assumption 3.1 holds since, for all $\theta^n \in X_n$, $a^1 \in A^1$ sincere relative to θ^n and $k, k' \in C$ with $k \neq k'$,

$$\pi_{(k,k')}(g_{a^1}(\theta^n)) \geq \pi_{(k,k')}(\theta^n) - \delta_n \geq \mu_{(k,k')} - \delta - \gamma \geq \frac{\mu_c}{2} - 3\zeta > \zeta > 0.$$

We next show that Assumption 3.2 holds. Let $\theta^n \in X_n$, $a^1 \in A^1$ be sincere relative to θ^n and (k_1, k_2, k_3) be a permutation of C. Let k_1 be such that $\mu_{k_1} > \mu_{k_2}$, which gives us three cases: $(k_1, k_2, k_3) = (a, b, c)$, $(k_1, k_2, k_3) = (a, c, b)$ and

$(k_1, k_2, k_3) = (b, c, a)$. Consider first the second and third cases. In this case, we have that $\pi_{k_1}(g_{a^1}(\theta^n)) \geq \pi_{k_1}(\theta^n) - \delta$, $\pi_{(k_3,k_1)}(g_{a^1}(\theta^n)) \geq \pi_{(k_3,k_1)}(\theta^n) - \delta$, $\pi_c(g_{a^1}(\theta^n)) \leq \pi_c(\theta^n) + \delta$ and $\pi_{(k_3,c)}(g_{a^1}(\theta^n)) \leq \pi_{(k_3,c)}(\theta^n) + \delta$. Therefore,

$$\pi_{k_1}(g_{a^1}(\theta^n)) + \pi_{(k_3,k_1)}(g_{a^1}(\theta^n)) \geq \pi_{k_1}(\theta^n) + \pi_{(k_3,k_1)}(\theta^n) - 2\delta \geq$$
$$\mu_{k_1} + \mu_{(k_3,k_1)} - 2\delta - 3\gamma > \mu_c + \mu_{(k_3,c)} + 9\zeta - 2\delta - 3\gamma \geq$$
$$\pi_c(\theta^n) + \pi_{(k_3,c)}(\theta^n) + 9\zeta - 2\delta - 6\gamma \geq \pi_c(g_{a^1}(\theta^n)) + \pi_{(k_3,c)}(g_{a^1}(\theta^n)) + 9\zeta - 4\delta - 6\gamma \geq$$
$$\pi_c(g_{a^1}(\theta^n)) + \pi_{(k_3,c)}(g_{a^1}(\theta^n))$$

since $\delta < 2\zeta$ and $8\gamma < \zeta$.

In the case of $(k_1, k_2, k_3) = (a, b, c)$, we need only to consider the case where $a_i^1 \in C$ for all $i \neq 1$. In this case, for all $k = a, b$, $\pi_k(g_{a^1}(\theta^n)) = \pi_k(\theta^n)$ and either $\pi_{(c,k)}(g_{a^1}(\theta^n)) = 0$ or $\pi_{(c,k)}(g_{a^1}(\theta^n)) = 1/n$ and $\theta_1^n = (c, k)$. Hence,

$$\pi_a(g_{a^1}(\theta^n)) + \pi_{(c,a)}(g_{a^1}(\theta^n)) \geq \pi_a(\theta^n) \geq \mu_a - 2\gamma = \mu_b + \zeta - 2\gamma \geq$$
$$\pi_b(\theta^n) + \zeta - 4\gamma \geq \pi_b(g_{a^1}(\theta^n)) + \pi_{(c,b)}(g_{a^1}(\theta^n)) + \zeta - 4\gamma - \frac{1}{n} >$$
$$\pi_b(g_{a^1}(\theta^n)) + \pi_{(c,b)}(g_{a^1}(\theta^n))$$

when n is sufficiently large (since the condition in Assumption 3.2 needs to hold for all n, we can let $X_n = \emptyset$ when n is not sufficiently large).

Let $\theta^n \in \Theta^n$ and a^1 be sincere relative to θ^n. Assumption 5.1 holds because $a_i^1 = k$, $\bar{a}^1 = (e, a_{-i}^1)$ and $k' \in A_j^2(\theta^n, a^1) \setminus \{k\}$ implies that $j \neq i$ and either $a_j^1 = k$ or $a_j^1 = e$. In the former case we have that $A_j^2(\theta^n, a^1) = \{k'\} = A_j^2(\theta^n, \bar{a}^1)$, while in the latter, $A_j^2(\theta^n, a^1) = C = A_j^2(\theta^n, \bar{a}^1)$. Assumption 5.2 holds because $A_i^2(\theta^n, a^1) = \{k\}$ implies either $a_i^1 = k$ or $a_1^1 = k$.

A.11 Robust strategic voting

Theorem 5 presents a stronger conclusion in the following two senses: First, the probability of observing a first period actions that induce equilibria with strategic voting is bounded away from zero and, second, the fraction of players voting strategically in those equilibria is also bounded away from zero.

This result requires stronger assumptions. Essentially, the properties that were assumed to hold for first period action profile a^1 that sincere relative to the profile θ^n of rankings need to hold when the fraction of those choosing sincerely in the first period is at least $1-\varepsilon$, for some $\varepsilon > 0$.

Let $n \in \mathbb{N}$, $\theta^n \in \Theta^n$, $a^1 \in A^1$ and $\varepsilon > 0$. We say that a^1 is ε-sincere relative to $\theta^n \in \Theta^n$ if $|\{i \in I_n : a_i^1 \notin \{e, f(\theta_i^n)\}\}|/n < \varepsilon$.

Assumption 7 *There exists $\varepsilon > 0$ such that, for all $\theta^n \in \Theta^n$, $i \in I_n$, $k \in C$ and $a^1 \in A^1$ that are ε-sincere relative to θ^n, if $A_i^2(\theta^n, a^1) \neq C$ and $\theta_{i,k}^n = 0$, then $k \notin A_i^2(\theta^n, a^1)$.*

Assumption 8 *There exists $\varepsilon > 0$ such that, for all $n \in \mathbb{N}$, there exist $X_n \subseteq \Omega^n$ such that $\lim_n \mu_n(X_n) = 1$ and the following conditions hold for all $\theta^n \in X_n$ and $a^1 \in A^1$ such that a^1 is ε-sincere relative to θ^n:*

1. *For all $k, k' \in C$ such that $k \neq k'$, if $\pi_{(k,k')}(\theta^n) > 0$, then $\pi_{(k,k')}(g_{a^1}(\theta^n)) > 0$.*
2. *For all permutations (k_1, k_2, k_3) of C, if $\pi_{k_1}(\theta^n) + \pi_{(k_3,k_1)}(\theta^n) \neq \pi_{k_2}(\theta^n) + \pi_{(k_3,k_2)}(\theta^n)$, then $\pi_{k_1}(g_{a^1}(\theta^n)) + \pi_{(k_3,k_1)}(g_{a^1}(\theta^n)) \neq \pi_{k_2}(g_{a^1}(\theta^n)) + \pi_{(k_3,k_2)}(g_{a^1}(\theta^n))$.*

Assumption 9 *There exists $\varepsilon > 0$ such that, for all $n \in \mathbb{N}$, there exist $Y_n \subseteq \Omega^n$ and $\delta_n \in (0,1)$ such that $\lim_n \mu_n(Y_n) = 1$, $\lim_n \delta_n = \delta \in (\mu_a - \mu_b, \mu_a)$ and the following condition holds for all $\theta^n \in Y_n$ and $a^1 \in A^1$ such that a^1 is ε-sincere relative to θ^n:*

If there is $k \in C$ such that $|\{j \in I_n : f(\theta_j^n) \neq k\}|/n \geq \delta_n$ and $a_i^1 = k$ for all $i \in I_n$ with $a_i^1 \in C$, then $|\{j \in I_n : f(\theta_j^n) \neq k$ and $A_j^2(\theta^n, a^1) = \{k\}\}|/n \geq \delta_n$.

Assumption 10 *There is $\varepsilon > 0$ such that the following conditions hold for all $n \in \mathbb{N}$, $\theta^n \in \Theta^n$ and $a^1 \in A^1$ such that a^1 is ε-sincere relative to θ^n:*

1. *If, for some $i \in I_n$ and $k \in C$, $a_i^1 = k$ and $\bar{a}^1 = (e, a_{-i}^1)$, then, for all $j \in I_n$ and all $k' \in C \setminus \{k\}$, $k' \in A_j^2(\theta^n, a^1)$ implies $k' \in A_j^2(\theta^n, \bar{a}^1)$.*
2. *If there exists $k \in C$ such that $a_i^1 \neq k$ for all $i \in I_n$, then $A_i^2(\theta^n, a^1) \neq \{k\}$ for all $i \in I_n$.*

Assumption 11 *There is $\varepsilon > 0$ such that, for all $n \in \mathbb{N}$, there exist $Z_n \subseteq \Omega^n$ such that $\lim_n \mu_n(Z_n) = 1$ and the following condition holds for all $\theta^n \in Z_n$ and $a^1 \in A^1$ such that a^1 is sincere relative to θ^n: There exists $k \in C$ such that $\pi_k(g_{a^1}(\theta^n)) > \pi_{k'}(g_{a^1}(\theta^n)) + 1/n$ for all $k' \neq k$.*

The following terminology is used in its statement. For all $n \in \mathbb{N}$, $\theta^n \in \Theta^n$, $\sigma \in \Sigma(\theta^n)$ and $\varepsilon > 0$, we say that σ is ε-robustly strategic if $\tilde{\sigma}(\{a^1 \in A^1 : |\{i \in I_n : \sigma_i^2(a^1) \neq f(\theta_i^n)\}| \geq n\varepsilon\}) \geq \varepsilon$. Thus, a strategy is ε-robustly strategic if the probability of a first period action that leads to a fraction of at least ε players voting strategically is at least ε. Let $\Sigma^\varepsilon(\theta^n)$ denote the set of mixed strategies σ such that $\sigma^2(a^1)$ is pure for all a^1 that are ε-sincere relative to θ^n and let $\bar{E}^\varepsilon(\theta^n) = \tilde{E}(\theta^n) \cap \Sigma^\varepsilon(\theta^n)$ be the set of mixed strategy equilibria of $G_n(\theta^n)$ that belong to $\Sigma^\varepsilon(\theta^n)$.

Theorem 5 *If Assumptions 1 and 7–10 hold, then there exists $\varepsilon > 0$ such that*

$$\lim_n \mu_n(\{\theta^n \in \Theta^n : \sigma \textit{ is } \varepsilon\textit{-robustly strategic relative to } \theta^n \textit{ for all } \sigma \in \bar{E}^\varepsilon(\theta^n)\}) = 1.$$

If, in addition, Assumption 11 holds, then there exists $\varepsilon > 0$ such that

$$\lim_n \mu_n(\{\theta^n \in \Theta^n : \bar{E}^\varepsilon(\theta^n) \neq \emptyset \textit{ and } \sigma \textit{ is } \varepsilon\textit{-robustly strategic relative to } \theta^n \textit{ for all } \sigma \in \bar{E}(\theta^n)\}) = 1.$$

References

Duverger, M. (1955): *Political Parties*. Wiley, New York.

Kalai, E. (2004): "Large Robust Games," *Econometrica*, 72, 1631–1665.