Independent Random Matching with Many Types

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Version: February 2010

Abstract

Random matching models with a continuum population are widely used in economics to model decentralized environments. A number of economic models—e.g., in evolutionary game theory and monetary theory—explicitly or implicitly assume pairwise random matching with convenient proportionality and independence properties. This paper provides foundations to random matching models of continuum populations with *infinitely* many types, which are currently used in the literature without an explicit justification. The approach of this paper uses tools of standard measure theory, as opposed to that in Duffie and Sun (Ann Appl Probab 17:386–419, 2007) which is based on nonstandard analysis.

JEL classification: C00, C02, C73, C78

Keywords: Random matching; Involution; Independence; Continuum population

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1 Introduction

Decentralized environments are relevant to a variety of fields, e.g., game theory, monetary economics, labor economics and experimental economics. Typically, random matching plays an important role in modeling decentralized environments and in generating a variety of frictions. For instance, in game theory, the matching technology has important implications for the sustainability of cooperation in dynamic games (see Kandori (1992)). Some monetary models are built on markets with frictions, where communication is limited and trade fragmented by assuming that agents interact in small coalitions (e.g., Kiyotaki and Wright (1989)). In experimental economics, the type of matching protocol has important behavioral implications (e.g., Andreoni and Croson (2008), Botelho et al. (2009)).

The typical approach in models with continuum of agents is to assume (implicitly or explicitly) that the matching technology satisfies a list of desired properties (e.g., proportionality properties, independence properties, anonymity properties, constant returns to scale, etc.). These properties are desirable as they are consistent with a limit approach. Specifically, under certain conditions (see Molzon and Puzzello (2010)), they arise naturally as limits of uniform random matching for finite populations. That is, these properties are consistent with the notion that continuum populations are a convenient idealization of finite large populations.

Proportionality and independence properties of random matching have also been employed in genetics to study the steady state frequencies of allelic types (See Hardy (1908)). The following example dates back to Hardy, and was introduced into economics by Boylan (1992). Consider a continuum population of randomly matched gametes consisting of a fraction p of alleles A and fraction $q = 1 - p$ of alleles B. Then, the Hardy-Weinberg approach (by implicitly invoking the law of large numbers) predicts that the new population will consist of a p^2 fraction whose parents are both of type A, a q^2 fraction whose parents are both of type B, and a $2pq$ fraction whose parents are mixed.

Even though the Hardy-Weinberg law has intuitive appeal, it has been quite challenging to provide its mathematical foundations also due to measurability issues associated with a continuum of i.i.d. random variables (for details, see Doob (1953) and Judd (1985)). Thus, a number of studies have constructed random matching technologies satisfying a list of desired properties by explicit constructions that do not invoke the law of large numbers (see Aliprantis et al. (2006), Alós-Ferrer (1999), Alós-Ferrer (2002), Boylan (1992), Boylan (1995), Gilboa and Matsui (1992)). For instance, Alós-Ferrer (1999) constructed an explicit random matching process satisfying proportionality properties for continuum populations. However, the process he proposed does not satisfy independence in types.[1] Only recently, Duffie and Sun (2007) have provided existence

[1]Molzon and Puzzello (2010) provide an existence result of random matching with propor-

results for random matching satisfying proportionality as well as independence properties. Their approach relies on tools from nonstandard analysis.

All the existence results available in the literature, including Duffie and Sun (2007), apply to models with *finitely* many types. As pointed out in Molzon and Puzzello (2010), this might be an issue. Indeed, for a wide class of economic models, it is not possible to capture all the relevant attributes of agents with a finite number of types. This is the case, e.g., for the models described in Green and Zhou (2002), Molico (2006), Lagos and Wright (2005), Shi (1997), Zhu (2005), where there are no upper bounds on money holdings or money holdings are perfectly divisible, or those described in Sandholm (2001), Oechssler and Riedel (2002), Hofbauer et al. (2008), where continuous strategy sets matter. Such models are currently missing explicit foundations for the random matching process.

Our paper provides several contributions to the literature on matching. In particular, it provides general existence and uniqueness results with the tools of standard measure theory. Specifically, it shows existence of random matching with proportionality and independence properties, thus providing explicit microfoundations to a wide class of economic models including ones with infinitely many types (e.g., Green and Zhou (2002), Molico (2006), Lagos and Wright (2005), Shi (1997), Zhu (2005), Sandholm (2001), Oechssler and Riedel (2002), Hofbauer et al. (2008)).

It has always been implicit in the literature that, in case of existence of random matching in some model, uniqueness will not hold in general; see in particular Molzon and Puzzello (2010) where it is clarified that uniqueness does not hold if the random matching satisfies only proportionality properties. However, in this paper we show that if a random matching satisfies both proportionality and independence properties, then, in terms of distributions on the set of matchings, uniqueness is guaranteed. We also discuss the relationships between different properties of a random matching and their relevance for economic models.

The paper is organized as follows. Section 2 provides some basic definitions and formalizes proportionality, independence and anonymity properties of random matching. The main result of our paper is on existence of independent random matching and can be found in Section 3. In Section 4 we discuss relationships between different properties of a random matching. Section 5 provides a uniqueness result. Note that our existence result is quite general and applies also to models with infinitely many types. Such models arise naturally in economics, as illustrated in Section 6 where some examples from the literature are discussed.

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tionality properties for finite populations. Clearly, in this setting, independence in types cannot be satisfied.

2 Definitions and properties of random matching

There are at least two ways that one can think of random matching. One approach works explicitly with the set of agents and with bilateral matches as mappings on this set. This is the approach usually followed in the literature on foundations of matching, e.g., Alós-Ferrer (1999), Duffie and Sun (2007), Molzon and Puzzello (2010), as well as in applications, e.g., Lagos and Wright (2005). The second approach avoids working directly with the population of agents and just takes measures on the set of characteristics/types of agents as the primitives. This is the approach followed by McLennan and Sonnenschein (1991).[2] We follow the first approach.

We first introduce some basic definitions that allow us to formalize random matching properties that are implicitly used in a variety of matching models.

Definition 1. Let X be a set. An *involution* on X is a bijection $f\colon X \to X$ which is self-inverse (i.e., such that the inverse f^{-1} satisfies $f^{-1} = f$); equivalently, an involution on X is a mapping $f\colon X \to X$ such that $f \circ f$ is the identity on X. A mapping $f\colon X \to X$ is said to be *fixed point free* if $f(x) \neq x$ for all $x \in X$.

Involutions provide a natural formalization for the notion of bilateral matching (e.g., Alós-Ferrer (1999), Aliprantis et al. (2006)). In this study, we focus on bilateral matching technologies where no agent remains unmatched.

Definition 2. Let A be a set of agents. A *bilateral matching* (or, for short, a *matching*) on A is a fixed point free involution on A.

We are now ready to give the definition of random bilateral matching.

Definition 3. Let $(A, \mathcal{A}, \mu)$ be a probability space of agents and let (Ω, Σ, ν) be a sample probability space. A *random bilateral matching* (or a *random matching*) is a mapping $f\colon A \times \Omega \to A$ such that

(a) $f(\cdot, y)$ is a matching on A for each $y \in \Omega$,

(b) the mappings $f(\cdot, y)\colon A \to A$ and $f(x, \cdot)\colon \Omega \to A$ are measurable for each $y \in \Omega$ and $x \in A$.

Notation. In the context of Definition 3, we also write f_x for the function $f(x, \cdot)$, and f_y for $f(\cdot, y)$.

Note that this definition involves two probability spaces: one associated with the population and the other associated with the set of bilateral matchings. This definition is general as it imposes only minimal measurability conditions necessary to formulate our definitions and results. Specifically, it leaves the door open

[2]See also Karavaev (2009), who proposes a model that avoids working directly with the population of agents and takes the distribution of idiosyncratic shocks as a primitive.

to a variety of properties including proportionality and independence properties that are consistent with the idea of "uniform" random matching.

Many economic models assume that agents of different types are randomly matched in pairs. For instance, in monetary theory buyers and sellers endowed with different amounts of money are randomly matched in pairs (e.g., Kiyotaki and Wright (1989)). Similarly, in evolutionary game theory, agents playing certain strategies are randomly matched to play a normal form game (e.g., Kandori et al. (1993)).

Next, we provide the definition of type assignment and its measurability properties.

Definition 4. A *type space* is a measurable space $(T, \mathcal{T})$. Given a probability space $(A, \mathcal{A}, \mu)$ of agents, a *type assignment* is a measurable mapping θ from $(A, \mathcal{A}, \mu)$ to a type space $(T, \mathcal{T})$, and the corresponding type distribution is the distribution of θ, i.e. the probability measure on T given by $\tau(B) = \mu(\theta^{-1}(B))$ for each $B \in \mathcal{T}$.

Proportionality and independence properties of random matching play an important role in existing matching models. For instance, in dynamic models, these properties are used to write down expected payoffs and law of motions in each period. These properties arise naturally as limits of bilateral uniform random matching for finite sets of agents (see Molzon and Puzzello (2010)). In other words, proportionality and independence properties are consistent with the notion that continuum populations are a convenient idealization of finite large populations. We formalize these properties next.

Let $(A, \mathcal{A}, \mu)$ be a probability space of agents, (Ω, Σ, ν) a sample probability space, $f\colon A \times \Omega \to A$ a random matching, $(T, \mathcal{T})$ a type space, $\theta\colon A \to T$ a type assignment, and τ the corresponding type distribution.

(P1) "Measure preservation:"[3] For all $y \in \Omega$, f_y is inverse-measure-preserving, i.e., $\mu(f_y^{-1}(E)) = \mu(E)$ for any $E \in \mathcal{A}$.

(P2) "General proportional law:" For all $x \in A$, f_x is inverse-measure-preserving, i.e., $\nu(f_x^{-1}(E)) = \mu(E)$ for any $E \in \mathcal{A}$.

(P3) "Strong mixing:" For any E_1, $E_2 \in \mathcal{A}$, $\mu(E_1 \cap f_y^{-1}(E_2)) = \mu(E_1)\mu(E_2)$ for almost all $y \in \Omega$.

(P4) "General independence:" The family $\langle f_x \rangle_{x \in A}$ is stochastically independent; that is, the family $\langle \Sigma_x \rangle_{x \in A}$ is stochastically independent, writing Σ_x for the sub-σ-algebra of Σ generated by f_x.

(P5) "Atomless:" For any two x, $x' \in A$, the set $\{y \in \Omega\colon f_x(y) = x'\}$ is a ν-null set.

[3] In the naming of (P1)-(P3) and (P5)-(P7) we follow Alós-Ferrer (1999).

(P6) "Types proportional law:" For all $x \in A$, the mapping $\theta \circ f_x$ from Ω to T has distribution τ, i.e., $\nu\big((\theta \circ f_x)^{-1}(B)\big) = \tau(B)$ for any $B \in \mathcal{T}$.

(P7) "Types mixing:" For any $B_1, B_2 \in \mathcal{T}$, $\mu\big(\theta^{-1}(B_1) \cap (\theta \circ f_y)^{-1}(B_2)\big) = \tau(B_1)\tau(B_2)$ for almost all $y \in \Omega$.

(P8) "Independence in types:" The family $\langle \theta \circ f_x \rangle_{x \in A}$ of mapping from Ω to T is stochastically independent; that is, the family $\langle \Sigma_x^\theta \rangle_{x \in A}$ is stochastically independent, writing Σ_x^θ for the sub-σ-algebra of Σ generated by $\theta \circ f_x$.

Property (P1) requires that, for all matchings, a given measurable set of agents must have the same measure as the set of their partners. Property (P2) establishes that the probability that a given agent is paired to an agent in a measurable set E is equal to the measure of the set of agents in E. This property plays an important role for expected payoff equations. Property (P3) states that the measure of the agents in a given measurable set which are matched with agents in another measurable set is equal to the product of the measures of the two sets for almost all matchings. This property holds if the stochastic dynamic system generated by uniform random matching can be approximated by a deterministic system (e.g., Alós-Ferrer (1999) or Boylan (1992)). In Section 3, we will show that this property may also be viewed as displaying some "large numbers" effect. The intuition behind property (P4) is that, for finitely many distinct agents in a continuum population, the events that these agents have partners in any given measurable sets should be independent.[4] Of course, (P4) cannot hold for random matchings in a finite population. However, by calculations it may be seen that in a sequence of finite populations where the size goes to infinity and the random matching in each of these populations is uniform, for any fixed integer $k \geq 2$ the deviation from independence that appears for any sets of k agents vanishes asymptotically.[5] Thus, in a model with a continuum population, viewed as idealization of a large finite population, it may be seen as natural to require a random matching to satisfy (P4). Property (P5) states that the probability that any two given agents are matched is zero. This property may also be seen as natural in a model with a continuum population. Moreover, this property

[4]The fact that matching agent x_i with agent x_j implies x_j is matched with x_i does not mean a contradiction to (P4) if the space of agents is atomless and the random matching satisfies (P2), because any two null sets in the sample space are trivially stochastically independent.

[5]Indeed, to capture also finite populations with an odd number of agents, modify Definition 1 to require that at most one agent remains unmatched. Then for each integer $n > 0$, let A^n be a finite population with n agents, and I^n the set of all matchings on A^n. Let P^n be the normalized counting measure on I^n. Suppose that the random matching is uniform for each n, i.e., that all elements of I^n are equally likely. Then, for each n, randomness of matching is captured by P^n. Fix an integer $k > 0$. For each $n > k$, let $A_1^n, \ldots, A_k^n$ be any subsets of A^n, let $x_1^n, \ldots, x_k^n$ be any distinct agents in A^n, and for each $i = 1, \ldots, k$, let F_i^n be the set of those elements of I^n which match agent x_i^n with an agent in A_i^n. A straightforward but a bit messy calculation shows that $\left| P^n(\bigcap_{i=1}^k F_i^n) - \prod_{i=1}^k P^n(F_i^n) \right| \to 0$ as $n \to \infty$, i.e., the deviation from independence that appears for any sets of k agents vanishes asymptotically.

is important in models with a continuum of agents as it captures the notion of "anonymity" (see Aliprantis et al. (2006) for the notion of "anonymity"). Properties (P6), (P7) and (P8) are analogs of (P2), (P3) and (P4) respectively, formulated in terms of type assignments. In fact, the following holds as may readily be seen.

Remark 1. For any type assignment, (P2) implies (P6), (P3) implies (P7), and (P4) implies (P8). If $(A, \mathcal{A}, \mu)$ is atomless, then (P2) also implies (P5).

See Section 4 for further relationships between the properties listed above.

3 The main existence result

In this section we state and prove our main result on existence of independent random matching.

Theorem 1. *There exists an atomless probability space* $(A, \mathcal{A}, \mu)$ *of agents, with* $\#(A) = \mathfrak{c}$ *(so that* A *may be identified with* $[0, 1]$*),*[6] *a sample probability space* (Ω, Σ, ν)*, and a random matching* $f : A \times \Omega \to A$ *such that the following hold.*

(a) (P1) *to* (P5) *are satisfied for* f*.*

(b) *Given any type space* $(T, \mathcal{T})$ *and type assignment* $\theta : A \to T$*,* (P6) *to* (P8) *are satisfied for* f*.*

(c) *Let* λ *be the product measure on* $A \times \Omega$ *defined from* μ *and* ν*. There is a Fubini extension* $\bar{\lambda}$ *of* λ *such that* f *is* $(\bar{\Lambda}, \mathcal{A})$*-measurable, writing* $\bar{\Lambda}$ *for the domain of* $\bar{\lambda}$*; in particular, given any type space* $(T, \mathcal{T})$ *and type assignment* $\theta : A \to T$*, the type process* $\theta \circ f$ *is* $(\bar{\Lambda}, \mathcal{T})$*-measurable.*[7]

Remark 2. Note that Theorem 1 gives a random matching that does not depend on the type assignment or type distribution. In particular, the random matching is independent in types against any type assignment.

Theorem 1 contributes to the literature in several ways. It provides a general existence result which differs from that in Alós-Ferrer (1999) as general independence is satisfied, and in particular, independence in types. As pointed out in the previous section, in a model with a continuum population used as idealization of large finite populations, it may be seen as natural to require that a random matching satisfies these independence properties.

In this regard, Theorem 1 also differs from the existence result in Duffie and Sun (2007, Theorem 2.4) which, concerning independence properties, makes a statement only about *pairwise* independence in types, i.e. property (P8) weakened to pairwise independence. However, given that one wants to have a random matching to satisfy independence properties, it seems more natural to us

[6] $\#(X)$ denotes the cardinal of a set X; $\mathfrak{c}$ denotes the cardinal of the continuum.
[7] See Remark 5 below for the notion of a Fubini extension.

to require these properties to hold already on the general level of the random matching, as in property (P4), and to require these properties to hold in the form of full stochastic independence, and not only in the form of pairwise independence.

Furthermore, unlike Duffie and Sun (2007), our result is not based on nonstandard analysis. In particular, it does not depend on Loeb space constructions.

Finally, our result applies also to the case of infinitely many types, currently missing explicit foundations in the literature. Indeed, recall that given any atomless probability space $(A, \mathcal{A}, \mu)$ and any Borel probability measure γ on a Polish space Z, there is a mapping $\theta\colon A \to Z$ which is inverse-measure-preserving for μ and γ; in other words, every such γ is the distribution of some measurable mapping from A to Z. Consequently, Theorem 1 allows for any Borel probability measure on a Polish space to be taken as type distribution.

Remark 3. In Theorem 1, for the probability space $(A, \mathcal{A}, \mu)$ of agents we can have $A = [0, 1]$; however, unlike Alós-Ferrer (1999), μ cannot be Lebesgue measure on $[0, 1]$. In fact, as noted in Alós-Ferrer (1999), if $[0, 1]$ with Lebesgue measure is taken as space of agents, there can be no random matching such that properties (P3) and (P1) hold together. However, in our opinion, having or not having $[0, 1]$ with Lebesgue measure as the space of agents is of no economic significance. As pointed out by Hildenbrand (1974, p. 113), if, in order to establish that any single agent has strictly no influence on aggregate levels, a large set of negligible agents is modelled as an atomless probability space, the σ-algebra should be considered as "only been introduced for technical reasons" and, conceptually, "be considered ... as the set of all subsets" of the set of agents.[8] Under this view, any atomless probability measure on $[0, 1]$ is as good as any other in modelling a large set of negligible agents, and a particular choice, e.g. according to our Theorem 1, of a σ-algebra, or probability measure, on the set of agents should not be discussed in terms of economic meaning, but should be seen as a technical device having to do some job. In this regard, note, for instance, that Theorem 1 gives a random matching with the property that types mixing holds for *any* type assignment, a property that cannot hold for any random matching when $[0, 1]$ with Lebesgue measure is the space of agents, by the next remark since this property implies (P3) as noted in Theorem 3 below. Similarly, by Theorem 4 below, if one wants to have a random matching satisfying independence and proportionality properties, as well as some joint measurability property with respect to agents and sample point, then $[0, 1]$ with Lebesgue measure is also not the appropriate choice of the probability space of agents.

Remark 4. Actually, if $[0, 1]$ with Lebesgue measure is taken as space of agents, then already (P3) alone cannot hold for any random matching. To see this, let

[8] In particular, it should not be considered an essential point whether or not the σ-algebra is countably generated. Further, the σ-algebra need not be derived from any topological structure on the set of agents, and therefore, in case of $[0, 1]$ as set of agents, it should also not be considered an essential point whether or not the σ-algebra contains the sub-intervals of $[0, 1]$.

$\mathcal{B}$ be the Borel σ-algebra of $[0,1]$, let μ be Lebesgue measure on $[0,1]$, and let $C \subset \mathcal{B}$ be a countable algebra generating $\mathcal{B}$. Suppose there would be a random matching $f\colon [0,1] \times \Omega \to [0,1]$ such that (P3) holds with respect to μ. Pick any $E_2 \in \mathcal{B}$ with $\mu(E_2) = 1/2$. Then (P3) implies that there is a $\bar{y} \in \Omega$ such that $\mu(E_1 \cap f_{\bar{y}}^{-1}(E_2)) = \mu(E_1)\mu(E_2)$ for all $E_1 \in C$. Note that for each $y \in \Omega$, $\{E_1 \in \mathcal{B}\colon \mu(E_1 \cap f_y^{-1}(E_2)) = \mu(E_1)\mu(E_2)\}$ is a monotone class. It follows that the function $f_{\bar{y}}\colon [0,1] \to [0,1]$ would satisfy $\mu(E_1 \cap f_{\bar{y}}^{-1}(E_2)) = \mu(E_1)\mu(E_2)$ for all $E_1 \in \mathcal{B}$. But such a function cannot exist. To see this, set $E_1 = f_{\bar{y}}^{-1}(E_2)$ if $\mu(f_{\bar{y}}^{-1}(E_2)) > 0$ and $E_1 = [0,1]$ otherwise.

Remark 5. Concerning part (c) of Theorem 1, recall first that a Fubini extension $\bar{\lambda}$ of the product measure λ defined from μ and ν is a probability measure on $A \times \Omega$ which extends λ but so that the *conclusion* of Fubini's theorem about repeated integrals with respect to the factor measure μ and ν continues to hold for $\bar{\lambda}$-integrable real-valued functions. (A formal definition of the notion of a Fubini extension is given prior to the statement of Theorem 2 below.) Now the relevance of part (c) of Theorem 1 comes from the following. In many applications, there is a need for the random matching f, and in particular for the type process $\theta \circ f$, to satisfy some joint measurability property with respect to agents and points in the sample space, e.g., to be in a position allowing to apply Fubini-type arguments to a function $R\colon A \times \Omega \to \mathbb{R}$ obtained by setting $R(x,y) = r(\theta(x), \theta(f(x,y)))$ for $x \in A$ and $y \in \Omega$, where $r\colon T \times T \to \mathbb{R}$ is a bounded $\mathcal{T} \otimes \mathcal{T}$-measurable function assigning some "payoff" to any two agents when they are matched and are of types t, $t' \in T$. But to be in such a position, it is sufficient that (c) of Theorem 1 holds for the random matching f. (Cf. the argument in the proof of Theorem 4 below.) Now it is important to note that if the random matching f satisfies properties (P2) and (P4), then measurability of f with respect to the domain of a Fubini extension $\bar{\lambda}$ of λ implies that $\bar{\lambda}$ must be a proper extension of λ.[9] Thus, in particular, if f satisfies (P2) and (P4), then f cannot be measurable already for the domain of the product measure λ itself. Thus the problem of establishing (c) of Theorem 1 is non-trivial.[10]

Proof of Theorem 1. Let ω_1 be the least uncountable cardinal. For each $\xi < \omega_1$, choose a subset $K_\xi \subset \omega_1$ with $\#(K_\xi) = \#(\xi)$ such that $\eta > \xi$ for each $\eta \in K_\xi$, and then choose a bijection $\rho_\xi\colon \xi \to K_\xi$. Define $h_\xi\colon \omega_1 \to \omega_1$ by setting

$$h_\xi(\eta) = \begin{cases} \rho(\eta) & \text{for } \eta < \xi \\ \rho^{-1}(\eta) & \text{for } \eta \in K_\xi \\ \eta & \text{for } \eta \notin \xi \cup K_\xi. \end{cases}$$

Then for each $\xi < \omega_1$, h_ξ is an involution on ω_1.

[9] See, e.g., Podczeck (2010, Remark 3), and note for this reference that, since $(A, \mathcal{A}, \mu)$ is atomless, (P4) implies that the family $\langle f_x \rangle_{x \in A}$ satisfies the property of essential pairwise independence which is considered in this reference.

[10] For a recent application of the notion of Fubini extension outside the scope of random matching models, see Sun and Yannelis (2008).

Consider the product space $\{0,1\}^{\omega_1}$. Let λ be the usual measure on $\{0,1\}^{\omega_1}$, and let Λ denote the domain of λ. Recall that λ is complete. For each $\xi < \omega_1$, define a mapping $\hat{\phi}_\xi\colon \{0,1\}^{\omega_1} \to \{0,1\}^{\omega_1}$ by setting, for each $x \in \{0,1\}^{\omega_1}$,

$$\hat{\phi}_\xi(x) = x \circ h_\xi .$$

(Thus $\hat{\phi}_\xi(x)$ is the element in $\{0,1\}^{\omega_1}$ that is given by $\hat{\phi}_\xi(x)(\eta) = x(h_\xi(\eta))$ for $\eta < \omega_1$.) Then for each $\xi < \omega_1$, $\hat{\phi}_\xi$ is inverse-measure-preserving for λ, and since h_ξ is an involution, $\hat{\phi}_\xi$ is an involution, too. (To see that $\hat{\phi}_\xi$ is an involution, observe that for each $x \in \{0,1\}^{\omega_1}$,

$$\hat{\phi}_\xi(\hat{\phi}_\xi(x)) = \hat{\phi}_\xi(x \circ h_\xi) = (x \circ h_\xi) \circ h_\xi = x \circ (h_\xi \circ h_\xi) = x.$$

To see that $\hat{\phi}_\xi$ is inverse-measure-preserving for λ, observe that whenever I is a finite subset of ω_1, we have

$$\lambda(\{x \in \{0,1\}^{\omega_1} : x(h_\xi(\eta)) = 1 \text{ for every } \eta \in I\}) = 2^{-\#(I)},$$

because h_ξ is an injection.)

We claim that given any E_1, $E_2 \in \Lambda$, for all but countably many $\xi < \omega_1$ the sets E_1 and $\hat{\phi}_\xi^{-1}(E_2)$ are stochastically independent, i.e., $\lambda(E_1 \cap \hat{\phi}_\xi^{-1}(E_2)) = \lambda(E_1)\lambda(\hat{\phi}_\xi^{-1}(E_2))$. To see this, pick any E_1, $E_2 \in \Lambda$. There is an $E_1' \in \Lambda$ which differs from E_1 by a null set and is determined by coordinates in a countable subset of ω_1, say D_1, and there is an $E_2' \in \Lambda$ which differs from E_2 by a null set and is determined by coordinates in a countable subset of ω_1, say D_2. Then by choice of $\hat{\phi}_\xi$, for each $\xi < \omega_1$ the set $\hat{\phi}_\xi^{-1}(E_2')$ is determined by coordinates in $h_\xi(D_2)$. As ω_1 has uncountable cofinality, we can find a $\beta < \omega_1$ such that $\eta < \beta$ for every $\eta \in D_1 \cup D_2$. Then by choice of h_ξ, for each $\xi < \omega_1$ with $\xi > \beta$, we have $\eta > \beta$ for every $\eta \in h_\xi(D_2)$. Hence for each $\xi < \omega_1$ with $\xi > \beta$, $D_1 \cap h_\xi(D_2) = \emptyset$, which implies that the sets E_1' and $\hat{\phi}_\xi^{-1}(E_2')$ are stochastically independent, E_1' being determined by coordinates in D_1, and $\hat{\phi}_\xi^{-1}(E_2')$ by coordinates in $h_\xi(D_2)$. Since $\hat{\phi}_\xi$ is inverse-measure-preserving for λ, the fact that E_1' and E_2' differ by a null set from E_1, E_2 respectively implies that $\hat{\phi}_\xi^{-1}(E_2')$ differs by a null set from $\hat{\phi}_\xi^{-1}(E_2)$, and $E_1' \cap \hat{\phi}_\xi^{-1}(E_2')$ by a null set from $E_1 \cap \hat{\phi}_\xi^{-1}(E_2)$. Consequently E_1 and $\hat{\phi}_\xi^{-1}(E_2)$ are stochastically independent for each $\xi < \omega_1$ with $\xi > \beta$, and thus the claim above is established.

Because each $\hat{\phi}_\xi$ is inverse-measure-preserving for λ, it follows that given any E_1, $E_2 \in \Lambda$ we have $\lambda(E_1 \cap \hat{\phi}_\xi^{-1}(E_2)) = \lambda(E_1)\lambda(E_2)$ for all but countably many $\xi < \omega_1$.

Let

$$A = \{x \in \{0,1\}^{\omega_1} : \text{ for some } \alpha < \omega_1,\ x(\xi) = 1 \text{ for all } \xi < \omega_1 \text{ with } \xi > \alpha\}.$$

Evidently A is the union of ω_1 many sets of cardinal $\mathfrak{c}$, so $\#(A) = \omega_1 \cdot \mathfrak{c} = \mathfrak{c}$. Also, A has full outer measure for λ, by the fact that every non-negligible member of Λ

includes a non-empty set that is determined by coordinates in some countable subset of ω_1, together with the fact that ω_1 has uncountable cofinality.

Let μ be the subspace measure on A induced from λ, and let $\mathcal{A}$ denote its domain. Then, as A has full outer measure for λ, $(A, \mathcal{A}, \mu)$ is a probability space. Clearly, as λ is complete and atomless, so is μ.

For each $\xi < \omega_1$ let $\tilde{\phi}_\xi$ be the restriction of $\hat{\phi}_\xi$ to A. Note that by construction, for each $\xi < \omega_1$ and each $x \in \{0,1\}^{\omega_1}$, $\hat{\phi}_\xi(x)$ and x agree in all but countably many coordinates in ω_1. Consequently, for each $\xi < \omega_1$, whenever $x \in A$ then $\tilde{\phi}_\xi(x) \in A$, again using the fact that ω_1 has uncountable cofinality. Thus since $\hat{\phi}_\xi$ is an involution on $\{0,1\}^{\omega_1}$, $\tilde{\phi}_\xi$ is an involution on A. By the fact that A has full outer measure for λ, the properties of the functions $\hat{\phi}_\xi$, $\xi < \omega_1$, also imply that, for each $\xi < \omega_1$, $\tilde{\phi}_\xi$ is inverse-measure-preserving for μ, and that, given any E_1, $E_2 \in \mathcal{A}$, for all but countably many $\xi < \omega_1$ we have $\mu(E_1 \cap \tilde{\phi}_\xi^{-1}(E_2)) = \mu(E_1)\mu(E_2)$.

We will now modify the mappings $\tilde{\phi}_\xi$ so as to make them fixed point free. Pick any $\xi < \omega_1$ with $\xi \geq \omega$. Let

$$\Delta_\xi = \{x \in \{0,1\}^{\omega_1} : x(\eta) = x(h_\xi(\eta)) \text{ for each } \eta < \omega_1\}$$

and let $\Delta_\xi^A = \Delta_\xi \cap A$. Then by the definitions of $\hat{\phi}_\xi$ and $\tilde{\phi}_\xi$, Δ_ξ^A is exactly the set of fixed points of $\tilde{\phi}_\xi$. Now by the definition of h_ξ,

$$\{\eta < \omega_1 : \eta < \xi\} \cap h_\xi(\{\eta < \omega_1 : \eta < \xi\}) = \emptyset.$$

Hence since $\xi \geq \omega$, Δ_ξ is a λ-null set in $\{0,1\}^{\omega_1}$ (directly from the definition of λ to be the usual measure on $\{0,1\}^{\omega_1}$), and thus Δ_ξ^A is a μ-null set in A. Finally, Δ_ξ^A is an infinite subset of A. (To see this, note that by definition of h_ξ, for some countable $D \subset \omega_1$ we have $h_\xi(\eta) = \eta$ for all $\eta < \omega_1$ with $\eta \notin D$, and let B be the set of those x in A for which $x(\eta) = 1$ for all $\eta < \omega_1$ with the exception of exactly one $\eta < \omega_1$ with $\eta \notin D$. Then B is an infinite subset of A, and since h_ξ is a bijection we must have $B \subset \Delta_\xi^A$.)

Now by the fact that any infinite set can be partitioned into two sets of the same cardinality, we can choose a fixed point free involution $\kappa_\xi : \Delta_\xi^A \to \Delta_\xi^A$. As Δ_ξ^A is the set of fixed points of $\tilde{\phi}_\xi$, the restriction of $\tilde{\phi}_\xi$ to $A \backslash \Delta_\xi^A$ is an involution on $A \backslash \Delta_\xi^A$. Therefore, defining $\phi_\xi : A \to A$ by

$$\phi_\xi(x) = \begin{cases} \kappa_\xi(x) & \text{if } x \in \Delta_\xi^A \\ \tilde{\phi}_\xi(x) & \text{if } x \in A \backslash \Delta_\xi^A, \end{cases}$$

ϕ_ξ is a fixed point free involution on A. As ϕ_ξ agrees with $\tilde{\phi}_\xi$ on the complement of a μ-null set, ϕ_ξ is inverse-measure-preserving for μ.

Doing this construction for all $\xi < \omega_1$ with $\xi \geq \omega$, and then letting $\phi_\xi = \phi_\omega$ for $\xi < \omega$, we get a family $\langle \phi_\xi \rangle_{\xi < \omega_1}$ of fixed point free involutions on A, each of them inverse-measure-preserving for μ. Moreover, given any E_1, $E_2 \in \mathcal{A}$, for all but countably many $\xi < \omega_1$ we have $\mu(E_1 \cap \phi_\xi^{-1}(E_2)) = \mu(E_1)\mu(E_2)$, by the corresponding property of the family $\langle \tilde{\phi}_\xi \rangle_{\xi<\omega_1}$, because ϕ_ξ agrees with $\tilde{\phi}_\xi$ on the complement of a μ-null set for $\omega \le \xi < \omega_1$.

Now choose a family $\langle x_\xi \rangle_{\xi<\omega_1}$ of elements of A so that given any countable $D \subset A$, for some $\xi < \omega_1$ we have both $x_\xi \notin D$ and $\phi_\xi(x_\xi) \notin D$. Such a choice is possible. Indeed, by transfinite recursion on ω_1 choose a family $\langle x_\xi \rangle_{\xi<\omega_1}$ as follows. Let x_0 be an arbitrary point of A. Given that $\langle x_\eta \rangle_{\eta<\xi}$ has been chosen, where $\xi < \omega_1$, consider the set $A_\xi = \{x_\eta, \phi_\eta(x_\eta) : \eta < \xi\}$. Then A_ξ is countable, so because A is uncountable and ϕ_ξ is a bijection, we can choose an x_ξ in A such that both $x_\xi \notin A_\xi$ and $\phi_\xi(x_\xi) \notin A_\xi$. This completes the recursion. The result is a family $\langle x_\xi \rangle_{\xi<\omega_1}$ of distinct elements of A such that the family $\langle \phi_\xi(x_\xi) \rangle_{\xi<\omega_1}$ also consists of distinct members. Thus $\langle x_\xi \rangle_{\xi<\omega_1}$ is a family as desired.

Let $\bar{\nu}$ be the complete product probability measure on A^A defined from μ, and let $\bar{\Sigma}$ denote the domain of $\bar{\nu}$. For each $\xi < \omega_1$ let

$$N_\xi = \Big\{ y \in A^A : \begin{array}{l} \text{(a) } y \text{ is a fixed point free involution on } A, \\ \text{(b) } y(x_\xi) = \phi_\xi(x_\xi), \\ \text{(c) } y \restriction A\setminus N = \phi_\xi \restriction A\setminus N \text{ for some } \mu\text{-null set } N \subset A \end{array} \Big\},$$

and then let $\Omega = \bigcup_{\xi<\kappa} N_\xi$.

From (c) in the definition of N_ξ, each $y \in \Omega$ is inverse-measure-preserving for μ. From (b) in that definition, each N_ξ is a $\bar{\nu}$-null set in A^A because, μ being atomless, singletons in A are μ-null sets.

On the other hand, Ω has full outer measure for $\bar{\nu}$. To see this, note first that it suffices to show that Ω intersects every non-negligible subset of A^A that is determined by coordinates in some countable subset of A (since every non-negligible element of $\bar{\Sigma}$ includes a such a set). Thus let E be a non-negligible subset of A^A, determined by coordinates in a countable subset of A, say D.

As D is countable and $(A, \mathcal{A}, \mu)$ is atomless, the set of all y in A^A such that $y \restriction D$ is injective is an element of $\bar{\Sigma}$ with $\bar{\nu}$-measure 1 (see Fremlin, 2001, 254V). Also, since a countable subset of A is a μ-null set in A, for each $x \in A$ the set of all y in A^A such that $y(x) \in D$ is a $\bar{\nu}$-null set in A^A, and hence (using again the fact that D is countable) the set of all y in A^A such that $D \cap y(D) = \emptyset$ belongs to $\bar{\Sigma}$ and has $\bar{\nu}$-measure 1. Consequently, because E is non-negligible, there is an element of E, say y_0, such that $y_0 \restriction D$ is a bijection onto $y_0(D)$ and such that $D \cap y_0(D) = \emptyset$.

Set $D' = y_0(D)$. Then $D \cup D'$ is countable, so we can choose a countably infinite subset H of A with $H \cap (D \cup D') = \emptyset$. Set $C = H \cup D \cup D'$. Then C is again countable, so by choice of the family $\langle x_\xi \rangle_{\xi<\omega_1}$, there is a $\xi < \omega_1$ such that $x_\xi \notin C$ as well as $\phi_\xi(x_\xi) \notin C$. Fix such a ξ and set $C' = C \cup \phi_\xi(C)$. Using the fact that ϕ_ξ is an involution, we may see that $x_\xi \notin C'$.

Also by the fact that ϕ_ξ is an involution, we have $\phi_\xi(C') = C'$ and therefore $\phi_\xi(A\setminus C') = A\setminus \phi_\xi(C') = A\setminus C'$. Thus $\phi_\xi \restriction A\setminus C'$ is a fixed point free involution on $A\setminus C'$.

Note that by choice of C, the set $C' \setminus (D \cup D')$ is infinite. Hence, since an infinite set can be partitioned into two sets of the same cardinality, we can choose a fixed point free involution $\zeta\colon C' \setminus (D \cup D') \to C' \setminus (D \cup D')$.

Now as $y_0 \restriction D$ is a bijection onto D', and $D \cap D' = \emptyset$, we get a fixed point free involution $y_1\colon A \to A$ by setting, for $x \in A$,

$$y_1(x) = \begin{cases} y_0(x) & \text{if } x \in D \\ y_0^{-1}(x) & \text{if } x \in D' \\ \zeta(x) & \text{if } x \in C' \setminus (D \cup D') \\ \phi_\xi(x) & \text{if } x \in A \setminus C'. \end{cases}$$

In particular, then, since $x_\xi \notin C'$, we have $y_1(x_\xi) = \phi_\xi(x_\xi)$. Thus $y_1 \in \Omega$, because the countable set C' is a μ-null set in A. On the other hand, y_1 agrees with y_0 on D, and since $y_0 \in E$ and E is determined by coordinates in D, we have $y_1 \in E$. Thus $\Omega \cap E \neq \emptyset$, proving that Ω has full outer measure for $\bar{\nu}$.

Let ν be the subspace measure on Ω induced from $\bar{\nu}$, and let Σ denote its domain. Then, as Ω has full outer measure for $\bar{\nu}$, (Ω, Σ, ν) is a probability space. Note that N_ξ is a ν-null set in Ω for each $\xi < \omega_1$.

Now let $f\colon A \times \Omega \to A$ be defined by setting

$$f(x, y) = y(x), \; x \in A, \; y \in \Omega.$$

Further, for each $x \in A$, let π_x be the coordinate projection $y \mapsto y(x)\colon A^A \to A$. Then, by definition of product measure, for each $x \in A$, π_x is inverse-measure-preserving for $\bar{\nu}$ and μ, and the family $\langle \pi_x \rangle_{x \in A}$ is stochastically independent. Evidently $f(x, \cdot)$ agrees with π_x on Ω for each $x \in A$, and since Ω has full outer measure for $\bar{\nu}$, it follows that for each $x \in A$, $f_x \equiv f(x, \cdot)$ is inverse-measure-preserving for ν and μ, and that the family $\langle f_x \rangle_{x \in A}$ is stochastically independent. On the other hand, for each $y \in \Omega$, f_y is the same as y. Hence, for each $y \in \Omega$, f_y is a fixed point free involution on A, and by what was noted following the definition of the sets N_ξ above, f_y is inverse-measure-preserving for μ. As was also noted above, given any $E_1, E_2 \in \mathcal{A}$, we have $\mu(E_1 \cap \phi_\xi^{-1}(E_2)) = \mu(E_1)\mu(E_2)$ for all but countably many $\xi < \omega_1$. By (c) in the definition of the sets N_ξ, this means that, given any $E_1, E_2 \in \mathcal{A}$, there is a countable $D \subset \omega_1$ such that whenever $y \in \Omega \setminus \bigcup_{\xi \in D} N_\xi$ then $\mu(E_1 \cap f_y^{-1}(E_2)) = \mu(E_1)\mu(E_2)$. As each N_ξ is a null set in Ω, it follows that, given any $E_1, E_2 \in \mathcal{A}$, we have $\mu(E_1 \cap f_y^{-1}(E_2)) = \mu(E_1)\mu(E_2)$ for almost all $y \in \Omega$.

Taken together, these properties of f mean that f is a random matching satisfying (P1) to (P4). By Remark 1 above, (P5) is also satisfied and, given any type space $(T, \mathcal{T})$ and type assignment $\theta\colon A \to T$, (P6) to (P8) are satisfied as well. Thus, (a) and (b) of the theorem hold for f. By Theorem 2 below, (c) of the theorem holds, too. This completes the proof. □

Here is a formal definition of the notion of a Fubini extension.

Definition 5. Let (X, Σ, μ) and (Y, T, ν) be probability spaces, and $(X \times Y, \Lambda, \lambda)$ the corresponding product probability space. Let $\bar{\lambda}$ be a probability measure on $X \times Y$, and $\bar{\Lambda}$ its domain. Then $\bar{\lambda}$ is said to be a *Fubini extension* of λ if (a) $\bar{\Lambda} \supset \Lambda$ and (b) for each $H \in \bar{\Lambda}$—denoting by χH the characteristic function of H—the integrals $\iint \chi H(x, y)\, d\nu(y)\, d\mu(x)$ and $\iint \chi H(x, y)\, d\mu(x)\, d\nu(y)$ are well-defined and $\iint \chi H(x, y)\, d\nu(y)\, d\mu(x) = \bar{\lambda}(H) = \iint \chi H(x, y)\, d\mu(x)\, d\nu(y)$.

The following notation will be used in the sequel.

Notation. If H is a subset of a product $X \times Y$ and $x \in X$, then H_x denotes the x-section of H, and if $y \in Y$ then H_y denotes the y-section of H. Thus, if $x \in X$, then $H_x = \{y \in Y\colon (x, y) \in H\}$; similarly, for $y \in Y$, $H_y = \{x \in X\colon (x, y) \in H\}$.

Theorem 2. *Let $(A, \mathcal{A}, \mu)$ be an atomless probability space of agents, (Ω, Σ, ν) a sample probability space, and $f\colon A \times \Omega \to A$ a random matching. Let λ be the product probability measure on $A \times \Omega$ defined from μ and ν. If f satisfies* (P1) *to* (P4)*, then λ has a Fubini extension $\bar{\lambda}$ such that f is $(\bar{\Lambda}, \mathcal{A})$-measurable, writing $\bar{\Lambda}$ for the domain of $\bar{\lambda}$.*

Proof. Using Maharam's theorem, we can choose a countable partition $\langle A_i \rangle_{i \in I}$ of A into non-negligible measurable sets so that for each $i \in I$ there is a family $\langle F^{i,j} \rangle_{j \in J_i}$ of measurable subsets of A, with $F^{i,j} \subset A_i$ for all $j \in J_i$, such that the following hold, writing μ_i for the probability measure on A_i obtained by normalizing the subspace measure induced by μ on A_i:

(i) For each $i \in I$, $\mu_i(F^{i,j}) = 1/2$ for all $j \in J_i$.

(ii) For each $i \in I$, the family $\langle F^{i,j} \rangle_{j \in J_i}$ is stochastically independent for μ_i.

(iii) Denoting by $\mathcal{A}'$ the sub-σ-algebra of $\mathcal{A}$ generated by $\{F^{i,j}\colon i \in I, j \in J_i\}$, for any $B \in \mathcal{A}$ there is a $B' \in \mathcal{A}'$ such that B' differs from B by a μ-null set.

For each $i \in I$ and $j \in J_i$, let $H^{i,j} = f^{-1}(F^{i,j})$. We will show that the family $\langle H^{i,j} \rangle_{i \in I, j \in J_i}$ satisfies the requirements in Lemma 1 below.

Clearly (c) of these requirements holds for $\langle H^{i,j} \rangle_{i \in I, j \in J_i}$. As earlier, write f_x for $f(x, \cdot)$ and f_y for $f(\cdot, y)$. Note that for each $i \in I$ and $j \in J_i$, the sections $H^{i,j}_x$ and $H^{i,j}_y$ satisfy

$$H^{i,j}_x = f_x^{-1}(F^{i,j}) \text{ and } H^{i,j}_y = f_y^{-1}(F^{i,j})$$

for all $x \in A$ and $y \in \Omega$, respectively. Thus, in particular, (a) of Lemma 1 holds for $\langle H^{i,j} \rangle_{i \in I, j \in J_i}$.

For each $i \in I$ set $\alpha_i = \mu(A_i)$. Fix any $i \in I$, and let $j_1, \ldots, j_n$ be distinct members of J_i. Note that (i) and (ii) imply:

$$(*) \qquad \mu(F^{i,j_1} \cap \cdots \cap F^{i,j_n}) = \alpha_i 2^{-n}.$$

Consider any $B \in \mathcal{A}$. As f satisfies (P3) by hypothesis, for almost all $y \in \Omega$ we have

$$\mu(B \cap f_y^{-1}(F^{i,j_1} \cap \cdots \cap F^{i,j_n})) = \mu(B)\mu(F^{i,j_1} \cap \cdots \cap F^{i,j_n}).$$

Using this fact together with $(*)$, we may see that for almost every $y \in \Omega$,

$$\begin{aligned}\mu(B \cap H_y^{i,j_1} \cap \cdots \cap H_y^{i,j_n}) =& \mu(B \cap f_y^{-1}(F^{i,j_1}) \cap \cdots \cap f_y^{-1}(F^{i,j_n})) \\ =& \mu(B \cap f_y^{-1}(F^{i,j_1} \cap \cdots \cap F^{i,j_n})) \\ =& \mu(B)\mu(F^{i,j_1} \cap \cdots \cap F^{i,j_n}) \\ =& \mu(B)\alpha_i 2^{-n}.\end{aligned}$$

Now consider any $C \in \Sigma$. For each $x \in A$ let Σ_x be the sub-σ-algebra of Σ generated by f_x, and let Σ_C be the sub-σ-algebra of Σ generated by C. By hypothesis, f satisfies (P4), i.e., the family $\langle \Sigma_x \rangle_{x \in A}$ is stochastically independent. By Fremlin (2008, 5A6-272W), it follows that there is a countable $D \subset A$ such that for each $x \in A \backslash D$, Σ_C and Σ_x are stochastically independent. Since $(A, \mathcal{A}, \nu)$ is atomless by hypothesis, this means that Σ_C and Σ_x are stochastically independent for almost every $x \in A$. Now for each $x \in A$, we have $f_x^{-1}(F^{i,j_1} \cap \cdots \cap F^{i,j_n}) \in \Sigma_x$, and it follows that for almost all $x \in A$,

$$\nu(C \cap f_x^{-1}(F^{i,j_1} \cap \cdots \cap F^{i,j_n})) = \nu(C)\nu(f_x^{-1}(F^{i,j_1} \cap \cdots \cap F^{i,j_n})).$$

Using this fact together with $(*)$ and the hypothesis that f satisfies (P2), i.e., that for each $x \in A$, f_x is inverse-measure-preserving for μ and ν, we may conclude that for almost all $x \in A$,

$$\begin{aligned}\nu(C \cap H_x^{i,j_1} \cap \cdots \cap H_x^{i,j_n}) =& \nu(C \cap f_x^{-1}(F^{i,j_1}) \cap \cdots \cap f_x^{-1}(F^{i,j_n})) \\ =& \nu(C \cap f_x^{-1}(F^{i,j_1} \cap \cdots \cap F^{i,j_n})) \\ =& \nu(C)\nu(f_x^{-1}(F^{i,j_1} \cap \cdots \cap F^{i,j_n})) \\ =& \nu(C)\mu(F^{i,j_1} \cap \cdots \cap F^{i,j_n}) \\ =& \nu(C)\alpha_i 2^{-n}.\end{aligned}$$

Thus also (b) of Lemma 1 holds for $\langle H^{i,j} \rangle_{i \in I, j \in J_i}$.

Now let

$$\mathcal{G} = \{f^{-1}(N) \colon N \text{ is a } \mu\text{-null set in } A\}.$$

Then for each $M \in \mathcal{G}$, and each $x \in A$, the section M_x is a ν-null set in Ω, by the facts that $M = f^{-1}(N)$ implies $M_x = f_x^{-1}(N)$ and f_x is inverse-measure-preserving. Similarly, as f satisfies (P1), i.e., for each $y \in \Omega$, f_y is inverse-measure-preserving, M_y is a μ-null set in A for each $M \in \mathcal{G}$ and each $y \in \Omega$.

We may now appeal to Lemma 1 to find a Fubini extension of λ such that, denoting by $\bar{\Lambda}$ its domain, $\bar{\Lambda}$ contains every member of $\mathcal{G}$ and every member of $\langle H^{i,j} \rangle_{i \in I, j \in J_i}$. In view of (iii) above, it follows that f is $(\bar{\Lambda}, \mathcal{A})$-measurable. This completes the proof. □

Lemma 1. *Let $(A, \mathcal{A}, \mu)$ and (Ω, Σ, ν) be probability spaces, and $(A \times \Omega, \Lambda, \lambda)$ the corresponding product probability space. Let $\mathcal{M}$ be the set of all $M \subset A \times \Omega$ for which M_x is a null set in Ω for almost all $x \in A$, and M_y a null set in A for almost all $y \in \Omega$. Further, let $\langle J_i \rangle_{i \in I}$ be a family of sets, and $\langle H^{i,j} \rangle_{i \in I, j \in J_i}$ a family of subsets of $A \times \Omega$. Suppose:*

(a) *For all $x \in A$ and all $y \in \Omega$, $H^{i,j}_x \in \Sigma$ and $H^{i,j}_y \in \mathcal{A}$ for each $i \in I$ and $j \in J_i$.*

(b) *For each $i \in I$ there is a real number $\alpha_i > 0$ such that whenever $j_1, \dots, j_n$ are finitely many distinct members of J_i, then given $B \in \mathcal{A}$,*

$$\mu(B \cap H^{i,j_1}_y \cap \cdots \cap H^{i,j_n}_y) = \mu(B) \alpha_i 2^{-n}$$

for almost all $y \in \Omega$, and given $C \in \Sigma$,

$$\nu(C \cap H^{i,j_1}_x \cap \cdots \cap H^{i,j_n}_x) = \nu(C) \alpha_i 2^{-n}$$

for almost all $x \in A$.

(c) *$H^{i,j} \cap H^{i',j'} = \emptyset$ whenever $i \neq i'$.*

Then λ has a Fubini extension $\bar{\lambda}$ such that $\mathcal{M} \cup \{H^{i,j} \colon i \in I, j \in J_i\} \subset \bar{\Lambda}$, writing $\bar{\Lambda}$ for the domain of $\bar{\lambda}$.

Proof. Let $\mathcal{F}$ denote the set of all subsets F of $A \times \Omega$ such that the integrals $\int_A \nu(F_x)\, d\mu(x)$ and $\int_\Omega \mu(F_y)\, d\nu(y)$ are well-defined and equal. Then $\mathcal{F}$ is a Dynkin class (i.e. $\emptyset \in \mathcal{F}$ and $\mathcal{F}$ is closed under complements and countable disjoint unions) as may easily be checked. In addition, (a) to (c) imply that whenever $B_1 \times C_1, \dots, B_n \times C_n$ are finitely many measurable rectangles in $A \times \Omega$ and $F_1, \dots, F_m$ are finitely many elements of $\mathcal{M} \cup \{H^{i,j} \colon i \in I, j \in J_i\}$, then the intersection

$$(B_1 \times C_1) \cap \cdots \cap (B_n \times C_n) \cap F_1 \cap \cdots \cap F_m$$

belongs to $\mathcal{F}$. Therefore, by the monotone class theorem, there is a σ-algebra $\Lambda' \subset \mathcal{F}$ which contains all measurable rectangles in $A \times \Omega$ and all members of $\mathcal{M} \cup \{H^{i,j} \colon i \in I, j \in J_i\}$. Define $\lambda' \colon \Lambda' \to \mathbb{R}$ by setting $\lambda'(F) = \int_A \nu(F_x)\, d\mu(x)$ for $F \in \Lambda'$. Using the monotone convergence theorem, we may see that λ' is a probability measure on $A \times \Omega$. Evidently λ' agrees with λ on all measurable rectangles in $A \times \Omega$.

Let $\bar{\lambda}$ be the completion of λ', and $\bar{\Lambda}$ the domain of $\bar{\lambda}$. Since λ' agrees with λ on the measurable rectangles in $A \times \Omega$, we must have $\bar{\Lambda} \supset \Lambda$. By construction, the Fubini property holds for the characteristic functions of the elements of Λ', which in particular implies that if N is a λ'-null set in $A \times \Omega$, then for μ-almost every $x \in A$, the x-section of N is a ν-null set in Ω, and for ν-almost every $y \in \Omega$, the y-section of N is a μ-null set in A. Hence, the Fubini property holds for the characteristic functions of the elements of $\bar{\Lambda}$. Thus $\bar{\lambda}$ is a Fubini extension of λ. By construction, the domain $\bar{\Lambda}$ of $\bar{\lambda}$ includes $\mathcal{M} \cup \{H^{i,j} \colon i \in I, j \in J_i\}$. $\square$

4 Relationship between properties of a random matching

As was noted in Remark 1, if a random matching satisfies "general independence," then it satisfies "independence in types" against any type assignment. The next theorem shows that the converse of this implication is also true. In fact, "general independence" already holds if "independence in types" holds against any type assignment with a finite type space. An analogous relationship holds for (P3) and (P7), and for (P2) and (P6).

Theorem 3. *Let* $(A, \mathcal{A}, \mu)$ *be an atomless probability space of agents,* (Ω, Σ, ν) *a sample probability space, and* $f: A \times \Omega \to A$ *a random matching.*

(a) *If* f *satisfies* (P8) *against any type assignment with a finite type space, then* f *satisfies* (P4)*.*

(b) *If* f *satisfies* (P7) *against any type assignment with a finite type space, then* f *satisfies* (P3)*.*

(c) *If* f *satisfies* (P6) *against any type assignment with a finite type space, then* f *satisfies* (P2)*.*

Proof. (a) We have to show that whenever $x_1, \ldots, x_n$ are distinct members of A and $E_1, \ldots, E_n$ are members of $\mathcal{A}$, then

$$\nu(f_{x_1}^{-1}(E_1) \cap \cdots \cap f_{x_n}^{-1}(E_n)) = \prod_{i=1}^{n} \nu(f_{x_i}^{-1}(E_i)).$$

Thus let such $x_1, \ldots, x_n$ and $E_1, \ldots, E_n$ be given. There is a finite partition $\mathcal{P}$ of A into measurable subsets such that for each $i = 1, \ldots, n$, E_i is the union of members of $\mathcal{P}$. Let the finite type space $(T, \mathcal{T})$ be given by setting $T = \mathcal{P}$ and $\mathcal{T} = 2^{\mathcal{P}}$, and let the type assignment $\theta: A \to T$ be the mapping that takes an $x \in A$ to that element of $\mathcal{P}$ which contains x. Evidently θ is $(\mathcal{A}, \mathcal{T})$-measurable and we have $\theta^{-1}(\theta(E_i)) = E_i$ for each $i = 1, \ldots, n$. Now (P8) implies that

$$\nu(f_{x_1}^{-1}(\theta^{-1}(\theta(E_1))) \cap \cdots \cap f_{x_n}^{-1}(\theta^{-1}(\theta(E_n)))) = \prod_{i=1}^{n} \nu(f_{x_i}^{-1}(\theta^{-1}(\theta(E_i)))),$$

and since $\theta^{-1}(\theta(E_i)) = E_i$ for each $i = 1, \ldots, n$, we have the desired conclusion.

(b) Consider any $E_1, E_2 \in \mathcal{A}$. Let a type space $(T, \mathcal{T})$ be given by setting $T = \{0, 1, 2, 3\}$ and $\mathcal{T} = 2^T$, and let the type assignment $\theta: A \to T$ be given by setting $\theta(x) = 0$ for $x \in E_1 \backslash E_2$, $\theta(x) = 1$ for $x \in E_1 \cap E_2$, $\theta(x) = 2$ for $x \in E_2 \backslash E_1$, and $\theta(x) = 3$ for $x \in A \backslash (E_1 \cup E_2)$. If (P7) holds against θ, then there is a ν-null set $N \subset \Omega$ such that for each $y \in \Omega \backslash N$,

$$\mu(\theta^{-1}(\{0, 1\}) \cap f_y^{-1}(\theta^{-1}(\{1, 2\}))) = \mu(\theta^{-1}(\{0, 1\}))\mu(\theta^{-1}(\{1, 2\})).$$

Thus $\mu(E_1 \cap f_y^{-1}(E_2)) = \mu(E_1)\mu(E_2)$ for each $y \in \Omega \backslash N$.

(c) Fix any $E \in \mathcal{A}$. Let $(T, \mathcal{T}) = (\{0, 1\}, 2^{\{0,1\}})$ and let $\theta: A \to T$ be given by $\theta(x) = 1$ if $x \in E$ and $\theta(x) = 0$ if $x \in A \backslash E$. If (P6) holds against f, then for each $x \in A$, $\nu(f_x^{-1}(\theta^{-1}(\{1\}))) = \mu(\theta^{-1}(\{1\}))$ and thus $\nu(f_x^{-1}(E)) = \mu(E)$. $\square$

The next theorem shows that if a random matching has the measurability property stated in (c) of Theorem 1, then (P2) and (P4) together imply (P3). Note that, given a random matching $f\colon A\times\Omega \to A$, (P2) and (P4) together mean that the family $\langle f_x \rangle_{x\in A}$ of functions from Ω to A is i.i.d.. Thus the next theorem shows, in particular, that the "strong mixing" property (P3) of the family $\langle f_y \rangle_{y\in\Omega}$ of sample functions may be viewed as manifestation of a law of large numbers.

Theorem 4. *Let $(A, \mathcal{A}, \mu)$ be an atomless probability space of agents, (Ω, Σ, ν) a sample probability space, and $f\colon A \times \Omega \to A$ a random matching. Let λ be the product probability measure on $A \times \Omega$ defined from μ and ν. Suppose:*

(i) *There is a Fubini extension $\bar\lambda$ of λ such that, writing $\bar\Lambda$ for the domain of $\bar\lambda$, f is $(\bar\Lambda, \mathcal{A})$-measurable.*

(ii) *f satisfies* (P2) *and* (P4)*.*

Then f satisfies (P3)*.*

Proof. Fix any $E_1, E_2 \in \mathcal{A}$ and consider any $B \in \Sigma$. Note first that by (i), we have $(E_1\times B)\cap f^{-1}(E_2) \in \bar\Lambda$, and therefore, from the definition of Fubini extension, the integrals $\int_{E_1} \nu(B\cap f_x^{-1}(E_2))\, d\mu(x)$ and $\int_B \mu(E_1\cap f_y^{-1}(E_2))\, d\nu(y)$ are well-defined and equal. Write Σ_B for the sub-σ-algebra of Σ generated by B, and Σ_x for that generated by f_x, $x \in A$. Now (P4) says that the family $\langle \Sigma_x \rangle_{x\in A}$ is stochastically independent. Using Fremlin (2008, 5A6-272W), it follows that there is a countable $D \subset A$ such that for each $x \in A\setminus D$, Σ_B and Σ_x are stochastically independent. Since $(A, \mathcal{A}, \nu)$ is atomless, this means that Σ_B and Σ_x are stochastically independent for almost every $x \in A$. Thus $\nu(B \cap f_x^{-1}(E_2)) = \nu(B)\nu(f_x^{-1}(E_2))$ for almost all $x \in A$. Finally, note that from (P2) we have $\nu(f_x^{-1}(E_2)) = \mu(E_2)$ for all $x \in A$.

Putting all these together, we may conclude that, for any $B \in \Sigma$,

$$\begin{aligned}\int_B \mu(E_1 \cap f_y^{-1}(E_2))\, d\nu(y) &= \int_{E_1} \nu(B \cap f_x^{-1}(E_2))\, d\mu(x)\\ &= \int_{E_1} \nu(B)\nu(f_x^{-1}(E_2))\, d\mu(x)\\ &= \int_{E_1} \nu(B)\mu(E_2)\, d\mu(x)\\ &= \nu(B)\mu(E_2)\mu(E_1).\end{aligned}$$

By the Radon-Nikodym theorem it follows that $\mu(E_1 \cap f_y^{-1}(E_2)) = \mu(E_1)\mu(E_2)$ for almost all $y \in \Sigma$. Thus, as $E_1, E_2 \in \mathcal{A}$ are arbitrary, (P3) holds for f. □

As noted in Remark 1, if a random matching satisfies (P3), then it satisfies (P7) against any type assignment. Hence the following corollary of Theorem 4 holds.

Corollary 1. *Let $(A, \mathcal{A}, \mu)$ be an atomless probability space of agents, (Ω, Σ, ν) a sample probability space, and $f \colon A \times \Omega \to A$ a random matching. Let λ be the product probability measure on $A \times \Omega$ defined from μ and ν. Suppose:*

(i) *There is a Fubini extension $\bar{\lambda}$ of λ such that, writing $\bar{\Lambda}$ for the domain of $\bar{\lambda}$, f is $(\bar{\Lambda}, \mathcal{A})$-measurable.*

(ii) *f satisfies* (P2) *and* (P4)*.*

Then f satisfies (P7) *against any type assignment.*

Remark 6. A widespread view in the literature seems to be that (P7) follows just by (P6). However, this is not the case, as illustrated in the following example, where (P6) holds but (P7) does not.

Example 1. Take the probability space $(A, \mathcal{A}, \mu)$ of agents to be $([0, 1], \mathcal{B}, \lambda)$, where λ is the Lebesgue measure, and $\mathcal{B}$ the Borel $\sigma-$algebra of $[0, 1]$. Partition $[0, 1]$ into eight measurable subsets $A_1, \ldots, A_8$, each with measure $1/8$. Let $\left(A_i, A_j\right)$ denote "the agents in A_i are matched with the agents in A_j." Recall that given any $C, C' \in \mathcal{B}$ with the same measure, there is an inverse measure preserving bijection from C onto C'. Using this fact, we can construct four matchings $f_1, \ldots, f_4$ on $[0, 1]$ such that each f_i is inverse measure-preserving and such that

f_1 satisfies	(A_1, A_2)	(A_3, A_7)	(A_4, A_8)	(A_5, A_6)
f_2 satisfies	(A_1, A_5)	(A_2, A_6)	(A_3, A_4)	(A_7, A_8)
f_3 satisfies	(A_1, A_2)	(A_3, A_4)	(A_5, A_6)	(A_7, A_8)
f_4 satisfies	(A_1, A_6)	(A_2, A_5)	(A_3, A_8)	(A_4, A_7)

Let the sample probability space (Ω, Σ, ν) be the set $\{1, 2, 3, 4\}$ with normalized counting measure and let a random matching $f \colon [0, 1] \times \Omega \to [0, 1]$ be given by $f(x, i) = f_i(x)$ for $x \in [0, 1]$ and $i \in \Omega$. Assume that there are just two types 0 and 1, and that the type assignment $\theta : [0, 1] \to \{0, 1\}$ is given by $\theta(x) = 0$ for $x \in \bigcup_{j=1}^{4} A_j$, and $\theta(x) = 1$ for $x \in \bigcup_{j=5}^{8} A_j$. Then, since in state 3 there is no match between any agents of different types, f fails to satisfy (P7). On the other hand, since each of the four matchings is equally likely, it is easy to check that f satisfies (P6). Moreover, f is $\mathcal{B} \otimes \Sigma-$measurable. However, since there are only finitely many sub-σ-algebras of Σ, f cannot satisfy the independence conditions (P4) or (P8).

5 A uniqueness result

Molzon and Puzzello (2010) show that if the random matching satisfies only measure preservation, proportional and mixing properties, then it is not unique (see also Example 2). However, if the random matching additionally satisfies independence, then it is possible to show that it is unique in the sense that the probability distribution on the set of bilateral matches is uniquely determined. In order to prove it, we first need to introduce some more notation.

Notation. Given a probability space $(A, \mathcal{A}, \mu)$ of agents, $M_A \subset A^A$ denotes the set of all matchings on A, i.e., the set of all fixed point free involutions on A; further, writing $\bar{\gamma}$ for the product probability measure on A^A defined from μ, γ denotes the restriction of $\bar{\gamma}$ to the σ-algebra generated by the measurable cylinders in A^A, γ_A the subspace measure on M_A induced from γ, and Γ_A the domain of γ_A. Given a sample probability space (Ω, Σ, ν) and a random matching $f\colon A \times \Omega \to A$ in addition, $\phi\colon \Omega \to M_A$ denotes the mapping defined by setting $\phi(y) = f_y$ for $y \in \Omega$.

The next theorem shows that, for a given probability space $(A, \mathcal{A}, \mu)$ of agents, if there exists a random matching such that (P2) and (P4) hold, then, in terms of distributions on (M_A, Γ_A), this random matching is unique.

Theorem 5. *Let $(A, \mathcal{A}, \mu)$ be an atomless probability space of agents. Then given any sample probability space (Ω, Σ, ν) and any random matching $f\colon A \times \Omega \to A$, the mapping ϕ is (Σ, Γ_A)-measurable, and if f satisfies* (P2) *and* (P4)*, then the distribution of ϕ on (M_A, Γ_A) is γ_A.*

Proof. It suffices to show that ϕ, viewed as mapping from Ω to A^A, has the property that whenever Z is a measurable cylinder in A^A, then $\phi^{-1}(Z) \in \Sigma$ and, if (P2) and (P4) hold, $\nu(\phi^{-1}(Z)) = \gamma(Z)$. Thus let Z be a measurable cylinder in A^A. Then for some finite collection $x_1, \ldots, x_n$ of distinct members of A, together with members $B_1, \ldots, B_n$ of $\mathcal{A}$, we have $Z = E_{B_1}^{x_1} \cap \cdots \cap E_{B_n}^{x_n}$ where $E_{B_i}^{x_i} = \{z \in A^A\colon z(x_i) \in B_i\}$, $i = 1, \ldots, n$. Note that for each $i = 1, \ldots, n$, $\phi^{-1}(E_{B_i}^{x_i}) = f_{x_i}^{-1}(B_i)$, because for any $y \in \Omega$,

$$\phi(y) \in E_{B_i}^{x_i} \Leftrightarrow \phi(y)(x_i) \in B_i \Leftrightarrow f_y(x_i) \in B_i \Leftrightarrow f_{x_i}(y) \in B_i.$$

Now by definition of random matching, f_x is $(\Sigma, \mathcal{A})$-measurable for any $x \in A$. It follows that $\phi^{-1}(E_{B_i}^{x_i}) \in \Sigma$ for each $i = 1, \ldots, n$, and thus $\phi^{-1}(Z) \in \Sigma$ as well. Moreover, if f satisfies (P2) and (P4), then

$$\begin{aligned}
\nu(\phi^{-1}(Z)) &= \nu(\phi^{-1}(E_{B_1}^{x_1}) \cap \cdots \cap \phi^{-1}(E_{B_n}^{x_n})) \\
&= \nu(f_{x_1}^{-1}(B_1) \cap \cdots \cap f_{x_n}^{-1}(B_n)) \\
&= \prod_{i=1}^{n} \nu(f_{x_i}^{-1}(B_i)) \quad \text{by (P4)} \\
&= \prod_{i=1}^{n} \mu(B_i) \quad \text{by (P2)} \\
&= \gamma(E_{B_1}^{x_1} \cap \cdots \cap E_{B_n}^{x_n}) = \gamma(Z),
\end{aligned}$$

the first equality in the previous line by the definition of product measure since the elements $x_1, \ldots, x_n$ of A are distinct. This completes the proof. $\square$

As noted in Theorem 3, if a random matching satisfies (P6) and (P8) against any type assignment, then it also satisfies (P2) and (P4). Therefore the above uniqueness result can be equivalently formulated in terms of type assignments in the following way.

Corollary 2. *Let $(A, \mathcal{A}, \mu)$ be an atomless probability space of agents. Then given any sample probability space (Ω, Σ, ν) and any random matching $f\colon A \times \Omega \to A$, if f satisfies* (P6) *and* (P8) *against any type assignment with a finite type space, then the distribution of ϕ on (M_A, Γ_A) is γ_A.*

6 Examples

The previous section provides conditions for the uniqueness of random matching. Next, Example 2 shows that it is crucial that the notion of type, as defined in Section 2, incorporates all payoff relevant characteristics of agents. In the first example, it is possible to capture all the payoff relevant characteristics with finitely many types. However, some economic models require a continuum of types, as illustrated in Examples 3 and 4.

Example 2. Production with finitely many types

This example indicates that non-uniqueness of the random matching can be a problem for economic models, if the notion of type does not capture all payoff relevant characteristics. The example is very simple but can be easily enriched to demonstrate that the same problem persists in the case of infinitely many agents (countable or uncountable).

Suppose our economy consists of an even number of agents, say 8, of two types, *"a"* and *"b"*. Denote the set of agents by

$$A = \{a_1, \dots, a_4, b_1, \dots, b_4\}\,.$$

Let M_A denote the set of all possible bilateral matches on this set of agents, and let elements of M_A be denoted by φ. The randomness of matching will be taken care of by placing a probability distribution on the set M_A.

Each agent $g \in A$ is endowed with a non-negative amount k_g of some input. Suppose that production of a certain good occurs only when agents of opposite type meet, and that in this case the production of agent g depends on his input and the input of the agent with whom agent g is matched. A simple specification capturing such a complementarity of inputs is, denoting by $F_g(\varphi)$ the production amount of agent g given φ,

$$F_g(\varphi) = \begin{cases} f(\min\{k_g, k_{\varphi(g)}\}) & \text{if} \quad g \text{ and } \varphi(g) \text{ have different types} \\ 0 & \text{if} \quad g \text{ and } \varphi(g) \text{ have the same type,} \end{cases}$$

where $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function with $f(0) = 0$.

We now consider two distinct probability distributions on M_A. The two distributions are described in the tables below, listing the bilateral matches and corresponding probabilities. Matches that do not appear are assigned probability 0. The notation (i, j) is used to denote that agent i is paired with agent j.

Distribution I

Match	Probability
$(a_1, a_2)(a_3, b_3)(a_4, b_4)(b_1, b_2)$	.5
$(a_1, b_1)(a_2, b_2)(a_3, a_4)(b_3, b_4)$	.5

Distribution II

Match	Probability
$(a_1, a_2)(a_3, b_3)(a_4, b_4)(b_1, b_2)$	.25
$(a_1, a_2)(a_3, b_1)(a_4, b_2)(b_3, b_4)$	.25
$(a_1, b_1)(a_2, b_2)(a_3, a_4)(b_3, b_4)$	.25
$(a_1, b_3)(a_2, b_4)(a_3, a_4)(b_1, b_2)$	.25

Note that both distributions satisfy property (P6) (Types proportional law) since any individual agent has probability .5 of being matched with a type *"a"* agent and probability .5 of being matched with a type *"b"* agent. Both distributions also satisfy the types mixing property (P7) since for each listed match, exactly one-half of the type "*a*" agents are matched with type "*a*" agents and one-half are matched with type "*b*" agents. Now, suppose that agents are given initial endowments as described in the following table:

Input endowments

Agent	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
Input	1	1	0	0	0	0	1	1

In the case of Distribution I, nothing can be produced. For both matches, either two agents of the same type are paired or a pair involves one agent with 0 resource.

In the case of Distribution II, if one of the first three distributions is realized, no production takes place because two agents of the same type meet or agents of opposite type meet but one of them has 0. However if the fourth match is realized (and this occurs with probability .25) then agents a_1, a_2, b_3, and b_4 all produce an amount $f(\min\{1, 1\}) = f(1)$.

The issue in the example is that the notion of type does not include payoff relevant attributes, namely input endowments. Note that in this example, it is easy to deal with non-uniqueness as it is possible to capture all payoff relevant characteristics with finitely many types and the existing literature provides foundations to these matching models (Boylan (1992), Duffie and Sun (2007), Alós-Ferrer (1999)).[11] Why then would non-uniqueness be an issue for economic models? As already mentioned, current existence results can accommodate finitely many types. However, for some models it is not possible to capture all the relevant attributes of agents with a finite number of types. This is the

[11] Examples of such models applied in the context of learning in games, monetary theory and asset pricing can be found in Ellison (1993), Kandori et al. (1993), Matsui and Matsuyama (1995), Kiyotaki and Wright (1989), Duffie et al. (2005).

case for the models described in Green and Zhou (2002), Molico (2006), Lagos and Wright (2005), Shi (1997), Zhu (2005), where there are no upper bounds on money holdings or money holdings are perfectly divisible, or those described in Sandholm (2001), Oechssler and Riedel (2002), Hofbauer et al. (2008), with continuous strategy sets. Thus, for these models, infinitely many attributes would not be captured by existing results. Our existence and uniqueness results allow one to entirely ignore these issues. Indeed, our results provide mathematical foundations also to random matching models with infinitely many types. We illustrate this point with examples from evolutionary game theory and monetary theory that employ an infinity of types.

Example 3. Evolutionary Game Theory

In economics, most work of evolutionary game theory focuses on populations of agents who are randomly matched to play a game with repeated rounds. In these environments, types are identified with strategies. Thus, games with continuous strategy spaces involve random matching with a continuum of types. Examples can be found in Sandholm (2001), Oechssler and Riedel (2002), Hofbauer et al. (2008). In these games, the distribution of strategies in the population is given by a probability distribution on the strategy space S, written as τ. Let $R(s, s')$ denote the payoff function to a player selecting strategy s when his partner/opponent chooses strategy s'. Then, the expected payoff to a player selecting strategy s is written as

$$E(s, \tau) = \int_S R(s, s') d\tau(s').$$

This expression makes implicit use of the types proportional law (P6) with a continuum of types.

Example 4. Monetary Theory

We start by describing the aspects of the model of Molico (2006) (see also Zhu (2005)) that are relevant to random matching. Time is discrete and the population $A = [0, 1]$ consists of a continuum of infinitely lived agents whose discount factor is $\beta \in (0, 1)$. Let $\tau_t(E)$ the measure of agents whose money holdings are in $E \subset [0, \infty)$ at the beginning of period t. In this model, the agent's type is given by his money holdings, and thus there are a continuum of types. In every period agents are randomly and bilaterally matched. An agent is the buyer in his match with probability α, the seller with probability α, and neither with probability $(1 - 2\alpha)$.

The trading rule is determined by means of Nash bargaining. We follow Molico (2006) and denote by $q_t(m_b, m_s)$ and $d_t(m_b, m_s)$ the amount of output and the amount of money determined by bargaining in a match where the buyer has m_b money holdings and the seller has m_s money holdings. Note that the payoff only depends on types.

The expected lifetime utility of an agent who enters period t with m money holdings is given by

$$\begin{aligned} V_t(m) =& \alpha \int_0^\infty \{u\left[q_t(m, m_s)\right] + \beta V_{t+1}\left[m - d_t(m, m_s)\right]\} \, d\tau_t(m_s) \\ &+ \alpha \int_0^\infty \{-c\left[q_t(m_b, m)\right] + \beta V_{t+1}\left[m + d_t(m_b, m)\right]\} \, d\tau_t(m_b) \\ &+ (1 - 2\alpha)\, \beta V_{t+1}(m). \end{aligned}$$

The state of the system at any time is defined by the distribution τ_t, whose law of motion depends on the proportion of sellers and the proportion of buyers. With x denoting the proportion of buyers and sellers during a period, the law of motion for the distribution of money in Molico (2006) can be written as

$$\begin{aligned} \tau_{t+1}(B) =& \alpha \int \int_{m_b - d_t(m_b, m_s) \in B} d\tau_t(m_b) d\tau_t(m_s) \\ &+ \alpha \int \int_{m_s + d_t(m_b, m_s) \in B} d\tau_t(m_b) d\tau_t(m_s) + (1 - 2\alpha)\tau_t(B). \end{aligned}$$

where the first and second terms are the measure of consumers and producers whose post-trade money holdings are in B, respectively (see also Zhu (2005)). The last term account for those agents who do not trade and thus their money holdings remain in B.

The expressions above suggest that the expected payoff and the law of motion equations implicitly postulate a matching process that satisfies properties (P6) and (P7) with a continuum of types. Our paper provides foundations to such matching process.

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