Economics and Space: Unified at Last

Costas Arkolakis¹

¹Yale University and NBER

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Economics and Space: A Love-Hate Relationship

- Economics and Space have had a love-hate relationship for a long time
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  - Even in spatial economics, space was rarely seriously considered

- **International trade**: Heckscher-Ohlin widespread use until mid-90’s
- **Geography**: Krugman model created an explosion of work in geography
- **Urban**: Rosen-Roback model main equilibrium framework
Economics and Space: The Challenge

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  - **Key challenge**: with rich spatial frictions models become intractable
    - ... and hard to combine with data
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  - **Key challenge**: with rich spatial frictions models become intractable
    - ... and hard to combine with data

- The spatial model with frictions is a formidable system!
  - Best case scenario, N locations equations/unknowns + interactions
    - Labor mobility (geography), knowledge spillovers (urban) make solution a true nightmare
Developing an Alternative

- Trade/geography economists recently developed a versatile alternative.
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- So-called ‘gravity framework’ and generalizations. It allows for
  1. Unified framework for **trade, geography and urban**
  2. Unified positive Analysis: A battery of mathematical tools can be used
     - e.g. non-linear/integral equations theory, perturbation theory etc.
  3. Robust comparative statics
  4. New Estimation Methods Robust Across Variations
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- Rapidly expanding literature:
  - Discussion based on results/model in Allen Arkolakis (AA) ’14, AA Takahashi ’14 (AAT), AA and Li ’14 (AAL), Allen Arkolakis (AA17), Adao, Arkolakis, Esposito (AAE) ’17, and earlier results by Arkolakis, Costinot Rodriguez-Clare (ACR) ’12
Roadmap

- A Simple Framework and the Unified Spatial Model
- Analytical Solution of Equilibrium
- Positive Properties and Computation of the Equilibrium
- Comparative Statics
- Welfare and Applications
Generalized Spatial Economy

- We first present a special case of the Generalized Spatial Competitive Economy developed in AAE

- $N$ locations each with differentiated commodity
  - Everything we say holds for sectors-locations

- Representative agent that allocates consumption and labor in space

- Competitive firms subject to Marshallian externalities
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- Spatial frictions:
  - Trade costs on consumption
  - Frictions on mobility of labor
  - Frictions on knowledge spillover
**Consumption**

- Agents in market $i$ solve

\[
\min_{C_{ij}} \sum_i p_{ij} C_{ij} \quad \text{s.t.} \quad \left[ \sum_i C_{ij}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} = 1
\]

- The spending share on goods of region $i$ in $j$ is

\[
\chi_{ij} \left( \{p_{ij}\}_{ij} \right) = \frac{p_{ij}^{1-\sigma}}{\sum_o p_{oj}^{1-\sigma}}
\]

where we define $P_j \equiv \sum_o p_{oj}^{1-\sigma}$
Labor Supply

We assume labor choice written as

\[ L_i \left( \left\{ \frac{w_i}{P_i} \right\}_i \right) = \frac{\nu_i^{1/\phi} \left( \frac{w_i}{P_i} \right)^{1/\phi}}{\sum_j \nu_j^{1/\phi} \left( \frac{w_j}{P_j} \right)^{1/\phi}} \]  

(2)

- Many ways to micro-found e.g. assuming worker mobility (see AA, AAT)
- \( w_i \) : wage rate, \( \nu_i \) : preference shifter
Firm Problem

- Perfect competition and cost minimization requires

\[ p_{ij} (w_i) = \frac{w_i \tau_{ij}}{A_i} \quad (3) \]

\( \tau_{ij} \): iceberg technological costs, agglomeration spillovers modeled as

\[ A_i = \bar{A}_i \Psi_i \left( \{L_j\}_j \right) . \]

For simplicity, \( \Psi \left( \{L_j\}_j \right) \equiv L^\psi_i \)
Closing the Model and Equilibrium

- Labor income is given by

\[ w_i L_i = \sum_j (x_{ij} w_j L_j) \] (4)

- Equilibrium in this model is characterized as \( \{w_i\} \) that satisfy (4) by substituting \( x_{ij}, L_i, p_{ij} \) using \( X_{ij}(\{p_{ij}\}), L_i(\{w_i/P_i\}), \Psi(\{L_i\}_i) \) (and a normalization)

- The model above can be massively generalized (see AAE)
  - Simply by considering general functions \( X_{ij}(\{p_{ij}\}), L_i(\{w_i/P_i\}), \Psi(\{L_i\}_i) \)
A Unified Spatial Model

**Economic Fundamentals:**

- a. Productivities $\{\zeta_i\}$
- b. Amenities $\{\nu_i\}$
- c. Trade Costs $\{\tau_i\}$

**Allocations:**

Prices, Consumer & Firm Choices

**General Equilibrium Aggregation**

- Consumers Eqs. (1), (2)
- Firms Eq. (3)

Routing problem, amenity or productivity spillovers, etc

$\chi, \Phi, \Psi$
The Simple Framework: Special Cases

1. No trade costs + No labor mobility: Neoclassical trade/macro/devo
   ▶ Many factors/sectors. H-O, Foster Rosenzweig '08, Bustos et al '16

2. No trade costs + labor mobility: The Rosen-Roback '82 model
   ▶ Version of celebrated Rosen Roback model, Glaeser '10, Kline Moretti '16

3. Trade costs + No labor mobility: The Gravity model and extensions
   ▶ Anderson '79, Ethier '82a, Eaton Kortum '02, Melitz '03/Chaney '08, Adao et al '17

4. Trade costs + labor mobility: New Economic Geography
   ▶ Helpman '98, Allen Arkolakis '14, Redding '16, Adao Arkolakis Esposito '18

5. Further extensions (define transfer of resources rule)
   ▶ Fiscal transfers: Nakamura-Stainsson '14, Chodorow-Reich '17. Assets of household: Su-Mian '13, Verner '17
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Ananytically Characterizing Spatial Models

- In general, analytical characterization of spatial models is hard
  - We need to solve variables as a function of all parameters (e.g. $\nu_i, \zeta_i, \tau_{ij}$)
  - Feasible with zero trade costs or with stylized geographies

- We will next proceed by allowing for labor mobility and start with the case of no trade costs
  - That will lead to the celebrated 'urban' Rosen-Roback’82 framework (e.g. Glaeser ’10, Kline Moretti ’16)
  - Our version has slightly different assumption but identical outcomes
  - Key similarity: no spatial frictions!
The ‘Urban Model’: No Trade Costs + Labor Mobility

 وعد The equilibrium is given by

\[ w_i L_i = \frac{(w_i/A_i)^{1-\sigma}}{\sum_o (w_o/A_o)^{1-\sigma}} Y \]

where \( Y \equiv \sum_j w_j L_j \). Normalize \( Y = 1 \).

 وعد You can prove that

\[ w_i = \nu_i \frac{\psi(\sigma-1)-1}{\gamma} A_i \frac{\phi(\sigma-1)}{\gamma} W^{1-\psi(\sigma-1)} \]

\[ L_i = \nu_i \frac{\sigma}{\gamma} A_i \frac{\sigma-1}{\gamma} \bar{L} \]

where \( \gamma \equiv 1 - \psi (\sigma - 1) - \phi \sigma \), \( W \) is welfare (we ll come back to that)

 وعد Intuition: population higher when productivity and amenity are higher. Related intuition for wages.
Recap: Economics but Not Yet Space...

- In both the macro and urban examples space implies a symmetric effect to all locations

- We imposed symmetry in either the trade costs or labor mobility
  - How do we introduce asymmetry on these links?
  - We will proceed with constant elasticity examples (e.g. AA, AAT)
    - AAE offer extensions to general mappings (1)-(3)

- Next: analytically characterize an example of non-zero trade costs
  - But assuming a stylized geography
Analytical Solution of a Geography Model

Consider trade on the line $S = [−\pi, \pi]$, 

- Global parameters: $\phi = \psi = 0$, $\sigma > 0$
- Kernel: $\nu (i) = \bar{A}(i) = 1$, $\tau (i, j) = e^{\tau|i−j|}$ for all $i, j \in S$. 

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- Consider trade on the line $S = [-\pi, \pi]$,
  - Global parameters: $\phi = \psi = 0$, $\sigma > 0$
  - Kernel: $\nu(i) = \bar{A}(i) = 1$, $\tau(i,j) = e^{\tau|i-j|}$ for all $i, j \in S$.

- Equilibrium written as an integral equation or a differential equation
  - Same differential equation in space as the pendulum in time
  - Like a **pendulum**, strength of agglomeration force proportional to distance from center and symmetric.

- In this special case, there exists a closed form solution (!):
  $$L(i) = c_1 \cos(ki) \frac{2\sigma - 1}{\sigma - 1}$$
  - $c_1, k$ depend on eigenvalue. Agglomeration force increases with $\tau$. 
Increasing trade costs $\tau$

Equilibrium Distribution of Labor
$\sigma=4$, $\alpha=0$, $\beta=0$, $\tau=0$
Increasing trade costs $\tau$

Equilibrium Distribution of Labor

$\sigma=4$, $\alpha=0$, $\beta=0$, $\tau=0.2$
Increasing trade costs $\tau$

Equilibrium Distribution of Labor
$\sigma=4$, $\alpha=0$, $\beta=0$, $\tau=0.4$
Increasing trade costs $\tau$

Equilibrium Distribution of Labor

$\sigma=4$, $\alpha=0$, $\beta=0$, $\tau=0.6$
Increasing trade costs $\tau$

Equilibrium Distribution of Labor
$\sigma=4$, $\alpha=0$, $\beta=0$, $\tau=0.8$
Increasing trade costs $\tau$

Equilibrium Distribution of Labor
$\sigma=4, \alpha=0, \beta=0, \tau=1$
Building a Border

- Now add a border in the middle (on top of trade cost)

- The solution becomes

\[ L(i) = (c_1 \cos(ki) + c_2 \sin(ki))^{\frac{2\sigma-1}{\sigma-1}} \]

- Same differential equation in space as the spring in time
  - Like a \textbf{spring}, strength of agglomeration force proportional to distance but border introduces \textit{asymmetry}. 
Building a border

Equilibrium Distribution of Labor
\( \sigma = 10, \alpha = 0, \beta = 0, \tau = 1, \text{ border cost} = 0 \)
Building a border

Equilibrium Distribution of Labor

$\sigma = 10, \alpha = 0, \beta = 0, \tau = 1, \text{border cost} = 0.1$
Building a border

Equilibrium Distribution of Labor

\( \sigma = 10, \alpha = 0, \beta = 0, \tau = 1, \text{ border cost} = 0.5 \)
Building a border

Equilibrium Distribution of Labor

$\sigma = 10, \alpha = 0, \beta = 0, \tau = 1, \text{ border cost} = 100$
Roadmap

- A Simple Framework and the Unified Spatial Model
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Spatial Models: Positive Analysis

- Having given intuition for the working elements of spatial models we next characterize positive properties
  - Existence, uniqueness, and equilibrium computation of spatial models

- For this, functional forms are essential, as we need to impose restrictions on parameters

- We will focus on the parametric examples
  - Workhorse analysis using the gravity model.
  - Combine consumer and firm decisions bilateral trade given by

\[
\chi_{ij} = \frac{\left( \frac{w_i \tau_{ij}}{A_i} \right)^{1-\sigma}}{\sum_o \rho_{oj}^{1-\sigma}} = \left( \frac{\tau_{ij}}{\tau_{ij}^e} \right)^{1-\sigma} \times \left( \frac{w_i}{A_i} \right)^{1-\sigma} \times \frac{1}{\sum_k \left( \frac{w_k}{A_k} \tau_{kj} \right)^{1-\sigma} \delta_{j}}
\]
Trade Model: Equilibrium Equations

- Equilibrium is trade gravity + market clearing.

\[ w_i L_i = \sum_i x_{ij} w_j L_j \implies w_i L_i = \sum_i \left( \frac{w_i \tau_{ij}}{A_i} \right)^{1-\sigma} w_j L_j \]

\[ w_i L_i = \sum_i \left( \frac{w_i \tau_{ij}}{A_i} \right)^{1-\sigma} w_j L_j \]
Trade Model: Equilibrium Equations

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$$w_i L_i = \sum_i x_{ij} w_j L_j \implies$$

$$w_i L_i = \sum_i \left( \frac{w_i \tau_{ij}}{A_i} \right)^{1-\sigma} \frac{w_j L_j}{\sum_o \left( \frac{w_o \tau_{oj}}{A_o} \right)^{1-\sigma}}$$

- Solve $w_i, P_i$ using

$$w_i^{\sigma} = \sum_j (\tau_{ij})^{1-\sigma} L_i^{-1} A_i^{\sigma-1} L_j w_j P_j^{\sigma-1}$$

$$P_i^{1-\sigma} = \sum_j (\tau_{ji})^{1-\sigma} A_j^{\sigma-1} (w_j)^{1-\sigma}$$
Trade Model: Equilibrium Equations

- In trade models (with no deficit) we have $E_i = Y_i$

- Equilibrium is trade gravity + market clearing + no labor mobility ($L_i = \bar{L}_i$)
  - Solve $w_i, P_i$ using

$$w_i^\sigma = \sum_j (\tau_{ij})^{1-\sigma} L_i^{-1} A_i^{\sigma-1} \nu_j^{\sigma-1} L_j w_j P_j^{\sigma-1}$$

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    $$P_i^{1-\sigma} = \sum_j (\tau_{ji})^{1-\sigma} A_j^{\sigma-1} (w_j)^{1-\sigma}$$

- We intentionally avoided substituting the price index.
  - Crucial to write it this way, as it is much easier to characterize
Geography Model: Equilibrium Equations

- Equilibrium is \textit{trade gravity + market clearing +}

\[ L_j = \frac{\nu_j^{1/\phi} (w_j / P_j)^{1/\phi}}{\sum_j \nu_j^{1/\phi} (w_j / P_j)^{1/\phi}} \]

- Solve \( w_i, L_i, W \) using

\[ W^{\sigma-1} w_i^\sigma L_i^{1-\psi(\sigma-1)} = \sum_{j=1}^{N} \tau_{ij}^{1-\sigma} \tilde{A}_i^{\sigma-1} \nu_j^{\sigma-1} w_j^\sigma L_j^{1+\phi(\sigma-1)} \]

\[ W^{\sigma-1} w_i^{1-\sigma} L_i^{\phi(1-\sigma)} = \sum_{j=1}^{N} \tau_{ji}^{1-\sigma} \nu_i^{\sigma-1} \tilde{A}_j^{\sigma-1} w_j^{1-\sigma} L_j^{\psi(\sigma-1)} \]

where \( W \equiv \left[ \sum_j \nu_j^{1/\phi} (w_j / P_j)^{1/\phi} \right]^{\phi(\sigma-1)}. \)

- Existence and uniqueness in AA and AAT: notice same mathematical structure as in the trade model.
  - Except now welfare is the eigenvalue of the system
Geography Model: The Linear Case

- Equilibrium is trade gravity + market clearing +

\[ L_j = \frac{\nu_j^{1/\phi} (w_j/P_j)^{1/\phi}}{\sum_j \nu_j^{1/\phi} (w_j/P_j)^{1/\phi}} \]

- Assume \( \phi = \psi \to 0 \)

\[ W^{\sigma-1} w_i^\sigma L_i = \sum_{j=1}^N \tau_{ij}^{1-\sigma} \bar{A}_i^{\sigma-1} \nu_j^{\sigma-1} w_j^\sigma L_j \]

\[ W^{\sigma-1} w_i^{1-\sigma} = \sum_{j=1}^N \tau_{ji}^{1-\sigma} \nu_i^{\sigma-1} \bar{A}_j^{\sigma-1} w_j^{1-\sigma} \]

where \( W \equiv \left[ \sum_j \nu_j^{1/\phi} (w_j/P_j)^{-1/\phi} \right]^{-\phi(\sigma-1)} \).

- (Practically) a linear system. Perron-Frobenius speaks to its solution
  - Unique positive solution. Notice 'eigenvalues' not guaranteed the same
Summary of GE Gravity Trade & Geography Models

- GE gravity trade (Anderson ’79: solve for $w_i, P_i$)

\[
w_i^\sigma = \sum_{j=1}^{N} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} L_j w_j P_j^{\sigma-1}
\]

\[
P_i^{1-\sigma} = \sum_{j=1}^{N} \tau_{ji}^{1-\sigma} A_j^{\sigma-1} w_j^{1-\sigma}
\]

- GE geography (AA: welfare equalizes, solve for $W, w_i, L_i$)

\[
W^{\sigma-1} w_i^\sigma L_i^{1-\psi(\sigma-1)} = \sum_{j=1}^{N} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} \nu_j^{\sigma-1} w_j^\sigma L_j^{1+\phi(\sigma-1)}
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\]

and total population constraint $\sum_j L_j = \bar{L}$
Comparison: Kernel

- GE gravity trade (Anderson ’79: solve for $w_i, P_i$)

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\[ W^{\sigma-1} w_i^{1-\sigma} L_i^{\phi(1-\sigma)} = \sum_{j=1}^{N} \tau_{ji}^{1-\sigma} A_j^{\sigma-1} \nu_i^{\sigma-1} w_j^{1-\sigma} L_j^{\phi(\sigma-1)} \]

and total population constraint $\sum_j L_j = \bar{L}$
Comparison: Global Parameters

- **GE gravity trade** (Anderson ’79: solve for $w_i, P_i$)

\[
w_i^{\sigma} = \sum_{j=1}^{N} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} L_j w_j P_j^{\sigma-1}
\]

\[
P_i^{1-\sigma} = \sum_{j=1}^{N} \tau_{ji}^{1-\sigma} A_j^{\sigma-1} w_j^{1-\sigma}
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- **GE geography** (AA: welfare equalizes, solve for $W, w_i, L_i$)

\[
W^{\sigma-1} w_i^{\sigma} L_i^{1-\psi(\sigma-1)} = \sum_{j=1}^{N} \tau_{ij}^{1-\sigma} \bar{A}_i^{\sigma-1} \nu_j^{\sigma-1} w_j^{1+\phi(\sigma-1)}
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\]

and total population constraint $\sum_j L_j = \bar{L}$
Comparison: Eigenvalues

- GE gravity trade (Anderson ’79: solve for $w_i, P_i$)

\[
\begin{align*}
1 \ w_i^\sigma & = \sum_{j=1}^{N} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} L_j w_j P_j^{\sigma-1} \\
1 \ P_i^{1-\sigma} & = \sum_{j=1}^{N} \tau_{ji}^{1-\sigma} A_j^{\sigma-1} w_j^{1-\sigma}
\end{align*}
\]

- GE geography (AA: welfare equalizes, solve for $W,w_i, L_i$)

\[
\begin{align*}
W^{\sigma-1} w_i^\sigma L_i^{1-\psi(\sigma-1)} & = \sum_{j=1}^{N} \tau_{ij}^{1-\sigma} \bar{A}_i^{\sigma-1} \nu_j^{\sigma-1} w_j^\sigma L_j^{1+\phi(\sigma-1)} \\
W^{\sigma-1} w_i^{1-\sigma} L_i^{\phi(1-\sigma)} & = \sum_{j=1}^{N} \tau_{ji}^{1-\sigma} \bar{A}_j^{\sigma-1} \nu_i^{\sigma-1} w_j^{1-\sigma} L_j^{\psi(\sigma-1)}
\end{align*}
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and total population constraint $\sum j \ L_j = \bar{L}$
Visualization of the Spatial Links
Visualization of Adding Locations
Visualization of Adding Spatial Links
Visualization of Adding Spatial Links

- Trade links
- Labor mobility links
- Technology spillover links
A Generalized Gravity ‘Model’

- Suppose equilibrium of a model reduces to a system of eqns where we denote locations (or sectors/location-sectors) with \( i,j \in \{1, ..., N\} \), eqns with \( k \), type of variable with \( h \); \( k, h \in \{1, ..., H\} \)

\[
\lambda^k \prod_{h=1}^{H} (x_i^h)^{\gamma_{kh}} = \sum_{j=1}^{N} K_{ij}^k \left[ \prod_{h=1}^{H} (x_j^h)^{\beta_{kh}} \right]
\]  

- Equilibrium variables \( x_i^h \): # to be solved \( H \times N \) (wage, price, labor etc)
- Eigenvalue \( \lambda^k \): Its role across models varies (typically welfare)
- Kernel \( K_{ij}^k \geq 0 \): spatial links (trade/commuting costs, productivity decay etc)
- Global parameters \( \gamma_{kh}, \beta_{kh} \geq 0 \):(EoS, Frechet elast., spillovers etc)

- \( \Gamma = \{\gamma_{kh}\}, \, B = \{\beta_{kh}\} \) are the corresponding matrices
Theorem: Allen Arkolakis Li ’15

Theorem

Consider the system of equations (5).

If $\Gamma$ is invertible then:

(i) If $K_{ij}^k > 0$, then there exists a strictly positive solution, $\{x_i^h, \lambda^k\}$

Define $A \equiv B\Gamma^{-1}$, with element $A_{ij}$ & $A^p \equiv \{|A_{ij}|\}$

(ii) If $K_{ij}^k \geq 0$ and the maximum of the eigenvalues of $A^p$, $\rho(A^p) \leq 1$, then there exists at most one strictly positive solution (up-to-scale)

(iii) If $\rho(A^p) > 1$ then there exists some $K_{ij}^k$ that generates multiple strictly positive solutions
Theorem: Allen Arkolakis Li '15

Consider the system of equations (5).

If $\Gamma$ is invertible then:

(i) If $K_{ij}^k > 0$, then there exists a strictly positive solution, $\{x^h_i, \lambda^k\}$

Define $A \equiv B\Gamma^{-1}$, with element $A_{ij} & A^p \equiv \{|A_{ij}|\}$

(ii) If $K_{ij}^k \geq 0$ and the maximum of the eigenvalues of $A^p$, $\rho(A^p) \leq 1$, then there exists at most one strictly positive solution (up-to-scale)

(iii) If $\rho(A^p) > 1$ then there exists some $K_{ij}^k$ that generates multiple strictly positive solutions

(iv) If $K_{ij}^k > 0$ and $\rho(A^p) < 1$ the unique (up-to-scale) solution can be computed by a simple iterative procedure
Application on Geography and Urban Model

- Note: Convenient conditions on global parameter vector not on Kernel
  - Can handle large dimensionality (many locations etc) like a charm

- The theorem is extremely powerful for economic geography model
  - In AA you can prove that equilibrium always exists; is unique if $\phi + \psi \leq 0$
  - With no trade costs, uniqueness holds under the same conditions
Roadmap

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- Analytical Characterization of the Equilibrium
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- Applications
How Changes in Fundamentals ('Economic Shocks') Affect Markets?

- **Question:** Characterize comparative statics/policy elasticities

\[
\epsilon_{W}^{ij} = \frac{d \ln W}{d \ln \tau_{ij}}, \quad \epsilon_{w}^{ij} = \frac{d \ln w_i}{d \ln \tau_{ij}}
\]
How Changes in Fundamentals (‘Economic Shocks’) Affect Markets?

▶ **Question:** Characterize comparative statics/policy elasticities

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\epsilon_{Wij} = \frac{d \ln W}{d \ln \tau_{ij}}, \quad \epsilon_{Wl} = \frac{d \ln w_l}{d \ln \tau_{ij}}
\]

▶ GE theory instills pessimism. Yet, we can obtain two results
  ▶ Express policy elasticities solely in terms of ‘deep’ elasticities, observed data
  ▶ Characterize counterfactuals solely in terms of deep elasticities, observed data, and economic shocks
    ▶ Characterization requires harnessing network effects in spatial models
Space and Comparative Statics

Let us consider richer spatial interactions

- We assume no trade cost but following, AAE

\[ \hat{L}_i = \sum_j \phi_{ij} \hat{w}_j, \quad \hat{A}_i = \hat{A}_i + \psi \hat{L}_i \]

Using (4) we obtain

\[ -\sigma \hat{w}_i + \sum_j \phi_{ij} \hat{w}_j + (\sigma - 1) \psi \sum_j \phi_{ij} \hat{w}_j = (1 - \sigma) \hat{\eta}_i + d \]

where \( d \) is a mixture of common GE terms and \( \hat{\eta}_i \equiv \hat{A}_i - \sum_o x_o \hat{A}_o \)
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- Multiple interactions: Space is kicking in!
  - Inverting implies \(w = M^{-1}A\) where \(M_{ij} = -1_{i\neq j} \sigma + [1 + (\sigma - 1) \psi] \phi_{ij}\)
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  - Fun (+useful) fact: \( \mathbf{M}^{-1} \) can be written as Neumann series of power terms of \( \mathbf{M} \): The network effects of trade!
Roadmap

- A Simple Framework and the Unified Spatial Model
- Analytical Solution of Equilibrium
- Positive Properties and Computation of the Equilibrium
- Comparative Statics
- Welfare and Applications
Welfare and Policy

▶ What about welfare?
  ▶ We may distinguish the ex-post and ex-ante evaluation of a policy change

▶ Ex-post: Evaluate welfare after policy is implemented looking at the two equilibria
  ▶ Robust ‘macro’ formula across trade geography models (Arkolakis, Costinot, Rodriguez-Clare ’12)
    ▶ Robust to changes in preferences, intermediate inputs/sectors, market structure (Costinot Rodriguez-Clare, ACDR, Midrigan Xu)
  ▶ Ex-post welfare $d \ln W_j = -\frac{d \ln \lambda_{ij}}{\epsilon}$ (note: welfare equalizes in econ geography)

▶ Ex-ante: Evaluate policy elasticity (counterfactuals)
Welfare Counterfactuals

- CES-demand trade models simple derivative (Atkeson Burstein, Lai et al)
  \[ \frac{d \ln W}{d \ln \tau_{ij}} = \frac{X_{ij}}{Y^W} \] (\(W\) here is expenditure weighted welfare, \(Y^W\) : world GDP)

- Much harder characterization in geography models because of eigenvalue
  - Need to use basics of perturbation theory (AA17)
    - If there is no spillovers (\(\psi + \phi \neq 0\)) we obtain the same result
    - With spillovers obtain a formula with an adjustment factor
Evaluating the Impact of Infrastructure Policies

- Now we can evaluate impact of real-world infrastructure policies
  - Consider a weighted graph with infrastructure matrix $T = \{t_{ij}\}$ denoting the cost of two connected points.
  - Bilateral trade costs $\tau_{ij}$ depends on $t_{kl}$ on the realized path
    - e.g. $\tau_{ij} = t_{i1} \times t_{1k} \times \ldots \times t_{lj}$

- Example: Infrastructure Investment. Want to measure
  \[
  \frac{d \ln W}{d \ln t_{ij}} = \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{d \ln W}{d \ln \tau_{kl}} \times \frac{d \ln \tau_{kl}}{d \ln t_{ij}}
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  1. Black box (but cool): Djikstra (Donaldson), Fast Marching Method (AA)
  2. Analytical characterization (but super cool): New AA
Applications

- Basically, hundreds of applications undertaken with this setup in trade.
  - New wave of applications in economic geography, urban (AA, Ahlfedlt et al '15, Monte et al, Redding 16, AAL15, Caliendo Parro Rossi-Hansberg '14, Faber Gaubert '15 etc)
- Fast marching method (AA) ideally fit for the job (Generalization of Dijkstra for continuous space).
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- Can we use this setup to think about trade cost/commuting costs etc?
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1. The Fast Marching Method for Spatial Economics
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1. Trade costs with FMM: transportation networks
Estimating trade costs with FMM: mode-specific trade
Removing the IHS: Estimated increase in $P$
Removing the IHS: Cost-benefit analysis

Estimated annual cost of the IHS: ≈ $100 billion
Annualized cost of construction: ≈ $30 billion ($560 billion @5%/year) (CBO, 1982)
Maintenance: ≈ $70 billion (FHA, 2008)

Estimated annual gain of the IHS: ≈ $150 – 200 billion
Welfare gain of IHS: 1.1 – 1.4%.
Given homothetic preferences and holding prices fixed, can multiply welfare gain by U.S. GDP.

Suggests gains from IHS substantially greater than costs.
Conclusion

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  - Tight connection to data
  - Many tools and methods to use!
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▶ We developed a unified spatial GE framework
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  ▶ Many tools and methods to use!
  ▶ Can combine with modern IO/macro/theory tools

▶ There is unbounded demand for good theorists to work on spatial topics!