Economics and Space: Unified at Last

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June, 2018

Economics and Space: A Love-Hate Relationship

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- **Geography**: Krugman model created an explosion of work in geography
- **Urban**: Rosen-Roback model main equilibrium framework

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  - **Key challenge:** with rich spatial frictions models become intractable
    - ... and hard to combine with data

- The spatial model with frictions is a formidable system!
  - Best case scenario, N locations equations/unknowns + interactions
    - Labor mobility (geography), knowledge spillovers (urban) make solution a true nightmare

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- So-called 'gravity framework' and generalizations. It allows for
  1. Unified framework for **trade, geography and urban**
  2. Unified positive Analysis: A battery of mathematical tools can be used
     - e.g. non-linear/integral equations theory, perturbation theory etc.
  3. Robust comparative statics
  4. New Estimation Methods Robust Across Variations

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  3. Robust comparative statics
  4. New Estimation Methods Robust Across Variations
- Rapidly expanding literature:
  - Discussion based on results/model in Allen Arkolakis (AA) '14, AA Takahashi '14 (AAT), AA and Li '14 (AAL), Allen Arkolakis (AA17), Adao, Arkolakis, Esposito (AAE) '17, and earlier results by Arkolakis, Costinot Rodriguez-Clare (ACR) '12

Roadmap

- A Simple Framework and the Unified Spatial Model
- Analytical Solution of Equilibrium
- Positive Properties and Computation of the Equilibrium
- Comparative Statics
- Welfare and Applications

Generalized Spatial Economy

- We first present a special case of the Generalized Spatial Competitive Economy developed in AAE
- N locations each with differentiated commodity
  - Everything we say holds for sectors-locations
- Representative agent that allocates consumption and labor in space
- Competitive firms subject to Marshallian externalities

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- N locations each with differentiated commodity
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- Representative agent that allocates consumption and labor in space
- Competitive firms subject to Marshallian externalities
- Spatial frictions:
  - Trade costs on consumption
  - Frictions on mobility of labor
  - Frictions on knowledge spillover

Consumption

- Agents in market i solve

$$\min_{C_{ij}} \sum_i p_{ij} C_{ij} \quad \text{s.t.} \quad \left[\sum_i C_{ij}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} = 1$$

  - The spending share on goods of region i in j is

$$x_{ij}\left(\{p_{ij}\}_{ij}\right) = \frac{p_{ij}^{1-\sigma}}{\sum_o p_{oj}^{1-\sigma}} \tag{1}$$

where we define $P_j \equiv \sum_o p_{oj}^{1-\sigma}$

Labor Supply

- We assume labor choice written as

$$L_i\left(\left\{\frac{w_i}{P_i}\right\}_i\right) = \frac{\nu_i^{1/\phi}\left(\frac{w_i}{P_i}\right)^{1/\phi}}{\sum_j \nu_i^{1/\phi}\left(\frac{w_j}{P_j}\right)^{1/\phi}} \tag{2}$$

  - Many ways to micro-found e.g. assuming worker mobility (see AA, AAT)
  - w_i : wage rate, ν_i : preference shifter

Firm Problem

- Perfect competition and cost minimization requires

$$p_{ij}\left(w_i\right) = \frac{w_i \tau_{ij}}{A_i} \tag{3}$$

τ_{ij} : iceberg technological costs, agglomeration spillovers modeled as

$$A_i = \bar{A}_i \Psi_i \left(\left\{L_j\right\}_j\right).$$

For simplicity, $\Psi\left(\left\{L_j\right\}_j\right) \equiv L_i^{\psi}$

Closing the Model and Equilibrium

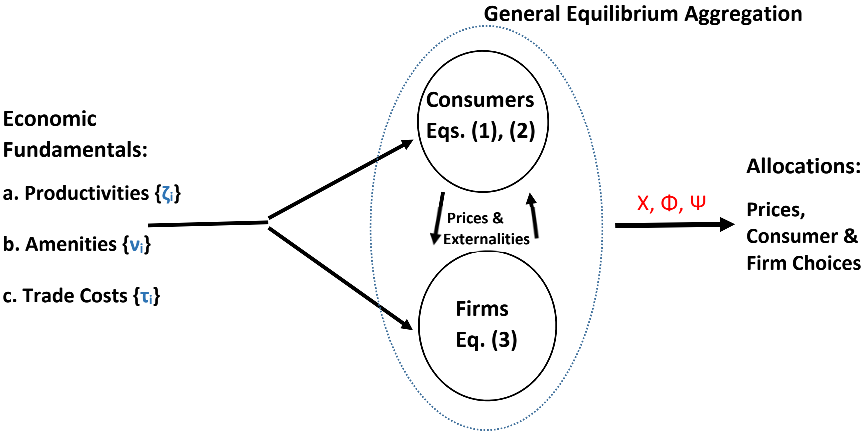
- Labor income is given by

$$w_i L_i = \sum_j \left(x_{ij} w_j L_j\right) \tag{4}$$

- Equilibrium in this model is characterized as $\{w_i\}$ that satisfy (4) by substituting x_{ij}, L_i, p_{ij} using $X_{ij}\left(\{p_{ij}\}\right)$, $L_i\left(\{w_i/P_i\}\right)$, $\Psi\left(\{L_i\}_i\right)$ (and a normalization)

- The model above can be massively generalized (see AAE)
  - Simply by considering general functions $X_{ij}\left(\{p_{ij}\}\right)$, $L_i\left(\{w_i/P_i\}\right)$, $\Psi\left(\{L_i\}_i\right)$

A Unified Spatial Model



The Simple Framework: Special Cases

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1. **No** trade costs + **No** labor mobility: Neoclassical trade/macro/devo
   - Many factors/sectors. H-O, Foster Rosenzweig '08, Bustos et al '16
2. **No** trade costs + labor mobility: The Rosen-Roback '82 model
   - Version of celebrated Rosen Roback model, Glaeser '10, Kline Moretti '16

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3. Trade costs + **No** labor mobility: The Gravity model and extensions
   - Anderson '79, Ethier '82a, Eaton Kortum '02, Melitz '03/Chaney '08, Adao et al '17
4. Trade costs + labor mobility: New Economic Geography
   - Helpman '98, Allen Arkolakis '14, Redding '16, Adao Arkolakis Esposito '18
5. Further extensions (define transfer of resources rule)
   - Fiscal transfers: Nakamura-Stainsson '14, Chodorow-Reich '17. Assets of household: Su-Mian '13, Verner '17

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Ananytically Characterizing Spatial Models

- In general, analytical characterization of spatial models is hard
  - We need to solve variables as a function of all parameters (e.g. $\nu_i, \zeta_i, \tau_{ij}$)
  - Feasible with zero trade costs or with stylized geographies

- We will next proceed by allowing for labor mobility and start with the case of no trade costs
  - That will lead to the celebrated 'urban' Rosen-Roback'82 framework (e.g. Glaeser '10, Kline Moretti '16)
  - Our version has slightly different assumption but identical outcomes
  - Key similarity: no spatial frictions!

The ‘Urban Model’: No Trade Costs + Labor Mobility

- The equilibrium is given by

$$w_i L_i = \frac{(w_i/A_i)^{1-\sigma}}{\sum_o (w_o/A_o)^{1-\sigma}} Y$$

where $Y \equiv \sum_j w_j L_j$. Normalize $Y = 1$.

- You can prove that

$$w_i = \nu_i^{\frac{\psi(\sigma-1)-1}{\gamma}} \bar{A}_i^{\frac{\phi(\sigma-1)}{\gamma}} W^{\frac{1-\psi(\sigma-1)}{\gamma}}$$

$$L_i = \frac{\nu_i^{\frac{\sigma}{\gamma}} \bar{A}_i^{\frac{\sigma-1}{\gamma}}}{\sum_o \nu_o^{\frac{\sigma}{\gamma}} \bar{A}_o^{\frac{\sigma-1}{\gamma}}} \bar{L}$$

where $\gamma \equiv 1 - \psi(\sigma - 1) - \phi\sigma$, W is welfare (we ll come back to that)

  - Intuition: population higher when productivity and amenity are higher. Related intuition for wages.

Recap: Economics but Not Yet Space...

- In both the macro and urban examples space implies a symmetric effect to all locations

- We imposed symmetry in either the trade costs or labor mobility
  - How do we introduce asymmetry on these links?
  - We will proceed with constant elasticity examples (e.g. AA, AAT)
    - AAE offer extensions to general mappings (1)-(3)

- Next: analytically characterize an example of non-zero trade costs
  - But assuming a stylized geography

Analytical Solution of a Geography Model

- Consider trade on the line $S = [-\pi, \pi]$,
  - Global parameters: $\phi = \psi = 0$, $\sigma > 0$
  - Kernel: $\nu(i) = \bar{A}(i) = 1$, $\tau(i, j) = e^{\tau|i-j|}$ for all $i, j \in S$.

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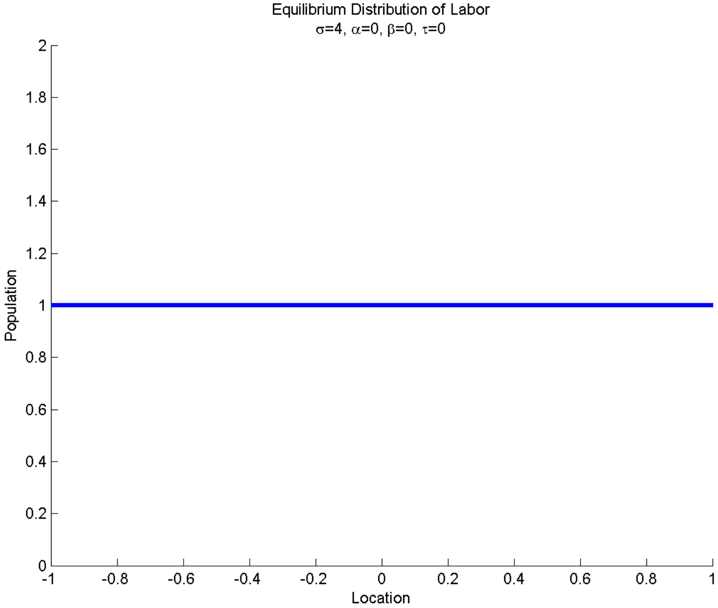
  - Kernel: $\nu(i) = \bar{A}(i) = 1$, $\tau(i,j) = e^{\tau|i-j|}$ for all $i, j \in S$.
- Equilibrium written as an *integral equation* or a differential equation
  - Same differential equation in space as the pendulum in time
  - Like a **pendulum**, strength of agglomeration force proportional to distance from center and *symmetric*.
- In this special case, there exists a closed form solution (!):

$$L(i) = c_1 \cos(ki)^{\frac{2\sigma-1}{\sigma-1}}$$

  - c_1, k depend on eigenvalue. Agglomeration force increases with τ.

Increasing trade costs τ



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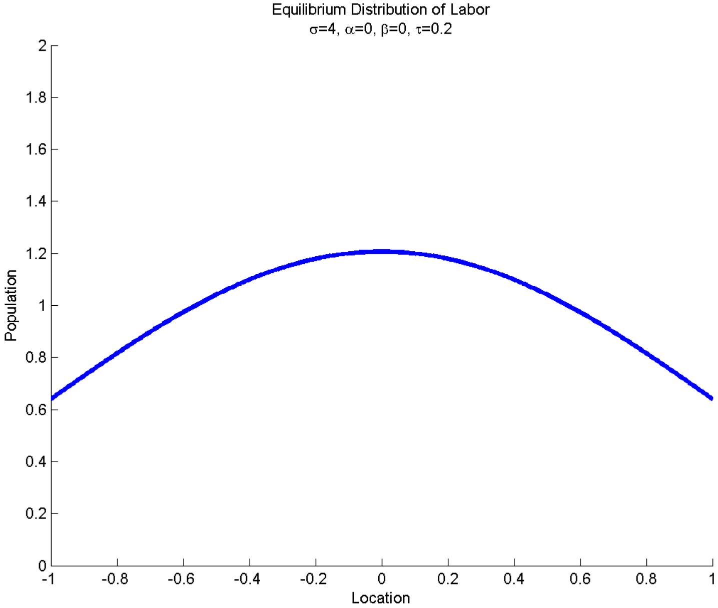

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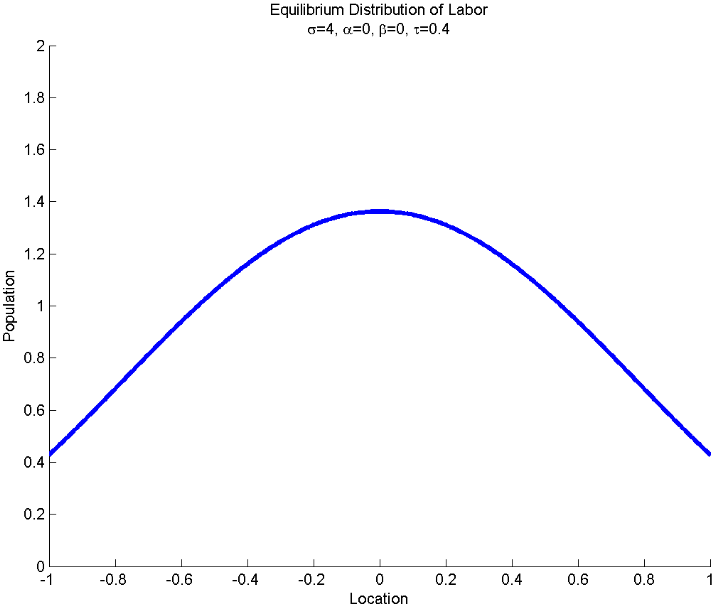

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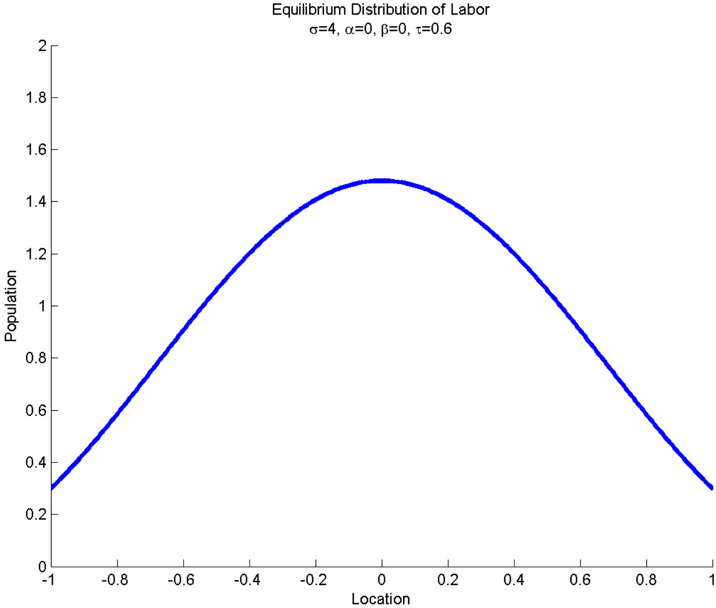

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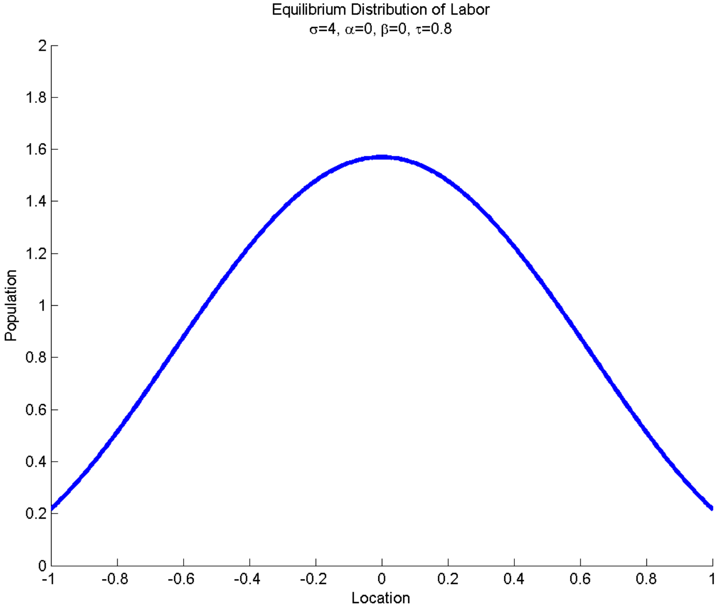

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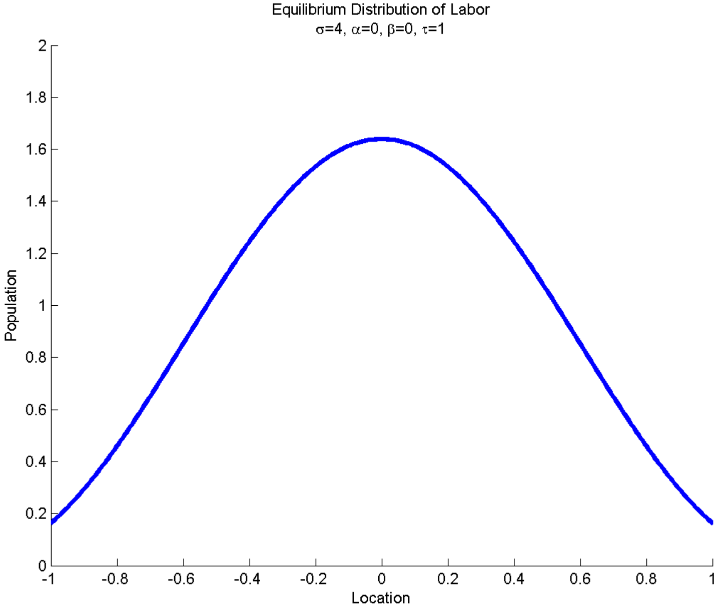


Building a Border

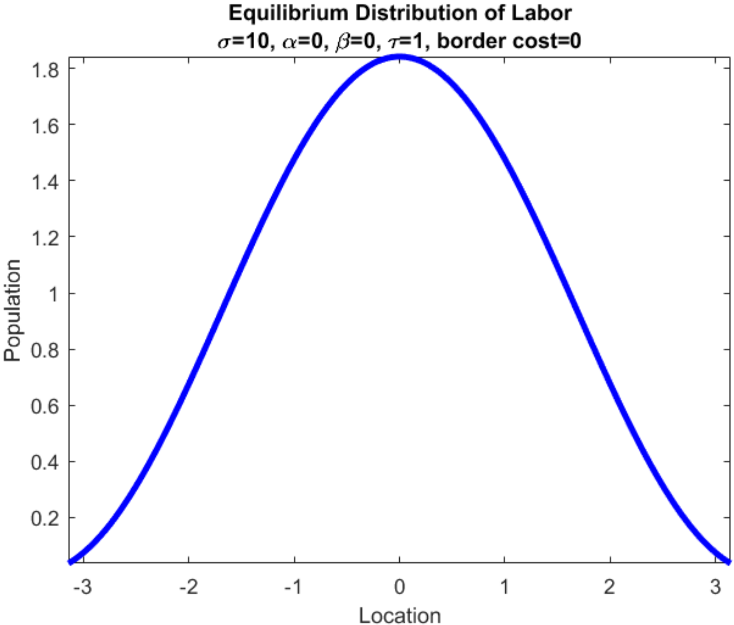
- Now add a border in the middle (on top of trade cost)

- The solution becomes

$$L(i) = \left(c_1 \cos(ki) + c_2 \sin(ki)\right)^{\frac{2\sigma-1}{\sigma-1}}$$

- Same differential equation in space as the spring in time
    - Like a **spring**, strength of agglomeration force proportional to distance but border introduces *assymetry*.

Building a border



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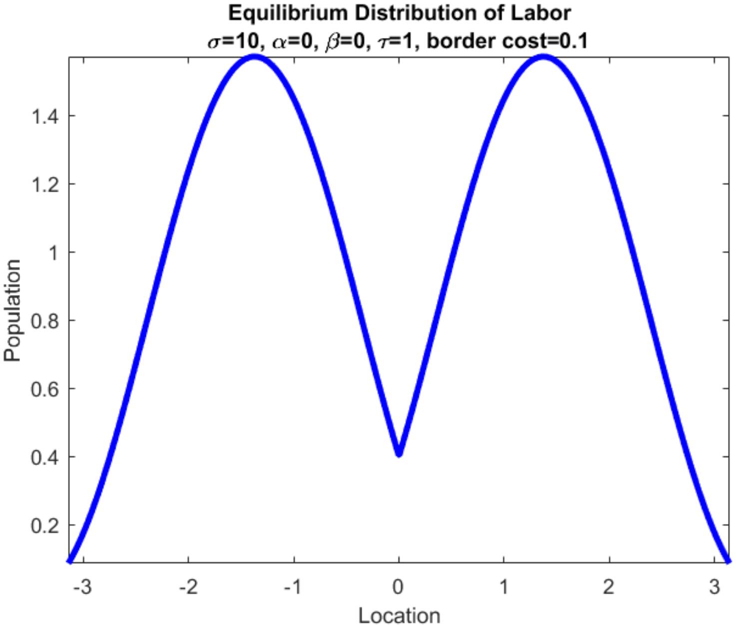

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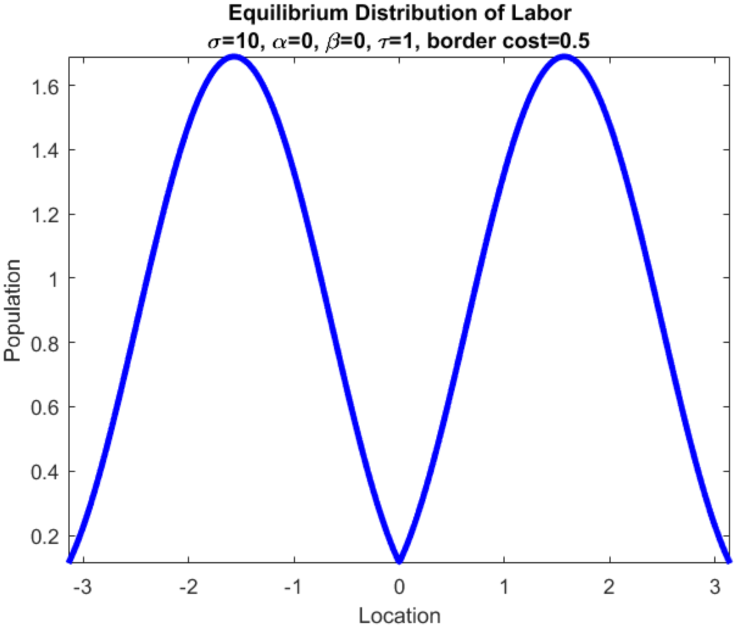

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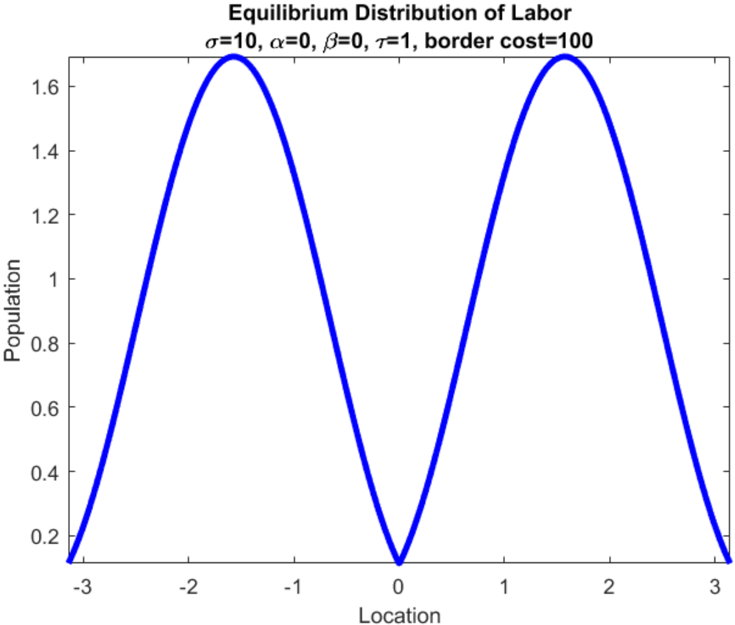

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Spatial Models: Positive Analysis

- Having given intuition for the working elements of spatial models we next characterize positive properties
  - Existence, uniqueness, and equilibrium computation of spatial models

- For this, functional forms are essential, as we need to impose restrictions on parameters

- We will focus on the parametric examples
  - Workhorse analysis using the gravity model.
  - Combine consumer and firm decisions bilateral trade given by

$$x_{ij} = \frac{\left(\frac{w_i \tau_{ij}}{A_i}\right)^{1-\sigma}}{\sum_o p_{oj}^{1-\sigma}} = \underbrace{\left(\tau_{ij}\right)^{1-\sigma}}_{\tau_{ij}^{\epsilon}} \times \underbrace{\left(\frac{w_i}{A_i}\right)^{1-\sigma}}_{\gamma_i} \times \underbrace{\frac{1}{\sum_k \left(\frac{w_k}{A_k}\tau_{kj}\right)^{1-\sigma}}}_{\delta_j}$$

Trade Model: Equilibrium Equations

- Equilibrium is trade gravity+market clearing.

$$w_i L_i = \sum_i x_{ij} w_j L_j \implies$$

$$w_i L_i = \sum_i \frac{\left(\frac{w_i \tau_{ij}}{A_i}\right)^{1-\sigma}}{\sum_o \left(\frac{w_o \tau_{oj}}{A_o}\right)^{1-\sigma}} w_j L_j$$

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- Solve w_i, P_i using

$$w_i^{\sigma} = \sum_j (\tau_{ij})^{1-\sigma} L_i^{-1} A_i^{\sigma-1} L_j w_j P_j^{\sigma-1}$$

$$P_i^{1-\sigma} = \sum_j (\tau_{ji})^{1-\sigma} A_j^{\sigma-1} (w_j)^{1-\sigma}$$

Trade Model: Equilibrium Equations

- In trade models (with no deficit) we have $E_i = Y_i$
- Equilibrium is trade gravity+market clearing+**no** labor mobility $(L_i = \bar{L}_i)$
  - Solve w_i, P_i using

$$w_i^{\sigma} = \sum_j (\tau_{ij})^{1-\sigma} L_i^{-1} A_i^{\sigma-1} \nu_j{}^{\sigma-1} L_j w_j P_j^{\sigma-1}$$

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- We intentionally avoided substituting the price index.
  - Crucial to write it this way, as it is much easier to characterize

Geography Model: Equilibrium Equations

- Equilibrium is trade gravity+market clearing+

$$L_j = \frac{\nu_j^{1/\phi} \left(w_j/P_j\right)^{1/\phi}}{\sum_j \nu_j^{1/\phi} \left(w_j/P_j\right)^{1/\phi}}$$

- Solve w_i, L_i, W using

$$W^{\sigma-1} w_i^{\sigma} L_i^{\,1-\psi(\sigma-1)} = \sum_{j=1}^{N} \tau_{ij}^{1-\sigma} \bar{A}_i^{\,\sigma-1} \nu_j^{\,\sigma-1} w_j^{\sigma} L_j^{\,1+\phi(\sigma-1)}$$

$$W^{\sigma-1} w_i^{\,1-\sigma} L_i^{\,\phi(1-\sigma)} = \sum_{j=1}^{N} \tau_{ji}^{1-\sigma} \nu_i^{\,\sigma-1} \bar{A}_j^{\,\sigma-1} w_j^{\,1-\sigma} L_j^{\psi(\sigma-1)}$$

where $W \equiv \left[\sum_j \nu_j^{1/\phi} \left(w_j/P_j\right)^{1/\phi}\right]^{\phi(\sigma-1)}$.

- Existence and uniqueness in AA and AAT: notice same mathematical structure as in the trade model.
  - Except now welfare is the eigenvalue of the system

Geography Model: The Linear Case

- Equilibrium is trade gravity+market clearing+

$$L_j = \frac{\nu_j^{1/\phi} \left(w_j/P_j\right)^{1/\phi}}{\sum_j \nu_j^{1/\phi} \left(w_j/P_j\right)^{1/\phi}}$$

- Assume $\phi = \psi \to 0$

$$W^{\sigma-1} w_i^{\sigma} L_i = \sum_{j=1}^{N} \tau_{ij}^{1-\sigma} \bar{A}_i^{\sigma-1} \nu_j^{\sigma-1} w_j^{\sigma} L_j$$

$$W^{\sigma-1} w_i^{1-\sigma} = \sum_{j=1}^{N} \tau_{ji}^{1-\sigma} \nu_i^{\sigma-1} \bar{A}_j^{\sigma-1} w_j^{1-\sigma}$$

where $W \equiv \left[\sum_j \nu_j^{1/\phi} \left(w_j/P_j\right)^{-1/\phi}\right]^{-\phi(\sigma-1)}$.

- (Practically) a linear system. Perron-Frobenius speaks to its solution
    - Unique positive solution. Notice 'eigenvalues' not guaranteed the same

Summary of GE Gravity Trade & Geography Models

- GE gravity trade (Anderson '79: solve for w_i, P_i)

$$w_i^{\sigma} = \sum_{j=1}^{N} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} L_j w_j P_j^{\sigma-1}$$

$$P_i^{1-\sigma} = \sum_{j=1}^{N} \tau_{ji}^{1-\sigma} A_j^{\sigma-1} w_j^{1-\sigma}$$

- GE geography (AA: welfare equalizes, solve for W, w_i, L_i)

$$W^{\sigma-1} w_i^{\sigma} L_i^{1-\psi(\sigma-1)} = \sum_{j=1}^{N} \tau_{ij}^{1-\sigma} \bar{A}_i^{\sigma-1} \nu_j^{\sigma-1} w_j^{\sigma} L_j^{1+\phi(\sigma-1)}$$

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and total population constraint $\sum_j L_j = \bar{L}$

Comparison: Kernel

- GE gravity trade (Anderson '79: solve for w_i, P_i)

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Comparison: Global Parameters

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and total population constraint $\sum_j L_j = \bar{L}$

Comparison: Eigenvalues

- GE gravity trade (Anderson '79: solve for w_i, P_i)

$$1\, w_i^{\sigma} = \sum_{j=1}^{N} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} L_j w_j P_j^{\sigma-1}$$

$$1\, P_i^{1-\sigma} = \sum_{j=1}^{N} \tau_{ji}^{1-\sigma} A_j^{\sigma-1} w_j^{1-\sigma}$$

- GE geography (AA: welfare equalizes, solve for W,w_i, L_i)

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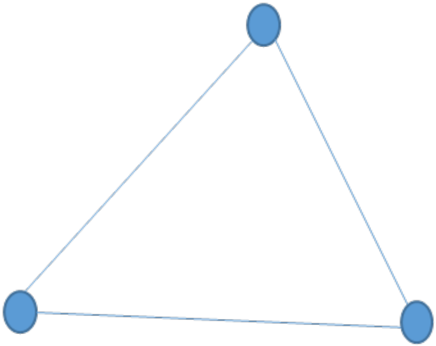
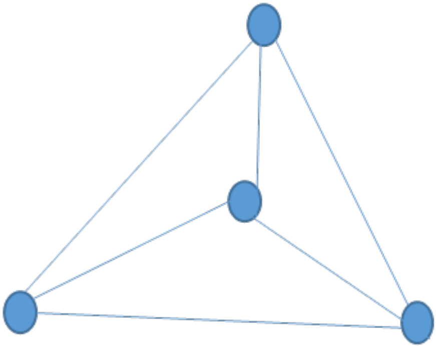
Visualization of the Spatial Links



Visualization of Adding Locations



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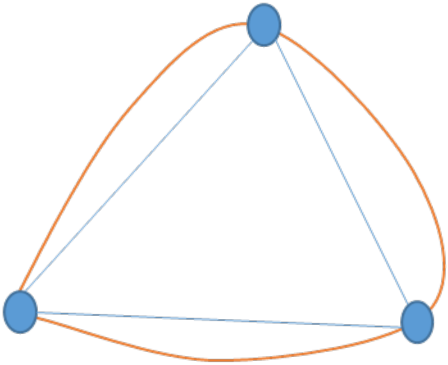
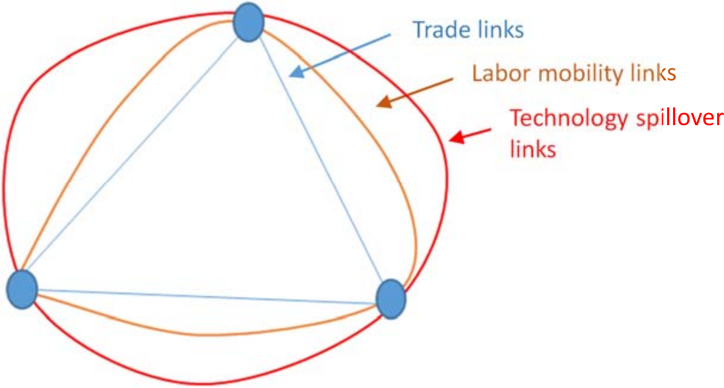


Visualization of Adding Spatial Links



A Generalized Gravity 'Model'

- Suppose equilibrium of a model reduces to a system of eqns where we denote locations (or sectors/location-sectors) with $i, j \in \{1, ..., N\}$, eqns with k, type of variable with h; $k, h \in \{1, ..., H\}$

$$\lambda^k \prod_{h=1}^{H} \left(x_i^h\right)^{\gamma_{kh}} = \sum_{j=1}^{N} K_{ij}^k \left[\prod_{h=1}^{H} \left(x_j^h\right)^{\beta_{kh}}\right] \tag{5}$$

  - Equilibrium variables x_i^h : # to be solved $H \times N$ (wage, price, labor etc)
  - Eigenvalue λ^k : Its role across models varies (typically welfare)
  - Kernel $K_{ij}^k \geq 0$: spatial links (trade/commuting costs, productivity decay etc)
  - Global parameters $\gamma_{kh}, \beta_{kh} \geq 0$:(EoS, Frechet elast., spillovers etc)

- $\mathbf{\Gamma} = \{\gamma_{kh}\}$, $\mathbf{B} = \{\beta_{kh}\}$ are the corresponding matrices

Theorem: Allen Arkolakis Li '15

Theorem

Consider the system of equations (5).
If $\mathbf{\Gamma}$ *is invertible then:*

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(ii) If $K_{ij}^{k} \geq 0$ *and the maximum of the eigenvalues of* $\mathbf{A}^p$*,* $\rho\left(\mathbf{A}^p\right) \leq 1$*, then there exists* **at most one** *strictly positive solution (up-to-scale)*
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Theorem: Allen Arkolakis Li '15

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(iv) If $K_{ij}^k > 0$ *and* $\rho\left(\mathbf{A}^p\right) < 1$ *the unique (up-to-scale) solution can be computed by a simple* **iterative** *procedure*

Application on Geography and Urban Model

- Note: Convenient conditions on global parameter vector not on Kernel
  - Can handle large dimensionality (many locations etc) like a charm

- The theorem is extremely powerful for economic geography model
  - In AA you can prove that equilibrium always exists; is unique if $\phi + \psi \leq 0$
  - With no trade costs, uniqueness holds under the same conditions

Roadmap

- A Simple Framework and the Unified Spatial Model
- Analytical Characterization of the Equilibrium
- Positive Properties and Computation of the Equilibrium
- **Comparative Statics**
- Welfare and Counterfactuals
- Applications

How Changes in Fundamentals ('Economic Shocks') Affect Markets?

- **Question:** Characterize comparative statics/*policy* elasticities

$$\epsilon_{ij}^{W} = \frac{d \ln W}{d \ln \tau_{ij}}, \ \epsilon_{ij}^{w_l} = \frac{d \ln w_l}{d \ln \tau_{ij}}$$

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- GE theory instills pessimism. Yet, we can obtain two results
  - Express policy elasticities solely in terms of *'deep'* elasticities, observed data
  - Characterize counterfactuals solely in terms of *deep* elasticities, observed data, and economic shocks
    - Characterization requires harnessing network effects in spatial models

Space and Comparative Statics

- Let us consider richer spatial interactions
  - We assume no trade cost but following, AAE

$$\hat{L}_i = \sum_j \phi_{ij} \hat{w}_j, \quad \hat{A}_i = \hat{\bar{A}}_i + \psi \hat{L}_i$$

  - Using (4) we obtain

$$-\sigma \hat{w}_i + \sum_j \phi_{ij} \hat{w}_j + (\sigma - 1)\, \psi \sum_j \phi_{ij} \hat{w}_j = (1 - \sigma)\, \hat{\eta}_i + d$$

  where d is a mixture of common GE terms and $\hat{\eta}_i \equiv \hat{\bar{A}}_i - \sum_o x_o \hat{A}_o$

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Welfare and Policy

- What about welfare?
  - We may distinguish the ex-post and ex-ante evaluation of a policy change
- **Ex-post:** Evaluate welfare after policy is implemented looking at the two equilibria
  - Robust 'macro' formula across trade geography models (Arkolakis, Costinot, Rodriguez-Clare '12)
    - Robust to changes in preferences, intermediate inputs/sectors, market structure (Costinot Rodriguez-Clare, ACDR, Midrigan Xu)
  - Ex-post welfare $d \ln W_j = -\frac{d \ln \lambda_{jj}}{\epsilon}$ (note: welfare equalizes in econ geography)
- **Ex-ante:** *Evaluate policy elasticity (counterfactuals)*

Welfare Counterfactuals

- CES-demand trade models simple derivative (Atkeson Burstein, Lai et al)
  - $\frac{d \ln W}{d \ln \tau_{ij}} = \frac{X_{ij}}{Y^W}$ (W here is expenditure weighted welfare, Y^W : world GDP)
- Much harder characterization in geography models because of eigenvalue
  - Need to use basics of perturbation theory (AA17)
    - If there is no spillovers ($\psi + \phi \neq 0$) we obtain the same result
    - With spillovers obtain a formula with an adjustment factor

Evaluating the Impact of Infrastructure Policies

- Now we can evaluate impact of real-world infrastructure policies
  - Consider a weighted graph with infrastructure matrix $T = \{t_{ij}\}$ denoting the cost of two connected points.
  - Bilateral trade costs τ_{ij} depends on t_{kl} on the realized path
    - e.g. $\tau_{ij} = t_{i1} \times t_{1k} \times ... \times t_{lj}$

- Example: Infrastructure Investment. Want to measure

$$\frac{d \ln W}{d \ln t_{ij}} = \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{d \ln W}{d \ln \tau_{kl}} \times \frac{d \ln \tau_{kl}}{d \ln t_{ij}}$$

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  2. Analytical characterization (but super cool): New AA

Applications

- Basically, hundreds of applications undertaken with this setup in trade.
  - New wave of applications in economic geography, urban (AA, Ahlfedlt et al '15, Monte et al, Redding 16, AAL15, Caliendo Parro Rossi-Hansberg '14, Faber Gaubert '15 etc)

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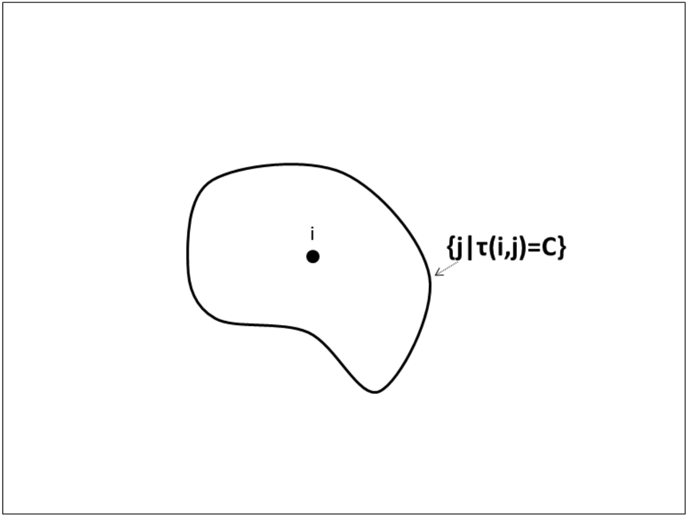
- Can we use this setup to think about trade cost/commuting costs etc?
  - Fast marching method (AA) ideally fit for the job (Generalization of Dijkstra for continuous space).

1. The Fast Marching Method for Spatial Economics



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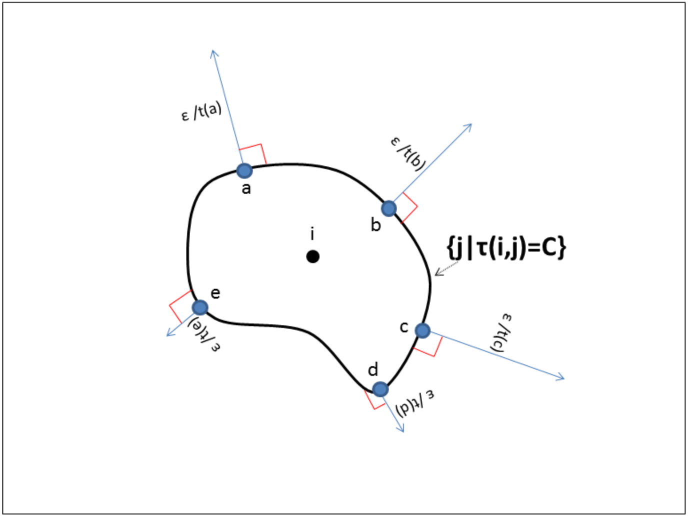

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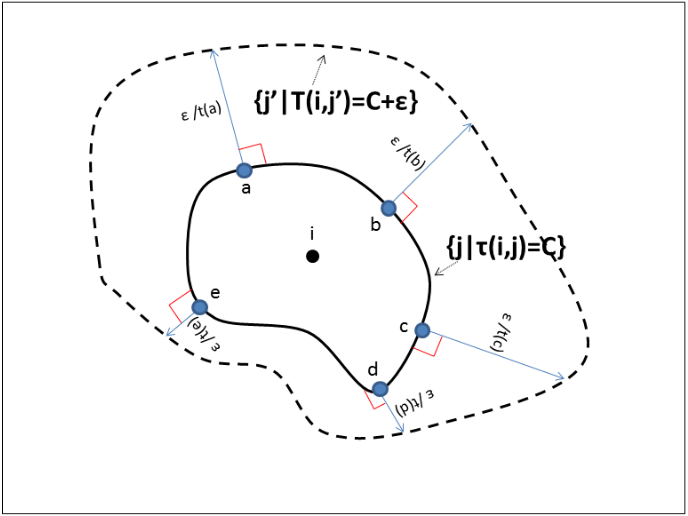

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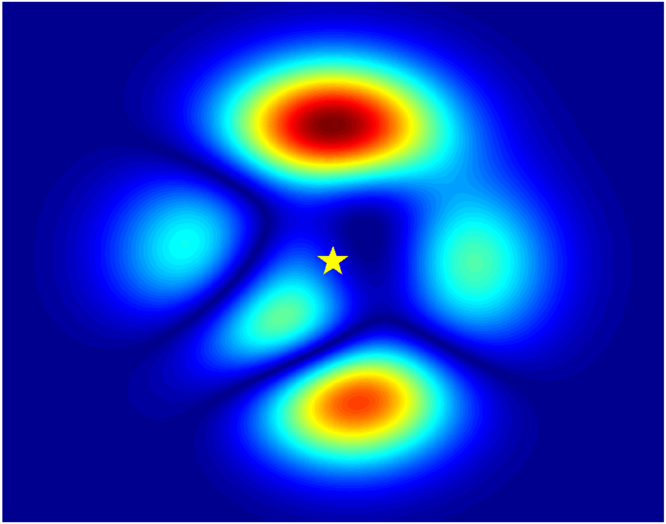

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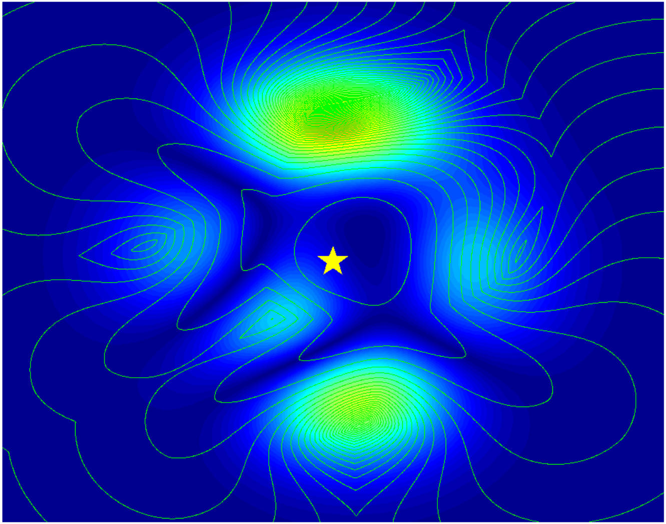
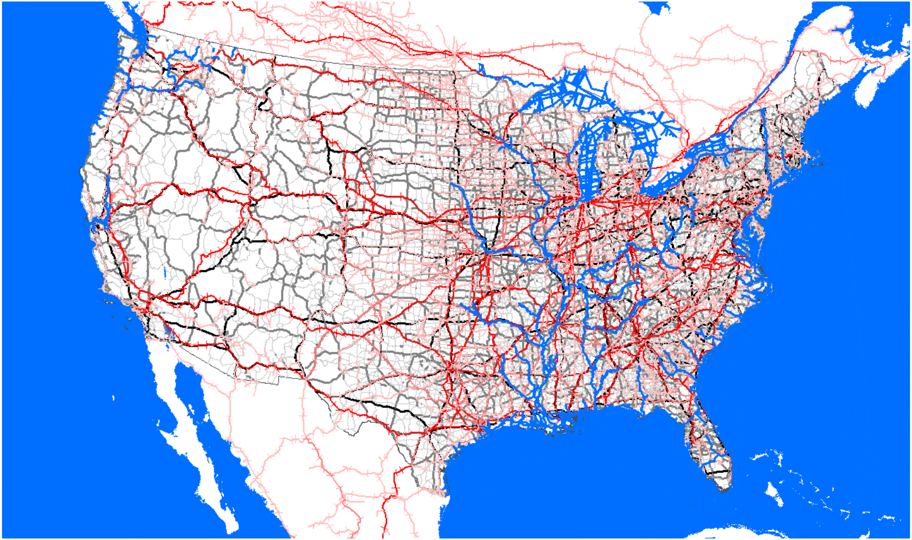
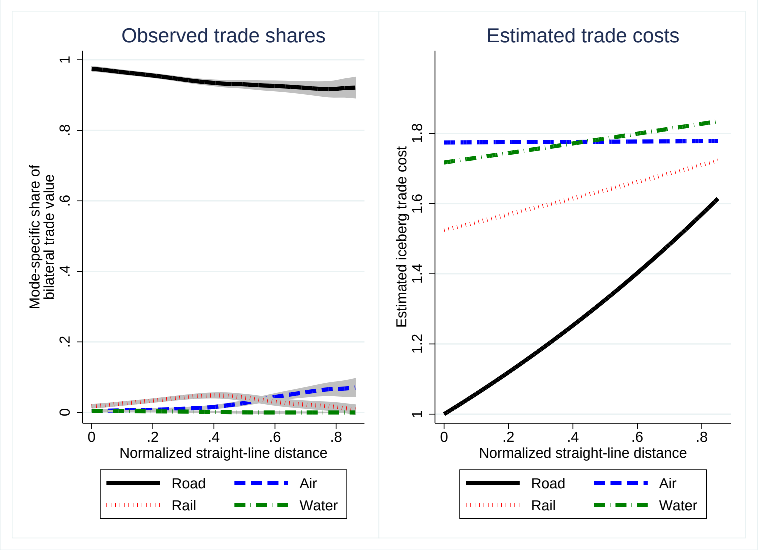


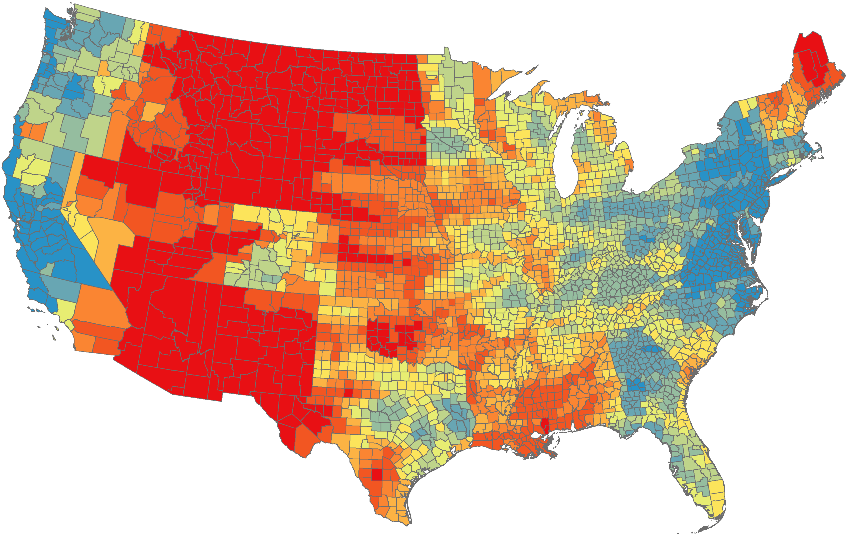
1. Trade costs with FMM: transportation networks



Estimating trade costs with FMM: mode-specific trade



Removing the IHS: Estimated increase in P



Removing the IHS: Cost-benefit analysis

Estimated annual cost of the IHS: $\approx$ \$100 billion

Annualized cost of construction: $\approx$ \$30 billion (\$560 billion @5%/year) (CBO, 1982)

Maintenance: $\approx$ \$70 billion (FHA, 2008)

Estimated annual gain of the IHS: $\approx$ \$150 – 200 billion

Welfare gain of IHS: $1.1 - 1.4\%$.
Given homothetic preferences and holding prices fixed, can multiply welfare gain by U.S. GDP.

Suggests gains from IHS substantially greater than costs.

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    - Tight connection to data
    - Many tools and methods to use!
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- There is unbounded demand for good theorists to work on spatial topics!