Dynamic Marriage Markets

Asefeh Salarinezhad
University of Ottawa

May 2, 2022

Abstract

The main focus of this paper is dynamic matching. I will consider a model to match agents on two sides of the market to each other while both sides have their preferences and the market environment is dynamic. That means the market is open for more than one period, new agents at the beginning of each period enter the market and the matched ones leave it. Each agent accepts a subset of the agents on the other side and she has an individual preference ranking over the agents on the other side who are in her acceptance set.

The main objective is to find an algorithm which considering the dynamic feature of the model, finds the matchings which are optimal for both sides if there exists such a matching and if not finds matchings which are fair for both sides (it does not favour any side). I introduce a new algorithm which is based on the DA (Deferred Acceptance) algorithm and its structure provides the opportunity to find the two-sided optimal matchings considering requirements and characteristics of dynamic environment. I present my model based on the marriage problem (couple match making) but it can be used in other similar matching markets. Compared to existing algorithms for dealing with marriage problem in static or dynamic environment, my algorithm is more realistic since it includes the real life marriage considerations and it is based on less unrealistic assumptions. Furthermore, it is more integrated regarding the optimality of two sides and avoids some of the issues of already existing algorithms like using a pre-defined list of agents for proposing procedure.

My algorithm, DM (Dynamic Marriage), considers a Marriage Market in a dynamic environment. The structure of the algorithm allows both sides to make offers simultaneously and selects a matching which is optimal for both sides in a defined dynamic structure if such a matching exists. Otherwise the algorithm finds a matching which lies somewhere in between the two sides optimal without favouring any side. This property makes the matching fair since it gives both sides a fair chance (fairness is a very important concept to be considered in marriage market, static or dynamic.) That is why my algorithm deals with the two sides offering alongside with the dynamic aspect of the model. The novelty of my paper is that my rule allows both sides to make offers simultaneously in a dynamic setting.

Within my framework, I also study the dynamic strategy-proofness of the algorithm and its fairness and stability. Furthermore, I will discuss finding the maximum matching in each period.

1 Introduction

While the majority of the matching literature deals with the static matching problems, there are lots of situations where the matching markets are actually operating in a dynamic environment. For instance, in a kidney exchange market, at any time, there are new patients entering the patients' list on one side of the market. On the other side, new kidneys may be available while the matched patients and kidneys leave the market. There is a similar situation in matching teachers (a new period begins at the beginning of a school year) or medical staff to positions. Another ongoing situation is the refugee problem which also can be considered a dynamic matching problem. Furthermore, dynamic matching could be easily applied to marriage markets. New men and women enter the matching market at the beginning of each period and they leave after finding their match. The purpose of this paper is to define a matching rule with desirable properties when the agents on both sides of the market have preferences over the other side in a dynamic environment.

Furthermore, it considers how each agent can select some of the agents on the other side as her special options by giving them extra waiting time. While the agent's preferences only shows the ranking of the other side agents for an agent, my model defines a way to reflect the fact that some options are much more preferred to others. This aspect rarely, if ever, has been discussed in the literature while it is highly realistic especially in a marriage market. Assume one agent is so important for another agent that not only has he ranked her as his top choice but also he is willing to risk being unmatched for a while and wait for her. In marriage market and many other similar matching problems, people want to be able to wait if they think somebody or something is worth it. The structure of my model and the dynamic environment of having more than one period make the waiting possible. It is also realistic to assume that each agent only accepts a subset of available agents. In addition, we cannot assume that an agent stays in the market forever after entering it if she does not find her match. Since new agents enter the market in each period and the situation changes over time, especially in a marriage market, agents must be able to update their status at the beginning of the new period if they are unmatched. They need to have some flexibility to change their mind over time about some of the previously declared statuses, such as staying in or out of the market or expanding the acceptance sets, as it definitely happens in real cases.

I propose an innovative new matching rule which finds a stable matching in a dynamic environment with realistic features of the model. This stable matching at the same time is more fair for both genders (two sides of the market) since it does not favour either side, minimizes the unrealistic assumptions and allows for both sides optimal matching outcomes. If a two-sided optimal matching does not exist, it aims for the best compromises possible to increase fairness in the sense of fair opportunities[1]. I also study the strategy-proofness of my algorithm and address the issue of maximum cardinality.

My intention is to make the model as realistic as possible which makes it somewhat complicated. However, for preserving the real aspects of the model, some complications are inevitable. I characterize the dynamic specifics of the model, and study real life considerations in marriage markets. This constitutes a substantial improvement over current results in the sense that my model leads to a realistic matching in marriage markets and avoids some constraints or unrealistic assumptions. Some already existing algorithms take the set of available agents on one side (Pereyra, 2013 [13] and Liu, 2020 [12]) or both sides (Kurino, 2020 [10]) fixed and only define multiple periods as the dynamic aspect of their model. Others fix the number of periods (Du and Livne, 2014 [6]) or agents' preferences (Pereyra, 2013 [13]). There are models which assume that everybody knows who will join the market in the future (Doval, 2020 [5]) or all agents can change their match in each period (Kurino, 2020 [10], Kotowski, 2019 [9] and Damiano and Lam, 2005 [4]). None of these assumptions are realistic for a marriage market and my model makes more realistic assumptions. Regarding the two-sided offering algorithms, my algorithm avoids complicated aspects of some existing algorithms such as making many offers by one agent at the same time (Kuvalekar, 2014 [11]). It also modifies some issues of other algorithms, since it lets the two sides make offers at the same time instead of only one side (Gale and Shapley, 1962 [16]) or one-by-one offering algorithms which are based on a pre-defined list of agents (Romero-Medina, 2005 [14] and Dworczak, 2021 [7]). If there is no two-sided optimal matching, my model does not favour one side nor uses other methods such as allowing men and women to propose alternatively or finding a median stable matching (Teo and Sethuraman, 1998 [17]). Instead it provides the possibility to compromise and choose a matching which is not optimal for either side and thus it is more fair because it gives both sides a fair chance. I also provide some ways to increase the number of matched pairs in each period. These methods make it possible for practical matching designers whose objective is maximizing the number of matches to get matchings with higher number of matches.

I introduce new dynamic concepts and prove that my matching rule is dynamically strategy-proof and stable. I also discuss dynamic optimality and show that my algorithm is dynamically Pareto-optimal.

[1]The meaning of fairness in this study is more like the common day to day meaning rather than fairness in matching theory.

This paper aims to provide a foundation for future research which will generalize the newly designed algorithm to other matching markets and extend my findings to more complex cases.

The structure of this chapter is as follows: In Section 3 I discuss the related literature. In Section 4 I introduce the model and its properties. In Section 5 I review some matching properties related to this study. Section 6 is dedicated to my new algorithm, Dynamic Marriage (DM) algorithm and new definitions related to it. In Section 7 I discuss the findings of my study and present theorems. In Section 8 I provide some ways to increase the number of matches in each period. Finally I conclude my study in Section 9.

2 Literature Review

In real life markets we can easily find lots of situations with multiple periods or repeated matching problems. Nevertheless, these kinds of markets have only started to receive some attention recently. Dynamic matching is generally a new topic in the matching literature and there are only a few papers on it. Furthermore, although the characteristics of these markets are different, all of them must have a main common feature; the next or previous periods have influence on the present period's matches. Otherwise they could be modeled as separate single period matchings.

One of the most recent papers on dynamic matching is "Stability in Repeated Matching Markets" by Liu (2020, [12]). He considers a fixed set of agents (hospitals) and calls them long-lived players. Hospitals are matched to a new generation of agents, short-lived players (medical students), in every period. In his setting, to make more students available for rural hospitals, urban hospitals need to decrease their hiring capacity. In this case, if a hospital does not respect the recommendation from the matching clearing house, it will be prohibited from future participation in the market. Therefore, the dynamic feature of the model has been used as a motivation or punishment tool to enforce more stable outcomes.

In another work, Arnosti and Shi (2020, [2]) consider a continuum of agents and a new object which must be immediately assigned after entering the market. They defined different types of lotteries and compared the results using different types of waiting lists in term of welfare and some other properties.

Kurino (2020, [10]) studies a dynamic marriage model. In his model there is no entry and exit of agents (women and men) who can change their partners at each period. In each period t, each agent has a utility function which defines the preferred outcome for her. He defines dynamic group stability for a matching μ when there is no group deviation: there is no group of agents who are better off by deviating from μ and choosing another matching μ'. He also assumes that when a group of agents deviate from a matching μ, the agents inside this group can only be matched with each other and the ones outside the group whose ex-partner is inside the group became unmatched.

There are three other recent studies on the dynamic context. The first one is "A Perfectly Robust Approach to Multiperiod Matching Problems" by Kotowski (2019, [9]). Kotowski's model lets agent change their assignment over multiple periods. Each agent has preferences over a sequence of assignments over time. The history and future of assignments can affect the current period preferences. The second one is "Static versus dynamic deferred acceptance in school choice: Theory and experiment" by Klijn, Pais and Vorsatz (2019, [8]). Their work compares the static and a dynamic student proposing DA by using an experiment. The last one is "Dynamically Stable Matching" by Doval (2019, [5]). She defines a dynamic stability concept based on the assumption that only agents available in the same period are allowed to form a blocking pair. Furthermore, she assumes that preferences are common knowledge and that every agent knows who will be available in the market in future periods.

Andersson et al. (2018, [1]) studies dynamic refugee matching markets. They propose a specific matching mechanism, Dynamic Order Mechanisms and show that any matching selected by this mechanism is Pareto-efficient and satisfies envy bounded by a single asylum seeker. In other words, envy between localities (such as states) is bounded by a single asylum seeker, i.e., whenever some locality m envies some locality m', the envy can be "eliminated" by removing a single asylum seeker either from the set of asylum seekers matched to locality m or from the ones matched to locality m'.

Bloch and Cantala (2017, [3]) worked on assigning objects to queuing agents in a dynamic context. They considered a constant size waiting list of agents while the objects arrive over time. Whenever a new object is available it is offered to the agents in the waiting list starting with the agent at the top of the list. If she rejects the object, it is offered to the next agent in the fixed sequence and if the object is rejected by all the agents, it will be wasted. They showed that using a lottery to offer the object decreases and even minimizes the waste while with both private and common values all agents prefer first-come first-served to lottery.

Du and Livne (2014, [6]) studies a two-period matching market. In their model, agents can decide to be matched in period 1 and leave the market or wait until period 2 when new agents enter the market. They showed that there is a stable matching for agents who are present in period 2. They also proved under some restrictive assumptions (such as a large number of agents in period 1 and a small number of new arrivals), that on average at least one quarter of all agents present in period 1 prefer to be matched before period 2, provided that they anticipate others are going to wait until the next period.

Pereyra (2013, [13]) studies "A dynamic school choice model". In his model the DA algorithm matches teachers to schools. From one period to another, they can remain in their current positions or apply for a more preferred one. In order to overcome the issue of respecting the teachers' improvement, he has moved each teacher who has been assigned to a school in the previous period to the top of the schools' priority list (which was originally based on teachers' grades from an evaluating test). This made the process manipulable. Therefore, he assumes that teachers cannot change their preferences from one period to another. He also assumes that the school positions are fixed, so actually one side of the market is not changing.

Damiano and Lam (2005, [4]) have presented "Stability in Dynamic Matching Markets". In their paper, they assume that each agent can be rematched at the end of each period and the payoff of the first period for each agent is equal to the sum of that period's payoff and the discounted payoffs of the next periods.

In addition, there are some studies that concentrate on two-sided offering. One of the early ones is Teo and Sethuraman (1998, [17]). They showed that when the total number of distinct stable marriage solutions, l, is odd, there is a stable marriage solution in which every person is assigned to a partner who is the "median" partner among all their possible mates. Their study shows a way to find a matching which compromises between the men-optimal and women-optimal matchings. It is neither men-optimal nor women-optimal, but more fair since it does not favour any side.

Romero-Medina (2005, [14]) introduced an algorithm, called the Equitable Algorithm which uses a fixed ordering of agents to compromise between both sides' ideal matchings. In his algorithm, agents propose based on a fixed ordering. At step k each person who receives the proposal accepts it if the offer is among her k best choices and/or is better than the proposal she has accepted in previous steps. In case of a rejection, the rejected agent proposes to his second most preferred choice among her k first choices.

Dworczak (2021, [7]) presents a class of algorithms, called $DACC$ (Deferred Acceptance with Compensation Chains) in which both sides of the market are allowed to make offers in an arbitrary order. Agents make their offers one at a time according to a pre-defined arbitrary order. Based on his work, when all agents are allowed to propose, it is possible that an agent rejects an offer from another agent from the opposite side but proposes to her later on. As a consequence, the agent might withdraw an offer he made to another agent. He uses a compensative system in a way that whenever some i deceives j, he compensates agent j by letting her make an offer in the current round irrespective of the pre-defined order.

Kuvalekar (2014, [10]) introduces an algorithm where both sides make proposals in each round to a set of their top agents. In each round k, agents expand their acceptable sets according to their preferences to their k top-ranked agents. The agents can only be matched when they both list each other as mutually acceptable. If the agent is matched, she proposes to only those agents that are better than her current match. Romero-Medina and Kuvalekar later completed this study (2021, [15]).

3 The Model

It is a model to match agents to each other on two sides of a dynamic market. In my model, the market runs for unlimited periods and at the beginning of each period new agents enter the market while the matched agents leave the market at the end of each period. Only the list of available agents on both sides of the market at the beginning of each period is public knowledge. Each agent accepts a subset of available agents on the other side and has preferences over the agents in her acceptance set. However, if one agent stays unmatched at the end of one period, her acceptance set may expand in the next period. It is realistic to assume that agents expand their acceptance set in fear of staying unmatched. Nevertheless, no agent can kick out of her acceptance set an unmatched agent from the previous period. Although the agents' preferences change at the beginning of each period, given arriving new agents on the other side or their expanding of the acceptance set, the preference ordering over the unmatched agents from the previous period is the same. Furthermore, if any agent adds other agents already present in the previous period to her acceptance set, she should rank them lower than the agents who were acceptable in the previous period. Each agent can only be assigned to one agent. These assumptions are all realistic in marriage markets.

Formally, the marriage market is given by $(M, W, T, L, \succ_M, \succ_W, POr)$.

It consists of:

- $M^t = \{m_1, m_2, \ldots, m_{n^t}\}$ is a set of n^t men on side 1 [2] at the beginning of period t.
- $W^t = \{w_1, w_2, \ldots, w_{m^t}\}$ is a set of m^t women on the other side at the beginning of period t.
- ${M_i}^t = \{w_j \in W^t\colon\ w_j \succ_{m_i} \emptyset\}$ is the set of acceptable women for man m_i in period t, where $i \in \{1, 2, \ldots, n^t\}$ and $j \in \{1, 2, \ldots, m^t\}$.
- ${W_j}^t = \{m_i \in M^t\colon\ m_i \succ_{w_j} \emptyset\}$ is the set of acceptable men for woman w_j in period t, where $i \in \{1, 2, \ldots, n^t\}$ and $j \in \{1, 2, \ldots, m^t\}$.
- T discrete periods of running time.

[2]Through this chapter, I consider side 1 as men and refer to a member of this side (a man) as a he and side 2 as women and refer to its members (women) as a she.

- $t \in T$ is a number assigned to each period, i.e., $t = 1$ is the first period of the market operation.
- $T_{m_i} \in T$ is the total number of consecutive periods that agent m_i is present in the market.
- $t_{m_i} \in T_{m_i}$ is the number of each period for agent m_i after joining the market; $t_{m_i} = 1$ is the first period that m_i enters the market and it increases as he moves unmatched to the next periods.

- L is the loyalty set.
  - ${L_{m_i}}^t = \{\{w_j : w_j \in {M_i}^t\} \subset {M_i}^t\}$ is the loyalty set of man m_i at the beginning of period t.
  - ${L_{w_j}}^t = \{\{m_i : m_i \in {W_j}^t\} \subset {W_j}^t\}$ is the loyalty set of woman w_j at the beginning of period t..

  A loyalty set consists of subsets of consecutive top members of an agent's preference ordering whom she/he is loyal to. That means the agent is ready to wait for them until the next period, provided that they are unmatched too.

- Each agent m_i has a strict preference relation $\succ_{m_i}$ over ${M_i}^t$.
- Each agent w_j has a strict preference relation $\succ_{w_j}$ over ${W_j}^t$.
- POr^t (Proposing Order[3]) is a dynamic ordering of all agents at the beginning of period t which will be used in case of a halt point[4]. It changes from one period to the other and also will be updated whenever it is going to be used based on the current situation of agents.[5]

Agents' preferences, acceptance and loyalty sets are private knowledge. Nobody knows who will enter the market in the future. At the beginning of each period, only the list of available agents on both sides of the market is public knowledge.

The main dynamic aspects of my model are:

1. Time: The market runs for an infinite number of periods.
2. Agents: At the beginning of each period new agents enter the market and matched ones leave it.
3. Preferences/Acceptance sets: The preferences/acceptance sets of agents can change/expand at each period due to arriving new agents and due to the fear of remaining unmatched.

[3]I use POr as an abbreviation for proposing order instead of PO to avoid the confusion with Pareto-Optimal (PO).
[4]No rejection, no possible match.
[5]I explain in detail how POr is updated in Section 6.

4. Loyalty sets: Agents have the possibility to wait for the agents on the other side whom they value much higher than others.

5. POr^t: Not only will POr^t change at the beginning of each period but also it will be updated at any time based on the current situation of the agents.

4 Properties of Matching Rules

The properties of matching rules that will be used in this chapter are introduced in this section. All of these properties are standard properties that have been studied extensively in various models.

Property 1: A **matching** μ is **individually rational** if all agents prefer their current mate under μ to being unmatched.

An **agent** m_i is **individually rational** if he prefers his current mate under μ to being unmatched.

Property 2: A matching μ is **stable** if there is no blocking pair, (m_i, w_j) who prefer each other to their current mate under μ, and each agent is individually rational.

Property 3: Agent $m_i \in M^t$ justifiably envies agent $m_{i'} \in M^t$ who is matched to agent $w_j \in W^t$ under an assignment μ if $w_j \succ_{m_i} \mu(m_i)$ and $m_i \succ_{w_j} m_{i'}$. Therefore, (m_i, w_j) is a blocking pair in period t.

Property 4: A matching μ is **fair** if there is no agent who justifiably envies another agent.

Property 5: A matching μ is **men/women-optimal** if it is the best matching for all men/women among all stable matchings.

Property 6: A matching rule f is **strategy-proof** if there do not exist $m_i, \succ$ and $\hat{\succ}_{m_i}$ such that $f_{m_i}(\hat{\succ}_{m_i}, \succ_{-m_i}) \succ_{m_i} f_{m_i}(\succ)$.

Property 7: A **matching** μ is **Pareto-optimal** if it is not Pareto-dominated in the sense that there is another matching which makes none of the agents worse off and at least makes one agent better off.

A **matching rule** f is **Pareto-optimal** if it specifies a Pareto-optimal matching for each preference profile.

Property 8: A matching μ is **maximum (cardinality)** if there is no other matching μ' such that $|\mu'| > |\mu|$.

5 Dynamic Marriage (DM) Algorithm

The agents' preferences, loyalty and acceptance sets are private information. Only the list of available agents at the beginning of each period is common knowledge.

Before introducing my algorithm, I provide some new definitions which will be needed in the algorithm's process:

Definition 1: Loyalty Set (L)

$L_{m_i} = \{\{w_j, \ldots\} \subset {M_i}^t\}$ is called the loyalty set of agent m_i if it includes subsets of consecutive top members of m_i's preference ordering whom m_i is loyal to and is ready to wait for until the next period, provided that they are unmatched too.

Each member of m_i's loyalty set is a subset of his acceptance set. The first member should start from the top agent in his preference ordering and it is not possible to skip agents.

Definition 2: Loyalty Loop (LL)

If a member of L_{m_i} consists of more than one agent, I call it a loyalty loop (LL_{m_i}).

$LL_{m_i} \in L_{m_i} = \{w_j, w_{j'}, w_{j''}, \ldots\}$ means that m_i is willing to wait for the most preferred member of this loop (let's say w_j) until the next period if w_j is also unmatched and m_i is not matched to the next agents in the loop ($w_{j'}, w_{j''}, \cdots$).

Therefore, loyalty loops are non-singleton members of loyalty sets.

Definition 3: Loyalty Loop Agents

For every $LL_{m_i} = \{w_j, w'_j, \cdots\}$, the agents $w_j, w'_j, \cdots$ are called Loyalty Loop Agents.

If $L_{m_i} = \{\{w_j\}\}$, the loyalty set of m_i includes only one subset of his acceptance set, M_i. This subset itself includes only one agent, w_j. The meaning of this loyalty set is that whenever m_i proposes to w_j, if w_j does not accept him, he will wait for her until the next period in the hope that she will include him in her acceptance set in the coming period (provided that w_j remains unmatched too).

If $L_{m_i} = \{\{w_j, w'_j, \cdots\}\}$, the loyalty set of m_i includes only one subset of his acceptance set, M_i. This subset itself includes more than one agent and should start from the top of m_i's preference ordering. This means that w_j is the top-ranked agent for m_i and $w_j, w_{j'}, \cdots$ are consecutive agents on m_i's preference ordering. Such a member of the loyalty set is called a loyalty loop. $LL_{m_i \in L_{m_i}} = \{w_j, w_j', \cdots\}$ means whenever m_i offers to w_j and w_j does not accept him, he will not wait for her but offers to the next agent in the loop, w_j'. If she rejects or does not accept him, then he goes to the next in the loop without waiting for w_j' until he gets rejected or not being accepted by all the agents in the loop, then he goes back to the top of the loop and reoffers (one offer per round) to the first agent who has not rejected him (the agent who does not accept him and he prefers her to all the other loop agents who have not rejected him) and waits for her until the next period. An agent may have more than one LL.

Let me clarify the definition of the loyalty set and loyalty loop and their difference with a simple example:

Example 1.

Assume that m_i has the following preference ordering over a set of five available women all of whom he accepts:

$w_2 \succ_{m_i} w_4 \succ_{m_i} w_5 \succ_{m_i} w_1 \succ_{m_i} w_3$

Let us assume that $L_{m_i} = \{\{w_2\}, \{w_4, w_5, w_1\}\}$ is the loyalty set of m_i. First of all, agents w_2, w_4, w_5 and w_1 are consecutive top members of m_i's preference ordering. So, as mentioned, all agents in m_i's loyalty set must be top consecutive members of his acceptance set starting with the top one who is w_2 in this case. m_i's loyalty set consists of two members, $\{w_2\}$ and $\{w_4, w_5, w_1\}$, which are subsets of her acceptance set. Furthermore, $\{w_4, w_5, w_1\}$ is a loyalty loop since it consists of more than one agent; $LL_{m_i} = \{w_4, w_5, w_1\}$.

When the algorithm starts, m_i proposes to his top-ranked woman, w_2. If w_2 rejects him in favour of a more preferred man, he will propose to his next most preferred woman, w_4. But if w_2 does not accept him simply because he is not in her acceptance set, then he will wait for her until the next period, hoping that w_2 will include him in her acceptance set in the next period. Admittedly, if w_2 does not include m_i in her acceptance set in period 1, he will wait for her until the next period provided that w_2 is unmatched too (she has not received any offer from her acceptable men) in hope of finding his way to her acceptance set. However, if w_2 receives offers from her acceptable men, she will reject m_i. Now let us say w_2 has m_i in her acceptance set but she receives a better offer in period 1. Then she rejects m_i in favour of the more preferred man. Then m_i will propose to his next preferred woman, w_4. If w_4 rejects him, he will propose to w_5. He also will propose to w_5 if he is not acceptable to w_4 (he does not wait for w_4). If he is not in w_5's acceptance set or w_5 rejects him in favour of a more preferred man, then he will not wait for her either and will propose to w_1 who is the last member of his loyalty loop. If w_1 does not accept him or rejects him then he will go back to the top of her loyalty loop, w_4. If w_4 has not rejected him before (otherwise he goes to the next one in the loop who has not rejected him yet) and will wait for her until the next period, provided that w_4 is still unmatched. The difference between $\{w_2\}$ and $\{w_4, w_5, w_1\}$ is that m_i moves to the next subset (here from $\{w_2\}$ to $\{w_4, w_5, w_1\}$) only if he has been rejected by all members of the subset, but inside the loyalty loop he moves between agents w_4, w_5 and w_1 if he is rejected or not acceptable. If w_4 rejects him, he is not in w_5's acceptance set, and w_1 rejects him too, then he will wait for the top member of the loop who has not rejected him, w_5. ▲

Remark: As mentioned before, if one agent stays unmatched at the end of one period, her acceptance set may expand in the next period (she cannot exclude any unmatched agents who were in her acceptable set in the previous period). The agents' preferences change at the beginning of each period, given arriving new agents on the other side or expanding the acceptance set. But their preferences over unmatched agents from the previous period do not change. If any agent adds other agents already present in the previous period to her acceptance set, she should rank them lower than the other agents in the previous period who were acceptable.

The next four definitions, 4 to 7, are needed to define the POr list, which is given in Definition 8.

Definition 4: Super Loyal Agent

An agent m_i is called super loyal to agent w_j in step t if $w_j \in L_{m_i}$ for all the periods that:

1. *w_j has been in the market such that $T_{w_j} > 1$ and;*
2. *m_i has been in the market such that $T_{m_i} > 1$.*

Agent m_i is super loyal to agent w_j if he includes her in his loyalty set in every period in which they are both in the market, provided that both of them are in the market for more than one period.

Definition 5: $TD_{m_i}^{t_{m_i}}$ (Tolerance Degree)

At period t, $TD_{m_i}^{t_{m_i}}$ is the number of agents in $L_{m_i}^{t_{m_i}}$.

In any period t, the Tolerance Degree of agent m_i is the number of agents from the other side of the market that he includes in his loyalty set and is willing to wait for until the next period if they are unmatched too.

Definition 6: $TTD_{m_i}^{t_{m_i}}$ (Total Tolerance Degree)

$TTD_{m_i}^{t_{m_i}} = \sum_1^{t_{m_i}} TD_{m_i}^{t_{m_i}}$.

The Total Tolerance Degree for agent m_i is the sum of all TDs for all the periods he has been in the market.

Definition 7: Waiting agent

Agent m_i is called a waiting agent if he has made an offer to one of the agents in his loyalty set and has been waiting for her to accept him or reject him in favour of a more preferred agent in subsequent periods.

Definition 8: POr^t (Proposing Order)

POr^t is a dynamic ordering of all available agents at the beginning of period t based on the number of periods they have been in the market and their TTD and TD, but it will be updated whenever it is going to be used.

- *Original POr^t: Generally at the beginning of period t, in the original POr[6]:*
  - *The most present agent in the market (agent m_i who has the highest t_{m_i}) goes to the top of the list and the more recent agents go after her/him from the older to the new comers.*
  - *In case of a tie, the one with the highest TTD comes first, if they have the same TTD, the one with the highest TD for that period goes first.*
  - *If there is still a tie then we need to use an arbitrary tie-breaker.*
- *Updated POr^t: Since the nature of my model is dynamic, POr is dynamic too. That means that not only does POr change at the beginning of each period but also whenever we want to use it we need to update it based on the current situation of all remaining agents:*
  - *If the top agent of POr is a waiting agent, then the agent who has proposed to her jumps before her on the list.*
  - *If there is no such an agent then the agent who is after the waiting agent, will go first.*
  - *The new waiting agents go before the ones transferred from previous periods.*

[6] Original POr^t is the POr defined at the beginning of each period t based on the seniority, TTD and TD.

– *An agent who is waiting on a super loyal offer which is an offer made by a super loyal agent to the one she/he is super loyal to, goes after agents who are waiting on a loyal offer which is an offer made by a loyal agent to the one she/he is loyal to.*

Remark: In case of a halt point at period t (no rejection, new offer or matching is possible), we take the original POr^t as a reference. The agent at the top of the POr^t is the first one who has to change her/his proposal and make a new offer to her/his next preferred agent unless she/he is waiting on a reactivated offer from the previous period or waiting on a new offer she/he has made to one of the agents in her/his loyalty set. In this case the POr will be updated and the agent who has made an offer to the waiting agent will jump in front of her/him. If there is no such an agent then simply the one who is after the waiting agent in the original POr will jump before her/him. Note that as mentioned in the POr definition, the new waiting agents go before the ones transferred from previous periods. That is because if an agent has been waiting for the other one until the next period specially for more than one period, it is not fair to make her/him deviate from that offer. In addition, an agent who is waiting on a super loyal offer goes after agents who are waiting on a loyal offer.

At the beginning of each period, agents report their loyalty set, based on their acceptable agents in the same way that they report their preferences. The loyalty sets are private knowledge and could be completely different from the previous period. Agents have the flexibility to change their mind regarding whether they want to wait for somebody any more. This is a realistic assumption since people naturally become inpatient when period after period they remain unmatched, and also because of new arrivals who might be more interesting. That is the way a human mind normally works.

5.1 Dynamic Marriage (DM) Algorithm

1. Agents on both sides of the market propose at the same time to their most preferred agent.

2. Agents receiving more than one offer keep the most preferred one and reject others.

3. If $w_j \in {M_i}^t$ and $m_i \in {W_j}^t$, when w_j rejects m_i, the rejected agent makes an offer to his next preferred agent based on his preference ordering, since he has been rejected in favour of a more preferred agent and he has no chance with w_j.

4. If $w_j \in {M_i}^t$ but $m_i \notin {W_j}^t$, whenever m_i proposes to w_j;

   (a) If $w_j \notin L_{m_i}$ then m_i will be automatically rejected by w_j and he will propose to his next preferred agent on his preference ordering.

(b) If $w_j \in L_{m_i}$ then m_i's offer will stay active until w_j rejects him in favour of another agent. Then m_i will propose to his next preferred agent.

(c) If $w_j \in L_{m_i}$ and w_j does not reject m_i in favour of another agent, it means that w_j has not received any offer from one of her acceptable agents and she will remain unmatched at the end of this period together with m_i who wants to wait for her until the next period, hoping that she will include him in her acceptance set in the next period. So, m_i will not make an offer to his next preferred agent at this period and his offer to w_j will stay valid until the next period.

(d) If $w_j \in LL_{m_i \in L_{m_i}}$ and w_j does not reject m_i in favour of another agent, it means w_j has not received any offer from one of her acceptable agents and she will remain unmatched at the end of this period. But m_i will not wait for her unless he gets rejected or is unacceptable by other members of the loop. In that case m_i will be unmatched too and waits for her until the next period provided that w_j is the most preferred mate in m_i's loyalty loop who has not rejected him.

5. If m_i enters a new period with a passive offer to w_j, all new arrivals and new added agents to m_i's acceptance set must be ranked under w_j, since she is an important agent who is worth waiting for. Agents can update their loyalty set at each period. As mentioned before, all members of loyalty set must be the top consecutive agents of m_i's acceptance set. Here there is a special case which must be considered carefully.

   Let us consider a case when m_i has been waiting on a passive offer to w_j moved to period $t+1$ from period t. This means m_i was not acceptable to w_j at period t but w_j was in loyalty set of m_i, then m_i waited for her until $t+1$. Now at $t+1$ he does not want to include her in his loyalty set again simply because if w_j does not include him in her acceptance set in $t+1$ or if does and then rejects him in favour of a more preferred man, then m_i will not want to wait for her for another period and will want to propose to his next preferred woman. Then assume some new agents enter the market and m_i wants to add them to his acceptance set. As mentioned before, they must be ranked under w_j because m_i is waiting on a passive offer to w_j, actually, w_j is the top agent in m_i preference ordering. While m_i does not want to wait for w_j any more (if w_j does not accept him or if she rejects him), he wants to put other women of his acceptance set in his loyalty set without including w_j, the rule that says a loyalty set must start with the top member of preference ordering (which is w_j here) will be violated. To cover this case, I add a condition to the loyalty set definition as follows:

   The agents in m_i's loyalty set are the top consecutive members of his preference ordering. This means that m_i's loyalty set must start from his most preferred women based on his preference ordering unless he is *waiting on a passive offer to this woman. In this case, if he does not want to include her in his loyalty set for another period, then the loyalty set will start from his second preferred woman according to his preference ordering.*

6. Agent m_i will be matched to w_j whenever they both have a proposal from the other one.

7. In case of a halt point (there are no mutual proposals and no rejection or new offer), based on the POr^t list, agents change their offers and propose to their next preferred agent. If the agent at the top of POr who needs to withdraw her current proposal and propose to her next preferred mate does not have any agent remained in her acceptance set and she is not waiting on any loyal proposal, she has to withdraw her current proposal and stay unmatched until the next period.

Important remarks:

1. Each agent who is matched is out of the market for good so they do not risk to end up with a less preferred match. If an agent comes back to the market she is considered a new agent.

2. Using DM algorithm we will automatically have the matching which is optimal for both sides if such a matching exists. But this two-sided optimal matching is different from the outcome of the DA algorithm. The difference comes from the fact that DM algorithm operates in a dynamic environment and allows agents to wait for other agents through the periods. It also allows both sides to propose at the same time.

3. In case of a halt point, each side which the top agent of POr belongs to will lose its optimality. If after the top agent in POr has changed her/his offer, still there is a halt point and the next agent in POr is from the other side, we will end up with mixed agents of both sides who lose their optimality unlike men-optimal and women-optimal matchings where all agents of the opposite side lose their optimal matches.

5.2 Example 2

I provide this example to clarify the procedure of the DM algorithm. For the sake of simplicity, I only show a two-period matching market. Furthermore, to show all details, I explain the example in separate rounds, although all the rounds could be done in one table.

Period 1:

${L_{m_1}}^1 = \{\{w_2\}, \{w_1, w_3\}\}$, ${L_{m_4}}^1 = \{\{w_4\}\}$

$t_{m_i} = t_{w_j} = 1$ for all $m_i \in M^1$ and $w_j \in W^1$ but, $TD^1_{M_1} = 3$ while $TD^1_{M_4} = 1$. Therefore, based on the POr definition, m_1 comes before m_4 in the POr list. The ordering of the remaining agents in POr is arbitrary:

$POr^1 = m_1, m_4, \cdots$

Men preferences:

m_1	m_2	m_3	m_4
w_2	w_2	w_2	w_4
w_1	w_4	w_3	w_1
w_3	w_3		
w_4	w_1		

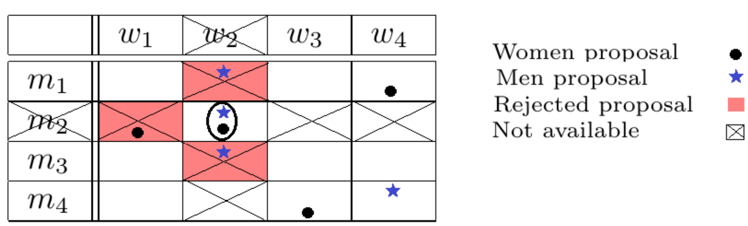
Women preferences:

w_1	w_2	w_3	w_4
m_2	m_2	m_4	m_1
m_4	m_3	m_3	m_2
m_3	m_1	m_1	
m_1	m_4	m_2	

DM **Algorithm:**

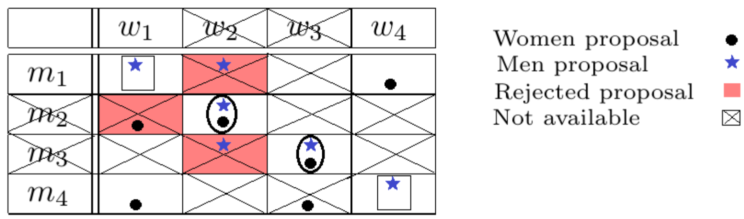
Round 1:

Two sides propose at the same time to their most preferred agent. Agents receiving more than one offer, reject the less preferred ones. At the end of this round, agents m_2 and w_2 are matched since they both have proposed to each other. As a result, w_2 rejects m_1 and m_3. Also, m_2 rejects w_1.



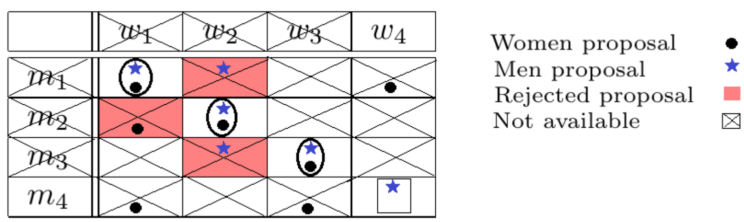
Round 2:

Agent m_4 has proposed to w_4 which is in his loyalty set. Since $m_4 \notin {W_4}^1$, and w_4 has not received any more preferred proposal, this offer will stay active which means m_4 is waiting for w_4. Rejected agents, m_1, m_3 and w_1 propose to their next agent in their preference ordering. Although $\{w_2\} \in L_{m_1}$ but since w_2 has rejected m_1 in favour of a more preferred one, he will propose to his next preferred agent. Agents receiving more than one offer reject the less preferred ones. Since m_4 is waiting for w_4 and $m_4 \notin L_{w_3}$ then w_3 does not wait for m_4 and withdraws her offer to m_4. She proposes to m_3. At the end of this round, m_3 and w_3 are matched and will leave the market while m_1 is waiting for w_1 and m_4 is waiting for w_4.



Round 3:

$m_1 \notin L_{w_4}$ and m_1 himself is waiting for w_1, so w_4 withdraws her proposal to m_1 and since her next preferred agent, m_2, is already matched she cannot make a new proposal. This means w_4 will leave this period unmatched. $m_4 \notin L_{w_1}$ and m_4 himself is waiting for w_4, so w_1 withdraws her proposal to m_4 and proposes to m_1 (m_3 is already matched). At the end of this round, m_1 is matched to w_1.



This is the end of period 1.

Matched pairs: $(m_2, w_2), (m_3, w_3), (m_1, w_1)$.

Unmatched agents: w_4 and m_4 (waiting for w_4).

Period 2:

New agents, m_5, m_6, w_5, w_6 enter the market.
There is no L_{m_i} or LL_{m_i}. Technically all L_{m_i} and LL_{m_i} sets are the empty sets.

Although m_4 and w_4 have been present in the market for the same number of periods, $t_{m_4} = t_{w_4} = 2$, m_4 comes before w_4 in POr since $TTD^2_{m_4} = 1$ while $TTD^2_{w_4} = 0$. The ordering of the rest agents in POr is arbitrary:

$POr^2 = m_4, w_4, \cdots$

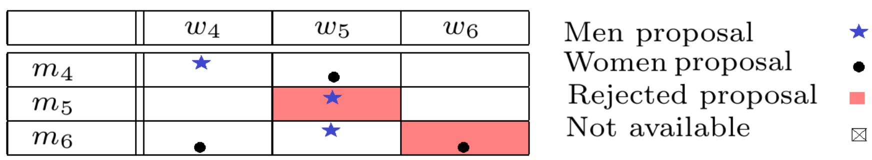
Men preferences: Women preferences:

m_4	m_5	m_6
w_4	w_5	w_5
w_6	w_4	w_4
w_5	w_6	

w_4	w_5	w_6
m_6	m_4	m_6
m_4	m_6	m_5
m_5		

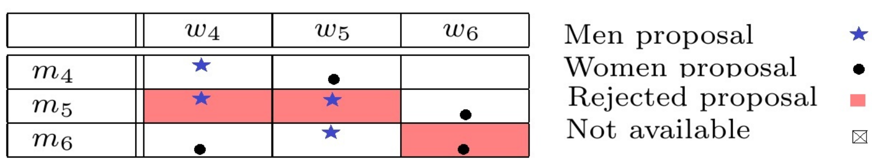
Round 1:

Two sides propose at the same time to their most preferred agent. Agents receiving more than one offer, reject the less preferred agents. At the end of this round there is no possible match.



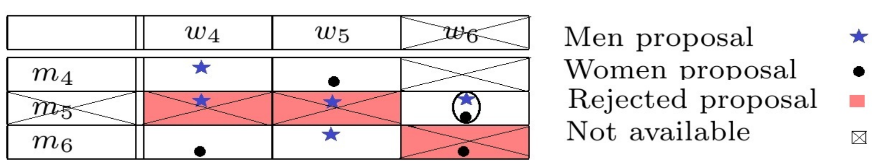
Round 2:

Rejected agents, m_5 and w_6 propose to their next preferred mates. Agents receiving more than one offer keep the most preferred one and reject others.



Round 3:

Rejected agent, m_5 proposes to his next preferred agent, w_6. At the end of this round, m_5 is matched to w_6.

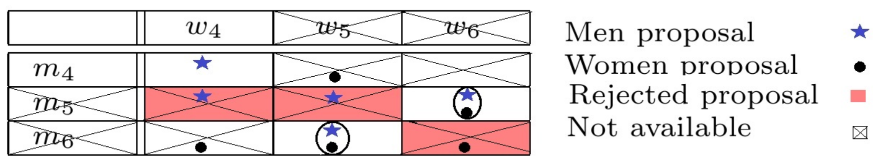


Round 4:

There is no possible match. We have a halt point and the *POr* list is required. As explained before, the *POr* list is dynamic and whenever it is going to be used, it will be updated based on the current situation of the market. Although m_4 was at the top of the POr list at the beginning of period 2 but because he is waiting on a reactivated offer from period 1 so he will not be the first agent in the list after updating POr. Based on the POr definition, w_5 who has proposed to waiting agent w_4 (the one waiting on a reactivated offer) moves to the top of the POr list:

Updated $POr^2 = w_5, m_4, w_4, \cdots$

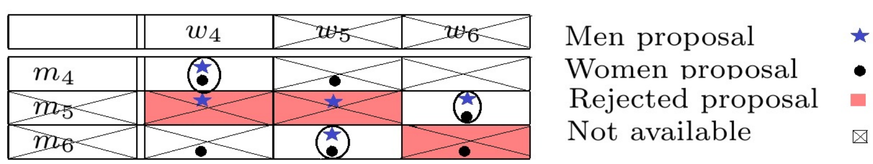
Therefore, w_5 has to withdraw her offer to m_4 and propose to her next preferred agent, m_6.



At the end of this round, agent m_6 is matched to w_5.

Round 5:

Rejected agent w_4 will propose to the last remaining agent in her preference ordering, m_4. Agent m_4 is matched to w_4.



This is the end of period 2 and the end of this matching problem.

Matched pairs: $(m_2, w_2), (m_3, w_3), (m_1, w_1), (m_4, w_4), (m_5, w_6), (m_6, w_5)$. ▲

6 Theorems and Results

In order to be able to state the results of this study, I will first introduce some new concepts:

Definition 9: Inter-periods blocking pair

An inter-periods blocking pair consists of a pair of agents m_i and w_j which prefer each other to their current match under μ but they have left the market in *different periods, provided that there were both in the market at least in period t when one of them who is matched first leaves the market. Furthermore, in one of the periods when they were both in the market, at least the one who leaves the market first, starts to include the other one in her/his acceptance set.*

Since in my model no agent knows who will join the market in the future and every agent who gets matched leaves the market, it is not possible for m_i and w_j to form a blocking pair if they have never met in the market. Furthermore, it is not logical that an agent m_i forms a blocking pair with another agent, w_j, whom he has never considered an acceptable mate while she was present in the market when he got matched. Also, let us recall that if $m_i \in {W_j}^t$, then $m_i \in {W_j}^{t'}$ $\forall$ $t' > t$ as long as m_i is in the market since no agent can kick out another agent from her/his acceptance set. Therefore, if m_i is the agent who leaves the market first, he should include the other agent, w_j, in his acceptance set at least in the last period that they are both in the market which is the period that m_i gets matched and leaves the market, i.e., there exists $t_{m_i} \in T_{m_i} : w_j \in {M_i}^{t'_{m_i}}$ $\forall$ $t'_{m_i} \geq t_{m_i}$ or/and there exists $t_{w_j} \in T_{w_j} : m_i \in {W_j}^{t'_{w_j}}$ $\forall$ $t'_{w_j} \geq t_{w_j}$.

Definition 10: Dynamic Individual Rationality

A matching μ is Dynamically Individual Rational if all agents prefer their match under μ in period t to staying unmatched until the next period, $t+1$.

Definition 11: Dynamic Stability

A matching μ is dynamically stable if:

1. *It is dynamically individual rational.*
2. *There is no blocking pair of agents (m_i, w_j) who leave the market at the same period t and prefer each other to their current match under μ.*
3. *There is no inter-periods blocking pair (m_i, w_j) who leave the market in different periods and prefer each other to their current match under μ unless one of them has been waiting on a passive offer to a more preferred agent outside the pair or did not include the other one in her/his acceptance set when the other one has been matched and the other one also has not waited for her/him.*

Theorem 1. *DM is dynamically stable.*

Theorem 1 is proved in Appendix *A*.

Definition 12: Dynamic two-sided optimality

A matching μ is dynamically two-sided optimal if it is the best dynamically stable matching for both sides.

The following simple example is a well known example which shows that matchings which are optimal for both sides may not exist.

Example 2.

Men preferences:

m_1	m_2
w_1	w_2
w_2	w_1

Women preferences:

w_1	w_2
m_2	m_1
m_1	m_2

Optimal matchings:

1. Men-optimal (men proposing): (m_1, w_1), (m_2, w_2).
2. Women-optimal (women proposing): (m_2, w_1), (m_1, w_2). ▲

Theorem 2. *DM finds dynamically two-sided optimal matchings if such a matching exist.*

Theorem 2 is proved in Appendix B.

We know that a matching which is optimal for both sides only exists if there is only one stable matching, in which case the unique stable matching is both men-optimal and women-optimal.

If there is no two-sided optimal matching, based on POr, DM finds matchings which are fair for both sides since it does not favour either side.

Definition 13: Dynamic Strategy-proofness

A matching rule is Dynamically Strategy-proof if no agent by misreporting her preferences at some period can get better results than what she could get using the dynamic features of the mechanism.

Theorem 3. *DM is dynamically strategy-proof.*

Theorem 3 is proved in Appendix C.

Definition 14: Dynamic Pareto-optimality

In a dynamic market a matching μ is Pareto-optimal if no agent can be made better off in period t without making somebody else worse off in some period $t'' : t'' \geq t$.

Theorem 4. *DM is dynamically Pareto-optimal.*

Theorem 4 is proved in Appendix D.

Let me recall the dynamic aspects of my model which are referred in Theorem 3 and Theorem 4.

1. Time which includes infinite number of periods.
2. Agents whose new comers enter the market at the beginning of each period and matched ones leave it.
3. Preferences/acceptance sets of agents which can change/expand at each period due to arriving new agents and due to the fear of remaining unmatched.
4. Loyalty sets which give agents the possibility of waiting for other agents on the other side whom they value much higher than others.
5. POr^t which changes at the beginning of each period and also is updated at any time based on the current situation of the agents.

7 Increasing the Matching Size in each Period

In case of marriage, maximizing the number of matched pairs in each period does not make sense since it deals with major life decisions. Therefore, taking some measures to put pressure on people for accepting mates whom they do not really want to does not seem right. If we use the algorithm for other cases, maximizing the number of matched pairs may be desirable.

Generally there are three reasons for staying unmatched in the DM algorithm:

1. Small acceptance set.
2. Being less preferred by others.
3. High TD and TTD (agents are waiting for others for long).

Since DM respects the agents' preferences on both sides of the market, it could be possible to increase the number of matches by motivating the agents to be less restricted when they are reporting their acceptance and loyalty sets.

There are restrictions which can be set up to motivate agents to become more generous about their acceptance set. These restrictions help to increase the number of matched pairs based on the preferences:

1. Agents remaining from previous period, go to the top of the POr. In case of ties, the ones with highest TTD and TD go first even if they are waiting on a reactivated offer. Therefore, in case of a halt point, they are the ones who have to change their offer and propose to their next preferred agent.

2. Time limit:

    - We can put a time limit for each agent to attend the market. When time is limited for each agent, they would expand their acceptance set in fear of being unmatched at the end of their time limit. For example each agent can only attend the market 3 times but the market runs forever. We can also motivate agents more to report generous acceptance sets by adding some conditions to the time limits. For example, if an agent is still unmatched after her time limit, she will leave the market forever, unless she has included all available agents in her acceptance set at least for half of her periods in the market, and an empty L set for the same amount of periods. The number of allowed periods for each of these restriction can vary based on the market situation and the designers goals. For example, when making as many matches as possible is a crucial goal, time limits will be tight.
    - The number of periods that the market operates is limited and at the last period, all agents are acceptable to each other.

These restrictions can be set up in a way to increase or even maximize the number of matchings.

8 Conclusion

In this project I introduced a matching model when both sides of the market have preferences over the other side and the market runs over multiple time periods. I have studied the different aspects of the model in-depth, to provide a fundamental understanding of relevant situations and to aid practical market designers. I have elaborated the idea, defined a novel model and studied its properties.

Furthermore, I introduced an algorithm, DM, which finds a matching which is dynamically optimal for both sides whenever it exists. Otherwise it selects a matching which is fair since it does not favour either side. In addition, I provided some restrictions to increase the number of matches at each period and I showed that the DM algorithm is fair in the sense of not favouring either side, dynamically stable, strategy-proof and Pareto-optimal.

My study addresses an important and overlooked issue of real-life matching situation in a dynamic environment. My goal is to provide a strong theoretical foundation for building a more prosperous society by ensuring that the human capital have been treated fairly and the mechanism of matching resources is stable and efficient.

9 Appendices

9.1 Appendix A

Theorem 1. *DM is dynamically stable.*

Proof.

- DM is dynamically individual rational:

  Each agent will be matched to one of her acceptable agents. Therefore, whoever she has been matched to is better than being unmatched. Furthermore, if she had somebody who is worth to wait for (although he did not accept her at the current period), she could put him in her loyalty set an take her chance to be matched to him in the next periods. If she has not done, then she prefers to be matched in the current period rather than waiting unmatched until the next periods.

- There is no blocking pair of agents (m_i, w_j) who leave the market in the same period t and prefer each other to their current match under μ:

  Assume (m_i, w_j) is a blocking pair of agents who has left the market in the same period and prefer each other to their match under μ. This simply means that they both have been acceptable to each other but they have not been matched to each other. Therefore:

  - There is a $w_{j'}$ whom m_i has been matched to while $w_j \succ_{m_i} w_{j'}$.
  - DM matches a man to a woman when both propose to each other at the same time.
  - While $m_i \in {W_j}^t$ and $w_j \succ_{m_i} w_{j'}$, then he has proposed to w_j before proposing to $w_{j'}$.
  - If he has not been matched to w_j that means w_j has rejected him in favor of a more preferred agent.

These contradict (m_i, w_j) being a blocking pair.

- There is no inter-periods blocking pair $(m_i; w_j)$ who leave the market in different periods and prefer each other to their match under μ unless one of them has been waiting on a passive offer to a more preferred agent outside the pair or did not include the other one in her/his acceptance set when the other one has been matched and the other one also has not waited for her/him:

  Assume (m_i, w_j) is an inter-period blocking pair of agents who prefer each other to their match under μ and m_i is the agent who leaves the market first[7] in period t.

  - DM matches a man to a woman when both propose to each other at the same time.
  - There is a $w_{j'}$ whom m_i has been matched to while $w_j \succ_{m_i} w_{j'}$.
  - Based on the definition of inter-periods blocking pair, at least in the m_i's last period, period t, both m_i and w_j were present in the market and w_j was acceptable by m_i.
  - m_i has left the market in period t. That means m_i has been matched to $w_{j'}$ in that period. Therefore, w_j and $w_{j'}$ have been both in his acceptance set in period t.
  - If $w_j \succ_{m_i} w_{j'}$, he has proposed to w_j before proposing to $w_{j'}$.
  - Therefore, if he has not been matched to w_j that means:
    - $m_i \in {W_j}^t$ but w_j has rejected him in favor of a more preferred agent. In this case they both leave the market in the same period and it is covered in blocking pair definition.
    - w_j did not include m_i in her acceptance set in period t when m_i has been matched and m_i has not waited for her (did not include her in his loyalty set).
    - w_j has included m_i in her acceptance set but she was waiting on a passive offer when m_i got matched and m_i itself did not wait for her.

  These contradict (m_i, w_j) being an inter-period blocking pair.

□

9.2 Appendix B

Theorem 2. *DM finds dynamically two-sided optimal matchings if such a matching exists.*

[7] This means m_i is the one who has been matched first.

Proof. DM matches a man to a woman when both of them propose to each other at the same time. Furthermore, we know a two-sided optimal only exist if there is a unique stable matching. Therefore, automatically if there is a two-sided optimal, DM will find it.

□

9.3 Appendix C

Theorem 3. *DM is dynamically strategy-proof.*

Proof. We know that DA is strategy-proof for the proposing side. DM is based on both sides proposing at the same time, therefore its results is different from the DA outcome even if it is used in one-period matching problem. On the other hand, since DM is a dynamic matching algorithm and static strategy-proofness does not apply to it, I show that DM is dynamically strategy-proof.

In static models, DA is not strategy-proof for both sides. The agents on non-proposing side can make themselves better off by manipulating the mechanism through shortening their acceptance set. However, in the dynamic setup of my model it is not possible for any agent to manipulate the algorithm and get better results than what she could get using the options that the model and corresponding algorithm provide her.

Generally, if any agent w_j gets rejected by m_i because m_i has received a proposal from a more preferred mate, no matter how w_j reports her preferences, she will not be matched to m_i. In other cases, shortening the acceptance set may improve the outcomes for w_j since it causes the automatic rejection of unacceptable agents. These rejected agents make new proposals and as a result the mate that w_j wanted more may get rejected from his more preferred choice and he will propose to w_j. The same goal can be achieved easily by defining a loyalty set including m_i. If m_i rejects w_j in favor of a preferred agent, as mentioned before no manipulation can help w_j. Otherwise, as long as w_j has not been rejected by m_i, she is not accepting any other proposals (receiving proposals from others is automatically rejected) and she is not forced to change her proposal. □

Let me recall Example 3 for further clarification. In a static men-optimal matching, both women w_1 and w_2 can manipulate the mechanism by reporting their second choice unacceptable. Men can do the same manipulation in the static women-optimal matching. Now, let me solve this matching problem using the DM algorithm:

Example 3.

If all agents report their true preferences then the results depend on the POr since we will have a halt point in Round 1. If either w_1 or w_2 is at the top of the POr[8] then women will end up with their second choice. The same happens for men if either of them is the first one who has to change his proposal based on the POr. Therefore, even without misreporting, there is an equal chance for both sides to have their optimal matching due to this fact that both sides are allowed to propose at the same time. Now, assume w_1 wants to increase her chance of getting m_2 by misreporting her preferences as follows:

$w_1{}'$	w_2
m_2	m_1
	m_2

The men's preference ordering is the same as in Example 3.

DM **algorithm:**

Period 1: m_1 and m_2 propose to their first choices, w_1 and w_2 respectively. At the same time, w_1 and w_2 propose to m_2 and m_1 respectively. The offer received by w_1 came from an unacceptable man but since she has not received any preferred proposal, she can not reject this proposal. Nevertheless, since m_1 has not included w_1 in his loyalty set, then his proposal to w_1 will be rejected automatically. Rejected m_1 proposes to w_2. Now w_2 who has received two proposals accepts the preferred one, m_1 and rejects m_2. Rejected m_2 proposes to w_1 and gets matched to her.

The fact is that she did not need to manipulate. She could easily have the same result only by including m_2 in her loyalty set. The manipulation is unnecessary especially since based on POr rule if an agent is at the top of this list and has to change her proposal but does not have any other agent left in her acceptance set, then she has to leave the period unmatched. Therefore, misreporting her preferences by shortening her acceptance set may even cause her staying unmatched while this situation will not happen if she includes m_2 in her loyalty set. That is because if w_1 includes m_2 in her loyalty set, although w_1 is at the top of the POr[9], since she is waiting on a loyal offer, then the agent proposing to her, m_1, jumps before her in the updated POr list. Therefore, m_1 should withdraw his proposal to w_1 and propose to w_2. Then, w_2 who now has two offers accepts the preferred one, m_1, and rejects m_2. Rejected m_2 proposes to his next preferred woman, w_1 and gets matched to her. Therefore, w_1 gets her best choice in period 1 which is the same result as misreporting the preferences. ▲

The structure of the DM algorithm which allows simultaneous proposals and possibility of waiting, makes manipulating unnecessary for agents.

[8] POr here is a random ordering of all agents since there is no seniority, TTD or TD.
[9] $TD_{w_j} = 1$ while it is zero for all other agents.

9.4 Appendix D

Theorem 4. *DM is dynamically Pareto-optimal.*

Proof. DM matches agents to each other when both propose to each other at the same time. An agent proposes to her next preferred agent if she has been rejected in favor of a more preferred agent or she is not acceptable by the other one and she did not want to wait for him. Therefore, when an agent gets matched, she has be assigned to her best possible option regarding her preferences and loyalty set.

Two agents can be matched to each other only if they have met in the market and at least in period t when they are matched to each other they both should be included in each other's acceptance sets. Assume w_j has been assigned to m_i in period t under μ and there exists another agent $m_{i'}$: $m_{i'} \succ_{w_j} m_i$. If $m_{i'}$ has left the market before period t then:

- $m_{i'} \in {W_j}^{t'}$ for some $t' < t$, then either he has rejected w_j in favour of a more preferred agent or did not accept w_j in t'.
- w_j did not accept $m_{i'}$ in t' or has been waiting on an offer to a more preferred agent and $m_{i'}$ has not waited for her. This means that $m_{i'}$ did not want to wait for her until period t.

Anyway, changing the assignment of w_j in period t and assigning $m_{i'}$ to w_j does not make sense since $m_{i'}$ is already matched and whoever is matched is out of the market. Moreover, no body knows who will join the market in the future, then changing the assignment of w_j and assigning her to an agent who has not joined the market in period t is not possible too. Therefore, if w_j has been matched to m_i in period t and there is an agent $m_{i'}$: $m_{i'} \succ_{w_j} m_i$ then $m_{i'}$ should be in ${W_j}^t$. In this case, if we assign $m_{i'}$ to w_j, we have made her better off. On the other hand, since $m_{i'} \succ_{w_j} m_i$, based on DM procedure, w_j has proposed to $m_{i'}$ before proposing to m_i. Therefore, if w_j has not been matched to $m_{i'}$ that means one of the following scenarios applies:

- $m_{i'}$ has rejected w_j in favor of a more preferred agent in period t, then assigning $m_{i'}$ to w_j makes $m_{i'}$ worse off.
- $w_j \notin {M_{i'}}^t$ and $m_{i'}$ has not been matched in period t but w_j did not want to wait for him. Thus, w_j prefers to be matched to m_i in period t instead of waiting for $m_{i'}$. It means for being matched to $m_{i'}$ she needs to stay unmatched in period t and this makes her worse off.
- $w_j \in {M_{i'}}^t$ but $m_{i'}$ is waiting on an offer to a more preferred agent in period t while w_j does not want to wait for him. If $m_{i'}$ leaves period t unmatched but does not get matched to the mate that he was waiting for and instead gets matched in period $t'' : t'' > t$ to $w_{j'}$ while $w_j \succ_{m_{i'}} w_{j'}$ then assigning $m_{i'}$ to w_j in t makes both of them better off but $m_{i'}$ is the best possible mate for $w_{j'}$ in period t''. Therefore, assigning w_j to $m_{i'}$ in period t makes $w_{j'}$ worse off in period $t'' : t'' > t$.

- $w_j \in {M_{i'}}^t$ but $m_{i'}$ is waiting on an offer to a more preferred agent in period t while w_j does not want to wait for him. If $m_{i'}$ leaves period t unmatched and gets matched in period $t'' : t'' > t$ to an agent $w_{j'}$ whom he prefers to w_j, then assigning $m_{i'}$ to w_j in period t makes both $m_{i'}$ and $w_{j'}$ worse off.

□

References

[1] Andersson, T., Ehlers, L., and Martinello, (2018). ”Dynamic refugee matching”. *Cahier de recherche, 10.*

[2] Arnosti, N., Shi, P. (2020). ”Design of Lotteries and Wait-Lists for Affordable Housing Allocation.” *Management Science.*

[3] Bloch, F., Cantala, D. (2017). Dynamic assignment of objects to queuing agents.” *American Economic Journal: Microeconomics, 9(1), 88-122.*

[4] Damiano, E., Lam, R. (2005). ”Stability in dynamic matching markets”. *Games and Economic Behavior, 52(1), 34-53.*

[5] Doval, L. (2019). ”Dynamically stable matching”. *Available at SSRN 3411717.*

[6] Du, S., Livne, Y. (2014). ”Rigidity of transfers and unraveling in matching markets”. *Available at SSRN 2203910.*

[7] Dworczak, P. (2016, July). ”Deferred acceptance with compensation chains.” *In Proceedings of the 2016 ACM Conference on Economics and Computation (pp. 65-66).*

[8] Klijn, F., Pais, J., Vorsatz, M. (2019). ”Static versus dynamic deferred acceptance in school choice: Theory and experiment.” *Games and Economic Behavior, 113, 147-163.*

[9] Kotowski, M. H. (2019). ”A Perfectly robust approach to multiperiod matching problems.”

[10] Kurino, M. (2020). ”Credibility, efficiency, and stability: A theory of dynamic matching markets.” *The Japanese Economic Review, 71(1), 135-165.*

[11] Kuvalekar, A. (2014). ”A Fair Procedure in a Marriage Market.” *Available at SSRN 2688630.*

[12] Liu, C. (2020). ”Stability in repeated matching markets”. *arXiv preprint arXiv:2007.03794.*

[13] Pereyra, J. S. (2013). ”A dynamic school choice model”. *Games and economic behavior, 80, 100-114.*

[14] Romero-Medina, A. (2005). "Equitable selection in bilateral matching markets." *Theory and Decision, 58(3), 305-324.*

[15] Romero-Medina, A., Kuvalekar, A. V (2021). "A fair procedure in a marriage market". *(No. 31711). Universidad Carlos III de Madrid. Departamento de Economía*

[16] Gale, D. and Shapley, L. S. (1962). "College admissions and the stability of marriage". *The American Mathematical Monthly, 69(1):9–15.*

[17] Teo, C. P., Sethuraman, J. (1998). "The geometry of fractional stable matchings and its applications." *Mathematics of Operations Research, 23(4), 874-891.*