On non-differentiable comparative statics in one-sided matching market ☆

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Abstract

This paper extends the one-sided matching market introduced by Shapley and Scarf (1974). In our setting the sets of agents and goods are measure spaces, this allows to present a unified model in which both sets can be continuous or discrete (finite or infinite). We present two models and establish conditions for the nonemptiness of the core. The first one introduces a concept of assignment as a measurable function that assigns to each type of agent a type of good. The second introduces a concept of assignment as a probability distribution which assigns a mass of agents to a mass of goods. In both models, we search a Pareto optimal assignment, using the optimal transport theory, to establish the nonemptiness of the core. We study the concept of ϵ-core to approximate the core of general models through finite models. Finally, we perform illustrative comparative statics.

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1. introduction

Assignment problems pioneered by Koopmans and Beckmann (1957), Gale and Shapley (1962), Shapley and Scarf (1974), and Shapley and Shubik (1971) are models where preferences of agents are heterogeneous, but fixed; in particular they ignore the underlying valuations of characteristics that leads to the preference orderings. Thus, from Kelso and Crawford (1982) and Crawford (1991) to Turhan (2019) and Pérez-Castrillo and Sotomayor (2019) comparative statics have mostly been restricted to changes in the sets of agents and objects, be they additions, subtractions, mergers or divisions. To tackle other comparative statics, however, the models require a generalization that distinguishes the idiosyncratic and the common components of the preferences.

An illustrative urban planning problem consists of a city where agents wish to live as close as possible to their office, there are two business districts in the city and the local government considers the possibility to build a new one. To evaluate the alternative locations of the new district, between two or more candidate locations, one wishes to computationally simulate the swapping of houses that it would trigger, for instance through the Top Trading Cycle. Modeling the behavior of the market requires to estimate the distribution of the idiosyncratic components of the preferences: the districts where they work before and after the creation of the district, and the location of their houses. The common component is the estimate of commuting times, considered homogeneous, i.e., such that it is the same for all agents covering a given travel. In the school choice model (Abdulkadiroğlu and Sönmez (2003)) the idiosyncratic parts of the preferences might be the location of the students and the enrollment of their siblings and the common part might be the ranking of the school and the transportation cost to the different schools. The comparative statics might consists in evaluating the localization of the new school.

Our objective in the paper is two-fold: first to propose a population model which allows to capture both the idiosyncratic and the common components of the preferences of agents, be they continuous or discrete; second to study the existence of Core assignments in these economies. Specifically, we model the sets of agents and indivisible goods as measure spaces, which allows to present in a unified manner both discrete sets of agents and indivisible goods, and a compact continuum of them. Concretely, types label indivisible goods and agents[1]. The type of an indivisible good describes the different characteristics that fully characterize a good and the type of an agent represents a preference relation over the set of goods. The number or mass of indivisible goods and agents of a certain type is measured by a probability distribution.

We propose two models that bridge the gap between one-sided problems and the optimal transport literature. The key result of the analysis consists in establishing that if an assignment is the solution of a Pareto optimization problem, specified as a linear maximization problem, then it is in the Core of the economy. Our first model, called type-exclusive assignment economy, introduces a natural concept of assignment as a measurable function that assigns to each type of agent a type of good. We establish specific continuity, differentiability and topological conditions for the nonemptiness of the core.

The type-exclusive assignment economy is embedded in the second model, called flexible assignment economy, which introduces a more general concept of assignment as a probability distribution that assigns a mass of agents to a mass of goods. In these settings the existence of a non-empty core is guaranteed under general conditions. Beside the existence of core allocation, comparative statics may require the use of numerical techniques as Monte Carlo simulations. In turn, they may be applied only to the core approximation. We study the concept of ϵ-core and establish conditions to approximate the core of general models through finite models. Finally, we present an illustrative example of Monte Carlo simulations in the housing market.

Currently, optimal transport theory is a very active research area of probability theory and optimization with several applications to economic theory (see e.g. Galichon (2016), Galichon (2017) and Xia (2015)). In the case of two-sided markets of one-to-one matching, stable matchings can be identified with solutions to the dual of the Kantorovich optimal transport problem, for models with transferable utility (TU) (Gretsky et al. (1992), Gretsky et al. (1999), Chiappori et al. (2010), Ekeland (2005), Ekeland (2010), Carlier and Ekeland (2010)); and for models with imperfectly transferable utility (ITU), Nöldeke and Samuelson (2018). These models use the dual of an optimal transportation problem to define and solve the nonemptiness of the core. In our one-sided matching market model, we use a different approach, we focus on the primal problem to define and solve the nonemptiness of the core for

[1] In this article, the word “type” does not have the use it is given in the Bayesian decision literature

one-sided matching market. Moreover, we categorize the sets of indivisible goods and agents in types and assign a probability distribution over each set of types (agents and goods). This allow us to study the sets of agents and goods as measure spaces. Thus, we establish a unified model in which continuous and discrete models (finite or infinite) are encompassed.

We establish conditions to approximate the core of general models through deterministic and finite models, i.e., models where the set of types of good and agents are finite, and the utility function is deterministic. This approximation through finite models differs from Greinecker and Kah (2018) and Menzel (2015) where the approximation for two-sided markets is through finite models with stochastic perturbations in the utility functions.

In our Illustrative example, we propose two numerical methods to calculate the core of finite economies that approximate the general model. In the first one, under a mass transfer scheme, and in the second one, under a simulation scheme and Monte Carlo integration techniques. This second numerical approximation allows us to perform comparative statics, which do not rely on differentiation techniques; indeed the change in the economy is discrete in nature, specifically if consists in evaluating the effects of the localization of a new business center. This method does not require the differentiability hypothesis (as in Graham (2013), Galichon and Salanié (2017)). On the other hand, the flexible economy model allows to capture idiosyncratic components for future econometric works, see for example, Chiappori (2020), Chiappori and Salanié (2016), and Galichon et al. (2019). While the econometrics of matching markets has experienced important developments, as far as we are aware, ours is first step toward a theory that support computational approximation of these markets.

The remainder of the paper is organized as follows. Section 2 presents the type-exclusive assignment economy in which the concept of assignment is a measurable function. Theorem 2.4 establishes a relation between the core of a type-exclusive assignment economy and an optimal transport problem. Section 3 presents the flexible assignment economy introducing a concept of assignment as a probability distribution which assigns a mass of agents to a mass of goods. We see that the type-exclusive assignment economy is embedded in the flexible assignment economy, and in Theorem 3.3 we establish conditions for the nonemptiness of the core in the flexible assignment economy.

Section 4 proves Theorem 3.3. Theorem 4.3 relates the nonemptiness of the core of the flexible assignment economy and an optimal transport problem. In Section 5 we study the concept of ϵ-core to approximate the core of general models through finite models. In Section 6 we present an housing market model. First we approach it as an optimal transport problem, second we examplify the use of the TTC in this population model, finally we adapt Monte Carlo integration techniques to solve a complexified version of the model.

We conclude in section 7 with some general comments. Appendix– A presents an extension of the Debreu's preferences representation theorem. Finally, Appendix– B proposes a model where there is non-atomic measure on the set of types of goods and agents and other conditions to ensure the nonemptiness of the core in the type-exclusive assignment economy.

2. Type-exclusive assignment model

In this section, we require assignments to be such that two or more agents of the same type are assigned to goods of the same type, and vice versa. An interpretation of this condition is that the assignment is anonymous, both for agents and goods, a normative criterion potentially incompatible with the existence of a core stable assignment.

2.1. Type-exclusive assignment economy

Consider an economy which has a population of agents, and a population of indivisible goods. Each indivisible good is labeled with a single type, g, which describes the different characteristics that fully characterize a good. The set of types of goods is G. Each agent is labeled with a type, a, where a particular type represents the preference relation $\precsim_a$ over G. The set of types of agents is A. Assume that A and G are compact Borel spaces, that is, they are separable and compact metric spaces.

We assume that preference relations $\{\precsim_a\}_{a\in A}$ are represented by a continuous utility function $u : A \times G \to [0, 1]$, that is

$$g \precsim_a g' \iff u(a, g) \leq u(a, g'). \tag{1}$$

Two comments are in order about the utility function: 1) continuity is a requirement one cannot dispense of for our results to hold (the definition of continuity should be understood in the context of general topology); and 2) the utility function $u(\cdot, \cdot)$ represents the preference relations of all types of agents, so when one considers transformations f of $u(\cdot, \cdot)$ that also represent the preference relations of all types, they might treat arguments a

and g differently. Specifically, since no cardinality of utility is assumed for our purpose, any transformation $u^f(\cdot,\cdot) = f(u(\cdot,\cdot))$ that is continuous also represents the original preference relations, that is, if $g \precsim_a g'$, then

$$u^f(a,g)) \leq u^f(a,g')) \Longleftrightarrow u(a,g) \leq u(a,g') \qquad \forall a \in A. \tag{2}$$

This is the only monotonicity restriction required on f.

For conditions in $\{\precsim_a\}_{a\in A}$ that guarantee the existence of $u(\cdot,\cdot)$, see for example Levin (1983), Rachev and Rüschendorf (1998) Theorem 5.5.18 page 337, or Bridges and Mehta (2013) Theorem 8.3.6 page 146 (see Appendix A).

Let $\mathcal{B}(A)$ and $\mathcal{B}(G)$ be the Borel σ-algebras of A and G, respectively. Probability measures η and ν assign a population distribution over the sets A and G, respectively. Finally, we denote the population of agents and the population of indivisible goods by the probability measure spaces

$$\begin{aligned} \mathbf{A} &:= (A,\ \mathcal{B}(A), \eta) \ \text{ and} \qquad (3)\\ \mathbf{G} &:= (G,\ \mathcal{B}(G), \nu), \qquad (4)\end{aligned}$$

respectively.

A *type-exclusive assignment economy* is a quadruple $\mathcal{E} := (\mathbf{A}, \mathbf{G}, u, \mu_0)$, where $\mathbf{A}$ is a population of agents as in (3), $\mathbf{G}$ is a population of indivisible goods as in (4), u is a continuous function that satisfies (1), and, finally, $\mu_0 : A \to G \cup \emptyset$ is a measurable function which assigns for each type of agent a in A the agent's initial endowment $\mu_0(a)$ in G and satisfies (5), or all gents have the empty set $\emptyset$ as initial endowment. The function μ_0 is called *the initial type-exclusive endowment*.

A *type-exclusive assignment* for the economy $\mathcal{E}$ is a measurable function $\mu : A \to G$. A type-exclusive assignment μ for $\mathcal{E}$ is *feasible* if for each set of types of indivisible goods E in $\mathcal{B}(G)$, the amount $\nu(E)$ of indivisible goods is proportional to the amount $\eta(\mu^{-1}(E))$ of agents. In other words, a type-exclusive assignment μ for $\mathcal{E}$ is feasible if

$$\eta(\mu^{-1}(E)) = \nu(E) \qquad \forall E \in \mathcal{B}(G). \tag{5}$$

The following are two examples of type-exclusive economies and assignments.

Example 2.1. *Consider an economy $\mathcal{E}$ where A and G are finite sets with the same cardinality n and the function u in (1) is represented as a square matrix*

$[u(a,g)]_{\{a\in A, g\in G\}}$ *of rank* n. *Let* η *and* ν *be uniform probability distributions over the sets* A *and* G, *respectively; that is,* $\eta(a) = \nu(b) = \frac{1}{n}$ *for all* $a \in A$ *and* $g \in G$. *In this case, any bijective function* $\mu : A \to G$ *is a feasible type-exclusive assignment.*

Remark: *The economy in Example 2.1 is a particular case of Shapley and Scarf (1974).*

Example 2.2. *Consider an economy* $\mathcal{E}$ *where* A *and* G *are both the interval* $[0,1]$ *and the function* u *in (1) is* $u(a,g) = a^2 + g^2$. *Let* η *and* ν *be uniform probability distribution over the sets* A *and* G, *respectively, that is,* $\eta(da) = \nu(dg) = 1$ *for all* $a \in A$ *and* $g \in G$. *In this case, the functions* $\mu_1(a) = a$ *and* $\mu_2(a) = 1 - a$ *are feasible type-exclusive assignments for* $\mathcal{E}$.

Now, we define the core for this economy.

Definition 2.3. *The* core $\mathbf{C}(\mathcal{E})$ *of an economy* $\mathcal{E}$ *is the set of all feasible type-exclusive assignment* μ *such that there is no coalition* $S \in \mathcal{B}(A)$ *with* $(\nu(S) > 0)$ *and type-exclusive assignment* γ *that satisfies the following three conditions:*

E1 *i)* $\eta(\gamma^{-1}(E)) = \nu(E)$ *for all* $E \in \mathcal{B}(G) \cap \overline{\gamma(S)}$; [2]

ii) $\overline{\mu(S)} = \overline{\gamma(S)}$;

E2 $u(a,\mu(a)) \leq u(a,\gamma(a))$ η*-almost everywhere in* S;

E3 *there is* $D \in \mathcal{B}(A) \cap S$ *with* $\eta(D) > 0$ *and* $u(a,\mu(a)) < u(a,\gamma(a))$ η*-almost everywhere in* D.

Condition **E1**-i) refers to the feasibility of the assignment, a mass of goods is assigned to a mass of agents in equivalent proportions. Condition **E1**-ii) ensures that the blocking coalition S does not require goods held by agents out of S. Conditions **E2** and **E3** refer to the incentives that individuals in coalition S have to improve with respect to their assignments. In conditions **E2** and **E3** the statement "η-almost everywhere" means that these conditions can fail only in a subset of η-measure zero.

[2] Symbols $\overline{\mu(S)}$ and $\overline{\gamma(S)}$ refer to the topological closures of $\mu(S)$ and $\gamma(S)$, respectively.

2.2. Looking for a Pareto optimal assignment

In this section we consider a social planner who searches a type-exclusive assignment that is Pareto optimal (when the initial matching is the empty one, i.e., $\mu_0(a) = \emptyset$ for all a in A). Let $\mathcal{L}$ be the set of all feasible type-exclusive assignment, i.e.,

$$\mathcal{L} := \left\{ \mu : A \to G \quad : \quad \eta(\mu^{-1}(E)) = \nu(E) \quad \forall E \in \mathcal{B}(G) \right\}. \tag{6}$$

Consider the social planner's problem in an economy $\mathcal{E}$:

$$\max_{\mu \in \mathcal{L}} \int_A u(a, \mu(a)) \eta(da) \tag{7}$$

with $\mathcal{L}$ as in (6).

If μ^* is a solution to the planner's problem, one cannot strictly increase the utility of a given type without lowering the others; thus μ^* is Pareto optimal. Obviously, it is not the only one and when u is transformed by a new utility function that satisfies (2), the new solution might be another Pareto optimal assignment. The dependence on the utility representation is not critical for our present purpose, which is to establish the relation between the core of the economy $\mathbf{C}(\mathcal{E})$ and the social planner's problem (7).

Theorem 2.4. *A type-exclusive assignment μ^* is solution to the social planner's problem (7), then μ^* is in $\mathbf{C}(\mathcal{E})$.*

Proof. Suppose that μ^* is solution to (7) and it is not in $\mathbf{C}(\mathcal{E})$. Then there exists $S \in \mathcal{B}(A)$ (with $\eta(S) > 0$) and a feasible exclusive assignment γ which satisfy **E1**-**E2** in Definition 2.3.

Now, consider the exclusive assignment

$$\mu(a) = \begin{cases} \mu^*(a) & \text{if} \quad a \notin S \\ \gamma(a) & \text{if} \quad a \in S \end{cases}$$

Note that $\overline{\gamma(S)} = \overline{\mu(S)} = \overline{\mu^*(S)}$. Let $E \in \mathcal{B}(G)$. Then by **E1**-*ii*), the properties of the inverse image and (5), we have that

$$\begin{aligned}
\eta(\mu^{-1}(E)) &= \eta\left(\mu^{-1}\left(E\cap\overline{\gamma(S)}\right)\right)+\eta\left(\mu^{-1}\left(E\cap\left(G\setminus\overline{\gamma(S)}\right)\right)\right)\\
&= \eta\left(\gamma^{-1}\left(E\cap\overline{\gamma(S)}\right)\right)+\eta\left(\mu^{*-1}\left(E\cap\left(G\setminus\overline{\gamma(S)}\right)\right)\right)\\
&= \eta\left(\gamma^{-1}\left(E\cap\overline{\gamma(S)}\right)\right)+\eta\left(\mu^{*-1}\left(E\setminus E\cap\overline{\gamma(S)}\right)\right)\\
&= \eta\left(\gamma^{-1}\left(E\cap\overline{\gamma(S)}\right)\right)+\eta\left(\mu^{*-1}(E)\setminus\mu^{*-1}\left(E\cap\overline{\gamma(S)}\right)\right)\\
&= \eta\left(\gamma^{-1}\left(E\cap\overline{\gamma(S)}\right)\right)+\eta(\mu^{*-1}(E))-\eta\left(\mu^{*-1}\left(E\cap\overline{\gamma(S)}\right)\right)\\
&= \nu\left(E\cap\overline{\gamma(S)}\right)+\nu(E)-\nu\left(E\cap\overline{\gamma(S)}\right)\\
&= \nu(E).
\end{aligned}$$

Thus, μ is a feasible exclusive assignment for $\mathcal{E}$. Moreover, by **E2** and **E3** it satisfies that

$$\begin{aligned}
\int_A u(a,\mu^*(a))\eta(da) &= \int_{A-S} u(a,\mu^*(a))\eta(da)+\int_S u(a,\mu^*(a))\eta(da)\\
&< \int_{A-S} u(a,\mu^*(a))\eta(da)+\int_S u(a,\gamma(a))\eta(da)\\
&= \int_{A-S} u(a,\mu(a))\eta(da)+\int_S u(a,\mu(a))\eta(da)\\
&= \int_A u(a,\mu(a))\eta(da).
\end{aligned}$$

Therefore μ^* is not optimal for (7), which is a contradiction. So, we conclude that μ^* is in in the core. □

Example 2.5. *Consider an economy $\mathcal{E}$ as in Example 2.1. In this case $\mathcal{L}$ (as in (6)) is the set of all bijective functions $\mu : A \to G$. The social planner's problem is given by the optimization problem:*

$$\max_{\mu\in\mathcal{L}} \frac{1}{n}\sum_{a\in A} u(a,\mu(a)).$$

2.3. The core and feasible allocations

Consider an economy $\mathcal{E}$. If μ^* is a solution to the problem (7), then by Theorem 2.4, μ^* is in $\mathbf{C}(\mathcal{E})$ and therefore the core of $\mathcal{E}$ is not empty.

Nevertheless, the set of feasible allocation $\mathcal{L}$ in (6) is not necessarily compact nor convex, in fact it may be empty (as in Example 2.6). In any case, (7) may have no solution. The following example provides a case where the set of feasible allocations $\mathcal{L}$ is empty.

Example 2.6. *Consider a population of agents* $\mathbf{A} := (A,\ \mathcal{B}(A), \delta_a)$ *and a population of goods* $\mathbf{G} := (G,\ \mathcal{B}(G), \nu)$*, where* δ_a *is Dirac probability measure at* $a \in A$ *and* ν *is defined by*

$$\nu(E) := \frac{1}{2}\delta_{g_1}(E) + \frac{1}{2}\delta_{g_2}(E) \quad \forall E \in \mathcal{B}(G),$$

where δ_{g_1} *and* δ_{g_2} *are Dirac probability measures on* G *with* $g_1 \neq g_2$*. This example describes a situation in which we only have two types of goods, and one type of agent. In this case* $\mathcal{L} = \emptyset$*.*

Remark 2.7. *Consider an economy as in Example 2.1 where* A *and* G *are finite sets with the same cardinality* n*, and* η *and* ν *are uniform probability distributions over the sets* A *and* G*, respectively. In this case, the social planner's problem has a solution; see, for example, Koopmans and Beckmann (1957) or Shapley and Scarf (1974). Moreover, by Theorem 2.4 the core in this economy is not empty.*

In Appendix B we establish conditions under which the core of an economy $\mathcal{E}$ is nonempty. In the next section we define a new concept of assignment that ensures that set of feasible allocations is not empty, under general conditions.

3. Flexible assignment model

In a type-exclusive assignment economy $\mathcal{E}$ presented in Section 2, we assume that any feasible type-exclusive assignment μ is such that two agents of the same type are assigned to goods of the same type, and the set of feasible type-exclusive assignments $\mathcal{L}$ in (6) may be empty. In this section, we relax this restriction and work with a more flexible concept of assignment. This flexibility is suitable in permitting the transport of masses of agents and goods that allows us to guarantee the existence of feasible allocations and, in addition, it ensures a solution to the social planner's problem.

In this second model, we also classify the population of agents and the population of goods in types. In the assignment, to each type of agent can correspond one or many types of products, although the assignment is one-to-one.

3.1. Flexible assignment economy

As in Section 2.1, we consider a population $\mathbf{A}$ of agents as in (3), and a population $\mathbf{G}$ of indivisible goods as in (4). We also assume, that the preference relations $\{\precsim_a\}_{a\in A}$ are represented by a continuous function u as in (1).

A *flexible assignment economy* is a quadruple $\mathcal{E}_\Pi := (\mathbf{A}, \mathbf{G}, u, \pi_0)$, where u is a continuous function that satisfies (1), and π_0 is a probability measure on $A \times G$, which satisfies (8) and assigns to each set of types of goods E (for E in $\mathcal{B}(G)$) the proportion $\pi_0(D \times E)$ of agents of types in D (for D in $\mathcal{B}(A)$). Alternatively, π_0 may be the empty matching, where all goods are unassigned. The probability measure π_0 is called the *initial flexible endowment*.

A *flexible assignment* for an economy $\mathcal{E}_\Pi$ is a probability measure π on $A \times G$. A flexible assignment π for $\mathcal{E}_\Pi$ is *feasible* if for all D in $\mathcal{B}(A)$ and E in $\mathcal{B}(G)$,

$$\pi(A \times E) = \nu(E) \quad \text{and} \quad \pi(D \times G) = \eta(D), \tag{8}$$

with ν and η as in (4) and (3), respectively.

Next is an example of a flexible assignment economy.

Example 3.1. *Let $\mathcal{E}_\Pi$ be an economy where $\mathbf{A}$, $\mathbf{G}$, and u are as in Example 2.6. The set of feasible flexible assignments Π for $\mathcal{E}_\Pi$ is not empty because it contains at least the product measure $\pi = \delta_a \times \nu$ defined by*

$$\pi(E) := \frac{1}{2}\delta_{(a,g_1)}(E) + \frac{1}{2}\delta_{(a,g_2)}(E) \quad \forall E \in \mathcal{B}(A \times G).$$

Definition 3.2. *The* core $\mathbf{C}(\mathcal{E}_\Pi)$ *of an economy $\mathcal{E}_\Pi$ is the set of all flexible assignments π such that there is no coalition $S \in \mathcal{B}(A)$ (with $\eta(S) > 0$), set $H \in \mathcal{B}(G)$ (with $\nu(H) > 0$) and flexible assignment κ that satisfy the following three conditions for any E in $\mathcal{B}(G) \cap H$, and D in $\mathcal{B}(A) \cap S$:*

F1 *i)* $\kappa(A \times E) = \nu(E)$ *and* $\kappa(D \times G) = \eta(D)$,

ii) $\kappa(S \times E) = \pi(S \times E)$ *and* $\kappa(D \times H) = \pi(D \times H)$*;*

F2 *if $\eta(D) > 0$ and $\nu(E) > 0$, then*

$$\int_{D\times E} u(a,g)\pi(da,dg) \le \int_{D\times E} u(a,g)\kappa(da,dg). \tag{9}$$

F3 *there exists D' in $\mathcal{B}(A)\cap S$ and E' in $\mathcal{B}(G)\cap H$, such that (9) is satisfied with inequality.*

Condition **F1**-*i*) refers to the feasibility of the assignment. Condition **F1**-*ii*) refers to preserving "the quantity" (mass) and "the types" of agents and indivisible goods in the change from π to κ. In other words, the mass $\pi(S \times E)$ of types of goods $E \subset H$ that are assigned to coalition S does no change ($\kappa(S \times E) = \pi(S \times E)$ for any E in $\mathcal{B}(G) \cap H$); conversely, the mass $\pi(D \times H)$ of types of agents $D \subset S$ that are assigned to H does no change ($\kappa(D \times H) = \pi(D \times H)$ for any D in $\mathcal{B}(A) \cap S$), thus, the coalition is on its own to block π. Conditions **F2** and **F3** refer to the incentives that individuals in coalition S have to improve their assignments.

The next theorem establishes conditions that are sufficient for the core of an economy $\mathcal{E}_\Pi$ to be nonempty.

Theorem 3.3. *For any economy $\mathcal{E}_\Pi$, the core $\mathbf{C}(\mathcal{E}_\Pi)$ is not empty.*

Theorem 3.3 is a consequence of Theorem 4.3 and Proposition 4.4, which are proven in section 4.

3.2. Assignments in $\mathcal{E}$ and $\mathcal{E}_\Pi$

In this section we compare the definitions of assignment for the economies $\mathcal{E}$ and $\mathcal{E}_\Pi$. Let μ be a type-exclusive assignment of an economy $\mathcal{E}$. We can rewrite μ as a flexible assignment π_μ for an economy $\mathcal{E}_\Pi$ as follows: for any $K \in \mathcal{B}(A \times B)$[3] let

$$\pi_\mu(K) := \eta(K_a) \quad \text{where} \quad K_a := \{a \in A \ : \ (a, \mu(a)) \in K\}. \tag{10}$$

Let μ be a flexible assignment in $\mathcal{E}$, note that if $K = S \times H$ in (10), where $S \in \mathcal{B}(A)$ and $H \in \mathcal{B}(G)$, then

$$\pi_\mu(S \times H) := \eta(S \cap \mu^{-1}(H)) = \eta(\mu^{-1}(H)) = \nu(H). \tag{11}$$

The following examples illustrate how a type-exclusive assignment μ can be rewritten as a flexible assignment π_μ.

Example 3.4. *Let $\mathcal{E}_\Pi$ be an economy where A, G, η, ν and u are as in Example 2.1. Then for any bijective function $\mu : A \to G$, the probability measure π_μ defined by*

$$\pi_\mu(a, g) = \begin{cases} \frac{1}{n} & \text{if} \quad g = \mu(a) \\ 0 & \text{otherwise} \end{cases}$$

is a feasible flexible assignment for the economy $\mathcal{E}_\Pi$.

[3] Where $\mathcal{B}(A \times B)$ is the σ-algebra product of $\mathcal{B}(A)$ and $\mathcal{B}(G)$.

Example 3.5. *Let $\mathcal{E}_\Pi$ be an economy where A, G, η, ν and u are as in Example 2.2. Consider the type-exclusive assignments $\mu_1(a) = a$ and $\mu_2(a) = 1 - a$. Then π_{μ_1} and π_{μ_2}, as in (10), are feasible flexible assignments in the economy $\mathcal{E}_\Pi$. That is, for any $(b, c] \subset [0, 1]$*

$$\begin{aligned}\pi_{\mu_2}((b,c],G) &= \int_{[b,c)\times G} \pi_{\mu_2}(da,dg) = c - b = \eta(b,c] \\ \pi_{\mu_2}(A,(b,c]) &= \int_{A\times[b,c)} \pi_{\mu_2}(da,dg) = (1-b) - (1-c) = c - b = \nu(b,c].\end{aligned}$$

Similarly for π_{μ_1}. For this economy $\mathcal{E}_\Pi$ the probability measure π_u with uniform density $\pi_u(da, dg) = 1$ is also a feasible assignment.

Note that if μ is a feasible type-exclusive assignment for $\mathcal{E}$, then π_μ in (10) satisfies (8), i.e., π_μ is a feasible flexible assignment of $\mathcal{E}_\Pi$.

Proposition 3.6. *Consider a feasible type-exclusive assignment μ for an economy $\mathcal{E}$ and a flexible assignment π_μ for an economy $\mathcal{E}_\Pi$, defined as in (10). Suppose that there exists a coalition $S \in \mathcal{B}(A)$ and assignment γ that satisfies* **E1**-**E3** *in Definition 2.3. Then the assignment π_γ in $\mathcal{E}_\Pi$ defined as (10) satisfies* **F1**-**F3** *in Definition 3.2 with $H = \overline{\gamma(S)}$.*

Proof. Let $H := \overline{\gamma(S)}$, then $H \in \mathcal{B}(G)$. Using the inverse image properties $S \subset \gamma^{-1}(\gamma(S)) \subset \gamma^{-1}(\overline{\gamma(S)})$ we have that

$$\nu(H) = \eta\left(\gamma^{-1}\left(\overline{\gamma(S)}\right)\right) \geq \eta(S) > 0.$$

If γ satisfies **E1**, then π_γ as in (10) satisfies **F1**-*i*), because if $E \in \mathcal{B}(G) \cap H$ and $D \in \mathcal{B}(A) \cap S$, then we have

$$\begin{aligned}\pi_\gamma(D \times G) &= \eta(D \cap \gamma^{-1}(G)) = \eta(D \cap A) = \eta(D), \\ \pi_\gamma(A \times E) &= \eta(\gamma^{-1}(E)) = \nu(E) \quad \text{(by (10) and (11))}.\end{aligned}$$

Moreover, if $E \in \mathcal{B}(G) \cap H$ and $D \in \mathcal{B}(A) \cap S$, then

$$\begin{aligned}\pi_\gamma(D \times H) &= \eta(D \cap \gamma^{-1}(H)) = \eta(D \cap S) = \eta(D) \\ \pi_\gamma(S \times E) &= \eta(S \cap \gamma^{-1}(E)) = \eta(\gamma^{-1}(E)) = \nu(E) \quad \text{(by (10) and (11))}.\end{aligned}$$

Since $\eta(\mu^{-1}(E)) = \nu(E)$ for all $E \in \mathcal{B}(G) \cap H$ and $D \in \mathcal{B}(A) \cap S$, we have that $\kappa(S \times E) = \pi(S \times E)$ and $\kappa(D \times H) = \pi(D \times H)$. Therefore **F1**-*ii*) is satisfied.

On the other hand, note that for any type-exclusive assignment γ of $\mathcal{E}$, if $E \in \mathcal{B}(G)$ and $D \in \mathcal{B}(A)$, then

$$\int_{D\times E} u(a,g)\pi_\gamma(da,dg) = \int_{D\cap\gamma^{-1}(E)} u(a,\gamma(a))\eta(da). \tag{12}$$

Since γ satisfies **E2**, if $E \in \mathcal{B}(G)\cap H$ and $D \in \mathcal{B}(A)\cap S$, with $\eta(E) > 0$ and $\nu(D) > 0$, then using (12) we have that

$$\int_{D\times E} u(a,g)\pi_\mu(da,dg) \leq \int_{D\times E} u(a,g)\pi_\gamma(da,dg), \tag{13}$$

which implies **F2**.

Finally, if there exist D' that satisfies **E3**, take $E' := \overline{\gamma(D')}$, we have that $E' \in \mathcal{B}(G)\cap H$. Using the inverse image properties $D' \subset \gamma^{-1}(\gamma(D')) \subset \gamma^{-1}(\overline{\gamma(D')})$, then

$$\nu(E') = \eta\left(\gamma^{-1}\left(\overline{\gamma(D')}\right)\right) \geq \eta(D') > 0.$$

Then (13) is satisfied with strictly inequality and this implies **F3**. □

The following theorems is a direct consequence of Proposition 3.6.

Theorem 3.7. *Consider an economy $\mathcal{E}_\Pi$, and π_μ in the core of $\mathcal{E}_\Pi$. If π_μ is as in (10), then the type-exclusive assignment μ is in the core of $\mathcal{E}$.*

Next example covers the classical housing market proposed by Shapley and Scarf (1974), Examples 2.1, 2.5, 2.6, 3.4, 3.1, and Remark 2.7.

Example 3.8. *Consider an economy $\mathcal{E}_\Pi$ where A and G are finite sets and the function u as in (1) is represented as a $m \times n$-matrix $[u(a,g)]_{\{a\in A, g\in G\}}$ where n is the cardinality of A and m is the cardinality of G. Let η and ν be any discrete probability distributions over the sets A and G, respectively. The product probability $\pi_{\eta,\nu} := \eta \times \nu$ is a feasible flexible assignment.*

4. Optimal transport theory and proof of Theorem 3.3, $\mathrm{C}(\mathcal{E}_\Pi) \neq \emptyset$

In this section we prove Theorem 4.3 and Proposition 4.4, which in turn are used to prove Theorem 3.3 in Section 4.2.

4.1. Looking for a Pareto optimal assignment

Consider (10), and let $\mathcal{L}$ be as in (6). We define the set

$$\Pi_{\mathcal{L}} := \{\pi \in \mathbb{P}(A \times G) : \pi = \pi_\mu, \quad \mu \in \mathcal{L}\}, \tag{14}$$

where $\mathbb{P}(A \times G)$ is the set of probability measures on $A \times G$.

In this case, we can rewrite problem (7) as

$$\max_{\pi \in \Pi_{\mathcal{L}}} \int_{A \times G} u(a, g)\pi(da, dg). \tag{15}$$

The set of feasible type-exclusive assignments (14) is not necessarily convex nor compact; in fact, it may be empty (as in the Example 2.6). In any case, (15) may have no solution. To solve the problem, we replace the set of feasible type-exclusive assignments $\Pi_{\mathcal{L}}$ by the convex set of Π which is the set of all feasible flexible assignments, i.e.,

$$\Pi := \{\pi \in \mathbb{P}(A \times G) \ : \ \pi \text{ satisfies } (8)\}, \tag{16}$$

where we have that $\Pi_{\mathcal{L}} \subset \Pi$.

As in Section 2.2, we consider a social planner who searches a Pareto optimal assignment for a flexible assignment economy $\mathcal{E}_\Pi$ as

$$\max_{\pi \in \Pi} \int_{A \times G} u(a, g)\pi(da, dg) \tag{17}$$

with Π as in (16).

Example 4.1. *Consider an economy $\mathcal{E}_\Pi$ as in Example 3.4. Then the set of feasible flexible assignments Π is given by the set of probabilities π that satisfy*

$$\sum_{a \in A} \pi(a, g) = 1/n, \qquad \sum_{g \in G} \pi(a, g) = 1/n.$$

Hence, the social planner's problem for this economy is

$$\max_{\pi \in \Pi} \sum_{a \in A} \sum_{a \in G} u(a, g)\pi(a, g).$$

The equivalence between the social planner's problem for an economy $\mathcal{E}$ in Examples 2.1 and 2.5, and the social planner's problem for an economy $\mathcal{E}_\pi$ can be seen in Koopmans and Beckmann (1957).

Example 4.2. *Consider an economy $\mathcal{E}_\Pi$ as in Example 3.8. Then the set of feasible flexible assignments Π is given by the set of probability measures π that satisfies*

$$\sum_{a\in A} \pi(a,g) = \eta(a), \qquad \sum_{g\in G} \pi(a,g) = \nu(a).$$

Hence, the social planner's problem for this economy $\mathcal{E}_\Pi$ is

$$\max_{\pi\in\Pi} \sum_{a\in A}\sum_{a\in G} u(a,g)\pi(a,g).$$

Let π^* be an optimal assignment for social planner's problem and let $S \subset A\times G$ be the support of π^*. Then for any sequence $\{(a_i,g_i)\}_{i=1}^k$ in S and any bijective function on the k elements $\mu : \{a_i\}_{i=1}^k \to \{g_i\}_{i=1}^k$, we have that

$$\sum_{i=1}^{k} u(a_i,\mu(a_i)) \leq \sum_{i=1}^{k} u(a_i,g_i);$$

see for example Gangbo and McCann (1996), Theorem 2.3. Hence, the solution of the social planner's problem for this economy $\mathcal{E}_\Pi$ also solves the social planner's problem for the economy $\mathcal{E}$ in Example 2.5 where the optimum assignment is searched in a set of permutations.

The following theorem establishes the relation between the core $\mathbf{C}(\mathcal{E}_\Pi)$ of an economy $\mathcal{E}_\Pi$ and the social planner's problem (17).

Theorem 4.3. *If the flexible assignment π^* is solution of the social planner's problem (17), then π^* is in $\mathbf{C}(\mathcal{E}_\Pi)$.*

Proof. Suppose that π^* is a solution to (17) and it is not in $\mathbf{C}(\mathcal{E}_\Pi)$. Then there exists a coalition $S \in \mathcal{B}(A)$ (with $\eta(S) > 0$), a flexible assignment κ, and a set $H \in \mathcal{B}(G)$ (with $\nu(H) > 0$), such that for any E in and $\mathcal{B}(G)\cap H$, and D in $\mathcal{B}(A)\cap S$ satified **F1**-**F3** in Definition 3.2.

Consider the flexible assignment π defined by

$$\pi(O) = \pi^*(O\cap(A\times G)) - \pi^*(O\cap(S\times H)) + \kappa(O\cap(S\times H))$$

for all O in $\mathcal{B}(A\times G)$. Let D be in $\mathcal{B}(A)$ and E in $\mathcal{B}(G)$. Then by **F1**

$$
\begin{aligned}
\pi(D\times G) &= \pi^*((D\times G)\cap(A\times G))-\pi^*((D\times G)\cap(S\times H))\\
&\quad +\kappa((D\times G)\cap(S\times H))\\
&= \pi^*(D\times G)-\pi^*((D\cap S)\times H)+\kappa((D\cap S)\times H)\\
&= \pi^*(D\times G)\\
&= \mu(D),
\end{aligned}
$$

$$
\begin{aligned}
\pi(A\times E) &= \pi^*((A\times E)\cap(A\times G))-\pi^*((A\times E)\cap(S\times H))\\
&\quad +\kappa((A\times E)\cap(S\times H))\\
&= \pi^*(A\times E)-\pi^*(S\times(E\cap H))+\kappa(S\times(E\cap H))\\
&= \pi^*(A\times E)\\
&= \mu(E).
\end{aligned}
$$

Hence π is a feasible flexible assignment satisfying (8) and Since **F2** and **F3** are satisfied, we have that

$$
\begin{aligned}
\int_{A\times G} u(a,g)\pi^*(da,dg) &< \int_{A\times G} u(a,g)\pi^*(da,dg)-\int_{S\times H} u(a,g)\pi^*(da,dg)\\
&\quad +\int_{S\times H} u(a,g)\kappa(da,dg)\\
&= \int_{A\times G} u(a,g)\pi(da,dg).
\end{aligned}
$$

Therefore, π^* is not optimal for problem (17), which is a contradiction. $\square$

4.2. Proof of Theorem 3.3, $\mathbf{C}(\mathcal{E}_\Pi)\neq\emptyset$

The optimization problem (7) is among the oldest and best known problems in probability theory, also known as the optimal transport problem. It was introduced by Gaspard Monge (1728), and posed as a mathematical linear problem (17) by L.V. Kantorovich (1942). The solvability of (17) has been studied under a wide variety of hypotheses on the underlying spaces A and G, and/or the utility function u. For instance see Hernández-Lerma and Gabriel (2002), Jiménez-Guerra and Rodríguez-Salinas (1996). A standard reference about this topic is Villani (2008).

Proposition 4.4. *Consider the economy $\mathcal{E}_\Pi$. Then there exists a solution to (17), i.e., there exists a flexible assignment $\pi^* \in \Pi$ (with Π as in (16)) such that*

$$\int_{A\times G} u(a,g)\pi^*(da,dg) = \max_{\pi\in\Pi} \int_{A\times G} u(a,g)\pi(da,dg). \tag{18}$$

Proof. See Santambrogio (2015) Pages 4-5, Theorem 1.4. $\square$

Proof of Theorem 3.3 Consider the hypotheses on the economy $\mathcal{E}_\Pi$. Then by Proposition 4.4, there exits $\pi^* \in \Pi$ that satisfies (18), i.e., π^* is a solution to (17). By Theorem 4.3, π^* is in $\mathbf{C}(\mathcal{E}_\Pi)$, and so Theorem 3.3 is satisfied. $\square$

5. The ϵ- core and finite approximations

In section 6 we carry out numerical approximations using the core as a solution concept. The aim of this sections is to justify the use of ϵ-core as a finite approximation of the core. We propose two results, Theorems 5.2 and 5.5, to approximate an assignment π in the core of the an economic $\mathcal{E}_\Pi$, through an essentially finite economy (Remark 5.3).

5.1. The ϵ- core and finite approximations

As in Section 2.1, we consider a population $\mathbf{A}$ of agents as in (3), and a population $\mathbf{G}$ of indivisible goods as in (4). We also assume that the preference relations $\{\precsim_a\}_{a\in A}$ are represented by a bounded function u as in (1). In this section we consider a *flexible assignment economy* $\mathcal{E}_\Pi := (\mathbf{A}, \mathbf{G}, u, \pi_0)$,.

Definition 5.1. *The* ϵ-core *of an economy $\mathcal{E}_\Pi$, ϵ-$\mathbf{C}(\mathcal{E}_\Pi)$ is the set of all flexible assignments π such that there is no coalition $S \in \mathcal{B}(A)$ (with $\eta(S) > 0$), set $H \in \mathcal{B}(G)$ (with $\nu(H) > 0$) and flexible assignment κ that satisfy the following three conditions for any E in $\mathcal{B}(G) \cap H$, and D in $\mathcal{B}(A) \cap S$:*

J1 *i)* $\kappa(A \times E) = \nu(E)$ *and* $\kappa(D \times G) = \eta(D)$,

ii) $\kappa(S \times E) = \pi(S \times E)$ *and* $\kappa(D \times H) = \pi(D \times H)$;

J2 *if* $\eta(D) > 0$ *and* $\nu(E) > 0$, *then*

$$\int_{D\times E} u(a,g)\pi(da,dg) - \epsilon \le \int_{D\times E} u(a,g)\kappa(da,dg). \tag{19}$$

J3 *there exists D' in $\mathcal{B}(A) \cap S$ and E' in $\mathcal{B}(G) \cap H$, such that (19) is satisfied with inequality.*

As a fact, $\mathbf{C}(\mathcal{E}_\Pi) \subseteq \epsilon\text{-}\mathbf{C}(\mathcal{E}_\Pi)$ for any $\epsilon > 0$. Moreover, if $\epsilon' \geq \epsilon > 0$, then $\epsilon\text{-}\mathbf{C}(\mathcal{E}_\Pi) \subseteq \epsilon'\text{-}\mathbf{C}(\mathcal{E}_\Pi)$.

Theorem 5.2. *Consider two economies*

$$\mathcal{E}_\Pi := (\mathbf{A}, \mathbf{G}, u, \pi_0), \quad \text{and} \quad \widehat{\mathcal{E}}_\Pi := (\mathbf{A}, \mathbf{G}, \widehat{u}, \pi_0),$$

and suppose that

$$\|u - \widehat{u}\|_\infty = \sup_{(a,g) \in (A \times G)} |u(a,g) - \widehat{u}(a,g)| < \alpha.$$

Then if π is in $\epsilon\text{-}\mathbf{C}(\widehat{\mathcal{E}}_\Pi)$, we have that π is in $\epsilon'\text{-}\mathbf{C}(\mathcal{E}_\Pi)$ where $\epsilon' = \epsilon + \alpha$.

Proof. Let π be in $\epsilon\text{-}\mathbf{C}(\widehat{\mathcal{E}}_\Pi)$, if it satisfies **J1** in Definition 5.1 for $\widehat{\mathcal{E}}_\Pi$, it is clear that π also satisfies **J1** for $\mathcal{E}_\Pi$. Now, if π satisfies **J2** for $\widehat{\mathcal{E}}_\Pi$, then there is no a *coalition* $S \in \mathcal{B}(A)$ (with $\eta(S) > 0$), a flexible assignment κ, and a set $H \in \mathcal{B}(G)$ (with $\nu(H) > 0$), such that for any E in and $\mathcal{B}(G) \cap H$, and D in $\mathcal{B}(A) \cap S$

$$\int_{D \times E} u(a,g)\pi(da,dg) - \int_{D \times E} u(a,g)\kappa(da,dg)$$

$$\begin{aligned}
\leq \; & \int_{D \times E} u(a,g)\pi(da,dg) - \int_{D \times E} \widehat{u}(a,g)\pi(da,dg) \\
& \quad + \int_{D \times E} \widehat{u}(a,g)\pi(da,dg) - \int_{D \times E} u(a,g)\kappa(da,dg) \\
\leq \; & \alpha + \epsilon = \epsilon'.
\end{aligned}$$

Then **J2** is satisfied, with $\epsilon' = \alpha + \epsilon$, for $\mathcal{E}_\Pi$. Finally, if π satisfies **J3** for $\widehat{\mathcal{E}}_\Pi$, it is clear that π also satisfies **J3** , with $\epsilon' = \alpha + \epsilon$, for $\mathcal{E}_\Pi$. □

Remark 5.3. *Consider an economy $\mathcal{E}_\Pi := (\mathbf{A}, \mathbf{G}, u, \pi_0)$ where, A (in $\mathbf{A}$) and G (in $\mathbf{G}$) are compact and separable metric spaces. Let u be a continuous function and consider any partitions $P_A^k := \{A_i\}_{i=0}^{k-1}$ over A and $P_G^r := \{G_j\}_{j=0}^{r-1}$ over G. Let $\widehat{u}$ be the discrete approximation of u defined by the function*

$$\widehat{u}(a,g) := u(a_i, g_j), \quad \text{if } (a,g) \in A_i \times G_j, \tag{20}$$

where $(a_i, g_j) \in inA_i \times G_j$ are fixed vectors.

Now, consider the economy

$$\widehat{\mathcal{E}}_\Pi := (\mathbf{A}, \mathbf{G}, \widehat{u}, \pi_0) \tag{21}$$

with $\widehat{u}$ as in (20). In this case, we say that the $\widehat{\mathcal{E}}_\Pi$ is essentially finite in the sense that for any O in $\mathcal{B}(A \times G) \cap \left(P_A^k \times P_G^r\right)$[4], we have that

$$\begin{aligned} \int_O \widehat{u}(a,g)\pi(da,dg) &= \sum_{(A_i \times G_j) \in O \cap P_A^k \times P_G^r} u(a_i, g_j)\pi(A_i \times G_j) \\ &\approx \sum_{(A_i \times G_j) \in O \cap P_A^k \times P_G^r} u(a_i, g_j)\widehat{\pi}(A_i \times G_j) \end{aligned} \tag{22}$$

where $\widehat{\pi}$ is a discrete approximation of π over the partitions[5]*. If we consider the population*

$$\mathbf{A}_k := (P_A^k,\ \mathcal{P}(P_A^k), \eta_k) \quad \text{and} \tag{23}$$

$$\mathbf{G}_r := (P_G^r,\ \mathcal{P}(P_G^r), \nu_r), \tag{24}$$

where $\mathcal{P}$ represents the potential set, and η_k and η_r are discrete approximation of η and ν over P_A^k, and P_G^r, respectively. Then the economy

$$\widehat{\mathcal{E}}_{\Pi_r^k} := (\mathbf{A_k}, \mathbf{G_r}, \widehat{u}, \widehat{\pi}_0), \tag{25}$$

approximates $\mathcal{E}_\Pi$ and $\widehat{\mathcal{E}}_\Pi$ in (21), see (22).

Using Theorem 5.2, we can approximate an assignment π in the core of an economic $\mathcal{E}_\Pi$ through an essentially finite economy $\widehat{\mathcal{E}}_{\Pi_r^k}$. That is, we can search an assignment $\widehat{\pi}$ in the core of $\widehat{\mathcal{E}}_{\Pi_r^k}$ which approximate some

$$\pi \in \epsilon - \mathcal{C}(\widehat{\mathcal{E}}_\Pi) \subset \epsilon' - \mathcal{C}(\mathcal{E}_\Pi) \quad \text{for} \quad \epsilon' \geq \epsilon > 0. \tag{26}$$

[4]Where $\mathcal{B}(A \times G)$ is the σ-algebra product of $\mathcal{B}(A)$ and $\mathcal{B}(G)$, and $P_A^k \times P_G^r$ is the Cartesian product of P_A^k and P_G^r

[5]If X is a separable metric space, then the set of all probability measures whose supports are finite are dense in the set of all all probability measures of X, (see Parthasarathy, 1967, Theorem 6.3, p. 44)

5.2. A simple random sample on $\mathcal{E}_\Pi$

In this section we propose a result, Theorem 5.5, that allows us to use the TTC algorithm to approximate an assignment $\widehat{\pi}$ in the core of $\widehat{\mathcal{E}}_{\Pi_r^k}$ in (25)-Remark 5.3, using Monte Carlo integration techniques (see section 6.2).

Consider a flexible assignment economy $\mathcal{E}_\Pi := (\mathbf{A}, \mathbf{G}, u, \pi_0)$, where the populations $\mathbf{A}$ and $\mathbf{G}$ are as in (3) and (4), respectively. Supposes that $A \subset \mathbb{R}^M$ and $G \subset \mathbb{R}^N$ are compact set, and let $A_n = \{\alpha_1, \alpha_2, ..., \alpha_n\}$, and $G_n = \{\beta_1, \beta_2, ..., \beta_n\}$, be simple random sample of populations $\mathbf{A}$ and $\mathbf{G}$, respectively. Note that for each $k, r = 1, 2, ..., n$, $\alpha_k \in A$ and $\beta_r \in G$. The preference relations $\{\precsim_{\alpha_k}\}_{k=1}^n$ over G_n, are represented as a square matrix $[u(\alpha_i, \beta_k)]_{k,r=1}^n$ of rank n.

Consider the classical housing market equivalent to the one of Shapley and Scarf (1974) described by the economy

$$\mathbf{E}_n =: \left(A_n, G_n, [u(\alpha_i, \beta_k)]_{k,r=1}^n, \mu_0\right),$$

where μ_0 is a bijective function. In this case, any bijective function $\mu : A_n \to G_n$ is a feasible assignment.

Definition 5.4. *The* core $\mathbf{C}(\mathbf{E}_n)$ *of an economy* $\mathbf{E}_n$ *is the set of all feasible assignment* μ *such that there is no coalition* $S_p = \{a_{i_1}, a_{i_2}, ..., a_{i_p}\}$*subset of* A_n *and assignment* γ *such that*

- **X1** *i)* γ *is a bijective ,*
  ii) $\gamma(a_{k_s}) \in \{\mu(a_{k_s}) : \ s = 1, ..., p\}$ *for all* $s = 1, 2, ..., p$*;*
- **X2** $u(a_{k_s}, \mu(a_{k_s})) \leq u(a_{k_s}, \gamma(a_{k_s}))$ *for all* $s = 1, 2, ..., p$*;*
- **X3** $u(a_{k_{s'}}, \mu(a_{k_{s'}})) < u(a_{k_{s'}}, \gamma(a_{k_{s'}}))$ *for some* $s' \in \{1, ..., p\}$.

Condition **X1**-*i)* refers to the feasibility of the assignment. Condition **X1**-*ii)* ensures that the blocking coalition S does not require goods held by agents out of S. Conditions **X2** and **X3** refer to the incentives that agents in coalition S have to improve their assignments.

The following theorem establishes a relation between the economies $\mathcal{E}_\Pi$ and $\mathbf{E}_n$.

Theorem 5.5. *Consider a flexible assignment economy* $\mathcal{E}_\Pi := (\mathbf{A}, \mathbf{G}, u, \pi_0)$, *and the economy* $\mathbf{E}_n =: \left(A_n, G_n, [u(\alpha_i, \beta_k)]_{k,r=1}^n, \mu_0\right)$ *, where* $A_n = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ *and* $G_n = \{\beta_1, \beta_2, ..., \beta_n\}$ *are simple random sample of populations* $\mathbf{A}$ *and* $\mathbf{G}$*,* *respectively. Suppose that $\mu \in \mathcal{C}(\mathbf{E}_n)$ and n is large enough, then there exists $\pi \in \mathbf{C}(\mathcal{E}_\Pi)$, such that*

$$\{(a_1, \mu(a_1)), (a_2, \mu(a_2)), ..., (a_n, \mu(a_n))\},$$

is a simple random sample where each element has a distribution π.

Proof. By Theorem 3.3 $\mathbf{C}(\mathcal{E}_\Pi)$ is nonempty. Let

$$\{(\alpha_1, \mu(\alpha_1)), (\alpha_2, \mu(\alpha_2)), ..., (\alpha_n, \mu(\alpha_n))\},$$

be a simple random sample where each element has a distribution π. Assume that π is not in $\mathbf{C}(\mathcal{E})$. Then there exists $S \in \mathcal{B}(A)$ (with $\eta(S) > 0$), flexible assignment κ, and set $H \in \mathcal{B}(G)$ (with $\nu(H) > 0$), such that satisfies $F1$-$F3$, in Definition (3.2). Since n is large enough, then there exists $S_p = \{\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_q}\} \subset A_n$ and $H_p = \{\beta_{k_1}, \beta_{k_2}, ..., \beta_{k_q}\} \subset G_n$ [6] such that

$$\{(\alpha_{i_1}, \beta_{k_1}), (\alpha_{i_2}, \beta_{k_2}), ..., (\alpha_{i_p}, \beta_{i_q})\}$$

is in the support of κ, and satisfies that if $D \subset \{1, 2, ..., q\}$, then

$$\sum_{s \in D} u(\alpha_{i_s}, \mu(\alpha_{i_s})) \leq \sum_{s \in D} u(\alpha_{i_s}, \beta_{k_s}). \tag{27}$$

with strictly inequality for some $D' \subset \{1, 2, ..., q\}$. Let $\gamma(\alpha_{i_s}) = \beta_{i_s}$ for $s = 1, ..., p$, and $\gamma(\alpha_i) = \mu(\beta_i)$ if $\alpha_i \notin S_p$. It is clear that γ satisfies **X1-X3**. Then μ is not in $\mathcal{C}(\mathbf{E}_n)$ which is a contradiction. Therefore we have that π is in $\mathbf{C}(\mathcal{E})$. □

6. An illustrative housing market

We establish the existence of a core allocation in the flexible assignment model and validated discrete approximations of the model. We now perform comparative statics in a stylized housing market, incorporating both the idiosyncratic parts of the preferences and the common parts. More specifically, in an empirical model we need to estimate the different parameters of u, μ and η. For example, if $\mathbf{G}$ is a population of houses as in (4), the type set

[6] Since n is large enough, if $S_p = \emptyset$ or $H_p = \emptyset$, then $\eta(S) = 0$ or $\nu(H) = 0$ which is a contradiction.

G may be a set of characteristics as: location, number of rooms, size, color, level of contamination in the area, among others. A characteristic can be modeled using a continuous distribution η over G. Similarly, for the population of agents $\mathbf{A}$ in (4), we may be interested in the preferences of agents with different characteristics such as income level, age, sex, workplace, among others. We illustrate the use of our model with a simple flexible assignment economy $\mathcal{E}_\Pi := (\mathbf{A}, \mathbf{G}, u, \pi_0)$ where:

i) $\mathbf{G}$ is a populations of houses as in (4), with $G = [0,1] \times [0,1]$, and ν is a uniform distribution on G; each point $g = (g_1, g_2) \in G$ represents the location of the house g. It is the idiosyncratic part of the preferences.

ii) Business center are concentrated in three locations

$$\{(0,0),\ (0.5, 0.5),\ (1,1)\}$$

called business center. We consider that each agent only works in one of this business centers.[7] The agents prefer the homes closest to their workplace. Then the populations of agents $\mathbf{A}$ as in (4), has a types set of the form $A = \{a_1, a_2, a_3\}$, where a_1, a_2, and a_3 are the types of agents that works in $(0,0)$, $(0.5, 0.5)$ and $(1,1)$, respectively. We assume that the utility function u has the form

$$u(a_i, g) = -d(a_i, g) \quad \forall\ i = 1, 2, 3,\ g \in G$$

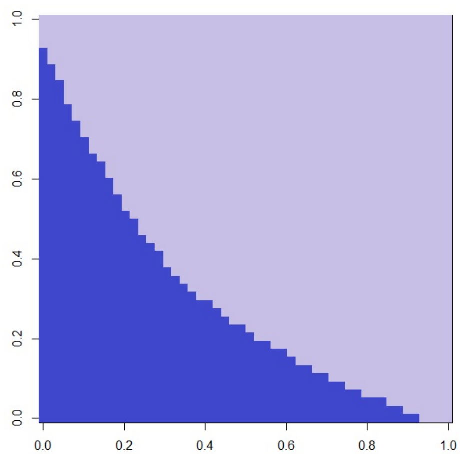
where $d(a_i, g)$ is the Euclidean distance that there exits of the localization or coordinated $g \in G$ of house at the location a_i of the business center, it is the common part of the preferences. Finally, we assume that η is a distribution on A of the form $\eta(a_1) = \eta(a_3) = 0.3$ and $\eta(a_2) = 0.4$.

6.1. Core approximation using optimal transport theory

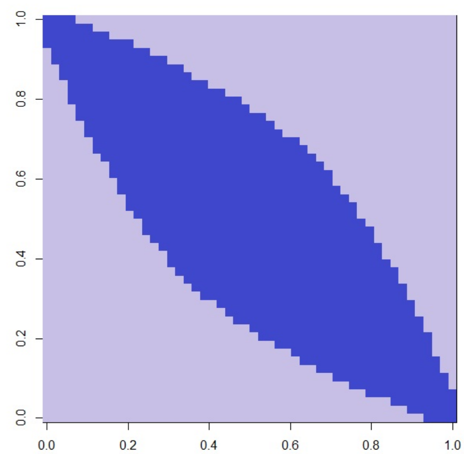
Using Theorem 5.2 and Remark 5.3 we search an assignment $\widehat{\pi}$ in the core of a finite economy $\widehat{\mathcal{E}}_{\Pi_r^k}$ (as in (25)) that approximate an assignment π in the core of an economy $\mathcal{E}_\Pi$, see (22) and (26).

[7]We can aggregate a continuous set of business centers (for example a line connecting the points $(0,0)$ and $(1,1)$), we use only the location of three business center to obtain illustrative graphics.

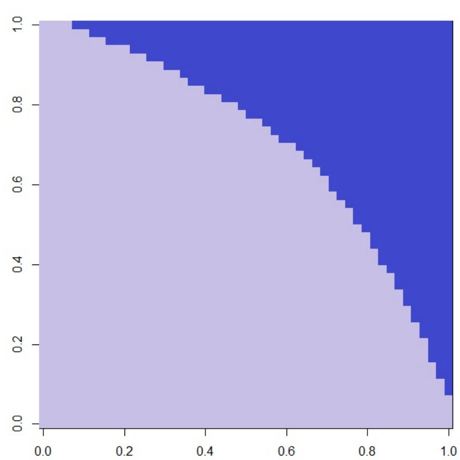
We approximate ν with a uniform discrete distribution $\widehat{\nu}$ on G. For this, we partition $[0,1]$, into 50 intervals I_j of the same length $l(I_j) = \frac{1}{50}$, then G



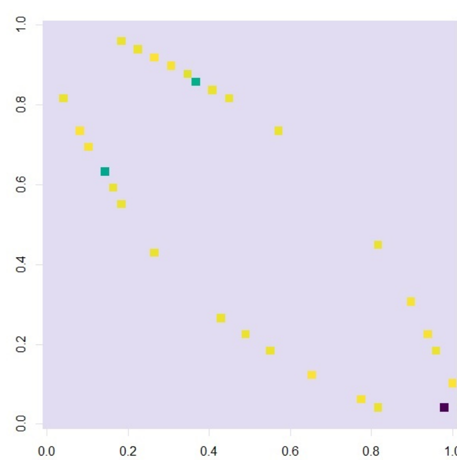
(a) $\widehat{\pi}(a_1, \cdot)$



(b) $\widehat{\pi}(a_2, \cdot)$



(c) $\widehat{\pi}(a_3, \cdot)$



(d) Overlap of the conditional distributions

Figure 1: Conditional distributions of the optimal $\widehat{\pi_{kr}}^*$.

is partitioned into $2,500$ cells of the form $I_h \times I_j$ with a volume or probability mass

$$\nu(I_h \times I_j) = \widehat{\nu}(I_h \times I_j) = l(I_h)l(I_j) = 1/2500.$$

In this case $I_h = [\frac{h}{50}, \frac{h+1}{50})$, $I_j = [\frac{j}{50}, \frac{j+1}{50})$ for $h, j = 0, ..., 48$ and for $I_{49} = [\frac{49}{50}, 1]$. Since η is a discrete distribution, it does not requires a discrete approximation.

To obtain a discrete approximation of u as in (20), let

$$\widehat{u}(a, g) := u(a_i, g_{hj}), \quad \text{if } a = a_i \text{ and } g \in I_h \times I_j,$$

where $i = 1, 2, 3$, and $g_{hj} = \left(\frac{h}{50} + \frac{1}{100}, \frac{j}{50} + \frac{1}{100}\right)$ for $h, j = 0, 1, ..., 49$. Now, we search $\widehat{\pi}$ in $\mathcal{C}(\widehat{\mathcal{E}}_{\Pi_r^k})$ as in (25) solving the optimization problem [8]

$$\max_{\pi} \sum_{i=1}^{3} \sum_{h,j=0}^{49} u(a_i, g_{h,j})\pi(\{a_i\}, I_h \times I_j) \tag{28}$$

Subject to

$$\sum_{i=1}^{3} \pi(\{a_i\}, I_h \times I_j) = \nu(I_h \times I_j) \text{ for } h, j = 0, 1, ...49, \tag{29}$$

$$\sum_{h=0}^{49} \sum_{j=0}^{49} \pi(\{a_i\}, I_h \times I_j) = \eta(\{a_i\}) \text{ for } i = 1, 2, 3. \tag{30}$$

This optimal $\widehat{\pi}$ approximates an assignment π in the ϵ'-core of $\mathcal{E}_\Pi$. Figure 1 shows the conditional distributions of the optimal $\widehat{\pi}$ that solves the optimal transport (28)-(30), as follows:

i) Figure 1-(a) shows the conditional distribution $\widehat{\pi}(a_1, \cdot)$ of the agents that works in the location $a1 = (0, 0)$.

ii) Figure 1-(b) shows the conditional distribution $\widehat{\pi}(a_2, \cdot)$ of the agents that works in the location $a_2 = (0.5, 0.5)$.

iii) Figure 1-(c) shows the conditional distribution $\widehat{\pi}(a_3, \cdot)$ of the agents that works in the location $a3 = (1, 1)$.

[8] We use the library "lpSolve" of the R software to solve (28)-(30), and also to obtain graphics of Figure 1

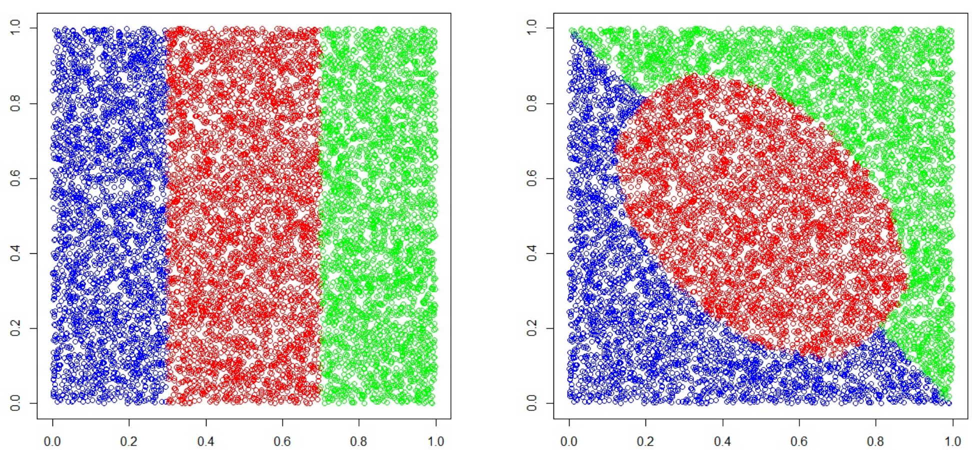
iv) The distributions $\widehat{\pi}(a_1, \cdot)$, $\widehat{\pi}(a_2, \cdot)$ and $\widehat{\pi}(a_3, \cdot)$ are almost uniform over the blue color, that is, each $I_h \times I_j$ of color blue have the same positive probability, and zero in other case .

v) Finally, Figure 1-(d) shows cells where previous conditional distributions overlap.

6.2. Core approximation using the TTC algorithm

Suppose that we have a sample of size n of the populations $\mathbf{G}$, and $\mathbf{A}$, by Theorem 5.5 we can use the TTC algorithm and Monte Carlo integration techniques to approximate an allocation $\widehat{\pi}$ in $\mathcal{C}(\widehat{\mathcal{E}}_{\Pi_r^k})$ as in (25)-Remark 5.3, (for $n \geq k, r$). This assignment $\widehat{\pi}$ is near to an assignment π in the core of an economy $\mathcal{E}_\Pi$, see (22) and (26). In this example we establish the following steps, where $n = 10,000$:

i) Let $\mathbf{A} = \mathbf{A_k}$ and consider an economy over the partitions $\widehat{\mathcal{E}}_{\Pi_r^k} := (\mathbf{A_k}, \mathbf{G_r}, \widehat{u}, \widehat{\pi}_0)$ as in (25), where $\widehat{\pi}_0 \approx \pi_0$.



(a) Simulation of a sample using π_0. (b) Assignment using the TTC algorithm

Figure 2: Assignment using the TTC algorithm of a sample of size $n = 10,000$.

ii) Take samples of size n of the populations $\mathbf{A}$ and $\mathbf{B}$, that is, $A_n =$

$\{\alpha_1, ..., \alpha_n\}$, $G_n = \{\beta_1, ..., \beta_n\}$, this sample satisfies the distribution π_0. For example if

$$\pi_0(da, dg) = 1_{\{a_1\}\times[0,0.3]\times[0,1]} 1_{\{a_2\}\times[0.3,0.7]\times[0,1]} 1_{\{a_3\}\times[0.7,1]\times[0,1]}$$

we generate with a uniform distribution, $3,000$ house location data corresponding to the cell $[0, 0.3] \times [0, 1]$ and assume that each house is assigned one agent that works in $a_1 = (0, 0)$, the blue points in Figure 2-(a). We have $3,000$ agents with the same preference relation $\precsim_{a_1}$. Similarly, we generate $4,000$ house location data in the cell $[0.3, 0, 7] \times [0, 1]$ and $3,000$ in $[0.7, 1] \times [0, 1]$, we suppose that each house is assigned one agent that works in $a_2 = (0.5, 0.5)$ and $a_3 = (1, 1)$, respectively (the red and green points in Figure 2-(a)). This initial assignment is μ_{π_0}

iii) We search for this finite economy $(A_n, G_n, \precsim_{\alpha_j}, \mu_{\pi_0})$, a location in the core using the TTC-algorithm and we obtain the new assignment μ^*. In Figure 2-(b), we illustrate this assignment using colors. Blue points represents the location of houses whose owners work at point $a_1 = (0, 0)$; red points represents the location of houses whose owners work at point $a_2 = (0.5, 0.5)$; and green points represents the location of houses whose owners work at point $a_3 = (1, 1)$.[9]

iv) **Monte Carlo integration.** Since, $\mu^* \in \mathbf{C}(\underline{\mathrm{E}}_n)$, from Theorem 5.5 there exists $\pi \in \mathbf{C}(\mathcal{E}_\Pi)$, such that

$$\{(a_1, \mu(a_1)), (a_2, \mu(a_2)), ..., (a_n, \mu(a_n))\},$$

is a simple random sample where each element has a distribution π. We can approximate the conditional distributions $\widehat{\pi}(a_1, \cdot)$, $\widehat{\pi}(a_2, \cdot)$, $\widehat{\pi}(a_3, \cdot)$ of $\widehat{\mathcal{E}}_{\Pi_r^k}$ in stage i), (similar to Figure 1) from the TTC-algorithm, following the steps:

a) we divide the sets A and G in partitions $P_A^k := \{A_i\}_{i=0}^{k-1}$ and $P_G^r := \{G_j\}_{j=0}^{r-1}$, as Remark 5.3. In the Figure 3 the number of parts in partitions P_A^k and P_G^r is the same, $r = k = 25$.

b) For each $A_i \times G_j$ we count the number of points by type of agent. Let NPB_{ij} and TPB be the number of point of color blue in $A_i \times G_j$ and $P_A^k \times P_G^r$ respectively, in this case $TPB = 3000$.

[9] We use "R package matchingMarkets" to get the assignments generated by the TTC-algorithm, see Klein (2018)

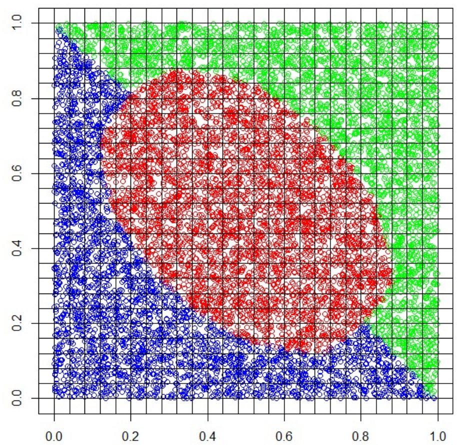


Figure 3: Counting the number of points by type of agent in each partition

c) To obtain the conditional distribution $\widehat{\pi}(a_1, \cdot)$, to each $A_i \times G_j$ we assign the probability $\frac{NPB_{i,j}}{TPB}$. We repeat this processes obtaining $\widehat{\pi}(a_2, \cdot)$, $\widehat{\pi}(a_3, \cdot)$.

To see more about Monte Carlo integration techniques, see for example Evans and Swartz (2000).

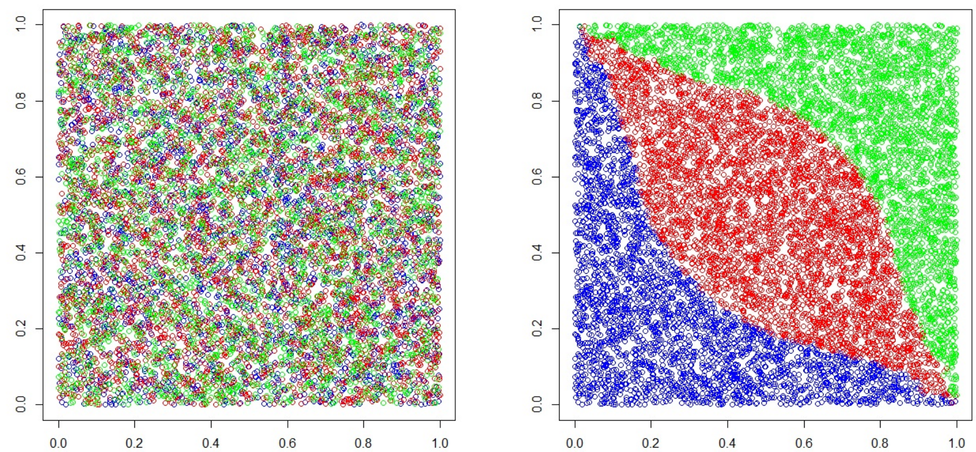
In partitions P_A^k and P_G^r with large numbers of elements k and r, respectively, the Monte Carlo integration technique requires a very large sample n to compute a good approximation of this conditional distribution $\widehat{\pi}(a_i, \cdot)$. In this example this approach is computationally more costly than the optimal transport approach, in particular the time required to perform the approximation is greater.

These techniques allow to numerically evaluate changes in different circumstances and answer, questions such as: How do assignments behave if we add a business center? How do the assignments respond to different compositions of the populations due to changes in ν and η?, among others.

If we change the initial condition π_0 for the TTC algorithm, we obtain different stable assignment. In figure 4 we have a sample with the same characteristics as in Figure 2 but with an initial condition

$$\pi_0'(da, dg) = (0.3)1_{\{a_1\}\times G}(0.4)1_{\{a_2\}\times G}(0.3)1_{\{a_3\}\times G}$$

Where $G = [0,1] \times [0,1]$. Compare Figure 1, 2 and 4 .



(a) Simulation of a sample using π_0'. (b) Assignment using the TTC algorithm

Figure 4: Assignment using the TTC algorithm of a sample of size $n = 10,000$.

Thus, our comparative statics establishes that the original distribution of houses impacts both the agents at an individual level, but also at an aggregate level since the blue, red, and green areas do not overlap with the one obtained at figure Figure 2-(b).

7. Comments

In this paper we present two models of a one-sided matching market. In our setting, the sets of agents and goods are measure spaces, which allows us to present an unified model in which both sets can be continuous or finite. The first model, called a type-exclusive assignment economy, introduces a concept of assignment as a measurable function that assigns to each type of agent a type of good. The second model, called a flexible assignment economy, introduces a concept of assignment as a probability distribution which assigns a mass of agents to a mass of goods. This approach allows us to use optimal transport theory (see Theorems 2.4 and 4.3) to establish conditions for the nonemptiness of the core in both models; see Theorems 3.3, B.3 and B.4. We show that the type-exclusive assignment economy is embedded in a flexible assignment economy; see Proposition 3.6, Theorem 3.7.

We study the concept of ϵ-core to approximate the core of general models through finite models. We proposed two results, Theorems 5.2 and 5.5, to approximate an assignment in the core of the an flexible assignment economic, through an essentially finite economy (Remark 5.3). Finally, we propose continuous housing allocation problem and solve it using optimal transport and other numerical techniques such as TTC algorithm, and Monte Carlo integration, for this use finite models.

Our approach complement the active research on econometric of matching, which allows to elaborate new competitive comparative statics in these markets. Particularly, the theoretical development of numerical and computational approximations of these markets will allow to expand the number of empirical applications. For example, using these approximation and simulation techniques under a comparative statics scheme (as developed in Section 6) it is possible to evaluate public policies of interest in these matching markets.

Appendix

A. An extension of the Debreu's preferences representation theorem

Consider an economy Assume that A and G are compact Borel spaces, that is, they are complete, separable and compact metric spaces. For the preference relations $\{\precsim_a\}_{a\in A}$, we assume that

H1 *rationality:* for each a in A, $\precsim_a$ is a is a complete and transitive order relation;

H2 *continuity in the goods:* for each a in A and g' in G the sets $\{g \in G : g' \precsim_a g\}$ and $\{g \in G : g \precsim_a g'\}$ are closed;

H3 *continuity in the agents:* for any $g', g \in G$ the set $\{a \in A : g' \precsim_a g\}$ is closed.

The following theorem is an extension of the Debreu's preferences representation theorem Debreu (1954). The proof can be see in Levin (1983), or in Rachev and Rüschendorf (1998) Theorem 5.5.18 page 337, or Bridges and Mehta (2013) Theorem 8.3.6 page 146.

Theorem A.1. *Let A be the set of type of agents, and G be the set of type of indivisible goods. Assume as in (1) and (3) that A and G are compact metric spaces, and let G, in addition, be separable. Suppose that the preference relations $\{\precsim_a\}_{a\in A}$ satisfy* **H1, H2** *and* **H3**. *Then there exits a continuous function $u : A \times G \to [0,1]$ such that*

$$\forall a \in A, \quad g \precsim_a g' \Longleftrightarrow u(a,g) \leq u(a,g').$$

B. Non-atomic sets of types an the non-emptiness of $\mathbf{C}(\boldsymbol{\mathcal{E}})$

In Section 2, we defined an economy where the set of feasible type-exclusive allocation $\mathcal{L}$ in (6) may be empty (see Example 2.6). In Section 3 we defined a general concept of assignment (the flexible-assignment) which ensures that the set of feasible assignments of a economy is not empty. Moreover, under general conditions the core of the economy is nonempty, as stated in Theorem 3.3.

In this section we establish particular conditions under which if we have non-atomic sets of types, then the core of economy $\mathcal{E}$ (in Section 2) is nonempty. We consider the following assumptions:

A1 *Non-atomic sets of types.* The set of types of agents A and the set of types of indivisible goods G are compact subsets of R^n, and η is a probability measure on $\mathcal{B}(A)$ which is absolutely continuous with respect to n-dimensional Lebesgue measure.

A2 *Heterogeneity on utility.* Let U be a differentiable function in $A \times G$. If $g', g \in \text{supp}(\nu)$ with $g \neq g'$, then

$$\frac{\partial u}{\partial a}(a,g) \neq \frac{\partial u}{\partial a}(a,g').$$

A3 *Convexity in types of agents.* The set A is convex, and for each $g \in \text{supp}(\nu)$ the function $a \to u(a,g)$ is concave or convex.

A4 *Smoothness on the heterogeneity of types of agents.* The set $\text{int}(\text{supp}(\eta))$ is not empty and its complement is Lebesgue negligible; for every $g \in \text{supp}(\nu)$, $a \to u(a,g)$ is differentiable and for any $a \in \text{supp}(\eta)$, there exits a neighborhood V of a and a number $c_a > 0$ such that

$$|u(a_1,g) - u(a_2,g)| \leq c_a \|a_1 - a_2\| \quad \forall a_1, a_2 \in V, g \in \text{supp}(\nu).$$

Proposition B.1. *Consider assumptions* **A1**, **A2**, **A3**. *Then the problems (7) and (17) admit at least one solution. Moreover, if* $\mu \in \mathcal{L}$ *(with* $\mathcal{L}$ *as in (6)) is a solution to (7), then* π_μ *as in (10) is a solution to (17), and*

$$\max_{\mu \in \mathcal{L}} \int_A u(a, \mu(a))\eta(da) = \max_{\pi \in \Pi} \int_A u(a, g)\pi(da, dg)$$

with Π *as in (16).*

Proof. See Levin (2004), Theorems 1.2 and 1.3. □

Proposition B.2. *Consider assumptions* **A1**, **A2**, **A4**. *Then the problems (7) and (17) admit at least one solution. Moreover, if* $\mu \in \mathcal{L}$, *(with* $\mathcal{L}$ *as in (6)) is a solution to (7), then* π_μ *as in (10) is a solution to (17), and*

$$\max_{\mu \in \mathcal{L}} \int_A u(a, \mu(a))\eta(da) = \max_{\pi \in \Pi} \int_A u(a, g)\pi(da, dg),$$

with Π *as in (16).*

Proof. See Levin (2004), Theorem 1.4. □

Carlier (2003) propose similar conditions of Proposition B.2, but for metric spaces.

Theorems B.3 and B.4 establish particular conditions under which the core of an economy $\mathcal{E}$ is nonempty, as well as the relation between the cores $\mathbf{C}(\mathcal{E})$ and $\mathbf{C}(\mathcal{E}_\Pi)$.

Theorem B.3. *Let* $\mathcal{E}$ *and* $\mathcal{E}_\Pi$ *be economies where* (u, A, G, ν, η) *satisfies assumptions* **A1**, **A2**, **A3**. *Then the cores* $\mathbf{C}(\mathcal{E})$ *and* $\mathbf{C}(\mathcal{E}_\Pi)$ *are not empty. Moreover, if* $\mu \in \mathbf{C}(\mathcal{E})$ *is a maximum in (7), then* π_μ *in (10) is a maximum in (17), and* $\pi_\mu \in \mathbf{C}(\mathcal{E}_\Pi)$.

Proof. The theorem follows from Theorem 2.4 and Proposition B.1. □

Theorem B.4. *Let* $\mathcal{E}$ *and* $\mathcal{E}_\Pi$ *be economies where* (u, A, G, ν, η) *satisfies assumptions* **A1**, **A2**, **A4**. *Then the cores* $\mathbf{C}(\mathcal{E})$ *and* $\mathbf{C}(\mathcal{E}_\Pi)$ *are not empty. Moreover, if* $\mu \in \mathbf{C}(\mathcal{E})$ *is a maximum in (7), then* π_μ *in (10) is a maximum in (17), and* $\pi_\mu \in \mathbf{C}(\mathcal{E}_\Pi)$.

Proof. The theorem follows from Theorem 2.4 and Proposition B.2. □

Next example satisfies the assumptions **A1**, **A2**, **A4**. To see the proof of the conclusion and other interesting examples, see Levin (2004).

Example B.5. *Let $\mathcal{E} := (\mathbf{A}, \mathbf{G}, u, \mu_0)$ be a type-exclusive assignment economy, where A and G are convex and compact subsets of $\mathbb{R}^n$; η is absolutely continuous with respect to the Lebesgue measure on A; $u(a, g) = -\sum(a_i - g_i)^2$ for $a := (a_1, ..., a_2)$ and $g := (g_1, ..., g_n)$ and μ_0 is any agent's initial endowment. Let $\mu^*(a) = Ha + b$ where H is symmetric and positive semidefinite matrix, and $b \in \mathbb{R}^n$. If $\nu(E) = \eta(\mu^{*-1}(E))$ for all $E \in \mathcal{B}(G)$, Then μ^* is the unique optimal solution of (7), and it is in the core of $\mathcal{E}$. Moreover, π_{μ^*} (as in (10)) is optimal solution of (17), and the core of the flexible assignment economy $\mathcal{E} := (\mathbf{A}, \mathbf{G}, u, \pi_{\mu_0})$ is not empty.*

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