# Simple dominance of fixed priority top trading cycles*

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**Abstract**

We consider assignment problems where agents are to be assigned at most one indivisible object and monetary transfers are not allowed. We study the implementation of fixed priority top trading cycles (FPTTC) rules via simply dominant mechanisms, and provide characterizations of all such FPTTC rules. We further introduce the notion of *simple strategy-proofness* to resolve the issue with agents being concerned about having time-inconsistent preferences, and discuss its relation with simple dominance.

**Keywords:** Fixed priority top trading cycles; Assignment problem; Simple dominance; Simple strategy-proofness; Indivisible goods

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# 1 Introduction

We consider the well-known *assignment problem* (also known as *house allocation problem* or *resource allocation problem*) where a set of heterogeneous indivisible objects are to be allocated among a group of agents so that each agent receives at most one object and monetary transfers are not allowed. Such problems arise when, for instance, the Government wants to assign houses to the citizens, or hospitals to doctors, or a manager wants to allocate offices to employees, or tasks to workers, or professor wants to assign projects to students. Agents are asked to report their preferences over the objects and the designer decides the allocation (of the objects among the agents) based on these reports. An important consideration while designing such decision processes is to implement desirable outcomes when the participating agents are strategic. The standard notion of *strategy-proofness* requires truth-telling to be a dominant strategy, that is, no agent can be strictly better-off by misreporting her (true) preference.

*Fixed priority top trading cycles (FPTTC) mechanism* (Abdulkadiroğlu and Sönmez, 2003) is a well-known strategy-proof way to assign objects in the absence of transfers. An FPTTC mechanism works in steps. At each step, the objects (available at that step) are owned by certain agents who then trade their objects by forming *top trading cycles (TTC)*.[1] Ownership of the objects at the start of each step is determined by a *priority structure*.[2] As observed in Troyan (2019), despite their appealing theoretical properties, the use of FPTTC mechanisms in practice is rare as participating agents find it difficult to understand them, particularly the fact that these mechanisms are strategy-proof.[3,4,5]

The notion of *obvious strategy-proofness (OSP)* (Li, 2017) has emerged as a remedy by strengthening strategy-proofness in a way so that it becomes transparent to the participating agents that a mechanism is not manipulable. The concept of OSP is based on the notion of *obvious dominance* in an *extensive-form game*. A strategy $s_i$ of an agent $i$ in an extensive-form game is obviously dominant if, whenever agent $i$ is called to play, even the worst possible final outcome from following $s_i$ is at least as good as the best possible outcome from following any deviating strategy $s_i'$ of agent $i$, where the best and worst cases are determined by considering all possible strategies that could be played by $i$'s opponents in the future, keeping her own strategy fixed. A mechanism is *OSP-implementable* if one can construct an extensive-form game that has an equilibrium in obviously dominant strategies.

While OSP relaxes the assumption that the participating agents fully comprehend how the strategies of opponents will affect outcomes, it still presumes that they understand how their own future actions will affect outcomes. In other words, when checking obvious dominance, the worst possible final outcome

[1] TTC is due to David Gale and discussed in Shapley and Scarf (1974).

[2] A priority structure is a collection of priorities (over the agents) – one for each object.

[3] Troyan (2019) uses the term "TTC rule" to refer to an FPTTC rule.

[4] Apart from strategy-proofness, FPTTC mechanisms also satisfy *group strategy-proofness*, *Pareto efficiency*, and *non-bossiness*. Group strategy-proofness ensures that no group of agents can be better-off by misreporting their preferences. Pareto efficiency ensures that there is no other way to allocate the objects so that each agent is weakly better-off. Non-bossiness says that an agent cannot change the assignment of another one without changing her own assignment.

[5] Similar phenomena is also observed in other settings, see Chen and Sönmez (2006), Hassidim et al. (2017), Hassidim et al. (2018), Rees-Jones (2018), and Shorrer and Sóvágó (2018) for details.

and the best possible final outcome are taken only over opponents' strategies, $s_{-i}$, fixing the agent's own strategy, $s_i$. Pycia and Troyan (2019) argue that in this case the agents might be concerned about having time-inconsistent preferences or making a mistake while performing backward induction over their own future actions. As a remedy, they introduce a natural strengthening of OSP called *strongly obvious strategy-proofness (SOSP)* by relaxing the assumption that the agents understand how their own future actions will affect outcomes. A strategic plan is strongly obviously dominant if, whenever an agent is called to play, even the worst possible final outcome from the prescribed action is at least as good as the best possible outcome from any other action, where what is possible may depend on all future actions, including actions by the agent's future-self. Thus, strongly obviously dominant strategies are those that are weakly better than all alternative actions even if the agent is concerned that she might have time-inconsistent preferences.[6]

### 1.1 Our motivation and contribution

Mandal and Roy (2022a,b) study the OSP-implementability of FPTTC mechanisms. They introduce the notion of *dual ownership* and show that it is both necessary and sufficient condition for an FPTTC mechanism to be OSP-implementable. An FPTTC mechanism satisfies dual ownership if for each preference profile and each step of the FPTTC mechanism at that preference profile, there are at most two agents who own all the objects available at that step. Although dual ownership is an intuitive property (and thereby, is quite helpful for explaining it to the participating agents), it is not so convenient for the designer to check whether a given FPTTC mechanism satisfies this property or not. This is because, technically, one needs to check at *every* preference profile and *every* step of the FPTTC mechanism at that preference profile, whether at most two agents are owning all the (available) objects at that stage or not. This motivates one natural question: *Is there any equivalent property to dual ownership, that is easier for the designer to check?*

To tackle this question, we present two conditions for FPTTC mechanisms, namely *acyclicity* and *strong acyclicity* (Troyan, 2019).[7] Both are technical properties, which, as the names suggest, ensure that certain type of cycles are *not* present in the associated priority structure of an FPTTC mechanism. The advantage of checking these properties for an FPTTC mechanism is that they only involve the priority structure, and not anything about the state of the FPTTC mechanism at different steps at different preference profiles. In a model without outside options, we show that dual ownership and acyclicity are equivalent properties (Theorem 4.2), while dual ownership and strong acyclicity are equivalent properties in a model with outside options (Theorem 4.3).[8] Since dual ownership property is intuitive but not convenient for the designer to check, whereas (strong) acyclicity property is technical but easier to check, these two equivalent properties, in a sense, complement each other.

[6] This verbal description of SOSP is adapted from Pycia and Troyan (2019).

[7] Troyan (2019) uses the term "weak acyclicity" to refer to the strong acyclicity property.

[8] In a model without outside options, every object is "acceptable" to every agent; in a model with outside options, each object need not be acceptable to an agent.

Next, we characterize the structure of SOSP-implementable FPTTC mechanisms. We introduce the notion of *weak serial dictatorship* for this purpose. An FPTTC mechanism satisfies weak serial dictatorship if, for any preference profile and any step of the FPTTC mechanism at that preference profile, if there are more than two objects available at that step, then there is exactly one agent who owns all those objects. We show that in a model without outside options, weak serial dictatorship is both necessary and sufficient condition for an FPTTC mechanism to be SOSP-implementable (Theorem 5.1). We obtain as a corollary (Corollary 5.2) of our result that in a model without outside options and with more objects than agents, the class of SOSP-implementable FPTTC mechanisms is characterized by the class of *serial dictatorships* (Satterthwaite and Sonnenschein, 1981). We further characterize all SOSP-implementable FPTTC rules as serial dictatorships in a model with outside options (Theorem 5.3).

Finally, we introduce the notion of *simple strategy-proofness* which strengthens OSP-implementation to resolve the issue with agents being concerned about having time-inconsistent preferences. Recall that Pycia and Troyan (2019) introduce SOSP-implementation for the same purpose. The concept of simple strategy-proofness is based on obvious dominance in a *simple* extensive-form game. An extensive-form game is simple if every agent is called to play at most once. A mechanism is simple strategy-proof if one can construct a simple extensive-form game that has an equilibrium in obviously dominant strategies. We show that simple strategy-proofness is even stronger than SOSP-implementability (Proposition 6.1). We further show that the class of simply strategy-proof FPTTC mechanisms is same as the class of SOSP-implementable FPTTC mechanisms in both models – with or without outside options (Theorem 6.1).

### 1.2 Additional related literature

Troyan (2019) shows that strong acyclicity is a sufficient condition for an FPTTC rule to be OSP-implementable when there are equal number of agents and objects.[9] Mandal and Roy (2022a) characterize OSP-implementable, Pareto efficient, and non-bossy assignment rules as *hierarchical exchange rules* (Pápai, 2000) satisfying dual ownership.[10] Bade and Gonczarowski (2017) *constructively* characterize OSP-implementable and Pareto efficient assignment rules as the ones that can be implemented via a mechanism they call *sequential barter with lurkers*. Pycia and Troyan (2019) characterize the full class of OSP mechanisms in environments without transfers as *millipede games with greedy strategies*. They also characterize the full class of of SOSP mechanisms as *sequential price mechanisms with greedy strategies*.

Ashlagi and Gonczarowski (2018) consider two-sided matching with one strategic side and show that for general preferences, no mechanism that implements the men-optimal stable matching (or any other

[9]Theorem 1 in Troyan (2019) says that in a model without outside options, strong acyclicity is both necessary and sufficient condition for an FPTTC rule to be OSP-implementable when there are equal number of individuals and objects. Later, Mandal and Roy (2022a) point out that while strong acyclicity is a sufficient condition for the same, it is not necessary (see Footnote 22 in Mandal and Roy (2022a) for details).

[10]Pápai (2000) characterizes all strategy-proof, Pareto efficient, non-bossy, and reallocation-proof assignment rules as hierarchical exchange rules. Later, Pycia and Ünver (2017) introduce the notion of *trading cycles rules* as generalization of hierarchical exchange rules and show that an assignment rule is strategy-proof, Pareto efficient, and non-bossy if and only if it is a trading cycles rule.

stable matching) is obviously strategy-proof for men. They also provide a sufficient condition for a deferred acceptance rule to be OSP-implementable. Later, Thomas (2020) provides a necessary and sufficient condition for the same.

### 1.3 Organization of the paper

The organization of this paper is as follows. In Section 2, we introduce basic notions and notations that we use throughout the paper, define assignment rules, and introduce the notion of simple dominance (OSP-implementation and SOSP-implementation). Section 3 introduces the notion of FPTTC rules. In Section 4, we introduce the dual ownership property of an FPTTC rule and present prior results on the OSP-implementability of FPTTC rules. We also introduce acyclicity property and strong acyclicity property of an FPTTC rule, and discuss their relation with the dual ownership property. In Section 5, we introduce the notion of weak serial dictatorship, and characterize all SOSP-implementable FPTTC rules. In Section 6, we introduce the notion of simple strategy-proofness, and discuss its relation with simple dominance. We further characterize all simply strategy-proof FPTTC rules. All omitted proofs are collected in the Appendix.

## 2 Preliminaries

### 2.1 Basic notions and notations

Let $N = \{1, \ldots, n\}$ be a finite set of agents, and $A$ be a non-empty and finite set of objects. Let $a_0$ denote the ***outside option***. An ***allocation*** is a function $\mu : N \to A \cup \{a_0\}$ such that $|\mu^{-1}(a)| \leq 1$ for all $a \in A$. Here, $\mu(i) = a$ means agent $i$ is assigned object $a$ under $\mu$, and $\mu(i) = a_0$ means agent $i$ is not assigned any object under $\mu$. We denote by $\mathcal{M}$ the set of all allocations.

Let $\mathbb{L}(A \cup \{a_0\})$ denote the set of all strict linear orders over $A \cup \{a_0\}$.[11] An element of $\mathbb{L}(A \cup \{a_0\})$ is called a ***preference*** over $A \cup \{a_0\}$. For a preference $P$, let $R$ denote the weak part of $P$.[12] For a preference $P \in \mathbb{L}(A \cup \{a_0\})$ and non-empty $A' \subseteq A \cup \{a_0\}$, let $\tau(P, A')$ denote the most-preferred element in $A'$ according to $P$.[13] For ease of presentation, we denote $\tau(P, A \cup \{a_0\})$ by $\tau(P)$.

For each object $a \in A$, we define the ***priority*** of $a$ as a "preference" $\succ_a$ over $N$.[14] Following our notational convention, for a priority $\succ \in \mathbb{L}(N)$ and non-empty $N' \subseteq N$, let $\tau(\succ, N')$ denote the most-preferred agent in $N'$ according to $\succ$. For ease of presentation, we denote $\tau(\succ, N)$ by $\tau(\succ)$. For a priority $\succ \in \mathbb{L}(N)$ and an agent $i \in N$, by $U(i, \succ)$ we denote the *(strict) upper contour set* $\{j \in N \mid j \succ i\}$ of $i$ at $\succ$. Furthermore, for a priority $\succ \in \mathbb{L}(N)$, an agent $i \in N$, and non-empty $N' \subseteq N \setminus \{i\}$, we write $i \succ N'$ to mean that $i \succ j$ for all $j \in N'$.

[11]A ***strict linear order*** is a semiconnex, asymmetric, and transitive binary relation.

[12]For all $a, b \in A \cup \{a_0\}$, $aRb$ if and only if $[aPb$ or $a = b]$

[13]For $P \in \mathbb{L}(A \cup \{a_0\})$ and non-empty $A' \subseteq A \cup \{a_0\}$, $\tau(P, A') = a$ if and only if $[a \in A'$ and $aPb$ for all $b \in A' \setminus \{a\}]$.

[14]That is, $\succ_a \in \mathbb{L}(N)$.

We call a collection $\succ_A = (\succ_a)_{a \in A}$ a ***priority structure***. Let $N' \subseteq N$, $A' \subseteq A$, and $\succ_A$ be a priority structure. The ***reduced priority structure*** $\succ^{N'}_{A'}$ is the collection $(\succ^{N'}_a)_{a \in A'}$ such that for all $a \in A'$, (i) $\succ^{N'}_a \in \mathbb{L}(N')$ and (ii) for all $i, j \in N'$, $i \succ^{N'}_a j$ if and only if $i \succ_a j$. Thus, the reduced priority structure $\succ^{N'}_{A'}$ is the restriction of $\succ_A$ to the submarket $(N', A')$.[15] Furthermore, let $\mathcal{T}(\succ^{N'}_{A'}) = \{i \mid \tau(\succ_a, N') = i \text{ for some } a \in A'\}$ be the set of agents who have the highest priority in $N'$ for at least one object in $A'$ according to $\succ_A$.

### 2.2 Types of domains

We denote by $\mathcal{P}_i \subseteq \mathbb{L}(A \cup \{a_0\})$ the set of admissible preferences of agent $i$. A ***preference profile***, denoted by $P_N = (P_1, \ldots, P_n)$, is an element of $\mathcal{P}_N = \prod_{i=1}^{n} \mathcal{P}_i$ that represents a collection of preferences – one for each agent. We say an object $a$ is ***acceptable*** to agent $i$ if $aP_i a_0$.

In this paper, we consider two well-known domains of preference profiles as follows.

(a) The unrestricted domain of preference profiles, which we denote by $\mathbb{L}^n(A \cup \{a_0\})$.

(b) The restricted domain of preference profiles, where every object is acceptable to every agent. With abuse of notation, we denote this domain by $\mathbb{L}^n(A)$.[16]

### 2.3 Assignment rules and simple dominance

An ***assignment rule*** is a function $f : \mathcal{P}_N \to \mathcal{M}$. For an assignment rule $f : \mathcal{P}_N \to \mathcal{M}$ and a preference profile $P_N \in \mathcal{P}_N$, let $f_i(P_N)$ denote the assignment of agent $i$ by $f$ at $P_N$.

Pycia and Troyan (2019) introduce the notion of *simple dominance*. In this paper, we discuss two types of simple dominance in the context of assignment rules, namely *obviously strategy-proofness (OSP)* (Li, 2017) and *strongly obviously strategy-proofness (SOSP)* (Pycia and Troyan, 2019). We use the following notions and notations to present these.

We denote a rooted (directed) tree by $T$. For a rooted tree $T$, we denote its set of nodes by $V(T)$, set of edges by $E(T)$, root by $r(T)$, and set of leaves (terminal nodes) by $L(T)$. For a node $v \in V(T)$, let $E^{out}(v)$ denote the set of outgoing edges from $v$. For an edge $e \in E(T)$, let $s(e)$ denote its source node. A *path* in a tree is a sequence of nodes such that every two consecutive nodes form an edge.

**Definition 2.1.** An ***extensive-form mechanism***, or simply a ***mechanism*** on $\mathcal{P}_N$, is defined as a tuple $G = \langle T, \eta^{LA}, \eta^{NI}, \eta^{EP} \rangle$, where

(i) $T$ is a rooted tree,

(ii) $\eta^{LA} : L(T) \to \mathcal{M}$ is a leaves-to-allocations function,

(iii) $\eta^{NI} : V(T) \setminus L(T) \to N$ is a nodes-to-agents function, and

[15]Thus, $\succ^N_A = \succ_A$.

[16]With abuse of notation, by $\mathbb{L}(A)$, we mean the set of preferences $\{P \in \mathbb{L}(A \cup \{a_0\}) \mid aPa_0 \text{ for all } a \in A\}$.

(iv) $\eta^{EP} : E(T) \to 2^{\mathbb{L}(A \cup \{a_0\})} \setminus \{\emptyset\}$ is an edges-to-preferences function such that

(a) for all distinct $e, e' \in E(T)$ with $s(e) = s(e')$, we have $\eta^{EP}(e) \cap \eta^{EP}(e') = \emptyset$, and

(b) for any $v \in V(T) \setminus L(T)$,

(1) if there exists a path $(v^1, \ldots, v^t)$ from $r(T)$ to $v$ and some $1 \leq r < t$ such that $\eta^{NI}(v^r) = \eta^{NI}(v)$ and $\eta^{NI}(v^s) \neq \eta^{NI}(v)$ for all $s = r+1, \ldots, t-1$, then $\bigcup\limits_{e \in E^{out}(v)} \eta^{EP}(e) = \eta^{EP}(v^r, v^{r+1})$, and

(2) if there is no such path, then $\bigcup\limits_{e \in E^{out}(v)} \eta^{EP}(e) = \mathcal{P}_{\eta^{NI}(v)}$.

For a given mechanism $G$ on $\mathcal{P}_N$, every preference profile $P_N \in \mathcal{P}_N$ identifies a unique path from the root to some leaf in $T$ in the following manner: from each node $v$, follow the outgoing edge $e$ from $v$ such that $\eta^{EP}(e)$ contains the preference $P_{\eta^{NI}(v)}$. If a node $v$ lies in such a path, then we say that the preference profile $P_N$ *passes through the node* $v$. Furthermore, we say two preferences $P_i$ and $P'_i$ of some agent $i$ *diverge at a node* $v \in V(T) \setminus L(T)$ if $\eta^{NI}(v) = i$ and there are two distinct outgoing edges $e$ and $e'$ in $E^{out}(v)$ such that $P_i \in \eta^{EP}(e)$ and $P'_i \in \eta^{EP}(e')$.

For a given mechanism $G$ on $\mathcal{P}_N$, the ***assignment rule*** $f^G : \mathcal{P}_N \to \mathcal{M}$ ***implemented by*** $G$ is defined as follows: for all preference profiles $P_N \in \mathcal{P}_N$, $f^G(P_N) = \eta^{LA}(l)$, where $l$ is the leaf that appears at the end of the unique path characterized by $P_N$.

**Definition 2.2.** A mechanism $G$ on $\mathcal{P}_N$ is ***OSP*** if for all $i \in N$, all nodes $v$ such that $\eta^{NI}(v) = i$, and all $P_N, \tilde{P}_N \in \mathcal{P}_N$ passing through $v$ such that $P_i$ and $\tilde{P}_i$ diverge at $v$, we have $f^G_i(P_N) R_i f^G_i(\tilde{P}_N)$.

An assignment rule $f : \mathcal{P}_N \to \mathcal{M}$ is ***OSP-implementable*** (on $\mathcal{P}_N$) if there exists an OSP mechanism $G$ on $\mathcal{P}_N$ such that $f = f^G$.

**Definition 2.3.** A mechanism $G$ on $\mathcal{P}_N$ is ***SOSP*** if for all $i \in N$, all nodes $v$ such that $\eta^{NI}(v) = i$, and all $P_N, P'_N, \tilde{P}_N \in \mathcal{P}_N$ passing through $v$ such that (i) $P_i$ and $P'_i$ do not diverge at $v$ and (ii) $P_i$ and $\tilde{P}_i$ diverge at $v$, we have $f^G_i(P'_N) R_i f^G_i(\tilde{P}_N)$.[17]

An assignment rule $f : \mathcal{P}_N \to \mathcal{M}$ is ***SOSP-implementable*** (on $\mathcal{P}_N$) if there exists an SOSP mechanism $G$ on $\mathcal{P}_N$ such that $f = f^G$.

**Remark 2.1.** For an arbitrary domain of preference profiles $\mathcal{P}_N$, every SOSP-implementable assignment rule is OSP-implementable.

## 3 Fixed priority top trading cycles rules

Fixed priority top trading cycles (FPTTC) rules are well-known in the literature; we present a brief description for the sake of completeness. For a given priority structure $\succ_A$, the ***FPTTC rule*** $T^{\succ_A}$ ***associated with*** $\succ_A$ is defined by an iterative procedure as follows. Consider an arbitrary preference profile $P_N \in \mathcal{P}_N$.

[17] In other words, there exists an edge $e \in E^{out}(v)$ such that $P_i, P'_i \in \eta^{EP}(e)$ and $\tilde{P}_i \notin \eta^{EP}(e)$.

***Step s.*** Let $N_s(P_N) \subseteq N$ be the set of agents that remain after Step $s-1$ and $A_s(P_N) \subseteq A$ be the set of objects that remain after Step $s-1$.[18]

We construct a directed graph with the set of nodes $N_s(P_N) \cup A_s(P_N) \cup \{a_0\}$. Each agent $i \in N_s(P_N)$ points to her most-preferred element in $A_s(P_N) \cup \{a_0\}$ according to $P_i$. Each object $a \in A_s(P_N)$ points to its most-preferred agent in $N_s(P_N)$ according to $\succ_a$. The outside option $a_0$ points to each agent in $N_s(P_N)$.

There is at least one cycle.[19] Each agent in a cycle is assigned the element she is pointing to (the element might be some object or the outside option $a_0$). Remove all agents and objects that appear in some cycle.

This procedure is repeated iteratively until either all agents are assigned or all objects are assigned. The final outcome is obtained by combining all the assignments at all steps.

# 4 OSP-implementability of FPTTC rules

## 4.1 Dual ownership and prior results

In this subsection, we present the notion of *dual ownership* (Mandal and Roy, 2022a), as well as the prior results on the OSP-implementability of FPTTC rules using this property.

The dual ownership property of FPTTC rules implies the following: for any preference profile and any step of the FPTTC rule at that preference profile, there are at most two agents who own all the objects that remain in the reduced market at that step.[20] In what follows, we present a formal definition. Recall the definitions of $N_s(P_N)$ and $A_s(P_N)$ given in Section 3.

**Definition 4.1.** The FPTTC rule $T^{\succ_A} : \mathcal{P}_N \to \mathcal{M}$ satisfies ***dual ownership*** (on $\mathcal{P}_N$) if for all $P_N \in \mathcal{P}_N$, we have $|\mathcal{T}(\succ^{N_s(P_N)}_{A_s(P_N)})| \leq 2$ for all $s$.

**Remark 4.1.** Note that if the FPTTC rule $T^{\succ_A} : \mathcal{P}_N \to \mathcal{M}$ satisfies dual ownership, then it satisfies dual ownership on some restricted domain $\tilde{\mathcal{P}}_N \subseteq \mathcal{P}_N$. However, the converse is not true.

**Theorem 4.1** (Mandal and Roy, 2022a,b)**.** *Suppose $\mathcal{P}_N \in \Big\{\mathbb{L}^n(A), \mathbb{L}^n(A \cup \{a_0\})\Big\}$. The FPTTC rule $T^{\succ_A} : \mathcal{P}_N \to \mathcal{M}$ is OSP-implementable if and only if it satisfies dual ownership.*

## 4.2 Results

Although dual ownership is an intuitive property, it is somewhat time consuming to check whether a given FPTTC rule satisfies this property. This is because, technically, one needs to check at every prefer-

[18] Note that for all $P_N \in \mathcal{P}_N$, $N_1(P_N) = N$ and $A_1(P_N) = A$.

[19] All the cycles we consider here are assumed to be "minimal", that is, no subset of nodes of such a cycle forms another cycle. In the model without outside options, cycles are always minimal. However, since there can be multiple outgoing edges from the outside option $a_0$, non-minimal cycles may appear in the model with outside options.

[20] For an FPTTC rule $T^{\succ_A}$ and a preference profile $P_N$, we say an agent $i$ *owns* an object $a$ at some step $s$ if $a \in A_s(P_N)$ and $\tau(\succ_a, N_s(P_N)) = i$.

ence profile whether at most two agents are owning all the (remaining) objects at every step of the FPTTC rule. In view of this observation, we introduce equivalent properties to dual ownership, which involve the priority structure only (and not the preference profiles), and thus, is more convenient to be checked.

### 4.2.1 Results on the restricted domain $\mathbb{L}^n(A)$

Here, we introduce the notion of *acyclicity* for FPTTC rules, and show that it is equivalent to the dual ownership property on the restricted domain $\mathbb{L}^n(A)$.

An FPTTC rule is acyclic if the associated priority structure does not contain any *priority cycle*. We begin with a verbal description of a priority cycle. A tuple $[(i_1, i_2, i_3), (a_1, a_2, a_3)]$ where $i_1, i_2, i_3 \in N$ and $a_1, a_2, a_3 \in A$ are all distinct, constitutes a priority cycle in two ways. In the first way, $i_h$ is the most-preferred agent of $\succ_{a_h}$ for all $h = 1, 2, 3$. To explain the second way, let us present a specific instance where agents $i_1, i_2, i_3$ and objects $a_1, a_2, a_3$ form a priority cycle. Suppose there exist distinct agents $i_4, i_5 \in N \setminus \{i_1, i_2, i_3\}$ and distinct objects $a_4, a_5 \in A \setminus \{a_1, a_2, a_3\}$. For $h = 1, \ldots, 5$, let $\succ_{a_h}$ be as given below (the dots indicate that all preferences for the corresponding parts are irrelevant and can be chosen arbitrarily).

| $\succ_{a_1}$ | $\succ_{a_2}$ | $\succ_{a_3}$ | $\succ_{a_4}$ | $\succ_{a_5}$ |
|---|---|---|---|---|
| $i_4$ | $i_4$ | $i_4$ | $i_5$ | $i_5$ |
| $i_1$ | $i_2$ | $i_3$ | $i_4$ | $\vdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |

Table 1: Priority structure with a priority cycle

The priority structure in Table 1 has the property that for all $h = 1, \ldots, 5$, the (strict) upper contour set of agent $i_h$ at $\succ_{a_h}$ is a subset of $\{i_4, i_5\}$. For instance, the (strict) upper contour set of agent $i_1$ is the singleton set $\{i_4\}$. In this case, the tuple $[(i_1, i_2, i_3), (a_1, a_2, a_3)]$ is called a priority cycle. In general, a tuple $[(i_1, i_2, i_3), (a_1, a_2, a_3)]$ is a priority cycle if one can get hold of agents $i_4, \ldots, i_t$ and objects $a_4, \ldots, a_t$ such that for all $h = 1, \ldots, t$, the (strict) upper contour set of agent $i_h$ at $\succ_{a_h}$ is a subset of $\{i_4, \ldots, i_t\}$. In what follows, we present a formal definition.

**Definition 4.2.** A tuple $[(i_1, i_2, i_3), (a_1, a_2, a_3)]$, where $i_1, i_2, i_3 \in N$ and $a_1, a_2, a_3 \in A$ are all distinct, is called a ***priority cycle*** at a priority structure $\succ_A$ if either $\tau(\succ_{a_h}) = i_h$ for all $h = 1, 2, 3$, or there exist distinct agents $i_4, \ldots, i_t \in N \setminus \{i_1, i_2, i_3\}$ and distinct objects $a_4, \ldots, a_t \in A \setminus \{a_1, a_2, a_3\}$ such that for all $h = 1, \ldots, t$, we have $U(i_h, \succ_{a_h}) \subseteq \{i_4, \ldots, i_t\}$.

We call a priority structure ***acyclic*** if it contains no priority cycles, and call an FPTTC rule ***acyclic*** if it is associated with an acyclic priority structure.

Our next theorem says that dual ownership and acyclicity are equivalent properties of an FPTTC rule on the restricted domain $\mathbb{L}^n(A)$.

**Theorem 4.2.** *The FPTTC rule $T^{\succ_A} : \mathbb{L}^n(A) \to \mathcal{M}$ satisfies dual ownership if and only if it is acyclic.*

The proof of this theorem is relegated to Appendix B.

Since dual ownership is both necessary and sufficient condition for an FPTTC rule to be OSP-implementable on the restricted domain $\mathbb{L}^n(A)$ (see Theorem 4.1), we obtain the following corollary from Theorem 4.2.[21]

**Corollary 4.1.** *The FPTTC rule $T^{\succ_A} : \mathbb{L}^n(A) \to \mathcal{M}$ is OSP-implementable if and only if it is acyclic.*

### 4.2.2 Results on the unrestricted domain $\mathbb{L}^n(A \cup \{a_0\})$

Troyan (2019) works on the OSP-implementable FPTTC rules on the restricted domain $\mathbb{L}^n(A)$ when there are equal number of agents and objects. For this purpose, he introduces the notion of *strong acyclicity*.[22] Here, we show that it is equivalent to the dual ownership property on the unrestricted domain.

**Definition 4.3.** A tuple $[(i_1, i_2, i_3), (a_1, a_2, a_3)]$, where $i_1, i_2, i_3 \in N$ and $a_1, a_2, a_3 \in A$ are all distinct, is called a ***weak cycle*** at a priority structure $\succ_A$ if $i_1 \succ_{a_1} \{i_2, i_3\}$, $i_2 \succ_{a_2} \{i_1, i_3\}$, and $i_3 \succ_{a_3} \{i_1, i_2\}$.

We call a priority structure ***strongly acyclic*** if it contains no weak cycles, and call an FPTTC rule ***strongly acyclic*** if it is associated with a strongly acyclic priority structure.

**Remark 4.2.** Note that if $\succ_A$ contains a priority cycle $[(i_1, i_2, i_3), (a_1, a_2, a_3)]$, then $i_1 \succ_{a_1} \{i_2, i_3\}$, $i_2 \succ_{a_2} \{i_1, i_3\}$, and $i_3 \succ_{a_3} \{i_1, i_2\}$. Therefore, every priority cycle is a weak cycle, and hence strong acyclicity implies acyclicity. However, the converse is not true (see Example 4.1 for details).

**Example 4.1.** Consider an allocation problem with four agents $N = \{i, j, k, l\}$ and four objects $A = \{a, b, c, d\}$. Let $\succ_A$ be as follows:

| $\succ_a$ | $\succ_b$ | $\succ_c$ | $\succ_d$ |
|---|---|---|---|
| $i$ | $l$ | $l$ | $i$ |
| $j$ | $j$ | $k$ | $j$ |
| $k$ | $k$ | $j$ | $k$ |
| $l$ | $i$ | $i$ | $l$ |

Table 2: Priority structure for Example 4.1

Note that $i \succ_a \{j, k\}$, $j \succ_b \{i, k\}$, and $k \succ_c \{i, j\}$, which means $[(i, j, k), (a, b, c)]$ is a weak cycle in $\succ_A$. Furthermore, it is straightforward to verify that $\succ_A$ is acyclic.

Our next theorem says that dual ownership and strong acyclicity are equivalent properties of an FPTTC rule on the unrestricted domain.

[21] As we have mentioned earlier, Theorem 1 in Troyan (2019) is not correct. Corollary 4.1 is a correct version of Theorem 1 in Troyan (2019) (in fact, our result is a general result for arbitrary (not necessarily equal) values of the number of agents and the number of objects).

[22] As we have mentioned, Troyan (2019) uses the term "weak acyclicity" to refer to the strong acyclicity property.

**Theorem 4.3.** *The FPTTC rule $T^{\succ_A} : \mathbb{L}^n(A \cup \{a_0\}) \to \mathcal{M}$ satisfies dual ownership if and only if it is strongly acyclic.*

The proof of this theorem is relegated to Appendix C.

Since dual ownership is both necessary and sufficient condition for an FPTTC rule to be OSP-implementable on the unrestricted domain (see Theorem 4.1), we obtain the following corollary from Theorem 4.3.

**Corollary 4.2.** *The FPTTC rule $T^{\succ_A} : \mathbb{L}^n(A \cup \{a_0\}) \to \mathcal{M}$ is OSP-implementable if and only if it is strongly acyclic.*

## 5 SOSP-implementability of FPTTC rules

Before proceeding with our next results, we first present a special class of assignment rules, namely *serial dictatorships* (Satterthwaite and Sonnenschein, 1981). In a ***serial dictatorship***, agents are ordered, and the first agent in the ordering gets her most-preferred element among all the objects and the outside option $a_0$, the second agent in the ordering gets her most-preferred element among the remaining objects and the outside option $a_0$, etc.

Note that serial dictatorships are special cases of FPTTC rules. For example, consider an allocation problem with three agents $N = \{1, 2, 3\}$ and three objects $A = \{a_1, a_2, a_3\}$. The priority structure associated with the FPTTC rule that corresponds to the serial dictatorship with the exogenously given ordering $(1 \succ 2 \succ 3)$ is as follows:

| $\succ_{a_1}$ | $\succ_{a_2}$ | $\succ_{a_3}$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 2 | 2 |
| 3 | 3 | 3 |

Further note that serial dictatorships are SOSP-implementable. For example, consider the allocation problem with three agents $N = \{1, 2, 3\}$ and three objects $A = \{a_1, a_2, a_3\}$. The SOSP mechanism in Figure 1 implements the serial dictatorship with the exogenously given ordering $(1 \succ 2 \succ 3)$.[23]

[23]We use the following notations in Figure 1: by $a_1a_2$ we denote the set of preferences where $a_1$ is preferred to $a_2$, and we denote an allocation $[(1, a_1), (2, a_2), (3, a_3)]$ by

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

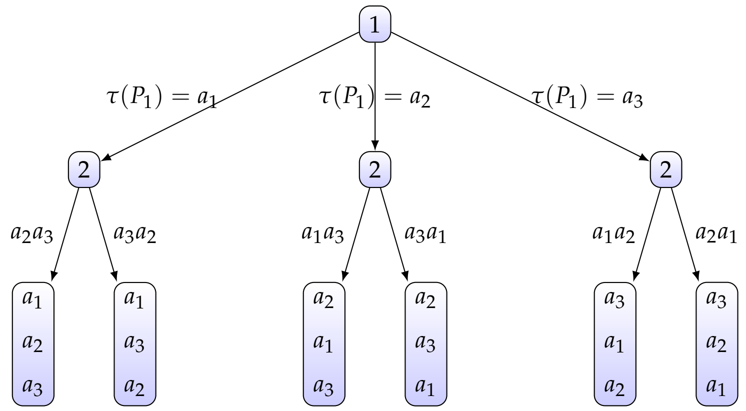


Figure 1: SOSP mechanism

We concisely sum up the above discussion as follows.

**Remark 5.1.** For an arbitrary domain of preference profiles $\mathcal{P}_N$, every serial dictatorship is an SOSP-implementable FPTTC rule.

Remark 5.1 arises a natural question: Apart from serial dictatorships, are there any other SOSP-implementable FPTTC rules? Theorem 5.1 and Theorem 5.3 provide an answer to this question for the domains $\mathbb{L}^n(A)$ and $\mathbb{L}^n(A \cup \{a_0\})$.

### 5.1 Results on the restricted domain $\mathbb{L}^n(A)$

In this subsection, we introduce a property called *weak serial dictatorship* of an FPTTC rule and provide a characterization of SOSP-implementable FPTTC rules on the restricted domain $\mathbb{L}^n(A)$ by means of this property. We further discuss the configuration of the priority structures associated with these FPTTC rules.

The weak serial dictatorship property of FPTTC rules implies the following: for any preference profile and any step of the FPTTC rule at that preference profile, if there are more than two objects remaining in the reduced market at that step, then there is exactly one agent who owns all those remaining objects. Clearly, serial dictatorships satisfy weak serial dictatorship. In what follows, we present a formal definition of weak serial dictatorship.

**Definition 5.1.** The FPTTC rule $T^{\succ_A} : \mathcal{P}_N \to \mathcal{M}$ satisfies ***weak serial dictatorship*** (on $\mathcal{P}_N$) if for all $P_N \in \mathcal{P}_N$ and all $s$,

$$|A_s(P_N)| > 2 \implies |\mathcal{T}(\succ^{N_s(P_N)}_{A_s(P_N)})| = 1.$$

Our next theorem says that weak serial dictatorship is both necessary and sufficient condition for an FPTTC rule to be SOSP-implementable on the restricted domain $\mathbb{L}^n(A)$.

**Theorem 5.1.** *The FPTTC rule $T^{\succ_A} : \mathbb{L}^n(A) \to \mathcal{M}$ is SOSP-implementable if and only if it satisfies weak serial dictatorship.*

The proof of this theorem is relegated to Appendix D.

Similarly as dual ownership, to verify whether a given FPTTC rule satisfies weak serial dictatorship or not, one needs to check its behavior at every step at every preference profile. In view of this observation, we present our next result regarding the configuration of the priority structures associated with these FPTTC rules. We use the following terminology to facilitate the result. For $\succ \in \mathbb{L}(N)$ and $i \in N$, we define $rank(i, \succ) = m$ if $|\{j \in N \mid j \succ i\}| = m - 1$.

**Theorem 5.2.** *The FPTTC rule $T^{\succ_A} : \mathbb{L}^n(A) \to \mathcal{M}$ satisfies weak serial dictatorship if and only $\succ_A$ has the following property: for all $a, b \in A$ and all $i \in N$,*

$$rank(i, \succ_a) \leq |A| - 2 \implies rank(i, \succ_a) = rank(i, \succ_b).$$

The proof of this theorem is relegated to Appendix E.

As a corollary of Theorem 5.1 and Theorem 5.2, we obtain the configuration of the priority structures associated with the SOSP-implementable FPTTC rules on the restricted domain $\mathbb{L}^n(A)$.

**Corollary 5.1.** *The FPTTC rule $T^{\succ_A} : \mathbb{L}^n(A) \to \mathcal{M}$ is SOSP-implementable if and only $\succ_A$ has the following property: for all $a, b \in A$ and all $i \in N$,*

$$rank(i, \succ_a) \leq |A| - 2 \implies rank(i, \succ_a) = rank(i, \succ_b).$$

We obtain the following corollary from Corollary 5.1. It says when there are more objects than agents, SOSP-implementable FPTTC rules on the restricted domain $\mathbb{L}^n(A)$ are characterized as serial dictatorships.

**Corollary 5.2.** *Suppose $|A| > |N|$. The FPTTC rule $T^{\succ_A} : \mathbb{L}^n(A) \to \mathcal{M}$ is SOSP-implementable if and only if it is a serial dictatorship.*

### 5.2 Results on the restricted domain $\mathbb{L}^n(A \cup \{a_0\})$

Recall that the serial dictatorships are SOSP-implementable FPTTC rules (see Remark 5.1). Here, we show that on the unrestricted domain, they are the *only* SOSP-implementable FPTTC rules.

**Theorem 5.3.** *The FPTTC rule $T^{\succ_A} : \mathbb{L}^n(A \cup \{a_0\}) \to \mathcal{M}$ is SOSP-implementable if and only if it is a serial dictatorship.*

The proof of this theorem is relegated to Appendix F.

## 6 Simple strategy-proofness

OSP-implementation presumes that the agents understand how their own future actions will affect outcomes (the worst and the best possible final outcomes are taken only over opponents' future actions), and consequently, they might be concerned about having time-inconsistent preferences or making a mistake while performing demanding backward induction over their own future actions. Pycia and Troyan (2019) introduces SOSP-implementation as a way to resolve this issue by relaxing the assumption that the agents fully comprehend how their own future actions will affect outcomes. Another way to resolve this issue will be by calling each agent to play at most once so that they need not be worried about their own future actions. In view of this observation, we introduce the notion of *simple strategy-proofness* in this section.

The concept of simple strategy-proofness is based on a *simple* OSP mechanism. A mechanism is simple if every agent is called to play at most once along a path. An assignment rule is *simply strategy-proof* if there exists a simple OSP mechanism that implements the assignment rule. In what follows, we present formal definitions of these. Recall the definition of a mechanism $G$ given in Section 2.3.

**Definition 6.1.** A mechanism $G = \langle T, \eta^{LA}, \eta^{NI}, \eta^{EP} \rangle$ on $\mathcal{P}_N$ is ***simple*** if $\eta^{NI}(v) \neq \eta^{NI}(v')$ for all distinct $v, v' \in V(T) \setminus L(T)$ that appear in same path.

**Definition 6.2.** An assignment rule $f : \mathcal{P}_N \to \mathcal{M}$ is ***simply strategy-proof*** (on $\mathcal{P}_N$) if there exists a simple OSP mechanism $G$ on $\mathcal{P}_N$ such that $f = f^G$.

By definition, simple strategy-proofness is stronger than OSP-implementability. Our next result shows that simple strategy-proofness is even stronger than SOSP-implementability.

**Proposition 6.1.** *For an arbitrary domain of preference profiles $\mathcal{P}_N$, every simply strategy-proof assignment rule is SOSP-implementable.*

The proof of this proposition is relegated to Appendix G.

It is worth mentioning that the converse of Proposition 6.1 is not true in general. Example 6.1 presents a domain of preference profiles and an FPTTC rule, which is SOSP-implementable but not simply strategy-proof on the given domain.

**Example 6.1.** Consider an allocation problem with two agents $N = \{1, 2\}$ and four objects $A = \{a_1, a_2, a_3, a_4\}$. Let $\tilde{\mathcal{P}} = \{a_1a_4a_3a_2a_0, a_2a_3a_4a_1a_0, a_3a_2a_4a_1a_0, a_4a_1a_3a_2a_0\}$. Let $\succ_A$ be as follows:

| $\succ_{a_1}$ | $\succ_{a_2}$ | $\succ_{a_3}$ | $\succ_{a_4}$ |
|---|---|---|---|
| 1 | 2 | 2 | 1 |
| 2 | 1 | 1 | 2 |

Table 3: Priority structure for Example 6.1

Consider the FPTTC rule $T^{\succ_A}$ on the domain $\tilde{\mathcal{P}}^2$. The SOSP mechanism in Figure 2 implements $T^{\succ_A}$ on $\tilde{\mathcal{P}}^2$.[24]

[24]We use the following notations in Figure 2: by $a_1a_2a_3$ we denote the set of preferences where $a_1$ is preferred to $a_2$ and $a_2$ is

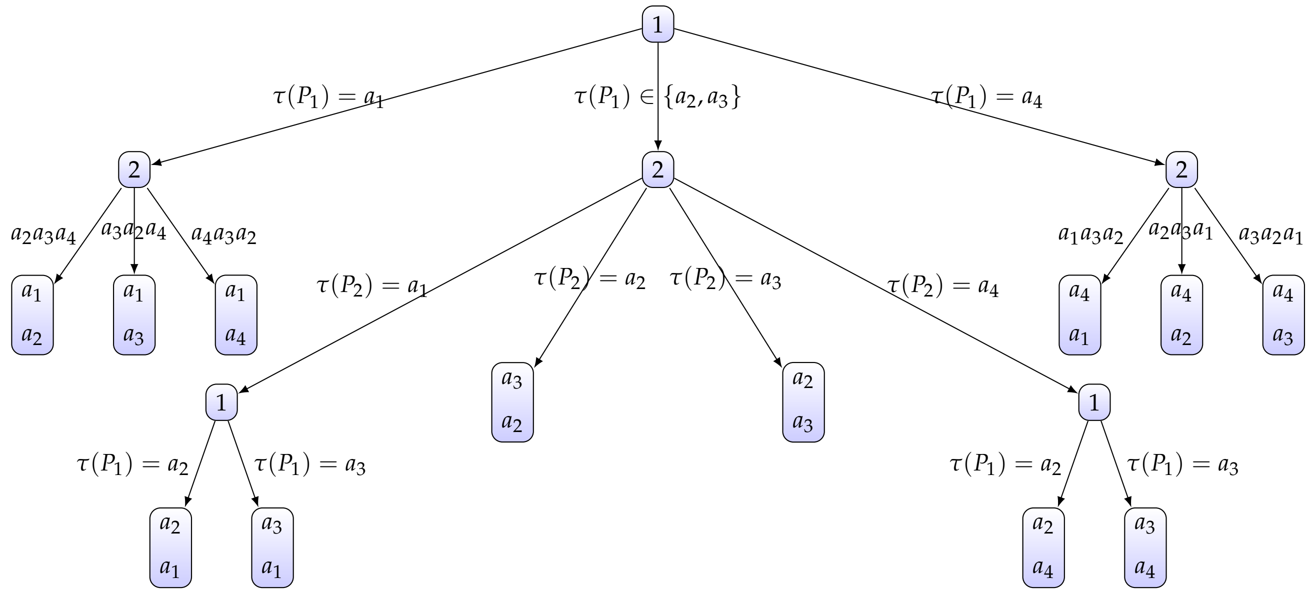


Figure 2: Tree Representation for Example 6.1

Now we argue that $T^{\succ_A}$ is not simply strategy-proof on $\tilde{\mathcal{P}}^2$. Consider the preference profiles presented (together with the outcome of $T^{\succ_A}$) in Table 4.

| Preference profiles | Agent 1 | Agent 2 | $T_1^{\succ_A}$ | $T_2^{\succ_A}$ |
|---|---|---|---|---|
| $\tilde{P}_N^1$ | $a_2a_3a_4a_1$ | $a_2a_3a_4a_1$ | $a_3$ | $a_2$ |
| $\tilde{P}_N^2$ | $a_3a_2a_4a_1$ | $a_3a_2a_4a_1$ | $a_2$ | $a_3$ |
| $\tilde{P}_N^3$ | $a_3a_2a_4a_1$ | $a_4a_1a_3a_2$ | $a_3$ | $a_4$ |
| $\tilde{P}_N^4$ | $a_2a_3a_4a_1$ | $a_4a_1a_3a_2$ | $a_2$ | $a_4$ |
| $\tilde{P}_N^5$ | $a_1a_4a_3a_2$ | $a_1a_4a_3a_2$ | $a_1$ | $a_4$ |
| $\tilde{P}_N^6$ | $a_2a_3a_4a_1$ | $a_1a_4a_3a_2$ | $a_2$ | $a_1$ |
| $\tilde{P}_N^7$ | $a_4a_1a_3a_2$ | $a_4a_1a_3a_2$ | $a_4$ | $a_1$ |

Table 4: Preference profiles for Example 6.1

Assume for contradiction that $T^{\succ_A}$ is simply strategy-proof on $\tilde{\mathcal{P}}^2$. So, there exists a simple OSP mechanism $\tilde{G}$ that implements $T^{\succ_A}$ on $\tilde{\mathcal{P}}_N$. Note that since $T^{\succ_A}(\tilde{P}_N^1) \neq T^{\succ_A}(\tilde{P}_N^2)$, there exists a node in the simple OSP mechanism $\tilde{G}$ that has at least two edges. Consider the first node (from the root) $v$ that has at least two edges. We distinguish two following two cases.

(i) Suppose $\eta^{NI}(v) = 1$.

By obvious strategy-proofness of $\tilde{G}$, the facts $a_2\tilde{P}_1^1a_3$, $T_1^{\succ_A}(\tilde{P}_N^1) = a_3$, and $T_1^{\succ_A}(\tilde{P}_N^2) = a_2$ together imply that $\tilde{P}_1^1$ and $\tilde{P}_1^2$ do not diverge at $v$. This, together with the facts that $\tilde{P}_1^1 = \tilde{P}_1^4$, $\tilde{P}_1^2 = \tilde{P}_1^3$,

preferred to $a_3$, and we denote an allocation $[(1, a_1), (2, a_2)]$ by

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

$\tilde{P}_2^3 = \tilde{P}_2^4$ and $T^{\succ_A}(\tilde{P}_N^3) \neq T^{\succ_A}(\tilde{P}_N^4)$, implies that there exists a node $v'$ at which $\tilde{P}_1^1$ and $\tilde{P}_1^2$ diverge. Clearly, $v$ and $v'$ are distinct nodes appearing in same path such that $\eta^{NI}(v) = \eta^{NI}(v') = 1$. This contradicts the fact that $\tilde{G}$ is a simple mechanism.

(ii) Suppose $\eta^{NI}(v) = 2$.

By obvious strategy-proofness of $\tilde{G}$, the facts $a_1\tilde{P}_2^5 a_4$, $T_2^{\succ_A}(\tilde{P}_N^5) = a_4$, and $T_2^{\succ_A}(\tilde{P}_N^7) = a_1$ together imply that $\tilde{P}_2^5$ and $\tilde{P}_2^7$ do not diverge at $v$. This, together with the facts that $\tilde{P}_2^5 = \tilde{P}_2^6$, $\tilde{P}_2^7 = \tilde{P}_2^4$, $\tilde{P}_1^4 = \tilde{P}_1^6$ and $T^{\succ_A}(\tilde{P}_N^4) \neq T^{\succ_A}(\tilde{P}_N^6)$, implies that there exists a node $v'$ at which $\tilde{P}_2^5$ and $\tilde{P}_2^7$ diverge. Clearly, $v$ and $v'$ are distinct nodes appearing in same path such that $\eta^{NI}(v) = \eta^{NI}(v') = 2$. This contradicts the fact that $\tilde{G}$ is a simple mechanism.

Since Cases (i) and (ii) are exhaustive, it follows that $T^{\succ_A}$ is not simply strategy-proof on $\tilde{\mathcal{P}}^2$.

Example 6.1 shows that on an arbitrary domain of preference profiles, every SOSP-implementable FPTTC rule might not be simply strategy-proof. However, on the restricted domain $\mathbb{L}^n(A)$ and the unrestricted domain $\mathbb{L}^n(A \cup \{a_0\})$, every SOSP-implementable FPTTC rule is simply strategy-proof.

**Theorem 6.1.** *Suppose $\mathcal{P}_N \in \left\{\mathbb{L}^n(A), \mathbb{L}^n(A \cup \{a_0\})\right\}$. The FPTTC rule $T^{\succ_A} : \mathcal{P}_N \to \mathcal{M}$ is simply strategy-proof if and only if it is SOSP-implementable.*

The proof of this proposition is relegated to Appendix H.

We obtain the following corollary from Theorem 5.1, Theorem 5.3, and Theorem 6.1.

**Corollary 6.1.** *(a) The FPTTC rule $T^{\succ_A} : \mathbb{L}^n(A) \to \mathcal{M}$ is simply strategy-proof if and only if it satisfies weak serial dictatorship.*

*(b) The FPTTC rule $T^{\succ_A} : \mathbb{L}^n(A \cup \{a_0\}) \to \mathcal{M}$ is simply strategy-proof if and only if it is a serial dictatorship.*

# Appendix A Some additional notations

It will be convenient to introduce some additional notations for the proofs. Following our notational terminology in Section 2.1, for a preference $P \in \mathbb{L}(A \cup \{a_0\})$ and two disjoint subsets $A'$ and $\hat{A}$ of $A \cup \{a_0\}$, we write $A'P\hat{A}$ to mean that $aPb$ for all $a \in A'$ and all $b \in \hat{A}$. Furthermore, for a preference profile $P_N$ and an FPTTC rule, let $I_s(P_N)$ be the set of assigned agents at Step $s$, $I^s(P_N)$ be the set of assigned agents up to Step $s$ (including Step $s$), $X_s(P_N)$ be the set of assigned objects at Step $s$, $X^s(P_N)$ be the set of assigned objects up to Step $s$ (including Step $s$), and $O_s(i, P_N)$ be the set of objects owned by agent $i$ at Step $s$.

# Appendix B Proof of Theorem 4.2

***(If part)*** Suppose $T^{\succ_A}$ does not satisfy dual ownership on $\mathbb{L}^n(A)$. We show that $\succ_A$ contains a priority cycle. Since $T^{\succ_A}$ does not satisfy dual ownership on $\mathbb{L}^n(A)$, there exist a preference profile $\tilde{P}_N \in \mathbb{L}^n(A)$

and a step $s^*$ of $T^{\succ_A}$ at $\tilde{P}_N$ such that $|\mathcal{T}(\succ_{A_{s^*}(\tilde{P}_N)}^{N_{s^*}(\tilde{P}_N)})| > 2$. This implies that there exist three agents $i_1, i_2, i_3 \in N_{s^*}(\tilde{P}_N)$ and three objects $a_1, a_2, a_3 \in A_{s^*}(\tilde{P}_N)$ such that for all $h = 1,2,3$, agent $i_h$ owns the object $a_h$ at Step $s^*$. We proceed to show that $[(i_1, i_2, i_3), (a_1, a_2, a_3)]$ is a priority cycle in $\succ_A$. We distinguish the following two cases.

**CASE 1**: Suppose $s^* = 1$.

Since for all $h = 1,2,3$, agent $i_h$ owns the object $a_h$ at Step 1, by the definition of $T^{\succ_A}$, it follows that $\tau(\succ_{a_h}) = i_h$ for all $h = 1,2,3$. This means $[(i_1, i_2, i_3), (a_1, a_2, a_3)]$ is a priority cycle in $\succ_A$.

**CASE 2**: Suppose $s^* > 1$.

Let $\{i_4, \ldots, i_t\} \subseteq N \setminus \{i_1, i_2, i_3\}$ and $\{a_4, \ldots, a_t\} \subseteq A \setminus \{a_1, a_2, a_3\}$ be as follows.

(i) $\{i_4, \ldots, i_t\} = I^{s^*-1}(\tilde{P}_N)$.

(ii) For all $h = 4, \ldots, t$, $\{a_h\} = \big(X_s(\tilde{P}_N) \cap O_s(i_h, \tilde{P}_N)\big)$ where $i_h \in I_s(\tilde{P}_N)$ for some $s < s^*$. To see that this is well-defined note that by the definition of $T^{\succ_A}$ and the fact $\tilde{P}_N \in \mathbb{L}^n(A)$, (a) for every $i_h \in I^{s^*-1}(\tilde{P}_N)$, there exists exactly one step $s$ with $s < s^*$ such that $i_h \in I_s(\tilde{P}_N)$, and (b) $O_s(i_h, \tilde{P}_N) \cap X_s(\tilde{P}_N)$ is a singleton set for all $i_h \in I_s(\tilde{P}_N)$ with $s < s^*$.

It follows from the definition of $T^{\succ_A}$ and the construction of $\{i_4, \ldots, i_t\}$ and $\{a_4, \ldots, a_t\}$ that $U(i_h, \succ_{a_h}) \subseteq \{i_4, \ldots, i_t\}$ for all $h = 1, \ldots, t$. This implies that $[(i_1, i_2, i_3), (a_1, a_2, a_3)]$ is a priority cycle in $\succ_A$, which completes the proof of the "if" part of Theorem 4.2.

***(Only-if part)*** Suppose $\succ_A$ contains a priority cycle $[(i_1, i_2, i_3), (a_1, a_2, a_3)]$. We show that $T^{\succ_A}$ does not satisfy dual ownership on $\mathbb{L}^n(A)$. By the definition of a priority cycle, one of the following two statements must hold.

(1) $\tau(\succ_{a_h}) = i_h$ for all $h = 1,2,3$.

(2) There exist distinct agents $i_4, \ldots, i_t \in N \setminus \{i_1, i_2, i_3\}$ and distinct objects $a_4, \ldots, a_t \in A \setminus \{a_1, a_2, a_3\}$ such that for all $h = 1, \ldots, t$, we have $U(i_h, \succ_{a_h}) \subseteq \{i_4, \ldots, i_t\}$.

We distinguish the following two cases.

**CASE 1**: Suppose (1) holds.

Since $\tau(\succ_{a_h}) = i_h$ for all $h = 1,2,3$, it must be that for any preference profile, agents $i_1, i_2$, and $i_3$ own objects $a_1, a_2$, and $a_3$, respectively, at Step 1 of $T^{\succ_A}$ at that preference profile. Therefore $T^{\succ_A}$ does not satisfy dual ownership on $\mathbb{L}^n(A)$.

**CASE 2**: Suppose (2) holds.

Consider the preference profile $\tilde{P}_N \in \mathbb{L}^n(A)$ defined as follows. Each $i_h \in \{i_4, \ldots, i_t\}$ has a preference $\tilde{P}_{i_h}$ such that $\tau(\tilde{P}_{i_h}) = a_h$ and each $j \in N \setminus \{i_4, \ldots, i_t\}$ has a preference $\tilde{P}_j$ such that $\{a_4, \ldots, a_t\} \tilde{P}_j (A \setminus \{a_4, \ldots, a_t\})$. The next claim establishes some properties of the outcome of $T^{\succ_A}$ at $\tilde{P}_N$ at Step 1.

**Claim B.1.** *(a)* $I_1(\tilde{P}_N) \subseteq \{i_4, \ldots, i_t\}$, *and (b)* $T_{i_h}^{\succ_A}(\tilde{P}_N) = a_h$ *for all* $h = 4, \ldots, t$ *with* $i_h \in I_1(\tilde{P}_N)$.

***Proof of Claim B.1.*** By the assumptions for Case 2, it follows that $\{a_4, \ldots, a_t\} \subseteq \bigcup_{h=4}^{t} O_1(i_h, \tilde{P}_N)$. Moreover, by the construction of $\tilde{P}_N$, we have $\tau(\tilde{P}_i) \in \{a_4, \ldots, a_t\}$ for all $i \in N$. Since $\{a_4, \ldots, a_t\} \subseteq \bigcup_{h=4}^{t} O_1(i_h, \tilde{P}_N)$ and $\tau(\tilde{P}_i) \in \{a_4, \ldots, a_t\}$ for all $i \in N$, it follows from the definition of $T^{\succ_A}$ that $I_1(\tilde{P}_N) \subseteq \{i_4, \ldots, i_t\}$ and $T_i^{\succ_A}(\tilde{P}_N) = \tau(\tilde{P}_i)$ for all $i \in I_1(\tilde{P}_N)$. These two facts, along with the construction of $\tilde{P}_N$, complete the proof of Claim B.1. □

By Claim B.1, $I_1(\tilde{P}_N) \subseteq \{i_4, \ldots, i_t\}$ and $T_{i_h}^{\succ_A}(\tilde{P}_N) = a_h$ for all $h = 4, \ldots, t$ with $i_h \in I_1(\tilde{P}_N)$. We proceed to show that there will be a step $s^*$ such that $I^{s^*}(\tilde{P}_N) = \{i_4, \ldots, i_t\}$ and $T_{i_h}^{\succ_A}(\tilde{P}_N) = a_h$ for all $h = 4, \ldots, t$. If $I_1(\tilde{P}_N) = \{i_4, \ldots, i_t\}$, then $s^* = 1$ and we are done. Suppose $I_1(\tilde{P}_N) \subsetneq \{i_4, \ldots, i_t\}$, that is, $I_1(\tilde{P}_N)$ is a proper subset of $\{i_4, \ldots, i_t\}$. Since $I_1(\tilde{P}_N) \subsetneq \{i_4, \ldots, i_t\}$ and $T_{i_h}^{\succ_A}(\tilde{P}_N) = a_h$ for all $h = 4, \ldots, t$ with $i_h \in I_1(\tilde{P}_N)$, using similar argument as for Claim B.1, it follows from the assumptions for Case 2 and the construction of $\tilde{P}_N$ that $I_2(\tilde{P}_N) \subseteq \big(\{i_4, \ldots, i_t\} \setminus I_1(\tilde{P}_N)\big)$ and $T_{i_h}^{\succ_A}(\tilde{P}_N) = a_h$ for all $h = 4, \ldots, t$ with $i_h \in I_2(\tilde{P}_N)$. If $I_1(\tilde{P}_N) \cup I_2(\tilde{P}_N) = \{i_4, \ldots, i_t\}$, then $s^* = 2$ and we are done. Otherwise, continuing in this manner, we obtain a step $s^* > 2$ of $T^{\succ_A}$ at $\tilde{P}_N$ such that $I^{s^*}(\tilde{P}_N) = \{i_4, \ldots, i_t\}$ and $T_{i_h}^{\succ_A}(\tilde{P}_N) = a_h$ for all $h = 4, \ldots, t$.

Since $I^{s^*}(\tilde{P}_N) = \{i_4, \ldots, i_t\}$ and $T_{i_h}^{\succ_A}(\tilde{P}_N) = a_h$ for all $h = 4, \ldots, t$, by the assumptions for Case 2, we have $a_h \in O_{s^*+1}(i_h, \tilde{P}_N)$ for all $h = 1, 2, 3$. This implies that agents $i_1$, $i_2$, and $i_3$ own the objects $a_1$, $a_2$, and $a_3$, respectively, at Step $s^* + 1$ of $T^{\succ_A}$ at $\tilde{P}_N$. Therefore $T^{\succ_A}$ does not satisfy dual ownership on $\mathbb{L}^n(A)$, which completes the proof of the "only-if" part of Theorem 4.2. ■

# Appendix C Proof of Theorem 4.3

Before we start proving Theorem 4.3, to facilitate the proof we present the notion of *dual dictatorship* (Troyan, 2019).

**Definition C.1.** The FPTTC rule $T^{\succ_A} : \mathcal{P}_N \to \mathcal{M}$ satisfies ***dual dictatorship*** if for all $N' \subseteq N$ and all $A' \subseteq A$, we have $|\mathcal{T}(\succ_{A'}^{N'})| \leq 2$.

Note that the dual dictatorship property does not depend on the choice of the domain.

***Completion of the proof of Theorem 4.3.*** Mandal and Roy (2022b) show that dual ownership and dual dictatorship are equivalent properties of an FPTTC rule on the unrestricted domain (see Theorem 4.1 in Mandal and Roy (2022b)).[25] In the model with *equal* number of agents and objects, Troyan (2019) shows that dual dictatorship and strong acyclicity are equivalent properties of an FPTTC rule on the restricted domain $\mathbb{L}^n(A)$ (see Theorem 2 in Troyan (2019)). His proof works verbatim on the unrestricted domain

[25]For an arbitrary domain of preference profiles $\mathcal{P}_N$, the set of FPTTC rules satisfying dual ownership is a superset of those satisfying dual dictatorship. See Mandal and Roy (2022b) for a detailed discussion about the relation between dual ownership and dual dictatorship.

in our model (that is, with arbitrary values of the number of agents and the number of objects), and his result still holds. Combining all these facts, we obtain that dual ownership, dual dictatorship, and strong acyclicity are equivalent properties of an FPTTC rule on the unrestricted domain. This completes the proof of Theorem 4.3. ■

## Appendix D Proof of Theorem 5.1

We first make a straightforward observation to facilitate the proof.

**Observation D.1.** *Suppose $|A| = 2$. Every FPTTC rule is SOSP-implementable on the restricted domain $\mathbb{L}^n(A)$.*

***Completion of the proof of Theorem 5.1. (If part)*** Using Observation D.1, it is straightforward to verify that on the restricted domain $\mathbb{L}^n(A)$, every FPTTC rule satisfying weak serial dictatorship is SOSP-implementable. This completes the proof of the "if" part of Theorem 5.1.

***(Only-if part)*** Let $T^{\succ_A}$ be an SOSP-implementable FPTTC rule on the restricted domain $\mathbb{L}^n(A)$. Since SOSP-implementability is stronger than OSP-implementability (see Remark 2.1), by Theorem 4.1, $T^{\succ_A}$ satisfies dual ownership. Assume for contradiction that $T^{\succ_A}$ does not satisfy weak serial dictatorship. Since $T^{\succ_A}$ satisfies dual ownership, but does not satisfy weak serial dictatorship, there exist a preference profile $P'_N \in \mathbb{L}^n(A)$ and a step $s^*$ of $T^{\succ_A}$ at $P'_N$ such that there are two agents $i, j$ and three objects $a, b, c$ in the reduced market at Step $s^*$ with the property that agent $i$ owns the object $a$, and agent $j$ owns the objects $b$ and $c$ at Step $s^*$. We distinguish the following two cases.

**Case A**: Suppose $s^* = 1$.

Consider the domain $\tilde{\mathcal{P}}_N \subseteq \mathbb{L}^n(A)$ with only four preference profiles presented in Table D.1.[26] Here, $l$ denotes an agent (might be empty) other than $i$ and $j$. Note that such an agent does not change her preference across the mentioned preference profiles.

| Preference profiles | Agent $i$ | Agent $j$ | $\ldots$ | Agent $l$ |
|---|---|---|---|---|
| $\tilde{P}^1_N$ | $abc\ldots$ | $acb\ldots$ | $\ldots$ | $P'_l$ |
| $\tilde{P}^2_N$ | $bac\ldots$ | $bac\ldots$ | $\ldots$ | $P'_l$ |
| $\tilde{P}^3_N$ | $bca\ldots$ | $abc\ldots$ | $\ldots$ | $P'_l$ |
| $\tilde{P}^4_N$ | $cab\ldots$ | $abc\ldots$ | $\ldots$ | $P'_l$ |

Table D.1: Preference profiles of $\tilde{\mathcal{P}}_N$

In Table D.2, we present some facts regarding the outcome of $T^{\succ_A}$ on the domain $\tilde{\mathcal{P}}_N$. These facts are deduced by the construction of $\tilde{\mathcal{P}}_N$ along with the assumptions for Case A.

[26] For instance, $abc\ldots$ indicates (any) preference that ranks $a$ first, $b$ second, and $c$ third.

| Preference profiles | Agent $i$ | Agent $j$ | $T_i^{\succ_A}$ | $T_j^{\succ_A}$ |
|---|---|---|---|---|
| $\tilde{P}_N^1$ | $abc\ldots$ | $acb\ldots$ | $a$ | $c$ |
| $\tilde{P}_N^2$ | $bac\ldots$ | $bac\ldots$ | $a$ | $b$ |
| $\tilde{P}_N^3$ | $bca\ldots$ | $abc\ldots$ | $b$ | $a$ |
| $\tilde{P}_N^4$ | $cab\ldots$ | $abc\ldots$ | $c$ | $a$ |

Table D.2: Partial outcome of $T^{\succ_A}$ on $\tilde{\mathcal{P}}_N$

Since $T^{\succ_A}$ is SOSP-implementable on $\mathbb{L}^n(A)$, it must be SOSP-implementable on the domain $\tilde{\mathcal{P}}_N$. Let $\tilde{G}$ be an SOSP mechanism that implements $T^{\succ_A}$ on $\tilde{\mathcal{P}}_N$.

Note that since $T^{\succ_A}(\tilde{P}_N^1) \neq T^{\succ_A}(\tilde{P}_N^2)$, there exists a node in the SOSP mechanism $\tilde{G}$ that has at least two edges. Also, note that since each agent $l \in N \setminus \{i,j\}$ has exactly one preference in $\tilde{\mathcal{P}}_l$, whenever there are at least two outgoing edges from a node, that node must be assigned to some agent in $\{i,j\}$. Consider the first node (from the root) $v$ that has at least two edges.

(i) Suppose $\eta^{NI}(v) = i$.
By SOSP-implementability, the facts $b\tilde{P}_i^2 a$, $T_i^{\succ_A}(\tilde{P}_N^2) = a$, and $T_i^{\succ_A}(\tilde{P}_N^3) = b$ together imply that $\tilde{P}_i^2$ and $\tilde{P}_i^3$ do not diverge at $v$. Since $\tilde{P}_i^2$ and $\tilde{P}_i^3$ do not diverge at $v$, by SOSP-implementability, the facts $c\tilde{P}_i^3 a$, $T_i^{\succ_A}(\tilde{P}_N^2) = a$, and $T_i^{\succ_A}(\tilde{P}_N^4) = c$ together imply that $\tilde{P}_i^3$ and $\tilde{P}_i^4$ do not diverge at $v$. Moreover, since $\tilde{P}_i^3$ and $\tilde{P}_i^4$ do not diverge at $v$, by SOSP-implementability, the facts $a\tilde{P}_i^4 b$, $T_i^{\succ_A}(\tilde{P}_N^1) = a$, and $T_i^{\succ_A}(\tilde{P}_N^3) = b$ together imply that $\tilde{P}_i^1$ and $\tilde{P}_i^4$ do not diverge at $v$. Combining all these observations, we have a contradiction to the fact that $v$ has at least two edges.

(ii) Suppose $\eta^{NI}(v) = j$.
By SOSP-implementability, the facts $a\tilde{P}_j^1 c$, $T_j^{\succ_A}(\tilde{P}_N^1) = c$, and $T_j^{\succ_A}(\tilde{P}_N^3) = a$ together imply that $\tilde{P}_j^1$ and $\tilde{P}_j^3$ do not diverge at $v$. Since $\tilde{P}_j^1$ and $\tilde{P}_j^3$ do not diverge at $v$, by SOSP-implementability, the facts $b\tilde{P}_j^3 c$, $T_j^{\succ_A}(\tilde{P}_N^1) = c$, and $T_j^{\succ_A}(\tilde{P}_N^2) = b$ together imply that $\tilde{P}_j^2$ and $\tilde{P}_j^3$ do not diverge at $v$. Combining all these observations, we have a contradiction to the fact that $v$ has at least two edges.

**CASE B**: Suppose $s^* > 1$.

Recall that $X^{s^*-1}(P'_N)$ is the set of assigned objects up to Stage $s^*-1$ (including Stage $s^*-1$) of $T^{\succ_A}$ at $P'_N$. Fix a preference $\hat{P} \in \mathbb{L}(X^{s^*-1}(P'_N))$ over these objects. Consider the domain $\tilde{\mathcal{P}}_N \subsetneq \mathbb{L}^n(A)$ with only four preference profiles presented in Table D.3.[27]

[27]For instance, $\hat{P}abc\ldots$ denotes a preference where objects in $X^{s^*-1}(P'_N)$ are ranked at the top according to the preference $\hat{P}$, objects $a$, $b$, and $c$ are ranked consecutively after that (in that order), and the ranking of the rest of the objects is arbitrarily.

| Preference profiles | Agent $i$ | Agent $j$ | ... | Agent $l$ |
|---|---|---|---|---|
| $\tilde{P}^1_N$ | $\hat{P}abc\ldots$ | $\hat{P}acb\ldots$ | ... | $P'_l$ |
| $\tilde{P}^2_N$ | $\hat{P}bac\ldots$ | $\hat{P}bac\ldots$ | ... | $P'_l$ |
| $\tilde{P}^3_N$ | $\hat{P}bca\ldots$ | $\hat{P}abc\ldots$ | ... | $P'_l$ |
| $\tilde{P}^4_N$ | $\hat{P}cab\ldots$ | $\hat{P}abc\ldots$ | ... | $P'_l$ |

Table D.3: Preference profiles of $\tilde{\mathcal{P}}_N$

In Table D.4, we present some facts regarding the outcome of $T^{\succ_A}$ on the domain $\tilde{\mathcal{P}}_N$ that can be deduced by the construction of the domain $\tilde{\mathcal{P}}_N$ along with the assumptions for Case B. The verification of these facts is left to the reader.

| Preference profiles | Agent $i$ | Agent $j$ | $T^{\succ_A}_i$ | $T^{\succ_A}_j$ |
|---|---|---|---|---|
| $\tilde{P}^1_N$ | $\hat{P}abc\ldots$ | $\hat{P}acb\ldots$ | $a$ | $c$ |
| $\tilde{P}^2_N$ | $\hat{P}bac\ldots$ | $\hat{P}bac\ldots$ | $a$ | $b$ |
| $\tilde{P}^3_N$ | $\hat{P}bca\ldots$ | $\hat{P}abc\ldots$ | $b$ | $a$ |
| $\tilde{P}^4_N$ | $\hat{P}cab\ldots$ | $\hat{P}abc\ldots$ | $c$ | $a$ |

Table D.4: Partial outcome of $T^{\succ_A}$ on $\tilde{\mathcal{P}}_N$

Using a similar argument as for Case A, we get a contradiction. This completes the proof of the "only-if" part of Theorem 5.1. ■

## Appendix E Proof of Theorem 5.2

The "if" part of the theorem is straightforward. We proceed to prove the "only-if" part. To do so, we prove the contrapositive. Suppose there exist an agent $i^* \in N$ and two objects $a^*, b^* \in A$ such that $rank(i^*, \succ_{a^*}) \leq |A| - 2$ and $rank(i^*, \succ_{a^*}) \neq rank(i^*, \succ_{b^*})$. Without loss of generality, assume that for all $l \in U(i^*, \succ_{a^*})$, $rank(l, \succ_{a^*}) = rank(l, \succ_b)$ for all $b \in A$. Let $rank(i^*, \succ_{a^*}) = m^*$ and let $A' \subseteq A \setminus \{a^*, b^*\}$ be such that $|A'| = m^* - 1$. Clearly, $m^* \leq |A| - 2$. Furthermore, $A'$ is well-defined since $m^* \leq |A| - 2$.

Fix a preference $\hat{P} \in \mathbb{L}(A)$ such that $A'\hat{P}(A \setminus A')$.[28] Consider the preference profile $\tilde{P}_N \in \mathbb{L}^n(A)$ such that $\tilde{P}_i = \hat{P}$ for all $i \in N$. Since $rank(l, \succ_{a^*}) = rank(l, \succ_b)$ for all $l \in U(i^*, \succ_{a^*})$ and all $b \in A$, and $rank(i^*, \succ_{a^*}) = m^*$, it follows from the construction of $\tilde{P}_N$ that $I^{m^*-1}(\tilde{P}_N) = U(i^*, \succ_{a^*})$ and $X^{m^*-1}(\tilde{P}_N) = A'$. The facts $A' \subseteq A \setminus \{a^*, b^*\}$ and $X^{m^*-1}(\tilde{P}_N) = A'$ together imply $a^*, b^* \in A_{m^*}(\tilde{P}_N)$. Since $I^{m^*-1}(\tilde{P}_N) = U(i^*, \succ_{a^*})$, $rank(l, \succ_{a^*}) = rank(l, \succ_{b^*})$ for all $l \in U(i^*, \succ_{a^*})$, $rank(i^*, \succ_{a^*}) \neq rank(i^*, \succ_{b^*})$, and $a^*, b^* \in A_{m^*}(\tilde{P}_N)$, it follows that $|N_{m^*}(\tilde{P}_N)| \geq 2$. Moreover, since $X^{m^*-1}(\tilde{P}_N) = A'$, $|A'| = m^* - 1$, and $m^* \leq |A| - 2$, we have $|A_{m^*}(\tilde{P}_N)| \geq 3$. However, the facts $|N_{m^*}(\tilde{P}_N)| \geq 2$ and $|A_{m^*}(\tilde{P}_N)| \geq 3$ together imply

[28] Recall that $\mathbb{L}(A) = \{P \in \mathbb{L}(A \cup \{a_0\}) \mid aPa_0 \text{ for all } a \in A\}$.

that $T^{\succ_A}$ does not satisfy weak serial dictatorship. This completes the proof of the "only-if" part of Theorem 5.2. ■

## Appendix F Proof of Theorem 5.3

The "if" part of the theorem is straightforward. We proceed to prove the "only-if" part. Let $T^{\succ_A}$ be an SOSP-implementable FPTTC rule on the unrestricted domain $\mathbb{L}^n(A \cup \{a_0\})$. Assume for contradiction that $T^{\succ_A}$ is not a serial dictatorship. Then, there exist two agents $i, j \in N$ and two objects $a, b \in A$ such that $i \succ_a j$ and $j \succ_b i$.

Fix a preference $\hat{P} \in \mathbb{L}(A \cup \{a_0\})$ such that $\tau(\hat{P}) = a_0$. Consider the domain $\tilde{\mathcal{P}}_N \subseteq \mathbb{L}^n(A \cup \{a_0\})$ with only three preference profiles presented in Table F.1.

| Preference profiles | Agent $i$ | Agent $j$ | $\ldots$ | Agent $l$ |
|---|---|---|---|---|
| $\tilde{P}^1_N$ | $aba_0 \ldots$ | $aa_0 \ldots$ | $\ldots$ | $\hat{P}$ |
| $\tilde{P}^2_N$ | $ba_0 \ldots$ | $baa_0 \ldots$ | $\ldots$ | $\hat{P}$ |
| $\tilde{P}^3_N$ | $baa_0 \ldots$ | $aba_0 \ldots$ | $\ldots$ | $\hat{P}$ |

Table F.1: Preference profiles of $\tilde{\mathcal{P}}_N$

In Table F.2, we present some facts regarding the outcome of $T^{\succ_A}$ on the domain $\tilde{\mathcal{P}}_N$. These facts are deduced by the construction of $\tilde{\mathcal{P}}_N$.

| Preference profiles | Agent $i$ | Agent $j$ | $T_i^{\succ_A}$ | $T_j^{\succ_A}$ |
|---|---|---|---|---|
| $\tilde{P}^1_N$ | $aba_0 \ldots$ | $aa_0 \ldots$ | $a$ | $a_0$ |
| $\tilde{P}^2_N$ | $ba_0 \ldots$ | $baa_0 \ldots$ | $a_0$ | $b$ |
| $\tilde{P}^3_N$ | $baa_0 \ldots$ | $aba_0 \ldots$ | $b$ | $a$ |

Table F.2: Partial outcome of $T^{\succ_A}$ on $\tilde{\mathcal{P}}_N$

Since $T^{\succ_A}$ is SOSP-implementable on $\mathbb{L}^n(A \cup \{a_0\})$, it must be SOSP-implementable on the domain $\tilde{\mathcal{P}}_N$. Let $\tilde{G}$ be an SOSP mechanism that implements $T^{\succ_A}$ on $\tilde{\mathcal{P}}_N$.

Note that since $T^{\succ_A}(\tilde{P}^1_N) \neq T^{\succ_A}(\tilde{P}^2_N)$, there exists a node in the SOSP mechanism $\tilde{G}$ that has at least two edges. Also, note that since each agent $l \in N \setminus \{i, j\}$ has exactly one preference in $\tilde{\mathcal{P}}_l$, whenever there are at least two outgoing edges from a node, that node must be assigned to some agent in $\{i, j\}$. Consider the first node (from the root) $v$ that has at least two edges. We distinguish the following two cases.

**Case A**: Suppose $\eta^{NI}(v) = i$.

By SOSP-implementability, the facts $b\tilde{P}^2_i a_0$, $T_i^{\succ_A}(\tilde{P}^2_N) = a_0$, and $T_i^{\succ_A}(\tilde{P}^3_N) = b$ together imply that $\tilde{P}^2_i$ and $\tilde{P}^3_i$ do not diverge at $v$. Since $\tilde{P}^2_i$ and $\tilde{P}^3_i$ do not diverge at $v$, by SOSP-implementability, the facts $a\tilde{P}^3_i a_0$, $T_i^{\succ_A}(\tilde{P}^2_N) = a_0$, and $T_i^{\succ_A}(\tilde{P}^1_N) = a$ together imply that $\tilde{P}^1_i$ and $\tilde{P}^3_i$ do not diverge at $v$. Combining all these observations, we have a contradiction to the fact that $v$ has at least two edges.

**CASE B**: Suppose $\eta^{NI}(v) = j$.

By SOSP-implementability, the facts $a\tilde{P}^1_j a_0$, $T^{\succ_A}_j(\tilde{P}^1_N) = a_0$, and $T^{\succ_A}_j(\tilde{P}^3_N) = a$ together imply that $\tilde{P}^1_j$ and $\tilde{P}^3_j$ do not diverge at $v$. Since $\tilde{P}^1_j$ and $\tilde{P}^3_j$ do not diverge at $v$, by SOSP-implementability, the facts $b\tilde{P}^3_j a_0$, $T^{\succ_A}_j(\tilde{P}^1_N) = a_0$, and $T^{\succ_A}_j(\tilde{P}^2_N) = b$ together imply that $\tilde{P}^2_j$ and $\tilde{P}^3_j$ do not diverge at $v$. Combining all these observations, we have a contradiction to the fact that $v$ has at least two edges.

Since Cases A and B are exhaustive, this completes the proof of the "only-if" part of Theorem 5.3. ■

# Appendix G Proofs of Proposition 6.1

Fix an arbitrary domain of preference profiles $\mathcal{P}_N$. Consider a simply strategy-proof assignment rule $f$ on $\mathcal{P}_N$. Let $G$ be the simple OSP mechanism that implements $f$ on $\mathcal{P}_N$. Consider an agent $i \in N$, a node $v$ such that $\eta^{NI}(v) = i$, and preference profiles $P_N, P'_N, \tilde{P}_N \in \mathcal{P}_N$ passing through $v$ such that (i) $P_i$ and $P'_i$ do not diverge at $v$ and (ii) $P_i$ and $\tilde{P}_i$ diverge at $v$. We show that $f_i(P'_N) R_i f_i(\tilde{P}_N)$.

Since $P_i$ and $P'_i$ do not diverge at $v$, the fact that $G$ is a simple mechanism implies that $f(P'_N) = f(P_i, P'_{-i})$. This, in particular, means

$$f_i(P'_N) = f_i(P_i, P'_{-i}). \tag{G.1}$$

The fact that both $P_N$ and $P'_N$ pass through $v$ implies that $(P_i, P'_{-i})$ passes through $v$. Consider the preference profiles $(P_i, P'_{-i})$ and $\tilde{P}_N$. Since both of them pass through $v$ at which $P_i$ and $\tilde{P}_i$ diverge, by obvious strategy-proofness of $G$, we have $f_i(P_i, P'_{-i}) R_i f_i(\tilde{P}_N)$. This, together with (G.1), implies $f_i(P'_N) R_i f_i(\tilde{P}_N)$. This completes the proof of Proposition 6.1. ■

# Appendix H Proof of Theorem 6.1

We first make a straightforward observation to facilitate the proof.

**Observation H.1.** *Suppose $|A| = 2$. Every FPTTC rule is simply strategy-proof on the restricted domain $\mathbb{L}^n(A)$.*

***Completion of the proof of Theorem 6.1.*** The "only-if" part of the theorem follows from Proposition 6.1, we proceed to prove the "if" part. Let $T^{\succ_A}$ be an SOSP-implementable FPTTC rule on $\mathcal{P}_N$. We distinguish the following two cases.

**CASE A**: Suppose $\mathcal{P}_N = \mathbb{L}^n(A)$.

From Theorem 5.1, it follows that $T^{\succ_A}$ satisfies weak serial dictatorship. This, along with Observation H.1, implies that $T^{\succ_A}$ is simply strategy-proof.

**CASE B**: Suppose $\mathcal{P}_N = \mathbb{L}^n(A \cup \{a_0\})$.

The proof of this case is straightforward. ■

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