Equilibrium under Ambiguity with Multiple Priors[*]

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Abstract

This paper studies normal form games with multiple prior preferences. We develop a new measure of ambiguity based on the concept that sets of priors which are translations of one another represent the same level of ambiguity. This is applied to non-cooperative games with ambiguity. We propose a solution concept for games where players have multiple priors preferences. The set of translations of a given set of priors is shown to be isomorphic to the simplex. This enable us to prove existence of equilibrium. Like conventional mixing, translations can convexify pure strategies.

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1 Introduction

Most issues in the social sciences involve uncertainty in some form. How to encourage people to be vaccinated or how to respond to threats from terrorism or rogue states are current policy examples. Often it is difficult or impossible to assign probabilities to the possible outcomes of this uncertainty. One reason for this is that the relevant uncertainty often involves the behaviour of other people, which is intrinsically difficult to predict. This type of uncertainty referred to as ambiguity.

In this paper we present a theory of how ambiguity affects the interactions between a group of agents. Strategic interactions between individuals are represented as normal form games. Ambiguity is modelled using multiple prior preferences. Gilboa and Schmeidler (1989) axiomatised ambiguity-averse multiple prior preferences which they refer to as Maxmin Expected utility, (henceforth MEU). Marinacci (2002) proposed an extension of multiple priors, which is not necessarily ambiguity averse which we shall refer to as α-MEU.

One advantage of using multiple prior preferences is that they can be applied to problems involving multiple time periods in a way which respects dynamic consistency, Sarin and Wakker (1998). Sarin and Wakker propose recursive multiple prior preferences which are dynamically consistent while also allowing for non-neutral attitudes to ambiguity.[1] This will allow an extension of our solution concept to extensive form games.

We present a new definition of comparative ambiguity based on the intuition that ambiguity should be the same for any set of priors of the same shape regardless of where it is located in the space of probabilities. In other words ambiguity is translation invariant within the probability simplex. We then apply this to strategic ambiguity in games. Consider a situation where a given player perceives his/her opponent's action to be ambiguous. Translation invariance enables us to model a situation where a player perceives similar levels of ambiguity regardless of which strategy his/her opponent plays.

We prove existence of equilibrium with new solution concept. Our existence proof is novel since it does not require preferences to be convex. This we are able to allow for ambiguity seeking as well as ambiguity-averse behaviour. In addition it does not impose any restrictions

[1] Epstein and Schneider (2003) develop the recursive multiple priors model and apply it to financial markets.

on pay-offs, (such as strategic complementarity, see Eichberger and Kelsey (2014)). This is achieved because we discover a novel isomorphism between the set of translations of a set of priors and the mixed strategy space of a related game. Thus translations have a convexifying effect analogous to the way mixed strategies convexify standard games. This enables us to demonstrate existence of equilibrium using fixed point theorems.

Organisation of the paper

In the next section we describe our framework and definitions. In Section 3 we argue that ambiguity is location invariant and use this to define a new measure of ambiguity. In Section 4 we apply this to normal form games. Section 5 extends the results to extensive form games and Section 6 concludes.

2 Preliminaries

We consider a finite state space S. Let $\Delta(S)$ denote the set of all probability distributions over S and 2^S the powerset of S. An *act* is a function $a : S \to \mathbb{R}$.. Let $A(S)$ denote the set of all acts. It is without loss of generality to assume that the pay-offs are utilities. A utility function may be derived for non-monetary outcomes in the usual way. Let $\mathcal{C}$ be a set of priors, i.e. a closed convex subset of $\Delta(S)$.

Definition 2.1 *The binary relation $\succcurlyeq$ is a multiple priors (or α-MEU) preference relation if there exists a set of priors $\mathcal{C}$ and $\alpha \in [0,1]$ such that:*

$$a \succcurlyeq b \Leftrightarrow \alpha \min_{p \in \mathcal{C}} \mathbf{E}_p a + (1-\alpha) \max_{p \in \mathcal{C}} \mathbf{E}_p a \geqslant \alpha \min_{p \in \mathcal{C}} \mathbf{E}_p b + (1-\alpha) \max_{p \in \mathcal{C}} \mathbf{E}_p b,$$

where $\mathbf{E}_p a$ denotes the expected value of a with respect to the probability distribution p.

Mulptiple prior preferences have been axiomatised by Hartmann, GMM, Seo et al,

to be completed

If $\alpha = 1$, these coincide with MEU preferences, Gilboa and Schmeidler (1989). If $\alpha = 0$ these would be maxmax preferences. For intermediate values of α these preferences are neither uniformly ambiguity-averse nor ambiguity loving. This is compatible with experimental evidence, see Kilka and Weber (2001).

We interpret $\mathcal{C}$ as the decision-maker's beliefs. However these are ambiguous beliefs. The decision-maker does not know which of the priors in $\mathcal{C}$ is the true probability distribution. His/her reaction to this ambiguity is in part pessimistic in the sense it values acts by the least favourable probability distribution. It is also partially optimistic since weight is also given to the most favourable probability in $\mathcal{C}$. The parameter α measures the individuals ambiguity-attitude. Higher values of α correspond to more ambiguity-averse preferences.

3 Location Invariance of Ambiguity

What does it mean for two prior sets to reflect the same ambiguity? This question is interesting in its own right. Moreover we believe that it is important for the study of ambiguity in games. We may wish to consider a situation where a player perceives his/her opponents' behaviour to be ambiguous. However we do not wish to specify which strategies the opponents will play before we have determined equilibrium. Translations allow us to do this.

We propose that ambiguity is *location invariant*, meaning that the exact "location" of the prior set within the set of probabilities over a state space is irrelevant for the level of ambiguity that it reflects. Location invariance is obvious when prior sets are singletons: If we consider two distinct probability distributions, these being two singleton prior sets differing only in "location", then they reflect the same ambiguity: None! In the following we argue that this location invariance generalizes to non-singleton prior sets as well. This leads to a natural measure which is intuitive and unifies many existing measures of ambiguity (see Epstein (1999), Ghirardato and Marinacci (2002)).

3.1 Location Invariance

Throughout this section we consider ambiguity in single-person decisions. In Section 4 we apply these concepts to games.

3.1.1 Translations

Our concept of location invariance builds on *translations of prior sets*, which are defined as follows.

Definition 3.1 (Translation) *Let H denote the hyperplane $H = p \in \mathbb{R}^S : \sum_{s \in S} p(s) = 1$. A translation is an additive function $\phi_t : H \to H$ defined by $\phi_t(p) = p + t$ where $t \in \mathbb{R}^S$ is such that $\sum_{s \in S} t(s) = 0$. Let Φ denote the space of all translations.*

Definition 3.2 (Translation of prior sets) *Two prior sets $\mathcal{C}, \mathcal{C}' \subseteq \Delta(S)$ are translations of each other if there exists a translation $\phi_t : 2^S \to \mathbb{R}$ such that*

$$\mathcal{C}' = \{p \in \Delta(S) | \exists q \in \mathcal{C} : p = \phi_t(q)\}.$$

In such a case we write $\mathcal{C}' = \phi_t(\mathcal{C})$ or $\mathcal{C}' = \mathcal{C} + t$. Below we present two examples to illustrate the concept of a translation. Firstly any single subjective probability is a translation of any other. Secondly all balls of a given size are translations of one another.

Example 3.1 (Translation)

- *$\mathcal{C} = \{p\}, \mathcal{C}' = \{p'\}$ with $p, p' \in \Delta(S)$. Define $t(s) = p(s) - p'(s)$ for all $s \in S$. Then $\mathcal{C}' + t = \mathcal{C}$.*
- *$\mathcal{C}' = B_\epsilon(p), \mathcal{C} = B_\epsilon(p') \subseteq \Delta(S)$ with $p, p' \in \Delta(S)$ and $B_\epsilon(p)$ the prior set, which is a ball with centre p and radius ϵ. Define $t(s) = p(s) - p'(s)$ for all $s \in S$. Then $\mathcal{C}' + t = \mathcal{C}$.*

3.1.2 Translations Keep Ambiguity Constant.

We claim that prior sets that are translations of one another reflect the same ambiguity. To see why consider a prior set $\mathcal{C}$ and an act $a : S \to \mathbb{R}$. Every prior p in $\mathcal{C}$ induces an expectation of a through $\sum_{s \in S} a(s)p(s)$. Since prior sets are compact, there exist priors in $\mathcal{C}$ that induce the (weakly) highest and lowest expectation. The difference between these two expectations is a measure of the ambiguity for a given $\mathcal{C}$. A translation of $\mathcal{C}$ keeps this ambiguity constant. The class of prior sets that reflect the same ambiguity thus corresponds precisely to the set of translations of a given set $\mathcal{C}$. These claims are justified by the results and definitions in this section.

Measure of Ambiguity

Next we present a measure of ambiguity for a prior set. For a given act a it is the normalised difference between the highest and lowest expected value of a.

Definition 3.3 (Ambiguity of Prior Sets) *Let $\mathcal{C}$ be a set of priors. The ambiguity for act a reflected by $\mathcal{C}$ is characterized by the function*

$$\delta_{\mathcal{C}}(a) = \begin{cases} \frac{\max_{p\in\mathcal{C}} E_p(a) - \min_{p\in\mathcal{C}} E_p(a)}{\max_{p\in\Delta(S)} E_p(a) - \min_{p\in\Delta(S)} E_p(a)}, & a \notin X \\ 0, & a \in X, \end{cases} \tag{1}$$

where $X = \{a \in A(S) : \max_{p\in\mathcal{C}} \mathbf{E}_p(a) = \min_{p\in\mathcal{C}} \mathbf{E}_p(a)\}$.

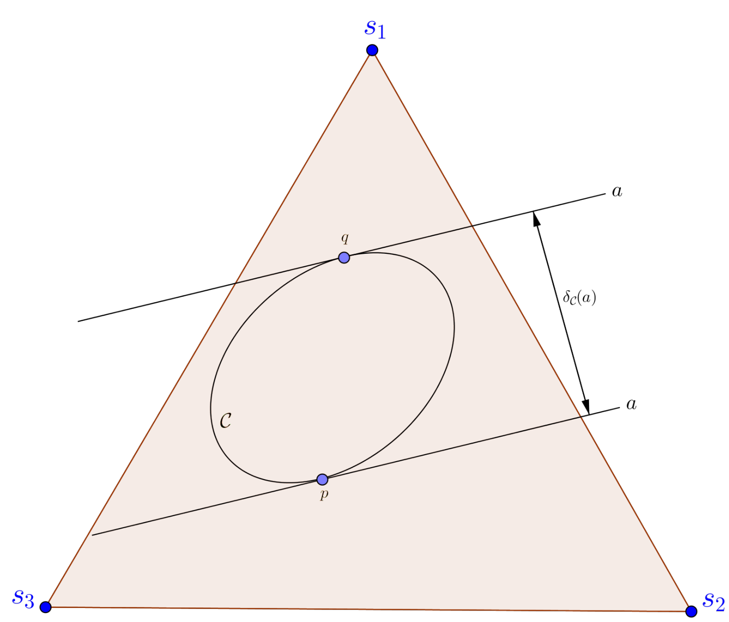

Figure 1: Worst and best scenario for the act a given prior set $\mathcal{C}$ and utility function u.

This measure of ambiguity is illustrated by Figure 1. There are 3 states $S = \{s_1, s_2, s_3\}$. Let $\mathcal{C}$ be a set of priors. The two parallel lines illustrate the best and worst case scenarios for some act $a : S \to \mathbb{R}$ given $\mathcal{C}$. The difference between these two lines is $\delta_{\mathcal{C}}(a) \in [0, 1]$ which measures the ambiguity reflected by $\mathcal{C}$ for the act a. The number $\delta_{\mathcal{C}}(a)$ thus measures the ambiguity of a given $\mathcal{C}$. If $\delta_{\mathcal{C}}(a) = 0$, then there is no ambiguity. The larger $\delta_{\mathcal{C}}(a)$ becomes, the more ambiguity there is. The following result shows that when two prior sets are translations, then their δ functions are identical. This illustrates why ambiguity does not change when a prior set is translated: A translation keeps the ambiguity constant for each act.

Proposition 3.1 *If two prior sets $\mathcal{C}$ and $\mathcal{C}'$ are translations of one another then $\delta_{\mathcal{C}}(a) = \delta_{\mathcal{C}'}(a)$ for all $a \in A(S)$.*

Proof. Let $a \in A(S)$ be an act, t a translation, and $p \in \Delta(S)$ a prior. Notice that $E_p(a) = p \centerdot a$. [2] Then

$$E_{p+t}(a) = (p+t) \centerdot a = p \centerdot a + t \centerdot a = E_p a + t \centerdot a$$

Now assume that the prior sets $\mathcal{C}$ and $\mathcal{C}'$ are translations, i.e. $\mathcal{C}' = \phi_t(\mathcal{C})$ for some translation t. The above implies that $q \in \operatorname{argmax}_{p \in \mathcal{C}'} E_p(a)$ if and only if $q + t \in \operatorname{argmax}_{p \in \mathcal{C}} E_p(a)$. In particular, $\min_{p \in \mathcal{C}'} E_p(a) = \min_{p \in \mathcal{C}} E_p(a) + t \centerdot a$. The same holds when "min" is replaced by "max". Thus

$$\begin{aligned} \max_{p \in \mathcal{C}'} E_p(a) - \min_{p \in \mathcal{C}'} E_p(a) &= \max_{p \in \mathcal{C}} E_p(a) + t \centerdot a - (\min_{p \in \mathcal{C}} (E_p(a) + t \centerdot a) \\ &= \max_{p \in \mathcal{C}} E_p(a) - \min_{p \in \mathcal{C}} E_p(a). \end{aligned}$$

This implies $\delta_{\mathcal{C}}(a) = \delta_{\mathcal{C}'}(a)$. ■

Remark 3.1 *This result illustrates why ambiguity does not change when a prior set is translated: A translation keeps the ambiguity constant for each act. The converse is not true, since rotations by 180^o also keep δ constant. To see this first note that a and $-a$ have the same ambiguity. Rotations would preserve ambiguity, if we rotated the acts as well as the priors. Moreover a 180^o rotation maps a to $-a$, hence ambiguity is unchanged by such a rotation. See Figure 2 for a demonstration. The two prior sets are 180^o rotations around the point p and obviously not translations of each other. Nonetheless we have $\delta_{\mathcal{C}} \equiv \delta_{\mathcal{C}'}$.*

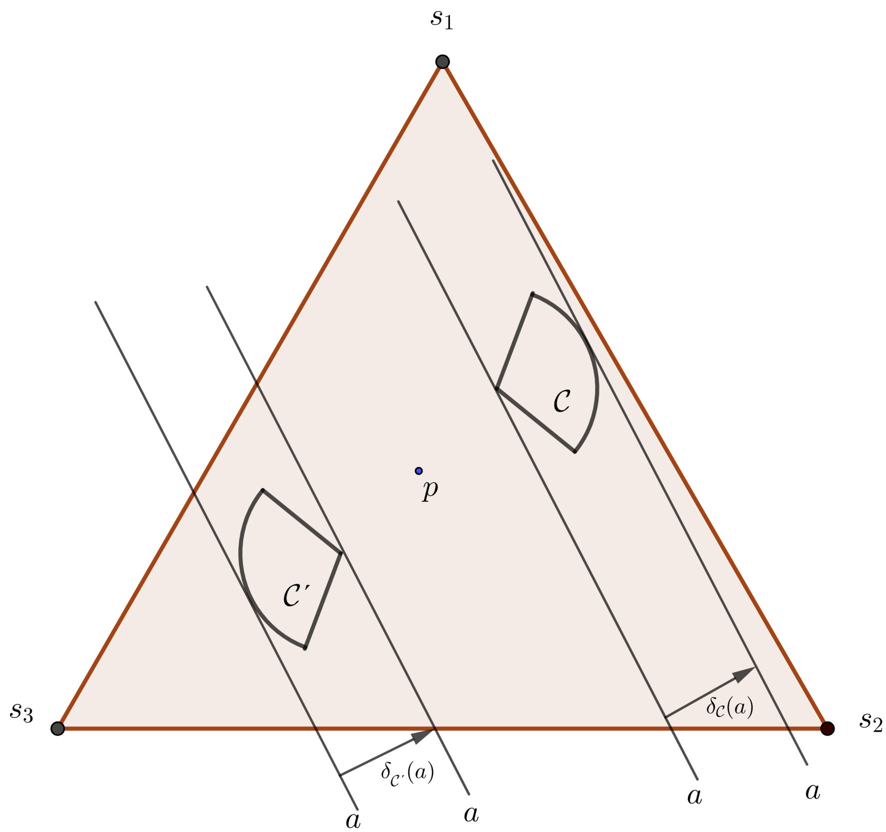

Figure 2: $\mathcal{C}$ and $\mathcal{C}'$ are 180^o rotations around the point p. The measures $\delta_{\mathcal{C}}(a)$ and $\delta_{\mathcal{C}'}(a)$ are the same for all acts $a \in A(S)$.

It is immediate that $\delta_{\mathcal{C}_1} \equiv \delta_{\mathcal{C}_2}$ holds if $\mathcal{C}_1$ and $\mathcal{C}_2$ are translations. We can thus define equivalence classes of prior sets that fix ambiguity.

Definition 3.4 (Ambiguity classes) *We define the ambiguity class of the set of priors $\mathcal{C}$ to be:*

$$[\mathcal{C}] = \{\mathcal{C}' \subseteq \Delta(S) | \exists \phi_t \in \Phi : \mathcal{C}' = \phi_t(\mathcal{C}).$$

The set $[\mathcal{C}]$ contains all the translations of $\mathcal{C}$ within $\Delta(S)$. By Proposition 3.1, it contains exactly those prior sets that reflect the same ambiguity according to Definition 3.3. The following definition now suggests itself.

Definition 3.5 (Comparative notion for ambiguity classes) *Let $\mathcal{C}, \mathcal{C}'$ be prior sets and $[\mathcal{C}], [\mathcal{C}']$ their respective ambiguity classes. Then $[\mathcal{C}]$ reflects more ambiguity than $[\mathcal{C}']$ if and only if $\mathcal{C}$ reflects more ambiguity than $\mathcal{C}'$, i.e. there exists a translation t such that $\mathcal{C}' \subseteq \mathcal{C} + t$.*

[2]Here $p \centerdot a = \sum_{s \in S} p(s)a(s)$ denotes the scalar product of vectors p and a.

This induces an order which is transitive but not complete. Since $[\mathcal{C}]$ is the set of translations of $\mathcal{C}$ and ambiguity is location invariant, $[\mathcal{C}]$ is exactly the set of prior sets that reflect the same ambiguity. We shall apply this concept to ambiguous beliefs in games. Ambiguity classes allow us to specify a level of ambiguity independent of the strategies actually played.

3.2 Isomorphism

Building on the concept of location invariance from Section 3.1, we now introduce a result which is crucial for existence of equilibrium. We show that any given ambiguity class is isomorphic to the simplex, hence ambiguity classes are convex. This enables us to use fixed point theorems. All claims made in this subsection are proved in the Appendix.

We define the support of a set of priors, $\mathcal{C}$, to be those states which are in the support of all of the priors in $\mathcal{C}$. This is a strong notion of support. It is the set of states in which the decision-maker "believes" in the strong sense that they receive positive probability no matter which of the priors is the true distribution.

Definition 3.6 *Let $\mathcal{C}$ be a set of priors. Define the support of $\mathcal{C}$ by* $\operatorname{supp}(\mathcal{C}) = \bigcap_{p\in\mathcal{C}} \operatorname{supp}(p)$.[3]

Let $\mathcal{C}$ be a prior set on a state space S and let $[\mathcal{C}]$ be its ambiguity class, (see Definition 3.4). Assume that the support of $\mathcal{C} = \operatorname{supp}(\mathcal{C})$ is non-empty. For each $s \in S$ there exists a unique element $\mathcal{C}^s$ in $[\mathcal{C}]$ such that $\operatorname{supp}(\mathcal{C}^s) = \{s\}$. This prior set is in the "s-corner" of Σ. We show in the appendix that this prior set always exists and is unique.

Lemma 3.1 *If S is a finite state space and $\mathcal{C} \subseteq \Delta(S)$ is a prior set with* $\operatorname{supp}(\mathcal{C}) \neq \emptyset$, *then for every state $s^* \in S$ there exists a unique prior set $\mathcal{C}^{s^*} \in [\mathcal{C}]$ such that* $\operatorname{supp}(\mathcal{C}^{s^*}) = \{s^*\}$.

Now consider a arbitrary $\sigma \in \Delta(S)$, i.e. a probability distribution over S. Below, we define $\mathcal{C}^\sigma$ as the σ-mix of the "corner" prior sets.

Definition 3.7 *Define a function $\psi : \Delta(S)$ to $\to [\mathcal{C}]$, by $\psi(\sigma) = \mathcal{C}^\sigma = \mathcal{C}^{s^*} + t^\sigma(s)$, where*

$$t^\sigma(s) = \begin{cases} \tau\sigma(s), & s \neq s^*, \\ \tau[\sigma(s) - 1], & s = s^*, \end{cases}$$

[3]This support notion is the logical one for prior sets and is introduced in Ryan (2002).

where $\tau = \min_{p \in \mathcal{C}^s} p(s) \in (0,1]$. [4] *The prior set* $\mathcal{C}^\sigma$ *is the "*σ *mix" of the sets* $\mathcal{C}^s$*,* $s \in S$*.*

The following proposition establishes the ψ is indeed an isomorphism.

Proposition 3.2 *The function* $\psi : \Delta(S) \to [\mathcal{C}]$*, is an isomorphism.*

Note that when the prior set $\mathcal{C}$ is a singleton, $[\mathcal{C}] = \Delta(S)$ and the isomorphism is the identity.

3.3 Comparison with other Measures of Ambiguity

In this section we compare our proposed measure of comparative ambiguity with related definitions in the previous literature.

3.3.1 Belief Hedges

Baillon et al. (2021) introduce two ambiguity indexes: an ambiguity-aversion index b and an (ambiguity generated-) insensitivity index a. They generalize many ambiguity indexes suggested in the literature. To introduce them some definitions are needed.

Definition 3.8 *A measurement design* $\mathcal{H}$ *is a finite collection of events,* $\{E_1, \ldots, E_n\}$ *denotes the smallest nonempty intersection of events in* $\mathcal{H}$*, called design atoms.* v *denotes the normalized event size, i.e.* $v(E) = \frac{|E|}{n}$*, where* $|E|$ *is the number of atoms in* E*.*

Definition 3.9 *A measurement design* $\mathcal{H}$ *is called l(evel)-hedged if each state* s *appears in exactly half of the events in* $\mathcal{H}$*. It is called v(ariation)-hedged if* $\sum_{s \in E} v(E)$ *is the same for all states* $s \in S$*. If* $\mathcal{H}$ *is both l-hedged and v-hedges, it is called a belief hedge.*

For a belief hedge, the design atoms form a partition of S. For a function $a : \mathcal{H} \to \mathbb{R}$, $\overline{a} = \frac{\sum_{E \in \mathcal{H}} a(E)}{|\mathcal{H}|}$ denotes the average of a. The indexes of Baillon et al. (2021) rely on the concept of probability matching. For each event $E \subseteq S$ there exists (by monotonicity) a unique number $m(E) \in [0,1]$ such that $x_E y \sim x_{m(E)} y$.[5] We can now introduce the indexes of Baillon et al. (2021).

[4] Lemma A.2, in the appendix, proves that τ is independent of s.

[5] As usual, $x_E y$ denotes the act which gives consequence x on E and y on E^c. $x_p y$ denotes the lottery which gives x with probability p and y with probability $1 - p$.

Definition 3.10 *If l-hedging holds then the index of ambiguity aversion is*

$$b = 1 - 2\overline{m}.$$

If in addition v-hedging holds (i.e. $\mathcal{H}$ is a belief hedge) then the index of a(mbiguity generated) insensitivity is

$$a = 1 - \frac{Cov(m, v)}{Var(v)}.$$

The following lemma shows that these indices are unaffected by translations.

Proposition 3.3 *Consider $\alpha \in [0,1]$ and prior sets $\mathcal{C}_1, \mathcal{C}_2$. Let $\succcurlyeq_i$ be the α-MEU preference relations induced by a and $\mathcal{C}_i$, for $i \in \{1,2\}$. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are translations, then $\succcurlyeq_1$ and $\succcurlyeq_2$ have the same ambiguity aversion and a-insensitivity index for any belief hedge $\mathcal{H}$.*

Proof. We first show that when $\mathcal{C}_1$ and $\mathcal{C}_2$ are translations, i.e. $\mathcal{C}_2 = \phi_t(\mathcal{C}_1)$, then $m_2 = m_1 + t$. First note that in the α-MEU model, $m(E) = \alpha \min_{p \in \mathcal{C}} p(E) + (1-\alpha) \max_{p \in \mathcal{C}} p(E)$. We have

$$\begin{aligned} m_2(E) &= \alpha \min_{p \in \mathcal{C}_2} p(E) + (1-\alpha) \max_{p \in \mathcal{C}_2} p(E) = \alpha \min_{p \in \mathcal{C}_1 + t} p(E) + (1-\alpha) \max_{p \in \mathcal{C}_1 + t} p(E) \\ &= \alpha [\min_{p \in \mathcal{C}_1} p(E) + t(E)] + (1-\alpha)[\max_{p \in \mathcal{C}_1} p(E) + t(E)] \\ &= \alpha \min_{p \in \mathcal{C}_1} p(E) + (1-\alpha) \max_{p \in \mathcal{C}_1} p(E) + t(E) = m_1(E) + t(E). \end{aligned}$$

Next we show that the average of the matching probability function is unaffected by translations, i.e. that $\overline{m}_1 = \overline{m}_2$.

First note that $\sum_{E \in \mathcal{H}} t(E) = \sum_{s \in S} t(s) \frac{|\mathcal{H}|}{2} = \frac{|\mathcal{H}|}{2} \sum_{s \in S} t(s) = 0$, where the first equality follows from $\mathcal{H}$ being l-hedged as well as the additivity of t and the last equality follows from t being a translation. Thus we have

$$\begin{aligned} \overline{m}_2 = \overline{m_1 + t} &= \frac{\sum_{E \in \mathcal{H}} [m_1(E) + t(E)]}{|\mathcal{H}|} \\ &= \frac{\sum_{E \in \mathcal{H}} m_1(E)}{|\mathcal{H}|} + \frac{\sum_{E \in \mathcal{H}} t(E)}{|\mathcal{H}|} = \frac{\sum_{E \in \mathcal{H}} m_1(E)}{|\mathcal{H}|} = \overline{m_1}. \end{aligned}$$

This directly implies that the ambiguity-aversion index is the same for both preferences.

We now turn to a-insensitivity, defined by $a = 1 - \frac{Cov(m,\nu)}{Var(\nu)}$. It suffices to show that $Cov(m_2, v) = Cov(m_1, v)$. We have

$$\begin{aligned} Cov(m_2, v) &= \overline{(m_2 - \overline{m_2})(v - \overline{v})} \\ &= \overline{m_2 v} - \overline{m_2}\ \overline{v} \\ &= \overline{(m_1 + t)v} - \overline{m_1 + t}\ \overline{v} \\ &= \overline{m_1 v} + \overline{tv} - \overline{m_1}\ \overline{v} - \overline{t}\ \overline{v} \\ &= \mathrm{Cov}(m_1, v) + \overline{tv} - \overline{v}\ \overline{t}. \end{aligned}$$

Since $\overline{t} = 0$ by the definition of translation it suffices to show that $\overline{tv} = 0$:

$$\begin{aligned} \overline{tv} &= \sum_{E \in \mathcal{H}} t(E)v(E) = \sum_{E \in \mathcal{H}} \sum_{E_i \in E} t(E_i) \underbrace{v(E_i)}_{=\frac{1}{|\mathcal{H}|}} \\ &= \frac{1}{|\mathcal{H}|} \sum_{E \in \mathcal{H}} \sum_{E_i \in E} \sum_{s \in E_i} t(s) \overset{(*)}{=} \frac{1}{|\mathcal{H}|} \frac{|\mathcal{H}|}{2} \underbrace{\sum_{s \in S} t(s)}_{=0} = 0, \end{aligned}$$

where $(*)$ holds due to l-hedging. We have thus shown that $\mathrm{Cov}(m_2, \nu) = \mathrm{Cov}(m_1, \nu)$ which implies that the a-insensitivity is uneffected by translations.

■

The reverse direction in Proposition 3.3 is not true, i.e. our measure is finer than the indexes from Baillon et al. (2021). There exist preferences with the same indexes but whose prior sets are not translations. This is not surprising as the indexes are one-dimensional and only depend on preferences over binary acts whereas there is much more variability in multiple prior models which depend on preferences over non-binary acts as well. The following example illustrates such a case.

Example 3.2 *Consider the state space $S = \{s_1, s_2, s_3\}$, the belief hedge $\mathcal{H} = \mathcal{P}(S)$, and let p_{unif} be the uniform distribution over S. Let $\alpha \in [0, 1]$ and consider the following two prior* *sets:*

$$\mathcal{C}_1 = B_{\frac{1}{6}}(p_{unif})$$
$$\mathcal{C}_2 = \{q \in \Delta(S) | q(s_i) \geq \frac{1}{6}, q(s_i, s_j) \geq \frac{1}{2}, i, j \in \{1,2,3\}, i \neq j\}.$$

The prior set $\mathcal{C}_1$ is the ball with radius $\frac{1}{6}$ and centre p_{unif}. The prior set $\mathcal{C}_2$ is the core of a convex capacity. Obviously they are not translations. However, it is immediate that

$$\min_{p \in \mathcal{C}_1} p(E) = \min_{p \in \mathcal{C}_2} p(E) = \begin{cases} \frac{1}{6}; |E| = 1, \\ \frac{1}{2}; |E| = 2; \end{cases}$$

$$\max_{p \in \mathcal{C}_1} p(E) = \max_{p \in \mathcal{C}_2} p(E) = \begin{cases} \frac{1}{2}; |E| = 1, \\ \frac{5}{6}; |E| = 2. \end{cases}$$

Let $\succcurlyeq_i$ be the α-MEU preference induced by α and $\mathcal{C}_i$, $i \in \{1,2\}$. For any α-MEU preference and all events E we have $m(E) = \alpha \min_{p \in \mathcal{C}} p(E) + (1-\alpha) \max_{p \in \mathcal{C}} p(E)$, thus $m_1 = m_2$ and in particular $\overline{m}_1 = \overline{m}_2$. This implies that the indexes coincide for $\succcurlyeq_1$ and $\succcurlyeq_2$.

The measure of Definition 3.3 is intuitively appealing.[6] In the following section we illustrate the definition by applying it to some examples.

3.3.2 Neo-Additive Preferences

Neo-Additive preferences, (Chateauneuf et al. (2007)) (henceforth CEG) satisfy both the multiple priors and Choquet Expected Utility (CEU) axioms. They are defined as follows.

Definition 3.11 *Let α, δ be real numbers such that $0 < \delta < 1, 0 < \alpha < 1$. A* neo-additive-capacity *ν on S is defined by $\nu(A) = \delta(1-\alpha) + (1-\delta)\pi(A)$, for $\emptyset \subsetneqq A \subsetneqq S$, where π is an additive probability distribution on S. A preference relation $\succcurlyeq$ is* Neo-Additive *if it can be represented by the Choquet integral*

$$W(a) = \int a d\nu.$$

[6] Eichberger and Kelsey (2014), Marinacci (2000), Chateauneuf et al. (2007) and Dominiak and Eichberger (2016) introduce measures that are all special cases of Definition 3.3.

CEG show that neo-additive capacities also have a multiple prior representation. Let ν be a neo-additive capacity characterized by π, δ and α. Then for any act $a \in A(S)$,

$$W(a) = \alpha \min_{\hat{p} \in \mathcal{D}} \mathbf{E}_p a + (1-\alpha) \max_{\hat{p} \in \mathcal{D}} \mathbf{E}_p a,$$

where $\mathcal{D} = \{\hat{p} \in \Delta(S) | \hat{p}(E) \geq (1-\delta)p(E), \forall\ E \in \Sigma\}$.

Neo-Additive preferences are represented by a weighted average of the highest expected pay-off and the lowest expected pay-off of the priors in $\mathcal{D}$. CEG interpret them as describing a situation where the decision maker's 'beliefs' are represented by the probability distribution π. However these are ambiguous beliefs. This ambiguity is captured by the parameter δ. The highest possible level of ambiguity corresponds to $\delta = 1$, while $\delta = 0$ corresponds to no ambiguity. The following proposition shows that two neo-additive capacities are translations of one another if and only if they have the same δ. Hence our measure of ambiguity coincides with that proposed by CEG.

Proposition 3.4 *Let ν and ν' be two neo-additive capacities characterized by π, δ, α and π', δ', α', respectively. Let $\mathcal{D}$ and $\mathcal{D}'$ be the respective prior sets from* (3.3.2). *Then $\delta = \delta'$ if and only if $\mathcal{D}$ and $\mathcal{D}'$ are translations.*

Proof. Assume that $\delta = \delta'$. Consider the translation $t = (1-\delta)(\pi' - \pi)$. We show that $\mathcal{D}' = \mathcal{D} + t$. Consider some $\hat{\pi} \in \Delta(S)$. Then

$$\hat{\pi} \in \mathcal{D} + t \iff \hat{\pi}(E) \geq (1-\delta)\pi(E) + (1-\delta)(\pi'(E) - \pi(E)), \forall E \subseteq S$$
$$\iff \hat{\pi}(E) \geq (1-\delta)\pi'(E), \forall E \subseteq S \iff \hat{\pi} \in \mathcal{D}'.$$

For the converse assume that $\delta > \delta'$. Consider an arbitrary state $s' \in S$ and the following translations t and t':

$$t(s) = \begin{cases} 1 - \pi(s), & s = s'; \\ -\pi(s), & s \neq s'; \end{cases} \qquad t'(s) = \begin{cases} 1 - \pi'(s), & s = s'; \\ -\pi'(s), & s \neq s'. \end{cases}$$

The prior sets $\mathcal{D} + t$ and $\mathcal{D}' + t'$ are the sets $\mathcal{D}$ and $\mathcal{D}'$ translated into the s'-corner of the simplex, i.e.

$$\mathcal{D}+t=\{\hat{\pi}\in\Delta(S)|\hat{\pi}(E)\geq(1-\delta)\mathbb{I}_{s'},\forall E\subseteq S\}$$
$$\mathcal{D}'+t'=\{\hat{\pi}\in\Delta(S)|\hat{\pi}(E)\geq(1-\delta')\mathbb{I}_{s'},\forall E\subseteq S\},$$

where $\mathbb{I}_{s'}$ is the degenerate distribution in $\Delta(S)$ which puts all weight on s'. Due to $\delta>\delta'$ we have that $\mathcal{D}+t\supsetneq\mathcal{D}'+t'$. Therefore $\mathcal{D}$ and $\mathcal{D}'$ are not translations. ■

3.4 Multiple dimensions

Here we extend our analysis to a multi-dimensional state space. The state space is a Cartesian product, $S=S_1\times...\times S_M$. The previous results imply the existence of an isomorphism between an ambiguity class over S and the set of probability distributions over $S,\Delta(S)$. However it is often useful to consider beliefs which are independent over the components. In particular if S is the space of possibly mixed strategies of one's opponents in a game it is usual to assume that they are independent. This can be achieved as follows.

Let Λ denote the set of all *independent* probability distributions over S, i.e. if $p\in\Sigma$, then $p(s_1,...,s_M)=\prod_{i=1}^{M}p_i(s_i)$. An *independent set of priors* $\mathcal{C}$ is a finite subset of Λ. Note that as defined $\mathcal{C}$ is not convex. However we could take the convex hull of $\mathcal{C}$. This would not change decisions since they only depend of the extremal points of $\mathcal{C}$.

Definition 3.12 *Let $\mathcal{C}_i$ be a set of priors over S_i,and let $[\mathcal{C}_i]$ denotes its ambiguity class for $1\leqslant i\leqslant M$. An* independent ambiguity class, *$\mathcal{I}$, is a family of sets of priors of the form $\mathcal{I}=\mathcal{C}_1'\times...\times\mathcal{C}_M'$, where $\mathcal{C}_i'\in[\mathcal{C}_i]$ for $1\leqslant i\leqslant M$. Alternatively $\mathcal{I}=\phi_{t_1}(\mathcal{C}_1)\times...\times\phi_{t_M}(\mathcal{C}_M)$, where ϕ_{t_i} is a translation for $1\leqslant i\leqslant M$.*

We can show that the independent ambiguity class, $\mathcal{I}$, is isomorphic to the set of independent probability distributions over S.

Proposition 3.5 *The function ξ defined by by $\xi(\pi_1,...,\pi_M)=\left\langle\psi^1(\pi_1),...,\psi^M(\pi_M)\right\rangle$ is an isomorphism. Here $\psi^i:\Delta(S_i)\rightarrow[\mathcal{C}^i]$ denotes the isomorphism from Definition 3.7.*

Proof. The fact that ψ^i is 1-1 for $1 \leqslant i \leqslant M$ implies that ξ is 1-1. Moreover ξ is onto since if $\langle \pi_1, ..., \pi_M \rangle \in \Lambda$, then $\langle \pi_1, ..., \pi_M \rangle = \xi \left\langle (\psi^1)^{-1} (\pi_1), ..., (\psi^M)^{-1} (\pi_M) \right\rangle$. ■

In the next section we apply the concept of translation invariance to ambiguity in games. The isomorphism which we have discovered will enable to prove existence of equilibrium with fixed point theorems.

4 Games with Multiple Priors

In this section we shall introduce our theory of games where players perceive *strategic ambiguity* about others behaviour. We model this ambiguity with a *multiple prior* approach. For comparability with the existing literature we shall assume that any given player views his/her opponents' actions to be independent. We shall also assume that beliefs are consistent in the sense that any two players have the same beliefs over the behaviour of a third party. These restrictions may be relaxed in applications if the context suggests it is desirable. We formally define our equilibrium concept and prove existence.

4.1 Framework

We consider normal-form games $\Gamma = \langle \mathcal{H}, S_i, u_i : 1 \leq i \leq N \rangle$, where $\mathcal{H}$ is a set of N players, S_i and u_i denote respectively the strategy set and utility function of player i. The pure strategy sets S_i for each of the N players are finite, Σ_i denotes the set of mixed strategies of player i (P_i). The sets of pure strategy and mixed strategy combinations of the game are denoted by $S = \times_{i=1}^N S_i$ and $\Sigma = \times_{i=1}^N \Sigma_i$. As usual, $S_{-i} = \times_{j \neq i} S_i$, and $\Sigma_{-i} = \times_{j \neq i} \Sigma_i$ denote the sets of pure and mixed strategies of i's opponents. For P_i, the function $u_i : \Sigma \to \mathbb{R}$ is the expected pay-off function.

The ambiguous belief of P_i about the strategy choice of the other players is reflected by a prior set $\mathcal{C}_{-i} \subseteq \Delta(S_{-i})$. The ambiguity-attitude of player P_i is represented by the parameter $\alpha_i \in [0, 1]$. Player P_i evaluates a strategy $s_i \in S_i$ by

$$V_i(s_i) = \alpha_i \min_{p \in \mathcal{C}_{-i}} u_i(s_i, p) + (1 - \alpha_i) \max_{p \in \mathcal{C}_{-i}} u_i(s_i, p). \tag{2}$$

Thus the player considers the worst and the best scenario over the beliefs and weights them according to his/her ambiguity attitude. The ambiguity-attitude parameter tells us how pessimistic or optimistic (s)he is towards the strategic ambiguity. If α_i is small (large), the player is optimistic (pessimistic).

4.2 Location invariance in games

When there are more than two players, strategic ambiguity is multi-dimensional. Thus we use the results from Section 3.4. The players perceive ambiguity about the behaviour of each opponent, each reflected by a prior set. The overall belief is reflected by the Cartesian product of these prior sets.

Definition 4.1 *Let $\mathcal{C}_i^j \subseteq \Sigma_j$ be the belief of player i about player j. The overall belief of player i is $\mathcal{C}_i = \times_{j \neq i} \mathcal{C}_i^j \subseteq \Sigma_{-i}$.*[7]

This definition incorporates the notion that any given player believes his/her opponents act independently. A translation of such a prior set $\mathcal{C}_i$ is the Cartesian product of translations of $\mathcal{C}_i^j$. The following definition says that any two players have the same beliefs about the behaviour of any third player. This is standard in game theory, however it needs to be imposed as an extra assumption in the present context.

Assumption 4.1 *We say that players have* consistent beliefs *if $\mathcal{C}_i^k = \mathcal{C}_j^k$ $i \neq j$, $i \neq k$, $j \neq k$.*

Next we extend the concept of an ambiguity class to games.

Definition 4.2 *Let $\mathcal{C}_i = \times_{j \neq i} \mathcal{C}_i^j$ be the belief of player i. Then $\hat{\mathcal{C}}_i = \times_{j \neq i} \hat{\mathcal{C}}_i^j$ is a translation of $\mathcal{C}_i$ if $\hat{\mathcal{C}}_i^j$ is a translation of $\mathcal{C}_i^j$ for all $j \neq i$. The ambiguity class of $\mathcal{C}_i$ denoted by $[\mathcal{C}_i]$, is the set of all such translations.*

Remark 4.1 *By Proposition 3.5 there exists an isomorphism ξ_i from $\Delta(S_{-i})$ to $[\mathcal{C}_i]$. This implies that any given ambiguity class is isomorphic to the set of mixed strategies in Γ.*

[7] This is not a convex set. However replacing the set by its convex hull will not affect preferences.

4.3 Definition of Equilibrium

In this section we present our definition of equilibrium for games with ambiguity, represented by multiple priors. Our equilibrium concept demands that players believe that others only play strategies that are optimal given their beliefs (Definition 4.3). Our equilibrium concept coincides with Nash Equilibrium when prior sets are singletons. Thus our model is the logical extension of Nash equilibrium to a multiple prior context.

Definition 4.3 (Equilibrium under Ambiguity) *Let* $\Gamma = \langle \mathcal{H}; S_i, u_i : 1 \leq i \leq N \rangle$ *be a normal-form game. The tuple* $(\mathcal{C}_{-i}, a_i)_{i=1}^{N}$ *is an Equilibrium Under Ambiguity (EUA) if for all* $1 \leq i \leq N$ *there exist* $\mathcal{C}^i \subseteq \Sigma(S_i)$ *such that* $\mathcal{C}_{-i} = \times_{j \neq i} \mathcal{C}_i^j$ *and*

$$\emptyset \neq \mathrm{supp}(\mathcal{C}_{-i}) \subseteq \times_{j \neq i} \mathrm{argmax}_{s_j \in S_j} V_j(s_j), \tag{3}$$

where $\mathrm{supp}(\mathcal{C}_{-i}) = \times_{j \neq i} \mathrm{supp}(\mathcal{C}_i^j)$*.*

Remark 4.2 *When prior sets are singletons, i.e.* $\mathcal{C}_{-i} = \sigma_i$ *for* $1 \leq i \leq N$*, then if* $\langle \sigma_1, ..., \sigma_N \rangle$ *is a EUA, it is also a Nash equilibrium of* Γ*.*

In light of Remark 4.2, EUA can be viewed as the natural extension of Nash Equilibrium (possibly in mixed strategies) to strategic ambiguity.

4.4 Equilibrium Existence

This section provides an equilibrium existence result. We achieve this for arbitrary perceived ambiguity and ambiguity-attitude for every player. This enables us to analyse the impact of ambiguity by conducting comparative static exercises. An *ambiguous normal-form game* is characterized by the players, their strategies and payouts as usual and in addition by an ambiguity class and ambiguity-attitude for each player. We rely in the concept of location invariance introduced in Section 3 and on the ambiguity classes $[\cdot]$ from Definition 3.4.

Definition 4.4 (Ambiguous Normal-form Game) *Consider a normal form game* $\Gamma = \langle \mathcal{H}, S_i, u_i : 1 \leq i \leq N \rangle$ *as well as ambiguity classes* $[\mathcal{C}^1], \ldots, [\mathcal{C}^N]$ *for the strategy of player*

i, $1 \leq i \leq N\}$ and ambiguity attitudes $a_1, \ldots, a_N$ for all players. Then we call this an ambiguous game and denote it by

$$\Gamma = \langle \mathcal{H}, S_i, u_i, [\mathcal{C}^i], a_i : 1 \leq i \leq N \rangle. \tag{4}$$

The following theorem shows that an EUA exists for every ambiguous game. The theorem thus guarantees existence for any ambiguity and ambiguity-attitude of the players.

Theorem 4.1 (Equilibrium Existence) *Let $\Gamma = \langle \mathcal{H}, S_i, u_i, [\mathcal{C}^i], a_i : 1 \leq i \leq N \rangle$ be an ambiguous normal-form game. Then there exist $\mathcal{C}^*_{-i} \in [\mathcal{C}_{-i}] := \times_{j \neq i} [\mathcal{C}^j]$ such that $(\mathcal{C}^*_{-i}, a_i)_{i=1}^N$ is an Equilibrium under Ambiguity.*

The existence of $\mathcal{C}^*_{-i} \in [\mathcal{C}_{-i}]$ means that no matter what shape and size we have fixed for each player i's ambiguity, reflected by $[\mathcal{C}^i]$, there always exist translations such that we have an equilibrium as in Definition 4.3. Such an equilibrium exists for any normal-form game and any possible ambiguity and ambiguity-attitude. This is a very general result and does not need any limiting assumptions such as convexity of preferences or restrictions on the pay-off functions.

5 Extensive form Games

I am not yet sure how the analysis of extensive form games can benefit from our new measure. I think however that translation invariance does play a role when updating for many important updating rules:

Proposition 5.1 *Let $\mathcal{C}$ and $\mathcal{C}'$ be prior sets and assume that $\mathcal{C}' = \phi_t(C)$. If $t(E^c) = 0$, then $\mathcal{C}'^{FB}_E = \phi_{t_E}(\mathcal{C}^{FB}_E)$ and $\mathcal{C}'^{ML}_E = \phi_{t_E}(\mathcal{C}^{ML}_E)$ for some translation $t_E : E \to \mathbb{R}$.*

If $t(E^c) > 0$, then $\mathcal{C}'^{FB}_E \supset \phi_{t_E}(\mathcal{C}^{FB}_E)$ and $\mathcal{C}'^{ML}_E \supset \phi_{t_E}(\mathcal{C}^{ML}_E)$ for some translation $t_E : E \to \mathbb{R}$. If $t(E^c) < 0$, then $\mathcal{C}'^{FB}_E \subset \phi_{t_E}(\mathcal{C}^{FB}_E)$ and $\mathcal{C}'^{ML}_E \subset \phi_{t_E}(\mathcal{C}^{ML}_E)$ for some translation $t_E : E \to \mathbb{R}$.

Proof. I'll add a proof once we decide that it is correct and that it's important.

Looks interesting

■

Some explanations are in order. If $t(E^c) > 0$, then $\mathcal{C}'$ is "closer" to E^c than $\mathcal{C}$. If now E happens then it is logical that $\mathcal{C}'_E$ reflects more ambiguity than $\mathcal{C}_E$ because event E happened which was perceived more unlikely for $\mathcal{C}'$ than for $\mathcal{C}$. So the update of $\mathcal{C}'$ reflects more ambiguity than the update of $\mathcal{C}$. If $t(E^c) = 0$, then both prior sets are equally "close" to E^c, thus the information E leads to the same ambiguity when the prior sets are updated.

6 Conclusion

This paper has two main contributions. Firstly we proposed a new measure of ambiguity based on the concept that ambiguity is translation invariant. Secondly we propose a solution for games where players have ambiguous beliefs as represented by the multiple priors model.

In this paper we assume that any two players have the same beliefs about the behaviour of a third party and that players believe that their opponents act independently. These assumptions are standard in the previous literature. It is useful to retain them, which enables us to isolate the effect of ambiguity in games. However we note these assumptions act as constraints. Therefore our proof of existence of equilibrium with these constraints implies existence when they are not required. In a context of ambiguity a given player may perceive ambiguity concerning whether or not his/her opponents are colluding. If the independence assumption is relaxed our framework would be suitable for analysing such questions

Our results can be extended to the case where the ambiguity attitude depends on the strategy chosen, as in the invariant biseparable model, see Hartmann.

Lorenz can you add a citation to ypur thesis.

A number of extensions are possible in future research. It would be relatively straightforward to include games of incomplete information by adding a type space. Our theory could be expanded to include extensive form games. Dynamic consistency can be ensured by requiring players to have preferences of the recursive multiple priors form, Sarin and Wakker (1998).

7 Extensions

7.1 The Smooth Model

In the smooth model of Klibanoff et al. (2005), μ is a second-order probability over the state space and reflects perceived ambiguity. Let μ' be another such second-order probability. When do μ and μ' reflect the same ambiguity? We show in the following that translations, suitable adapted to their framework, gives an intuitive solution to this question.

As Klibanoff et al. (2005) we denote by μ_f the distribution over expected utility values induced by μ and $f \in \mathcal{F}$. It is natural to say that μ and μ' reflect the same ambiguity for f if (and only if?) the cumulative distribution functions of μ_f and μ'_f are shifts of each other, i.e. there exists an $a \in \mathbb{R}$ such that $F'(x) = F(x + a)$ for all $x \in \mathbb{R}$ and where F, F' are the cumulative distribution functions of μ_f and μ'_f, respectively. If this holds then all probability mass is simply shifted uniformly by the constant a. We argue that such a shift does not change ambiguity.

The following result states that μ and μ' are translations, i.e. $\mu' = \phi_t(\mu)$, if and only if the above holds for all acts.

Proposition 7.1 *Let μ and μ' be two second-order probabilities over some finite state space S. Then μ and μ' are translations if and only if for all $f \in \mathcal{F}$ there exist $a_f \in \mathbb{R}$ such that $F'(x) = F(x + a_f)$ for all $x \in \mathbb{R}$.*

Proof. I'll add a proof once we decide that it is correct and that it's important. ■

Looks interesting but not sure that it fits with the theme of the present paper.

7.2 Applications

Possible applications include:

- Herding in Financial Markets,
  - Conjecture. If individuals view the behaviour of others as ambiguous they will be less sensitive to changes in their behaviour. As a result ambiguity will make

herding and bubbles less likely. Maybe increases in ambiguity can cause the collapse of bubbles.

- Entry Games see Tirole Ch.8
- Repeated Games.

Appendix

The proof of Theorem 4.1 can be divided into three steps. Firstly we show that for a state space S and a set of priors $\mathcal{C}$ there is an isomorphism between $\Delta(S)$ and $[\mathcal{C}]$. We use this to define an isomorphism between the space of mixed strategies and an ambiguity class of beliefs about the opponent's behaviour in a non-cooperative game. In the second step we show that this isomorphism induces what we call an "ambiguity perturbed game" of Γ. In the third step we show that every Nash equilibrium in the perturbed game induces an EUA in Γ. Since the perturbed game is a conventional normal-form game and thus has a Nash equilibrium, we are finished.

A Ambiguity Classes

This appendix shows that any given ambiguity class is isomorphic to the simplex. We introduce the isomorphism for a finite state space and then extend this to a state space with a product structure. This can then be directly applied to games.

Lemma A.1 *Consider a finite state space S and prior sets $\mathcal{C}, \mathcal{C}' \subseteq \Delta(S)$ with $\mathcal{C}' \in [\mathcal{C}]$. Then* $\operatorname{supp}(\mathcal{C}) \neq \emptyset$ *if and only if* $\operatorname{supp}(\mathcal{C}') \neq \emptyset$*. Furthermore, if both supports are empty, then the sets are identical, i.e. $\mathcal{C} = \mathcal{C}'$.*

Proof. Since $\mathcal{C}' \in [\mathcal{C}]$, there exists some translation ϕ_t such that $\mathcal{C}' = \phi_t(\mathcal{C})$. First assume that $\operatorname{supp}(\mathcal{C}) \neq \emptyset$, i.e. there exists some $\hat{s} \in S$ such that $\min_{p\in\mathcal{C}} p(\hat{s}) > 0$. If $t(\hat{s}) > -\min_{p\in\mathcal{C}} p(\hat{s})$, then $\hat{s} \in \operatorname{supp}(\mathcal{C}')$. Otherwise $t(\hat{s}) = -\min_{p\in\mathcal{C}} p(\hat{s}) < 0$. Hence there exists some $s' \in S$ such that $t(s') > 0$. It holds that $\min_{q\in\mathcal{C}'} q(s') = \min_{p\in\mathcal{C}} p(s') + t(s') \geq t(s') > 0$. Thus $s' \in \operatorname{supp}(\mathcal{C}') \neq \emptyset$.

Now assume that $\text{supp}(\mathcal{C}) = \emptyset$, which implies $\min_{p\in\mathcal{C}} p(s) = 0$ for all $s \in S$. Assume for contradiction that $t(s) < 0$ for some $s \in S$. Then $\min_{q\in\mathcal{C}'} q(s) = \min_{p\in\mathcal{C}} p(s) + t(s) = t(s) < 0$. But this cannot happen since $q(s) \geq 0$. Thus $t \equiv 0$ must hold, so $\mathcal{C}' = \mathcal{C}$. ■

Lemma 3.1 *If S is a finite state space and $\mathcal{C} \subseteq \Delta(S)$ is a prior set with* $\text{supp}(\mathcal{C}) \neq \emptyset$*, then for every state $s^* \in S$ there exists a unique prior set $\mathcal{C}^{s^*} \in [\mathcal{C}]$ such that* $\text{supp}(\mathcal{C}^{s^*}) = \{s^*\}$*.*

Proof. Consider a prior set $\mathcal{C}^{s^*} = \phi_t(\mathcal{C})$ where t is defined by

$$t(s) = -\min_{p\in\mathcal{C}} p(s), s \neq s^*; \qquad t(s^*) = \sum_{s\in S\backslash\{s^*\}} \min_{p\in\mathcal{C}} p(s).$$

By construction, $\min_{p\in\mathcal{C}^{s^*}} p(s) = \min_{p\in\mathcal{C}} p(s) + t(s) = 0$ for all $s \in S\backslash\{s^*\}$, implying that $\text{supp}(\mathcal{C}^{s^*}) \subseteq \{s^*\}$. By Lemma A.1, $\text{supp}(\mathcal{C}^{s^*}) \neq \emptyset$ so we can conclude that $\text{supp}(\mathcal{C}^{s^*}) = \{s^*\}$.

Now let $\mathcal{C}' = \phi_{\hat{t}}$ be a translation of $\mathcal{C}$ such that $\text{supp}(\mathcal{C}') = \{s^*\}$. There does not exist $\hat{s} \neq s^*$ such that $\hat{t}(\hat{s}) > -\min_{p\in\mathcal{C}} p(\hat{s})$. This would imply $\min_{p\in\mathcal{C}'} p(\hat{s}) > 0$, and hence $\hat{s} \in \text{supp}(\mathcal{C}')$. For $\hat{s} \neq s^*$, we cannot have $\hat{t}(\hat{s}) < -\min_{p\in\mathcal{C}} p(\hat{s})$, since this would imply the existence of $\tilde{p} \in \mathcal{C}'$ such that $\tilde{p}(\hat{s}) < 0$. Hence $\hat{t}(s) = -\min_{p\in\mathcal{C}} p(s), s \neq s^*$. Since $\sum_{s\in S} t(s) = \sum_{s\in S} \hat{t}(s) = 0$ $\hat{t}(s^*) = t(s^*)$, which implies $\hat{t} = t$. This establishes uniqueness of $\mathcal{C}^{s^*}$. ■

In what follows, for $s \in S$, we denote the prior set derived in Lemma 3.1 by $\mathcal{C}^s$.

Lemma A.2 *The number $\tau(s)$ defined by $\tau(s) = \min_{p\in\mathcal{C}^s} p(s) \in (0, 1]$ does not depend on s.*

Proof. Let $\hat{s}, \tilde{s} \in S, \hat{s} \neq \tilde{s}$. Note that $\min_{p\in\mathcal{C}^{\hat{s}}} p(\hat{s}) = \tau(\hat{s})$ and $\min_{p\in\mathcal{C}} p(s) = 0, s \neq \hat{s}$. By construction $\mathcal{C}^{\tilde{s}} = \mathcal{C}^{\hat{s}} + t'$, where $t'(\hat{s}) = -\tau(\hat{s}), t'(\tilde{s}) = \tau(\hat{s})$ and $t(s) = 0$ otherwise. It is now clear that $\tau(\tilde{s}) = \min_{p\in\mathcal{C}^{\tilde{s}}} p(s) = \tau(\hat{s})$. The result follows. ■

Remark A.1 *This result enables us to define $\tau = \tau(s)$. The number τ can be interpreted as a measure for the space that the prior sets in $[\mathcal{C}]$ have to move around in $\Delta(S)$. It is 0 if and only if* $\text{supp}(\mathcal{C}) = \emptyset$ *and 1 if and only if* $|\mathcal{C}| = 1$*.*

Next note that for a given $s^* \in S$,

$$[\mathcal{C}] = \{\mathcal{C} \subseteq \Delta(S) | \mathcal{C} = \mathcal{C}^{s^*} + t, t(s) \geq 0 \ \forall \ s \neq s^*, \sum_{s\in S\backslash\{s^*\}} t(s) \leq \tau, t \in T\}.$$

Lemma 3.1 defines a 1-1 map from S to a subset of $[\mathcal{C}]$. We shall now extend this to an isomorphism between $\Delta(S)$ and $[\mathcal{C}]$.

Proposition 3.2 *The function* $\psi : \Delta(S) \rightarrow [\mathcal{C}]$, *is an isomorphism.*

LH: Where do we define the function ψ?

Proof. First note that ψ is well defined, since $\forall p \in \mathcal{C}^{s^*}, p(s^*) \geq \tau, p + t(s) \geq 0$. Moreover ψ is 1-1, since if $\hat{\sigma} \neq \tilde{\sigma}$ then $t^{\hat{\sigma}}(s) \neq t^{\tilde{\sigma}}(s)$, hence $\mathcal{C}^{\hat{\sigma}} \neq \mathcal{C}^{\tilde{\sigma}}$.

To show that ψ is onto if $\mathcal{C}' \in [\mathcal{C}]$ then $C' = \mathcal{C}^{s^*} + t'$ for some t', such that $\sum_{s \in S} t'(s) = 1$. Notice that if $\tilde{s} \neq s^*, \exists \tilde{p} \in C$ such that $\tilde{p}(\tilde{s}) = 0$. Thus we must have $t'(\tilde{s}) \geq 0$ to ensure that $\tilde{p} + t'$ is a well defined probability distribution.

Define a probability distribution $\sigma' \in S$ by:

$$\sigma'(s) \quad = \quad \begin{cases} \frac{t'(s)}{\tau}, & s \neq s^*, \\ \frac{t'(s)}{\tau} + 1, & s = s^*. \end{cases}$$

We can show that σ' so defined in a probability distribution since:

$$\sum_{s \in S} \sigma'(s) = \sum_{s \in S} \frac{t'(s)}{\tau} + 1 = 1.$$

By construction $\psi(\sigma') = C'$, which establishes that ψ is onto. The result follows. ■

Corollary A.1 *If* $\sigma \in \Sigma$ *then* $\text{supp}(\mathcal{C}^\sigma) = \text{supp}(\sigma)$.

Proof. From the proof of Proposition 3.2 we see that $\mathcal{C}^\sigma = \mathcal{C}^{s^*} + t'$, where $t'(s) > 0 \iff \sigma(s) > 0$. Since $\forall p \in \mathcal{C}^\sigma), p(s) \geq t'(s)$, this implies $\text{supp}(\sigma) \subseteq \text{supp}(\mathcal{C}^\sigma)$.

For the converse assume that $\tilde{s} \notin \text{supp}(\mathcal{C}^\sigma)$, then $\exists p \in \mathcal{C}^\sigma$ such that $p(\tilde{s}) = 0$. This implies that $t'(\tilde{s}) = 0$ hence $\tilde{s} \notin \text{supp}(\sigma)$. Thus we may conclude $\text{supp}(\mathcal{C}^\sigma) \subseteq \text{supp}(\sigma)$, which establishes the result. ■

Remark A.2 *The isomorphism* ψ *implies that for every prior set* $\mathcal{C}$ *we have*

$$[\mathcal{C}] = \{\mathcal{C}^\sigma | \sigma \in \Delta(S)\}.$$

Figure 3 illustrates the work done for the case $S = \{s_1, s_2, s_3\}$. The isomorphism maps the strategy s_1 to the set $\mathcal{C}^{s_1}$. It maps the mixed strategy $\frac{1}{2}s_1 + \frac{1}{2}s_3$ to the set $\mathcal{C}^{\frac{1}{2}s_1+\frac{1}{2}s_3}$.

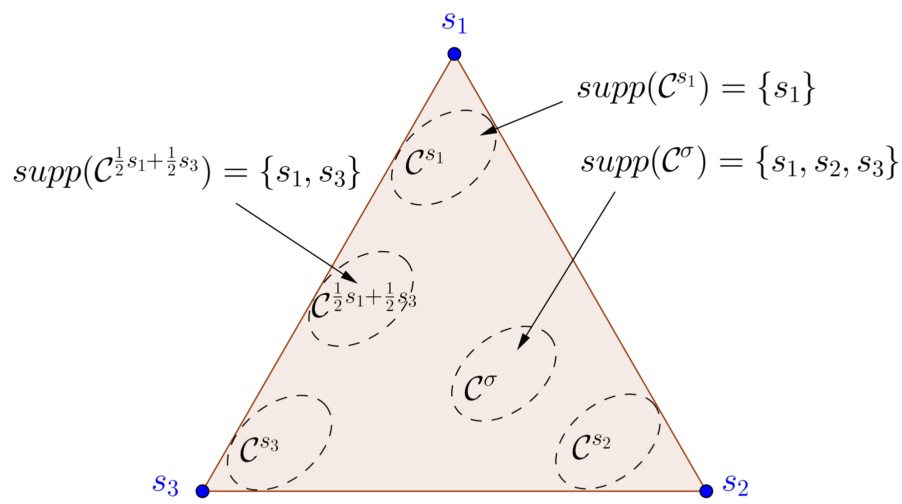

Figure 3: The sets $\mathcal{C}^{\sigma}$.

The next result shows that if beliefs are represented by $\mathcal{C}^q$, then preferences are linear in q. This implies that if s corresponds to the strategies of an opponent in a non-cooperative game, then mixed strategies will convexify pay-offs in the usual way.

For consistency with the above should we say The evaluation functional V is linear in $\sigma \in \Delta(S)$ i.e. replace q by σ,? Maybe we should replace σ by q instead? Since we are here still in the non-game context. I can change the picture accordingly if you agree.

Lemma A.3 *The evaluation functional V is linear in $q \in \Delta(S)$, i.e. for $a \in A(S)$.*

$$V(s|\mathcal{C}^q, a) = \sum_{s \in S} q(s) V(s|\mathcal{C}^s, a).$$

Proof. Consider $q \in \Delta(S)$. For $\tilde{s} \in S$ if we define the translation $t^{q-\tilde{s}s'}$ by

$$t^{q-\tilde{s}}(s) = \begin{cases} \tau q(s), & s \neq \tilde{s}; \\ \tau\left[q(\tilde{s}) - 1\right], & s = \tilde{s}, \end{cases}$$

then holds that $\mathcal{C}^q = \mathcal{C}^{\tilde{s}} + t^{q-\tilde{s}}$. Next we show that for all $a \in A(S)$ and $q \in \Delta(S)$, $\sum_{s \in S} q(s) t^{q-s} \centerdot a = 0$. Here $t^{q-s} \centerdot a$ denotes the scalar product of vectors t^{q-s} and a.

$$\begin{aligned}
\sum_{s\in S} q(s) t^{q-s} \centerdot a &= \sum_{s\in S} q(s) \left[\sum_{s'\in S} t^{q-s} \centerdot a \right] \\
&= \sum_{s\in S} q(s)\tau \left[\sum_{s'\neq s} q(s')a(s') + (q(s)-1)a(s) \right] \\
&= \sum_{s\in S} q(s)\tau \left[\sum_{s'\in S} q(s')a(s') - a(s) \right] \\
&= \tau \left[\sum_{s'\in S} \underbrace{\sum_{s\in S} q(s)}_{=1} q(s')a(s') \right] - \tau \sum_{s\in S} q(s)a(s) \\
&= \tau \sum_{s'\in S} q(s')a(s') - \tau \sum_{s\in S} q(s)a(s) = 0.
\end{aligned}$$

Now we have

$$\begin{aligned}
\min_{p\in\mathcal{C}^q} \mathbf{E}_p a &= \sum_{s\in S} q(s) \min_{p\in\mathcal{C}^q} \mathbf{E}_p a \\
&= \sum_{s\in S} q(s) \min_{p\in\mathcal{C}^s} \mathbf{E}_p a + \sum_{s\in S} q(s) t^{q-s} \centerdot a \\
&= \sum_{s\in S} q(s) \min_{p\in\mathcal{C}^s} \mathbf{E}_p a).
\end{aligned}$$

Analogously we can show that $\max_{p\in\mathcal{C}^q} \mathbf{E}_p a = \sum_{s\in S} q(s) \max_{p\in\mathcal{C}^s} \mathbf{E}_p a$. This implies

$$\begin{aligned}
V(a|\mathcal{C}^q, \alpha) &= \alpha(a) \min_{p\in\mathcal{C}^q} \mathbf{E}_p a + (1-\alpha(a)) \max_{p\in\mathcal{C}^q} E_p a \\
&= \alpha(a) \sum_{s\in S} q(s) \min_{p\in\mathcal{C}^s} \mathbf{E}_p a + (1-\alpha(a)) \sum_{s\in S} q(s) \max_{p\in\mathcal{C}^s} E_p a \\
&= \sum_{s\in S} q(s) \left[\alpha(a) \min_{p\in\mathcal{C}^s} \mathbf{E}_p a + (1-\alpha(a)) \max_{p\in\mathcal{C}^s} \mathbf{E}_p a \right] \\
&= \sum_{s\in S} q(s) V(a|\mathcal{C}^s, \alpha).
\end{aligned}$$

This shows the required linearity. ■

B Existence of Equilibrium in Games

In this appendix we prove existence of equilibrium with ambiguity. Our strategy is to construct a related game without ambiguity which we refer to as the perturbed game. We use the isomorphism constructed above to map a mixed strategy equilibrium of the perturbed game to an equilibrium with ambiguity in the original game. Existence of equilibrium then follows by applying Nash's Theorem to the perturbed game.

The perturbed game Γ^*

The isomorphism is crucial for the following definition, the key step towards proving Theorem 4.1. For any ambiguous normal-form game $\Gamma = \langle \mathcal{H}, S_i, u_i, [\mathcal{C}_i], \alpha_i : 1 \leq i \leq N \rangle$ we can define what we call the perturbed game of Γ. This is a standard normal form game with the same players and strategy sets but modified pay-off functions.

Definition B.1 (Perturbed Game) *Let $\Gamma = \langle \mathcal{H}, S_i, u_i, [\mathcal{C}_i], \alpha_i : 1 \leq i \leq N \rangle$ be an ambiguous normal-form game with independent beliefs. The independent perturbed game is a normal form game $\Gamma^* = \langle \mathcal{H}; S_i, w_i : 1 \leq i \leq N \rangle$ defined by*

$$\omega_i(s_i, s_{-i}) = \alpha_i \min_{p \in \mathcal{C}^{s_{-i}}} \mathbf{E}_p u_i(s_i, s_{-i}) + (1 - \alpha_i) \max_{p \in \mathcal{C}^{s_{-i}}} \mathbf{E}_p u_i(s_i, s_{-i}).$$

The game Γ^* is well-defined. It is a normal-form game with exactly the same players and strategy sets as Γ. The payoff functions will differ unless the players do not perceive any ambiguity, i.e. the $[\mathcal{C}_i^j]$'s are singletons. Let $\mathcal{C}_i^{s_{-i}}$ be the member of $[\mathcal{C}_i]$ such that $\text{supp}(\mathcal{C}_i^{s_{-i}}) = s_{-i}$. Existence and uniqueness of $\mathcal{C}_i^{s_{-i}}$ is ensured by Lemma 3.1. If $\sigma_{-i} \in \Sigma_{-i}$, define $\mathcal{C}_i^{\sigma_{-i}} = \xi_i^{-1}(\sigma_{-i})$, where ξ_i is the isomorphism defined in Proposition 3.5. The following Lemma is the crucial step for proving Theorem 4.1. It shows that the Nash Equilibria of Γ^* induces an Equilibrium under Ambiguity of Γ.

Lemma B.1 *Let $\Gamma = \langle \mathcal{H}, S_i, u_i, [\mathcal{C}_i], \alpha_i : 1 \leq i \leq N \rangle$ be an ambiguous normal-form game with consistent beliefs. If $\sigma \in \Sigma$ is a Nash Equilibrium of the independent perturbed game Γ^* then $(\mathcal{C}^{\sigma_{-i}}, \alpha_i)_{i=1}^N = (\xi_i(\sigma_{-i}), \alpha_i)_{i=1}^N$ is an Equilibrium under Ambiguity of Γ.*

Proof. Assume that $\sigma = (\sigma_i)_{i=1}^N$ is a Nash Equilibrium of Γ^*. By Lemma A.3, V_i is linear in σ_i, hence

$$V_i(s_i|\mathcal{C}^{\sigma_{-i}}, \alpha_i) = \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) V_i(s_i|\mathcal{C}^{s_{-i}}, \alpha_i).$$

This implies that $V_i(s_i|\mathcal{C}^{\sigma_{-i}}, \alpha_i) = \omega_i(s_i, \sigma_{-i})$.

Assume that $s_i' \in \text{supp}(\sigma_i)$. Then $s_i' \in \text{argmax}_{s_i \in S_i} \omega_i(s_i, \sigma_{-i})$. Therefore $s_i' \in \text{argmax}_{s_i \in S_i} V_i(s_i|\mathcal{C}^{\sigma_{-i}}, \alpha_i)$. So s_i' is a best response given belief $\mathcal{C}^{\sigma_{-i}}$ and a_i. Then

$$\emptyset \neq \text{supp}(\sigma_{-i}) \subseteq \times_{j \neq i} \text{argmax}_{s_j \in S_j} \omega_j(s_j, \sigma_{-j}).$$

By Corollary A.1, $\text{supp}(\sigma_{-i}) = \text{supp}(\mathcal{C}^{\sigma_{-i}})$ and by the definition of ω_j this is equivalent to

$$\emptyset \neq \text{supp}(\mathcal{C}^{\sigma_{-i}}) \subseteq \times_{j \neq i} \text{argmax}_{s_j \in S_j} V(s_j|\mathcal{C}^{\sigma_{-j}}, \alpha_j).$$

This implies that $(\mathcal{C}_i^{\sigma_{-i}}, \alpha_i)_{i=1}^N$ is an Equilibrium under Ambiguity. ■

Proof of Theorem 4.1. The proof follows directly from Lemma B.1. By Nash's theorem, the game Γ^* always has a Nash Equilibrium which induces an Equilibrium under Ambiguity in Γ. ■

A direct proof may be more elegant than defining a perturbed game.

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