Far-Sighted Clustering with Group-Size Effects and Reputations*

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Abstract

We formulate a new model of strategic group formation by far-sighted players, in a seller-buyer setting. In each period, sellers are partitioned into groups/brands. At the end of each period, one seller may fail and exit from the market by exogenous shock. When there is a vacant slot in the market, an entrant seller comes and chooses which existing group to join or to create a new group. There is a trade-off: larger groups enjoy more-than-proportional benefits of group size thanks to, for example, their visibility to attract customers and their negotiation power in factor markets. However, larger groups are more likely to experience member failure, which is a reputation loss. We find that when the rate of reputation loss is small, clustering is inevitable, but as the rate of reputation loss increases, the largest group with a bad reputation does not attract an entrant, dissolving a cluster. With a limited group-size benefit and a high rate of reputation loss, all entrants create a new group, that is, no clustering occurs. A mathematically interesting result is that, even though the model itself is stationary and symmetric, depending on the parameters, there may be multiple pure-strategy, symmetric stationary equilibria, or there may be no such an equilibrium. Economic implications include that group reputation may prevent clustering and that similar markets can have different cluster structures.

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Key words: clustering, reputation, group size, far-sighted, dynamic game.

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1 Introduction

Many markets are occupied by brands or chain stores these days. For example, the fast-food market around the world is dominated by a few gigantic brands (McDonald's, KFC, etc.). The grocery markets in many countries are dominated by a few supermarket chains. In other markets, however, large and small brands may co-exist. In movie theater markets, in addition to large chains, there are small, independent theaters that show hand-picked films from small studios. Therefore, not all markets have clusters consisting of a few large groups of sellers.

There are at least two competing forces on clustering by far-sighted sellers. A force that induces sellers to cluster is the more-than-proportional benefit of the size of the brand, due to market visibility (e.g., Aaker, 1995), negotiation power in factor markets, and so forth. In a marketing situation, Lanchester's laws (Lanchester, 1916) are often quoted: the effect of an "attack" (e.g., marketing) is more than proportional to the size of the attack. That is, the number of consumers attracted to a group of size $2x$ is more than double the consumers attracted to a group of size x. The bargaining power in factor markets can be also more-than-proportional, because, for suppliers, the threat of losing a large customer is much more serious than the threat of losing a small customer.

An opposing force for sellers to make a cluster is the risk of reputation loss. Reputation exists not only for individual agents but also for groups of agents (e.g., Tirole, 1996). In a large brand, it is more likely that some reputation loss occurs to one of the members, which affects all other members in the same brand. Possible causes of reputation loss may include mistakes (e.g., advertisements are unexpectedly perceived as discriminatory) and bad luck (e.g., ingredients unexpectedly deteriorated). There is empirical evidence that a reputation loss of a seller negatively affects purchases from "similar name" sellers (Fujiwara-Greve et al., 2016). Hence we can take the "groups" as a wide concept.

To address these forces and determine when and what kind of clustering is stable under the trade-off, we construct a new model of far-sighted group formation with disproportionate size effects and group reputations. This is a dynamic model over a discrete-time horizon with the following features: (i) In each period, with a small probability, one player "fails" and leaves the game. (ii) The group that experienced the failure of one of its members carries a bad reputation for one period, which affects their payoffs. (iii) When the exit occurs, an entrant approaches the game to choose which group to join or to create a new group, after observing the interim state of the player groups. (iv) After the current state of the seller groups and the reputation for each group is determined, sales take place according to the state, which gives one-shot payoffs to each player.

Because consumers' attention span is often short-lived, we take a "period" as the time span of reputation loss.

Each entrant operates over some finite periods, but the end is uncertain. Therefore we assume that each entrant optimizes over the uncertain horizon, instead of choosing a (myopic) best-response (Bala and Goyal, 2000) or optimizes over a fixed finite horizon (e.g., Jackson and Watts, 2010).[1] Because only one player makes a decision at a time, with the information of only the immediate past, it is natural to use the notion of Markov Perfect Equilibrium. Therefore we do not use the core-based stability notions (cf. Herings et al., 2009, Konishi and Ray, 2003, Mauleon et al., 2014, and Page Jr. et al., 2005).

In terms of the model similarly, the closest work is by Dutta et al. (2005). They consider a dynamic network formation game such that, for each period, a pair of players are randomly selected to choose unilateral link separation (of any link they have) and bilateral link formation (among them). Their notion of a strategy profile, as ours, describes how a new graph (a new partition, in our case) is chosen based on the long-run expected payoffs, except that they allow a probabilistic choice of links. As we do, they also analyze a Markov process of (undirected) graphs induced by a strategy profile. The underlying idea of the (non-cooperative) stability is also the same across our paper and their paper: no active player deviates from the equilibrium strategy, if all later players stick to the equilibrium strategy.

In terms of the economic problems considered, there is no theoretical work that addresses a dynamic process of group formation with the trade-off of group size advantage and group reputation concern. This is the first paper to explicitly analyze such a problem. The underlying consumer behavior is similar to the one assumed in Cabral (2009): consumers react the same way to "similar sellers".

Our main findings (for the three seller case)[2] are as follows. When the "rate of reputation loss" (the fraction of consumers who observe a failure in a group times the probability that a failure occurs) is small, the "size-conscious" strategy, under which an entrant always joins the largest size group regardless of its reputation is an equilibrium strategy, because of the more-than-proportional group size benefit. Therefore clustering is inevitable. As the rate of reputation loss increases, the

[1]To focus on the novel trade-off, we do not include the option of existing group members to refuse or request an entrant to join, an entrant's cost of joining an existing group or creating a new group, and so on. Empirical evidence (Fujiwara-Greve et al., 2016) suggests that a group can be just similar names, and in such cases we do not need to consider agreement/cost to join a group. We also do not consider information transmission from surviving consumers to newcomers (cf. Bloch et al., 2018). Our reputation externality from a failed seller to other members in the same group is also different from alliance formation externalities. See the surveys by Bloch (2012) and Yi (2003).

[2]The four seller case is similar but tedious to write out. A general analysis is quite difficult, which will be explained in the text.

"join the largest without F" strategy, under which an entrant joins the largest group except when that group has a reputation loss, becomes an equilibrium. Hence reputation concern can dissolve a monopoly with a bad reputation. When the group-size effect is limited and the rate of reputation loss is high, all entrants create a new group, that is, no clustering happens.

Interestingly, although our model allows at most one player to change the group structure in each period, multiple symmetric, stationary equilibria exist for some parameters. Thus, ex-ante homogeneous players may coordinate on different group-formation strategies. Specifically, when the rate of reputation loss is not too small but not too large, the "size-conscious" strategy and the "join the largest without F" strategy are both equilibria. Moreover, under some parameters, there is no pure-strategy, symmetric, stationary equilibrium. Therefore, even though the model is stationary and symmetric for all players, we need to look for either a probabilistic group formation or a cyclic behavior.[3]

This paper is organized as follows. In Section 2, we introduce a new model for group formation by far-sighted players in a seller-buyer setting. In Section 3 we completely analyze stationary, symmetric strategies in the three seller model with group reputations. It also shows why a general analysis is very difficult. Section 4 concludes.

2 Model

We consider a stationary-size market, with $N \geqq 3$ homogeneous sellers[4] and a continuum (of measure 1) of homogeneous consumers over a discrete-time horizon $t = 1, 2, \ldots$. Although this paper explains only the $N = 3$ case analytically, we describe the model for a general N for future reference.

At the end of each period, $(1 - \delta)$ (where $0 < \delta < 1$) of the consumers exit from the market for some exogenous reason, which we call "death". In the next period, the same number of new consumers approach the market, keeping the size of consumers constant. Consumers only observe the seller-group reputations while they participate in the game. Alternatively, the δ parameter captures the share of "reputation-conscious" consumers per period.[5]

As for sellers, we assume that each seller is hit by stochastic failure with $\frac{\varepsilon}{N}$ where $\varepsilon \in (0, 1)$. With probability $1-\varepsilon$, no seller fails. If a seller experiences a failure, it must exit the market but, in

[3]In Online Appendix, we present a cyclic equilibrium for a parameter combination in which no pure-strategy, stationary equilibrium exists.

[4]If $N = 2$, then we cannot address the more-than-proportional, group-size effect.

[5]Although we do not address each consumer's optimization in this paper, if consumers are also active players, the δ is their effective discount factor (e.g., Fujiwara-Greve et al., 2016).

the next period, a new seller enters the market, keeping the number of sellers constant. Therefore, we can interpret N as the number of "seller slots" available in the market.

Denote by $(G_1, \ldots, G_n)$ a partition of seller slots $\{1, 2, \ldots, N\}$. In each period, the sellers are partitioned into groups with group reputations. To describe the groups' reputations, we write *a group distribution with reputation* as $s = (G_{1z_1}, G_{2z_2}, \ldots, G_{nz_n})$ where $z_i = F$ is the reputation attached to group G_i such that $z_i = F$ if and only if one of its members failed at the end of the previous period. For example, when $N = 3$, a group distribution with reputations is $s = (\{2\}_F, \{1, 3\})$. This means that the singleton group of seller 2 has the failure reputation, and the two-seller group $\{1, 3\}$ does not have the failure reputation. In the dynamic of group structures, group distributions with reputations are the "states".

The initial group distribution with reputations is exogenous (but arbitrary), and after that, entrants' group choice strategies and stochastic failures determine the dynamic process of group distribution and reputation changes. We allow each entering seller to either join one of the existing groups or to create a new group (a group of size 1). The entrant seller learns the (interim) group-size distribution and reputation without cost before choosing the join/create action. The current group reputation does not change with an entrant's choice, since it is determined by a previous failure. However, it disappears after one period, since it is known that Nature chooses a new stochastic failure every period. This is plausible, for example, in the restaurant industry with possible food poisoning and for any industry in which some employees may incur great damage to their company's reputation by misbehaving on social networks.[6] Such incidents are likely to be iid over time and across sellers and would not make a long-term damage to the sellers that are not the directly affected ones.

After the current period's group distribution with (previous-period failure) reputation is determined, newcomer consumers and surviving consumers choose a seller simultaneously. Surviving consumers remember the group reputations, while newcomers only observe the current group distribution. The consumers' choice can be interpreted as purchasing an experience good, investing money, hiring a professional to perform some task, and so on. The measure of consumers who choose a seller is this period's payoff to that seller. At the end of a period, $(1 - \delta)$ of consumers exit the game by "death" and are replaced by newcomers.

The consumer turnover is not only realistic, but also provides a "necessary dollar cushion"

[6] In Japan, there have been many incidents where workers posted a scandalous video taken at the workplace on the social networks, for example spitting in the kitchen pots or lying down in an ice cream freezer. Some shops were forced to go out of business because of such posts.

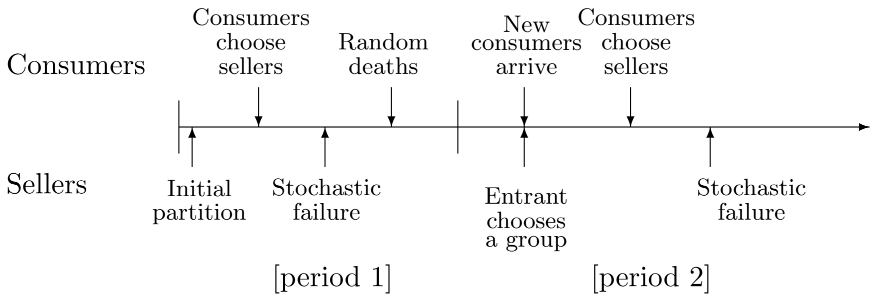

Figure 1: The time line of the dynamic game

(Hirschman, 1972) for sellers that are hit in a group that had a failure, because newcomers may patronize any seller. The timeline of actions of sellers and consumers and stochastic events (consumer "death" and seller failure) is depicted in Figure 1.

In this paper, we focus on the far-sighted sellers' strategic group choices. Thus, we assume that the consumers' behaviors are given as follows.

Assumption 1. *Let n be the number of groups in a period. For each $i = 1, 2, \ldots, n$, new consumers in that period choose group G_i with a probability of*

$$p^k_{new}(G_i) = \frac{|G_i|^k}{\sum_{j=1}^n |G_j|^k}$$

and a seller in G_i with equal probability.

Let $j \in \{1, 2, \ldots, n\}$ be the group with an F record (if it exists). For each $i \neq j$, the surviving consumers in that period choose group G_i with a probability of

$$p^k(G_i) = \frac{|G_i|^k}{\sum_{h \neq j} |G_h|^k}$$

and a seller in G_i with equal probability. (Hence sellers in group j are not chosen by any surviving consumer.)

The parameter $k > 1$ captures the more-than-proportional effect of the group size to attract consumers. An axiomatization of the probability structure of the consumers' choice is given by Skaperdas (1996). It is also an extension of Lanchester's Square Law (which is $k = 2$). Assumption 1 determines the one-shot payoff of each seller after the group distribution with reputations is determined.[7]

From a seller's point of view, the payoff-relevant information is the distribution of the group sizes and attached reputation only. Let us define a *social state from a seller's point of view* as

[7] Alternatively, we can adapt the model of Fujiwara-Greve et al. (2016) to the current environment so that consumers also choose sellers strategically. If all sellers' products (and prices) are the same on the equilibrium path, then the behavior in Assumption 1 is an optimal one for consumers.

$(g_{1z_1}, \ldots, g_{nz_n})$ such that the first entry g_{1z_1} consists of the size of the group $g_1 \in \{1, 2, \ldots, N\}$ that this seller belongs and $z_1 = F$ if and only if this group had a Failure, and, for other $i = 2, 3, \ldots, n$, the size of other groups (if exists)[8] are denoted $g_i \in \{1, 2, \ldots, N-1\}$ (so that $\sum_{i=1}^{n} g_i = N$) and $z_i = F$ if and only if the group i had a Failure.

For example, if the group distribution with reputations is $s = (\{2\}_F, \{1, 3\})$, then the social state from seller 2's point of view is written as $(1_F, 2)$ and from the viewpoint of seller 1 or 3 is $(2, 1_F)$. The relevant seller's group size and its reputation comes first. This notation is convenient for the sellers' optimization analysis below. At this social state, Assumption 1 implies that surviving consumers would choose sellers 1 and 3 with probability $\frac{1}{2}$ and do not choose seller 2. By contrast, new consumers would choose seller 2 with probability $\frac{1^k}{1^k+2^k}$ and choose each of sellers 1 and 3 with probability $(\frac{2^k}{1^k+2^k}) \cdot (\frac{1}{2})$. Thus, the total measure of consumers choosing seller 1 or 3 is $\delta \cdot \frac{1}{2} + (1-\delta)(\frac{2^k}{1^k+2^k}) \cdot (\frac{1}{2})$, which is the value of $U(21_F)$ in the system of equations (1) below.

We assume that each entrant seller maximizes its long-run expected payoff by choosing a strategy that prescribes which group to join or to create a new group, for each possible history. If there are groups which are the same in size and reputation, we assume that entrants behave the same way towards them, for simplicity. (This allows for an entrant to not join any of them.)

Note that, in terms of the existing sellers' payoff combinations, any group distribution (partition) is efficient. This is because each seller's payoff is its share of the total measure 1 of the consumers, and thus the sum of all sellers' payoffs in each period is always 1.

3 Three sellers with group reputation

In this section, we give a complete analysis of a market of $N = 3$. For notational simplicity, let us write 11_F1 instead of $(1, 1_F, 1)$ and so on. Then we can write the set of possible *social states* for a seller as

$$S_3 := \{111, 1_F11, 11_F1, 12, 1_F2, 12_F, 21, 2_F1, 21_F, 3, 3_F\}.$$

(Strictly speaking, there is also an additional absorbing state "exit". It comes with payoff 0 forever after.) Recall that, for each state in S_3, the first entry is the focal seller's group size with the reputation from the previous period (if no failure, no subscript), and the second and the third entries (if they exist) describe other sellers' groups.

Under Assumption 1, the one-shot payoff of a seller in each state in S_3 is as follows. Recall that

[8] If the relevant seller belongs to the monopoly group, i.e., $g_1 = N$, then $i \neq 1$ does not exist.

$\delta \in (0,1)$ is the measure of surviving consumers who remember a failure in the previous period.

$$\begin{aligned}
U(111) &= \frac{1^k}{1^k+1^k+1^k} = \frac{1}{3},\\
U(1_F11) &= (1-\delta)\frac{1^k}{1^k+1^k+1^k},\\
U(11_F1) &= \delta \cdot \frac{1^k}{1^k+1^k} + (1-\delta)\frac{1^k}{1^k+1^k+1^k},\\
U(12) &= \frac{1^k}{1^k+2^k},\\
U(1_F2) &= (1-\delta)\frac{1^k}{1^k+2^k},\\
U(12_F) &= \delta + (1-\delta)\frac{1^k}{1^k+2^k},\\
U(21) &= \frac{2^k}{1^k+2^k} \cdot (\frac{1}{2}),\\
U(2_F1) &= (1-\delta)\frac{2^k}{1^k+2^k} \cdot (\frac{1}{2}),\\
U(21_F) &= \delta \cdot \frac{1}{2} + (1-\delta)\frac{2^k}{1^k+2^k} \cdot (\frac{1}{2}),\\
U(3) &= \frac{1}{3},\\
U(3_F) &= \frac{1}{3}. \qquad (1)
\end{aligned}$$

For later reference, let

$$U := \begin{pmatrix} U(111) \\ U(1_F11) \\ \vdots \\ U(3_F) \end{pmatrix}.$$

If no failure happens in the current period, then the social state moves to the same size distribution state without the subscript F (e.g., 111, 1_F11 and 11_F1 all become 111). If a seller fails, (i) a new F record is given to the group to which the failed seller belonged, if the group itself did not disappear (an interim state), and (ii) after that, an entrant's strategy determines the next social state. The possible *interim states* from the viewpoint of an entrant are $11, 1_F1, 2$, and 2_F.[9]

For each $s \in S_3$ and each $t = 1, 2, \ldots$, let $V_t(s)$ be the optimal value function (the optimal total expected payoff) of a seller whose group size and reputation is the first entry of s. Then $V_t(s)$ and $V_{t+1}(s)$ have a stationary relationship under the stationary reputation loss process. For example, for any $t = 1, 2, \ldots$, starting at the state 111, the value function evolves as follows.

$$\begin{aligned}
V_t(111) = U(111) &+ \frac{2\varepsilon}{3}\left\{\frac{1}{2}V_{t+1}(21) + \frac{1}{2}V_{t+1}(12)\right\} \cdot \texttt{IF}\big(V_{t+1}(21) \geqq V_{t+1}(111)\big)\\
&+ \frac{2\varepsilon}{3}V_{t+1}(111) \cdot \texttt{IF}\big(V_{t+1}(21) < V_{t+1}(111)\big) + (1-\varepsilon)V_{t+1}(111), \qquad (2)
\end{aligned}$$

[9] From the viewpoint of an entrant, interim states 1_F1 and 11_F are the same.

where IF is a function such that for any statement A, $\texttt{IF}(A) = 1$ if A is true, and $\texttt{IF}(A) = 0$ if A is false.

To explain the equation (2), it looks at a seller in a singleton group with no Failure while there are two other singleton groups both without failure as well. (Recall that the first 1 is this seller's group size and reputation, which is $\emptyset$.) The seller receives the one-shot payoff $U(111)$ in period t. If no failure happens (with probability $1-\varepsilon$), the state continues to be 111 and the continuation value is thus $V_{t+1}(111)$. This is the last term of the RHS. If this seller fails and exits, the continuation value is 0, which is omitted from the RHS. Finally, one of the other two sellers may fail. The resulting interim state is 11. The entrant in $t+1$ chooses to join one of the existing groups (with equal probability by our assumption) if $V_{t+1}(21) \geqq V_{t+1}(111)$, and creates a new group otherwise. These are the second and the third terms in the RHS.

From states 1_F11 or 11_F1 (in which the relevant seller's group with reputation is the first entry), the optimal value function evolves very similarly.

$$\begin{aligned} V_t(1_F11) &= U(1_F11) + \frac{2\varepsilon}{3}\left\{\frac{1}{2}V_{t+1}(21) + \frac{1}{2}V_{t+1}(12)\right\} \cdot \texttt{IF}\big(V_{t+1}(21) \geqq V_{t+1}(111)\big) \\ &\quad + \frac{2\varepsilon}{3}V_{t+1}(111) \cdot \texttt{IF}\big(V_{t+1}(21) < V_{t+1}(111)\big) + (1-\varepsilon)V_{t+1}(111), \qquad (3) \\ V_t(11_F1) &= U(11_F1) + \frac{2\varepsilon}{3}\left\{\frac{1}{2}V_{t+1}(21) + \frac{1}{2}V_{t+1}(12)\right\} \cdot \texttt{IF}\big(V_{t+1}(21) \geqq V_{t+1}(111)\big) \\ &\quad + \frac{2\varepsilon}{3}V_{t+1}(111) \cdot \texttt{IF}\big(V_{t+1}(21) < V_{t+1}(111)\big) + (1-\varepsilon)V_{t+1}(111). \qquad (4) \end{aligned}$$

The only difference is the one-shot payoff, and the continuation value structure is the same across equations (2), (3) and (4). In Appendix A.1, we show the relationships of the optimal value functions over two periods for all other states.

We can simplify our analysis substantially by the next lemma.

Lemma 1. *For any $t = 1, 2, \ldots$, any $k > 1$, any $\delta \in (0,1)$ and any $\varepsilon \in (0,1)$,*

$$V_t(21_F) > V_t(2_F1);$$

$$V_t(21_F) \geqq V_t(11_F1) \Rightarrow V_t(21) \geqq V_t(111);$$

$$V_t(12) \geqq V_t(3) \Rightarrow V_t(12_F) > V_t(3_F).$$

Proof. See Appendix A.1. □

The intuition for the first inequality of Lemma 1 is as follows. Take an entrant who faces the interim state of 1_F1, that is, there are two singleton groups, one with a Failure record and one without it. The first inequality says that it is always better to join the group without F than to join the group with F (of the same size 1). (Note that, it does not say that the former is optimal, since there is a third option to create a new group.) This is because (i) the one-shot payoff is such that $U(21_F) > U(2_F 1)$, and (ii) after this period, the transition of the states are the same. The intuition of other two results are similar.

The first inequality implies that, although there are three possible states 21_F, $2_F 1$, and $11_F 1$ that may be reached from the interim state $1_F 1$, in any equilibrium, $2_F 1$ will not happen. Hence, regardless of the strategies of players, the possible social states that can be reached from each interim state by far-sighted entrants are as shown in Table 1. This fact simplifies our analysis.

interim state	reachable social states
11	21, 111
$1_F 1$	21_F, $11_F 1$
2	3, 12
2_F	3_F, 12_F

Table 1: Reachable social states by far-sighted entrants

In the following, we consider symmetric, stationary Markov perfect equilibria in which all entrants use the same Markov strategy which maps each possible interim state to a social sate, in any period. Since a group's reputation lasts only one period, and we assumed that an entrant seller only observes the current interim state, focusing on Markov strategies is not restrictive.[10]

The second and the third result in Lemma 1 imply that there are only nine possible strategies that may constitute a symmetric, stationary equilibrium, as shown in Table 2. (See also Table 4 in Appendix A.1, which classifies the value functions.) To explain, if moving from $1_F 1$ to 21_F is chosen (as in $f1$, $f2$, and $f3$), then $V_t(21_F) \geqq V_t(11_F 1)$ must be the case, so that $11 \to 21$ must be chosen by the second result in Lemma 1. Hence we do not need to consider a strategy which moves $1_F 1 \to 21_F$ and $11 \to 111$. In this way, we can restrict candidate strategies to those in Table 2.

To see the trade-off between the group-size effect and reputation concern, an interesting strategy is the "join-the-largest-without-F" strategy (denoted $f1$). This strategy prescribes that an entrant joins the largest group without F, except when the interim state is the monopoly group but with F, 2_F. In this case, an entrant is supposed to create a new group so that the next state becomes 12_F. Hence if all entrants use this $f1$-strategy, clustering happens, but the monopoly group always gets destroyed after a failure, due to the reputation concern.

By contrast, the "size-conscious" strategy $f2$ ignores the group reputation and always joins the

[10]However, there may be non-stationary Markov equilibria in which entrants take into account both the interim state and the period in the game. See Online Appendix.

Strategy name	Transition[11] from each interim state			
Join-the-largest-without-F strategy ($f1$)	$11 \to 21$	$1_F1 \to 21_F$	$2 \to 3$	$2_F \to 12_F$
Size-conscious strategy ($f2$)	$11 \to 21$	$1_F1 \to 21_F$	$2 \to 3$	$2_F \to 3_F$
Join-the-smallest strategy ($f3$)	$11 \to 21$	$1_F1 \to 21_F$	$2 \to 12$	$2_F \to 12_F$
Weakly entrepreneurial strategy ($f4$)	$11 \to 21$	$1_F1 \to 11_F1$	$2 \to 3$	$2_F \to 12_F$
Intermediately size-conscious strategy ($f5$)	$11 \to 21$	$1_F1 \to 11_F1$	$2 \to 3$	$2_F \to 3_F$
Intermediately entrepreneurial strategy ($f6$)	$11 \to 21$	$1_F1 \to 11_F1$	$2 \to 12$	$2_F \to 12_F$
Entrepreneurial with join-the-largest strategy ($f7$)	$11 \to 111$	$1_F1 \to 11_F1$	$2 \to 3$	$2_F \to 12_F$
Small competition but size-conscious strategy ($f8$)	$11 \to 111$	$1_F1 \to 11_F1$	$2 \to 3$	$2_F \to 3_F$
Strongly entrepreneurial strategy ($f9$)	$11 \to 111$	$1_F1 \to 11_F1$	$2 \to 12$	$2_F \to 12_F$

Table 2: Possible strategies based on Lemma 1

largest group. If this constitutes an equilibrium, monopoly regardless of reputation is absorbing. Another economically interesting strategy is $f9$, under which an entrant always creates a new group so that, from any initial state, no clustering occurs over time. Detailed interpretations of each strategy are given below.

For each of the above possible strategy, we derive the parameter condition under which the strategy constitutes a (symmetric, stationary) Markov perfect equilibrium (MPE). We give all the details of the analysis for the focal strategy $f1$ in the text and Appendix A.2, and the derivation of the equilibrium conditions of other strategies are relegated to Online Appendix.

3.1 Join-the-largest-without-F strategy $f1$

Consider the case such that

$$V(21) \geqq V(111),\ V(21_F) \geqq V(11_F1),\ V(3) \geqq V(12),\ \text{and } V(12_F) \geqq V(3_F). \tag{5}$$

(Henceforth we drop the time subscript, since we focus on stationary equilibria.)

If the optimal value function satisfies (5), each entrant uses the following strategy (strategy $f1$): if the interim state is 11 or 1_F1, join one of the groups with no F with equal probability. If the interim state is 2, join that group. Finally, if the interim state is 2_F, then create a new group (of size 1). This strategy avoids any group with F but otherwise prefers to join the largest group. We think that this strategy represents a good middle ground to utilize the group-size effect as much as possible while taking into account the reputation concern.

Once we determine the value relationships as in (5), we can specify the `IF` part of equations (2), (3), (4) and so forth, for all $s \in S_3$, and solve the resulting system of equations of $(V(s))_{s \in S_3}$ as a recursive system of equations. The system can be written with a state-transition matrix[12] T

[11] The first entry in the social state is the relevant player's group size with reputation.
[12] Since we omit the state "exit", T is not a probability matrix. Each row sums to $1 - \frac{\varepsilon}{3}$.

as follows.[13]

$$V = U + T \cdot V,$$

where

$$V := \begin{pmatrix} V(111) \\ V(1_F 11) \\ \vdots \\ V(3_F) \end{pmatrix}.$$

Under $f1$, the state-transition matrix T_{f1} (the subscript is the name of the strategy) for an existing seller is as follows. (For clarity, we added the state names.)

$$T_{f1} = \begin{array}{c} \\ 111 \\ 11_F1 \\ 111_F \\ 12 \\ 1_F2 \\ 12_F \\ 21 \\ 2_F1 \\ 21_F \\ 3 \\ 3_F \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 111 & 1_F11 & 11_F1 & 12 & 1_F2 & 12_F & 21 & 2_F1 & 21_F & 3 & 3_F \end{array} \\ \left[\begin{array}{ccccccccccc} 1-\varepsilon & 0 & 0 & \frac{\varepsilon}{3} & 0 & 0 & \frac{\varepsilon}{3} & 0 & 0 & 0 & 0 \\ 1-\varepsilon & 0 & 0 & \frac{\varepsilon}{3} & 0 & 0 & \frac{\varepsilon}{3} & 0 & 0 & 0 & 0 \\ 1-\varepsilon & 0 & 0 & \frac{\varepsilon}{3} & 0 & 0 & \frac{\varepsilon}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\varepsilon & 0 & 0 & 0 & 0 & \frac{2\varepsilon}{3} & 0 & 0 \\ 0 & 0 & 0 & 1-\varepsilon & 0 & 0 & 0 & 0 & \frac{2\varepsilon}{3} & 0 & 0 \\ 0 & 0 & 0 & 1-\varepsilon & 0 & 0 & 0 & 0 & \frac{2\varepsilon}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\varepsilon}{3} & 0 & 1-\varepsilon & 0 & 0 & \frac{\varepsilon}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{\varepsilon}{3} & 0 & 1-\varepsilon & 0 & 0 & \frac{\varepsilon}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{\varepsilon}{3} & 0 & 1-\varepsilon & 0 & 0 & \frac{\varepsilon}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2\varepsilon}{3} & 0 & 1-\varepsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2\varepsilon}{3} & 0 & 1-\varepsilon & 0 \end{array} \right] \end{array}.$$

To explain, with probability $1 - \varepsilon$, no failure happens and hence any state moves to the same group-size distribution without F. (This property is common to all state-transition matrices for any strategy.) Under the strategy $f1$ used by all entrants, if the state was in $\{111, 1_F11, 11_F1\}$ and failure occurs to one of the other groups (which happens with probability $2\frac{\varepsilon}{3}$), the interim state is 11 for sure. Then the next period entrant joins one of the existing groups with equal probability so that either this seller's group size becomes 2 (the next state is 21) or is still 1 (the next state is 12) with the same probability $2\frac{\varepsilon}{3} \cdot \frac{1}{2} = \frac{\varepsilon}{3}$. If the state was in $\{12, 1_F2, 12_F\}$ and failure happened to the other group, the focal seller's group receives the entrant with the conditional probability 1 since the interim state is 11_F, with probability $2\frac{\varepsilon}{3}$. If the state was in $\{21, 2_F1, 21_F\}$ and the focal seller survives but a failure has occurred, then either the other group failed or a fellow seller in the same group failed (with the same probability $\frac{\varepsilon}{3}$ each). If the other group has disappeared, the entrant joins the focal seller's group so that the next state is 3. If a fellow seller in the focal seller's group has failed, by contrast, the interim state is 1_F1 and the other group gets the entrant so that

[13]This formula essentially corresponds to equation (1) of Konishi and Ray (2003). However, we omit the state "exit" so that T is not the probability matrix, as noted in the above footnote 12, and the effective discount factor ε is embedded in T.

the state becomes 1_F2. Finally, if the state was in $\{3, 3_F\}$ and the focal seller survives but a failure has occurred, its group has the F record and the next state is 2_F1.

Using T_{f1} and the one-shot payoff vector U, the value function vector of strategy $f1$

$$V_{f1} := \begin{pmatrix} V_{f1}(111) \\ V_{f1}(1_F11) \\ \vdots \\ V_{f1}(3_F) \end{pmatrix}$$

is the solution to the recursive equation

$$V_{f1} = U + T_{f1} \cdot V_{f1}. \tag{6}$$

For $N = 3$, we can solve (6) explicitly to derive the long-run expected payoff $(V_{f1}(s))_{s \in S_3}$ (see Appendix A.2). Then we can determine the parameter conditions for the four inequalities of (5) to be satisfied, i.e., when $f1$ constitutes an MPE. But it should be clear that for general N, the qualitative comparison of different rows of the vector V_{f1} would be very difficult.

Let $x := \delta \cdot \epsilon$. This parameter can be called the *rate of reputation loss* on the payoff of the sellers. It is the fraction of the consumers who remember the Failure times the probability of a Failure in the market. It turns out that all equilibrium conditions are classified by this x and the group-size effect parameter k.

If the strategy $f1$ is optimal for any entrant at any interim state, the following must be satisfied simultaneously.

$$\begin{aligned}
V_{f1}(21) \geqq V_{f1}(111) &\iff x \leqq \frac{7\left(2^k - 2\right)}{2^{k+2} + 13}, \\
V_{f1}(21_F) \geqq V_{f1}(11_F1) &\iff x \leqq \frac{14\left(2^k - 2\right)}{23 \cdot 2^k - 4}, \\
V_{f1}(3) \geqq V_{f1}(12) &\iff x \leqq \frac{2^k - 2}{2^k + 1}, \\
V_{f1}(12_F) \geqq V_{f1}(3_F) &\iff x \geqq \frac{2^k - 2}{2^{k+2} + 1}.
\end{aligned}$$

By computation (see Appendix A.2), the binding conditions are $V_{f1}(21_F) \geqq V_{f1}(11_F1)$ and $V_{f1}(12_F) \geqq V_{f1}(3_F)$. Therefore, the strategy $f1$ used by all entrants constitutes an MPE if and only if

$$(\underline{x}_1 :=) \frac{2^k - 2}{2^{k+2} + 1} \leqq x \leqq \frac{14\left(2^k - 2\right)}{23 \cdot 2^k - 4} (=: \overline{x}_1).$$

3.2 Size-conscious strategy

Consider the case such that

$$V(21) \geqq V(111),\ V(21_F) \geqq V(11_F1),\ V(3) \geqq V(12),\ \text{and}\ V(12_F) \leqq V(3_F).$$

This case corresponds to the situation that each entrant uses the following strategy $f2$: if the interim state is 11 or 1_F1, join one of the groups with no F with equal probability. If the interim state is 2 or 2_F, join that group. Hence in this strategy, an entrant ignores the group's reputation if the existing group is the monopoly.

By solving the system of equations (see Online Appendix), the equilibrium condition is

$$\begin{aligned}
&V_{f2}(21) \geqq V_{f2}(111) \iff x \leqq \frac{3}{5}\left(2^k-2\right),\\
&V_{f2}(21_F) \geqq V_{f2}(11_F1) \iff x \leqq \frac{6\left(2^k-2\right)}{7\cdot 2^k-4},\\
&V_{f2}(3) \geqq V_{f2}(12) \iff x \leqq 2\left(2^k-2\right),\\
&V_{f2}(3_F) \geqq V_{f2}(12_F) \iff x \leqq \frac{2\left(2^k-2\right)}{7\cdot 2^k+1}.
\end{aligned}$$

By computation, the last inequality is binding, i.e., this strategy constitutes an MPE if and only if

$$x \leqq \frac{2\left(2^k-2\right)}{7\cdot 2^k+1}(=: \overline{x}_2).$$

3.3 Join-the-smallest strategy

Consider the case such that

$$V(21) \geqq V(111),\ V(21_F) \geqq V(11_F1),\ V(3) \leqq V(12),\ \text{and } V(12_F) \geqq V(3_F).$$

This corresponds to the following strategy $f3$: if the interim state is 11 or 1_F1, join one of the groups with no F with equal probability. If the interim state is 2 or 2_F, create a new group.

The equilibrium condition is

$$\begin{aligned}
&V_{f3}(21) \geqq V_{f3}(111) \iff x \leqq \frac{1}{2}(-2+2^k),\\
&V_{f3}(21_F) \geqq V_{f3}(11_F1) \iff x \leqq 2^{-k}(-2+2^k),\\
&V_{f3}(12) \geqq V_{f3}(3) \iff x \geqq \frac{2^k-2}{2^k+1},\\
&V_{f3}(12_F) \geqq V_{f3}(3_F) \iff x \geqq \frac{2^k-2}{2^{k+2}+1}.
\end{aligned}$$

By computation, the binding conditions are $V_{f3}(21_F) \geqq V_{f3}(11_F1)$ and $V_{f3}(12) \geqq V_{f3}(3)$, i.e., $f3$ constitutes an MPE if and only if

$$(\underline{x}_3 :=)\frac{2^k-2}{2^k+1} \leqq x \leqq 2^{-k}(-2+2^k)(=: \overline{x}_3).$$

3.4 Weakly entrepreneurial strategy

Consider the case such that

$$V(21) \geqq V(111),\ V(21_F) \leqq V(11_F1),\ V(3) \geqq V(12),\ \text{and}\ V(12_F) \geqq V(3_F).$$

This corresponds to the following strategy $f4$: if the interim state is 11 or 2, join one of the existing groups with equal probability. Otherwise, create a new group. Under this strategy, entrants are entrepreneurial (to create a new group) as long as there is a group with F.

The equilibrium condition is

$$\begin{aligned}
V_{f4}(21) \geqq V_{f4}(111) &\iff x \leqq \frac{15\left(2^k-2\right)}{2\left(7\cdot 2^k+4\right)}, \\
V_{f4}(11_F1) \geqq V_{f4}(21_F) &\iff x \geqq \frac{5\left(2^k-2\right)}{4\left(2^{k+1}-1\right)}, \\
V_{f4}(3) \geqq V_{f4}(12) &\iff x \leqq \frac{5\left(2^k-2\right)}{2\left(3\cdot 2^k+1\right)}, \\
V_{f4}(12_F) \geqq V_{f4}(3_F) &\iff x \geqq \frac{5\left(2^k-2\right)}{2\left(2^{k+3}+1\right)}.
\end{aligned}$$

The binding conditions are $V_{f4}(11_F1) \geqq V_{f4}(21_F)$ and $V_{F4}(12_F) \geqq V_{f4}(3_F)$. To have an equilibrium, the region

$$\frac{5\left(2^k-2\right)}{4\left(2^{k+1}-1\right)} \leqq x \leqq \frac{5\left(2^k-2\right)}{2\left(3\cdot 2^k+1\right)}$$

should be nonempty. This holds if and only if

$$\frac{5\left(2^k-2\right)}{4\left(2^{k+1}-1\right)} \leqq \frac{5\left(2^k-2\right)}{2\left(3\cdot 2^k+1\right)} \iff k \geqq \frac{\ln 3}{\ln 2} \approx 1.58496.$$

In sum, the equilibrium condition is

$$k \geqq \frac{\ln 3}{\ln 2}\ \text{and}\ (\underline{x}_4 :=)\frac{5\left(2^k-2\right)}{4\left(2^{k+1}-1\right)} \leqq x \leqq \frac{5\left(2^k-2\right)}{2\left(3\cdot 2^k+1\right)}(=: \overline{x}_4).$$

Interestingly, this weakly entrepreneurial strategy requires a high k which favors larger groups. This is a consequence of far-sighted optimization.

3.5 Intermediately size-conscious strategy

Suppose that

$$V(21) \geqq V(111),\ V(21_F) \leqq V(11_F1),\ V(3) \geqq V(12),\ \text{and}\ V(12_F) \leqq V(3_F).$$

This case corresponds to the following strategy $f5$: if the interim state is 11, 2, or 2_F, join one of the existing groups with equal probability. If the interim state is 11_F, then create a new group.

The equilibrium condition is

$$V_{f5}(21) \geqq V_{f5}(111) \iff x \leqq \frac{2\left(2^k-2\right)}{2^k+1},$$
$$V_{f5}(11_F1) \geqq V_{f5}(21_F) \iff x \geqq \frac{4\left(2^k-2\right)}{5\cdot 2^k-4},$$
$$V_{f5}(3) \geqq V_{f5}(12) \iff x \leqq \frac{7\left(2^k-2\right)}{2\left(2^k+1\right)},$$
$$V_{f5}(3_F) \geqq V_{f5}(12_F) \iff x \leqq \frac{7\left(2^k-2\right)}{2\left(5\cdot 2^{k+1}+1\right)}.$$

By computation, it is impossible to satisfy $V_{f5}(11_F1) \geqq V_{f5}(21_F)$ and $V_{f5}(3_F) \geqq V_{f5}(12_F)$ simultaneously. Hence this strategy cannot constitute an MPE. Namely, joining the monopoly group with F and creating a new group when the interim state is 11_F are contradictory.

3.6 Intermediately entrepreneurial strategy

Consider the case such that

$$V(21) \geqq V(111),\ V(21_F) \leqq V(11_F1),\ V(3) \leqq V(12),\ \text{and } V(12_F) \geqq V(3_F).$$

This corresponds to a strategy (call it $f6$) which differs from $f4$ in that it creates a new group at all interim states other than 11. If the interim state is 11, this strategy prescribes to join one of the existing groups with equal probability.

The equilibrium condition is

$$V_{f6}(21) \geqq V_{f6}(111) \iff x \leqq \frac{9\left(2^k-2\right)}{5\left(2^k+1\right)},$$
$$V_{f6}(11_F1) \geqq V_{f6}(21_F) \iff x \geqq \frac{6\left(2^k-2\right)}{7\cdot 2^k-4},$$
$$V_{f6}(12) \geqq V_{f6}6(3) \iff x \geqq \frac{17\left(2^k-2\right)}{9\ 2^{k+1}+7},$$
$$V_{f6}(12_F) \geqq V_{f6}(3_F) \iff x \geqq \frac{17\left(2^k-2\right)}{51\cdot 2^k+7}.$$

By computation, this strategy constitutes an MPE if and only if

$$(\underline{x}_6 :=) \max\{\frac{6\left(2^k-2\right)}{7\cdot 2^k-4}, \frac{17\left(2^k-2\right)}{9\cdot 2^{k+1}+7}\} \leqq x \leqq \frac{9\left(2^k-2\right)}{5\left(2^k+1\right)} (=: \overline{x}_6).$$

3.7 Entrepreneurial with join-the-largest strategy

Consider the case such that

$$V(21) \leqq V(111),\ V(21_F) \leqq V(11_F1),\ V(3) \geqq V(12),\ \text{and } V(12_F) \geqq V(3_F).$$

This corresponds to the following strategy $f7$ which also differs from $f4$ in one place: it creates a new group at all interim states other than 2. If the interim state is 2, this strategy prescribes joining the existing group to become the monopoly.

The equilibrium condition is

$$\begin{aligned}
V_{f7}(111) \geqq V_{f7}(21) &\iff x \geqq \frac{3\left(2^k-2\right)}{2\left(2^{k+1}+1\right)}, \\
V_{f7}(11_F1) \geqq V_{f7}(21_F) &\iff x \geqq \frac{9\left(2^k-2\right)}{19\ 2^k-8}, \\
V_{f7}(3) \geqq V_{f7}(12) &\iff x \leqq \frac{30\left(2^k-2\right)}{5\cdot 2^{k+3}+13}, \\
V_{f7}(12_F) \geqq V_{f7}(3_F) &\iff x \geqq \frac{30\left(2^k-2\right)}{103\ 2^k+13}.
\end{aligned}$$

By computation, this strategy constitutes an MPE if and only if

$$(\underline{x}_7 :=)\frac{3\left(2^k-2\right)}{2\left(2^{k+1}+1\right)} \leqq x \leqq \frac{30\left(2^k-2\right)}{5\cdot 2^{k+3}+13}(=: \overline{x}_7).$$

3.8 Small-competition but size-conscious strategy

Suppose that

$$V(21) \leqq V(111),\ V(21_F) \leqq V(11_F1),\ V(3) \geqq V(12),\ \text{and } V(12_F) \leqq V(3_F).$$

This corresponds to the strategy $f8$ such that a new group is created at interim states $11, 1_F1$ but an entrant joins the existing group of size 2 regardless of the reputation.

The equilibrium condition is

$$\begin{aligned}
V_{f8}(111) \geqq V_{f8}(21) &\iff x \geqq \frac{3\left(2^k-2\right)}{2\left(2^k+1\right)}, \\
V_{f8}(11_F1) \geqq V_{f8}(21_F) &\iff x \geqq \frac{3\left(2^k-2\right)}{5\cdot 2^k-4}, \\
V_{f8}(3) \geqq V_{f8}(12) &\iff x \leqq \frac{3\left(2^k-2\right)}{2^k+1}, \\
V_{f8}(3_F) \geqq V_{f8}(12_F) &\iff x \leqq \frac{3\left(2^k-2\right)}{5\cdot 2^{k+1}+1}.
\end{aligned}$$

This is impossible, and the reason is the same as that of strategy $f5$.

3.9 Strongly entrepreneurial strategy

Finally consider the case such that

$$V(21) \leqq V(111),\ V(21_F) \leqq V(11_F1),\ V(3) \leqq V(12),\ \text{and } V(12_F) \geqq V(3_F).$$

This corresponds to the following simple strategy $f9$: create a new group at any interim state.

The equilibrium condition is

$$\begin{aligned} V_{f9}(111) \geqq V_{f9}(21) &\iff x \geqq \frac{3\left(2^k-2\right)}{2\left(2^k+1\right)}, \\ V_{f9}(11_F1) \geqq V_{f9}(21_F) &\iff x \geqq \frac{3\left(2^k-2\right)}{2\left(2^{k+1}-1\right)}, \\ V_{f9}(12) \geqq V_{f9}(3) &\iff x \geqq \frac{9\left(2^k-2\right)}{2\left(5\cdot 2^k+2\right)}, \\ V_{f9}(12_F) \geqq V_{f9}(3_F) &\iff x \geqq \frac{9\left(2^k-2\right)}{4\left(7\cdot 2^k+1\right)}. \end{aligned}$$

This strategy constitutes an MPE if and only if

$$V_{f9}(111) \geqq V_{f9}(21) \iff x \geqq \frac{3\left(2^k-2\right)}{2\left(2^k+1\right)} (=: \underline{x}_9),$$

and

$$\frac{3\left(2^k-2\right)}{2\left(2^k+1\right)} < 1 \iff k < 3.$$

3.10 Non-Existence and Multiplicity of Stationary Equilibria

We now show that the parameter space $(1,\infty)\times(0,1)$ of (k,x) is **not** partitioned for different MPEs. Rather, multiple stationary MPEs may co-exist for the same (k,x). Moreover, although the model is symmetric and stationary, in some regions of (k,x), there is no symmetric, stationary MPE.

Let the lower (resp. upper) bound of $x=\delta\cdot\varepsilon$ for strategy fi $(i=1,2,3,4,6,7,9)$ to constitute an MPE be $\underline{x}_i$ (resp. $\overline{x}_i$). From the above analysis, we have Table 3. Note that for each $i = 1,3,4,6,7,9$, $\underline{x}_i > 0$ and $\underline{x}_2 = 0$.

bounds on $x=\delta\cdot\varepsilon$	$\underline{x}_i$	$\overline{x}_i$
$f1$	$\frac{2^k-2}{2^{k+2}+1}$	$\frac{14\left(2^k-2\right)}{23\cdot 2^k-4}$
$f2$	0	$\frac{2\left(2^k-2\right)}{7\cdot 2^k+1}$
$f3$	$\frac{2^k-2}{2^k+1}$	$2^{-k}(-2+2^k)$
$f4$ (if $k \geqq \ln 3/\ln 2$)	$\frac{5\left(2^k-2\right)}{4\left(2^{k+1}-1\right)}$	$\frac{5\left(2^k-2\right)}{2\left(3\cdot 2^k+1\right)}$
$f6$	$\max\{\frac{6\left(2^k-2\right)}{7\cdot 2^k-4}, \frac{17\left(2^k-2\right)}{9\cdot 2^{k+1}+7}\}$	$\frac{9\left(2^k-2\right)}{5\left(2^k+1\right)}$
$f7$	$\frac{3\left(2^k-2\right)}{2\left(2^{k+1}+1\right)}$	$\frac{30\left(2^k-2\right)}{5\cdot 2^{k+3}+13}$
$f9$ (if $k<3$)	$\frac{3\left(2^k-2\right)}{2\left(2^k+1\right)}$	1

Table 3: Equilibrium bounds for $N=3$

Proposition 1. *(i) For any $k > 1$, the strategies $f1$, $f2$, $f3$, $f6$, and $f7$ can constitute an MPE for some $x = \delta \cdot \varepsilon \in (0, 1)$, that is*

$$\underline{x}_1 < 1,\ \underline{x}_3 < 1,\ \underline{x}_6 < 1,\ \underline{x}_7 < 1$$

hold.

(ii) For any $k > 1$,the following relationships among the bounds to $x = \delta \cdot \varepsilon$ hold.

$$\underline{x}_1 < \overline{x}_2 < \overline{x}_1,$$

$$\underline{x}_1 < \underline{x}_7 < \underline{x}_6,$$

$$\underline{x}_1 < \underline{x}_3 < \underline{x}_9,$$

$$\underline{x}_1 < \underline{x}_4.$$

Proof. (i) By computation, the necessary and sufficient conditions for the inequalities are as follows, and they all hold.

$$\begin{aligned}
\underline{x}_1 < 1 &\iff 2^k - 2 < 4 \cdot 2^k + 1, \\
\underline{x}_3 < 1 &\iff 2^k - 2 < 2^k + 1, \\
\underline{x}_7 < 1 &\iff 3 \cdot 2^k - 6 < 4 \cdot 2^k + 2, \\
\underline{x}_6 < 1 &\iff 6 \cdot 2^k - 12 < 7 \cdot 2^k - 4 \text{ and} \\
&\qquad 17 \cdot 2^k - 34 < 18 \cdot 2^k + 7.
\end{aligned}$$

(ii) This is also by computation.

$$\begin{aligned}
\underline{x}_1 < \overline{x}_2 &\iff \frac{\left(2^k - 2\right)}{\left(2^{k+2} + 1\right)} < \frac{2\left(2^k - 2\right)}{7 \cdot 2^k + 1} \iff 0 < 2^k + 1, \\
\overline{x}_2 < \overline{x}_1 &\iff \frac{2\left(2^k - 2\right)}{7 \cdot 2^k + 1} < \frac{14\left(2^k - 2\right)}{23 \cdot 2^k - 4} \iff 0 < 26 \cdot 2^k + 11, \\
\underline{x}_1 < \underline{x}_7 &\iff \frac{\left(2^k - 2\right)}{\left(2^{k+2} + 1\right)} < \frac{3\left(2^k - 2\right)}{2\left(2^{k+1} + 1\right)} \iff 0 < 8 \cdot 2^k + 1, \\
\underline{x}_7 < \underline{x}_6 \Leftarrow \frac{3\left(2^k - 2\right)}{2\left(2^{k+1} + 1\right)} &< \frac{6\left(2^k - 2\right)}{7 \cdot 2^k - 4} \iff 0 < 2^k + 8, \\
\underline{x}_1 < \underline{x}_3 < \underline{x}_9 &\iff \frac{\left(2^k - 2\right)}{\left(2^{k+2} + 1\right)} < \frac{\left(2^k - 2\right)}{\left(2^k + 1\right)} < \frac{3\left(2^k - 2\right)}{2\left(2^k + 1\right)}, \\
\underline{x}_1 < \underline{x}_4 &\iff 0 < 12 \cdot 2^k + 9.
\end{aligned}$$

□

Table 3 and Proposition 1 imply that for small $x = \delta \cdot \varepsilon < \overline{x}_2$, $f2$ constitutes an MPE, for slightly higher $x = \delta \cdot \varepsilon \in [\underline{x}_1, \overline{x}_1]$, $f1$ constitutes an MPE, and the intervals $(0, \overline{x}_2]$ and $[\underline{x}_1, \overline{x}_1]$ intersect (see Figure 2). Hence we have multiplicity of symmetric, stationary MPEs.

Corollary 1. *For any $k > 1$, there are two symmetric, stationary MPEs consisting of $f1$ and $f2$, when $\underline{x}_1 < \delta \cdot \varepsilon < \overline{x}_2$.*

Proposition 2. *When $\frac{40}{13} < 2^k$, for any $x = \delta \cdot \varepsilon$ such that $\overline{x}_1 < x < \min\{\underline{x}_4, \underline{x}_7\}$, there is no stationary, symmetric MPE.*

Proof. Recall that $f4$-equilibrium may not always exist, but $\frac{\ln 3}{\ln 2} < \frac{\ln(\frac{40}{13})}{\ln 2}$ so that $\frac{40}{13} < 2^k$ implies that $\underline{x}_4 < \overline{x}_4$ holds. Therefore we prove that

$$\overline{x}_2 < \overline{x}_1 \text{ and } \overline{x}_1 < \min\{\underline{x}_4, \underline{x}_7\} < \min\{\underline{x}_i; i = 3, 6, 9\}.$$

(See also Figure 2.)

First, by Proposition 1 (ii), we have

$$\overline{x}_2 < \overline{x}_1;$$

$$\underline{x}_7 < \underline{x}_6;$$

$$\underline{x}_3 < \underline{x}_9.$$

By computation (see Online Appendix), it always holds that

$$\overline{x}_1 < \underline{x}_4;$$

$$\underline{x}_7 < \underline{x}_3.$$

Therefore, we have that

$$\underline{x}_7 < \min\{\underline{x}_i; i = 3, 6, 9\}.$$

The relationship between $\underline{x}_4$ and $\underline{x}_7$ varies. However,

$$\overline{x}_1 < \underline{x}_7 \iff 2^k > \frac{40}{13},$$

hence we conclude that

$$2^k > \frac{40}{13} \Rightarrow \overline{x}_1 < \min\{\underline{x}_4, \underline{x}_7\}.$$

□

To interpret Proposition 2, fix any k such that $\frac{40}{13} < 2^k$. For the intermediate values of the rate of reputation loss such that $\overline{x}_1 < x < \min\{\underline{x}_4, \underline{x}_7\}$, if all entrants are following $f1$, $f4$, or $f7$, Proposition 2 says that an entrant wants to deviate from it. On one hand, the fact that $x > \overline{x}_1$ means that the rate of reputation loss is not too small, and an entrant (if all others use $f1$) has $V_{f1}(11_F1) > V_{f1}(21_F)$, i.e., becoming a member of a group of size 2 is worse than creating a new group of size 1. On the other hand, $x < \underline{x}_7$ (resp. $x < \underline{x}_4$) implies that, if all others use $f7$ (resp. $f4$), $V_{f7}(21) > V_{f7}(111)$ (resp. $V_{f4}(21_F) > V_{f4}(11_F1)$). That is, even though all future entrants would create a new group at the interim state 11 (resp. 1_F1), the rate of reputation loss is not large enough so that the group-size effect is attractive.

Therefore, when $\overline{x}_1 < x < \min\{\underline{x}_4, \underline{x}_7\}$, it is not optimal to stick to one of the Markov strategies **for all periods**, even though the environment is symmetric and stationary.

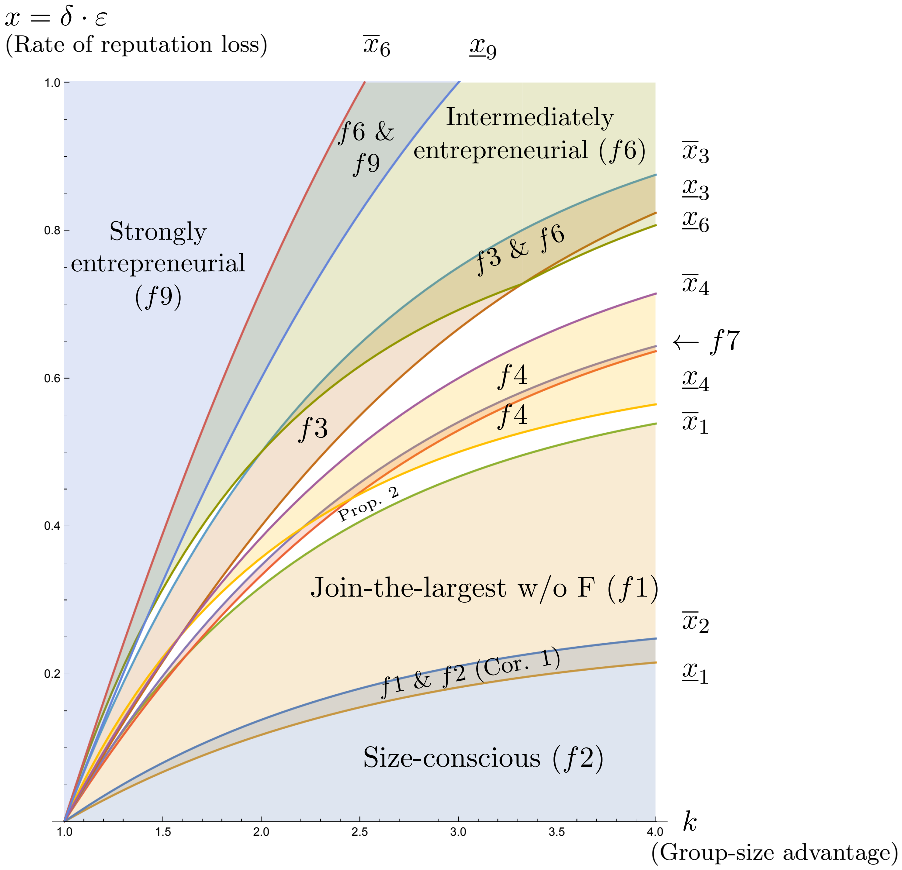

Figure 2: Parameter regions of various MPE's

Figure 2 summarizes the above results and shows all the bounds for various symmetric, stationary strategies to become an equilibrium. Online Appendix gives a complete classification of the orders of the bounds for all $k > 1$. We see that there are many parameter regions in which no symmetric, stationary MPE exists, in addition to the region that we found in Proposition 2. For the region of Proposition 2, we present a symmetric but cyclic equilibrium in which $f1$ is used for 6 consecutive periods and $f4$ is used for one period, in Online Appendix.

To illustrate how multiple, symmetric and stationary MPEs co-exist, Figure 3 may be also useful. This figure shows how various equilibria overlap for some region of k (which is too narrow to see easily in Figure 2) in addition to Corollary 1.[14]

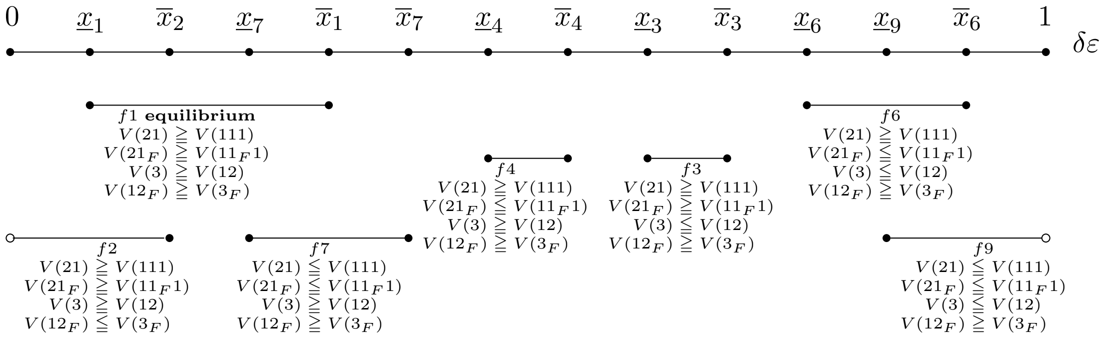

Figure 3: Overlapping equilibria when $3 < 2^k < \frac{40}{13}$

Let us consider comparative statics from Figure 2. Fix an arbitrary $k > 1$. We see that, when $x = \delta \cdot \varepsilon$ is close to 0 (e.g., for a fixed δ, we let the failure rate ε be very small), then the size-conscious strategy $f2$ is the unique stationary equilibrium strategy. Under $f2$ by all entrants, eventually, the set of states $\{3, 3_F\}$ is reached and is recurrent. Since reputation loss is negligible ($x \approx 0$) and the group-size effect $k > 1$ is present, the complete clustering emerges. This may give a rationale to the single ruling political party in some countries. Although new politicians can choose parties to join or to create a new party, the conservative society (with a very small probability of reputation loss) induces them to join the ruling party.

If δ is close to 1 while $\varepsilon \in (\underline{x}_1, \overline{x}_1)$ (under a fixed $k > 1$), then the Join-the-largest-without-F strategy $f1$ is also an equilibrium strategy. Under this strategy $f1$, new groups are formed infinitely often (see T_{f1}). This is because consumers are mostly "informed" customers so that, if the existing group is 2_F, an entrant can immediately get a high payoff by creating a new group. This equilibrium has the limit cycle of two groups and a monopoly group, and thus endogenous "de-monopolization" occurs. However, the size effect keeps the industry oligopolistic, in the sense that 111 is unstable. Possible examples include the fashion industry and theaters, where consumers actively switch sellers based on reputations.

When $\delta \cdot \varepsilon$ is very large and $k < 3$, then the strongly entrepreneurial equilibrium (by $f9$ strategy which creates a new group at any interim state) exists, and the social state converges to 111. This means that even when there is a size advantage ($k > 1$) to attract customers, high probabilities of reputation loss prevent endogenous clustering. This equilibrium may correspond to street-food vendors that are vulnerable to the volatile tastes of customers and potential food poisoning and

[14]Similar figures for all $k > 1$ are available upon request.

other mistakes.

Let us also consider the limit cases of k. Figure 2 shows that, given a value of $x = \delta\cdot\varepsilon > 0$, when k goes to 1, the strongly entrepreneurial strategy $f9$ is the only symmetric equilibrium strategy. This is intuitive since the group-size advantage disappears, but reputation is still a concern. By contrast, when k is very large, the existence of each symmetric equilibrium still depends on the level of $x = \delta \cdot \varepsilon$. For markets with very rare reputation loss (with very small $x = \delta \cdot \varepsilon$), the size-conscious strategy $f2$ is the only symmetric equilibrium strategy. This is also intuitive because it is primarily the group-size advantage which matters.

4 Concluding Remarks

In this paper, we presented two ways to describe the system of optimal value functions and solved for symmetric, stationary Markov perfect equilibria when there are three sellers, facing a continuum of consumers.[15] However, each of them has its own difficulty to find an equilibrium for a general N seller's model. First, if the system of equations (2), (3), (4), etc. has a solution, we have an equilibrium. However, the "IF" parts make it very difficult to determine whether there is a solution to the simultaneous equations. Second, the matrix formulation such as (6) requires a guess of an equilibrium strategy to specify the transition matrix T. Moreover, it gives only the $V(s)$ vector, and whether the $V(s)$ vector satisfies the equilibrium conditions is difficult to verify.

Next, let us discuss possible modifications of the model. We could separate the reputation loss of a group and the sellers' turnover. Let ε/N be the probability of reputation loss of each (existing) seller as before, so that the probability of reputation loss of a group of size n is $n\cdot\varepsilon/N$. Introduce another parameter $\beta \in (0,1)$ such that each seller is replaced by an entrant with probability β/N. With probability $1-\beta$, no seller exits. Assume that exiting does not affect a group's reputation. An example is that an "undeserved loss" of reputation happens (see Fujiwara-Greve et al., 2016) with probability ε/N and each seller may experience an unlucky real hazard (e.g., fire) with probability β/N. The new model only complicates the state-transition matrix but essentially a similar analysis can be done.[16]

We could also include the sellers' effort choices, which may affect the individual failure rate, as in Fujiwara-Greve et al. (2016). As long as we focus on stationary equilibria in both effort choice and entry choice, a similar analysis to this paper can be done. However, to conduct an efficiency

[15]We have also similar results for $N = 4$.

[16]A variable seller-population model where N changes over time can be also formulated if there is a fixed upper bound to N.

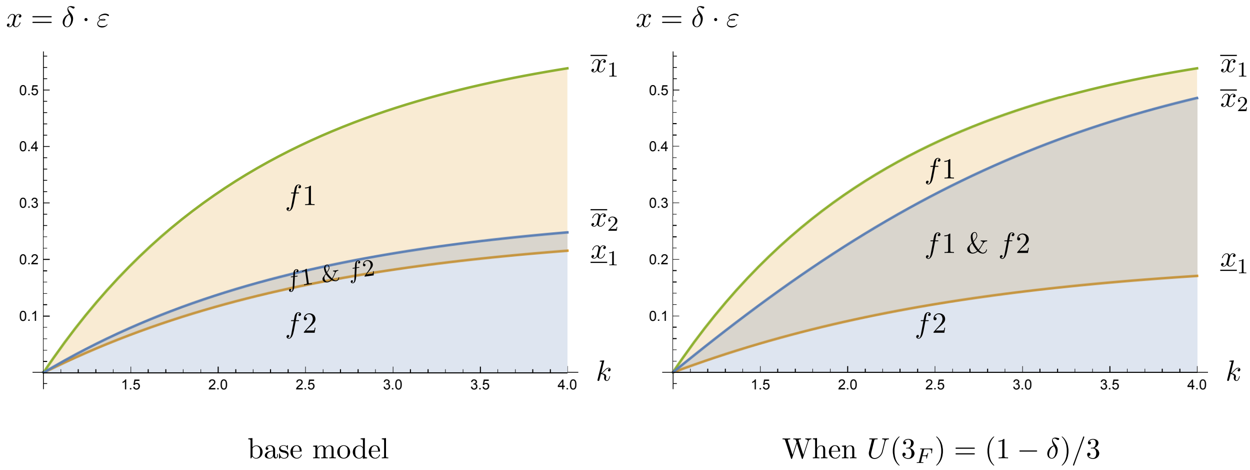

Figure 4: Comparison of $f1$ and $f2$ equilibrium regions

analysis, we need to specify the cost of effort, consumers' utility, and so on.

The failure/reputation loss probabilities may be controlled by consumer strategies and government regulations. In particular, if the quality of trade is endogenously chosen by the sellers' effort, then consumers and the government have choices on how much to react for various degrees of quality decline, etc. The group-size effect k may also be controlled by the consumers' information processing strategy and the platforms that intermediate sellers and buyers. From these perspectives, a good extension to formulate the underlying mechanism of the parameter values of the current model will give many important economic implications.

When there is a monopoly group with the Failure record, surviving consumers may want to refrain from purchasing from that group in that period. This modification can be incorporated by setting $U(3_F) = (1-\delta)\frac{1}{3}$. It is straightforward to prove that Lemma 1 continues to hold. Hence the qualitative results do not change. See Figure 4. Note that, under this formulation, the social state 3_F is the unique inefficient state because the total sales across all sellers is less than 1. Nonetheless, it may happen infinitely often in the $f2$-equilibrium. Hence inefficient clustering can occur.

Declarations

The authors have no relevant financial or non-financial interests to disclose.

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Appendix

A.1. Optimal value functions and the proof of Lemma 1

The stationary relationships of the optimal value functions at all states $s \in S_3$ other than (2), (3), (4) are as follows.

$$
\begin{aligned}
V_t(12) &= U(12) + \frac{2\varepsilon}{3} V_{t+1}(21_F) \cdot \texttt{AND}\big(V_{t+1}(21_F) \geqq V_{t+1}(2_F1), V_{t+1}(21_F) \geqq V_{t+1}(11_F1)\big) \\
&\quad + \frac{2\varepsilon}{3} V_{t+1}(12_F) \cdot \texttt{AND}\big(V_{t+1}(21_F) < V_{t+1}(2_F1), V_{t+1}(2_F1) \geqq V_{t+1}(11_F1)\big) \\
&\quad + \frac{2\varepsilon}{3} V_{t+1}(11_F1) \cdot \texttt{AND}\big(V_{t+1}(21_F) < V_{t+1}(11_F1), V_{t+1}(2_F1) < V_{t+1}(11_F1)\big) \\
&\quad + (1-\varepsilon) V_{t+1}(12), \\
V_t(12_F) &= U(12_F) + \frac{2\varepsilon}{3} V_{t+1}(21_F) \cdot \texttt{AND}\big(V_{t+1}(21_F) \geqq V_{t+1}(2_F1), V_{t+1}(21_F) \geqq V_{t+1}(11_F1)\big) \\
&\quad + \frac{2\varepsilon}{3} V_{t+1}(12_F) \cdot \texttt{AND}\big(V_{t+1}(21_F) < V_{t+1}(2_F1), V_{t+1}(2_F1) \geqq V_{t+1}(11_F1)\big) \\
&\quad + \frac{2\varepsilon}{3} V_{t+1}(11_F1) \cdot \texttt{AND}\big(V_{t+1}(21_F) < V_{t+1}(11_F1), V_{t+1}(2_F1) < V_{t+1}(11_F1)\big) \\
&\quad + (1-\varepsilon) V_{t+1}(12), \\
V_t(1_F2) &= U(1_F2) + \frac{2\varepsilon}{3} V_{t+1}(21_F) \cdot \texttt{AND}\big(V_{t+1}(21_F) \geqq V_{t+1}(2_F1), V(21_F) \geqq V_{t+1}(11_F1)\big) \\
&\quad + \frac{2\varepsilon}{3} V_{t+1}(12_F) \cdot \texttt{AND}\big(V_{t+1}(21_F) < V_{t+1}(2_F1), V_{t+1}(2_F1) \geqq V_{t+1}(11_F1)\big) \\
&\quad + \frac{2\varepsilon}{3} V_{t+1}(11_F1) \cdot \texttt{AND}\big(V_{t+1}(21_F) < V_{t+1}(11_F1), V_{t+1}(2_F1) < V_{t+1}(11_F1)\big) \\
&\quad + (1-\varepsilon) V_{t+1}(12), \\
V_t(21) &= U(21) + \frac{\varepsilon}{3} V(3) \cdot \texttt{IF}\big(V_{t+1}(3) \geqq V_{t+1}(12)\big) + \frac{\varepsilon}{3} V_{t+1}(21) \cdot \texttt{IF}\big(V_{t+1}(3) < V_{t+1}(12)\big) \\
&\quad + \frac{\varepsilon}{3} V_{t+1}(1_F2) \cdot \texttt{AND}\big(V_{t+1}(21_F) \geqq V_{t+1}(2_F1), V_{t+1}(21_F) \geqq V_{t+1}(11_F1)\big) \\
&\quad + \frac{\varepsilon}{3} V_{t+1}(2_F1) \cdot \texttt{AND}\big(V_{t+1}(21_F) < V_{t+1}(2_F1), V_{t+1}(2_F1) \geqq V_{t+1}(11_F1)\big) \\
&\quad + \frac{\varepsilon}{3} V_{t+1}(1_F11) \cdot \texttt{AND}\big(V_{t+1}(21_F) < V_{t+1}(11_F1), V_{t+1}(2_F1) < V_{t+1}(11_F1)\big) \\
&\quad + (1-\varepsilon) V_{t+1}(21),
\end{aligned}
$$

$$
\begin{aligned}
V_t(2_F1) \;&=\; U(2_F1) + \frac{\varepsilon}{3}V_{t+1}(3)\cdot \texttt{IF}\big(V_{t+1}(3) \geqq V_{t+1}(12)\big) + \frac{\varepsilon}{3}V_{t+1}(21)\cdot \texttt{IF}\big(V_{t+1}(3) < V_{t+1}(12)\big)\\
&\quad + \frac{\varepsilon}{3}V_{t+1}(1_F2)\cdot \texttt{AND}\big(V_{t+1}(21_F) \geqq V_{t+1}(2_F1), V_{t+1}(21_F) \geqq V_{t+1}(11_F1)\big)\\
&\quad + \frac{\varepsilon}{3}V_{t+1}(2_F1)\cdot \texttt{AND}\big(V_{t+1}(21_F) < V_{t+1}(2_F1), V_{t+1}(2_F1) \geqq V_{t+1}(11_F1)\big)\\
&\quad + \frac{\varepsilon}{3}V_{t+1}(1_F11)\cdot \texttt{AND}\big(V_{t+1}(21_F) < V_{t+1}(11_F1), V_{t+1}(2_F1) < V_{t+1}(11_F1)\big)\\
&\quad + (1-\varepsilon)V_{t+1}(21),\\
V_t(21_F) \;&=\; U(21_F) + \frac{\varepsilon}{3}V_{t+1}(3)\cdot \texttt{IF}\big(V_{t+1}(3) \geqq V_{t+1}(12)\big) + \frac{\varepsilon}{3}V_{t+1}(21)\cdot \texttt{IF}\big(V_{t+1}(3) < V_{t+1}(12)\big)\\
&\quad + \frac{\varepsilon}{3}V_{t+1}(1_F2)\cdot \texttt{AND}\big(V_{t+1}(21_F) \geqq V_{t+1}(2_F1), V_{t+1}(21_F) \geqq V_{t+1}(11_F1)\big)\\
&\quad + \frac{\varepsilon}{3}V_{t+1}(2_F1)\cdot \texttt{AND}\big(V_{t+1}(21_F) < V_{t+1}(2_F1), V_{t+1}(2_F1) \geqq V_{t+1}(11_F1)\big)\\
&\quad + \frac{\varepsilon}{3}V_{t+1}(1_F11)\cdot \texttt{AND}\big(V_{t+1}(21_F) < V_{t+1}(11_F1), V_{t+1}(2_F1) < V_{t+1}(11_F1)\big)\\
&\quad + (1-\varepsilon)V_{t+1}(21),\\
V_t(3) \;&=\; U(3) + \frac{2\varepsilon}{3}V_{t+1}(2_F1)\cdot \texttt{IF}\big(V_{t+1}(12_F) \geqq V_{t+1}(3_F)\big)\\
&\quad + \frac{2\varepsilon}{3}V_{t+1}(3_F)\cdot \texttt{IF}\big(V_{t+1}(12_F) < V_{t+1}(3_F)\big) + (1-\varepsilon)V_{t+1}(3),\\
V_t(3_F) \;&=\; U(3_F) + \frac{2\varepsilon}{3}V_{t+1}(2_F1)\cdot \texttt{IF}\big(V_{t+1}(12_F) \geqq V_{t+1}(3_F)\big)\\
&\quad + \frac{2\varepsilon}{3}V_{t+1}(3_F)\cdot \texttt{IF}\big(V_{t+1}(12_F) < V_{t+1}(3_F)\big) + (1-\varepsilon)V_{t+1}(3),
\end{aligned}
$$

where AND is a function such that for any two statements A and B, $\texttt{AND}(A,B) = 1$ if both A and B are true, and $\texttt{AND}(A,B) = 0$ otherwise.

Proof of Lemma 1. We can simplify some of the above equations.

$$
\begin{aligned}
V_t(1_F11) &= V_t(111) - U(111) + U(1_F11),\\
V_t(11_F1) &= V_t(111) - U(111) + U(11_F1),\\
V_t(12_F) &= V_t(12) - U(12) + U(12_F),\\
V_t(1_F2) &= V_t(12) - U(12) + U(1_F2),\\
V_t(2_F1) &= V_t(21) - U(21) + U(2_F1),\\
V_t(21_F) &= V_t(21) - U(21) + U(21_F),\\
V_t(3_F) &= V_t(3) - U(3) + U(3_F) = V_t(3).
\end{aligned}
$$

Hence (dropping the time subscript),

$$
V(21_F) - V(2_F1) \;=\; U(21_F) - U(2_F1) \;=\; \frac{\delta}{2} \;>\; 0.
$$

We also have

$$\begin{aligned} V(21_F) - V(11_F1) &= V(21) - U(21) + U(21_F) - V(111) + U(111) - U(11_F1) \\ &= V(21) - V(111) - \frac{2^{k-1}\delta}{1+2^k} + \frac{\delta}{2} - \frac{\delta}{6} \\ &= V(21) - V(111) + \frac{(1-2^{k-1})\delta}{3(1+2^k)}. \end{aligned}$$

Since $k > 1$, $V(21_F) \geqq V(11_F1)$ implies $V(21) > V(111)$. Similarly,

$$\begin{aligned} V(12_F) - V(3_F) &= V(12) - U(12) + U(12_F) - V(3) \\ &= V(12) - V(3) + \frac{2^{k-1}\delta}{1+2^k}. \end{aligned}$$

Thus, $V(12) \geqq V(3)$ implies $V(12_F) > V(3_F)$. □

In view of Lemma 1, we have the following classification of possible relationships of value functions.[17]

$V(21_F) \geqq V(11_F1)$ and $V(12) \geqq V(3)$	$V(21) \geqq V(111)$ and $V(12_F) \geqq V(3_F)$ must hold. (Lemma 1)
$V(21_F) \geqq V(11_F1)$ but $V(12) \leqq V(3)$	$V(21) \geqq V(111)$ and [either $V(12_F) \geqq V(3_F)$ or $V(12_F) \leqq V(3_F)$] (2 cases)
$V(21_F) \leqq V(11_F1)$ and $V(12) \geqq V(3)$	[either $V(21) \geqq V(111)$ or $V(21) < V(111)$] and $V(12_F) \geqq V(3_F)$ (2 cases)
$V(21_F) \leqq V(11_F1)$ and $V(12) \leqq V(3)$	[either $V(21) \geqq V(111)$ or $V(21) \leqq V(111)$] and [either$V(12_F) \geqq V(3_F)$ or $V(12_F) \leqq V(3_F)$] (4 cases)

Table 4: Possible relationships of long-run values

Based on this classification, we have only nine possible strategies as candidates of symmetric, stationary equilibrium strategies as listed in Table 2.

A.2. Derivation of the $f1$-equilibrium

We only show the detailed computations for the equilibrium of the "Join-the-largest-without-F" strategy ($f1$). The derivation of the bounds for other strategies as well as the relationships among the bounds are given in Online Appendix.

Suppose that

$$V(21) \geqq V(111),\ V(21_F) \geqq V(11_F1),\ V(3) \geqq V(12),\ \text{and } V(12_F) \geqq V(3_F).$$

[17]Since a Nash equilibrium holds under weak inequalities, we did not partition the cases. Thus the "either" statement is not an exclusive one.

Then the equations that the value function satisfies (shown in the text and in Appendix A.1) become

$$\begin{aligned}
V(111) &= U(111) + \frac{2\varepsilon}{3}\left\{\frac{1}{2}V(21) + \frac{1}{2}V(12)\right\} + (1-\varepsilon)V(111), \\
V(1_F11) &= V(111) - U(111) + U(1_F11), \\
V(11_F1) &= V(111) - U(111) + U(11_F1), \\
V_(12) &= U(12) + \frac{2\varepsilon}{3}V(21_F) + (1-\varepsilon)V(12), \\
V(12_F) &= V(12) - U(12) + U(12_F), \\
V(1_F2) &= V(12) - U(12) + U(1_F2), \\
V(21) &= U(21) + \frac{\varepsilon}{3}V(3) + \frac{\varepsilon}{3}V(1_F2) + (1-\varepsilon)V(21), \\
V(2_F1) &= V(21) - U(21) + U(2_F1), \\
V(21_F) &= V(21) - U(21) + U(21_F), \\
V(3) &= U(3) + \frac{2\varepsilon}{3}V(2_F1) + (1-\varepsilon)V(3), \\
V(3_F) &= V(3).
\end{aligned}$$

We have the following solution for this system of equations:[18]

$$\begin{aligned}
V(21) &= \frac{4 - 2\delta\varepsilon + 11\cdot 2^{k-1} - 2^k\delta\varepsilon}{5(1+2^k)\varepsilon}, \\
V(2_F1) &= \frac{4 - 2\delta\varepsilon + 11\cdot 2^{k-1} - 7\cdot 2^{k-1}\delta\varepsilon}{5(1+2^k)\varepsilon}, \\
V(21_F) &= \frac{8 + \delta\varepsilon + 11\cdot 2^k - 2^{k+1}\delta\varepsilon}{10(1+2^k)\varepsilon}, \\
V(12) &= \frac{23 + \delta\varepsilon + 11\cdot 2^k - 2^{k+1}\delta\varepsilon}{15(1+2^k)\varepsilon}, \\
V(12_F) &= \frac{23 + \delta\varepsilon + 11\cdot 2^k + 13\cdot 2^k\delta\varepsilon}{15(1+2^k)\varepsilon}, \\
V(1_F2) &= \frac{23 - 14\delta\varepsilon + 11\cdot 2^k - 2^{k+1}\delta\varepsilon}{15(1+2^k)\varepsilon}, \\
V(111) &= \frac{10 - \delta\varepsilon + 17\cdot 2^{k-1} - 2^k\delta\varepsilon}{9(1+2^k)\varepsilon}, \\
V(1_F11) &= \frac{10 - 4\delta\varepsilon + 17\cdot 2^{k-1} - 2^{k+2}\delta\varepsilon}{9(1+2^k)\varepsilon}, \\
V(11_F1) &= \frac{20 + \delta\varepsilon + 17\cdot 2^k + 2^k\delta\varepsilon}{18(1+2^k)\varepsilon},
\end{aligned}$$

[18]We solved the system of equations in two ways: by using Mathematica for the recursive equation (6) and by hand. The detailed calculations (and the Mathematica codes) are available from the authors on request.

$$\begin{aligned} V(3) &= \frac{13 - 4\delta\varepsilon + 2^{k+4} - 7 \cdot 2^k \delta\varepsilon}{15(1+2^k)\varepsilon}, \\ V(3_F) &= \frac{13 - 4\delta\varepsilon + 2^{k+4} - 7 \cdot 2^k \delta\varepsilon}{15(1+2^k)\varepsilon}. \end{aligned}$$

Note that

$$\begin{aligned} V(21_F) - V(11_F1) &= \frac{8 + \delta\varepsilon + 11 \cdot 2^k - 2^{k+1}\delta\varepsilon}{10(1+2^k)\varepsilon} - \frac{20 + \delta\varepsilon + 17 \cdot 2^k + 2^k\delta\varepsilon}{18(1+2^k)\varepsilon} \\ &= \frac{-28 + 4\delta\varepsilon + 14 \cdot 2^k - 23 \cdot 2^k\delta\varepsilon}{90(1+2^k)\varepsilon}, \\ V(3) - V(12) &= \frac{13 - 4\delta\varepsilon + 2^{k+4} - 7 \cdot 2^k\delta\varepsilon}{15(1+2^k)\varepsilon} - \frac{23 + \delta\varepsilon + 11 \cdot 2^k - 2^{k+1}\delta\varepsilon}{15(1+2^k)\varepsilon} \\ &= \frac{-2 - \delta\varepsilon + 2^k - 2^k\delta\varepsilon}{3(1+2^k)\varepsilon}. \end{aligned}$$

Thus,

$$\begin{aligned} V(21_F) \geqq V(11_F1) &\quad \Leftrightarrow \quad \delta\varepsilon \leqq \frac{28(2^{k-1}-1)}{23 \cdot 2^k - 4}, \\ V(3) \geqq V(12) &\quad \Leftrightarrow \quad \delta\varepsilon \leqq \frac{2(2^{k-1}-1)}{2^k + 1}. \end{aligned}$$

Since

$$\frac{2}{2^k+1} - \frac{28}{23 \cdot 2^k - 4} \quad = \quad \frac{36(2^{k-1}-1)}{(2^k+1)(23 \cdot 2^k - 4} \ > \ 0,$$

we have

$$V(21_F) \geqq V(11_F1) \text{ and } V(2) \geqq V(12) \quad \Leftrightarrow \quad \delta\varepsilon \leqq \frac{28(2^{k-1}-1)}{23 \cdot 2^k - 4}.$$

Note also that

$$V(12_F) - V(3_F) = \frac{23 + \delta\varepsilon + 11 \cdot 2^k + 13 \cdot 2^k\delta\varepsilon}{15(1+2^k)\varepsilon} - \frac{13 - 4\delta\varepsilon + 2^{k+4} - 7 \cdot 2^k\delta\varepsilon}{15(1+2^k)\varepsilon} = \frac{2 + \delta - 2^k + 2^{k+2}\delta\varepsilon}{3(1+2^k)\varepsilon}.$$

Thus,

$$V(12_F) \geqq V(3_F) \quad \Leftrightarrow \quad \delta\varepsilon \geqq \frac{2(2^{k-1}-1)}{2^{k+2}+1}.$$

Since

$$\frac{28}{23 \cdot 2^k - 4} - \frac{2}{2^{k+2}+1} \quad = \quad \frac{33 \cdot 2^{k+1} + 36}{(23 \cdot 2^k - 4)(2^{k+2}+1)} \ > \ 0,$$

we have

$$\left. \begin{aligned} V(21) &\geqq V(111) \\ V(21_F) &\geqq V(11_F1) \\ V(3) &\geqq V(12) \\ V(12_F) &\geqq V(3_F) \end{aligned} \right\} \quad \Leftrightarrow \quad \underline{x}_1 \leqq \delta\varepsilon \leqq \overline{x}_1,$$

where

$$\underline{x}_1 := \frac{2(2^{k-1}-1)}{2^{k+2}+1},$$
$$\overline{x}_1 := \frac{28(2^{k-1}-1)}{23\cdot 2^k-4}.$$

Online Appendix to “Far-Sighted Clustering with Group-Size Effects and Reputations”

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OA.1. Details of Section 3 computations

In this section we provide detailed derivation of equilibrium conditions for strategies $f2$ to $f9$. For notational simplicity, we omit the subscripts of strategy names to V.

Strategy $f2$: Size-conscious strategy

Suppose that

$$V(21) \geqq V(111),\ V(21_F) \geqq V(11_F1),\ V(3) \geqq V(12),\ \text{and } V(12_F) \leqq V(3_F).$$

The state-transition matrix is

$$T_{f2} = \begin{array}{c} \\ 111 \\ 1_F11 \\ 11_F1 \\ 12 \\ 1_F2 \\ 12_F \\ 21 \\ 2_F1 \\ 21_F \\ 3 \\ 3_F \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 111 & 1_F11 & 11_F1 & 12 & 1_F2 & 12_F & 21 & 2_F1 & 21_F & 3 & 3_F \end{array} \\ \left[\begin{array}{ccccccccccc} 1-\varepsilon & 0 & 0 & \frac{\varepsilon}{3} & 0 & 0 & \frac{\varepsilon}{3} & 0 & 0 & 0 & 0 \\ 1-\varepsilon & 0 & 0 & \frac{\varepsilon}{3} & 0 & 0 & \frac{\varepsilon}{3} & 0 & 0 & 0 & 0 \\ 1-\varepsilon & 0 & 0 & \frac{\varepsilon}{3} & 0 & 0 & \frac{\varepsilon}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\varepsilon & 0 & 0 & 0 & 0 & \frac{2\varepsilon}{3} & 0 & 0 \\ 0 & 0 & 0 & 1-\varepsilon & 0 & 0 & 0 & 0 & \frac{2\varepsilon}{3} & 0 & 0 \\ 0 & 0 & 0 & 1-\varepsilon & 0 & 0 & 0 & 0 & \frac{2\varepsilon}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\varepsilon}{3} & 0 & 1-\varepsilon & 0 & 0 & \frac{\varepsilon}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{\varepsilon}{3} & 0 & 1-\varepsilon & 0 & 0 & \frac{\varepsilon}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{\varepsilon}{3} & 0 & 1-\varepsilon & 0 & 0 & \frac{\varepsilon}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\varepsilon & \frac{2\varepsilon}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\varepsilon & \frac{2\varepsilon}{3} \end{array}\right] \end{array}.$$

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Alternatively, we can explicitly write the optimal value functions as follows.

$$\begin{aligned}
V(111) &= U(111) + \frac{2\varepsilon}{3}\left\{\frac{1}{2}V(21) + \frac{1}{2}V(12)\right\} + (1-\varepsilon)V(111), \\
V(1_F11) &= V(111) - U(111) + U(1_F11), \\
V(11_F1) &= V(111) - U(111) + U(11_F1), \\
V(12) &= U(12) + \frac{2\varepsilon}{3}V(21_F) + (1-\varepsilon)V(12), \\
V(12_F) &= V(12) - U(12) + U(12_F), \\
V(1_F2) &= V(12) - U(12) + U(1_F2), \\
V(21) &= U(21) + \frac{\varepsilon}{3}V(3) + \frac{\varepsilon}{3}V(1_F2) + (1-\varepsilon)V(21), \\
V(2_F1) &= V(21) - U(21) + U(2_F1), \\
V(21_F) &= V(21) - U(21) + U(21_F), \\
V(3) &= U(3) + \frac{2\varepsilon}{3}V(3_F) + (1-\varepsilon)V(3), \\
V(3_F) &= V(3).
\end{aligned}$$

By computation, we have the following solution:

$$\begin{aligned}
V(21) &= \frac{6 - 2\delta\varepsilon + 15\cdot 2^{k-1}}{7(1+2^k)\varepsilon}, \\
V(2_F1) &= \frac{6 - 2\delta\varepsilon + 15\cdot 2^{k-1} - 7\cdot 2^{k-1}\delta\varepsilon}{7(1+2^k)\varepsilon}, \\
V(21_F) &= \frac{12 + 3\delta\varepsilon + 15\cdot 2^{k}}{14(1+2^k)\varepsilon}, \\
V(12) &= \frac{11 + \delta\varepsilon + 5\cdot 2^{k}}{7(1+2^k)\varepsilon}, \\
V(12_F) &= \frac{11 + \delta\varepsilon + 5\cdot 2^{k} + 7\cdot 2^k\delta\varepsilon}{7(1+2^k)\varepsilon}, \\
V(1_F2) &= \frac{11 - 6\delta\varepsilon + 5\cdot 2^{k}}{7(1+2^k)\varepsilon}, \\
V(111) &= \frac{24 - \delta\varepsilon + 39\cdot 2^{k-1}}{21(1+2^k)\varepsilon}, \\
V(1_F11) &= \frac{24 - 8\delta\varepsilon + 39\cdot 2^{k-1} - 7\cdot 2^k\delta\varepsilon}{21(1+2^k)\varepsilon}, \\
V(11_F1) &= \frac{48 + 5\delta\varepsilon + 39\cdot 2^{k} + 7\cdot 2^k\delta\varepsilon}{42(1+2^k)\varepsilon}, \\
V(3) &= \frac{1}{\varepsilon}, \\
V(3_F) &= \frac{1}{\varepsilon}.
\end{aligned}$$

Note that

$$V(21_F) - V(11_F1) = \frac{12+3\delta\varepsilon+15\cdot 2^k}{14(1+2^k)\varepsilon} - \frac{48+5\delta\varepsilon+39\cdot 2^k}{42(1+2^k)\varepsilon} = \frac{-12+4\delta\varepsilon+6\cdot 2^k - 7\cdot 2^k\delta\varepsilon}{42(1+2^k)\varepsilon},$$

$$V(3_F) - V(12_F) = \frac{1}{\varepsilon} - \frac{11+\delta\varepsilon+5\cdot 2^k+7\cdot 2^k\delta\varepsilon}{7(1+2^k)\varepsilon} = \frac{-4-\delta\varepsilon+2^{k+1}-7\cdot 2^k\delta\varepsilon}{7(1+2^k)\varepsilon}.$$

Thus,

$$V(21_F) \geqq V(11_F1) \quad \Leftrightarrow \quad \delta\varepsilon \leqq \frac{12(2^{k-1}-1)}{7\cdot 2^k - 4},$$

$$V(12_F) \leqq V(3_F) \quad \Leftrightarrow \quad \delta\varepsilon \leqq \frac{4(2^{k-1}-1)}{7\cdot 2^k + 1}.$$

Since

$$\frac{12}{7\cdot 2^k - 4} - \frac{4}{7\cdot 2^k + 1} = \frac{28(2^{k+1}+1)}{(7\cdot 2^k-4)(7\cdot 2^k+1)} > 0,$$

we have

$$\left.\begin{array}{rcl} V(21) & \geqq & V(111) \\ V(21_F) & \geqq & V(11_F1) \\ V(3) & \geqq & V(12) \\ V(12_F) & \leqq & V(3_F) \end{array}\right\} \Leftrightarrow \quad \delta\varepsilon \leqq \overline{x}_2,$$

where

$$\overline{x}_2 := \frac{2(2^k-2)}{7\cdot 2^k+1}.$$

Strategy $f3$: Join-the-smallest strategy

Suppose that

$$V(21) \geqq V(111),\ V(21_F) \geqq V(11_F1),\ V(3) \leqq V(12),\ \text{and } V(12_F) \geqq V(3_F).$$

Then

$$V(111) = U(111) + \frac{2\varepsilon}{3}\left\{\frac{1}{2}V(21) + \frac{1}{2}V(12)\right\} + (1-\varepsilon)V(111),$$

$$V(1_F11) = V(111) - U(111) + U(1_F11),$$

$$V(11_F1) = V(111) - U(111) + U(11_F1),$$

$$V(12) = U(12) + \frac{2\varepsilon}{3}V(21_F) + (1-\varepsilon)V(12),$$

$$V(12_F) = V(12) - U(12) + U(12_F),$$

$$V(1_F2) = V(12) - U(12) + U(1_F2),$$

$$V(21) = U(21) + \frac{\varepsilon}{3}V(21) + \frac{\varepsilon}{3}V(1_F2) + (1-\varepsilon)V(21),$$

$$V(2_F1) = V(21) - U(21) + U(2_F1),$$

$$V(21_F) = V(21) - U(21) + U(21_F),$$

$$
\begin{aligned}
V(3) &= U(3) + \frac{2\varepsilon}{3}V(2_F1) + (1-\varepsilon)V(3),\\
V(3_F) &= V(3).
\end{aligned}
$$

Then we have the following solution:

$$
\begin{aligned}
V(21) &= \frac{3 - 2\delta\varepsilon + 9\cdot 2^{k-1}}{4(1+2^k)\varepsilon},\\
V(2_F1) &= \frac{3 - 2\delta\varepsilon + 9\cdot 2^{k-1} - 2^{k+1}\delta\varepsilon}{4\varepsilon(1+2^k)},\\
V(21_F) &= \frac{3 + 9\cdot 2^{k-1}}{4(1+2^k)\varepsilon},\\
V(12) &= \frac{3 + 3\cdot 2^{k-1}}{2(1+2^k)\varepsilon},\\
V(12_F) &= \frac{3 - 2\delta\varepsilon + 3\cdot 2^{k-1}}{2(1+2^k)\varepsilon},\\
V(111) &= \frac{13 - 2\delta\varepsilon + 23\cdot 2^{k-1}}{12(1+2^k)\varepsilon},\\
V(1_F11) &= \frac{13 - 6\delta\varepsilon + 23\cdot 2^{k-1} - 2^{k+2}\delta\varepsilon}{12(1+2^k)\varepsilon},\\
V(11_F1) &= \frac{13 + 23\cdot 2^{k-1} + 2^{k+1}\delta\varepsilon}{12(1+2^k)\varepsilon},\\
V(3) &= \frac{5 - 2\delta\varepsilon + 13\cdot 2^{k-1} - 2^{k+1}\delta\varepsilon}{6(1+2^k)\varepsilon},\\
V(3_F) &= \frac{5 - 2\delta\varepsilon + 13\cdot 2^{k-1} - 2^{k+1}\delta\varepsilon}{6(1+2^k)\varepsilon}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
V(21_F) - V(11_F1) &= \frac{3 + 9\cdot 2^{k-1}}{4(1+2^k)\varepsilon} - \frac{13 + 23\cdot 2^{k-1} + 2^{k+1}\delta\varepsilon}{12(1+2^k)\varepsilon} = \frac{-1 + 2^{k-1} - 2^{k-1}\delta\varepsilon}{3(1+2^k)\varepsilon},\\
V(12) - V(3) &= \frac{3 + 3\cdot 2^{k-1}}{2(1+2^k)\varepsilon} - \frac{5 - 2\delta\varepsilon + 13\cdot 2^{k-1} - 2^{k+1}\delta\varepsilon}{6(1+2^k)\varepsilon}\\
&= \frac{2 + \delta\varepsilon - 2^k + 2^k\delta\varepsilon}{3(1+2^k)\varepsilon}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
V(21_F) \geqq V(11_F1) &\Leftrightarrow \delta\varepsilon \leqq \frac{2^{k-1}-1}{2^{k-1}},\\
V(3) \leqq V(12) &\Leftrightarrow \delta\varepsilon \geqq \frac{2(2^{k-1}-1)}{2^k+1}.
\end{aligned}
$$

Since

$$
\frac{1}{2^{k-1}} - \frac{2}{2^k+1} = \frac{1}{2^{k-1}(2^k+1)} > 0,
$$

we have

$$\left.\begin{array}{rcl} V(21) & \geqq & V(111) \\ V(21_F) & \geqq & V(11_F1) \\ V(3) & \leqq & V(12) \\ V(12_F) & \geqq & V(3_F) \end{array}\right\} \Leftrightarrow \quad \underline{x}_3 \leqq \delta\varepsilon \leqq \overline{x}_3,$$

where

$$\begin{aligned} \underline{x}_3 &:= \frac{2(2^{k-1}-1)}{2^k+1}, \\ \overline{x}_3 &:= \frac{2^{k-1}-1}{2^{k-1}}. \end{aligned}$$

Strategy $f4$: Weakly entrepreneurial strategy

Suppose that

$$V(21) \geqq V(111),\ V(21_F) \leqq V(11_F1),\ V(3) \geqq V(12),\ \text{and } V(12_F) \geqq V(3_F).$$

Then

$$\begin{aligned} V(111) &= U(111) + \frac{2\varepsilon}{3}\left\{\frac{1}{2}V(21) + \frac{1}{2}V(12)\right\} + (1-\varepsilon)V(111), \\ V(1_F11) &= V(111) - U(111) + U(1_F11), \\ V(11_F1) &= V(111) - U(111) + U(11_F1), \\ V(12) &= U(12) + \frac{2\varepsilon}{3}V(11_F1) + (1-\varepsilon)V(12), \\ V(12_F) &= V(12) - U(12) + U(12_F), \\ V(1_F2) &= V(12) - U(12) + U(1_F2), \\ V(21) &= U(21) + \frac{\varepsilon}{3}V(3) + \frac{\varepsilon}{3}V(1_F11) + (1-\varepsilon)V(21), \\ V(2_F1) &= V(21) - U(21) + U(2_F1), \\ V(21_F) &= V(21) - U(21) + U(21_F), \\ V(3) &= U(3) + \frac{2\varepsilon}{3}V(2_F1) + (1-\varepsilon)V(3), \\ V(3_F) &= V(3). \end{aligned}$$

Then we have the following solution:

$$\begin{aligned} V(21) &= \frac{25 + 95 \cdot 2^{k-1} - 6\delta\varepsilon - 13 \cdot 2^k\delta\varepsilon}{40(1+2^k)\varepsilon}, \\ V(2_F1) &= \frac{25 + 95 \cdot 2^{k-1} - 6\delta\varepsilon - 33 \cdot 2^k\delta\varepsilon}{40(1+2^k)\varepsilon}, \\ V(21_F) &= \frac{25 + 95 \cdot 2^{k-1} + 14\delta\varepsilon - 13 \cdot 2^k\delta\varepsilon}{40(1+2^k)\varepsilon}, \end{aligned}$$

$$
\begin{aligned}
V(111) &= \frac{135 + 225 \cdot 2^{k-1} - 2\delta\varepsilon - 11 \cdot 2^k \delta\varepsilon}{120(1+2^k)\varepsilon}, \\
V(1_F 11) &= \frac{45 + 75 \cdot 2^{k-1} - 14\delta\varepsilon - 17 \cdot 2^k \delta\varepsilon}{40(1+2^k)\varepsilon}, \\
V(11_F 1) &= \frac{45 + 75 \cdot 2^{k-1} + 6\delta\varepsilon + 3 \cdot 2^k \delta\varepsilon}{40(1+2^k)\varepsilon}, \\
V(12) &= \frac{35 + 25 \cdot 2^{k-1} + 2\delta\varepsilon + 2^k \delta\varepsilon}{20(1+2^k)\varepsilon}, \\
V(12_F) &= \frac{35 + 25 \cdot 2^{k-1} + 2\delta\varepsilon + 21 \cdot 2^k \delta\varepsilon}{20(1+2^k)\varepsilon}, \\
V(1_F 2) &= \frac{35 + 25 \cdot 2^{k-1} - 18\delta\varepsilon + 2^k \delta\varepsilon}{20(1+2^k)\varepsilon}, \\
V(3) &= \frac{15 + 45 \cdot 2^{k-1} - 2\delta\varepsilon - 11 \cdot 2^k \delta\varepsilon}{20(1+2^k)\varepsilon}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
V(21) - V(111) &= \frac{25 + 95 \cdot 2^{k-1} - 6\delta\varepsilon - 13 \cdot 2^k \delta\varepsilon}{40(1+2^k)\varepsilon} - \frac{135 + 225 \cdot 2^{k-1} - 2\delta\varepsilon - 11 \cdot 2^k \delta\varepsilon}{120(1+2^k)\varepsilon} \\
&= \frac{-15 + 15 \cdot 2^{k-1} - 4\delta\varepsilon - 7 \cdot 2^k \delta\varepsilon}{30(1+2^k)\varepsilon}, \\
V(11_F 1) - V(21_F) &= \frac{45 + 75 \cdot 2^{k-1} + 6\delta\varepsilon + 3 \cdot 2^k \delta\varepsilon}{40(1+2^k)\varepsilon} - \frac{25 + 95 \cdot 2^{k-1} + 14\delta\varepsilon - 13 \cdot 2^k \delta\varepsilon}{40(1+2^k)\varepsilon} \\
&= \frac{5 - 5 \cdot 2^{k-1} - 2\delta\varepsilon + 2^{k+2} \delta\varepsilon}{10(1+2^k)\varepsilon}, \\
V(3) - V(12) &= \frac{15 + 45 \cdot 2^{k-1} - 2\delta\varepsilon - 11 \cdot 2^k \delta\varepsilon}{20(1+2^k)\varepsilon} - \frac{35 + 25 \cdot 2^{k-1} + 2\delta\varepsilon + 2^k \delta\varepsilon}{20(1+2^k)\varepsilon} \\
&= \frac{5(-1 + 2^{k-1}) - (1 + 3 \cdot 2^k)\delta\varepsilon}{5(1+2^k)\varepsilon}, \\
V(12_F) - V(3_F) &= \frac{35 + 25 \cdot 2^{k-1} + 2\delta\varepsilon + 21 \cdot 2^k \delta\varepsilon}{20(1+2^k)\varepsilon} - \frac{15 + 45 \cdot 2^{k-1} - 2\delta\varepsilon - 11 \cdot 2^k \delta\varepsilon}{20(1+2^k)\varepsilon} \\
&= \frac{5(1 - 2^{k-1}) + (1 + 2^{k+3})\delta\varepsilon}{5(1+2^k)\varepsilon}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
V(21) \geqq V(111) &\Leftrightarrow \delta\varepsilon \leqq \frac{15 \cdot (2^{k-1} - 1)}{7 \cdot 2^k + 4}, \\
V(11_F 1) \geqq V(21_F) &\Leftrightarrow \delta\varepsilon \geqq \frac{5(2^{k-1} - 1)}{2^{k+2} - 2}, \\
V(3) \geqq V(12) &\Leftrightarrow \delta\varepsilon \leqq \frac{5(2^{k-1} - 1)}{3 \cdot 2^k + 1}, \\
V(12_F) - V(3_F) &\Leftrightarrow \delta\varepsilon \geqq \frac{5(2^{k-1} - 1)}{2^{k+3} + 1}.
\end{aligned}
$$

Since

$$\begin{aligned}
\frac{15(2^{k-1}-1)}{7\cdot 2^k+4}-\frac{5(2^{k-1}-1)}{3\cdot 2^k+1} &= \frac{(2^{k+1}-1)(2^{k-1}-1)}{(7\cdot 2^k+4)(3\cdot 2^k+1)}>0,\\
\frac{5(2^{k-1}-1)}{2^{k+2}-2}-\frac{5(2^{k-1}-1)}{2^{k+3}+1} &= \frac{5(2^{k+2}+3)(2^{k-1}-1)}{(2^{k+2}-2)(2^{k+3}+1)}>0,\\
\frac{5(2^{k-1}-1)}{3\cdot 2^k+1}-\frac{5(2^{k-1}-1)}{2^{k+2}-2} &= \frac{5(2^k-3)(2^{k-1}-1)}{(3\cdot 2^k+1)(2^{k+2}-2)},
\end{aligned}$$

we have

$$\left.\begin{array}{rcl}
V(21) &\geqq& V(111)\\
V(21_F) &\leqq& V(11_F1)\\
V(3) &\geqq& V(12)\\
V(12_F) &\geqq& V(3_F)
\end{array}\right\} \Leftrightarrow \quad 2^k\geqq 3 \text{ and } \underline{x}_4\leqq \delta\varepsilon\leqq \overline{x}_4,$$

where

$$\begin{aligned}
\underline{x}_4 &:= \frac{5(2^{k-1}-1)}{2^{k+2}-2},\\
\overline{x}_4 &:= \frac{5(2^{k-1}-1)}{3\cdot 2^k+1}.
\end{aligned}$$

Strategy $f5$: Intermediately size-conscious strategy

Suppose that

$$V(21)\geqq V(111),\ V(21_F)\leqq V(11_F1),\ V(3)\geqq V(12),\ \text{and } V(12_F)\leqq V(3_F).$$

Then

$$\begin{aligned}
V(111) &= U(111)+\frac{2\varepsilon}{3}\left\{\frac{1}{2}V(21)+\frac{1}{2}V(12)\right\}+(1-\varepsilon)V(111),\\
V(1_F11) &= V(111)-U(111)+U(1_F11),\\
V(11_F1) &= V(111)-U(111)+U(11_F1),\\
V(12) &= U(12)+\frac{2\varepsilon}{3}V(11_F1)+(1-\varepsilon)V(12),\\
V(12_F) &= V(12)-U(12)+U(12_F),\\
V(1_F2) &= V(12)-U(12)+U(1_F2),\\
V(21) &= U(21)+\frac{\varepsilon}{3}V(3)+\frac{\varepsilon}{3}V(1_F11)+(1-\varepsilon)V(21),\\
V(2_F1) &= V(21)-U(21)+U(2_F1),\\
V(21_F) &= V(21)-U(21)+U(21_F),\\
V(3) &= U(3)+\frac{2\varepsilon}{3}V(3_F)+(1-\varepsilon)V(3),\\
V(3_F) &= V(3)-U(3)+U(3_F)=V_t(3).
\end{aligned}$$

Then we have the following solution:

$$\begin{aligned}
V(111) &= \frac{7+11\cdot 2^{k-1}}{6(1+2^k)\varepsilon},\\
V(1_F11) &= \frac{7+11\cdot 2^{k-1}-2(1+2^k)\delta\varepsilon}{6(1+2^k)\varepsilon},\\
V(11_F1) &= \frac{7+11\cdot 2^{k-1}+(1+2^k)\delta\varepsilon}{6(1+2^k)\varepsilon},\\
V(12) &= \frac{16+11\cdot 2^{k-1}+(1+2^k)\delta\varepsilon}{9(1+2^k)\varepsilon},\\
V(12_F) &= \frac{16+11\cdot 2^{k-1}+(1+5\cdot 2^{k+1})\delta\varepsilon}{9(1+2^k)\varepsilon},\\
V(1_F2) &= \frac{16+11\cdot 2^{k-1}+(-8+2^k)\delta\varepsilon}{9(1+2^k)\varepsilon},\\
V(21) &= \frac{13+41\cdot 2^{k-1}-2(1+2^k)\delta\varepsilon}{18(1+2^k)\varepsilon},\\
V(2_F1) &= \frac{13+41\cdot 2^{k-1}-(2+11\cdot 2^k)\delta\varepsilon}{18(1+2^k)\varepsilon},\\
V(21_F) &= \frac{13+41\cdot 2^{k-1}+(7-2^{k+1})\delta\varepsilon}{18(1+2^k)\varepsilon},\\
V(3) &= V(3_F) = \frac{1}{\varepsilon}.
\end{aligned}$$

Note that

$$\begin{aligned}
V(21)-V(111) &= \frac{13+41\cdot 2^{k-1}-2(1+2^k)\delta\varepsilon}{18(1+2^k)\varepsilon} - \frac{7+11\cdot 2^{k-1}}{6(1+2^k)\varepsilon} = \frac{-4+2^{k+1}-(1+2^k)\delta\varepsilon}{9(1+2^k)\varepsilon},\\
V(11_F1)-V(21_F) &= \frac{7+11\cdot 2^{k-1}+(1+2^k)\delta\varepsilon}{6(1+2^k)\varepsilon} - \frac{13+41\cdot 2^{k-1}+(7-2^{k+1})\delta\varepsilon}{18(1+2^k)\varepsilon}\\
&= \frac{8-8\cdot 2^{k-1}+(-4+5\cdot 2^k)\delta\varepsilon}{18(1+2^k)\varepsilon},\\
V(3_F)-V(12_F) &= \frac{1}{\varepsilon} - \frac{16+11\cdot 2^{k-1}+(1+5\cdot 2^{k+1})\delta\varepsilon}{9(1+2^k)\varepsilon} = \frac{-7+7\cdot 2^{k-1}-(1+5\cdot 2^{k+1})\delta\varepsilon}{9(1+2^k)\varepsilon}.
\end{aligned}$$

Thus,

$$\begin{aligned}
V(21) \geqq V(111) &\Leftrightarrow \delta\varepsilon \leqq \frac{4(2^{k-1}-1)}{2^k+1},\\
V(11_F1) \geqq V(21_F) &\Leftrightarrow \delta\varepsilon \geqq \frac{8(2^{k-1}-1)}{5\cdot 2^k-4},\\
V(3_F) \geqq V(12_F) &\Leftrightarrow \delta\varepsilon \leqq \frac{7(2^{k-1}-1)}{5\cdot 2^{k+1}+1}.
\end{aligned}$$

Note that

$$\frac{8(2^{k-1}-1)}{5\cdot 2^k-4} - \frac{7(2^{k-1}-1)}{5\cdot 2^{k+1}+1} = \frac{9(5\cdot 2^k+4)(2^{k-1}-1)}{(5\cdot 2^k-4)(5\cdot 2^{k+1}+1)} > 0.$$

Thus, $V(11_F1) \geqq V(21_F)$ and $V(3_F) \geqq V(12_F)$ are incompatible. This means that there is no symmetric, stationary equilibrium that consists of strategy $f5$.

Strategy $f6$: Intermediately entrepreneurial strategy

Suppose that

$$V(21) \geqq V(111),\ V(21_F) \leqq V(11_F1),\ V(3) \leqq V(12),\ \text{and } V(12_F) \geqq V(3_F).$$

Then

$$\begin{aligned}
V(111) &= U(111) + \frac{2\varepsilon}{3}\left\{\frac{1}{2}V(21) + \frac{1}{2}V(12)\right\} + (1-\varepsilon)V(111),\\
V(1_F11) &= V(111) - U(111) + U(1_F11),\\
V(11_F1) &= V(111) - U(111) + U(11_F1),\\
V(12) &= U(12) + \frac{2\varepsilon}{3}V(11_F1) + (1-\varepsilon)V(12),\\
V(12_F) &= V(12) - U(12) + U(12_F),\\
V(1_F2) &= V(12) - U(12) + U(1_F2),\\
V(21) &= U(21) + \frac{\varepsilon}{3}V(21) + \frac{\varepsilon}{3}V(1_F11) + (1-\varepsilon)V(21),\\
V(2_F1) &= V(21) - U(21) + U(2_F1),\\
V(21_F) &= V(21) - U(21) + U(21_F),\\
V(3) &= U(3) + \frac{2\varepsilon}{3}V(2_F1) + (1-\varepsilon)V(3),\\
V(3_F) &= V(3).
\end{aligned}$$

Then we have the following solution:

$$\begin{aligned}
V(111) &= \frac{36 + 63\cdot 2^{k-1} - (1+2^k)\delta\varepsilon}{33(1+2^k)\varepsilon},\\
V(1_F11) &= \frac{12 + 21\cdot 2^{k-1} - 4(1+2^k)\delta\varepsilon}{11(1+2^k)\varepsilon},\\
V(11_F1) &= \frac{24 + 21\cdot 2^{k} + 3(1+2^k)\delta\varepsilon}{22(1+2^k)\varepsilon},\\
V(12) &= \frac{19 + 7\cdot 2^{k} + (1+2^k)\delta\varepsilon}{11(1+2^k)\varepsilon},\\
V(12_F) &= \frac{19 + 7\cdot 2^{k} + (1+3\cdot 2^{k+2})\delta\varepsilon}{11(1+2^k)\varepsilon},\\
V(1_F2) &= \frac{19 + 7\cdot 2^{k} + 2(-5+\cdot 2^{k-1})\delta\varepsilon}{11(1+2^k)\varepsilon},\\
V(21) &= \frac{6 + 27\cdot 2^{k-1} - 2(1+2^k)\delta\varepsilon}{11(1+2^k)\varepsilon},\\
V(2_F1) &= \frac{6 + 27\cdot 2^{k-1} - (2+15\cdot 2^{k-1})\delta\varepsilon}{11(1+2^k)\varepsilon},\\
V(21_F) &= \frac{12 + 27\cdot 2^{k} + (7-2^{k+2})\delta\varepsilon}{22(1+2^k)\varepsilon},
\end{aligned}$$

$$
\begin{aligned}
V(3) &= \frac{23 + 19\cdot 2^{k+1} - (4 + 15\cdot 2^k)\delta\varepsilon}{33(1+2^k)\varepsilon}, \\
V(3_F) &= \frac{23 + 19\cdot 2^{k+1} - (4 + 15\cdot 2^k)\delta\varepsilon}{33(1+2^k)\varepsilon}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
V(21) - V(111) &= \frac{6 + 27\cdot 2^{k-1} - 2(1+2^k)\delta\varepsilon}{11(1+2^k)\varepsilon} - \frac{36 + 63\cdot 2^{k-1} - (1+2^k)\delta\varepsilon}{33(1+2^k)\varepsilon} \\
&= \frac{18(-1+2^{k-1}) - 5(1+2^k)\delta\varepsilon}{33(1+2^k)\varepsilon}, \\
V(11_F1) - V(21_F) &= \frac{24 + 21\cdot 2^k + 3(1+2^k)\delta\varepsilon}{22(1+2^k)\varepsilon} - \frac{12 + 27\cdot 2^k + (7 - 2^{k+2})\delta\varepsilon}{22(1+2^k)\varepsilon} \\
&= \frac{12(1-2^{k-1}) + (-4 + 7\cdot 2^k)\delta\varepsilon}{22(1+2^k)\varepsilon}, \\
V(12) - V(3) &= \frac{19 + 7\cdot 2^k + (1+2^k)\delta\varepsilon}{11(1+2^k)\varepsilon} - \frac{23 + 19\cdot 2^{k+1} - (4 + 15\cdot 2^k)\delta\varepsilon}{33(1+2^k)\varepsilon} \\
&= \frac{34(1-2^{k-1}) + (7 + 9\cdot 2^{k+1})\delta\varepsilon}{33(1+2^k)\varepsilon}, \\
\frac{6}{7\cdot 2^k - 4} - \frac{17}{9\cdot 2^{k+1}+7} &= \frac{22(5 - 2^{k-1})}{(7\cdot 2^k - 4)(9\cdot 2^{k+1}+7)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
V(21) \geqq V(111) \quad &\Leftrightarrow \quad \delta\varepsilon \leqq \frac{18(2^{k-1}-1)}{5(2^k+1)}, \\
V(11_F1) \geqq V(21_F) \quad &\Leftrightarrow \quad \delta\varepsilon \geqq \frac{12(2^{k-1}-1)}{7\cdot 2^k - 4}, \\
V(12) \geqq V(3) \quad &\Leftrightarrow \quad \delta\varepsilon \geqq \frac{34(2^{k-1}-1)}{9\cdot 2^{k+1}+7}, \\
\frac{12(2^{k-1}-1)}{7\cdot 2^k - 4} \geqq \frac{34(2^{k-1}-1)}{9\cdot 2^{k+1}+7} \quad &\Leftrightarrow \quad 5 \geqq 2^{k-1}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\frac{3}{5(2^k+1)} - \frac{2}{7\cdot 2^k - 4} &= \frac{22(2^{k-1}-1)}{5(2^k+1)(7\cdot 2^k - 4)} > 0, \\
\frac{9}{5(2^k+1)} - \frac{17}{9\cdot 2^{k+1}+7} &= \frac{22(7\cdot 2^{k-1}-1)}{5(2^k+1)(9\cdot 2^{k+1}+7)} > 0,
\end{aligned}
$$

we have

$$
\begin{aligned}
\frac{18(2^{k-1}-1)}{5(2^k+1)} &> \frac{12(2^{k-1}-1)}{7\cdot 2^k - 4}, \\
\frac{18(2^{k-1}-1)}{5(2^k+1)} &> \frac{34(2^{k-1}-1)}{9\cdot 2^{k+1}+7}.
\end{aligned}
$$

Thus,

$$
\left.
\begin{array}{rcl}
V(21) & \geqq & V(111) \\
V(21_F) & \leqq & V(11_F1) \\
V(3) & \leqq & V(12) \\
V(12_F) & \geqq & V(3_F)
\end{array}
\right\} \quad \Leftrightarrow \quad \underline{x}_6 \leqq \delta\varepsilon \leqq \overline{x}_6,
$$

where

$$\begin{aligned}
\underline{x}_6 &:= \max\left\{\frac{12(2^{k-1}-1)}{7\cdot 2^k-4}, \frac{34(2^{k-1}-1)}{9\cdot 2^{k+1}+7}\right\},\\
\overline{x}_6 &= \frac{18(2^{k-1}-1)}{5(2^k+1)}.
\end{aligned}$$

Strategy $f7$: Entrepreneurial with join-the-largest strategy

Suppose that

$$V(21) \leqq V(111),\ V(21_F) \leqq V(11_F1),\ V(3) \geqq V(12),\ \text{and } V(12_F) \geqq V(3_F).$$

Then

$$\begin{aligned}
V(111) &= U(111) + \frac{2\varepsilon}{3}V(111) + (1-\varepsilon)V(111),\\
V(1_F11) &= V(111) - U(111) + U(1_F11),\\
V(11_F1) &= V(111) - U(111) + U(11_F1),\\
V(12) &= U(12) + \frac{2\varepsilon}{3}V(11_F1) + (1-\varepsilon)V(12),\\
V(12_F) &= V(12) - U(12) + U(12_F),\\
V(1_F2) &= V(12) - U(12) + U(1_F2),\\
V(21) &= U(21) + \frac{\varepsilon}{3}V(3) + \frac{\varepsilon}{3}V(1_F11) + (1-\varepsilon)V(21),\\
V(2_F1) &= V(21) - U(21) + U(2_F1),\\
V(21_F) &= V(21) - U(21) + U(21_F),\\
V(3) &= U(3) + \frac{2\varepsilon}{3}V(2_F1) + (1-\varepsilon)V(3),\\
V(3_F) &= V(3).
\end{aligned}$$

Then we have the following solution:

$$\begin{aligned}
V(21) &= \frac{4-\delta\varepsilon+17\cdot 2^{k-1}-2^{k+1}\delta\varepsilon}{7(1+2^k)\varepsilon},\\
V(2_F1) &= \frac{4-\delta\varepsilon+17\cdot 2^{k-1}-11\cdot 2^{k-1}\delta\varepsilon}{7(1+2^k)\varepsilon},\\
V(21_F) &= \frac{8+5\delta\varepsilon+17\cdot 2^k-2^{k+3}\delta\varepsilon}{14(1+2^k)\varepsilon},\\
V(111) &= \frac{1}{\varepsilon},\\
V(1_F11) &= \frac{3-\delta\varepsilon}{3\varepsilon},\\
V(11_F1) &= \frac{6+\delta\varepsilon}{6\varepsilon},\\
V(12) &= \frac{15+\delta\varepsilon+3\cdot 2^{k+1}+2^k\delta\varepsilon}{9(1+2^k)\varepsilon},
\end{aligned}$$

$$
\begin{aligned}
V(12_F) &= \frac{15+\delta\varepsilon+3\cdot 2^{k+1}+5\cdot 2^{k+1}\delta\varepsilon}{9(1+2^k)\varepsilon},\\
V(1_F2) &= \frac{15-8\delta\varepsilon+3\cdot 2^{k+1}+2^k\delta\varepsilon}{9(1+2^k)\varepsilon},\\
V(3) &= \frac{15-2\delta\varepsilon+3\cdot 2^{k+3}-11\cdot 2^k\delta\varepsilon}{21(1+2^k)\varepsilon},\\
V(3_F) &= \frac{15-2\delta\varepsilon+3\cdot 2^{k+3}-11\cdot 2^k\delta\varepsilon}{21(1+2^k)\varepsilon}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
V(111)-V(21) &= \frac{1}{\varepsilon}-\frac{4-\delta\varepsilon+17\cdot 2^{k-1}-2^{k+1}\delta\varepsilon}{7(1+2^k)\varepsilon} = \frac{3+\delta\varepsilon-3\cdot 2^{k-1}+2^{k+1}\delta\varepsilon}{7(1+2^k)\varepsilon},\\
V(3)-V(12) &= \frac{15-2\delta\varepsilon+3\cdot 2^{k+3}-11\cdot 2^k\delta\varepsilon}{21(1+2^k)\varepsilon}-\frac{15+\delta\varepsilon+3\cdot 2^{k+1}+2^k\delta\varepsilon}{9(1+2^k)\varepsilon}\\
&= \frac{-60-13\delta\varepsilon+15\cdot 2^{k+1}-5\cdot 2^{k+3}\delta\varepsilon}{63(1+2^k)\varepsilon},\\
V(12_F)-V(3_F) &= \frac{15+\delta\varepsilon+3\cdot 2^{k+1}+5\cdot 2^{k+1}\delta\varepsilon}{9(1+2^k)\varepsilon}-\frac{15-2\delta\varepsilon+3\cdot 2^{k+3}-11\cdot 2^k\delta\varepsilon}{21(1+2^k)\varepsilon}\\
&= \frac{60+13\delta\varepsilon-15\cdot 2^{k+1}+103\cdot 2^k\delta\varepsilon}{63(1+2^k)\varepsilon}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
V(21)\leqq V(111) &\Leftrightarrow \delta\varepsilon \geqq \frac{3(2^{k-1}-1)}{2^{k+1}+1},\\
V(3)\geqq V(12) &\Leftrightarrow \delta\varepsilon \leqq \frac{60(2^{k-1}-1)}{5\cdot 2^{k+3}+13},\\
V(12_F)\geqq V(3_F) &\Leftrightarrow \delta\varepsilon \geqq \frac{60(2^{k-1}-1)}{103\cdot 2^k+13}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\frac{3}{2^{k+1}+1}-\frac{60}{103\cdot 2^k+13} &= \frac{3(63\cdot 2^k-7)}{(2^{k+1}+1)(103\cdot 2^k+13)} > 0,\\
\frac{60}{5\cdot 2^{k+3}+13}-\frac{3}{2^{k+1}+1} &= \frac{21}{(5\cdot 2^{k+3}+13)(2^{k+1}+1)} > 0,
\end{aligned}
$$

we have

$$
\left.\begin{array}{rcl}
V(21) &\leqq& V(111)\\
V(21_F) &\leqq& V(11_F1)\\
V(3) &\geqq& V(12)\\
V(12_F) &\geqq& V(3_F)
\end{array}\right\} \Leftrightarrow \underline{x}_7 \leqq \delta\varepsilon \leqq \overline{x}_7,
$$

where

$$
\begin{aligned}
\underline{x}_7 &:= \frac{3(2^{k-1}-1)}{2^{k+1}+1},\\
\overline{x}_7 &:= \frac{60(2^{k-1}-1)}{5\cdot 2^{k+3}+13}.
\end{aligned}
$$

Strategy $f8$: Small-competition but size-conscious strategy

Suppose that

$$V(21) \leqq V(111),\ V(21_F) \leqq V(11_F1),\ V(3) \geqq V(12),\ \text{and } V(12_F) \leqq V(3_F).$$

Then

$$\begin{aligned}
V(111) &= U(111) + \frac{2\varepsilon}{3}V(111) + (1-\varepsilon)V(111),\\
V(1_F11) &= V(111) - U(111) + U(1_F11),\\
V(11_F1) &= V(111) - U(111) + U(11_F1),\\
V(12) &= U(12) + \frac{2\varepsilon}{3}V(11_F1) + (1-\varepsilon)V(12),\\
V(12_F) &= V(12) - U(12) + U(12_F),\\
V(1_F2) &= V(12) - U(12) + U(1_F2),\\
V(21) &= U(21) + \frac{\varepsilon}{3}V(3) + \frac{\varepsilon}{3}V(1_F11) + (1-\varepsilon)V(21),\\
V(2_F1) &= V(21) - U(21) + U(2_F1),\\
V(21_F) &= V(21) - U(21) + U(21_F),\\
V(3) &= U(3) + \frac{2\varepsilon}{3}V(3_F) + (1-\varepsilon)V(3),\\
V(3_F) &= V(3).
\end{aligned}$$

Then we have the following solution:

$$\begin{aligned}
V(111) &= \frac{3-\delta\varepsilon}{3\varepsilon},\\
V(11_F1) &= \frac{6+\delta\varepsilon}{6\varepsilon},\\
V(12) &= \frac{15+\delta\varepsilon+3\cdot 2^{k+1}+2^k\delta\varepsilon}{9(1+2^k)\varepsilon},\\
V(12_F) &= \frac{15+\delta\varepsilon+3\cdot 2^{k+1}+5\cdot 2^{k+1}\delta\varepsilon}{9(1+2^k)\varepsilon},\\
V(1_F2) &= \frac{15-8\delta\varepsilon+3\cdot 2^{k+1}+2^k\delta\varepsilon}{9(1+2^k)\varepsilon},\\
V(3) &= \frac{1}{\varepsilon},\\
V(3_F) &= \frac{1}{\varepsilon},\\
V(21) &= \frac{6-\delta\varepsilon+21\cdot 2^{k-1}-2^k\delta\varepsilon}{9(1+2^k)\varepsilon},\\
V(2_F1) &= \frac{6-\delta\varepsilon+21\cdot 2^{k-1}-11\cdot 2^{k-1}\delta\varepsilon}{9(1+2^k)\varepsilon},\\
V(21_F) &= \frac{12+7\delta\varepsilon+21\cdot 2^k-2^{k+1}\delta\varepsilon}{18(1+2^k)\varepsilon}.
\end{aligned}$$

Note that

$$\begin{aligned} V(111) - V(21) &= \frac{1}{\varepsilon} - \frac{6 - \delta\varepsilon + 21 \cdot 2^{k-1} - 2^k \delta\varepsilon}{9(1+2^k)\varepsilon} \\ &= \frac{3 + \delta\varepsilon - 3 \cdot 2^{k-1} + 2^k \delta\varepsilon}{9(1+2^k)\varepsilon}, \\ V(3_F) - V(12_F) &= \frac{1}{\varepsilon} - \frac{15 + \delta\varepsilon + 3 \cdot 2^{k+1} + 5 \cdot 2^{k+1}\delta\varepsilon}{9(1+2^k)\varepsilon} \\ &= \frac{-6 - \delta\varepsilon + 3 \cdot 2^k - 5 \cdot 2^{k+1}\delta\varepsilon}{9(1+2^k)\varepsilon}. \end{aligned}$$

Thus,

$$\begin{aligned} V(21) \leqq V(111) \quad &\Leftrightarrow \quad \delta\varepsilon \geqq \frac{3(2^{k-1}-1)}{2^k+1}, \\ V(12_F) \leqq V(3_F) \quad &\Leftrightarrow \quad \delta\varepsilon \leqq \frac{6(2^{k-1}-1)}{5 \cdot 2^{k+1}+1}. \end{aligned}$$

Since

$$\frac{3}{2^k+1} - \frac{6}{5 \cdot 2^{k+1}+1} = \frac{3(2^{k+3}-1)}{(2^k+1)(5 \cdot 2^k - 4)} > 0,$$

$V(21) \leqq V(111)$ and $V(12_F) \leqq V(3_F)$ are not compatible. This means that there is no symmetric, stationary equilibrium that consists of strategy $f8$.

Strategy $f9$: Strongly entrepreneurial strategy

Suppose that

$$V(21) \leqq V(111), \; V(21_F) \leqq V(11_F1), \; V(3) \leqq V(12), \text{ and } V(12_F) \geqq V(3_F).$$

Then

$$\begin{aligned} V(111) &= U(111) + \frac{2\varepsilon}{3}V(111) + (1-\varepsilon)V(111), \\ V(1_F11) &= V(111) - U(111) + U(1_F11), \\ V(11_F1) &= V(111) - U(111) + U(11_F1), \\ V(12) &= U(12) + \frac{2\varepsilon}{3}V(11_F1) + (1-\varepsilon)V(12), \\ V(12_F) &= V(12) - U(12) + U(12_F), \\ V(1_F2) &= V(12) - U(12) + U(1_F2), \\ V(21) &= U(21) + \frac{\varepsilon}{3}V(21) + \frac{\varepsilon}{3}V(1_F11) + (1-\varepsilon)V(21), \\ V(2_F1) &= V(21) - U(21) + U(2_F1), \\ V(21_F) &= V(21) - U(21) + U(21_F), \\ V(3) &= U(3) + \frac{2\varepsilon}{3}V(2_F1) + (1-\varepsilon)V(3), \\ V(3_F) &= V(3). \end{aligned}$$

Then we have

$$
\begin{aligned}
V(111) &= \frac{1}{\varepsilon}, \\
V(1_F11) &= \frac{3-\delta\varepsilon}{3\varepsilon}, \\
V(11_F1) &= \frac{6+\delta\varepsilon}{6\varepsilon}, \\
V(12) &= \frac{15+\delta\varepsilon+3\cdot 2^{k+1}+2^k\delta\varepsilon}{9(1+2^k)\varepsilon}, \\
V(12_F) &= \frac{15+\delta\varepsilon+3\cdot 2^{k+1}+5\cdot 2^{k+1}\delta\varepsilon}{9(1+2^k)\varepsilon}, \\
V(1_F2) &= \frac{15+8\delta\varepsilon+3\cdot 2^{k+1}+2^k\delta\varepsilon}{9(1+2^k)\varepsilon}, \\
V(21) &= \frac{3-\delta\varepsilon+15\cdot 2^{k-1}-2^k\delta\varepsilon}{6(1+2^k)\varepsilon}, \\
V(2_F1) &= \frac{3-\delta\varepsilon+15\cdot 2^{k-1}-2^{k+2}\delta\varepsilon}{6(1+2^k)\varepsilon}, \\
V(21_F) &= \frac{3+2\delta\varepsilon+15\cdot 2^{k-1}-2^k\delta\varepsilon}{6(1+2^k)\varepsilon}, \\
V(3) &= \frac{6-\delta\varepsilon+21\cdot 2^{k-1}-2^{k+2}\delta\varepsilon}{9(1+2^k)\varepsilon}, \\
V(3_F) &= \frac{6-\delta\varepsilon+21\cdot 2^{k-1}-2^{k+2}\delta\varepsilon}{9(1+2^k)\varepsilon}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
V(111)-V(21) &= \frac{1}{\varepsilon}-\frac{3-\delta\varepsilon+15\cdot 2^{k-1}-2^k\delta\varepsilon}{6(1+2^k)\varepsilon} \\
&= \frac{3+\delta\varepsilon-3\cdot 2^{k-1}+2^k\delta\varepsilon}{6(1+2^k)\varepsilon}, \\
V(12)-V(3) &= \frac{15+\delta\varepsilon+3\cdot 2^{k+1}+2^k\delta\varepsilon}{9(1+2^k)\varepsilon}-\frac{6-\delta\varepsilon+21\cdot 2^{k-1}-2^{k+2}\delta\varepsilon}{9(1+2^k)\varepsilon} \\
&= \frac{9+2\delta\varepsilon-9\cdot 2^{k-1}+5\cdot 2^k\delta\varepsilon}{9(1+2^k)\varepsilon}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
V(21) \leqq V(111) \quad &\Leftrightarrow \quad \delta\varepsilon \geqq \frac{3(2^{k-1}-1)}{2^k-1}, \\
V(3) \leqq V(12) \quad &\Leftrightarrow \quad \delta\varepsilon \geqq \frac{9(2^{k-1}-1)}{5\cdot 2^k+2}.
\end{aligned}
$$

Since

$$
\frac{3}{2^k+1}-\frac{9}{5\cdot 2^k+2} \;=\; \frac{3(2^{k+1}-1)}{(2^k+1)(5\cdot 2^k+2)} \;>\; 0,
$$

we have

$$\left.\begin{array}{rcl} V(21) & \leqq & V(111) \\ V(21_F) & \leqq & V(11_F1) \\ V(3) & \leqq & V(12) \\ V(12_F) & \geqq & V(3_F) \end{array}\right\} \quad \Leftrightarrow \quad \delta\varepsilon \geqq \underline{x}_9,$$

where

$$\underline{x}_9 \;:=\; \frac{3(2^{k-1}-1)}{2^k+1}.$$

OA.2. Classification of the equilibrium bounds for all k

In this section, we classify the regions of $x = \delta \cdot \varepsilon$ for strategies $f1$, $f2$, $f3$, $f4$, $f6$, $f7$, and $f9$ to constitute an MPE, depending on the value of k.

By computation, some bounds are ordered regardless of k. In addition to those in Proposition 1, we have the following relationships for any $k > 1$.

$$\begin{aligned}
\underline{x}_7 - \overline{x}_2 &= \frac{3(2^{k-1}-1)}{2^{k+1}+1} - \frac{4(2^{k-1}-1)}{7\cdot 2^k+1} = \frac{(13\cdot 2^k-1)(2^{k-1}-1)}{(2^{k+1}+1)(7\cdot 2^k+1)} > 0,\\
\underline{x}_3 - \overline{x}_7 &= \frac{2(2^{k-1}-1)}{2^k+1} - \frac{60(2^{k-1}-1)}{5\cdot 2^{k+3}+13} = \frac{2(10\cdot 2^k-17)(2^{k-1}-1)}{(2^k+1)(5\cdot 2^{k+3}+13)} > 0,\\
\underline{x}_3 - \overline{x}_1 &= \frac{2(2^{k-1}-1)}{2^k+1} - \frac{28(2^{k-1}-1)}{23\cdot 2^k-4} = \frac{36(2^{k-1}-1)^2}{(2^k+1)(23\cdot 2^k-4)} > 0,\\
\underline{x}_4 - \overline{x}_1 &= \frac{5(2^{k-1}-1)}{2^{k+2}-2} - \frac{28(2^{k-1}-1)}{23\cdot 2^k-4} = \frac{(3\cdot 2^k+36)(2^{k-1}-1)}{(2^{k+2}-2)(23\cdot 2^k-4)} > 0,\\
\overline{x}_4 - \overline{x}_7 &= \frac{5(2^{k-1}-1)}{3\cdot 2^k+1} - \frac{60(2^{k-1}-1)}{5\cdot 2^{k+3}+13} = \frac{5(2^{k+2}+1)(2^{k-1}-1)}{(3\cdot 2^k+1)(5\cdot 2^{k+3}+3)} > 0,\\
\overline{x}_6 - \overline{x}_4 &= \frac{18(2^{k-1}-1)}{5(2^k+1)} - \frac{5}{3\cdot 2^k+1} = \frac{(29\cdot 2^k-7)(2^{k-1}-1)}{5(2^k+1)(3\cdot 2^k+1)} > 0,\\
\underline{x}_6 - \overline{x}_7 &\geqq \frac{34(2^{k-1}-1)}{9\cdot 2^{k+1}+7} - \frac{60(2^{k-1}-1)}{5\cdot 2^{k+3}+13} = \frac{(140\cdot 2^{k+1}+22)(2^{k-1}-1)}{(9\cdot 2^{k+1}+7)(5\cdot 2^{k+3}+13)} > 0,\\
\underline{x}_9 - \overline{x}_4 &= \frac{3(2^{k-1}-1)}{2^k+1} - \frac{5(2^{k-1}-1)}{3\cdot 2^k+1} = \frac{(2^{k+2}-2)(2^{k-1}-1)}{(2^k+1)(5\cdot 2^k+2)} > 0,\\
1 - \overline{x}_1 &= 1 - \frac{28(2^{k-1}-1)}{23\cdot 2^k-4} = \frac{9\cdot 2^k+24}{23\cdot 2^k-4} > 0,\\
\underline{x}_9 - \overline{x}_3 &= \frac{3(2^{k-1}-1)}{2^k+1} - \frac{(2^{k-1}-1)}{2^{k-1}} = \frac{(2^{k-1}-1)^2}{(2^k+1)2^{k-1}} > 0.
\end{aligned}$$

Note that we do not have clear orders among some bounds.

$$\begin{aligned}
\underline{x}_7 - \underline{x}_4 &= \frac{3(2^{k-1}-1)}{2^{k+1}+1} - \frac{5(2^{k-1}-1)}{2^{k+2}-2} = \frac{(2^{k+1}-11)(2^{k-1}-1)}{(2^{k+1}+1)(2^{k+2}-2)},\\
\underline{x}_3 - \overline{x}_4 &= \frac{2(2^{k-1}-1)}{2^k+1} - \frac{5(2^{k-1}-1)}{3\cdot 2^k+1} = \frac{(2^k-3)(2^{k-1}-1)}{(2^k+1)(3\cdot 2^k+1)},\\
\overline{x}_7 - \overline{x}_1 &= \frac{60(2^{k-1}-1)}{5\cdot 2^{k+3}+13} - \frac{28(2^{k-1}-1)}{23\cdot 2^k-4} = \frac{4(65\cdot 2^k-151)(2^{k-1}-1)}{(5\cdot 2^{k+3}+13)(23\cdot 2^k-4)},
\end{aligned}$$

$$\begin{aligned}
\underline{x}_7 - \overline{x}_1 &= \frac{3(2^{k-1}-1)}{2^{k+1}+1} - \frac{28(2^{k-1}-1)}{23\cdot 2^k - 4} = \frac{(13\cdot 2^k - 40)(2^{k-1}-1)}{(2^{k+1}+1)(23\cdot 2^k-4)}, \\
\overline{x}_7 - \underline{x}_4 &= \frac{60(2^{k-1}-1)}{5\cdot 2^{k+3}+13} - \frac{5(2^{k-1}-1)}{2^{k+2}-2} = \frac{5(2^{k+3}-37)(2^{k-1}-1)}{(5\cdot 2^{k+3}+13)(2^{k+2}-2)}, \\
\overline{x}_6 - 1 &= \frac{18(2^{k-1}-1)}{5(2^k+1)} - 1 = \frac{4\cdot 2^k - 23}{5(2^k+1)}, \\
\underline{x}_9 - 1 &= \frac{3(2^{k-1}-1)}{2^k+1} - 1 = \frac{2^{k-1}-4}{2^k+1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\underline{x}_7 \geqq \underline{x}_4 &\Leftrightarrow 2^k \geqq \frac{11}{2}, \\
\underline{x}_3 \geqq \overline{x}_4 &\Leftrightarrow 2^k \geqq 3, \\
\overline{x}_7 \geqq \overline{x}_1 &\Leftrightarrow 2^k \geqq \frac{151}{65}, \\
\underline{x}_7 \geqq \overline{x}_1 &\Leftrightarrow 2^k \geqq \frac{40}{13}, \\
\overline{x}_7 \geqq \underline{x}_4 &\Leftrightarrow 2^k \geqq \frac{37}{8}, \\
\overline{x}_6 \geqq 1 &\Leftrightarrow 2^k \geqq \frac{23}{4}, \\
\underline{x}_9 \geqq 1 &\Leftrightarrow 2^k \geqq 8.
\end{aligned}$$

Recall that

$$\underline{x}_6 = \max\left\{\frac{12(2^{k-1}-1)}{7\cdot 2^k-4}, \frac{34(2^{k-1}-1)}{9\cdot 2^{k+1}+7}\right\}.$$

To compare this with $\underline{x}_3$, note that

$$\begin{aligned}
\underline{x}_3 - \frac{12(2^{k-1}-1)}{7\cdot 2^k-4} &= \frac{2(2^{k-1}-1)}{2^k+1} - \frac{12(2^{k-1}-1)}{7\cdot 2^k-4} = \frac{4(2^{k-1}-5)(2^{k-1}-1)}{(7\cdot 2^k-4)(2^k+1)}, \\
\underline{x}_3 - \frac{34(2^{k-1}-1)}{9\cdot 2^{k+1}+7} &= \frac{2(2^{k-1}-1)}{2^k+1} - \frac{34(2^{k-1}-1)}{9\cdot 2^{k+1}+7} = \frac{4(2^{k-1}-5)(2^{k-1}-1)}{(2^k+1)(9\cdot 2^{k+1}+7)}.
\end{aligned}$$

Then

$$\begin{aligned}
\underline{x}_3 \geqq \frac{12(2^{k-1}-1)}{7\cdot 2^k-4} &\Leftrightarrow 2^k \geqq 10, \\
\underline{x}_3 \geqq \frac{34(2^{k-1}-1)}{9\cdot 2^{k+1}+7} &\Leftrightarrow 2^k \geqq 10.
\end{aligned}$$

Therefore we have

$$\underline{x}_3 \geqq \underline{x}_6 \Leftrightarrow 2^k \geqq 10.$$

Next, we compare $\underline{x}_6$ with $\underline{x}_9$ and $\overline{x}_3$. Notice that

$$\begin{aligned}
\underline{x}_9 - \frac{34(2^{k-1}-1)}{9\cdot 2^{k+1}+7} &= \frac{3(2^{k-1}-1)}{2^k+1} - \frac{34(2^{k-1}-1)}{9\cdot 2^{k+1}+7} = \frac{(5\cdot 2k+2-27)(2^{k-1}-1)}{(2^k+1)(9\cdot 2^{k+1}+7)} > 0 \\
\underline{x}_9 - \frac{12(2^{k-1}-1)}{7\cdot 2^k-4} &= \frac{3(2^{k-1}-1)}{2^k+1} - \frac{12(2^{k-1}-1)}{7\cdot 2^k-4} = \frac{3(3\cdot 2^k-8)(2^{k-1}-1)}{(2^k+1)(7\cdot 2^k-4)}, \\
\overline{x}_3 - \frac{12(2^{k-1}-1)}{7\cdot 2^k-4} &= \frac{2^{k-1}-1}{2^{k-1}} - \frac{12(2^{k-1}-1)}{7\cdot 2^k-4} = \frac{(2^k-4)(2^{k-1}-1)}{2^{k-1}(7\cdot 2^k-4)}, \\
\overline{x}_3 - \frac{34(2^{k-1}-1)}{9\cdot 2^{k+1}+7} &= \frac{2^{k-1}-1}{2^{k-1}} - \frac{34(2^{k-1}-1)}{9\cdot 2^{k+1}+7} = \frac{(2^k+7)(2^{k-1}-1)}{2^{k-1}(9\cdot 2^{k+1}+7)} > 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
\underline{x}_9 \geqq \underline{x}_6 &\Leftrightarrow 2^k \geqq \frac{8}{3}, \\
\overline{x}_3 \geqq \underline{x}_6 &\Leftrightarrow 2^k \geqq 4.
\end{aligned}$$

Finally, because

$$\begin{aligned}
1 - \frac{12(2^{k-1}-1)}{7\cdot 2^k-4} &= \frac{2^k+8}{7\cdot 2^k-4} > 0, \\
1 - \frac{34(2^k-1)}{9\cdot 2^{k+1}+7} &= \frac{2^k+41}{9\cdot 2^{k+1}+7} > 0,
\end{aligned}$$

we have that

$$1 > \underline{x}_6.$$

We now have all orders among various bounds.

OA.3 An example of non-stationary symmetric equilibrium

In this section we use the time subscripts, as V_t, for the value functions. Suppose that the equilibrium conditions of strategy $f1$ hold for six consecutive periods, starting in $t = 2$:

$$\begin{array}{llllllll}
V_2(21) \geqq V_2(111), & V_2(21_F) \geqq V_2(11_F1), & V_2(3) \geqq V_2(12), & V_2(12_F) \geq V_2(3_F), \\
V_3(21) \geqq V_3(111), & V_3(21_F) \geqq V_3(11_F1), & V_3(3) \geqq V_3(12), & V_3(12_F) \geqq V_3(3_F), \\
V_4(21) \geqq V_4(111), & V_4(21_F) \geqq V_4(11_F1), & V_4(3) \geqq V_4(12), & V_4(12_F) \geqq V_4(3_F), \\
V_5(21) \geqq V_5(111), & V_5(21_F) \geqq V_5(11_F1), & V_5(3) \geqq V_5(12), & V_5(12_F) \geqq V_5(3_F), \\
V_6(21) \geqq V_6(111), & V_6(21_F) \geqq V_6(11_F1), & V_6(3) \geqq V_6(12), & V_6(12_F) \geqq V_6(3_F), \\
V_7(21) \geqq V_7(111), & V_7(21_F) \geqq V_7(11_F1), & V_7(3) \geqq V_7(12), & V_7(12_F) \geqq V_7(3_F),
\end{array}$$

Suppose also that the equilibrium conditions of strategy $f4$ hold in the next period, the conditions of strategy $f1$ hold in the next 6 periods, and so on:

$$\begin{aligned}
V_1(21) = V_8(21) &\geqq V_8(111) = V_1(111), \\
V_1(21_F) = V_8(21_F) &\leqq V_8(11_F1) = V_1(11_F1), \\
V_1(3) = V_8(3) &\geqq V_8(12) = V_1(12), \\
V_1(12_F) = V_8(12_F) &\geqq V_8(3_F) = V_8(2_F).
\end{aligned}$$

Then the optimal value functions satisfy the following system of equations.

$$
\begin{aligned}
V_1(111) &= U(111) + \frac{2\varepsilon}{3}\left\{\frac{1}{2}V_2(21) + \frac{1}{2}V_2(12)\right\} + (1-\varepsilon)V_2(111),\\
V_1(12) &= U(12) + \frac{2\varepsilon}{3}\{V_2(21) - U(21) + U(21_F)\} + (1-\varepsilon)V_2(12),\\
V_1(21) &= U(21) + \frac{\varepsilon}{3}V_2(3) + \frac{\varepsilon}{3}\{V_2(12) - U(12) + U(1_F2)\} + (1-\varepsilon)V_2(21),\\
V_1(3) &= U(3) + \frac{2\varepsilon}{3}\{V_2(21) - U(21) + U(2_F1)\} + (1-\varepsilon)V_2(3),\\
V_2(111) &= U(111) + \frac{2\varepsilon}{3}\left\{\frac{1}{2}V_3(21) + \frac{1}{2}V_3(12)\right\} + (1-\varepsilon)V_3(111),\\
V_2(12) &= U(12) + \frac{2\varepsilon}{3}\{V_3(21) - U(21) + U(21_F)\} + (1-\varepsilon)V_3(12),\\
V_2(21) &= U(21) + \frac{\varepsilon}{3}V_3(3) + \frac{\varepsilon}{3}\{V_3(12) - U(12) + U(1_F2)\} + (1-\varepsilon)V_3(21),\\
V_2(3) &= U(3) + \frac{2\varepsilon}{3}\{V_3(21) - U(21) + U(2_F1)\} + (1-\varepsilon)V_3(3),\\
V_3(111) &= U(111) + \frac{2\varepsilon}{3}\left\{\frac{1}{2}V_4(21) + \frac{1}{2}V_4(12)\right\} + (1-\varepsilon)V_4(111),\\
V_3(12) &= U(12) + \frac{2\varepsilon}{3}\{V_4(21) - U(21) + U(21_F)\} + (1-\varepsilon)V_4(12),\\
V_3(21) &= U(21) + \frac{\varepsilon}{3}V_4(3) + \frac{\varepsilon}{3}\{V_4(12) - U(12) + U(1_F2)\} + (1-\varepsilon)V_4(21),\\
V_3(3) &= U(3) + \frac{2\varepsilon}{3}\{V_4(21) - U(21) + U(2_F1)\} + (1-\varepsilon)V_4(3),\\
V_4(111) &= U(111) + \frac{2\varepsilon}{3}\left\{\frac{1}{2}V_5(21) + \frac{1}{2}V_5(12)\right\} + (1-\varepsilon)V_5(111),\\
V_4(12) &= U(12) + \frac{2\varepsilon}{3}\{V_5(21) - U(21) + U(21_F)\} + (1-\varepsilon)V_5(12),\\
V_4(21) &= U(21) + \frac{\varepsilon}{3}V_2(3) + \frac{\varepsilon}{3}\{V_5(12) - U(12) + U(1_F2)\} + (1-\varepsilon)V_5(21),\\
V_4(3) &= U(3) + \frac{2\varepsilon}{3}\{V_5(21) - U(21) + U(2_F1)\} + (1-\varepsilon)V_5(3),\\
V_5(111) &= U(111) + \frac{2\varepsilon}{3}\left\{\frac{1}{2}V_6(21) + \frac{1}{2}V_6(12)\right\} + (1-\varepsilon)V_6(111),\\
V_5(12) &= U(12) + \frac{2\varepsilon}{3}\{V_6(21) - U(21) + U(21_F)\} + (1-\varepsilon)V_6(12),\\
V_5(21) &= U(21) + \frac{\varepsilon}{3}V_6(3) + \frac{\varepsilon}{3}\{V_6(12) - U(12) + U(1_F2)\} + (1-\varepsilon)V_6(21),\\
V_5(3) &= U(3) + \frac{2\varepsilon}{3}\{V_6(21) - U(21) + U(2_F1)\} + (1-\varepsilon)V_6(3),\\
V_6(111) &= U(111) + \frac{2\varepsilon}{3}\left\{\frac{1}{2}V_7(21) + \frac{1}{2}V_7(12)\right\} + (1-\varepsilon)V_7(111),\\
V_6(12) &= U(12) + \frac{2\varepsilon}{3}\{V_7(21) - U(21) + U(21_F)\} + (1-\varepsilon)V_7(12),\\
V_6(21) &= U(21) + \frac{\varepsilon}{3}V_7(3) + \frac{\varepsilon}{3}\{V_7(12) - U(12) + U(1_F2)\} + (1-\varepsilon)V_7(21),\\
V_6(3) &= U(3) + \frac{2\varepsilon}{3}\{V_7(21) - U(21) + U(2_F1)\} + (1-\varepsilon)V_7(3),\\
V_7(111) &= U(111) + \frac{2\varepsilon}{3}\left\{\frac{1}{2}V_1(21) + \frac{1}{2}V_1(12)\right\} + (1-\varepsilon)V_1(111),
\end{aligned}
$$

$$V_7(12) = U(12) + \frac{2\varepsilon}{3}\{V_1(111) - U(111) + U(11_F1)\} + (1-\varepsilon)V_1(12),$$
$$V_7(21) = U(21) + \frac{\varepsilon}{3}V_1(3) + \frac{\varepsilon}{3}\{V_1(111) - U(111) + U(1_F11)\} + (1-\varepsilon)V_1(21),$$
$$V_7(3) = U(3) + \frac{2\varepsilon}{3}\{V_1(21) - U(21) + U(2_F1)\} + (1-\varepsilon)V_1(3).$$

Consider the following particular parameter values.

δ	ϵ	k
0.7	0.355	1.7

For these parameter values, we can find the solution for the above system of equations as follows.

$V_1(111)$	2.69694	$V_2(111)$	2.69728	$V_3(111)$	2.69759	$V_4(111)$	2.69779
$V_1(12)$	2.53989	$V_2(12)$	2.54042	$V_3(12)$	2.54098	$V_4(12)$	2.5415
$V_1(21)$	2.73099	$V_2(21)$	2.73159	$V_3(21)$	2.73231	$V_4(21)$	2.73325
$V_1(3)$	2.58257	$V_2(3)$	2.5831	$V_3(3)$	2.5831	$V_4(3)$	2.58416

$V_5(111)$	2.6978	$V_6(111)$	2.69747	$V_7(111)$	2.69658
$V_5(12)$	2.54181	$V_6(12)$	2.54149	$V_7(12)$	2.53947
$V_5(21)$	2.73459	$V_6(21)$	2.73679	$V_7(21)$	2.74094
$V_5(3)$	2.58446	$V_6(3)$	2.58413	$V_7(3)$	2.58208

This solution satisfies those 28 inequality conditions stated above. Hence we have a cyclic equilibrium.