On manipulability in financial systems

by

Pedro Calleja, Francesc Llerena, Peter Sudhölter

Discussion Papers on Economics
No. 8/2021

FURTHER INFORMATION
Department of Economics
Faculty of Business and Social Sciences
University of Southern Denmark
Campusvej 55, DK – 5230 Odense M
Denmark
Email: astni@sam.sdu.dk
https://www.sdu.dk/en/om_sdu/institutter_centre/oekonomiskinstitut

ISSN 2596-8157

On manipulability in financial systems

Pedro Calleja[a], Francesc Llerena[b], Peter Sudhölter[c,*]

[a]*Departament de Matemàtica Econòmica, Financera i Actuarial, Universitat de Barcelona-BEAT,*
Av. Diagonal, 690, 08034 Barcelona, Spain
[b]*Departament de Gestió d'Empreses, Universitat Rovira i Virgili-ECO-SOS,*
Av. de la Universitat, 1, 43204 Reus, Spain
[c]*Department of Economics, University of Southern Denmark,*
Campusvej 55, 5230 Odense M, Denmark

Abstract

We investigate manipulability in the setting of financial systems by considering two weak forms of immunity: non-manipulability via merging and non-manipulability via splitting. Not surprinsingly, non-manipulability via splitting is incompatible with some basic axioms: claim boundedness, limited liability, and absolute priority. Outstandingly, we introduce a large class of financial rules that are immune to manipulations via merging. This class includes the proportional financial rule but also financial rules in accordance with parametric bankruptcy rules fulfilling non-manipulability via merging.

Keywords: financial systems, manipulability via merging, manipulability via splitting, parametric bankruptcy rules
JEL: C71

1. Introduction

A financial system is a network of different institutions (banks, individual investors, hedge funds, insurance companies, etc) linked to each other through financial contracts. The failure of Lehman Brothers in September 2008 puts the stability of the global financial system at risk, producing the bankruptcy of

[*] *Corresponding author:* calleja@ub.edu (P. Calleja), francesc.llerena@urv.cat (F. Llerena), psu@sam.sdu.dk (P. Sudhölter).

Preprint submitted to Elsevier *December 16, 2021*

other institutions. Since then, the articles on contagion in financial networks has grown significantly, being the seminal paper of Eisenberg and Noe (2001) the basis for subsequent research. The reader is referred to the works of Glasserman and Young (2016), Caccioli et al. (2018), and Jackson and Pernoud (2021) for reviews of the extensive literature on this topic.

Among the different aspects addressed in this setting, an important issue is how to clear the mutual obligations between institutions when the financial network structure collapses. That is, when the insolvency of an agent results in the default of other agents from contagion, how should the total cash in the system be distributed among the agents? This problem can be seen as a non-trivial generalization of a classical bankruptcy problem (O'Neill, 1982), where the value of the estate of a single insolvent firm is exogenous and should be distributed among a group of creditors. On the contrary, a financial system comprises a group of firms that simultaneously play the role of debtors and creditors and, consequently, the value of the estate of each of them is endogenous and depends on the extent to which other firms afford their obligations. An approach that connects both frameworks is to extend bankruptcy rules to financial systems computing the asset value of each firm and making payments according to the chosen bankruptcy rules. Indeed, Eisenberg and Noe (2001) determine the existence of payment matrices, which prescribe mutual payments among entities so as to clear the system, based on the *proportional* bankruptcy rule. They also propose an algorithm not only for identifying *clearing payment matrices* but also for analyzing the vulnerability of the system. This point is addressed, among others, in Chen et al. (2013) and Demange (2018) which focus on measuring the systemic risk. Instead of imposing proportionality, Groote Schaarsberg et al. (2018) allow for other bankruptcy rules, but not for agent-specific bankruptcy rules, while Csóska and Herings (2018) extend the aforementioned works, in a discrete and continuous setting, allowing each entity to employ a different bankruptcy rule. In line with Eisenberg and Noe, these authors show that clearing payment matrices supported by bankruptcy rules always exist and might not be unique, although the resulting entities' utilities, measured by means of their *value of equity*, are invariant.

Moreover, they also introduce different procedures to compute them.[1] All these works impose three basic requirements: *claim boundedness* (**CB**), specifying that no payment should be over the corresponding liability, *limited liability* of equity (**LL**), and *absolute priority* of debt over equity (**AP**).

A few works focus on the axiomatic grounds of financial rules, that recommend, for each financial system, a set of clearing payment matrices. Groote Schaarsberg et al. (2018) characterize the financial rule based on the *Talmud* bankruptcy rule (Aumann and Maschler, 1985). Ketelaars et al. (2021) extent the characterization of *priority* bankruptcy rules provided by Moulin (2000) to the setup of financial systems and Csóska and Herings (2021) characterize the *proportional financial* rule. In the context of classical bankruptcy situations, de Frutos (1999) proves that the proportional rule is the unique solution satisfying *non-manipulability* [2] that enforces to accomplish two weak forms of immunity to misrepresentations of claims: *non-manipulability via splitting*, which roughly speaking requires that no agent should have incentives to split into several ones, and *non-manipulability via merging*, meaning that a group of agents neither have incentives to merge into a single one. De frutos (1999) and Ju (2003) identify the families of parametric rules (Young, 1987) satisfying either non-manipulability via merging or non-manipulability via splitting. In the setting of financial systems, non-manipulability has been studied by Csóska and Herings (2021). Their definition of non-manipulability relays on the invariance of clearing payment matrices rather than on the comparison of the utilities of the agents, and does not distinguish between non-manipulability via merging or splitting. Contrary to bankruptcy problems, they find out that together with the requirements of **CB**, **LL**, and **AP**, no financial rule is immune to manipulability; in particular, the proportional rule. This impossibility result is not surprising considering that, as stressed in Csóska and Herings (2021), if a firm can create a new entity that inherits only all its liabilities while keeping the obligations of others, then the firm will end up in paying none of its initial debts. At this point, two natural questions emerge. Firstly, since maximizing utility drives the incentives of

[1] See also Ketelaars et al. (2020) and Ketelaars and Borm (2021).

[2] Different axiomatizations of the proportional rule can be found in O'Neill (1982), Curiel et al. (1987), Chun (1988), Moreno-Ternero (2006), and Ju et al. (2007)

decision-makers, it seems more suitable to define non-manipulability in terms of equity value. Secondly, as in real life entities not only use to create spin-offs but also merging companies, it is relevant to analyze these two weak forms of strategic manipulation separately. Although comparing equity values yield to weaker immunity conditions, as expected non-manipulability via splitting is still incompatible with the aforementioned basic requirements. Outstandingly, in this paper we introduce a large class of financial rules, not necessarily induced by bankruptcy rules, that are immune to manipulations via merging. This class includes those based on bankruptcy rules satisfying *strong non-manipulability via merging*, a new property inspired by *additivity* (or *strong non-manipulability*) as introduced in Curiel et al. (1987) and Moreno-Ternero (2006). Specifically, it contains the proportional financial rule but also financial rules in accordance with parametric rules fulfilling non-manipulability via merging (see Young, 1987 and Ju, 2003).

The rest of the paper is organized as follows. Section 2 contains Tarski's fixed-point theorem, a key result in our analysis. Section 3 handles with non-manipulability in the setting of bankruptcy problems and introduces some new results. Section 4 is devoted to financial systems induced by bankruptcy rules and by *division schemes*, a new approach that allows to recognize financial rules from a more general perspective by considering all the liabilities in the network. Section 5 presents the main results concerning the non-manipulability in the financial system setting. Section 6 concludes. The Appendix contains some technical proofs.

2. Preliminaries

An important result in our analysis of manipulability in financial systems is Tarski's fixed-point theorem (Tarski, 1955). In order to formulate it, it is useful to recall some definitions. A *lattice* is a pair $(A, \leq)$ formed by a non-empty set A and a transitive and antisymmetric binary relation $\leq$ on[3] A that determines a partial order on A such that, for any two elements $x, y \in A$, there is a *supremum*

[3] We write $x < y$ if $x \leq y$ but $x \neq y$.

(join), denoted by $x \vee y$, and an *infimum (meet)*, denoted by $x \wedge y$. The *supremum* $x \vee y$ is the unique element of A such that $x, y \leq x \vee y$ and if $z \in A$ is such that $z \geq x, y$, then $z \geq x \vee y$. The *infimum* $x \wedge y$ is the unique element of A such that $x, y \geq x \wedge y$ and if $z \in A$ is such that $z \leq x, y$, then $z \leq x \wedge y$. The lattice $(A, \leq)$ is called *complete* if every non-empty subset $B \subseteq A$ has a *supremum*, denoted by $\bigvee B$, and an *infimum*, denoted by $\bigwedge B$. In this case we use the convention that $\bigvee \emptyset = \bigwedge A$ and $\bigwedge \emptyset = \bigvee A$. Given two elements $x, y \in A$ with $x \leq y$, we denote by $[x, y]$ the interval with the endpoints x and y, i.e., $[x, y] = \{z \in A \mid x \leq z \leq y\}$. Clearly, $([x, y], \leq)$ is a lattice, and it is a complete lattice if $(A, \leq)$ is complete. We shall consider functions $f : B \to C$, where $B, C \subseteq A$. Such a function f is called *non-decreasing* if, for any pair of elements $x, y \in B$, $x \leq y$ implies $f(x) \leq f(y)$. A *fixed point* of f is an element x of B such that $x = f(x)$. Let $FIX(f)$ denote the set of fixed points of f. Now, we have all the tools to state Tarski's fixed-point theorem.

Theorem 1. (Tarski, 1955) *If $(A, \leq)$ is a complete lattice and $f : A \to A$ is a non-decreasing function, then $(FIX(f), \leq)$ is a complete lattice, $\bigvee FIX(f) = \bigvee \{z \in A \mid z \geq f(z)\}$, and $\bigwedge FIX(f) = \bigwedge \{z \in A \mid z \leq f(z)\}$.*

3. Bankruptcy problems

Financial systems can be regarded as generalized bankruptcy problems. Indeed, among others, Eisenberg and Noe (2001), Groote Schaarsberg et al. (2018), Csóska and Herings (2018, 2021), Ketelaars at al. (2020), and Ketelaars and Borm (2021) propose solutions to financial systems (i.e., clearing payment matrices) that are inspired by bankruptcy rules. In order to introduce the aforementioned approach formally, we recall some standard definitions and well-known results in the framework of bankruptcy problems.

Let $\mathbb{N}$ (the set of natural numbers) represent the set of all potential agents (claimants) and let $\mathcal{N}$ be the collection of all non-empty finite subsets of $\mathbb{N}$. An element $N \in \mathcal{N}$ describes a finite set of agents where $|N| = n$. A *bankruptcy problem is a* triple (N, E, c) such that $N \in \mathcal{N}, c \in \mathbb{R}^N_+, E \geq 0$, and $\sum_{i \in N} c_i \geq E$. If (N, E, c) is a bankruptcy problem, then each agent in the set of creditors N has a claim c_i to the net worth or estate $E \geq 0$ of a bankrupt firm. By $\mathcal{B}$ we denote the set of all bankruptcy problems. A *bankruptcy rule* is a function $\beta : \mathcal{B} \longrightarrow \bigcup_{N\in\mathcal{N}} \mathbb{R}_+^N$ that associates with every $(N,E,c) \in \mathcal{B}$ a unique vector $\beta(N,E,c) \in \mathbb{R}_+^N$ satisfying $\sum\limits_{i\in N} \beta_i(N,E,c) = E$ (*budget balance* (BB)) and $\beta_i(N,E,c) \leq c_i$ for all $i \in N$ (*claim boundedness* (CB)).

There is a wide range of well-behaved bankruptcy rules studied in the literature, among which we focus on those satisfying the following axioms. A bankruptcy rule β satisfies *equal treatment of equals* (ETE) if for all $(N,E,c) \in \mathcal{B}$ and all $i,j \in N$, if $c_i = c_j$ then $\beta_i(N,E,c) = \beta_j(N,E,c)$. That is, ETE requires that agents with the same claim receive the same amount. It satisfies *consistency* (CONS) if for all $(N,E,c) \in \mathcal{B}$ and all $\emptyset \neq N' \subseteq N$, $\beta_{N'}(N,E,c) = \beta\left(N', \sum_{i\in N'} \beta_i(N,E,c), c_{N'}\right)$.[4] CONS requires that in the reduced bankruptcy problem, which arises when some players leave with their share, each of the remaining players receives the same amount as in the original problem. It satisfies *continuity* (CONT) if for all sequences of bankruptcy problems $\left((N,E^n,c^n)\right)_{n\in\mathbb{N}}$ converging to (N,E,c), the sequence $\left(\beta(N,E^n,c^n)\right)_{n\in\mathbb{N}}$ converges to $\beta(N,E,c)$. Young (1987) characterizes the family of bankruptcy rules satisfying ETE, CONS, and CONT, called *parametric rules*. Let $[-\infty,+\infty] = \mathbb{R}\cup\{-\infty,\infty\}$ be the extended real line ($-\infty < t < +\infty$ for all $t \in \mathbb{R}$ and $-\infty < +\infty$ by convention) and let H be the set of functions $h : [a,b]\times\mathbb{R}_+ \to \mathbb{R}_+$, where $a,b \in [-\infty,+\infty]$, $a \leq b$, such that h is *continuous*, *non-decreasing in the first argument*, and for each $\bar{c} \in \mathbb{R}_+$, $h(a,\bar{c}) = 0$ and $h(b,\bar{c}) = \bar{c}$. A rule β is *parametric* if there exists $h \in H$ such that for all $(N,E,c) \in \mathcal{B}$ there exists $\lambda \in [a,b]$ satisfying $\beta_i(N,E,c) = h(\lambda,c_i)$ for all $i \in N$ and $\sum_{i\in N} h(\lambda,c_i) = E$. In this case, h is called a *representation* of β. In fact, there are infinitely many representations of a parametric rule (Thomson 1995). This family of rules satisfies, additionally, resource monotonicity. A bankruptcy rule β is said to satisfy *resource monotonicity* (RM) if for all $(N,E,c),(N,E',c) \in \mathcal{B}$ such that $E' > E$, it holds that $\beta(N,E,c) \leq \beta(N,E',c)$. Instances of well studied parametric rules are the *proportional* rule (PR), the *constrained equal awards* rule (CEA), and the *constrained equal losses* rule (CEL). The PR rule makes awards proportional to the claims and it is probably the most commonly used rule when a firm goes

[4] Given $N \in \mathcal{N}$, $\emptyset \neq S \subset N$, and $x \in \mathbb{R}^N$, $x_S = (x_i)_{i\in S} \in \mathbb{R}^S$.

bankrupt. Formally, for all $(N, E, c) \in \mathcal{B}$ and all $i \in N$, $PR_i(N, E, c) = \lambda\, c_i$ where $\lambda \in \mathbb{R}_+$ is such that $\sum\limits_{j \in N} \lambda\, c_j = E$. The CEA rule rewards equally to all claimants subject to no one receiving more than her claim. Formally, for all $(N, E, c) \in \mathcal{B}$ and all $i \in N$, $CEA_i(N, E, c) = \min\{c_i, \lambda\}$ where $\lambda \in \mathbb{R}_+$ is such that $\sum\limits_{j \in N} \min\{c_j, \lambda\} = E$. In contrast, the CEL rule equalizes the losses of claimants subject to no one receiving a negative amount. That is, for all $(N, E, c) \in \mathcal{B}$ and all $i \in N$, $CEL_i(N, E, c) = \max\{c_i - \lambda, 0\}$ where $\lambda \in \mathbb{R}_+$ is such that $\sum\limits_{j \in N} \max\{c_j - \lambda, 0\} = E$.

Given a bankruptcy rule β, its dual β^d is defined by setting, for all $(N, E, c) \in \mathcal{B}$ and all $i \in N$, $\beta^d_i(N, E, c) = c_i - \beta_i\left(N, \sum_{i \in N} c_i - E, c\right)$. The CEA and the CEL are dual rules and the PR is self-dual, i.e., $PR = PR^d$. We say that two properties P and P' are dual to each other if β satisfies P if and only if its dual β^d satisfies P'. If, moreover, P coincides with P', then P is self-dual. For instance, RM, ETE, CONS, and CONT are self-dual properties.

A large fraction of the literature on bankruptcy problems is devoted to study the strategic incentives of claimants to misrepresent claims, either by merging or splitting their respective claims in order to obtain some extra profits. De Frutos (1999) introduces two different "immunity" properties so as to separate these two types of incentives. A bankruptcy rule β on $\mathcal{B}$ satisfies

- *non-manipulability* (NM) if for all $N, N' \in \mathcal{N}$ and all (N, E, c), $(N', E, c') \in \mathcal{B}$, if $N' \subset N$ and there is $m \in N'$ such that $c'_m = c_m + \sum_{j \in N \setminus N'} c_j$ and $c'_j = c_j$ for all $j \in N' \setminus \{m\}$, then

$$\beta_m(N', E, c') = \beta_m(N, E, c) + \sum_{j \in N \setminus N'} \beta_j(N, E, c).$$

NM can be divided into *non-manipulability via merging* (NMM) imposing

$$\beta_m(N', E, c') \leq \beta_m(N, E, c) + \sum_{j \in N \setminus N'} \beta_j(N, E, c),$$

and *non-manipulability via splitting* (NMS) requiring the inverse inequality

$$\beta_m(N', E, c') \geq \beta_m(N, E, c) + \sum_{j \in N \setminus N'} \beta_j(N, E, c).$$

NM requires NMM and NMS simultaneously. NMM stipulates that no group of claimants can benefit from consolidating claims. On the contrary, NMS guarantees that no claimant can benefit from dividing its claim into claims of a group of claimants. NMM and NMS are dual properties (see de Frutos, 1999). De Frutos (1999) and Ju and Miyagawa (2002) show that the PR rule is the only rule satisfying NM. Moreno-Ternero (2006) proves that *strong non-manipulability*, introduced as *additivity* by Curiel et al. (1987), is equivalent to NM. A bankruptcy rule β on $\mathcal{B}$ satisfies

- *strong non-manipulability* (SNM) if for all $N, N' \in \mathcal{N}$ and all (N, E, c), $(N', E, c') \in \mathcal{B}$, if $N' \subset N$ and there is $m \in N'$ such that $c'_m = c_m + \sum_{j \in N \setminus N'} c_j$ and $c'_j = c_j$ for all $j \in N' \setminus \{m\}$ then, for all $j \in N' \setminus \{m\}$,

$$\beta_j(N', E, c') = \beta_j(N, E, c).$$

SNM can be separated into *strong non-manipulability via merging* (SNMM) requiring, for all $j \in N' \setminus \{m\}$,

$$\beta_j(N', E, c') \geq \beta_j(N, E, c),$$

and *strong non-manipulability via splitting* (SNMS) imposing, for all $j \in N' \setminus \{m\}$, the reverse inequalities,

$$\beta_j(N', E, c') \leq \beta_j(N, E, c).$$

SNM guarantees that merging or splitting the agents' claims do not affect the amounts received by each other agent involved in the problem. SNMM imposes that merging the claims of a group of creditors does not damage the rest of agents, while SNMS requests that if a creditor divides its claim into claims of several claimants, then none of the remaining agents is worse off. It is not difficult, and it is left to the reader, to check that SNMM and SNMS are dual properties. Clearly, SNMM implies NMM and SNMS implies NMS. The following theorem, that is proved in the Appendix, states that the reverse implications do not hold.

Theorem 2. *Neither* NMM *implies* SNMM*, nor* NMS *implies* SNMS*.*

Nevertheless, under the standard requirements of RM and CONS, NMM and NMS are equivalent to SNMM and SNMS, respectively.

Proposition 1. *Let β be a bankruptcy rule satisfying* RM *and* CONS*. Then, β satisfies* NMM (NMS) *if and only if it satisfies* SNMM (SNMS)*.*

Proof. Since NMM (SNMM) and NMS (SNMS) are dual properties to each other, it is enough to see that, under RM and CONS, a bankruptcy rule β satisfies NMM if and only if it satisfies SNMM. Clearly, SNMM implies NMM. To show the reverse implication, consider a bankruptcy rule β satisfying RM, CONS, and NMM. Let (N, E, c), $(N', E, c') \in \mathcal{B}$ such that $N' \subset N$ and there is $m \in N'$ with $c'_m = c_m + \sum_{j \in N \setminus N'} c_j$ and $c_j = c'_j$ for all $j \in N' \setminus \{m\}$. By NMM, $\beta_m(N'E, c') \leq \beta_m(N, E, c) + \sum_{j \in N \setminus N'} \beta_j(N, E, c)$ or, equivalently,

$$E - \beta_m(N', E, c') \geq E - \sum_{j \in \{m\} \cup N \setminus N'} \beta_j(N, E, c). \tag{1}$$

From (1), and taking into account that $c_i = c'_i$ for all $i \in N' \setminus \{m\}$, by RM and CONS we obtain

$$\begin{aligned}
\beta_i(N', E, c') &\underset{\text{CONS}}{=} \beta_i\left(N' \setminus \{m\}, E - \beta_m(N', E, c'), c'_{|N' \setminus \{m\}}\right) \\
&\underset{\text{RM}}{\geq} \beta_i\left(N' \setminus \{m\}, E - \sum_{j \in \{m\} \cup N \setminus N'} \beta_j(N, E, c), c_{|N' \setminus \{m\}}\right) \\
&\underset{\text{CONS}}{=} \beta_i(N, E, c),
\end{aligned}$$

which proves SNMM of β. □

Since parametric rules satisfy RM and CONS, a direct consequence of Proposition 1 is the following.

Corollary 1. *A parametric rule is* NMM (NMS) *if and only if it is* SNMM (SNMS)*.*

Ju (2003) characterizes the set of parametric bankruptcy rules that are NMM or NMS making use of the following properties of representations. A representation $h : [a, b] \times \mathbb{R}_+ \to \mathbb{R}_+$ is *superadditive in claims* if for all $\lambda \in [a, b]$ and all $\bar{c}, \bar{c'} \in \mathbb{R}_+, h(\lambda, \bar{c} + \bar{c'}) \geq h(\lambda, \bar{c}) + h(\lambda, \bar{c'})$. If the inequality holds in the opposite direction, we say that the representation h is *subadditive in claims*. Although there are many representations of a parametric rule, Ju (2003) shows that superadditivity (or subadditivity) in claims is an invariance property.

Proposition 2. (Ju, 2003) *A parametric rule is* NMM (NMS) *if and only if its representations are subadditive (superadditive) in claims.*

It is well known that the representations of the CEA rule and the PR rule are subadditive in claims, while the representations of the CEL rule and the PR rule are superadditive in claims.

4. Financial systems

In order to define financial systems we first introduce some notation. For $N \in \mathcal{N}$ we denote by $\mathcal{M}(N)$ the set of all non-negative real $N \times N$ matrices $M = (M_{ij})_{i,j \in N}$ with a zero diagonal. Moreover, we define $\mathcal{M} = \bigcup\limits_{N \in \mathcal{N}} \mathcal{M}(N)$. For $M \in \mathcal{M}(N)$ and $i \in N$, we denote the row i of M by M_i, i.e., $M_i = (M_{ij})_{j \in N} \in \mathbb{R}^N_+$. Finally, we abbreviate $\bar{M}_i = \sum\limits_{j \in N} M_{ij}$.

A *financial system* is a triple (N, L, e) such that $N \in \mathcal{N}, L \in \mathcal{M}(N)$, and $e \in \mathbb{R}^N_+$. In such a financial system, N is the set of distinct economic entities, firms, or agents. The matrix L represents the structure of *liabilities*, where L_{ij} stands for the liability of entity $i \in N$ to entity $j \in N$, or equivalently, the claim of entity j against entity i, and $L_{ii} = 0$ for all $i \in N$ means that no entity has a claim against itself. So, two agents may have mutual positive claims against each other in the system. Here, $\bar{L} = (\bar{L}_i)_{i \in N} \in \mathbb{R}^N$ is the vector of total obligations in the system. The vector $e \in \mathbb{R}^N_+$ represents the *initial endowments* of the agents that are assumed to be non-negative, and includes all the assets of each entity excluding the claims on other entities in the system, both together constitute the resources to afford its liabilities. In this framework, the default of an entity, that does not have enough resources to satisfy all its liabilities, can induce the default of other initially healthy entities, due to its connections with the first one. Depending on how strong the interconnections of the entities in the system are, this fact may put the system in systemic risk. By $\mathcal{F}$ we denote the set of all financial systems. We assume that the system is closed and there are not outside resources available to the entities in the system to meet their obligations. For each $(N, L, e) \in \mathcal{F}$, a *payment matrix* $P \in \mathcal{M}(N)$ specifies what monetary amount P_{ij} should be paid by entity $i \in N$ to entity $j \in N$, with $P_{ii} = 0$ for all $i \in N$. Associated to a payment matrix P and an endowment vector $e \in \mathbb{R}^N_+$, we define the *asset value* of entity $i \in N$ by

$$a_i(P, e) = e_i + \sum_{k \in N} P_{ki}, \tag{2}$$

the amount of resources of i to clear its debts. The *value of equity*, or utility, of entity $i \in N$ is defined by

$$E_i(P, e) = a_i(P, e) - \bar{P}_i, \tag{3}$$

where the $\bar{P}_i$ is the total payment by entity i to the other agents in the system according to P.

Definition 1. *A* financial rule σ *assigns a subset* $\sigma(N, L, e)$ *of* $\mathcal{M}(N)$ *to each* $(N, L, e) \in \mathcal{F}$*.*

Hence, a financial rule associates to each financial system a possibly empty set of payment matrices. The recent literature typically focusses on financial rules induced by certain bankruptcy rules as discussed in Sect. 4.1. We explore, in addition, financial rules induced by *division schemes* (see Sect. 4.2). This completely novel approach allows to consider all the interactions among agents when proposing payment matrices.

Interestingly, for a given $\varepsilon = (N, L, e) \in \mathcal{F}$, and a payment matrix $P \in \mathcal{M}(N)$ it holds that $\sum_{i \in N} e_i = \sum_{i \in N} E_i(P, e)$. Hence, a recommendation on payment matrices can be interpreted as a recommendation on the distribution of the equity values of the firms in the system, that in total equals the total initial endowments.

A financial rule σ is said to be *single-valuedness* (**SIVA**) if, for all $(N, L, e) \in \mathcal{F}$, $\sigma(N, L, e)$ consist of a single payment matrix, i.e., $|\sigma(N, L, e)| = 1$. Hence, **SIVA** implies non-emptiness, which requires that, for all $(N, L, e) \in \mathcal{F}$, $\sigma(N, L, e) \neq \emptyset$. We are interested in financial rules supplying “clearing payment matrices” providing feasible payments among agents in the sense of Eisenberg and Noe (2001). Formally, they should satisfy the following three properties. A financial rule σ satisfies

- *claims boundedness* (**CB**) if, for all $(N, L, e) \in \mathcal{F}$, all $P \in \sigma(N, L, e)$, and all $i, j \in N$, $P_{ij} \leq L_{ij}$;
- *limited liability* (**LL**) if, for all $(N, L, e) \in \mathcal{F}$, all $P \in \sigma(N, L, e)$, and all $i \in N$, $E_i(P, e) \geq 0$;
- *absolute priority* (**AP**) if, for all $(N, L, e) \in \mathcal{F}$, all $P \in \sigma(N, L, e)$, and all $i \in N$, $E_i(P, e) > 0$ implies $P_{ij} = L_{ij}$ for all $j \in N$.

CB imposes that no agent pays to any other entity more than the liability to it. **LL** requires that all entities can afford the payments established by the payment matrices in the sense that the liabilities of the firm to others are limited to the asset value of the firm, thereby guaranteeing that there will be no need to get resources from outside the system. **AP** demands that stockholders of each entity can not receive a positive value unless all obligations have been completely paid.

4.1. Financial rules induced by bankruptcy rules

The result of a financial rule, when applied to a financial system, can be interpreted as a recommendation to a single decision-maker on how to solve this conflicting mutual liabilities problem. However, in a global world, this problem may cause an international insolvency issue, in which different courts, at most as many as different economic entities are involved, take part, each one as a different decision-maker. In such a case one may ask whether the recommendation of the financial rule is compatible with the recommendations provided by traditional bankruptcy rules applied to each different entity. Therefore, we consider financial rules that arise from a collection of different bankruptcy rules, one for each economic entity, that abide a different court or law principle, which motivates the following definition.

Definition 2. *Let $(\beta^i)_{i\in\mathbb{N}}$ be a collection of bankruptcy rules, i.e., β^i is a bankruptcy rule for each $i \in \mathbb{N}$. We say that a financial rule σ is* induced by $(\beta^i)_{i\in\mathbb{N}}$ *if, for all $(N, L, e) \in \mathcal{F}$, all $P \in \sigma(N, L, e)$, and all $i \in N$, it holds that $P_{ij} = \beta^i_j(N \setminus \{i\}, E, c)$ for all $j \in N \setminus \{i\}$ where $E = \bar{P}_i$ and $c \in \mathbb{R}_+^{N\setminus\{i\}}$ with $c_j = L_{ij}$ for all $j \in N \setminus \{i\}$.*

For the sake of convenience, we say that a financial rule σ is *induced by bankruptcy rules* if it is induced by some collection of bankruptcy rules. At this point it should be emphasized that, in contrast to the case of a bankruptcy problem, in a financial system the value of the estate distributed by each firm is endogenously determined, since it depends on the initial endowment of the firm, but also on the claims of such firm against other agents, that may not be fulfilled. Eisenberg and Noe (2001) show that there are non-empty financial rules satisfying **CB**, **LL**, and **AP** that are induced by the proportional bankruptcy rule. Groote Schaarsberg et al. (2018) prove non-emptyness following a different approach and for the more general case in which all entities impose the same bankruptcy rule, but not necessarily the proportional rule. Csóka and Herings (2018) extend this result to the case in which each entity may use a distinct bankruptcy rule.

The next lemma estates that under **CB**, independently of whether or not the financial rule is in accordance with bankruptcy rules, **LL** and **AP** are equivalent to require that every firm pays the minimum between its total funds (the asset value) and its total debt obligations.

Lemma 1. *Let σ be a financial rule satisfying* **CB**. *Then, the following statements are equivalent:*

1. *σ satisfies* **LL** *and* **AP**.
2. *For all $\varepsilon = (N, L, e) \in \mathcal{F}$, all $P \in \sigma(\varepsilon)$, and all $i \in N$,*

$$\bar{P}_i = \min\left\{e_i + \sum_{k\in N} P_{ki}, \bar{L}_i\right\} \tag{4}$$

Proof. ($1 \Rightarrow 2$) Let σ be a financial rule satisfying **CB**, **LL**, and **AP**. Let $\varepsilon = (N, L, e) \in \mathcal{F}$ and $P \in \sigma(\varepsilon)$. By **LL**, $E_i(P, e) \geq 0$ for all $i \in N$. If $E_i(P, e) = 0$, $e_i + \sum_{k\in N} P_{ki} = \bar{P}_i \leq \bar{L}_i$, where the inequality comes from **CB** . If $E_i(P, e) > 0$, by **AP**, $\bar{P}_i = \bar{L}_i$ and thus $e_i + \sum_{k\in N} P_{ki} > \bar{P}_i = \bar{L}_i$. Hence, in both cases expression (4) holds.

($2 \Rightarrow 1$) Let $\varepsilon = (N, L, e) \in \mathcal{F}$ and $P \in \sigma(\varepsilon)$. By hypothesis, expression (4) holds and hence, for all $i \in N$, $E_i(P, e) = e_i + \sum_{k\in N} P_{ki} - \bar{P}_i \geq 0$, which proves **LL**. To check **AP**, let $i \in N$ and suppose that $E_i(P, e) > 0$. Then, $e_i + \sum_{k\in N} P_{ki} > \bar{P}_i$ and from (4) we conclude that $\bar{P}_i = \bar{L}_i$, which, by **CB**, is equivalent to $P_{ik} = L_{ik}$ for all $k \in N$. $\square$

Employing Lemma 1, and from Definition 2, examples of non-empty financial rules generated by each entity applying the PR rule, the CEA rule, and the CEL rule and that, additionally, satisfy **CB**, **LL**, and **AP**, can be introduced. The *proportional financial rule*, σ^{PR}, is defined by setting: for all $\varepsilon = (N, L, e) \in \mathcal{F}$, all $P \in \sigma^{PR}(\varepsilon)$, and all $i, j \in N$, $P_{ij} = \lambda_i L_{ij}$, where $\lambda_i \in \mathbb{R}_+$ satisfies $\bar{P}_i = \min\{e_i + \sum_{k \in N} P_{ki}, \bar{L}_i\}$. The *CEA financial rule*, σ^{CEA}, is defined by setting: for all $\varepsilon = (N, L, e) \in \mathcal{F}$, all $P \in \sigma^{CEA}(\varepsilon)$, and all $i, j \in N$, $P_{ij} = \min\{\lambda_i, L_{ij}\}$, where $\lambda_i \in \mathbb{R}_+$ satisfies $\bar{P}_i = \min\{e_i + \sum_{k \in N} P_{ki}, \bar{L}_i\}$. The *CEL financial rule*, σ^{CEL}, is defined by setting: for all $\varepsilon = (N, L, e) \in \mathcal{F}$, all $P \in \sigma^{CEL}(\varepsilon)$, and all $i, j \in N$, $P_{ij} = \max\{0, L_{ij} - \lambda_i\}$, where $\lambda_i \in \mathbb{R}_+$ satisfies $\bar{P}_i = \min\{e_i + \sum_{k \in N} P_{ki}, \bar{L}_i\}$.

Example 1 below highlights that, from the point of view of a single (central) decision-maker, restricting oneself to financial rules induced by a list of bankruptcy rules, one for each economic entity, might be a myopic approach. Actually, according to a financial rule of this type, each entity distributes any amount among its creditors taking only into account the corresponding liabilities to them, but not any other liabilities in the system.

Example 1. *Let $\varepsilon = (N, L, e)$ and $\varepsilon' = (N, L', e')$ be two financial systems with agent set $N = \{1, 2, 3\}$, vectors of initial endowments $e = e' = (1, 0, 0)$, and liability matrices:*

$$L = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1000 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad L' = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1000 & 0 \end{pmatrix}.$$

In the financial system ε, firm 3 seems healthier than firm 2 as a result of more responsible credit decisions, and in ε' is exactly the contrary. Under **LL** *and* **AP***, debtor 1 faces the same bankruptcy situation in both ε and ε' described by an estate $E = E' = 1$, since she has an initial endowment $e_1 = e'_1 = 1$ but, under* **CB***, she will receive nothing from 2 and 3 as they have no liabilities to 1; and a vector of claims $c = (1, 1)$. When clearing the system, a central decision-maker could prioritize the most needy firm or, on the contrary, the healthiest (more responsible) one.*

A route to overcome this problem is to enrich the notion of bankruptcy rules by means of *division schemes.*

4.2. Financial rules induced by division schemes

Here we introduce division schemes as a tool to resolve financial systems from a more general perspective by recognizing all the features of the financial network. Consequently, new financial rules can be generated and several existing rules, as those induced by bankruptcy rules, are obtained as particular cases. For a given financial system, a *division scheme* associates a list of functions each one describing how each entity divides a non-negative amount representing its available resource among her creditors.

Definition 3. *Given $\varepsilon = (N, L, e) \in \mathcal{F}$, a division scheme $f^\varepsilon = (f_i^\varepsilon)_{i \in N}$ associates to each $i \in N$ a function $f_i^\varepsilon : \mathbb{R}_+ \longrightarrow \mathbb{R}_+^N$ satisfying:*

- *Budget Balance* ($\mathbb{BB}$)*: for all $t \geq 0$, $\sum_{k \in N} f_{ik}^\varepsilon(t) = t$;*
- *Claim Boundedness* ($\mathbb{CB}$)*: for all $t \geq 0$ and all $k \in N$, $f_{ik}^\varepsilon(t) \leq L_{ik}$.*

Note that $\mathbb{BB}$ together with $\mathbb{CB}$ imply $0 \leq t \leq \bar{L}_i$ and $f_{ii}^\varepsilon(t) = 0$, for all $i \in N$. Let us denote by Λ^ε the set of all division schemes on $\varepsilon = (N, L, e)$. A *complete division scheme*, $f \in \bigcup_{\varepsilon \in \mathcal{F}} \Lambda^\varepsilon$, provides a division scheme for any financial system. By $\mathcal{DS}$ we denote the set of all complete division schemes.

Obviously, making use of complete division schemes we will enrich the notion of financial rules supported by bankruptcy rules. Next definition connects an inventory of bankruptcy rules with complete division scheme.

Definition 4. *Let $(\beta^i)_{i \in \mathbb{N}}$ be a collection of bankruptcy rules and $f \in \mathcal{DS}$. We say that f represents $(\beta^i)_{i \in \mathbb{N}}$ if, for all $\varepsilon = (N, L, e) \in \mathcal{F}$, all $i \in N$, and all $j \in N \setminus \{i\}$, $f_{ij}^\varepsilon(t) = \beta_j^i \left(N \setminus \{i\}, t, (L_{ij})_{j \in N \setminus \{i\}}\right)$, for all $0 \leq t \leq \bar{L}_i$.*

Nevertheless, division schemes allows for a much wider approach taking into consideration the entire financial system permitting, for instance, that an economic debtor establishes payments depending on the full network of interconnections among entities as well as on the initial endowments rather than simply on her liabilities to others. That is, the payments made by agent i to any other agent k need not only depend on the claims of those agents toward agent i but also on the mutual claims among them. Therefore, agent i might treat differently agents that have equal claims towards it. We come back to Example 1 to point out the differences between these two concepts.

Example 1(*revisited*) *According to the different features of ε and ε', we may define a division scheme f such that $f_1^{\varepsilon}(1) = (0,0,1)$ and $f_1^{\varepsilon'}(1) = (0,1,0)$, in favor of responsible firms. Another argument could be to prioritize the most needy firm. In that case, we might define a different division scheme g such that $g_1^{\varepsilon}(1) = (0,1,0)$ and $g_1^{\varepsilon'}(1) = (0,0,1)$. As we observed before, debtor 1 faces the same bankruptcy problem in both ε and ε'. However, $f_1^{\varepsilon}(1) \neq f_1^{\varepsilon'}(1)$ and $g_1^{\varepsilon}(1) \neq g_1^{\varepsilon'}(1)$, which shows that f and g do not admit an interpretation as bankruptcy rules.*

As shown in Example 1, not all complete division schemes f can be represented by bankruptcy rules. Actually, f admits an interpretation as bankruptcy rules whenever any entity distributes an amount among its creditors taking only into account the corresponding liabilities to them, but not any other liability in the system. That is, given $\varepsilon = (N, L, e)$ and $\varepsilon' = (N, L', e')$, if $L_i = L'_i$ for some $i \in N$, then $f_i^{\varepsilon}(t) = f_i^{\varepsilon'}(t)$, for all $0 \leq t \leq \bar{L}_i$.

Next, we generate financial rules making use of complete division schemes.

Definition 5. *A financial rule σ is said to be supported by $f \in \mathcal{DS}$ if for all $\varepsilon \in \mathcal{F}$, all $P \in \sigma(\varepsilon)$, and all $i \in N$, it hods that $P_i = f_i^{\varepsilon}(\bar{P}_i)$.*

Regarding uniqueness of clearing payment matrices, it is worth to stress that any financial rule satisfying **SIVA** and **CB** is supported by a complete division scheme. Indeed, let σ be a financial rule satisfying **SIVA** and **CB**. It is enough to define $f \in \mathcal{DS}$ as follows: for all $\varepsilon = (N, L, e) \in \mathcal{F}$, $\sigma(\varepsilon) = \{P\}$, and all $i \in N$,

$$f_{ij}^{\varepsilon}(t) = \begin{cases} 0 & \text{if} \quad 0 = t = \bar{P}_i \\ \lambda P_{ij} & \text{if} \quad 0 < t \leq \bar{P}_i \\ \lambda P_{ij} + (1-\lambda) L_{ij} & \text{if} \quad \bar{P}_i < t \leq \bar{L}_i \end{cases}$$

for all $j \in N$, where $\lambda \in [0,1]$ is chosen so as to satisfy $\sum_{k \in N} f_{ik}^{\varepsilon}(t) = t$. Note that $f_{ij}^{\varepsilon}(\bar{P}_i) = P_{ij}$. However, in general, not all financial rule is supported by a complete division scheme. Consider the financial system $\varepsilon' = (N', L', e')$ being

$$N' = \{1,2,3\}, \quad L' = \begin{pmatrix} 0 & 5 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad e' = (5,0,0).$$

Now define the financial rule σ as follows: for all $\varepsilon \in \mathcal{F}$,

$$\sigma(\varepsilon) = \begin{cases} \sigma^{PR}(\varepsilon) & \text{if} \quad \varepsilon \neq \varepsilon' \\ \left\{ \begin{pmatrix} 0 & \lambda & 5-\lambda \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid \lambda \in [0,5] \right\} & \text{if} \quad \varepsilon = \varepsilon' \end{cases}.$$

Suppose that σ is supported by some division scheme f. Then,

$$P = \begin{pmatrix} 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \sigma(\varepsilon') \quad \text{and} \quad P' = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \sigma(\varepsilon')$$

but $P_{12} = f^{\varepsilon}_{12}(5) = 1 \neq 2 = P'_{12} = f^{\varepsilon}_{12}(5)$, getting a contradiction.

The question of uniqueness is addressed, among others, by Eisenberg and Noe (2001), Groote Schaarsberg et al. (2018), Koster (2019), and Ketelaars et al. (2020).

Since a financial rule σ supported by $f \in \mathcal{DS}$ satisfies **CB**, as a direct consequence of Lemma 1 we obtain the following corollary.

Corollary 2. *Let σ be a financial rule supported by $f \in \mathcal{DS}$. Then, the following statements are equivalent:*

1. *σ satisfies* **LL** *and* **AP**.
2. *For all $\varepsilon = (N, L, e) \in \mathcal{F}$ and all $P \in \sigma(\varepsilon)$, $\bar{P} \in \mathbb{R}^N$ is a fixed-point of the function $\Phi^{\varepsilon,f} : [\mathbf{0}, \bar{L}] \longrightarrow [\mathbf{0}, \bar{L}]$ defined by*

$$\Phi^{\varepsilon,f}_i(\mathbf{t}) = \min\left\{ e_i + \sum_{k \in N} f^{\varepsilon}_{ki}(\mathbf{t}_k), \bar{L}_i \right\}, \tag{5}$$

for all $i \in N$ and all $\mathbf{t} \in [\mathbf{0}, \bar{L}]$, being $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^N$.

In view of Corollary 2, we might apply Tarski's fixed-point theorem to the function $\Phi^{\varepsilon,f}$ to guarantee the existence of financial rules satisfying **LL** and **AP**. To this aim, we study conditions ensuring the monotonicity of $\Phi^{\varepsilon,f}$. A complete division scheme $f \in \mathcal{DS}$ satisfies

- *Resource Monotonicity* ($\mathbb{RM}$) if for all $\varepsilon = (N, L, e) \in \mathcal{F}$, all $i \in N$, and all $t, t' \in \mathbb{R}_+$ with $\bar{L}_i \geq t > t'$, $f_i^\varepsilon(t) \geq f_i^\varepsilon(t')$.

$\mathbb{RM}$ requires that each entity rewards the rest of players non-decreasingly when the total amount distributed by such entity rises. As a direct implication of Tarski's fixed-point theorem, the next proposition establishes the existence of non-empty financial rules compatible with the standard requirements of **LL** and **AP**, and supported by division schemes.

Proposition 3. *Let $f \in \mathcal{DS}$ satisfy $\mathbb{RM}$. Then, there exists non-empty financial rules σ supported by f satisfying* **LL** *and* **AP**.

Proof. Let $f \in \mathcal{DS}$ satisfy $\mathbb{RM}$. Then, $\Phi^{\varepsilon,f}$ is non-decreasing for all $\varepsilon \in \mathcal{F}$. Now, from Corollary 2 we can infer a *procedure* to construct non-empty financial rules satisfying **LL** and **AP**. Indeed, by Tarski's theorem, the set of fixed-points $FIX(\Phi^{\varepsilon,f})$ is non-empty and forms a complete lattice. We now may define a non-empty financial rule σ supported by f as follow: given $\varepsilon = (N, L, e) \in \mathcal{F}$ choose a non-empty subset $\mathcal{V} \subseteq FIX(\Phi^{\varepsilon,f})$. For all $\mathbf{t} \in \mathcal{V}$, define the matrix $P^{\mathbf{t}} \in \mathcal{M}(N)$ as $P_{ij}^{\mathbf{t}} = f_{ij}^\varepsilon(\mathbf{t}_i)$, for all $i, j \in N$, and $\sigma(\varepsilon) = \{P^{\mathbf{t}} \mid \mathbf{t} \in \mathcal{V}\}$. Notice that, for all $\mathbf{t} \in \mathcal{V}$ and all $i \in N$, by $\mathbb{BB}$ of f we have that $\bar{P}_i^{\mathbf{t}} = \sum_{k \in N} f_{ik}^\varepsilon(\mathbf{t}_i) = \mathbf{t}_i$, that is, $\bar{P}^{\mathbf{t}}$ is a fixed-point of $\Phi^{\varepsilon,f}$. Hence, from Corollary 2 we may conclude that σ satisfies **LL** and **AP**. □

Proposition 3 generalizes analogous results in Groote Schaarsberg et al. (2018) that applies for financial rules induced by the same bankruptcy rule for all players, as well as the extension to financial rules induced by an inventory of bankruptcy rules, one for each player (see Csóka and Hearings, 2018).

The next remark emphasizes that $\mathbb{RM}$ can not be weakened, in view of expression (5), imposing monotonicity in aggregate terms.

Remark 1. *$\mathbb{RM}$ is equivalent to require that whenever the whole set of players distribute larger amounts, then every single player receives at least as initially. Formally, $f \in \mathcal{DS}$ satisfies $\mathbb{RM}$ if and only if, for all $\varepsilon = (N, L, e) \in \mathcal{F}$, all $i \in N$, and all $\mathbf{t}, \mathbf{t}' \in \mathbb{R}^N_+$ with $\bar{L} \geq \mathbf{t} \geq \mathbf{t}'$, $\sum_{k \in N} f^\varepsilon_{ki}(\mathbf{t}_k) \geq \sum_{k \in N} f^\varepsilon_{ki}(\mathbf{t}'_k)$, which, in view of expression (5), it is sufficient for the monotonicity of $\Phi^{\varepsilon,f}$. The* only if *part is straightforward. To show the reverse implication suppose that f is not $\mathbb{RM}$, that is, there is $\varepsilon = (N, L, e) \in \mathcal{F}$, $i, j \in N$, and $t, t' \in \mathbb{R}_+$ with $\bar{L}_j \geq t > t'$ but $f^\varepsilon_{ji}(t) < f^\varepsilon_{ji}(t')$. Now consider $\mathbf{t}, \mathbf{t}' \in \mathbb{R}^N$ being $\mathbf{t}_j = t$, $\mathbf{t}'_j = t'$, and $\mathbf{t}_k = \mathbf{t'}_k = 0$ for all $k \in N \setminus \{j\}$. Then, $\bar{L} \geq \mathbf{t} > \mathbf{t}'$ but*

$$\sum_{k \in N} f^\varepsilon_{ki}(\mathbf{t}_k) = f^\varepsilon_{ji}(t_j) + \sum_{k \in N \setminus \{j\}} f^\varepsilon_{ki}(0) < f^\varepsilon_{ji}(t'_j) + \sum_{k \in N \setminus \{j\}} f^\varepsilon_{ki}(0) = \sum_{k \in N} f^\varepsilon_{ki}(\mathbf{t}'_k),$$

which proves the if *part.*

Another direct consequence of Tarski's fixed-point theorem is that if a financial rule σ supported by a resource monotonic division scheme f meets **LL** and **AP** then, for any financial system, the equity value of any economic entity does not depend on the chosen clearing payment matrix in σ. Hence, although in general a financial rule may propose different clearing payment matrices for the same financial system, uniqueness in terms of utility (net worth) is guaranteed.

Proposition 4. *Let $f \in \mathcal{DS}$ satisfying $\mathbb{RM}$ and σ be a financial rule supported by f. If σ satisfies **LL** and **AP** then, for all $\varepsilon = (N, L, e) \in \mathcal{F}$ and all $P, P' \in \sigma(\varepsilon)$, $E_i(P, e) = E_i(P', e)$ for all $i \in N$.*

Proof. Let σ be a financial rule that meets **LL** and **AP** and supported by $f \in \mathcal{DS}$ satisfying $\mathbb{RM}$. Let $\varepsilon = (N, L, e) \in \mathcal{F}$. By Tarski's theorem applied to $\Phi^{\varepsilon,f}$ on $[\mathbf{0}, \bar{L}]$ the set of fixed-points $FIX(\Phi^{\varepsilon,f})$ is non-empty and forms a complete lattice. Let $\mathbf{t}^+ \in [\mathbf{0}, \bar{L}]$ be the supremum of $FIX(\Phi^{\varepsilon,f})$. Now, define the financial rule σ' as $\sigma'(\varepsilon) = \{P^+\}$, where the clearing payment matrix $P^+ \in \mathcal{M}(N)$ is given by $P^+_{ij} = f^\varepsilon_{ij}(\mathbf{t}^+_i)$ for all $i, j \in N$. By Corollary 2, σ' satisfies **LL** and **AP** and thus, for all $i \in N$, we have that

$$\begin{aligned} E_i(P^+, e) &= e_i + \sum_{k\in N} P^+_{ki} - \bar{P}^+_i \\ &= e_i + \sum_{k\in N} P^+_{ki} - \min\left\{e_i + \sum_{k\in N} P^+_{ki}, \bar{L}_i\right\} \\ &= \max\left\{0, e_i + \sum_{k\in N} P^+_{ki} - \bar{L}_i\right\}. \end{aligned} \tag{6}$$

Similarly, for all $P \in \sigma(\varepsilon)$ and all $i \in N$,

$$E_i(P, e) = \max\left\{0, e_i + \sum_{k\in N} P_{ki} - \bar{L}_i\right\}. \tag{7}$$

By Corollary 2, $\bar{P}$ is a fixed-point of $\Phi^{\varepsilon,f}$ and thus $\mathbf{t}^+ \geq \bar{P}$. By $\mathbb{RM}$ of f, $\sum_{k\in N} P^+_{ki} = \sum_{k\in N} f^\varepsilon_{ki}(\mathbf{t}^+_k) \geq \sum_{k\in N} f^\varepsilon_{ki}(\bar{P}_k) = \sum_{k\in N} P_{ki}$. Hence, for all $i \in N$, from (6) and (7) we obtain,

$$E_i(P^+, e) \geq E_i(P, e).$$

If there is $i \in N$ such that $E_i(P^+, e) > E_i(P, e)$, then $\sum_{i\in N} e_i = \sum_{i\in N} E_i(P^+, e) > \sum_{i\in N} E_i(P, e) = \sum_{i\in N} e_i$ which leads to a contradiction. Thus, $E_i(P^+, e) = E_i(P, e)$ for all $i \in N$, which finishes the proof. □

Remark 2. *If a non-empty financial rule σ, supported or not by a division scheme, satisfies* **CB**, **LL** *and* **AP**, *a sufficient condition to ensure the same value of equity for all players and all clearing payment matrices in σ is that, for all $\varepsilon = (N, L, e) \in \mathcal{F}$ and all $P, P' \in \sigma(\varepsilon)$, either $\sum_{j\in N} P_{ji} \geq \sum_{j\in N} P'_{ji}$ or $\sum_{j\in N} P_{ji} \leq \sum_{j\in N} P'_{ji}$ for all $i \in N$. Another sufficient condition is that the set of clearing payment matrices in $\sigma(\varepsilon)$ forms either a join-semilattice or a meet-semilattice, i.e., $P \vee P' \in \sigma(\varepsilon)$ or $P \wedge P' \in \sigma(\varepsilon)$.*

Proposition 4 extends similar results in Eisenberg and Noe (2001), Groote Schaarsberg et. al (2018), Csóka and Herings (2018) (for the perfectly divisible setup), and Koster (2019). From the proof of Proposition 4 it follows immediately that, in terms of the equity values, financial rules supported by the same division scheme are equivalent.

Corollary 3. *Let $f \in \mathcal{DF}$ satisfy $\mathbb{RM}$ and σ, σ' be two different financial rules supported by f meeting* **LL** *and* **AP**. *Then, for all $\varepsilon = (N, L, e)$, all $P \in \sigma(\varepsilon)$, and all $P' \in \sigma'(\varepsilon)$, it holds that $E_i(P, e) = E_i(P', e)$ for all $i \in N$.*

5. On non-manipulability of financial rules

The aim of this section is to extend the study of strategic incentives in the setting of bankruptcy problems to the financial systems setup. In particular, it is of our interest to investigate the existence of financial rules that are immune to strategic manipulations of the economic entities in the system via merging or splittings. An intuitive way to extend non-manipulability to financial systems is to enforce the financial rule to avoid giving incentives to firms either to split or to merge in order to obtain larger profits, that is, larger equity values. In contrast, Csóka and Herings (2021) impose some invariance on the payment matrices. Another distinct aspect is that, as in de Frutos (1999), we study merging and splitting incentives separately. Formally, a financial rule σ satisfies

- *non-manipulability* (**NM**) if for all $N, N' \in \mathcal{N}$ and all (N, L, e), $(N', L', e') \in \mathcal{F}$, if $N' \subset N$ and there is $m \in N'$ such that

$$\begin{array}{rcl} e'_m & = & e_m + \sum_{k \in N \setminus N'} e_k \\ e'_i & = & e_i \text{ for all } i \in N' \setminus \{m\} \\ L'_{mj} & = & L_{mj} + \sum_{k \in N \setminus N'} L_{kj} \text{ for all } j \in N' \setminus \{m\} \\ L'_{jm} & = & L_{jm} + \sum_{k \in N \setminus N'} L_{jk} \text{ for all } j \in N' \setminus \{m\} \\ L'_{ij} & = & L_{ij} \text{ for all } i, j \in N' \setminus \{m\} \end{array} \tag{8}$$

then, for all $P \in \sigma(N, L, e)$ and all $P' \in \sigma(N', L', e')$, we have

$$E_m(P', e') = E_m(P, e) + \sum_{k \in N \setminus N'} E_k(P, e); \tag{9}$$

NM can be divided into *non-manipulability via merging* (**NMM**) requiring

$$E_m(P', e') \leq E_m(P, e) + \sum_{k \in N \setminus N'} E_k(P, e), \tag{10}$$

and *non-manipulability via splitting* (**NMS**) imposing the reverse inequality

$$E_m(P', e') \geq E_m(P, e) + \sum_{k \in N \setminus N'} E_k(P, e). \tag{11}$$

NM requires **NMM** and **NMS** together. **NMM** imposes that no group of entities has incentives to merge liabilities to others, liabilities against others and initial endowments, by means of comparing equity values. On the other hand, **NMS** says that no entity has incentives to split in a new group of entities dividing among them the initial liabilities to others, liabilities against others and the initial endowments, in terms of equity values.

It is worth to mention that if a financial rule satisfies non-manipulability as defined in Csóka and Herings (2021) then it satisfies **NM**, but the reverse implication is not true. The underlying reason is that the conditions imposed in their definition of non-manipulability concern some invariance on payments, not only for the group of firms merging or splitting, but also for those that are not involved (in the spirit of strong non-manipulability). This conditions imply that the equity values of the agents do not change.

Not surprisingly, in our first result we prove that **NMS** is incompatible with **LL**, **AP**, and **CB**. As expected, the value of equity of a firm may rise when such firm splits into a "good" firm keeping the full endowment and the liabilities of others to it, and a "bad" firm that inherits uniquely the liabilities of the original firm to others.

Theorem 3. *There is no a financial rule satisfying* **CB***,* **LL***,* **AP***, and* **NMS***.*

Proof. Let σ be a financial rule satisfying **CB**, **LL**, **AP**, and **NMS**. Let $\varepsilon' = (N', L', e') \in \mathcal{F}$ with set of players $N' = \{1, 2\}$, vector of endowments $e' = (1, 0)$, and matrix of liabilities

$$L' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

By **CB**, **LL**, and **AP** we have that $\sigma(\varepsilon) = \{L'\}$

Now, assume that firm 1 splits into firms 1 and 3, keeping the full initial endowment in firm 1, and the initial liability of firm 1 to firm 2, is conveyed to the new firm 3. So, we take the new financial system $\varepsilon = (N, L, e) \in \mathcal{F}$ with

$N = \{1, 2, 3\}$, $e = (1, 0, 0)$ and

$$L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Observe that, by **CB**, players 1 and 2 pay nothing because they have no liabilities to other firms. Moreover, since $e_3 = 0$, **LL** implies that the payoffs of player 3 are also zero. Hence,

$$\sigma(\varepsilon) = \left\{ P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Finally, $E_1(L', e') = e'_1 + L'_{21} - L'_{12} = 0$ while $E_1(P, e) = e_1 + P_{21} + P_{31} - P_{12} - P_{13} = 1$ and $E_3(P, e) = e_3 + P_{13} + P_{23} - P_{31} - P_{32} = 0$, in contradiction with **NMS**. $\square$

It is worth to mention that for the case of splitting firms we allow the possibility to create fictitious liabilities among them. Nevertheless, Theorem 3 still hold if we no longer admit this type of malpractices, forcing all liabilities among those firms splitting to be zero.

The rest of this section is devoted to show that, contrary to Theorem 3, there is a large family of financial rules for which **CB**, **LL**, **AP** and **NMM** are compatible. A natural approach is to investigate if an inventory of bankruptcy rules satisfying non-manipulability via merging for bankruptcy problems produces a financial rule that is non-manipulable via merging in the setting of financial systems. Unfortunately, as the next proposition states, this is not the case.

Proposition 5. *A financial rule σ satisfying* **LL** *and* **AP***, and induced by a collection of bankruptcy rules $(\beta^i)_{i\in\mathbb{N}}$ satisfying* NMM*, does not need to satisfy* **NMM***.*

The proof of Proposition 5 can be found in the Appendix.

To overcome this problem, we extend SNMM from bankruptcy rules to complete division schemes. A complete division scheme $f \in \mathcal{DS}$ on $\mathcal{F}$ satisfies

- *strong non-manipulability via Merging* ($\mathbb{SNMM}$) if for all $N, N' \in \mathcal{N}$ and all $\varepsilon = (N, L, e), \varepsilon' = (N', L', e') \in \mathcal{F}$, if $N' \subset N$ and there is $m \in N'$ such that all conditions in (8) hold then, for all $j \in N' \setminus \{m\}$, it holds that

$$f_{jk}^{\varepsilon'}(t_j) \geq f_{jk}^{\varepsilon}(t_j), \tag{12}$$

  for all $k \in N' \setminus \{m\}$ and all $t_j \in \mathbb{R}_+$ such that $0 \leq t_j \leq \bar{L}'_j = \bar{L}_j$.

$\mathbb{SNMM}$ imposes that whenever a group of firms merge, not only receive from each other firm in $N' \setminus \{m\}$ at most as when they are separated, but also those firms that do not merge are rewarded at least as initially by any other firm in $N' \setminus \{m\}$.

Remark 3. *Let us point out that demanding that players not merging are not payed less by each one of the non-merging entities is equivalent to require that they receive in total at least as initially. That is, $f \in \mathcal{DS}$ satisfies $\mathbb{SNMM}$ if and only if $\sum_{j \in N' \setminus \{m\}} f_{jk}^{\varepsilon'}(\mathbf{t}_j) \geq \sum_{j \in N' \setminus \{m\}} f_{jk}^{\varepsilon}(\mathbf{t}_j)$ for all $k \in N' \setminus \{m\}$ and all $\mathbf{t} \in \mathbb{R}_+^N$ such that $\mathbf{t} \leq \bar{L} = \bar{L}'$, where $\varepsilon = (N, L, e)$ and $\varepsilon' = (N', L', e')$ are defined as in (8). The if part is straightforward. To see the reverse implication, suppose, on the contrary, there are $\varepsilon = (N, L, e)$ and $\varepsilon' = (N'L', e')$ as defined in (8), $j^*, k^* \in N' \setminus \{m\}$, and $\bar{L}_{j^*} = \bar{L}'_{j^*} \geq t_{j^*} > 0$ such that $f_{j^*k^*}^{\varepsilon'}(t_{j^*}) < f_{j^*k^*}^{\varepsilon}(t_{j^*})$. Now, define $\mathbf{t} \in \mathbb{R}_+^N$ as $\mathbf{t}_i = \mathbf{t}_{j^*}$ if $i = j^*$ and $\mathbf{t}_i = 0$ otherwise. Hence, since $\sum_{j \in N' \setminus \{m, j^*\}} f_{jk^*}^{\varepsilon'}(0) = \sum_{j \in N' \setminus \{m, j^*\}} f_{jk^*}^{\varepsilon}(0) = 0$, we obtain $\sum_{j \in N' \setminus \{m\}} f_{jk^*}^{\varepsilon'}(\mathbf{t}_j) = f_{j^*k^*}^{\varepsilon'}(\mathbf{t}_{j^*}) < f_{j^*k^*}^{\varepsilon}(\mathbf{t}_{j^*}) = \sum_{j \in N' \setminus \{m\}} f_{jk^*}^{\varepsilon}(\mathbf{t}_j)$, which conclude the arguments.*

In fact, if f describes an inventory of bankruptcy rules (see Definition 4), $\mathbb{SNMM}$ corresponds to SNMM. Formally,

Proposition 6. *Let $f \in \mathcal{DS}$ represent a collection of bankruptcy rules $(\beta^i)_{i \in \mathbb{N}}$. Then, f satisfies $\mathbb{SNMM}$ if and only if β^i satisfies* SNMM *for all $i \in \mathbb{N}$.*

Proof. Let $f \in \mathcal{DS}$ represent a collection of bankruptcy rules $(\beta^i)_{i \in \mathbb{N}}$ satisfying SNMM for all $i \in \mathbb{N}$. Let $\varepsilon = (N, L, e), \varepsilon' = (N', L'e') \in \mathcal{F}$, with $m \in N' \subset N$ such that all conditions in (8) hold. Let $j, k \in N' \setminus \{m\}$ and $t_j \in \mathbb{R}_+$ with $t_j \leq \bar{L}_j = \bar{L}'_j$, then $f_{jk}^{\varepsilon'}(t_j) = \beta_k^j(N' \setminus \{j\}, t_j, (L'_{ji})_{i \in N' \setminus \{j\}}) \geq \beta_k^j(N \setminus \{j\}, t_j, (L_{ji})_{i \in N \setminus \{j\}}) = f_{jk}^{\varepsilon}(t_j)$, where the inequality follows from SNMM of β^j.

To show the *only if* part, let $f \in \mathcal{DS}$ satisfy $\mathbb{SNMM}$ and represent a collection of bankruptcy rules $(\beta^i)_{i\in\mathbb{N}}$. Let $j \in \mathbb{N}$, and let $\delta = (N, E, c), \delta' = (N', E, c') \in \mathcal{B}$, with $N' \subset N$, $j \notin N$ and $m \in N'$ such that $c'_m = c_m + \sum_{i\in N\setminus N'} c_i$ and $c'_i = c_i$ for all $i \in N' \setminus \{m\}$. Define $\varepsilon = (N \cup \{j\}, L, e)$, $\varepsilon' = (N' \cup \{j\}, L', e') \in \mathcal{F}$ by

$$e_k = \begin{cases} E & \text{if} \quad k = j \\ 0 & \text{if} \quad k \in N \setminus \{j\} \end{cases} \qquad \text{and} \qquad e'_k = \begin{cases} E & \text{if} \quad k = j \\ 0 & \text{if} \quad k \in N' \setminus \{j\} \end{cases},$$

and the matrices of liabilities as follows:

$$L_{ki} = \begin{cases} c_i & \text{if} \quad k = j \text{ and } i \neq j \\ 0 & \qquad \text{otherwise} \end{cases} \qquad \text{and} \qquad L'_{ki} = \begin{cases} c'_i & \text{if} \quad k = j \text{ and } i \neq j \\ 0 & \qquad \text{otherwise} \end{cases}.$$

Now, for $k \in N' \setminus \{m\}$, $\beta^j_k(\delta') = f^{\varepsilon'}_{jk}(E) \geq f^{\varepsilon}_{jk}(E) = \beta^j_k(\delta)$ where the inequality follows from $\mathbb{SNMM}$ of f. □

Next, we remark that $\mathbb{SNMM}$ complete division schemes substantially improve the class of SNMM bankruptcy rules. For instance, it allows to establish a priority list of creditors on the basis of the debts of such creditors to an institution, rather than on the claims of the creditors on the institution.[5]

Remark 4. *There are complete division schemes satisfying* $\mathbb{SNMM}$ *that can not be obtained from an inventory of bankruptcy rules. To make it clear, for a given* $\varepsilon = (N, L, e)$ *and* $i \in N$*, let* $\prec^{\varepsilon}_i$ *be the strict total order on* $N \setminus \{i\}$ *defined as follows: for all* $j, k \in N \setminus \{i\}$*,*

$$k \prec^{\varepsilon}_i j \textit{ if either } L_{ki} < L_{ji} \textit{ or } L_{ki} = L_{ji} \textit{ and } k < j. \tag{13}$$

The debts priority complete division scheme, $f^d \in \mathcal{DS}$*, is defined as*

$$(f^d)^{\varepsilon}_{ij}(t) = \begin{cases} \max\left\{0, \min\left\{t - \sum_{\{k\in N\setminus\{i\} | k\prec^{\varepsilon}_i j\}} L_{ik},\ L_{ij}\right\}\right\} & \textit{if} \quad j \in N \setminus \{i\} \\ 0 & \textit{if} \quad j = i \end{cases},$$

for all $0 \leq t \leq \bar{L}_i$.

[5]See Moulin (2000) for a study of priority rules in bankruptcy problems.

According to f^d, for a given financial system and to clear liabilities, any entity prioritizes those other entities having lower debts to it. Clearly, it satisfies $\mathbb{RM}$. To check $\mathbb{SNMM}$, consider $\varepsilon = (N, L, e)$, $\varepsilon' = (N', L', e') \in \mathcal{F}$, with $N' \subset N$ and there is $m \in N'$ such that all conditions in (8) hold. Choose an arbitrary firm $i \in N' \setminus \{m\}$, then when entities in $\{m\} \cup N \setminus N'$ merge into m, any other entity $j \in N' \setminus \{m\}$ achieves a higher priority according to $\prec_i^{\varepsilon'}$ than according to $\prec_i^{\varepsilon}$ and, moreover, it does not have any new predecessor. Additionally, all predecessor of j with respect to $\prec_i^{\varepsilon'}$ present the same liabilities as initially. So, for all $j \in N' \setminus \{m\}$, we have $(f^d)_{ij}^{\varepsilon'}(t_i) \geq (f^d)_{ij}^{\varepsilon}(t_i)$ for all $0 \leq t_i \leq \bar{L}_i = \bar{L}'_i$ and thus f^d satisfies $\mathbb{SNMM}$.

To finish, we make clear that f^d does not represent an inventory of bankruptcy rules. Let $\varepsilon = (N, L, e)$, $\varepsilon' = (N, L', e') \in \mathcal{F}$, with agent set $N = \{1, 2, 3\}$, vectors of initial endowments $e = e' = (1, 0, 0)$, and the following liability matrices:

$$L = \begin{pmatrix} 0 & 1 & 1 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad L' = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}.$$

It is not difficult to check that $3 \prec_1^{\varepsilon} 2$ and, on the contrary, $2 \prec_1^{\varepsilon'} 3$. Under **LL** *and* **AP***, for any financial rule σ supported by f^d we have that*

$$\sigma(\varepsilon) = \left\{ P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \quad \text{and} \quad \sigma(\varepsilon') = \left\{ P' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Hence, $(f^d)_1^{\varepsilon}(1) = (0, 0, 1)$ while $(f^d)_1^{\varepsilon'}(1) = (0, 1, 0)$. However, debtor 1 faces the same bankruptcy situation in both ε and ε' described by an estate $E = 1$ and vector of claims $c = (1, 1)$, which shows that f^d does not admit a representation as bankruptcy rules.

In the main result of the paper we show that complete division schemes satisfying $\mathbb{RM}$ and $\mathbb{SNMM}$, which in view of Remark 4 is much richer than inventories of bankruptcy rules meeting the two analogous properties, lead to **LL** and **AP** financial rules satisfying **NMM**.

Theorem 4. *Let $f \in \mathcal{DS}$ satisfy $\mathbb{RM}$ and σ be a financial rule supported by f satisfying* **LL** *and* **AP**. *If f satisfies $\mathbb{SNMM}$, then σ satisfies* **NMM**.

Proof. Let $f \in \mathcal{DS}$ satisfying $\mathbb{RM}$ and $\mathbb{SNMM}$. Since f satisfies $\mathbb{RM}$, for all $\varepsilon \in \mathcal{F}$, by Tarski's fixed-point theorem $FIX\left(\Phi^{\varepsilon,f}\right)$ is non-empty and forms a complete lattice. Given $\varepsilon = (N, L, e)$, let $\mathbf{t}^- \in [\mathbf{0}, \bar{L}]$ be the infimum of $FIX\left(\Phi^{\varepsilon,f}\right)$. Now, define the financial rule σ' as $\sigma'(\varepsilon) = \{P^-\}$, where the payment matrix $P^- \in \mathcal{M}(N)$ is given by $P^-_{ij} = f^{\varepsilon}_{ij}(\mathbf{t}^-_i)$ for all $i, j \in N$. Hence, σ' is supported by f and, by Corollary 2, σ' satisfies **LL** and **AP**.

Assume, by contradiction, that there is a financial rule σ supported by f that meets **LL** and **AP** but not **NMM**. Then, there exist $\varepsilon = (N, L, e)$ and $\varepsilon' = (N', L', e')$ with $N' \subset N$ and $m \in N'$ such that all conditions in (8) hold, and there exists $P \in \sigma(N, L, e)$ and $P' \in \sigma(N', L', e')$, such that

$$E_m(P', e') > E_m(P, e) + \sum_{k \in N \setminus N'} E_k(P, e). \tag{14}$$

By **LL** and (14), $E_m(P', e') > 0$. Thus, by **AP**, $\bar{P}'_m = \bar{L}'_m$. Since σ is supported by f, it satisfies **CB** and hence

$$P'_{mk} = L'_{mk} \text{ for all } k \in N' \setminus \{m\}. \tag{15}$$

Claim 1 : Let $\mathbf{q} \in \mathbb{R}^N$ be defined as follows:

$$\mathbf{q}_k = \begin{cases} \bar{P}'_k & \text{if} \quad k \in N' \setminus \{m\} \\ \bar{L}_k & \text{if} \quad k \in \{m\} \cup N \setminus N' \end{cases}. \tag{16}$$

Then, $\mathbf{q} \in \left\{\mathbf{t} \in [\mathbf{0}, \bar{L}] \mid \Phi^{\varepsilon,f}(\mathbf{t}) \leq \mathbf{t}\right\}$. Moreover, $\mathbf{t}^- \leq \mathbf{q}$.

To prove it, let us consider two cases:

- $k \in \{m\} \cup N \setminus N'$. In this situation, $\mathbf{q}_k = \bar{L}_k$ and

$$\Phi^{\varepsilon,f}_k(\mathbf{q}) = \min\left\{e_k + \sum_{j \in N} f^{\varepsilon}_{jk}(\mathbf{q}_j), \bar{L}_k\right\} \leq \bar{L}_k = \mathbf{q}_k.$$

- $k \in N' \setminus \{m\}$. In this situation, $\mathbf{q}_k = \bar{P'}_k$ and

$$\begin{aligned}
\Phi_k^{\varepsilon,f}(\mathbf{q}) &= \min\left\{e_k + \sum_{j\in N'\setminus\{m\}} f_{jk}^{\varepsilon}(\mathbf{q}_j) + \sum_{j\in\{m\}\cup N\setminus N'} f_{jk}^{\varepsilon}(\mathbf{q}_j), \bar{L}_k\right\} \\
&\underset{(16)}{=} \min\left\{e_k + \sum_{j\in N'\setminus\{m\}} f_{jk}^{\varepsilon}(\bar{P'}_j) + \sum_{j\in\{m\}\cup N\setminus N'} f_{jk}^{\varepsilon}(\bar{L}_j), \bar{L}_k\right\} \\
&\underset{\mathbb{CB}+\mathbb{BB}}{=} \min\left\{e_k + \sum_{j\in N'\setminus\{m\}} f_{jk}^{\varepsilon}(\bar{P'}_j) + \sum_{j\in\{m\}\cup N\setminus N'} L_{jk}, \bar{L}_k\right\} \\
&= \min\left\{e_k + \sum_{j\in N'\setminus\{m\}} f_{jk}^{\varepsilon}(\bar{P'}_j) + L'_{mk}, \bar{L}_k\right\} \\
&\underset{(15)}{=} \min\left\{e_k + \sum_{j\in N'\setminus\{m\}} f_{jk}^{\varepsilon}(\bar{P'}_j) + P'_{mk}, \bar{L}_k\right\} \\
&\underset{P'_{mk}=f_{mk}^{\varepsilon'}(\bar{P'}_m)}{=} \min\left\{e_k + \sum_{j\in N'\setminus\{m\}} f_{jk}^{\varepsilon}(\bar{P'}_j) + f_{mk}^{\varepsilon'}(\bar{P'}_m), \bar{L}_k\right\} \\
&\underset{\mathbb{SNMM}}{\leq} \min\left\{e_k + \sum_{j\in N'\setminus\{m\}} f_{jk}^{\varepsilon'}(\bar{P'}_j) + f_{mk}^{\varepsilon'}(\bar{P'}_m), \bar{L}_k\right\} \\
&= \min\left\{e_k + \sum_{j\in N'} f_{jk}^{\varepsilon'}(\bar{P'}_j), \bar{L}_k\right\} \\
&= \Phi^{\varepsilon',f}(\bar{P'}) \underset{\bar{P'}\in FIX(\Phi^{\varepsilon',f})}{=} \bar{P'}_k \underset{(16)}{=} \mathbf{q}_k,
\end{aligned}$$

where the last but one equality follows from Corollary (2). By Tarski's fixed-point theorem, $\mathbf{t}^-$ is also the infimum of $\left\{\mathbf{t} \in [\mathbf{0}, \bar{L}] \mid \Phi^{\varepsilon,f}(\mathbf{t}) \leq \mathbf{t}\right\}$ and thus $\mathbf{t}^- \leq \mathbf{q}$. This concludes the proof of Claim 1.

Claim 2 : Let $k \in N' \setminus \{m\}$, then $E_k(P, e) = E_k(P^-, e) \leq E_k(P', e')$.

Since σ and σ' satisfy **LL** and **AP** and are supported by f meeting $\mathbb{RM}$, from Corollary 3 we obtain $E_k(P, e) = E_k(P^-, e)$.

From Claim 1, **CB**, and (16), for all $k \in N' \setminus \{m\}$,

$$\mathbf{t}_k^- \leq \mathbf{q}_k = \bar{P'}_k \leq \bar{L'}_k = \bar{L}_k.$$

Observe that

$$E_k(P^-, e) = e_k + \sum_{j\in N} f_{jk}^{\varepsilon}(\mathbf{t}_j^-) - \mathbf{t}_k^-.$$

Let us consider the following two cases:

- $\mathbf{t}_k^- < \bar{L}_k$. Then,
$$E_k(P^-, e) \underset{\mathbf{AP}}{=} 0 \underset{\mathbf{LL}}{\leq} E_k(P', e').$$

- $\mathbf{t}_k^- = \mathbf{q}_k = \bar{P'}_k = \bar{L'}_k = \bar{L}_k$. In this case,

$$\begin{aligned}
E_k(P^-, e) &\underset{\mathbf{t}^- \in FIX(\Phi^{\varepsilon,f})}{=} e_k + \sum_{j \in N} f_{jk}^{\varepsilon}(\mathbf{t}_j^-) - \min\left\{ e_k + \sum_{j \in N} f_{jk}^{\varepsilon}(\mathbf{t}_j^-), \bar{L}_k \right\} \\
&= \max\left\{ e_k + \sum_{j \in N} f_{jk}^{\varepsilon}(\mathbf{t}_j^-) - \bar{L}_k, 0 \right\} \\
&= \max\left\{ e'_k + \sum_{j \in N' \setminus \{m\}} f_{jk}^{\varepsilon}(\mathbf{t}_j^-) + \sum_{j \in \{m\} \cup N \setminus N'} f_{jk}^{\varepsilon}(\mathbf{t}_j^-) - \bar{L}_k, 0 \right\} \\
&\underset{\mathbb{CB}}{\leq} \max\left\{ e'_k + \sum_{j \in N' \setminus \{m\}} f_{jk}^{\varepsilon}(\mathbf{t}_j^-) + \sum_{j \in \{m\} \cup N \setminus N'} L_{jk} - \bar{L}_k, 0 \right\} \\
&\underset{(*)}{\leq} \max\left\{ e'_k + \sum_{j \in N' \setminus \{m\}} f_{jk}^{\varepsilon}(\bar{P'}_j) + L'_{mk} - \bar{L'}_k, 0 \right\} \\
&\underset{(15)}{=} \max\left\{ e'_k + \sum_{j \in N' \setminus \{m\}} f_{jk}^{\varepsilon}(\bar{P'}_j) + f_{mk}^{\varepsilon'}(\bar{P'}_m) - \bar{L'}_k, 0 \right\} \\
&\underset{\mathbb{SNMM}}{\leq} \max\left\{ e'_k + \sum_{j \in N' \setminus \{m\}} f_{jk}^{\varepsilon'}(\bar{P'}_j) + f_{mk}^{\varepsilon'}(\bar{P'}_m) - \bar{L'}_k, 0 \right\} \\
&= E_k(P', e'),
\end{aligned}$$

where the inequality $(*)$ follows from (16), Claim 1, which implies $\bar{P'}_j = \mathbf{q}_j \geq \mathbf{t}_j^-$ for all $j \in N' \setminus \{m\}$, and $\mathbb{RM}$ of f. This concludes the proof of Claim 2.

Hence,

$$\sum_{j \in N' \setminus \{m\}} E_j(P, e) = \sum_{j \in N' \setminus \{m\}} E_j(P^-, e) \leq \sum_{j \in N' \setminus \{m\}} E_j(P', e'). \tag{17}$$

Since

$$\sum_{j \in N} E_j(P, e) = \sum_{j \in N} e_j = \sum_{j \in N'} e'_j = \sum_{j \in N'} E_j(P', e'),$$

from (17) it comes that

$$E_m(P,e) + \sum_{j\in N\setminus N'} E_j(P,e) \geq E_m(P',e'),$$

in contradiction with (14). $\square$

We highlight the important class of bankruptcy rules formed by parametric rules whose representations are subadditive in claims containing, among others, the CEA and the PR solutions. These rules satisfies SNMM (see Corollary 1 and Proposition 2). Hence, as a consequence of Theorem 4, and taking into account Proposition 6, we obtain the following corollary.

Corollary 4. *Let σ be a financial rule induced by a collection of bankruptcy rules $(\beta^i)_{i\in\mathbb{N}}$ where for all $i \in \mathbb{N}$, β^i is a parametric rule such that its representations are sudadditive in claims. Then, σ satisfies* **NMM**.

A natural question is to ask if the reverse implication in Theorem 4 holds. Unfortunately, as the next example shows, this is not the case.

Example 2. *Introduce first the following financial system, $\varepsilon_1 = (N, L, e)$, with agent set $N = \{1,2,3\}$, vectors of initial endowments $e = (1,0,0)$, and the following liability matrix:*

$$L = \begin{pmatrix} 0 & 5 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now define the complete division scheme g as follows

- *If $\varepsilon \neq \varepsilon_1$ and $i \in N$ or $\varepsilon = \varepsilon_1$ and $i \in N \setminus \{1\}$ then,*

  $$(g)^{\varepsilon}_{ij}(t) = \begin{cases} PR_j\left(N\setminus\{i\}, t, (L_{ij})_{j\in N\setminus\{i\}}\right) & \textit{if} \quad j \in N\setminus\{i\} \\ 0 & \textit{if} \quad j = i \end{cases},$$

  for all $0 \leq t \leq \bar{L}_i$.

- *If $\varepsilon = \varepsilon_1$ and for agent 1,*

$$(g)^{\varepsilon}_{1j}(t) = \begin{cases} PR_j\left(N \setminus \{1\}, t, (L_{1j})_{j \in N\setminus\{1\}}\right) & \text{if} \quad j \in N \setminus \{i\} \text{ and } t \leq 1 \\ 1/3 + CEL_j\left(N \setminus \{1\}, t-1, (L_{1j})_{j \in N\setminus\{1\}}\right) & \text{if} \quad j = 2 \text{ and } t > 1 \\ 2/3 + CEL_j\left(N \setminus \{1\}, t-1, (L_{1j})_{j \in N\setminus\{1\}}\right) & \text{if} \quad j = 3 \text{ and } t > 1 \\ 0 & \text{if} \quad j = i \end{cases},$$

To see that g satisfies $\mathbb{RM}$, it is enough to verify that when $\varepsilon = \varepsilon_1$, for agent 1, $PR(N \setminus \{1\}, 1, (5, 10)) = (1/3, 2/3)$. On the contrary, g is not $\mathbb{SNMM}$. To show it, we introduce the new financial system $\varepsilon_2 = (N', L', e')$ by setting: $N' = \{1, 2, 3, 4\}$, $e' = (1, 0, 0, 0)$, and the following liability matrix:

$$L = \begin{pmatrix} 0 & 5 & 5 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Notice that from ε_2 firms 3 and 4 merge into firm 3 in ε_1. Now, $(g)^{\varepsilon_1}_{13}(3) = 2/3 + 2 = 8/3$ and, by $\mathbb{BB}$ of g, $(g)^{\varepsilon_1}_{12}(3) = 3 - (g)^{\varepsilon_1}_{13}(3) = 3 - 8/3 = 1/3$. Moreover, $(g)^{\varepsilon_2}_{13}(3) = (g)^{\varepsilon_2}_{14}(3) = 1$ and, again by $\mathbb{BB}$ of g, $(g)^{\varepsilon_2}_{12}(3) = 1$. Thus, $(g)^{\varepsilon_1}_{12}(3) < (g)^{\varepsilon_2}_{12}(3)$ and, consequently, g is not $\mathbb{SNMM}$. In fact, extending in a natural way the definition on non manipulability by merging from the setting of bankruptcy rules to division schemes, it can be checked that the g is neither $\mathbb{NMM}$.

To finish, we prove that any financial rule σ on $\mathcal{F}$ meeting **LL** *and* **AP***, and supported by g, is* **NMM***. It is enough to observe that for all $P \in \sigma(\varepsilon_1)$ it holds that $P_{1j} = (g)^{\varepsilon_1}_{1j}(\bar{P}_1) = PR_j\left(N \setminus \{1\}, \bar{P}_1, (L_{1j})_{j \in N\setminus\{1\}}\right)$. Notice that, by* **LL** *and* **AP***, $\bar{P}_1 \leq 1$ since $e_1 = 1$ and firm 1 does not present any liability against any other firm.*

In Example 2, the key factor in order to show that the complete division scheme g is not $\mathbb{NMM}$ relies on the fact that g cannot be represented as a bankruptcy rule. Interestingly, if a financial rule induced by set of bankruptcy rules $(\beta^i)_{i \in \mathbb{N}}$ is **NMM**, then β^i is NMM, for all $i \in \mathbb{N}$.

Theorem 5. *Let $(\beta^i)_{i\in\mathbb{N}}$ be a collection of bankruptcy rules and σ be a financial rule induced by $(\beta^i)_{i\in\mathbb{N}}$ satisfying* **LL** *and* **AP**. *If σ satisfies* **NMM**, *then β^i satisfies* NMM, *for all $i \in \mathbb{N}$.*

Proof. Let $(\beta^i)_{i\in\mathbb{N}}$ be a collection of bankruptcy rules and σ be a financial rule induced by $(\beta^i)_{i\in\mathbb{N}}$ satisfying **LL** and **AP**. Then σ also satisfies **CB**. Take $i \in \mathbb{N}$ and the corresponding bankruptcy rule β^i. Let $N, N' \in \mathcal{N}$, (N, E, c), $(N', E, c') \in \mathcal{B}$ with $m \in N' \subset N$ and $i \notin N$, such that $c'_m = c_m + \sum_{j\in N\setminus N'} c_j$ and $c'_j = c_j$ for all $j \in N' \setminus \{m\}$, we need to show that

$$\beta_m(N', E, c') \leq \beta_m(N, E, c) + \sum_{j\in N\setminus N'} \beta_j(N, E, c). \tag{18}$$

Define the financial systems $\varepsilon = (N \cup \{i\}, L, e)$ and $\varepsilon' = (N' \cup \{i\}, L', e')$ as follows:

$$L_{jk} = \begin{cases} c_k & \text{if } j = i \text{ and } k \neq i \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad L'_{jk} = \begin{cases} c'_k & \text{if } j = i \text{ and } k \neq i \\ 0 & \text{otherwise,} \end{cases}$$

and $e_i = e'_i = E$, $e_j = 0$ for all $j \in N \setminus \{i\}$ and $e'_j = 0$ for all $j \in N' \setminus \{i\}$. It is not difficult to check that ε and ε' satisfy all the conditions in (8).

Let $P \in \sigma(\varepsilon)$ and $P' \in \sigma(\varepsilon')$. By **CB**, if $j \neq i$ then $P_{jk} = 0$ for all $k \in N \cup \{i\}$, and $P'_{jk} = 0$ for all $k \in N' \cup \{i\}$. Under **LL**, **AP**, the equity value for every agent is independent on the chosen payment matrices P and P' (see Proposition 4). Then we have:

$$\begin{aligned} E_m(P', e') &= e'_m + \textstyle\sum_{k\in N\cup\{i\}} P'_{km} - \sum_{k\in N\cup\{i\}} P'_{mk} = P'_{im} \\ &= \beta^i_m(N', \bar{P'}_i, (L'_{ij})_{j\in N'}) \\ &= \beta^i_m(N', E, c'), \end{aligned}$$

where the last equality follows from the definition of L', the fact that $P'_{ji} = 0$ for all $j \in N'$, $e'_i = E \leq \bar{L'}_i = \sum_{k\in N'} c'_k$ since $(N', E, c') \in \mathcal{B}$, and σ satisfies **AP**. Similarly,

$$E_m(P, e) = \beta^i_m(N, E, c)$$

and, for all $k \in N \setminus N'$,

$$E_k(P, e) = \beta^i_k(N, E, c).$$

Finally, by **NMM** of σ, (18) holds. $\square$

Combining Proposition 1 with Theorems 4 and 5, we conclude with a characterization result. For the case of financial rules fulfilling the basic requirements of limited liability and absolute priority that are produced by bankruptcy rules that meet resource monotonicity and consistency, non manipulability via merging is equivalent in both settings.

Theorem 6. *Let $(\beta^i)_{i\in\mathbb{N}}$ be a collection of bankruptcy rules satisfying* RM *and* CONS *and σ be a financial rule induced by $(\beta^i)_{i\in\mathbb{N}}$ satisfying* **LL** *and* **AP**. *Then, σ satisfies* **NMM** *if and only if β^i satisfies* NMM, *for all $i \in \mathbb{N}$.*

6. Concluding remarks

In this paper, we investigate manipulability in financial systems. The property of non-manipulability requires that the merger of a group of agents or the split of an agent into multiple agents does not affect the utility (equity value) of the agents. In line with de Frutos (1999), we consider separately manipulability via merging and manipulability via splitting. To the best of our knowledge, this is the first paper that address these two weak forms of manipulability in the setup of financial systems. We also propose a novel approach to generate financial rules based on division schemes, a notion that allows to take into consideration all the links among agents when clearing the system.

We show that non-manipulability via splitting is incompatible with the basic requirements of claim boundedness, limited liability, and absolute priority while, on the contrary, a large class of financial rules reconcile these conditions with non-manipulability via merging. Indeed, we prove that strong non-manipulability via merging of the underlying division scheme is the keystone to generate financial rules immune to manipulations via merging. We highlight that bankruptcy parametric rules (Young 1987) whose representations are subadditive in claims (Ju, 2003), including the well-established proportional rule, produce financial rules unaffected by manipulability via merging when the financial network collapses. Moreover the set of bankruptcy rules for which the requirement of non advantageous merging is necessary and sufficient to generate a non advantageous merging financial rule is much larger, in fact only consistency, together with the mild condition of resource monotonicity is needed.

Appendix

Proof. (**Theorem 2**) Since SNMM and SNMS are dual properties, it is enough to prove that NMM does not imply SNMM. To do it, we first introduce the following bankruptcy rule. Given $\delta = (N, E, c)$, let $\prec^\delta$ be the strict total order on N defined as follows: for all $i, j \in N$,

$$i \prec^\delta j \text{ if either } c_i < c_j \text{ or } c_i = c_j \text{ and } i < j. \tag{19}$$

The *claims priority bankruptcy rule*, P^c, is defined as follows: let $\delta = (N, E, c) \in \mathcal{B}$ and $\prec^\delta$ be the corresponding order as defined in (19),

$$P_i^c(N, E, c) = \max\left\{0, \min\left\{E - \sum_{\{j \in N \mid j \prec^\delta i\}} c_j,\ c_i\right\}\right\} \tag{20}$$

for all $i \in N$.

Let us consider the following subclass of bankruptcy problems:

$$\mathcal{C}^* = \left\{ \begin{array}{c} \delta = (N, E, c) \in \mathcal{B} \text{ such that } \{1,2\} \subset N, E = 1, c_1 = c_2 = c_{k^\delta} = 1 \\ \text{for some } k^\delta \in N \setminus \{1,2\}, \text{ and } c_i = 0 \text{ for all } i \in N \setminus \{1, 2, k^\delta\} \end{array} \right\}.$$

Now define the bankruptcy rule β^* as follows: let $\delta = (N, E, c) \in \mathcal{B}$ and $i \in N$

$$\beta_i^*(\delta) = \left\{ \begin{array}{ccl} P_i^c(\delta) & \text{if} & \delta \notin \mathcal{C}^* \\ 1 & \text{if} & \delta \in \mathcal{C}^* \text{ and } i = k^\delta \\ 0 & \text{if} & \delta \in \mathcal{C}^* \text{ and } i \neq k^\delta. \end{array} \right. \tag{21}$$

Claim 1: β^* satisfies NMM.

To prove that β^* satifies NMM, let $\delta = (N, E, c)$ and $\delta' = (N', E, c')$ be two bankruptcy problems such that $N' \subset N$ and there is $m \in N'$ with $c'_m = c_m + \sum_{j \in N \setminus N'} c_j$ and $c'_j = c_j$, for all $j \in N' \setminus \{m\}$. We consider the following cases:

Case 1: $\delta, \delta' \notin \mathcal{C}^*$.

Then, $\beta^*(\delta) = P^c(\delta)$ and $\beta^*(\delta') = P^c(\delta')$. Observe that when players in $\{m\} \cup N \setminus N'$ merge into m, then for a given $j \in N' \setminus \{m\}$ the position according to $\prec^{\delta'}$ is less than or equal to the position according to $\prec^\delta$, and moreover, she does not have any new predecessor. Additionally, all predecessors of j with respect to $\prec^{\delta'}$ have the same claim. So, for all $j \in N' \setminus \{m\}$, we have $\beta^*_j(\delta') \geq \beta^*_j(\delta)$ and thus, by budget balance of β^*, NMM holds.

Case 2: $\delta, \delta' \in \mathcal{C}^*$.

If $m \neq k^{\delta'}$, then $\beta^*_m(\delta') = 0$ and hence, by non-negativity of β^*, $0 = \beta^*_m(\delta') \leq \beta^*_m(\delta) + \sum_{j \in N \setminus N'} \beta^*_j(\delta)$.

If $m = k^{\delta'}$, then $\beta^*_m(\delta') = 1$. To see that $\beta^*_m(\delta) + \sum_{j \in N \setminus N'} \beta^*_j(\delta) = 1$ it is enough to check that $k^\delta \in \{m\} \cup N \setminus N'$. Indeed, if not, $k^\delta \in N' \setminus \{m\}$ with $k^\delta \neq k^{\delta'}$ and hence $c_{k^{\delta'}} = c_{k^\delta} = 1$, in contradiction with $\delta' \in \mathcal{C}^*$. Thus, NMM holds.

Case 3: $\delta \notin \mathcal{C}^*$ and $\delta' \in \mathcal{C}^*$.

If $m \neq k^{\delta'}$, as in Case 2 by non-negativity of β^*, $0 = \beta^*_m(\delta') \leq \beta^*_m(\delta) + \sum_{j \in N \setminus N'} \beta^*_j(\delta)$.

If $m = k^{\delta'}$, then $\beta^*_m(\delta') = 1$. Since $\delta' \in \mathcal{C}^*$,

$$1 = c'_{k^{\delta'}} = c_{k^{\delta'}} + \sum_{j \in N \setminus N'} c_j, \tag{22}$$

and hence $|A| \geq 1$, where $A = \{j \in \{k^{\delta'}\} \cup N \setminus N' \text{ such that } c_j > 0\}$. Suppose that $|A| = 1$. In this situation, $1, 2 \in N' \setminus \{m\}$, $c_1 = c'_1 = 1$, $c_2 = c'_2 = 1$ and there is a unique $k \in A$ with $c_k = 1$. Moreover, from the definition of δ and δ', $c_i = 0$ for all $i \in N \setminus \{1, 2, k\}$. But then, $\delta \in \mathcal{C}^*$ getting a contradiction. Consequently, $|A| \geq 2$ and, in view of (22), $0 < c_j < 1$ for all $j \in A$. Now, $c_1 = c_2 = 1$ and $c_i = 0$ for all $i \in N \setminus A \cup \{1, 2\}$, and the order $\prec^\delta$ on N place all players in A immediately after zero-claimants, and by (22) we obtain $\beta^*_{k^{\delta'}}(\delta) + \sum_{j \in N \setminus N'} \beta^*_j(\delta) = 1 = \beta^*_m(\delta')$, which proves NMM.

Case 4: $\delta \in \mathcal{C}^*$ and $\delta' \notin \mathcal{C}^*$.

If $k^\delta \in \{m\} \cup N \setminus N'$, since $\delta \in \mathcal{C}^*$, $\beta^*_{k^\delta}(\delta) = 1$ and, for all $i \in N \setminus \{k^\delta\}$, $\beta^*_i(\delta) = 0$. Thus, $1 = \beta^*_{k^\delta}(\delta) = \beta^*_m(\delta) + \sum_{j \in N \setminus N'} \beta^*_j(\delta) \geq \beta^*_m(\delta')$, where the inequality comes from BB and non-negativity of β^*.

If $k^\delta \notin \{m\} \cup N \setminus N'$. Recall that, since $\delta \in \mathcal{C}^*$, $c_j = 0$ for all $j \in N \setminus \{1, 2, k^\delta\}$. We distinguish the following sub-cases:

(a) $1, 2 \notin \{m\} \cup N \setminus N'$.
Hence, $m \notin \{1, 2, k^\delta\}$. Consequently, $c'_m = c_m + \sum_{j \in N \setminus N'} c_j = 0$. By CB and non-negativity of β^*, we conclude that $0 = \beta^*_m(\delta') \leq \beta^*_m(\delta) + \sum_{j \in N \setminus N'} \beta^*_j(\delta)$.

(b) $1, 2 \in \{m\} \cup N \setminus N'$.
Here, $c'_m = c_1 + c_2 = 2$, $c'_{k^\delta} = c_{k^\delta} = 1$, and $c'_j = 0$ for all $j \in N' \setminus \{m, k^\delta\}$. Hence, $\beta^*_m(\delta') = P^c_m(\delta') = 0$, and by non-negativity of β^*, we conclude that $\beta^*_m(\delta') \leq \beta^*_m(\delta) + \sum_{j \in N \setminus N'} \beta^*_j(\delta)$.

(c) $1 \in \{m\} \cup N \setminus N'$ but $2 \notin \{m\} \cup N \setminus N'$.
In this case, $c'_m = c_1 = 1$, $c'_2 = c_2 = 1$, $c'_{k^\delta} = c_{k^\delta} = 1$, and $c'_j = 0$ for all $j \in N' \setminus \{m, 2, k^\delta\}$. Since $\delta' \notin \mathcal{C}^*$, $m \neq 1$ and thus $m \geq 3$. Hence, $\beta^*_m(\delta') = P^c_m(\delta') = 0$ and, by non-negativity of β^*, we obtain $\beta^*_m(\delta') \leq \beta^*_m(\delta) + \sum_{j \in N \setminus N'} \beta^*_j(\delta)$.

(d) $1 \notin \{m\} \cup N \setminus N'$ but $2 \in \{m\} \cup N \setminus N'$.
In this situation, $c'_m = c_2 = 1$, $c'_1 = c_1 = 1$, $c'_{k^\delta} = c_{k^\delta} = 1$, and $c'_j = 0$ for all $j \in N' \setminus \{1, m, k^\delta\}$. Since $\delta' \notin \mathcal{C}^*$, $m \neq 2$ and thus $m \geq 3$. Hence, $\beta^*_m(\delta') = P^c_m(\delta') = 0$ and, by non-negativity of β^*, we obtain $\beta^*_m(\delta') \leq \beta^*_m(\delta) + \sum_{j \in N \setminus N'} \beta^*_j(\delta)$.

Claim 2: β^* does not satisfy SNMM.

To show that β^* does not meet SNMM, consider the bankruptcy problem $\delta = (N, E, c)$ with set of players $N = \{1, 2, 3, 4\}$, estate $E = 1$, and vector of claims $c = (1, 0, 1, 1)$. Now let $\delta' = (N', E, c')$ with $N' = \{1, 2, 3\}$, where agents 2 and 4 have merged into agent 2, and the vector of claims is $c' = (1, 1, 1)$. Since $\delta \notin \mathcal{C}^*$, $\beta^*(\delta) = (1, 0, 0, 0)$. On the other hand, $\delta' \in \mathcal{C}^*$ and thus $\beta^*(\delta') = (0, 0, 1)$. Hence, $\beta^*_1(\delta') = 0 < \beta^*_1(\delta) = 1$, and β^* does not satisfy SNMM. $\square$

Proof. (**Proposition** 5) We use the bankruptcy rule β^* introduced in (21), that satisfies NMM, to define a financial rule that does not meet **NMM**.

Let $\varepsilon = (N, L, e), \varepsilon' = (N', L', e') \in \mathcal{F}$ with $N = \{1, 2, 3, 4, 5\}$ and $N' = \{1, 2, 3, 4\}$, vectors of endowments $e = (0.1, 0.1, 0.1, 0.7, 0)$ and $e' = (0.1, 0.1, 0.1, 0.7)$, and matrices of liabilities:

$$L = \begin{pmatrix} 0 & 0.1 & 0.1 & 0.1 & 0 \\ 0.1 & 0 & 0.1 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0.1 & 0.1 & 0 & 0.1 & 0 \end{pmatrix} \quad \text{and} \quad L' = \begin{pmatrix} 0 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0 & 0.1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Now define the financial rule σ as follows:[6]

$$\sigma(\varepsilon) = \begin{cases} \emptyset & \text{if} \quad \varepsilon \notin \{\varepsilon, \varepsilon'\} \\ P & \text{if} \quad \varepsilon = \varepsilon \\ P' & \text{if} \quad \varepsilon = \varepsilon' \end{cases}$$

where

$$P = \begin{pmatrix} 0 & 0.1 & 0.1 & 0.1 & 0 \\ 0.1 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad P' = \begin{pmatrix} 0 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0 & 0.1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Note that σ is induced by $(\beta^*)^i$ for all $i \in \mathbb{N}$ as defined in (21). In fact, to obtain the rows in P only claims priority bankruptcy rules apply (see expression (20)). Moreover, to obtain the rows in P' claims priority bankruptcy rules apply except for row 4, where $(P')_4 = \big((\beta^*)^4(N \setminus \{4\}, 1, (1,1,1)), 0\big) = ((0,0,1),0)$, since $(N \setminus \{4\}, 1, (1,1,1)) \in \mathcal{C}^*$.

It is easy to check that σ satisfies **LL** and **AP**. To see that it does not satisfy **NMM** observe first that ε' can be obtained from ε when entities 3 and 5 merge into entity 3. Now, $E_3(P', e') = 0.1 + 1.2 - 0.3 = 1$ while $E_3(P, e) = 0.1 + 0.2 - 0 = 0.3$ and $E_5(P, e) = 0$, in contradiction with **NMM**. $\square$

[6]It can be extended to a non-empty financial rule.

References

[1] Aumann, R. and Maschler M. (1985) Game theoretic analysis of a bankruptcy problem from the Talmud. Journal of Economic Theory, 36: 195-213.

[2] Caccioli, F., Barucca, P. and Kobayashi, T. Network models of financial systemic risk: a review. Journal of Computational Social Science 1:81-114.

[3] Chen, C., Iyengar, G. and Moallemi, C.C. (2013) An axiomatic approach to systemic risk. Management Science, 59(6): 1373-1388.

[4] Chun, Y. (1988) The proportional solution for rights problems. Mathematical Social Sciences, 15: 231-246.

[5] Csóka, P., Herings, PJ-J. (2018) Decentralized clearing in financial networks. Management Science, 64(10): 4681-4699.

[6] Csóka, P., Herings, PJ-J. (2021) An Axiomatization of the proportional rule in financial networks. Management Science, 67(5): 2799-2812.

[7] Curiel, I.J., Maschler, M. and Tijs, S.H. (1987) Bankruptcy games. Zeitschrift für Operations Research 31: A 143-A 159.

[8] Demange, G. (2018) Contagion in financial networks: A threat index. Management Science, 64(2): 955-970.

[9] Eisenberg, L. and Noe, T.H. (2001) Systemic risk in financial systems. Management Science, 47(2): 236-249.

[10] de Frutos, M.A. (1999) Coalitional manipulations in a bankruptcy problem. Review of Economic Design, 4: 255-272.

[11] Glasserman, P. and Young. H.P. (2016) Contagion in financial networks. Journal of Economic Literature, 54(3): 779-831.

[12] Groote Schaarsberg, M., Reijnierse, H. and Borm, P. (2018) On solving mutual liability problems. Mathematical Methods of Operations Research, 87: 383-409.

[13] Jackson, M.O. and Pernoud, A. (2021) Systemic risk in financial networks: a survey. Annual Review of Economics, 13: 171-202.

[14] Ju, B.-G. and Miyagawa, E. (2002) Generalized proportional rules and pairwise reallocation-proofness. Mimeo.

[15] Ju, B.-G. (2003) Manipulation via merging and splitting in claims problems. Review of Economic Design, 8: 205-215.

[16] Ju, B.-G., Miyagawa, E. and Sakai, T. (2007) Non-manipulable division rules in claims problems and generalizations. Journal of Economic Theory, 132: 1-26.

[17] Ketelaars, M., Borm, P. and Quant, M. (2020) Descentralization and mutual liability rules. Mathematical Methods of Operations Research, 92: 577-599.

[18] Ketelaars, M. and Borm, P. (2021) On the unification of centralized and descentralized clearing mechanisms in financial networks. (CentER Discussion Paper, Vol 2021-015) CentER, Center for Economic Research.

[19] Koster, M. (2019). A note on uniqueness of clearing prices in financial systems. SSRN. https://doi.org/10.2139/ssrn.3427039.

[20] Moreno-Ternero, J.D. (2006) Proportionality and non-manipulability in bankruptcy problems. International Game Theory Review, 1: 127-139.

[21] Moulin, H. (2000) Priority rules and other asymmetric rationing methods. Econometrica, 68(3): 643-684.

[22] O'Neill, B. (1982) A problem of rights arbitration from the Talmud, Mathematical Social Sciences, 2: 345-371.

[23] Tarski, A. (1955) A lattice-theoretical fixpoint theorem and its applications. Pacific Journal of Mathematics, 5 (2): 285-309.

[24] Thomson, W. (1995) Consistent allocation rules. University of Rochester, Mimeo.

[25] Young, H.P. (1987) On dividing an amount according to individual claims or liabilities. Mathematics of Operations Research 12(3): 398-414.