On uniqueness of equilibrium prices in a Bayesian Assignment Game

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Abstract

In the Assignment Game, introduced by Shapley and Shubik (1971), most solution concepts yield a multiplicity of solutions. We study the Assignment Game in a Bayesian environment where neither buyers nor sellers know the valuation of other players, and derive conditions on the distribution of valuations to guarantee the uniqueness of equilibrium. We also provide a closed-form solution when valuations follow an exponential distribution. Finally, we observe that the Intermediate Value Theorem is pervasive in auction settings.

Keywords: Assignment Game, Uniqueness, Newton's Method, Contraction mapping, Reverse Auction

1 Introduction

1.1 Uniqueness of equilibrium in the Assignment Game

In the Assignment Game, introduced in [1], the literature focuses on analyzing the conditions under which a competitive equilibrium does exist and the mechanisms to compute them. These studies present characterizations of such equilibria based on core and efficiency features as in [2] and [3]. This approach establishes that multiple equilibrium prices may prevail in the Assignment Game and does not propose refinements to select a particular equilibrium. Our objective, in contrast, is to look at conditions under which comparative statics naturally develop in the Assignment Game. Explicitly, our approach guarantees the uniqueness of equilibrium and a closed-form solution of equilibrium pricing strategies when valuations are exponentially distributed.

We analyze the set of equilibria in the Assignment Game in a Bayesian framework where buyers and sellers have private information. We consider that sellers have valuations over their goods, and buyers have heterogeneous valuations over the set of goods. Still, players do not know the valuation of other sellers and buyers. Considering the previous setting, we study the Assignment Game as a two-stage game. In the first stage, nature draws the valuation of each agent, namely over the goods they own for sellers and all goods for buyers. At stage two, sellers simultaneously set prices. A critical feature of the game is that each good is sold to a buyer for whom it maximizes her surplus (the difference between her valuation and the price set at stage two). Thus, if a good is not maximal for any buyer, it is left unsold. In the previous assignment mechanism, we observe that sellers' best response correspondence is independent of other sellers' valuations/strategies, in opposition to price formation in typical approaches to the assignment game [1, 4]. Furthermore, equilibrium prices are characterized by the inverse hazard rate function of buyers' valuations distribution.

Our main result establishes a sufficient condition for the existence of a unique price vector at equilibrium; specifically, it requires the inverse hazard rate to be a contraction. Geometrically, this means that the inverse hazard rate behaves similarly to a constant function, i.e., we can say that the tail distribution is almost a multiple of the density function. The condition, however, is not necessary since uniqueness is also guaranteed when valuations are uniformly distributed, although their associated inverse hazard rate is not a contraction.

Since sellers behave as bidders, it is worth emphasizing that our modeling approach resembles a reverse auction [5]. Reverse auctions are applied on electricity markets [6], pricing of spacecraft commodities [7], and the transition to renewable energies [8]. From a theoretical point of view, reverse auctions have focused on analyzing procurement problems where sellers compete among themselves to provide goods or services to a single buyer (the government), who is interested in avoiding inefficiencies to guarantee the provision of public services [9]. When there is a single buyer and symmetric sellers, [10] point out the existence of multiple symmetric equilibria in a sealed-bid reverse auction, in opposition to the well-known sealed-bid auction. Hence, the literature on reverse auctions has focused on comparing their results with classical auctions [11, 12] and investigating the instances where the revenue equivalence theorem applies to reverse auctions [13]. We analyze the uniqueness of a reverse auction with multiple buyers and sellers, where sellers' bids do not rely on their cost structure since their goods are indivisible. Our main result provides sufficient conditions under which the best responses are independent of other sellers' strategies.

1.2 Auction Mechanism and the Intermediate Value Theorem

We carry out our analyses by applying the Intermediate Value Theorem (IVT) for infinite integrals to establish that equilibrium prices are independent of each other. We observe that the IVT is pervasive in auctions when there are modeled as Bayesian games. In such models, a bidder competes against players whose distributions of valuations are well-known, and although she does not know the specific bidding strategy of the competitors, the fact that these strategies exist means that their expected values exist: this is what the IVT captures. In Appendix A, we apply the IVT in a typical classroom exercise consisting in looking for equilibria in a first-price sealed bid auction.

Although pervasive, [14] and [15] are the few papers we are aware of that make explicit use of the IVT in an auction setting. As [15], we rely on the IVT's power to analyze the uniqueness of equilibrium for any possible distribution of valuations. Particularly, the IVT allows us to demonstrate the existence of a unique equilibrium the existence without considering a symmetric behavior from other sellers. In contrast, following symmetric strategies, [16] establish the uniqueness of equilibrium in sealed-bid auctions for the case of two buyers and valuations' distribution with finite support and positive mass at the lower endpoint. The result is a special case of Theorem 15 when we consider that buyers set prices for homogeneous goods, as we show in Appendix D.

Furthermore, we show that the equilibrium prices are the fixed points of buyers' inverse hazard rate function. So, assuming that the inverse hazard rate function is a contraction mapping is sufficient to guarantee the existence and uniqueness of equilibrium prices by Banach's Fixed-Point Theorem. The sufficient condition is tight; we discuss variations under which the condition does not hold (the case of the uniform distribution). The use of the Contraction Map Theorem in game theory is not new; [17] and [18] also use this approach to guarantee a unique Nash equilibrium in Cournot games and weighted potential games, respectively.

1.3 Review of the literature

Our paper is closely related to the literature that analyses the characteristics of particular classes of games to guarantee the uniqueness of Nash equilibrium like [19], [20], [21], and [22]. We determine the valuation probability distribution's features that guarantee a unique price in the Bayesian Assignment Game we analyze. By assuming statistical independence between distributions of valuations, the first-order conditions state that the direct effect of increasing the price is equivalent to the indirect effect provided by the probability of not selling the good. So, we get an equation that implicitly determines the seller's best responses regarding a buyer's inverse hazard rate.

As far as we know, ours is the first approach to establishing the uniqueness of equilibrium prices in the Assignment Game. In contrast, the multiplicity of competitive equilibria and core allocations are pervasive in the Assignment Game [23]. Assuming that agents cannot have more than one indivisible good, [24] and [25] show that the economy's core is non-empty and not necessarily unique. Also, Quinzii analyses the conditions under which the core allocations coincide with competitive equilibrium allocations. Similarly, [26] proves the existence of at least one competitive equilibrium, which is not always unique, in a model with externalities. In similar models where the multiplicity of fair allocations prevail, [27] characterizes the set of fair allocation rules that are strategy-proof, and [28] study when fair allocations satisfy consistency. A generalization of the assignment game is made by [29] by analyzing the production of indivisible goods. He emphasizes the problems of performing comparative statistics when indivisibilities cause the failure of competitive prices. [30] show that the set of equilibria has a lattice structure in the Assignment game, and the set of efficient and envy-free allocations is non-empty. Even in the multiplicity of fair assignments, they show that it is possible to do some comparative statistics when money increases.

In our cardinal model, buyers are non-strategic. Incomplete information is introduced in two-sided ordinal settings by [31], who shows that results concerning dominant and dominated strategies extend from the complete information setting while the one related to Nash equilibria does not extend in a stable mechanism. [32] establish that truth-telling is an ordinal Bayesian Nash equilibrium of the revelation game induced by a common belief if and only if all the profiles in support of the common belief have a singleton core. In a cardinal setting where the quality of the agents on one side of the market is unknown to agents on the other side of the market, [33] introduce a stability notion and show that the set of incomplete information stable outcomes is a superset of the complete information ones and a subset of the set of sustainable price allocations. Another significant difference with our setting is that their model does not encompass the case where agents on both sides of the markets have complementarities since the remuneration values of agents are independent of their match. The previous result mimics seller's behavior in the real estate market where the empirical evidence suggests that sellers set prices based on demand features [34], such as buyers' valuations [35], and property's observable features [36] instead of taking into account the prices of other properties [37].

To illustrate the IVT's implementation, we search for the equilibrium prices when valuations are exponentially distributed. In such a case, the inverse hazard rate is constant and satisfies the contraction property. Hence, there is a unique equilibrium where prices have a positive relationship with sellers' valuations, while their relationship with the distribution's parameter is negative. Since the parameter can be interpreted as the average time required to buy a good, this last result means that the more buyers are in a hurry to buy a good, the higher the price. The previous example echoes empirical works in the real estate market. [35], [38], [39], and [40] suggest that buying a house is an exponentially distributed event in the sense that it depends on finding a buyer who is willing to pay a specific price. Furthermore, the distribution's parameter has a positive relationship with the number of sellers since buyers need more time to compare all available options [41, 42]. By the previous observation, we conclude that prices increase when the number of sellers increases, which also happens in markets where sellers have monopolistic power [43] or seller's costs and consumer's preferences are private information [44].

The paper is organized as follows. Section 2 presents the model and the two-stage game. Section 3 analyses the set of equilibria and the sufficient condition over the valuation distributions to guarantee a unique price vector at equilibrium. In Section 4, we show that valuations exponentially distributed satisfy the conditions for the existence of a single price, which allows computing a closed form for it. Also, we perform some comparative statics. Conclusions are presented in Section 5.

2 The Model

2.1 Buyers and Sellers

We consider a market with indivisible goods, money, and two disjoint sets of agents: a set of sellers, S, and a set of buyers, B. Let r be a generic agent in $S \cup B$. Money is a perfectly divisible good $\omega \in \mathbb{R}$ that agents use to pay the bill. We assume that all agents r initially own a certain amount of money $\omega_r \in \mathbb{R}_+$, and that all sellers initially own one and only one indivisible good, while buyers initially do not own any good. We use $\varnothing$ whenever an agent does not own any good; we refer to $\varnothing$ as the outside option for buyers.

Let S be the set of m sellers, we use s_j to denote a generic seller with $j = 1, 2, \ldots, m$. Since each seller s_j initially owns an indivisible good, we identify this good with s_j to avoid extra notation. By simplicity, s_j initially owns an amount of money $\omega_j = 0$. So, the initial endowment of s_j is the money/indivisible good basket $(0, s_j)$. Also, seller s_j has a valuation (type) $v_j \in \mathbb{R}$ of her good. Let $\tilde{V}_j$ be the set of all possible types of seller s_j. We consider that seller s_j has a preference relation over baskets $(\omega, s) \in \mathcal{R} \times \{\varnothing, s_j\}$. Given a valuation v_j, this preference relation is represented by a utility function u_{s_j}

that maps baskets (ω, s) into real numbers. We assume the following quasi-linear utility function for each seller s_j

$$u_{s_j}(\omega, s;\ v_j) = \begin{cases} \omega + v_j & \text{if } s = s_j, \\ \omega & \text{if } s = \varnothing. \end{cases}$$

Consider B, the set of n buyers. We identify a generic buyer by i. Each buyer i initially owns an amount of money $\omega_i \geq 0$, and no indivisible good. Thus, the initial endowment of buyer i is the basket $(\omega_i, \varnothing)$. Also, each buyer i has a valuation v_{ji} of good s_j, for all $s_j \in S$. So, the type of buyer i is a vector $\hat{v}_i = (v_{1i}, \ldots, v_{mi}, \omega_i) \in \mathbb{R}^m \times \mathbb{R}_+$. We denote by $\hat{V}_i$ the set of all possible types of buyer i. Also, each buyer i has a preference relation over baskets $(\omega, s) \in \mathbb{R} \times (S \cup \{\varnothing\})$. Given a type $\hat{v}_i$, this preference relation is represented by the utility function $u_i(\cdot)$ that maps baskets (ω, s) into real numbers. We assume the following quasi-linear utility function

$$u_i(\omega, s;\ \hat{v}_i) = \begin{cases} \omega + v_{ji} & \text{if } s = s_j, \\ \omega & \text{if } s = \varnothing. \end{cases}$$

The **state of the market** is the vector of all agents types $v = (v_1, \ldots, v_m, \hat{v}_1, \ldots, \hat{v}_n) \in \prod_j^m \tilde{V}_j \times \prod_{i=1}^n \hat{V}_i$. Let V be the set of all possible states of the market, i.e. $V = \prod_j^m V_j \times \prod_{i=1}^n \hat{V}_i$. We assume that the state of the market v is drawn according to a probability function f from $\tilde{V}$ to $\mathbb{R}$, of common knowledge. In other words, $v = (v_1, \ldots, v_m, \hat{v}_1, \ldots, \hat{v}_n)$ is the realization of the random vector $V = (V_1, \ldots, V_m, \hat{V}_1, \ldots, \hat{V}_n)$; we assume that all variables in V are statistically independent.

An **assignment** is a function Γ from $S \cup B$ to $\mathbb{R} \times (S \cup \{\varnothing\})$. We use $\Gamma(r) = (\Gamma_\omega(r), \Gamma_s(r))$ to describe the allocation of r under the assignment Γ, for all $r \in S \cup B$. That is to say, Γ assigns to each member of the market r a basket composed of an amount of money, $\Gamma_\omega(r)$, and an element in $S \times \{\varnothing\}$, $\Gamma_s(r)$. We say that an assigment Γ is an **individually rational (IR)** assignment if each member of the market weakly prefers her allocation under Γ to her initial endowment.

An assignment Γ is **feasible** if it satisfies the following three conditions:

1. $\sum_{r \in S \cup B} \Gamma_\omega(r) \leq \sum_{i=1}^n \omega_i$,
2. Let r, $r' \in S \cup B$. If $\Gamma_s(r) = \Gamma_s(r') \in S$, then $r = r'$, and
3. For all $s \in S$ there exists some $r \in \{s\} \cup B$ such that $\Gamma_s(r) = s$.

Conditions 2 and 3 tell us that at Γ, any good in the market is assigned to one and only one agent.

2.2 The Game

Agents interact in a two-stage game. Nature moves first and determines the type of each member of the market according to the probability distribution f. All members of the market observe their type but do not observe others' type.

In stage 2, sellers simultaneously decide to sell their goods. If a seller s_j decides to sell her good, she sets a non-negative price p_j. Otherwise, she sets a price $p_j = +\infty$. Thus, $A_j = \mathbb{R}_+ \cup \{+\infty\}$ is the set of actions of seller s_j. Consequently, a **price vector** $p = (p_1, p_2, \dots, p_m)$ is an element of $A_1 \times A_2 \times \cdots \times A_m$. We say that a basket $(w_i - p_j, s_j)$ is **maximal** for buyer i if $u_i(\omega_i - p_j, s_j) \geq u_i(\omega_i - p_\tau, s_\tau)$ for all $s_\tau \in S$.

At the end of the game, payoffs are determined by the final assignment procedure $\Lambda[p]$, which sequentially assigns goods whenever they are maximal for some buyer i.

Step 1 A buyer i gets a basket $(\omega_i - p_j, s_j)$ only if the basket is individually rational and maximal, among all baskets $(\omega_i - p_\tau, s_\tau)$, for buyer i; and seller s_j is assigned to basket $(p_j, \varnothing)$. Then, the pair (i, s_j) is removed from the market. In the case of a tie (i is indifferent between two goods, or s_j is the most preferred good for two or more buyers), the mechanism randomly breaks ties in a way that the maximum number of buyers gets one and only one good. If all goods are assigned, or all buyers get some good or no good is individually rational for buyers, the mechanism stops; otherwise, the mechanism goes to the following step.

Step **t** Each buyer i gets a basket $(\omega_i - p_j, s_j)$ only if $u_i(\omega_i - p_j, s_j)$ is individually rational for i and maximal concerning the goods that remain in the market; and seller s_j is assigned to basket $(p_j, \varnothing)$. Hence, the pair (i, s_j) is removed from the market. In the case of a tie (i is indifferent between two goods, or s_j is the most preferred good for two or more buyers), the mechanism randomly breaks ties in a way that the maximum number of buyers gets one and only one good. If all goods are assigned, or all buyers get some good or no no good is individually rational for buyers, the mechanism stops; otherwise, the mechanism goes to the following step.

The final assignment is denoted by $\Lambda[p]$. We write $\Lambda[p](r)$ to denote the allocation of agent r at price p for all $r \in S \cup B$.

2.3 The solution concept

Before presenting the solution concept, we introduce the following notation. A decision rule for seller s_j is a function $\sigma_j : \tilde{V}_j \to A_j$ mapping a type into a price. Thus, a **pure strategy** for seller s_j is an element $\sigma_j \in \Sigma_j = \{\sigma_j : \sigma_j \text{ is a decision rule}\}$. So, a profile of sellers' pure strategies is a vector $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_m)$; as usual, σ_{-s} denotes the profile of decision rules different from σ_s. Given a realization v of the market state, a price vector is $p = (\sigma_1(v_1), \dots, \sigma_m(v_m))$.

Sellers' payoffs depend on the final allocation $\Lambda[p]$. Since sellers do not know the valuation of other sellers and buyers, they may not sell their goods; hence, the payoff of s_j is the expected utility denoted by $E[u_s(\Lambda[p](s))]$. Consequently, we consider a Bayesian Nash equilibrium as a solution concept for our game.

Definition 1 Let $\sigma^* = (\sigma_1^*, \ldots, \sigma_m^*)$ be a profile of sellers' pure strategies. We say that σ^* is a **Bayesian Nash equilibrium** if

$$E[u_{s_j}(\Lambda(\sigma_j^*, \sigma_{-j}^*)(s_j))] \geq E[u_{s_j}(\Lambda(\sigma_j, \sigma_{-j}^*)(s_j))],$$

for all $s_j \in S$ and $\sigma_j \in \Sigma_j$.

3 Equilibrium Analysis

In this section, we investigate the existence and uniqueness of a Bayesian Nash equilibria σ^*. We know that sellers' payoffs are determined by the final assignment $\Lambda[p]$. So, if buyer i is assigned to basket $(\omega_i - p_j, s_j)$, this means that s_j is individually rational for i, i.e., the surplus $v_{ji} - p_j$ is positive. With respect to sellers, if s_j is assigned to the basket $(p_j, \varnothing)$, her payoff is p_j.

The assignment procedure $\Lambda[p]$ may generate different allocations since it randomly breaks ties. So, a good s may be assigned to buyers i and i', with $i \neq i'$, at assignments $\Lambda_1[p]$ and $\Lambda_2[p]$, respectively. Regardless the buyer who gets good s, the payoff of seller s is the same in both assignments.

Proposition 1 *Consider assignments $\Lambda_1[p]$ and $\Lambda_2[p]$. Whenever a seller s_j sells her good either at $\Lambda_1[p]$ or at $\Lambda_2[p]$, her payoff is independent of the buyer who buys good s_j.*

Proof All proof are in Appendix B. □

By Proposition 1, the payoff function of each seller s_j is

$$u_{s_j}(\Lambda[p](s_j);\ v_j) = \begin{cases} p_j & \text{if } s_j \text{ sells her good,} \\ v_j & \text{otherwise.} \end{cases}$$

To simplify the algebra, from now on, we consider that the payoff function of s_j is $\overline{u}_{s_j}(\Lambda[p](s_j);\ v_j) = u_{s_j}(\Lambda[p](s_j);\ v_j) - v_j$ which is a monotonic transformation of u_{s_j}. So, we have that

$$\overline{u}_{s_j}((\Lambda[p](s_j);\ v_j) = \begin{cases} p_j - v_j & \text{if } s_j \text{ sells her good,} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that seller s_j is not certain about her final payoff because the final assignment $\Lambda[p]$ depends on the profile of decision rules $p = (\sigma_1(v_1), \sigma_2(v_2), \ldots, \sigma_m(v_m))$ and valuations are not common knowledge. Thus, sellers make decisions concerning their expected utility function. Let S_j be the probability event where s_j sells her good, and N_j the probability event of not selling good s_j; note that $S_j, N_j \subset \tilde{V}$. Thus, the expected utility function of s_j is

$$E[\overline{u}_{s_j}(\Lambda[p](s_j);\ v_j)] = (p_j - v_j)Pr[S_j] + 0Pr[N_j].$$

3.1 The probability of selling

Note that, if some good s_j is sold to some buyer i, the assignment procedure is such that the basket $(\omega_j - p_j, s_j)$ is an individually rational basket for i. It is also worth emphasizing that buyers do not necessarily get their top basket at $\Lambda[p]$. So, i can get one of her k-th most preferred goods during the assignment procedure. Let $S_j^k \subset \tilde{V}$ be the probability event where s_j is assigned to some buyer i from which s_j is one of i's k-th most preferred good. For any $x \in S_j^k$, buyer i gets her $k-$th largest surplus, which we denote by z_{ji}^k, among all positive surpluses $v_{\tau i} - p_\tau$. So, whenever a seller sells her good to some buyer, the seller is uncertain about how large the buyer's surplus is. The following establishes that the family of events $\{S_j^1, S_j^2, \ldots, S_j^m\}$ is a partition of the event S_j, which allows us to compute the probability that seller s_j sells her good.

Lemma 2 *For all $s_j \in S$, we have that*

$$S_j = \bigcup_{k=1}^{m} S_j^k.$$

Lemma 3 *For any price vector $p \in \mathbb{R}_+^m$. For any $k, k' \in \{1, 2, \ldots, m\}$ such that $k \neq k'$, we have that $S_j^k \cap S_j^{k'} = \varnothing$.*

The previous lemmas imply that the probability of selling is the sum of probabilities $Pr[S_j^k]$.

Proposition 4 *For all $s_j \in S$, the probability that s_j sells his good is*

$$Pr[S_j] = \sum_{k=1}^{m} Pr[S_j^k]. \tag{1}$$

Now, notice that S_j^k is the event where some buyer i gets her k-th largest surplus by buying s_j. Hence, surpluses' ordering depends on buyer i's valuation vector $\hat{v}_i$ and the price vector p. Remembering that s_j only observes her valuation v_j, we have that $\hat{v}_i = (v_{1i}, v_{2i}, \ldots, v_{mi})$ is a realization of the random vector $\hat{V}_i = (V_{1i}, V_{2i}, \ldots, V_{mi})$ for all $i \in B$. Also, price $p_\tau = \sigma_\tau(v_\tau)$ is the realization of the random variable $p_\tau = \sigma_\tau(V_\tau)$ for all $\tau \in S - \{s_j\}$; to simplify the notation, we consider that $p_\tau = p_\tau(V_\tau)$.

By the above discussion, surplus z_{si} is a realization of the random variable $Z_{si} = V_{si} - p_s(V_s)$ for all $s \in S$. We use $g_{Z_{si}}$ and $G_{Z_{si}}$ to denote the marginal and cumulative density functions of the variable Z_{si}, respectively. We use $Z_{(k)}$ to denote the k-th largest surplus that i gets from buying a good in $S - \{s_j\}$. So, $Z_{(k)}$ denotes the k-th order statistics of the statistical sample $\{Z_{si} \mid s \neq s_j\}$. From now on, we only refer to surpluses Z_{si} since the name of the buyer that gets s_j does not impact sellers' payoff, as for Proposition 1. So, $Z_{(1)}$ is the largest surplus that some buyer i can get from buying a good in $S - \{s_j\}$, that is to say, $Z_{(1)} = \max_{s_\tau \in S}\{Z_{\tau i} \mid \tau \neq j\}$. Since z_{ji} is not in the statistical sample $\{Z_{si} \mid s \neq s_j\}$, the variable $Z_{(m-1)}$ represents the minimum surplus that i can get from buying a good in $S - \{s_j\}$.

The following proposition establishes that probabilities $Pr[S_j^k]$, for all $k \in \{1, 2, \dots, m\}$, depend on the order statistics of the sample $\{Z_{si} \mid s \neq s_j\}$.

Proposition 5 *Let $\mathcal{Z} = \{Z_{(1)}, Z_{(2)}, ..., Z_{(m-1)}\}$ be the orders' statistics family of sample $\{Z_{\tau i} \mid s_\tau \neq s_j\}$. Then*

1. *$Pr[S_j^1] = Pr[Z_{ji} > Z_{(1)} \geq 0]$,*
2. *For all $k \in \{2, \dots, m-1\}$, we have that $Pr[S_j^k] = Pr[Z_{(k-1)} > Z_{ji} \geq Z_{(k)} \geq 0]$.*
3. *$Pr[S_j^m] = Pr[Z_{(m-1)} \geq Z_{ji} \geq 0]$.*

Based on the previous proposition, below, we develop a general formulation to compute the probability of event S_j^k, for all $k \in \{1, 2, \dots, m\}$.

Proposition 6 *Consider that $V = (V_1, V_2, \dots, V_m, \hat{V}_1, \dots, \hat{V}_n)$ is a vector of independent and identically distributed random variables. Then*

1. *$Pr[S_j^1] = \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) F_{Z_{(1)}}(v_{ji} - p_j) dv_{ji}$.*
2. *$Pr[S_j^m] = \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) \left(1 - F_{Z_{(m-1)}}(v_{ji} - p_j)\right) dv_{ji}$.*
3. *$Pr[S_j^k] = \frac{(m-1)!}{k!(m-k)!(m-k-1)} \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji})[1 - G_{Z_{\tau i}}(v_{ji} - p_j)]^{m-k-1}[G_{Z_{\tau i}}(v_{ji} - p_j) d_{v_{ji}}]^k$, for all $k \in \{2, \dots, m-1\}$.*

3.2 Sellers best responses

For seller s_j, a decision rule σ_j maps a valuation v_j into a price p_j, i.e. $\sigma_j(v_j) = p_j$. So, the **best response correspondence** $b_{s_j}(\sigma_{-j})$ of seller s_j to a pure strategies profile σ_{-j} is a set of decision rules such that s_j gets the largest possible payoff when other sellers choose decision rules σ_{-j}. By Proposition 4, best responses of s_j to σ_{-j} maximize the expected utility

$$E[\overline{u}_{s_j}(\Lambda[p_j, p_{-j}](s_j))] = (p_j - v_j)Pr[S_j] = (p_j - v_j)\sum_{k=1}^{m} Pr[S_j^k].$$

Moreover, Proposition 6 implies that cumulative density functions $F_{Z_{(1)}}$, $F_{Z_{(m-1)}}$ and $G_{Z_{\tau i}}$ are necessary to compute the probability of event S_j^k, for all $k \in \{1,,2,\ldots,m\}$.

Remember that $Z_{(k)}$ are the $k-th$ order statistics concerning the sample of surpluses $\{V_{\tau i} - p(V_\tau) \mid s_\tau \neq s_j\}$. Thus, the random variable $Z_{(k)}$ is a transformation of the variable $Z_{\tau i} = V_{\tau i} - p_\tau(V_\tau)$, i.e., the k-th order statistic is a linear transformation of variables $V_{\tau i}$ and $p_\tau(V_\tau)$. Since seller s_j does not know the responses of other sellers (decision rules $p_\tau = \sigma_\tau$ for all $s_\tau \neq s_j$), we cannot directly get the density function of any order statistics. The following lemma establishes the conditions under which $F_{Z_{(1)}}$, $F_{Z_{(m-1)}}$ and $G_{Z_{\tau i}}$ are cumulative density functions.

Lemma 7 *Consider that* $V = (V_1, V_2, \ldots, V_m, \hat{V}_1, \ldots, \hat{V}_n)$ *is a vector of independent and identically distributed random variables. Let* $p_{-j} = (p_\tau)_{\tau \neq j}$ *be a profile of decision rules such that* $p_\tau = \sigma_\tau$ *is differentiable over* V_j *for all* $s_\tau \neq s_j$*. Then,* $Z_{(1)}$, $Z_{(m-1)}$ *and* $Z_{\tau i}$ *are random variables whose cumulative density functions are* $F_{Z_{(1)}}$, $F_{Z_{(m-1)}}$ *and* $G_{Z_{\tau i}}$*, respectively.*

Corollary 8 *For all* $k \in \{1, 2, \ldots, m\}$*, we have that* $Pr[S_j^k]$ *is independent of* p_τ *for all* $\tau \neq j$*.*

However, the properties of the cumulative density functions $F_{Z_{(1)}}$, $F_{Z_{(m-1)}}$ and $G_{Z_{\tau i}}$ are enough to compute the probability of selling without explicitly computing them.

Theorem 9 *Consider that* $V = (V_1, V_2, \ldots, V_m, \hat{V}_1, \ldots, \hat{V}_n)$ *is a vector of independent and identically distributed random variables. Let* $p_{-j} = (p_\tau)_{\tau \neq j}$ *be a profile of decision rules such that* $p_\tau = \sigma_\tau$ *is differentiable over* $\tilde{V}_j$ *for all* $s_\tau \neq s_j$*. Given a realization of the market state* v*, consider that* s_j *sells her good to buyer* i *at the assignment* $\Lambda[p_j, \sigma_{-j}]$*. Then, there exists constants* $\mu_1, \mu_2, \ldots, \mu_{m-1}, \mu_m \in [0, 1]$ *such that*

1. $Pr[S_j^1] = \mu_1(1 - F_{V_{ji}}(p_j))$,
2. $Pr[S_j^m] = \mu_m(1 - F_{V_{ji}}(p_j))$, *and*
3. $Pr[S_j^k] = \frac{\mu_k (m-1)!}{k!(m-k)!(m-k-1)}(1 - F_{V_{ji}}(p_j))$ *for all* $1 < k < m$.

Proof Due to the complexity of establishing cumulative density functions $F_{Z_{(1)}}$, $F_{Z_{(m-1)}}$ and $G_{Z_{\tau i}}$, we compute the probability of selling by using the Proposition 6. To simplify the calculation of such probability, we first consider the change of variable $x_{ij} = v_{ji} - p_j$. So, we have that $dx_{ij} = dv_{ji}$ and $v_{ji} = x_{ij} + p_j$. We substitute the previous variables into the integrals 1, 2, and 3 of Proposition 6, and we get that

$$Pr[S_j^1] = \int_0^\infty f_{V_{ji}}(x_{ij} + p_j) F_{Z_{(1)}}(x_{ij}) dx_{ij},$$

$$\begin{aligned} Pr[S_j^m] &= \int_0^\infty f_{V_{ji}}(x+p_j)\left[1-F_{Z_{(m-1)}}(x_{ij})\right]dx_{ij}, \\ Pr[S_j^k] &= \Delta_k \int_0^\infty f_{V_{ji}}(x_{ij}+p_j)\left[1-G_{Z_{\tau i}}(x_{ij})\right]^{m-k-1}\left[G_{Z_{\tau i}}(x)\right]^k dx_{ij}, \end{aligned} \tag{2}$$

where

$$\Delta_k = \frac{(m-1)!}{k!(m-k)!(m-k-1)}.$$

Now, we proceed to analyze the previous integrals through the First Mean Value Theorem for Infinite Integrals (FMVTII) [45] to avoid a direct calculation of cumulative density functions.

FMVTII. Let h be a function such that $m \le h(x) \le M$ for $x \ge a$, and integrable in any interval $[a,b]$. Also, consider g a function such that $g(x) \ge 0$ for all $x \ge a$ and $\int_a^\infty g(x)dx$ is finite. Then,

$$\int_a^\infty h(x)g(x)dx = \mu \int_a^\infty g(x)dx,$$

where $m \le \mu \le M$.

Lemma 7 states that functions $F_{Z_{(1)}}$, $F_{Z_{(m-1)}}$ and $G_{Z_{\tau i}}$ are cumulative density functions of variables $Z_{(1)}$, $Z_{(m-1)}$ and $Z_{\tau i}$. Then, these functions are right continuous and, consequently, integrable on any interval $[a,b] \subset (-\infty,\infty)$ with $a \ge 0$. Moreover, the previous functions are non-negative with a lower bound (0) and an upper bound (1). In other words, $F_{Z_{(1)}}$, $F_{Z_{(m-1)}}$ and $[G_{Z_{\tau i}}]^k[1-G_{Z_{\tau i}}]^{m-k-1}$ satisfy the necessary conditions to apply the FMVTII on expression (2). That is to say, we consider that $F_{Z_{(1)}}$, $F_{Z_{(m-1)}}$ and $[G_{Z_{\tau i}}]^k[1-G_{Z_{\tau i}}]^{m-k-1}$ play the role of the h function, while $f_{V_{ji}}(x+p_j)$ refers to function g in the FMVTII. So, there exists constants $\mu_1, \mu_2, \ldots, \mu_m \in [0,1]$ such that

$$\begin{aligned} \int_0^\infty f_{V_{ji}}(x_{ij}+p_j)F_{Z_{(1)}}(x_{ij})dx_{ij} &= \mu_1 \int_0^\infty f_{V_{ji}}(x_{ij}+pj)dx, \\ \int_0^\infty f_{V_{ji}}(x_{ij}+p_j)\left[1-F_{Z_{(m-1)}}(x_{ij})\right]dx_{ij} &= \mu_m \int_0^\infty f_{V_{ji}}(x_{ij}+pj)dx_{ij}, \\ \int_0^\infty f_{V_{ji}}(x_{ij}+p_j)\left[1-G_{Z_{\tau i}}(x_{ij})\right]^{m-k-1}\left[G_{Z_{\tau i}}(x_{ij})\right]^k dx_{ij} &= \mu_k \int_0^\infty f_{V_{ji}}(x_{ij}+pj)dx_{ij}, \end{aligned} \tag{3}$$

for all $k \in \{2,3,\ldots,m-1\}$.

Finally, note that $\int_0^\infty f_{V_{ji}}(x_{ij}+p_j)dx = 1 - F_{V_{ji}}(p_j)$, which we substitute into expression (3). Hence, we conclude that

$$\begin{aligned} Pr[S_j^1] &= \mu_1(1-F_{V_{ji}}(p_j)) \\ Pr[S_j^m] &= \mu_m(1-F_{V_{ji}}(p_j)) \\ Pr[S_j^k] &= \frac{\mu_k(m-1)!}{k!(m-k)!(m-k-1)}(1-F_{V_{ji}}(p_j)) \text{ for } 1<k<m. \end{aligned} \tag{4}$$

□

The previous result states that probabilities $Pr[S_j^k]$ are weighted by a mean value of the unknown functions $F_{Z_{(1)}}$, $F_{Z_{(m-1)}}$ and $G_{Z_{\tau i}}[1-G_{Z_{\tau i}}]$. In the following corollary we show that such constants are related to the conditional probability of being the k-th most preferred good of buyer i given that i buys such good during the assignment procedure Λ, for all $k \in \{1,,2,\ldots,m\}$.

Corollary 10 ***The interpretation of constants*** $\mu_{\mathbf{k}}$. *Let S_{ji} be the probability event of all market states $v \in \tilde{V}$ such that the basket $(\omega_i - p_j, s_j)$ is IR for buyer i. Then*

1. $\mu_1 = Pr[S_j^1 \mid S_{ji}]$,
2. $\mu_m = Pr[S_j^m \mid S_{ji}]$,
3. $\mu_k = \frac{k!(m-k)!(m-k-1)}{(m-1)!} Pr[S_j^k \mid S_{ji}]$ *for all* $k \in \{2, \ldots, m-1\}$.

Proof Note that $Pr[S_{ji}] = Pr[v_{ji} \geq p_j] = 1 - F_{v_{ji}}(p_j)$. By Theorem 9, we have that

$$\begin{aligned} Pr[S_j^1] &= \mu_1 Pr[S_{ji}] \\ Pr[S_j^m] &= \mu_m Pr[S_{ji}] \\ Pr[S_j^k] &= \frac{\mu_k (m-1)!}{k!(m-k)!(m-k-1)} Pr[S_{ji}] \text{ for } 1 < k < m. \end{aligned} \tag{5}$$

Now, consider $v \in S_j^k$. By the definition of the event S_j^k, we have that s_j is one of the $k-th$ most preferred goods of buyer i for some $k \in \{1, 2, \ldots, m\}$; that is to say $(\omega_i - p_j, s_j)$ is IR for buyer i. Thus, $v_{ji} \geq p_j$, which means that $v \in S_{ji}$, which implies that $S_j^k \subseteq S_{ji}$ for all $k \in \{1, , 2, \ldots, m\}$.

By the previous discussion, note that $Pr[S_j^k] = Pr[S_j^k \cap S_{ji}]$. If $Pr[S_{ji}] > 0$, we can rewrite the equations in expression (5) in the following way

$$\begin{aligned} \mu_1 &= \frac{Pr[S_j^1 \cap S_{ji}]}{Pr[S_{ji}]}, \\ \mu_m &= \frac{Pr[S_j^m \cap S_{ji}]}{Pr[S_{ji}]}, \\ \mu_k &= \frac{k!(m-k)!(m-k-1)}{(m-1)!} \frac{Pr[S_j^k \cap S_{ji}]}{Pr[S_{ji}]}. \end{aligned}$$

Remembering that the conditional probability of event A given that event B has occurred is $P[A \mid B] = Pr[A \cap B]/Pr[B]$, we conclude that

$$\begin{aligned} \mu_1 &= Pr[S_j^1 \mid S_{ji}], \\ \mu_m &= Pr[S_j^m \mid S_{ji}], \\ \mu_k &= \frac{k!(m-k)!(m-k-1)}{(m-1)!} Pr[S_j^k \mid S_{ji}] \text{ for all } k \in \{2, \ldots, m-1\}. \end{aligned}$$

□

In words, constants μ^1 and μ_m are the conditional probabilities of i getting the largest and the smallest surplus, respectively, given that the basket $(\omega_i - p_j, s_j)$ is IR for i. Concerning μ_k for all $k \in \{2, \ldots, m-1\}$, the conditional probability of i getting her $k-th$ largest surplus, given that $(\omega_i - p_j, s_j)$ is IR for i, is weighted by

$$\frac{k!(m-k)!(m-k-1)}{(m-1)!} = \left(\frac{(m-1)!}{k!(m-k)!(m-k-1)} \right)^{-1}.$$

Note that

$$\frac{(m-1)!}{k!(m-k)!(m-k-1)}$$

is the number of preference lists of buyer i under she strictly prefers other $m-k$ goods to s_j, and the good s_j is strictly better to k goods. The term $m-k-1$ represents the possible ties of s_j being the $k-th$ most preferred good.

Example 1 **The Intermediate Value Theorem**. To illustrate the FMVTII, we consider that V_{ji} and $Z_{\tau i}$ are identically and exponentially distributed with parameter λ for all $\tau \in S - \{s_j\}$. Then, we have that

$$F_{Z_{(1)}}(x) = [F_{Z_{\tau i}}(x)]^{m-1} = \begin{cases} \left[1-e^{-\lambda x}\right]^{m-1} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

By Proposition 6 and Theorem 9, we have that

$$\begin{aligned} Pr[S_j^1] &= \int_0^\infty f_{V_{ji}}(x+p_j) F_{Z_{(1)}}(x) dx \\ &= \int_0^\infty \lambda e^{-(x+p_j)} \left[1-e^{-\lambda x}\right]^{m-1} dx. \end{aligned}$$

If $W = 1 - e^{-\lambda x}$, then $dW = \lambda e^{-\lambda}$. So, we integrate the previous expression by using the substitution method, i.e., we get that

$$\begin{aligned} \int_0^\infty \lambda e^{-\lambda(x+p_j)} \left[1-e^{-\lambda x}\right]^{m-1} dx &= \frac{e^{-\lambda p_j}}{m} \left[1-e^{-\lambda x}\right]^m \Big|_0^\infty \\ &= \frac{\lambda e^{-\lambda p_j}}{m\lambda} = \frac{e^{-\lambda p_j}}{m}. \end{aligned}$$

Given that $f_{V_{ji}}$ is positive in any interval $[a,b] \subset [0,\infty)$, we apply the FMVTII by considering $g = f_{V_{ji}}$ and $h = F_{V_{(1)}}$. Then

$$\begin{aligned} \int_0^\infty \lambda e^{-\lambda(x+p_j)} \left[1-e^{-\lambda x}\right]^{m-1} dx &= \mu_1 \int_0^\infty \lambda e^{-\lambda(x+p_j)} dx \\ &= \mu_1 e^{-\lambda p_j} \int_0^\infty \lambda e^{-\lambda x} dx, \end{aligned}$$

where $\int_0^\infty \lambda e^{-\lambda x} dx = 1$. Consequently, we have that

$$\frac{e^{-\lambda p_j}}{m} = \mu_1 e^{-\lambda p_j}.$$

So, $\mu_1 = \frac{1}{m}$, which we can interpret as the probability of s_j being the most preferred good of buyer i given that all baskets $(\omega - p_\tau, s_\tau)$ are IR for i.

□

Under the conditions of Theorem 9, it is not necessary to compute the density functions of the random variables $Z_{(k)}$. In other words, based on the previous results, it is possible to provide a general formulation for the probability of selling.

Theorem 11 *Consider that $V = (V_1, V_2, \dots, V_m, \hat{V}_1, \dots, \hat{V}_n)$ is a vector of independent and identically distributed random variables. Let $p_{-j} = (p_\mathcal{T})_{\mathcal{T} \neq j}$ be a profile of decision rules such that $p_\mathcal{T} = \sigma_\mathcal{T}$ is differentiable over $\tilde{V}_j$ for all $s_\mathcal{T} \neq s_j$. Given a realization of the market state v, consider that s_j sells her good to buyer i during the assignment $\Lambda[p_j, \sigma_{-j}]$. Then, there exists constants $\mu_1, \mu_2, \dots, \mu_m \in [0, 1]$ such that*

$$Pr[S_j] = (1 - F_{V_{ji}}(p_j)) \left(\mu_1 + \sum_{k=2}^{m-1} \frac{\mu_k (m-1)!}{k!(m-k)!(m-k-1)} + \mu_m \right).$$

Proof It follows from Proposition 4 and Theorem 9. □

It is worth recalling that Theorem 9 holds for any profile of increasing and continuous decision rules $p_{-j} = (\sigma_\tau)_{\tau \neq j}$.

If we assume that all goods in S are homogeneous, the previous result allows us to establish a connection between our Bayesian Assignment Game and the sealed-bid auction. In such a case, see Appendix D, we show that the probability of winning depends on s_j being the most preferred good, the one with the minimum price, whose probability function is independent of other sellers' prices.

It is worth noticing that Theorem 11 indicates that the probability of selling does not directly depend on other sellers' prices, unlike in sealed-bid auction. Specifically, the previous result also implies that it is not necessary to assume that sellers follow a symmetric behavior in the search of a best response, which is a typical procedure in the analysis of auctions. Thus, for example, our results are independent of considering that $P_\tau = \alpha V_\tau$ for all sellers $s_\tau \neq s_j$ when we search for the best response of seller s_j.

Theorem 11 allows to write the probability of selling as $Pr[S_j] = M(1 - F_{V_{ji}}(p_j))$, where $M = \mu_1 + \sum_{k=2}^{m-1} \frac{\mu_k(m-1)!}{k!(m-k)!(m-k-1)} + \mu_m$ is a positive constant. In words, the probability of selling is the product between the average of being one of the most k-th preferred goods of buyer i and the probability that $(\omega_i - p_j, s_j)$ is individually rational for i. Hence, we rewrite the expected utility function of seller s_j as follows

$$E[\overline{u}_{s_j}(\Lambda[p_j, p_{-j}])(s_j)] = M(p_j - v_j)\left(1 - F_{V_{ji}}(p_j)\right). \tag{6}$$

Theorem 12 *Consider that $V = (V_1, V_2, \dots, V_m, \hat{V}_1, \dots, \hat{V}_n)$ is a vector of independent and identically distributed random variables. Let $p_{-j} = (p_\mathcal{T})_{\mathcal{T} \neq j}$ be a profile of decision rules such that $p_\mathcal{T} = \sigma_\mathcal{T}$ is differentiable over $\tilde{V}_j$ for all $s_\mathcal{T} \neq s_j$. Best responses of seller s_j are independent of other sellers prices.*

Proof Since the cumulative function $F_{V_{ji}}$ is differentiable, we search for s_j' best responses by maximizing her expected utility function. Thus, we solve the first order condition, which is given by the equation $\partial E[\overline{u}s_j]/\partial p_j = 0$. Taking the derivative of expression (6) with respect to p_j, best responses of s_j are solutions of the equation

$$M\left(1 - F_{V_{ji}}(p_j)\right) - M(p_j - v_j)f_{v_{ji}}(p_j) = 0.$$

Rearranging the previous expression, we get that

$$1 - F_{V_{ji}}(p_j) = (p_j - v_j)f_{V_{ji}}(p_j). \tag{7}$$

By expression (7), a best response p_j^* of s_j balances the indirect effect of not selling the good $(1 - F_{V_{ji}}(p_j^*))$ with the direct effect of increasing the price $((p_j^* - v_j)f_{V_{ji}}(p_j^*))$. Moreover, note that the critical points of the expected utility function of s_j are implicitly defined as solutions of an equation that depends on how the random variable V_{ji} behaves. Furthermore, we can rearrange this expression in the following way

$$p_j = \frac{1 - F_{V_{ji}}(p_j)}{f_{Vji}(p_j)} + v_j. \tag{8}$$

By the previous expression, the best responses of s_j depend on functions $F_{V_{ji}}$ and $f_{V_{ji}}$. Therefore, the best response correspondence of s_j is independent from other sellers' prices.

□

3.3 Existence and uniqueness conditions

The best response correspondence $b_j(\sigma_{-j})$ of seller s_j is the set of all decision rules $p_j(v_j)$ that satisfy expression (8). Remembering that $f_{V_{ji}}(p_j)/(1 - F_{V_{ji}}(p_j))$ is the hazard rate, we note that the solutions of equation (8) are implicitly defined by the inverse hazard rate of buyer i. Let γ be the following function γ

$$\gamma_j(p_j) = \frac{1 - F_{V_{ji}}(p_j)}{f_{Vji}(p_j)} + v_j. \tag{9}$$

Then, a best response of s_j is a decision rules $b_j(-\sigma_{-j}) = \{p_j \in \Sigma_j \mid p_j = \gamma(p_j)\}$. In other words, a best response of s_j is a fixed point of γ_j. Since γ_j is not necessarily a linear function, numerical techniques are commonly used to find fixed points of a non-linear function. We part from Newton's method to determine the condition that guarantees the existence and uniqueness of a unique selling price, at equilibrium, in our Bayesian version of the assignment game.

Below, we describe the Newton's Method to compute the solutions of a non-linear equation, and later we explain its relation with fixed points.

Consider an equation $g(x) = 0$, where g is a non-linear function. Suppose that this equation has at least one **root** $x^* \in \mathbb{R}$, i.e. $g(x^*) = 0$. The **Newton's Method** proceeds as follows

Step 0. Start with an initial guess, $x_0 \in \mathbb{R}$, for the location of the root.

Step t. To find a root of equation $g(x) = 0$, we improve the initial guessing by iterating repeatedly the next expression

$$x_{t+1} = x_t - \frac{g(x_t)}{g'(x_t)}.$$

Previous procedure generates the set $\{x_t\}_{t=0}^{\infty} = \{x_0, x_1, \ldots x_t, \ldots\}$, which is called the **Newton's succession**. It is possible to demonstrate that $x^* = \lim_{t\to\infty} x_t$ is a root of the non-linear equation $g(x) = 0$, see [46].

Two immediate questions arise about the application of the Newton's method. The first one is related to the convergence of Newton's succession. The second one is about the independence of the initial guess, i.e., do different initial guesses convergence to the same point? To answer these questions, it is important to note that, if Newton's succession $\{x_t\}_{t\in\mathbb{N}}$ converges to some x^*, we have that

$$x^* = x^* - \frac{g(x^*)}{g'(x^*)}.$$

In other words, the point x^* is a **fixed point** of the function

$$h(x) = x - \frac{g(x)}{g'(x)}.$$

This means that finding a unique root for equation $g(x) = 0$ is equivalent to finding a unique fixed point of function h. To analyze the uniqueness of fixed points, it is necessary to introduce the definition of contractions.

Definition 2 A function $g : \mathbb{R} \to \mathbb{R}$ is a **contraction** if there exists a constant L such that $0 < L < 1$ for any $x, y \in \mathbb{R}$:

$$\mid g(x) - g(y) \mid \leq L \mid x - y \mid .$$

If a function f is a contraction, we say that it satisfies the **contraction property**; this property is a sufficient condition to guarantee the convergence of Newton's succession. Moreover, this property makes the convergence process independent of the initial guess.

Theorem 13 ***(Contracting Map Theorem, [47]).*** *Consider a function $h : X \subset \mathbb{R} \to X$ that satisfies the contraction property. Then, there exists a unique point $x^* \in X$ such that $h(x^*) = x^*$. Moreover, the Newton's succession converges to x^* as $n \to \infty$ for any $x_0 \in X$.*

Proof A sketch of the proof is shown in the Appendix. For more details, see [46].
□

Remark 1 **Geometrical Interpretation of the Contraction Property**. If a function g is a contraction, this means that the distances between two points is larger than the distance between their images. Moreover, we can rewrite this condition as follows

$$\frac{\mid g(x) - g(y) \mid}{\mid x - y \mid} \leq L, \text{ for some } 0 \leq L < 1.$$

Hence, the slope of the tangent line of $g(x)$ is bounded by L, which makes them very similar to a constant or linear functions.

The Contracting Map Theorem guarantees the existence of a unique fixed point when functions are contractions. Furthermore, note that Newton's Method establishes a link between finding fixed points, and the roots of a non-linear equation. Thus, if the function γ in expression (9) is a contraction, Theorem 13 implies that equation (8) has a unique solution. Consequently, the expected utility of seller s_j has a unique critical point.

Lemma 14 *If*

$$\delta_j(p_j) = \frac{1 - F_{V_{ji}}(p_j)}{f_{V_{ji}}(p_j)}$$

is a contracting map, then $\delta_j(\cdot)$ *has a unique fixed point* p_j^*.

Proof It is a consequence of Theorem 13. □

Theorem 15 *Given a price vector* p *and a market state* v*, consider that goods* $s \in S$ *are assigned by following the procedure* $\Lambda[p]$. *If*

$$\gamma_j(p_j) = \frac{1 - F_{V_{ji}}(p_j)}{f_{V_{ji}}(p_j)} + v_j$$

is a contraction map on a subset X *of* V_j*, then* γ_j *has a unique fixed point* p_j^*. *In addition, the Newton's succession*

$$p_{n+1} = p_n - \frac{\gamma_j(p_n) - p_n}{\gamma_j'(p_n) - 1}$$

converges to p_j^* *regardless the initial guess* $p_0 \in X$.

Proof Note that $\gamma_j = \delta_j + v_j$. So, γ_j is a vertical translation of δ since v_j is constant. If δ_j is a contraction mapping, we have that

$$\begin{aligned} \frac{\mid \gamma_j(x) - \gamma_j(y) \mid}{\mid x - y \mid} &= \frac{\mid \delta(x) + v_j - (\delta(y) + v_j) \mid}{\mid x - y \mid} \\ &= \frac{\mid \delta_j(x) - \delta_j(y) \mid}{\mid x - y \mid} \\ &\leq L, \end{aligned}$$

for some $0 \leq L < 1$. So, γ_j is a contraction. By Theorem 13, γ_j has a unique fixed point p_j^*. □

Remark 2 It is important to emphasize that sellers set a price by considering the existence of a buyer who is willing to pay that price. The existence of such buyer, however, is not certain, thus goods may remain unsold.

Therefore, it is sufficient that the inverse hazard rate satisfies the contraction property to guarantee the existence of a unique critical point of seller s_j' expected utility function, which is a decision rule $p_j^*(V_j)$. However, we do not know if $p_j^*(V_j)$ provides the largest possible payoff to s_j. In other words, we need to check the second-order condition.

Assuming that $f_{V_{ji}}$ is differentiable, the second derivative of $E[\overline{u}_{s_j}(\Lambda[p_j, p_{-j}](s_j))]$ with respect to p_j is

$$\frac{\partial^2 E[\overline{u}_{s_j}]}{\partial p_j^2} = M\left(-f_{V_{ji}}(p_j) - f_{V_{ji}}(p_j) - (p_j - v_j)f'_{V_{ji}}(p_j)\right).$$

Thus, we have that

$$\frac{\partial^2 E[\overline{u}_{s_j}]}{\partial p_j^2} = -M\left(2f_{V_{ji}}(p_j) + (p_j - v_j)f'_{V_{ji}}(p_j)\right).$$

Hence, we have that the decision rule p_j^* maximizes $E[\overline{u}_{s_j}]$ if $f_{V_{ji}}(p_j) + (p_j - v_j)f'_{V_{ji}}(p_j) > 0$. We summarize the previous discussion in the following proposition.

Proposition 16 *Consider that $f_{V_{ji}}$ is differentiable for all $j \in S$ and $i \in B$ and γ_j is a contraction. Given a state of the market ν and a profile of decision rules p_{-j}, the best response $b_j(p_{-j})$ is a function, i.e., $b_j(p_{-j}) = p_j^*(v_j)$ that maximizes the expected utility of s_j by considering that p_{-j} is the profile of other sellers decision rules. Furthermore, p_j^* is the unique fixed point of γ_j.*

By Proposition 16 and expression (8), we know that $p_j^*(v_j)$ is the best response to any profile of decision rules p_{-j}. In particular, note that $b_j((p_\tau^*(v_\tau))_{\tau \neq j}) = p_j^*(v_j)$ since function γ_j is independent to the decision rules of other sellers. Therefore, $p^* = (p_1^*(v_1), p_2^*(v_2), \ldots, p_m^*(v_m))$ is the Bayesian Nash equilibrium of the game described in Section 2. Moreover, Theorem 14 guarantees that p^* is the unique Bayesian Nash equilibrium, and we find them using the Newton's method since p_j^* is the unique fixed point of γ_j when this function satisfies the contraction property.

Although the equilibrium price is implicitly defined by expression (9), assuming that such function is a contraction allows us to analyze the relationship between p_j^* and v_j. This relationship is positive when the density function of V_{ji} is non-negative.

Proposition 17 *If $\partial f_{V_{ji}}(p_j)/\partial p_j \geq 0$, then the relationship between p_j^* and v_j is positive.*

Proof We know that p_j^* is implicitly defined by the equation

$$(1 - F_{V_{ji}}(p_j^*)) = (p_j^* - v_j) f_{V_{ji}}(p_j^*).$$

The implicit derivative with respect to v_j

$$-f_{V_{ji}}(p_j^*)\frac{dp_j^*}{dv_j} = \left(\frac{dp_j^*}{v_j} - 1\right) f_{V_{ji}}(p_j^*) + (p_j^* - v_j)\frac{df_{V_{ji}}(p_j^*)}{dp_j^*}\frac{dp_j^*}{dv_j}$$

which we can rewrite as follows

$$f_{V_{ji}}(p_j^*) = \frac{dp_j^*}{dv_j}\left(2f_{V_{ji}}(p_j) + (p_j^* - v_j)\frac{df_{V_{ji}}(p_j^*)}{dp_j^*}\right)$$

In case of selling, we know that $(p_j^* - v_j) > 0$. When $\partial f_{V_{ji}}(p_j)/\partial p_j \geq 0$, we have that

$$2f_{V_{ji}}(p_j) + (p_j^* - v_j)\frac{df_{V_{ji}}(p_j^*)}{dp_j^*} > 0$$

because $f_{V_{ji}}$ is a density function. So, the relationship between p_j^* and v_j is positive, i.e. $\partial p_j^*/\partial v_j > 0$. □

The following example shows that multiple equilibria may arise when the function γ is not a contraction.

Example 2 **Multiple equilibria**. By Theorem 14, the bests responses of seller s_j depend on the distribution $f_{V_{ji}}$. We consider the following distribution function:

$$f_{V_{ji}}(v) = \begin{cases} \frac{1+v^4}{630} & \text{if } 0 \leq v \leq 5, \\ 0 & \text{otherwise.} \end{cases}.$$

Hence, the cumulative distribution function is

$$F_{V_{ji}}(v) = \begin{cases} 0 & \text{if } v \leq 0, \\ \frac{v}{630} + \frac{v^5}{3150} & \text{if } 0 \leq v \leq 5, \\ 0 & \text{if } v \geq 5. \end{cases}. \tag{10}$$

Considering $f_{V_{ji}}$ and $F_{V_{ji}}$, the equation that implicitly defines best responses of seller s_j is

$$\frac{p_j^5}{525} - \frac{v_j p_j^4}{630} + \frac{p_j}{315} - \frac{v_j}{630} - 1 = 0.$$

Then, bests responses are solutions of a five degree equation, which at most have five roots. According to the Descartes' Rule, it is possible to approximate the number of positive roots by counting the number of sign changes. In the previous equation, we have three changes of sign, which means that seller s_j has at most three positive real roots, and at least one real root. We analyse the multiplicity of positive roots through the contraction condition. By doing some algebra, we get that

$$\gamma(p_j) = \frac{1 - F_{V_{ji}}(p_j)}{f_{v_{ji}}(p_j)} = \frac{3150 - 5p_j - p_j^5}{5(1 + p_j^4)} \text{ for all } p_j \in [0, 5].$$

Now note that

$$\mid \gamma(0) - \gamma(1) \mid = \frac{1578}{5} > 1,$$

this means that γ is not a contraction on the interval $[0, 5]$, where it is defined. Hence, the probability distribution (10) does not induce a unique best response for seller s_j. Therefore, equilibrium prices are not unique.

□

3.4 The uniform distribution

Remark 1 leads us to verify if the uniform distribution, a constant distribution, induces a unique price vector at equilibrium. Assuming that V_{ji} is uniformly distributed on an interval $[a, b]$, we have that

$$f_{V_{ji}}(v) = \begin{cases} \frac{1}{b-a} & \text{for all } v \in [a, b], \\ 0 & \text{otherwise.} \end{cases} \quad \text{and } F_{V_{ji}}(v) = \begin{cases} 0 & \text{if } x \le a \\ \frac{x-a}{b-a} & \text{if } x \in [a, b] \\ 1 & \text{if } x \ge b. \end{cases}$$

Substituting in expression (8), we get that

$$p_j = \frac{1 - \frac{b_j - a}{b-a}}{\frac{1}{b-a}} + v_j = b - p_j + v_j.$$

Therefore, the equilibrium price is $p_j^* = b/2 + v_j$. However, considering γ as it is defined in expression (9), we note the following

$$\mid \gamma(x) - \gamma(y) \mid = \mid b - x - (b - y) \mid = \mid x - y \mid .$$

Hence, γ is not a contraction when buyers' valuations are uniformly distributed. This means that the contraction requirement over γ is sufficient, but not necessary to guarantee the uniqueness of equilibrium prices.

4 Equilibrium Characterization for the Exponential Case

In this section, we show that an exponential distribution induces a function γ that satisfies the contraction condition, and consequently behaves like in explanation (1). Moreover, this distribution function allows computing a closed-form solution to the Bayesian Nash equilibrium, which is suitable to perform some comparative statistics for different probability distribution assumptions.

Consider that V_j, V_{ji} are independent and exponentially distributed with parameter $\lambda > 0$. So, their probability distributions are

$$f_{V_{ji}}(x) = f_{V_j}(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0, \\ 0 & \text{otherwise,} \end{cases}$$

for all $j \in \{1, 2, \dots, m\}$ and $i \in \{1, 2, \dots, n\}$. The mean of this distribution, $1/\lambda$, is the occurrence of selling the indivisible good. Thus, if λ increases, buying the object happens more quickly. By expression (6), we have that

$$E[\overline{u}_{s_j}] = (p_j - v_j)\mu e^{-\lambda p_j}. \tag{11}$$

Although it is not difficult to solve the first order condition of $E[\overline{u}_{s_j}]$ when V_{ji} follows an exponential distribution, in the following proposition we show that $\gamma = (1 - F_{V_{ji}})/f_{V_{ji}}$ is a contraction map to show the application of Theorem 14.

Proposition 18 *The function γ is a contraction map when V_{ji} is exponentially distributed.*

Proof We have that

$$\gamma = \frac{1 - F_{V_{ji}}}{f_{V_{ji}}} = \frac{1 - (1 - e^{-\lambda x})}{\lambda e^{-\lambda}} = \frac{1}{\lambda}.$$

Hence, γ is a constant function when V_{ji} is exponentially distributed. Therefore, γ is a contraction.

□

By Proposition 18 and Theorem 14, we can derive a closed form solution for the equilibrium price by using expression (8). We get that $p_j = 1/\lambda + v_j$. The following theorem summarizes the previous discussion.

Theorem 19 *Suppose that V_j and V_{ji} are independent and exponentially distributed with parameter $\lambda > 0$. The price that each seller s_j sets at equilibrium is*

$$p_j^*(v_j) = \frac{1}{\lambda} + v_j,$$

for all $s_j \in S$.

Since the decision rule at equilibrium is unique, we can do some comparative statics.

Corollary 20 *Let p_j^* be the unique price at the symmetric equilibrium found. Then*

1. *The relation between p_j^* and v_j is positive, and*
2. *the relation between p_j^* and λ is negative.*

Proof By Theorem 19, we know that $p_j^* = 1/(m + 1)\lambda + v_j$. Taking the derivatives of p_j^* with respect to v_j, λ and m, we get that

$$\frac{\partial p_j^*}{\partial v_j} = 1 > 0,$$

$$\frac{\partial p_j^*}{\partial \lambda} = -\frac{1}{\lambda^2} < 0.$$

□

In other words, the price increases when the valuation of seller s_j increases, and decreases when the parameter λ increases. This last point implies that prices increase when buyers are in a hurry to buy an indivisible good.

5 Concluding Remarks

We analyze the uniqueness of equilibrium prices in a Bayesian version of the Assignment Game. Although our assignment procedure casts similarities with the one used in First-Price auctions, we observe that sellers' best responses only depend on buyers' valuations, i.e., sellers' best response ignore the price of other goods set by other sellers. Thus, equilibrium prices are determined by the distribution function of buyers' valuations; we find that the inverse hazard rate function, which is the quotient between the cumulative distribution function and the density function, must be a contracting map to induce the existence of unique prices at equilibrium. This condition allows us to show the existence of a positive relationship between prices and sellers valuation.

We also show that the exponential probability functions satisfy the contraction's requirements. Hence, we get a closed-form solution for this specific distributions. In words, there is a unique equilibrium, even if the occurrence of selling each good differs from one good to another. Finally, comparative statistics are naturally performed with this closed-form solution, reflecting empirical evidence such as the fact that price increases even if a good has remained unsold for a long time.

Acknowledgments.

Declarations

Not applicable.

Appendix A Intermediate Value Theorem and the first price sealed bid auction

The IVT is pervasive in auction theory. Consider a classroom exercise consisting in looking for equilibria in a first-price sealed-bid auction. Typically, one assumes symmetric linear bidding functions for agents, which simplifies the search for the probability of winning the auction (the first-order statistic), thus the objective function of the agents. Finally, one validates the procedure by checking those specific parameters of the linear bidding functions that entail the best responses of agents for all possible valuations. The IVT opens another route. It allows searching for the best decision rule for bidders without assuming that the linear bidding functions are symmetric.

The set of bidders is $A = \{1, \ldots, n\}$. Each bidder i has a private valuation v_i drawn from a distribution $f : T_i \to \mathbb{R}$ where $0 = \min\{T_i\}$. Here, we assume that valuations are independent and identically distributed. Also, each bidder observes her type $v_i \in T_i$, but does not observe other bidders type. The types' vector $v = (v_1, \ldots, v_n)$ is drawn from the joint distribution $F : T = \prod_{i=1}^{n} T_i \to \mathbb{R}^n$, with density function f, which we assume of common knowledge.

A bid of agent i is denoted by b_i. Let B_i be the set of all possible bids of agent i; so, a decision rule β_i is function from T_i to B_i. By considering a first-price, sealed-bid auction, bidder i wins if $b_i > \beta_j(v_j)$ for all $j \in A - \{i\}$. Hence, the payoff function of bidder i is

$$u_i(b_i, b_{-i};\ v) = \begin{cases} v_i - b_i & \text{if } b_i > \max_{j\neq i}\{b_j\}, \\ \frac{v_i - b_i}{k} & \text{if } \mid \{j \in M \mid b_j = b_i = \max\{b_k \mid k \in \mathcal{M}\}\} \mid = k, \\ 0 & \text{if } b_i < \max_{j\neq i}\{b_j\}. \end{cases}$$

Consider the probability events

- $Win = \{v \in T \mid \beta_i(v_i) > \max_{j\neq i}\{\beta_j(v_j)\}\}$,
- $Tie = \{v \in T : |\{j \in M \mid \beta_j(v_j) = \beta_i(v_i) = \max\{\beta_k(v_k) \mid k \in \mathcal{M}\}\}| = k\}$, and
- $Lose = \{v \in T \mid \beta_i(v_i) < \max_{j\neq i}\{\beta_j(v_j)\}\}$.

Then, the expected payoff of bidder i is

$$\begin{aligned} E[u_i(b_i, \beta_{-i});\ v] &= Pr[Win](v_i - b_i) + Pr[Tie]\left(\frac{v_i - b_i}{k}\right) + Pr[Lose]0 \\ &= (v_i - b_i)Pr\left[b_i > \max_{j\neq i}\{\beta_j(v_j)\}\right]. \end{aligned}$$

To compute the probability of event Win, note that $\max_{j\neq i}\{\beta_j(v_j)\} = Y_{MAX}(v)$ is the maximum order statistic, which implies that

$$\begin{aligned} Pr\left[b_i > \max_{j\neq i}\{\beta_j(v_j)\}\right] &= Pr\left[b_i > Y_{MAX}(v)\right] \\ &= (Pr[b_i > \beta_j(v_j)])^{(n-1)} \\ &= \left(\int_0^{b_i} f(v_j)dv_j\right)^{(n-1)}. \end{aligned}$$

By the intermediate value theorem for integrals, there exists $c \in [0, b_i]$ such that

$$\int_0^{b_i} f(v_j)dv_j = f(c)(b_i - 0).$$

Then, the expected utility of bidder i can be rewritten as follows

$$E[u_i] = (v_i - b_i)\left(f(c)b_i\right)^{n-1} = f(c)^{n-1}(v_i b_i^{n-1} - b_i^n).$$

Consequently, the critical points of $E[u_i]$ are the solutions of

$$\begin{aligned} 0 &= \frac{\partial E}{\partial b_i} \\ &= f(c)^{n-1}((n-1)v_i b_i^{n-2} - n b_i^{n-1}). \end{aligned}$$

Therefore, the best response of bidder i, to the profile of decision rules β_{-i}, is

$$b_i = \frac{(n-1)v_i}{n}.$$

The previous expression is no other than the unique symmetric equilibrium for the sealed first price auction [48].

Appendix B Proofs

Proof of Proposition 1. Given that seller s_j sells her good at $\Lambda_1[p]$ and $\Lambda_2[p]$, there exist buyers $i, i' \in B$ such that

$$\Lambda_1[p](i) = (v_{ji} - p_j, s_j) \text{ and } \Lambda_2[p](i') = (v_{ji'} - p_j, s_j).$$

Thus, the allocation of s_j is

$$\Lambda_1[p](s_j) = (p_j, \varnothing) \text{ and } \Lambda_2[p](s_j) = (p_j, \varnothing),$$

respectively. In any case, the payoff of s_j is

$$u_{s_j}(\Lambda_1[p](s_j)) = p_j = u_{s_j}(\Lambda_2[p](s_j)).$$

Consequently, the payoff of s_j does not depend on the buyer who buys her good.

Proof of Lemma 2. Consider $x \in S_j$, then seller s_j sells her good to some buyer i at market state x. By the assignment procedure $\Lambda[p]$, i gets the basket $(v_{ji} - p_j, s_j)$ at some step t. We have the following cases:

Case I . Buyer i gets s_j at step $t = 1$. Hence, good s_j is a top good for buyer i which implies that $x \in S_j^1$.
Case II . Buyer i gets s_j at step $t > 1$. Then, s_j is not a top good of i, but $(v_{ji} - p_j, s_j)$ is an IR basket for buyer i since it is the most preferred good for i among the ones that remain in the market at step t. Then, s_j is one of the k-th most preferred goods of i, for some $k \in \{2, \ldots, m\}$. Consequently, $x \in S_j^k$.

In any case, $x \in S_j^k$ for some $k \in \{1, 2, \ldots, m\}$. Thus, we have that

$$S_j \subset \bigcup_{k=1}^{m} S_j^k. \tag{B1}$$

If $x \in \cup_{k=1}^{m} S_j^k$, then $x \in S_j^k$ for some $k \in \{1, 2, \ldots, m\}$. Hence, there exists some buyer i such that $\Lambda[p](i) = (v_{ji} - p_j, s_j)$ and $v_{ji} - p_j = z_{ji}^k$. In other words, seller s_j sells her good to i, which implies that $x \in S_j$. Then

$$\bigcup_{k=1}^{m} S_j^k \subset S_j. \tag{B2}$$

By expressions (B1) and (B2), we conclude that

$$S_j = \bigcup_{k=1}^{m} S_j^k.$$

Proof of Lemma 3

We proceed by contradiction. So, consider a state of the market x such that $x \in S_j^k \cap S_j^{k'}$. Hence, there exist buyers i and i' that buy s_j. We have the following cases:

Case I. $i = i'$. So, i simultaneously gets z_{ji}^k and $z_{ji}^{k'}$ by buying s_j. Without loss of generality, we assume that $k < k'$; then, the assignment procedure $\Lambda[p]$ implies that $z_{ji}^k > z_{ji}^{k'}$ at x. However, $z_{ji}^k = v_{ji} - p_j = z_{ji}^{k'}$ because p and x are fix. Therefore, we have that $v_{ji} - p_j > v_{ji} - p_j$, which is a contradiction.
Case II. $i \neq i'$. By Lemma 2, we have that $S_j^k, S_j^{k'} \subset S_j$. Thus, s_j sells her good to i and i'; in other words, the assignment mechanism $\Lambda[p]$ assigns s_j to two different buyers, which is not possible due to $\Lambda[p]$ randomly breaking ties.

In any case, we get a contradiction since we assume that $S_j^k \cap S_j^{k'}$ is not empty. Therefore, we conclude that $S_j^k \cap S_j^{k'} = \varnothing$.

Proof of Proposition 4. By Lemma 2, we know that $S_j = \cup_{k=1}^{m} S_j^k$. Also, Lemma 3 points out that probability events in $\{S_j^1, S_j^2, \ldots, S_j^m\}$ are disjoint. Consequently, we have that

$$Pr[S_j] = Pr\left[\bigcup_{k=1}^{m} S_j^k\right] = \sum_{k=1}^{m} Pr[S_j^k].$$

Proof of Proposition 5. We know that $\mathcal{Z}$ is the set of all order statistics concerning the sample of surpluses $Z_{\tau i}$ such that $s_\tau \neq s_j$. So, we have that $Z_{(1)} > Z_{(2)} > \cdots > Z_{(m-1)}$. Note that we can compare the surplus Z_{ji} with the previous variables.

Remember that S_j^1 is the event where the good s_j is the most preferred good for the buyer i who gets s_j at the end of the game. Moreover, the assignment procedure $\Lambda[p]$ is IR. Then, we have that $Z_{ji} \geq Z_{(1)}$ and $Z_{(1)} \geq 0$; consequently, we conclude that

$$Pr[S_j^1] = Pr[Z_{ji} \geq Z_{(1)} \geq 0].$$

Concerning S_j^k, this is the event where some buyer i gets s_j at $\Lambda[p]$, and s_j is the k-th most preferred good for i. Then, s_j is not the $(k-1)$-th good for i which means that $Z_{(k-1)} > Z_{ji}$. Also, we have that $Z_{ji} \geq Z_{(k)}$ because i may be indifferent between goods s_j and s_τ for some $\tau \in S - \{s_j\}$. Since the assignment $\Lambda(p)$ is IR, we get that

$$Pr[S_j^k] = Pr[Z_{(k-1)} > Z_{ji} \geq Z_{(k)} \geq 0],$$

for all $k \in \{2, 3, \ldots, m-1\}$.

Finally, consider that Z_{ji} is the m-th largest surplus of i among all IR baskets $(v_{si} - p_s, s)$. In other words, s_j is the less preferred good of buyer i. So, all surpluses in $\mathcal{Z}$ are greater than Z_{ji}, which implies that $Z_{(m-1)} > Z_{ji}$. Therefore, we conclude that

$$Pr[S_j^m] = Pr[Z_{(m-1)} \geq Z_{ji} \geq 0].$$

Proof of Proposition 6.
The probability of $\mathbf{S_j^1}$.
Let $f_{V_{ji}Z_{(1)}}$ be the joint distribution of V_{ji} and $Z_{(1)}$. Since $Z_{(1)}$ is the largest surplus in $\{Z_{\tau i} \mid s_\tau \in S - \{s_j\}\}$, we have that $Z_{(1)}$ is a transformation of variables $Z_{\tau i}$ that does not include the random variable V_{ji}. Then, V_{ji} and $Z_{(1)}$ are statistically independent because $\hat{V}_i$ is a vector of statistically independent random variables. So, the joint distribution between V_{ji} and $Z_{(1)}$ is equal to the product of their marginal distributions; that is to say, $f_{V_{ji}Z_{(1)}} = f_{V_{ji}} f_{Z_{(1)}}$. Now, by Proposition 5, note that

$$\begin{aligned} Pr[S_j^1] &= Pr[Z_{ji} \geq Z_{(1)} \geq 0] \\ &= Pr[V_{ji} - p_j \geq Z_{(1)} \text{ and } Z_{(1)} \geq 0]. \end{aligned}$$

Consequently, the probability of S_j^k is the integral of $F_{V_{ji}Z_{(1)}}$ over the region $R = \{(z, v_{ji}) \in \mathbb{R}^2 \mid v_{ji} - p_j > z \text{ and } z \geq 0\}$. Thus, we have that

$$\begin{aligned} Pr[S_j^1] &= \int_{p_j}^{\infty} \int_0^{v_{ji}-p_j} f_{V_{ji}}(v_{ji}) f_{Z_{(1)}}(z) dz dv_{ji} \\ &= \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) \left(\int_0^{v_{ji}-p_j} f_{Z_{(1)}}(z) dz \right) dv_{ji} \\ &= \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) \left(F_{Z_{(1)}}(v_{ji} - p_j) - F_{Z_{(1)}}(0) \right) dv_{ji}, \end{aligned}$$

where $F_{Z_{(1)}}$ is the cumulative density function of $Z_{(1)}$.

Remember that $\mathcal{Z}$ is the family of order statistics that represents that buyer i has $m-1$ IR baskets $(v_{\tau i} - p_j, s_\tau)$ different from $(v_{ji} - p_j, s_j)$. Thus,

$F_{Z_{(1)}}(0) = Pr[z \leq 0] = 0$ because the assignment procedure $\Delta[p]$ only assigns individually rational baskets to each buyer. Therefore, the probability of selling is

$$Pr[S_j^1] = \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) F_{Z_{(1)}}(v_{ji} - p_j) dv_{ji}. \tag{B3}$$

The probability of $\mathbf{S_j^m}$.

Consider that $f_{V_{ji}Z_{(m-1)}}$ is the joint probability density of variables V_{ji} and $Z_{(m-1)}$. As before, the variables $Z_{(m-1)}$ and V_{ji} are statistically independent; so, their joint probability distribution is $f_{V_{ji}Z_{(m-1)}} = f_{V_{ji}} f_{Z_{(m-1)}}$, where $f_{V_{ji}}$ and $f_{Z_{(m-1)}}$ are the marginal distributions of V_{ji} and $Z_{(m-1)}$, respectively. By Proposition 5, the probability of event S_j^m is

$$\begin{aligned} Pr[S_j^m] &= Pr[Z_{(m-1)} \geq Z_{ji} \geq 0] \\ &= Pr[Z_{(m-1)} \geq V_{ji} - p_j \text{ and } V_{ji} - p_j \geq 0]. \end{aligned}$$

Consequently, the probability of the event S_j^m is the integral of $f_{V_{ji}Z_{(m-1)}}$ over the region $R = \{(v_{ji}, z \in \mathbb{R}^2 \mid z > v_{ji} - p_j \text{ and } v_{ji} \geq p_j\}$. Hence, we have that

$$\begin{aligned} Pr[S_j^m] &= \int_{p_j}^{\infty} \int_{v_{ji}-p_j}^{\infty} f_{V_{ji}}(v_{ji}) f_{Z_{(m-1)}}(z) dz dv_{ji} \\ &= \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) \left(\int_{v_{ji}-p_j}^{\infty} f_{Z_{(m-1)}}(z) dz \right) dv_{ji} \\ &= \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) \left(\lim_{z \to \infty} F_{Z_{(m-1)}}(z) - F_{Z_{(m-1)}}(v_{ji} - p_j) \right) dv_{ji}, \end{aligned}$$

where $F_{Z_{(m-1)}}$ is the cumulative density function of $Z_{(m-1)}$. Then $\lim_{z \to \infty} F_{Z_{(m-1)}}(z) = 1$. Therefore, the probability of event S_j^m is

$$Pr[S_j^m] = \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) \left(1 - F_{Z_{(m-1)}}(v_{ji} - p_j)\right) dv_{ji}.$$

The Probability of S_j^k.

It is worth remembering that $\mathcal{Z}$ is the family of order statistics of the sample $\{z_{\tau i} \mid \tau \in S - \{s_j\}\}$, which implies that V_{ji} is statistically independent of variables $Z_{(k-1)}$ and $Z_{(k)}$. Then, the joint probability density of V_{ji}, $Z_{(k-1)}$ and $Z_{(k)}$ can be written as $f_{V_{ji}Z_{(k-1)}Z_{(k)}} = f_{V_{ji}} f_{Z_{(k-1)}Z_{(k)}}$, where $f_{Z_{(k-1)}Z_{(k)}}$ is the joint distribution of variables $Z_{(k-1)}$ and $Z_{(k)}$.

Proposition 5 establishes that $Pr[S_j^k] = Pr[Z_{(k-1)} \geq V_{ji} - p_j \geq Z_{(k)} \geq 0]$, i.e., we can compute such probability as the integral of the joint distribution

$f_{V_{ji}Z_{(k-1)}Z_{(k)}}$ over the region $R = \{(x, v_{ji}, y) \in \mathbb{R}^3 \mid x \geq v_{ji} - p_j, v_{ji} - p_j \geq y \text{ and } y \geq 0\}$. We have that

$$\begin{aligned} Pr[S_j^k] &= \int_{p_j}^{\infty} \int_{v_{ji}-p_j}^{\infty} \int_{0}^{v_{ji}-p_j} f_{V_{ji}}(v_{ji}) f_{Z_{(k-1)}Z_{(k)}}(x,y)dydxdv_{ji} \\ &= \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) \left(\int_{v_{ji}-p_j}^{\infty} \int_{0}^{v_{ji}-p_j} f_{Z_{(k-1)}Z_{(k)}}(x,y)dydx \right) dv_{ji}. \end{aligned}$$

To compute the previous integral, we first calculate

$$\Delta = \int_{v_{ji}-p_j}^{\infty} \int_{0}^{v_{ji}-p_j} f_{Z_{(k-1)}Z_{(k)}}(x,y)dydx.$$

Given that the market state V is a random vector of independent and identically distributed random variables, we have that Z_{si} are statistically independent and identically distributed, with cumulative density function is $F_{Z_{si}}$, for all $s \in S$. Also, the joint probability between the k and $k-1$ largest order statistics is

$$f_{Z_{(k-1)}Z_{(k)}}(x,y) = \frac{(m-1)!}{(k-1)!(m-k)!} \left[G_{Z_{\tau i}}(x)\right]^{k-1} \left[1 - G_{Z_{\tau i}}(y)\right]^{m-k-2} g(x)g(y) \tag{B4}$$

By substituting expression (B4) into Δ, we get that

$$\Delta = \frac{(m-1)!}{(k-1)!(m-k)!} \int_{0}^{v_{ji}-p_j} \left[G_{Z_{\tau i}}(x)\right]^{k-1} g_{Z_{\tau i}}(x)dx \int_{v_{ji}-p_j}^{\infty} \left[1 - G_{Z_{\tau i}}(y)\right]^{m-k-2} g_{Z_{\tau i}}(y)d$$

To solve the previous integral, we consider that $U = G_{Z_{\tau i}}(x)$ and $W = 1 - G_{Z_{\tau i}}(y)$ because $dU = g_{Z_{\tau i}}(x)dx$ and $dW = -g_{Z_{\tau i}}(y)d(y)$. Thus

$$\begin{aligned} &\int_{v_{ji}-p_j}^{\infty} \left[1 - G_{Z_{\tau i}}(y)\right]^{m-k-2} g_{Z_{\tau i}}(y)dy = \\ &= \frac{-1}{m-k-1} \left[\lim_{y \to \infty} (1 - G_{Z_{\tau i}}(y))^{m-k-1} - (1 - G_{Z_{\tau i}}(v_{ji} - p_j))^{m-k-1} \right] \\ &= \frac{1}{m-k-1} \left[1 - G_{Z_{\tau i}}(v_{ji} - p_j)\right]^{m-k-1}. \end{aligned} \tag{B5}$$

By considering that $V = G_{Z_{\tau i}}(x)$, i.e., $dV = g_{z\tau i}$, the second integral is

$$\int_{0}^{v_{ji}-p_j} \left[G_{Z_{\tau i}}(x)\right]^{k-1} g_{Z_{\tau i}}(x)dx = \frac{1}{k} \left[G_{Z_{\tau i}}(v_{ji} - p_j) - G_{Z_{\tau i}}(0)\right]^{k}$$

$$= \frac{1}{k}\left[G_{Z_{\tau i}}(v_{ji} - p_j)\right]^k. \qquad \text{(B6)}$$

Expressions (B5) and (B6) imply that

$$\Delta = \frac{(m-1)!}{k!(m-k)!(m-k-1)}[1 - G_{Z_{\tau i}}(v_{ji} - p_j)]G_{Z_{\tau i}}(v_{ji} - p_j).$$

Therefore, we conclude that

$$Pr[S_j^k] = \frac{(m-1)!}{k!(m-k)!(m-k-1)} \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji})[1-G_{Z_{\tau i}}(v_{ji}-p_j)]^{m-k-1}[G_{Z_{\tau i}}(v_{ji}-p_j)dv_{ji}^k.$$

Proof of Lemma 8.

Note that $F_{Z_{(1)}}$, $F_{Z_{(m-1)}}$ and $G_{Z_{\tau i}}$ are transformations of the density function $f_{Z_{\tau i}}$. By Lemma 7, we know that $f_{Z_{\tau i}}$ is independent of prices p_τ for all $\tau \neq j$. Therefore, $Pr[S_j^k]$ is independent of p_τ for all $\tau \neq j$.

Proof of Lemma 7.

Consider the random vector $Z_\tau = (Z_{\tau i}, V_\tau) = (V_{\tau i} - p_\tau(V_\tau), V_\tau)$, that is to say, Z_τ is a transformation of the random vector $(V_{\tau i}, V_\tau)$. Note that we can write $(V_{\tau i}, V_\tau)$ in terms of $(Z_{\tau i}, V_\tau)$ as follows

$$(V_{\tau i}, V_\tau) = (Z_{\tau i} + p_\tau(V_\tau), V_\tau).$$

Moreover, since p_τ is differentiable over V_j, the Jacobian of $(Z_{\tau i}, V_\tau)$ is

$$J = \begin{vmatrix} \frac{\partial V_{\tau i}}{\partial Z_{\tau i}} & \frac{\partial V_{\tau i}}{\partial V_\tau} \\ \frac{\partial V_\tau}{\partial Z_{\tau i}} & \frac{\partial V_\tau}{\partial V_\tau} \end{vmatrix} = \begin{vmatrix} 1 & p'_\tau(V_\tau) \\ 0 & 1 \end{vmatrix} = 1.$$

In other words, Z_τ is a invertible and derivable transformation of $(V_{\tau i}, V_\tau)$. Then, the joint distribution of $Z_{\tau i}$ and V_τ is

$$f_{Z_{\tau i}V_\tau} = f_{V_{\tau i}V_\tau}(V_{\tau i}(Z_\tau), V_\tau(Z_\tau)) \mid J \mid = f_{V_{\tau i}V_\tau}(Z_{\tau i} + p_\tau(V_\tau), V_\tau).$$

From the previous expression, the probability density function of $Z_{\tau i}$ is

$$f_{Z_{\tau i}}(z) = \int_{-\infty}^{\infty} Z_{\tau i}, V_\tau(z, v)dv.$$

So, given that $f_{Z_{\tau i}}$ exists and it is well defined, we can find the cumulative density functions of $Z_{(k)}$ for all $k \in \{1, 2, \ldots, m\}$.

Appendix C Contracting Map Theorem

Theorem 21 *(**Contraction Mapping**) Assume that $g(x)$ is a continuous function on $[a, b]$. Also, suppose that $g(x)$ satisfies the Lipschitz condition (2), and that $g([a, b]) \subseteq [a, b]$. Then $g(x)$ has a unique fixed point $c \in [a, b]$. Also, the Newton's succession $\{x_n\}$ defined in the main text converges to c as $n \to \infty$ for any $x_0 \in [a, b]$.*

Proof By the Brower's Theorem, we know that $g(x)$ has at least one fixed point. So, to prove the uniqueness of the fixed point, we assume that there are two fixed points c_1 and c_2. We will prove that these two points must be identical. We know that

$$\mid c_1 - c_2 \mid = \mid g(c_1) - g(c_2) \mid \leq L \mid c_1 - c_2 \mid \text{ and } 0 < L < 1,$$

consequently, c_1 must be equal to c_2.
Finally, we need to prove that the succession described in the main text converge to c, for any $x_0 \in [a, b]$. note that

$$\mid x_{n+1} - c \mid = \mid g(x_n) - g(c) \mid \leq L \mid x_n - c \mid \leq \ldots \leq L^{n+1} \mid x_0 - c \mid .$$

Since $0 < L < 1$, we have that $\mid x_{n+1} - c \mid \to 0$, as $n \to \infty$. The succession converges to the fixed point of $g(x)$, independently of the starting point x_0.

□

Appendix D The case of homogeneous goods

Expression (B3) allows us to write the probability of selling as $Pr[S_j] = \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) F_{V_M}(v_{ji} - p_j) dv_{ji}$, where $V_M = \max_{\tau \in S - \{s_j\}} \{Z_i^\tau\}$. By the definition of the cumulative density function, we have that

$$\begin{aligned} Pr[S_j] &= \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) Pr[v_M \leq v_{ji} - p_j] dv_{ji} \\ &= \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) Pr[\max_{s \neq j}\{v_{si} - p(v_s)\} \leq v_{ji} - p_j] dv_{ji} \\ &= \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) Pr[\max_{s \neq j}\{v_{si} - p(v_s)\} - v_{ji} \leq -p_j] dv_{ji} \\ &= \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) Pr[\max_{s \neq j}\{v_{si} - p(v_s) - v_{ji}\} \leq -p_j] dv_{ji}. \end{aligned} \tag{D7}$$

Now, if all goods are homogeneous for each buyer i, we have that $v_{ji} = v_{si}$ for all $s \in S$. Thus, we can rewrite expression (D7) in the following way

$$\begin{aligned} Pr[S_j] &= \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) Pr[\max_{s \neq j}\{-p(v_s)\} \leq -p_j] dv_{ji} \\ &= \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) Pr[p_j \leq \min_{s \neq j}\{p(v_s)\}] dv_{ji} \end{aligned}$$

By considering that $v_m = \min_{s \neq j}\{p(v_s)\}$, we have that

$$Pr[S_j] = \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) Pr[p_j \leq v_m] dv_{ji}, \tag{D8}$$

In expression (D8), note that $Pr[p_j \leq v_m]$ is independent from the variable v_{ji}, then

$$Pr[S_j] = Pr[p_j \leq v_m] \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) dv_{ji}, \tag{D9}$$

where $\int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) dv_{ji} = Pr[v_j \geq p_j]$. Given that sellers are rational, they do not set a price lower that their valuation, i.e., $Pr[v_j \geq p_j] = 1$. So, in the case of homogeneous goods, the expected utility of s_j is

$$E[u_{s_j}] = (v_j - p_j) Pr[S_j] = (v_j - p_j) Pr[p_j \leq v_m].$$

The previous expression indicates that our assignment procedure is similar to the one used in first-price auctions. Below, we establish the similarities and differences, at equilibrium, between our game and first-price auctions.

Proposition 22 *Consider that all goods in S are homogeneous. If there exists a unique symmetric equilibrium $\sigma^* = (\sigma_j^*)_{s_j \in S}$ where all decision rules are increasing, the equilibrium strategy σ_j is the expectation of V_m conditional to those values greater than v_j, i.e.*

$$\sigma_j^*(v_j) = E[V_m \mid V_m > v_j] \text{ for all } v_j \in V_j.$$

Proof Let $p^* = (p_j^*)_{s_j \in S}$ be a unique symmetric equilibrium of the Bayesian Assignment Game; hence, p_j^* is best response of seller s_j to other sellers best responses $p_k^* = \sigma_k^*(v_k)$ for all $k \neq j$. Note that, if some buyer i buys the good s_j, then the good s_j provides i the largest possible surplus, which means that $v_{ji} - p_j > v_{ki} - p_k$ for all $s_k \in S - \{s_j\}$. Together with the fact that goods are homogeneous, the previous expression implies that $p_j^* < \min_{s_k \neq s_j}\{p_k^* = \sigma^*(v_k)\}$ (for more details, see expressions (D7) and (D8)).
Given that $\sigma^* = (\sigma_k^*)_{k \neq j}$ is a symmetric profile of increasing best responses, these decision rules are also injective function, and we have that

$$\min_{s_k \neq s_j}\{p_k^* = \sigma^*\} = \sigma^*(V_m = \min_{s_k \neq s_j}\{v_k\}).$$

Moreover, the inverse function $(\sigma^*)^{-1}$ exists because σ_j^* is increasing. Consequently, s_j sells her good to i if and only if $(\sigma^*)^{-1}(p_j^*) < V_m$. Consequently, at equilibrium,

$$Pr[S_j] = Pr[(\sigma^*)^{-1}(p_j^*) < V_m].$$

Substituting the previous expression in the expected utility of s_j, we get that

$$E[\overline{u}_{s_j}] = (v_j - p_j)Pr[(\sigma^*)^{-1}(p_j^*) < V_m]. \tag{D10}$$

As before, we denote by F_{V_m} the cumulative density function of V_m. So, we rewrite expression (D10) as follows

$$E[\overline{u}_{s_j}] = (v_j - p_j)(1 - F_{V_m}(\sigma^*)^{-1}(p_j^*)). \tag{D11}$$

Now, remember that best response at equilibrium are solutions of $\partial E[\overline{u}_{s_j}]/\partial p_j = 0$. To get the first derivative of the expected utility, we apply the Inverse Function Theorem; so, we have that

$$-(1 - F_{V_m}((\sigma^*)^{-1}(p_j^*)) + (v_j - p_j)\left(-\frac{f_{V_m}((\sigma^*)^{-1}(p_j))}{(\sigma^*)'((\sigma^*)^{-1}(p_j))}\right) = 0.$$

Since $p_j^* = \sigma_j^*(v_i)$ is the best response of s_j at equilibrium, it satisfies the first order condition. Then, we have that

$$\begin{aligned} -(1 - F_{V_m}(v_j)) + (v_j - \sigma_j^*(p_j))\frac{-f_{V_m}(v_j)}{(\sigma_j^*)'(v_j)} &= 0 \\ -(\sigma_j^*)'(v_j)(1 - F_{V_m}(v_j)) + (v_j - \sigma_j^*(p_j))(-f_{V_m}(v_j)) &= 0 \\ -(\sigma_j^*)'(v_j)(1 - F_{V_m}(v_j)) - \sigma_j^*(p_j)(-f_{V_m}(v_j)) &= v_j f_{V_m}(v_j), \end{aligned} \tag{D12}$$

where we note that

$$-(\sigma_j^*)'(v_j)(1 - F_{V_m}(v_j)) - \sigma_j^*(p_j)(-f_{V_m}(v_j) = \frac{d[-\sigma_j^*(v_j)(1 - F_{V_m}(v_j))]}{dv_j}.$$

Thus, expression (D12) to

$$\frac{d[-\sigma_j^*(v_j)(1 - F_{V_m}(v_j))]}{dv_j} = v_j f_{V_m}(v_j). \tag{D13}$$

By integrating expression (D13) with respect to v_j, we get that

$$\sigma_j^*(v_j) = \frac{-1}{1 - F_{V_m}(vm)} \int_0^{v_j} \tau f_{V_m}(\tau) d\tau. \tag{D14}$$

From expression (D14), we conclude that

$$\sigma_j^*(v_j) = E[V_m \mid V_m > v_j].$$

□

Corollary 23 *At the unique symmetric equilibrium $\sigma^* = (\sigma_j^*)_{s_j \in S}$, all sellers $s_j \in S$ set prices above their valuation v_j.*

Proof We know that expression (D14) hold because σ^* is the unique symmetric equilibrium. Thus, we integrate it by parts

$$\begin{aligned} \sigma^*(v_j) &= \frac{-1}{1 - F_{V_m}(v_j)}\left(\tau\, F_{V_j}(\tau)\Big|_0^{v_j} - \int_0^{v_j} F_{V_m}(\tau) d\tau\right) \\ &= \frac{-1}{1 - F_{V_m}(v_j)}\left(v_j F_{V_j}(v_j) + v_j - v_j - \int_0^{v_j} F_{V_m}(\tau) d\tau\right) \end{aligned}$$

$$= \frac{-1}{1-F_{V_m}(v_j)}\left(v_j F_{V_m}(v_j) + v_j - v_j - \int_0^{v_j} F_{V_m}(\tau)d\tau\right)$$
$$= \frac{1}{1-F_{V_m}(v_j)}\left(v_j(1-F_{V_m}(v_j)) + \int_0^{v_j} F_{V_m}(\tau)d\tau - \int_0^{v_j} 1d\tau\right)$$
$$= v_j - \frac{1}{1-F_{V_m}(v_j)}\int_{v_j}^{0}(F_{V_m}(\tau)-1)d\tau.$$

Note that $F_{V_m}(v_j) - 1 < 0$, hence $-\int_{v_j}^{0}(F_{V_m}(\tau)-1)d\tau > 0$.

□

Proposition 22 establishes that s_j sells her good when she expects that other sellers valuations are greater than her valuation. Moreover, we observe that sellers sell their goods if they set the lowest price for some buyer i. It is worth noticing that previous observations hold in the opposite sense for equilibrium decision rules of a first-price auction; that is to say, each bidder gets a good when they expect that other bidders' valuations are less than her valuation and they bid the greatest price. However, in both cases, s_j sets a price p_j^*, at equilibrium, above her valuation. So, by considering homogeneous goods, our Assignment Game resembles a reverse first-price auction under which sellers search for buyers.

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