Preferences with Costly Bayesian Learning

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Abstract

In this paper, we study a general model of information acquisition: costly Bayesian learning. Using a menu choice framework, we provide an axiomatic characterization of the model, identify its parameters (a utility function, an increasing transformation, a second-order prior belief, and an information cost function), and behaviorally compare the costs. Our results show that the rational inattention model, which has found various applications in the literature, is a special case of the costly Bayesian learning model. We identify several behavioral conditions each of which can be used to test if the decision maker is rationally inattentive or is of a more general type Bayesian learner including those who exhibit aversion to uncertainty. We argue that our decision makers can have flexible attitudes towards the timing of resolution of uncertainty.

1 Introduction

Information acquisition is ubiquitous in economics. To lower the uncertainty about economic variables and improve the quality of decision making, agents acquire information all the time. A prominent model of information acquisition in economics

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is the rational inattention (RI) model. A rationally inattentive agent balances the benefits and costs of information by choosing optimally the type and quantity of information about the *true state* of the world. It has been shown by numerous authors that rational inattention models can have fruitful applications.[1] Rational inattention has been given solid axiomatic foundations with testable implications.[2]

There are, however, other important models of costly information acquisition, where learning is not about the states (unlike rational inattention), but rather about the *distribution of states*. These models are natural to apply especially when the decision maker (DM) does not know the true model of the economy, but rather have a set of possibilities.[3] Following Baaley and Veldkamp [2021], we call the general class of information acquisition models with learning about distributions, Bayesian Learning (BL). In fact, as we discuss later, RI models can be seen as part of this general class of BL models. But then what exactly are the axiomatic foundations for this general class? How can we distinguish them from RI models in general? Can we elicit information costs for the BL models? Is it possible to compare information costs by using choice data?

In this paper, our objective is to answer these and similar questions by considering a general costly Bayesian learning (CBL) model by way of an axiomatic analysis. To do this, following de Oliveira et al. [2017], we use a menu choice framework. However, unlike de Oliveira et al. [2017] who consider menus of acts, we consider menus of act-lotteries (i.e., simple lotteries over acts). In this relatively richer framework, we show that a set of simple and plausible axioms are enough to characterize a general CBL model. We also show that by using menu choice

[1]See, e.g., Mackowiak et al. [2021] for a comprehensive review of the literature on economic applications of rational inattention.

[2]See, e.g., Caplin and Dean [2015]; de Oliveira et al. [2017]; Ellis [2018]; Hebert and Woodford [2019] on axiomatic foundations of rational inattention.

[3]For instance, this is the case for decision makers in finance settings (see, e.g., Pastor and Veronesi [2009] for a review article on parameter learning in finance models); or in labor market settings (see, e.g., Borovickova [2016] on learning about firm characteristics, or Moscarini [2001] on learning about innate worker skills in different occupations); or in IO settings (see, e.g., Jovanovic and Nyarko [1996] on learning about parameters for technology choice); or in policy making settings (see, e.g., Kelly and Kolstad [1999] on learning about climate change parameters, or Cogley and Sargent [2005] on learning about the trade-off between inflation and unemployment). See Baaley and Veldkamp [2021] for a review of the literature on Bayesian learning including models of learning about parameters.

data, all parameters of the model (a utility function; an increasing transformation; a second-order prior belief; and an information cost function) can be uniquely identified. Moreover, we provide a comparative statics result about information costs, and show that special cases of the CBL model (including RI) can be given a characterization in our setting.

In accomplishing the above objectives, we utilize some of the techniques that de Oliveira et al. [2017] developed for characterizing, identifying, and comparing a general model of rational inattention. The main point of departure that we have from de Oliveira et al. [2017]'s work is that they consider a DM with a first-order expected utility and we consider a DM with a second-order expected utility. Similar to Seo [2009]'s analysis of second-order expected utility in an Anscombe-Aumann choice setting, we weaken some of the conditions that de Oliveira et al. [2017] consider to model the DM's uncertainty about the states as a second-order prior instead of a first-order prior. Naturally, a second-order expected utility extends a first-order expected utility, and therefore it can allow a richer set of choice behavior. For instance, we show that while the RI model allows for unambiguous comparisons only when one alternative dominates the other state by state, the CBL model can allow for more general comparisons, where comparisons can be done when one alternative dominates the other posterior by posterior. This means the CBL model can accommodate a richer set of choice behavior by requiring weaker dominance.

There are other distinguishing features of the CBL model from the RI model. One of which is, for instance, about the attitudes towards the timing of resolution of randomizations. Randomizations between possible actions of choice are a natural part of economic activities (e.g., mixed strategies). According to the RI model, the DM is neutral to the timing of resolutions. In particular, the DM is indifferent whether the randomization between two acts is resolved before or after the state of the world is revealed. There are, however, plausible reasons to expect the DM not to be indifferent between the timing of resolutions. For instance, this is true whenever the DM has non-linear attitudes towards uncertainty, as in the cases of uncertainty aversion or uncertainty seeking. These more general attitudes towards the resolution of uncertainty are well-documented in the literature since the seminal work of Ellsberg [1961] and they have been utilized in explaining many puzzling behavior in various economic settings. Our CBL model can accommodate these richer attitudes unlike the RI model.

The RI model that de Oliveira et al. [2017] consider can be embedded into our setting as a special case of the CBL model. By imposing either (i) a stronger dominance axiom, or (ii) an indifference to the timing of resolution type axiom, or (iii) an independence over acts axiom, the RI model can be characterized in our setting as a class of CBL preferences. Importantly the characterization we provide shows that a DM seemingly following the RI model can in fact be someone who does follow the CBL model with multiple first-order prior beliefs, but due to the linear transformation function her behavior is seen as if she has a unique first-order prior belief. This means there is indeed a strong connection between the information acquisition models where learning is about the distribution of states or about the actual state. We believe, therefore, our results in general help better understand the relation of these two seemingly different strands of the literature.

The paper is organized as follows. In Section 2, we present our menu-choice framework, introduce a general model of information acquisition, and define a set of canonical properties for the information cost function. Section 3 provides a set of testable axioms for menu choice data that are implied by the CBL model. Section 4.1 characterizes the CBL model by showing that these testable implications of the model are the only implications for menu choice data. In Section 4.2, we present our identification results for: a utility function, an increasing transformation, a second-order prior belief, and an information cost function. In particular, we show that the canonical properties of an information cost function are necessary and sufficient to uniquely elicit the information costs. Section 4.3 provides a comparative statics result. Section 5 characterizes some special cases of the general CBL model, where the information acquisition model is either the RI model, or a CBL model with a concave (or convex) transformation function capturing attitude towards uncertainty, or a constrained Bayesian learning (ConBL) model capturing costless but constrained information acquisition, or a passive Bayesian learning (PBL) model capturing costless but fixed information acquisition. Section 6 provides a brief review of the related literature. Section 7 concludes. Proofs are given in an Appendix.

2 Preliminaries

In this section, we introduce our choice framework of menus of lotteries over state-contingent acts. We then describe a general information acquisition problem, and define the induced preference relation over menus.

2.1 Framework

Let $\Omega = \{\omega_1, ..., \omega_n\}$ be a finite set of *states* and let Z be an arbitrary set of *prizes* with generic elements x, y, z.[4] A lottery is called *simple* if it has a finite support. Let $\Delta(Z)$ denote the set of simple lotteries over Z with typical elements p, q, r. We call a lottery in $\Delta(Z)$ a *prize-lottery*. An *act* is a map from the set of states Ω into the set of prize-lotteries $\Delta(Z)$. We denote by $\mathcal{F}$ the set of all acts with generic elements f, g, h. Let $\Delta(\mathcal{F})$ denote the set of simple lotteries over $\mathcal{F}$ with typical elements P, Q, R. We call a lottery in $\Delta(\mathcal{F})$ an *act-lottery*. A *menu* $F \subset \Delta(\mathcal{F})$ is a finite set of act-lotteries. Let $\mathbb{F}$ denote the collection of all menus with generic elements F, G, H.

For all $\alpha \in [0,1]$, prize-lotteries $p, q \in \Delta(Z)$, acts $f, g \in \mathcal{F}$, act-lotteries $P, Q \in \Delta(\mathcal{F})$, and menus $F, G \in \mathbb{F}$, we denote (i) by $\alpha p + (1-\alpha) q$ the *mixed prize-lottery* $r \in \Delta(Z)$ such that $r(x) = \alpha p(x) + (1-\alpha) q(x)$ for all $x \in Z$, (ii) by $\alpha f + (1-\alpha) g$ the *mixed act* $h \in \mathcal{F}$ such that $h(\omega) = \alpha f(\omega) + (1-\alpha) g(\omega)$ for all $\omega \in \Omega$, (iii) by $\alpha P + (1-\alpha) Q$ the *mixed act-lottery* $R \in \Delta(\mathcal{F})$ such that $R(f) = \alpha P(f) + (1-\alpha)\, Q(f)$ for all $f \in \mathcal{F}$, and (iv) by $\alpha F + (1-\alpha) G$ the *mixed menu* $H \in \mathbb{F}$ such that $H = \{\alpha P + (1-\alpha)\, Q \,:\, \forall P \in F \text{ and } \forall Q \in G\}$.

With some abuse of notation, we identify a singleton menu $\{P\} \in \mathbb{F}$ with the act-lottery $P \in \Delta(\mathcal{F})$, and a constant act $f \in \mathcal{F}$ with the prize-lottery $p \in \Delta(Z)$ given that $f(\omega) = p$ for all $\omega \in \Omega$. An act $f \in \mathcal{F}$ can be identified with a degenerate act-lottery $\delta_f \in \Delta(\mathcal{F})$ and a prize $x \in Z$ can be identified with a degenerate prize-lottery $\delta_x \in \Delta(Z)$. For an act-lottery $P \in \Delta(\mathcal{F})$, let $f_P \in \mathcal{F}$ denote its induced act, where the resolution of the act-lottery is delayed by the determination

[4] We assume that Ω is finite to simplify the exposition. Our analysis can be extended to a general measurable space by straightforward modifications.

of a state; that is, $f_P(\omega) = \sum P(f)f(\omega)$ for each $\omega \in \Omega$. Notice that any act-lottery $P \in \Delta(\mathcal{F})$ with only constant acts in its support can be identified as a (simple) two-stage compound lottery $P \in \Delta(\Delta(Z))$.

Let $\Delta(\Omega)$ denote the set of all possible beliefs about the likelihood of states in Ω with typical elements μ, ν, τ. Given a belief $\mu \in \Delta(\Omega)$, (i) an act $f \in \mathcal{F}$ induces a (simple) one-stage lottery $f_\mu \in \Delta(Z)$ such that $f_u = \sum_{\omega\in\Omega} \mu(\omega)f(\omega)$ and (ii) an act-lottery $P \in \Delta(\mathcal{F})$ induces a (simple) two-stage compound lottery $P_\mu \in \Delta(\Delta(Z))$ such that $P_\mu(p) = \sum_{f_\mu=p} P(f)$ for all $p \in \Delta(Z)$.

Our primitive is a binary relation $\succsim$ over the set of menus, which represents the preferences of a decision-maker (henceforth, DM). The asymmetric and symmetric parts of $\succsim$ are denoted $\succ$ and $\sim$, respectively. We assume that both $\Delta(\Omega)$ and $\Delta(\Delta(\Omega))$ are measurable spaces, while both $\Delta(\Delta(\Omega))$ and $\Delta(\Delta(\Delta(\Omega)))$ are endowed with the weak* topology. Finally, for any given measurable space $(X, \mathcal{A})$, we denote by $E_\theta[w(.)]$ the average value $\int_X w(x)\theta(dx)$ of a measurable function $w : X \to \mathbb{R}$ by a given measure $\theta : \mathcal{A} \to \mathbb{R}$.

2.2 The information acquisition problem

We consider a general information acquisition problem under uncertainty. Below we first describe how the DM receives utility from each act-lottery when information is not relevant. We then formalize the notion of information in our setting. Finally, we describe the general problem of information acquisition under uncertainty by discussing benefits and costs of information.

Second-order expected utility. The DM receives utility from any given act $f \in \mathcal{F}$ according to the second-order expected utility model

$$E_{\bar{m}}[v(E_{f_\mu}[u(x)])] = \int_{\Delta(\Omega)} v\left(\int_Z u(x)f_\mu(dx)\right)\bar{m}(d\mu), \tag{1}$$

where $u : Z \to \mathbb{R}$ is an unbounded utility function, $v : u(Z) \to \mathbb{R}$ is an unbounded increasing transformation, and $\bar{m} \in \Delta(\Delta(\Omega))$ is a second-order prior.[5] The inter-

[5]Klibanoff et al. [2005] were the first to propose the model given in equation (1), which they call

pretation of this model is that the DM believes there is a true probability model in $\Delta(\Omega)$ which governs the likelihood of each state, but she is uncertain about which probability model $\mu \in \Delta(\Omega)$ is the correct one. However, she has enough information to form a prior belief $\bar{m} \in \Delta(\Delta(\Omega))$ on the likelihood of relevant probability models. Hence, using her second order prior belief $\bar{m}$, she aggregates each possible expected utility $v(E_{f_\mu}[u(x)])$ that she can gain under each probability model μ with her utility function u and a transformation v.

Furthermore, the DM evaluates an act-lottery by taking the expectation of second-order expected utility of each act in its support, and so

$$U_P^{u,v}(\bar{m}) = E_P[E_{\bar{m}}[v(E_{f_\mu}[u(x)])]] \tag{2}$$

gives the second-order expected utility of each act-lottery $P \in \Delta(\mathcal{F})$.

Information and Blackwell ordering. We consider the possibility that the DM acquires information in order to improve her choices under uncertainty. In particular, the DM can acquire a noisy signal that conveys additional information about the true probability model. For instance, she can do this by sampling previous realizations of states. Each such sampling will result in a signal which would induce a posterior belief $m \in \Delta(\Delta(\Omega))$ from the prior $\bar{m}$ according to Bayes rule. Thus, a signal would lead to a *distribution over posteriors* $\pi \in \Delta(\Delta(\Delta(\Omega)))$ such that the expected posterior is equal to the prior. As a result, the collection of all possible signals can be given by the set

$$\Pi(\bar{m}) = \left\{ \pi \in \Delta(\Delta(\Delta(\Omega))) \,:\, \bar{m} = \int_{\Delta(\Delta(\Omega))} m\, \pi(dm) \right\}.$$

The set of signals $\Pi(\bar{m})$ is partially ordered in terms of their informativeness by the well-known Blackwell [1951] order, which in this context can be defined as follows:

Definition 1 (Blackwell order)**.** Signal $\pi \in \Pi(\bar{m})$ is *more informative* than signal

the smooth ambiguity model, to study attitudes towards ambiguity. They provide an axiomatic characterization of this model by using a rich choice setting of first-order and second-order acts. Seo [2009] provides an axiomatic characterization of the second-order expected utility model by using a lotteries of acts choice setting.

$\rho \in \Pi(\bar{m})$, denoted $\pi \trianglerighteq \rho$, if

$$\int_{\Delta(\Delta(\Omega))} \phi(m)\,\pi(dm) \geq \int_{\Delta(\Delta(\Omega))} \phi(m)\,\rho(dm)$$

for every convex continuous function $\phi : \Delta(\Delta(\Omega)) \to \mathbb{R}$.

Benefits and costs of information. Given a menu F, extracting a signal allows the DM to make a more informed choice from F because she can choose an act-lottery to maximize her second-order expected utility for each posterior $m \in \Delta(\Delta(\Omega))$. With a utility function $u : Z \to \mathbb{R}$ and an increasing transformation function $v : u(Z) \to \mathbb{R}$, the *benefit of information* for a given signal $\pi \in \Pi(\bar{m})$ is therefore,

$$b_F^{u,v}(\pi) = \int_{\Delta(\Delta(\Omega))} \left[\max_{P \in F} U_P^{u,v}(m)\right] \pi(dm).$$

Since the integrand in square brackets is a convex continuous function on $\Delta(\Delta(\Omega))$, the benefits of information are increasing in the Blackwell order $\trianglerighteq$.

A rational DM balances the benefit of information from a signal π against the cost for acquiring that signal. These costs are measured by an information cost function $c : \Pi(\bar{m}) \to [0, \infty]$, which associates a cost $c(\pi)$ to each signal $\pi \in \Pi(\bar{m})$. In the information acquisition problem we consider, the DM therefore chooses a signal π that maximizes the difference between benefits and costs of information $b_F^{u,v}(\pi) - c(\pi)$. We say a cost function is *proper* if it can assume a finite value and it is lower-semicontinuous.

2.3 Costly Bayesian learning preferences

In our framework, the DM chooses a menu with the expectation of acquiring information before she selects an act-lottery. We model information acquisition as illustrated above, and study the induced preference relation over menus.

Definition 2. A binary relation $\succsim$ over menus is a *costly Bayesian learning (CBL)* preference if it is represented by a functional $V : \mathbb{F} \to \mathbb{R}$, defined by

$$V(F) = \max_{\pi \in \Pi(\bar{m})} \left[b_F^{u,v}(\pi) - c(\pi)\right], \tag{3}$$

where $u : Z \to \mathbb{R}$ is an unbounded utility function, $v : u(Z) \to \mathbb{R}$ is an unbounded increasing transformation, $\bar{m} \in \Delta(\Delta(\Omega))$ is a second-order prior, and $c : \Pi(\bar{m}) \to [0, \infty]$ is a proper information cost function. In this case, we also say $\succsim$ is represented by the parameters $(u, v, \bar{m}, c)$.

The assumptions on parameters $(u, v, \bar{m}, c)$ are standard. Unboundedness of u and v implies that the benefits of information are not bounded, which will be useful for unique identification of parameters. Monotonicity of v is natural property that ensures that more is better. Properness of c is a minimal assumption required to ensure that the maximization over costly signals is well-defined.

2.4 Canonical information costs

Properness is the only restriction that we impose on the cost function to define the information acquisition problem. On the other hand, there are a number of intuitive properties that, without loss of generality, can be imposed on an information cost function (see Corollary 1 in section 4.2).

Definition 3. An information cost function $c : \Pi(\bar{m}) \to [0, \infty]$ is canonical if

(i) $c(\delta_{\bar{m}}) = 0$, where $\delta_{\bar{m}} \in \Pi(\bar{m})$ assigns probability 1 to the second-order prior $\bar{m}$ [groundedness: no information no cost],

(ii) $\pi \trianglerighteq \rho$ implies $c(\pi) \geq c(\rho)$ [monotonicity: more information more cost],

(iii) $\alpha c(\pi) + (1-\alpha)c(\rho) \geq c(\alpha\pi + (1-\alpha)\rho)$ [convexity: average cost of information exceeds cost of average information].

It is well known that mutual information satisfies properties (i)–(iii), and so the cost functions based on mutual information–which are frequently used in the literature–are canonical.[6]

[6]See, e.g., Cover and Thomas [2006, Chapter 2] for a comprehensive analysis on mutual information and its use in economics.

3 Axioms

In the following, we consider eight axioms for the DM's preferences over menus. The first three axioms are standard in the menu-choice literature:

Axiom 1 (Weak order)**.** *For all menus* F, G *and* H*, (i)* $F \succsim G$ *or* $G \succsim F$*, and (ii) if* $F \succsim G$ *and* $G \succsim H$*, then* $F \succsim H$*.*

Axiom 2 (Continuity)**.** *For all menus* F*,* G *and* H*, the following sets are closed:* $\{\alpha \in [0,1] : \alpha F + (1-\alpha)G \succsim H\}$ *and* $\{\alpha \in [0,1] : H \succsim \alpha F + (1-\alpha)G\}$*.*

Axiom 3 (Unboundedness)**.** *There are prizes* x *and* y*, with* $x \succ y$*, such that for all* $\alpha \in (0,1)$ *there is a prize* z *satisfying either* $y \succ \alpha z + (1-\alpha)x$ *or* $\alpha z + (1-\alpha)y \succ x$*.*

Axioms 1 and 2 ensure that preferences are complete, transitive and continuous. Axiom 3 implies that preferences over outcomes are unbounded (see, e.g., Maccheroni et al. [2006, Lemma 29]). The remaining axioms reflect more distinctive features of the information acquisition problem which defines the CBL preferences.

The DM chooses an act-lottery conditional on signal realizations by considering all possible alternatives in her menu. As such, when choosing a menu, the DM exhibits a *desire for flexibility* (Kreps [1979]); that is, adding an alternative to a menu can only make the DM better off since she can always ignore the added alternative if it does bring no added value.[7]

Axiom 4 (Preference for flexibility)**.** *For all menus* F *and* G*, if* $F \supset G$ *then* $F \succsim G$*.*

The DM solves an optimal signal extraction problem by balancing the benefits and costs of information that may differ from menu to menu. As a result, when choosing a menu, the DM exhibits a *desire for early resolution of uncertainty* (Kreps and Porteus [1978]) about which menu is relevant for her payoff. For instance, when the DM faces the menu F for sure, then she can focus her signal extraction on

[7] Desire for flexibility distinguishes CBL preferences from models where the DM may miss some of the alternatives in a menu due to her bounded rationality (see, e.g., Masatlioglu, Nakajima, and Ozbay [2012], Manzini and Mariotti [2014], or Ortoleva [2013]).

F, but if she faces a mixture of F with another (equally good) menu G, then she cannot focus her signal extraction on F or on G causing her potentially to loose value. This behavior reflects in our framework as a preference towards having one of the equally good menus F or G for sure rather than having a mixed menu $\alpha F + (1-\alpha)G$, a behavior which is called *aversion to contingent planning* (Ergin and Sarver [2010]).

Axiom 5 (Aversion to contingent planning). *For all menus F and G, if $F \sim G$ then $F \succsim \alpha F + (1-\alpha)G$ for all $\alpha \in (0,1)$.*

Information is redundant for singleton menus since signal realizations do not help to choose a best alternative in a singleton menu and since the expected posterior belief is equal to the prior. Hence, the optimal information in a mixed menu $\alpha F + (1-\alpha)P$ depends only on α and F, and does not change if P is replaced by an alternative act-lottery Q. Thus, the DM's preferences exhibit an *independence of degenerate decisions* (Ergin and Sarver [2010]).

Axiom 6 (Independence of degenerate decisions). *For all menus F and G, act-lotteries P and Q, and $\alpha \in (0,1)$, if $\alpha F + (1-\alpha)P \succsim \alpha G + (1-\alpha)P$, then $\alpha F + (1-\alpha)Q \succsim \alpha G + (1-\alpha)Q$.*

The DM acquires information only about the true probability model in $\Delta(\Omega)$. As a result, adding an act-lottery Q to a menu F can make the DM strictly better off only when there is some optimal information about the true probability model that would lead the DM to choose Q from $F \cup \{Q\}$. Thus, if F already contains an act-lottery that is preferred to Q posterior by posterior, adding Q to her opportunity set can make her neither better off nor worse off.[8]

Axiom 7 (Posteriorwise dominance). *For all menus F and act-lottery Q, if there exists $P \in F$ such that $P_\mu \succsim Q_\mu$ for all $\mu \in \Delta(\Omega)$, then $F \sim F \cup \{Q\}$.*

The DM is an expected utility maximizer. Thus, her preferences satisfy an independence axiom when they are reduced to the set of prize-lotteries, where neither uncertainty about states nor information acquisition is relevant.

[8] A counterpart of our stronger dominance axiom in the original Anscombe-Aumann choice setting was introduced by Seo [2009] in order to characterize the second-order expected utility model.

Axiom 8 (Independence of prize-lotteries). *For all prize-lotteries p, q, r and $\alpha \in (0,1)$, $p \succsim q$ if and only if $\alpha p + (1-\alpha) r \succsim \alpha q + (1-\alpha) r$.*

Discussion. In a menu of act choice setting, de Oliveira et al. [2017] consider the first six axioms which we adapt to our menu of act-lottery choice setting. Our seventh axiom is a dominance condition which, in our choice setting, strengthens de Oliveira et al. [2017]'s counterpart, a state-wise dominance (their seventh and last axiom). de Oliveira et al. [2017]'s state-wise dominance axiom can be extended to our choice setting. We argue in section 5 that the (extended) state-wise dominance axiom implies our posterior-wise dominance condition. Moreover, we show that an extension of de Oliveira et al. [2017]'s RI preferences can be characterized by this stronger dominance axiom in our setting showing that the RI model can be embedded into our setting as a special case of the CBL model.

Our eighth and last axiom is a standard independence of prize-lotteries axiom adapted to our menu of act-lottery choice setting. de Oliveira et al. [2017]'s Axioms 5 and 6 imply an independence of acts axiom in their setting. Moreover, by the definition of mixture acts in their setting, their RI preferences implicitly satisfy a reversal of order axiom (see section 5). As a result, the independence of acts axiom they have together with their implicit reversal of order axiom imply independence of prize-lotteries axiom. However, we do not assume an independence of act-lotteries axiom nor a reversal of order axiom. Thus, our CBL preferences are more general than the RI preferences. In fact, we show in section 5 that de Oliveira et al. [2017]'s RI model can be characterized in our setting by imposing on our CBL preferences either the independence of act-lotteries axiom or the reversal of order style axioms.

4 Analysis

In this section, we show that CBL preferences can be characterized by the set of axioms that we discussed in the previous section. Moreover, we show that the model parameters can be uniquely identified and compared across decision-makers using menu-choice data.

4.1 Characterization

The following result shows that Axioms 1–8 characterize all observable implications of CBL preferences.

Theorem 1. *A binary relation $\succsim$ over menus is a costly Bayesian learning preference if and only if it satisfies Axioms 1–8.*

Theorem 1 shows that the information acquisition problem in Section 2.2 implies a set of intuitive choice behavior that can be observed in our framework. It also establishes a formal connection between the literature on information acquisition about the distribution of the state and the decision-theory literature on menu-choice.

Proof sketch: Necessity part of the proof is straightforward to show. For sufficiency, Lemma 1 (Appendix A.2) shows that if a binary relation $\succsim$ satisfies Axioms 1–8, then there exist an unbounded utility function $u : Z \to \mathbb{R}$, an unbounded increasing transformation $v : u(Z) \to \mathbb{R}$, a second-order prior $\bar{m}$, and an information cost function $c : \Pi(\bar{m}) \to [0, \infty]$ such that $(u, v, \bar{m}, c)$ represents $\succsim$. In particular, Axioms 1, 2, 4, and 7 imply that every menu $F \in \mathbb{F}$ has a *singleton equivalent* $P_F \in \Delta(\mathcal{F})$ such that $P_F \sim F$. By considering the induced axioms over the set of act-lotteries, we then obtain–as explained in more detail in the next section–a second-order expected utility representation for the restriction of the preferences on act-lotteries with parameters $(u, v, \bar{m})$. Using singleton equivalents and the second-order expected utility representation, (with a slight abuse of notation) a functional V over the set $\Phi_{\mathbb{F}} = \{\phi_F : \Delta(\Delta(\Omega)) \to \mathbb{R} \,|\, \phi_F(m) = \max_{P \in F} E_P E_m v(E_{f_\mu} u(x)),\ \forall m,\ \forall F\}$ can be defined such that $F \succsim G$ if and only if $V(\phi_F) \geq V(\phi_G)$ for all menus F and G. The remainder of the proof uses Axioms 1–8 to show that V is monotone, continuous and convex, and employs duality arguments to establish the desired representation.

4.2 Identification

In this section, we show how the parameters $(u, v, \bar{m}, c)$ in the information acquisition problem can be identified from menu-choice data.

Identifying a utility function. The restriction of the DM's preferences $\succsim$ over the set of first-order prize-lotteries $\Delta(Z)$ satisfies the axioms of expected utility (with unboundedness), and so they can be represented by the expected utility model with an unbounded utility function $u : Z \to \mathbb{R}$ such that $p \succsim q$ if and only if $E_p u(x) \geq E_q u(x)$. Moreover, u can be normalized such that $E_{p'} u(x) = 1$ and $E_{q'} u(x) = 0$ for some $p' \succ q'$.

Identifying an increasing transformation. The restriction of the DM's preferences $\succsim$ over the set of second-order prize-lotteries $\Delta(\Delta(Z))$ satisfies the axioms of expected utility (with unboundedness). Thus, there exists an unbounded function $\bar{U} : \Delta(Z) \to \mathbb{R}$ such that $\bar{P} \succsim \bar{Q}$ if and only if $E_{\bar{P}} \bar{U}(r) \geq E_{\bar{Q}} \bar{U}(r)$ for all $\bar{P}, \bar{Q} \in \Delta(\Delta(Z))$. Moreover, $\bar{U}$ can be normalized such that $E_{\bar{P}'} \bar{U}(r) = 1$ and $E_{\bar{Q}'} \bar{U}(r) = 0$ for some $\bar{P}' \succ \bar{Q}'$. Without loss of generality, let $\bar{P}' = \delta_{p'}$ and $\bar{Q}' = \delta_{q'}$. Since $\Delta(Z)$ can be embedded into $\Delta(\Delta(Z))$, we have $E_{\delta_p} \bar{U}(r) \geq E_{\delta_q} \bar{U}(r)$ if and only if $E_p u(x) \geq E_q u(x)$. This means there exists an unbounded increasing transformation $v : u(Z) \to \mathbb{R}$ such that $\bar{U}(p) = v(E_p u(x))$ for all $p \in \Delta(Z)$.

Identifying a second-order prior. The restriction of the DM's preferences $\succsim$ over the set of act-lotteries $\Delta(\mathcal{F})$ satisfies the axioms of expected utility. Thus, there exists a function $U : \mathcal{F} \to \mathbb{R}$ such that $P \succsim Q$ if and only if $E_P[U(f)] \geq E_Q[U(f)]$ for all $P, Q \in \Delta(\mathcal{F})$. Note that we can let U such that $U(p) = \bar{U}(p) = v(E_p u(x))$ for all $p \in \Delta(Z)$. But then, since $\succsim$ satisfies Axiom 7, following Seo [2009, Lemma B.3-B10], there exists a second-order belief $\bar{m} \in \Delta(\Delta(\Omega))$ such that $U(f) = E_{\bar{m}}[v(E_{f_\mu} u(x))]$ yields the value of each act $f \in \mathcal{F}$.

Identifying an information cost function. Given all other parameters $(u, v, \bar{m})$, we can obtain a unique canonical information cost function c by using menu-choice data. Note that for any given menu $F \in \mathbb{F}$, there exists an equivalent act-lottery $P_F \in \Delta(\mathcal{F})$ such that $F \sim P_F$.

Theorem 2. *Let $\succsim$ be a costly Bayesian learning preference such that the restriction of $\succsim$ to singleton menus is represented by $(u, v, \bar{m})$. Then the cost function*

$c : \Pi(\bar{m}) \to [0, \infty]$, defined by

$$c(\pi) = \sup_{F \in \mathbb{F}} \left[b_F^{u,v}(\pi) - U_{P_F}^{u,v}(\bar{m}) \right], \tag{4}$$

is the unique canonical cost function such that $(u, \phi, \bar{m}, c)$ represents $\succsim$.

As an implication of Theorem 2, we can identify all parameters of the costly information acquisition model up to standard positive affine transformations. For any two measures $m, m' \in \Delta(\Delta(\Omega))$, we say they are essentially equivalent (denoted by $m \approx m'$) if for all $P \in \Delta(\mathcal{F})$,

$$E_P[E_m[v(E_{f_\mu}[u(x)])]] = E_P[E_{m'}[v(E_{f_\mu}[u(x)])]].$$

Corollary 1. *If $(u, v, \bar{m}, c)$ and $(u', v', \bar{m}', c')$ represent the same CBL preferences $\succsim$ with canonical costs c and c', then there exists $\alpha, \lambda > 0$ and $\beta, \gamma \in \mathbb{R}$ such that $u' = \alpha u + \beta$, $v'(\alpha u + \beta) = \lambda v(u) + \gamma$, $\bar{m}' \approx \bar{m}$ and $c' = \lambda c$.*

4.3 Comparative statics

As an application of our identification results, we now consider a comparative measure of flexibility. Let DM1 and DM2 be two individuals who have CBL preferences $\succsim_1$ and $\succsim_2$, respectively, with canonical representations agreeing on the utilities, transformations, second-order priors, but the costs. We say that DM2 is less able to acquire information than DM1 when acquiring information is costlier for DM2 than DM1; that is, $c_2^* \geq c_1^*$. Intuitively, when DM2 is less able to acquire information, she should find the option of committing to a singleton menu–which eliminates the need of acquiring information–more valuable than DM1. The following comparative defines when DM2 finds singleton menus more valuable than DM1.

Definition 4. [Comparative desire for singletons] Let $\succsim_1$ and $\succsim_2$ be two binary relations on the set of menus $\mathbb{F}$. Then $\succsim_2$ has a *stronger desire for singletons* than $\succsim_1$ if, for all F and P, whenever $P \succ_1 F$, then $P \succ_2 F$.

The following result shows that the comparative in Definition 4 characterizes when DM2 is less able to acquire information than DM1.

Theorem 3. *Let $\succsim_1$ and $\succsim_2$ be costly Bayesian learning preferences with canonical representations $(u_1, v_1, \bar{m}_1, c_1)$ and $(u_2, v_2, \bar{m}_2, c_2)$, respectively. Then, $\succsim_2$ has a* stronger desire for singleton menus *than $\succsim_1$ if and only if $(u_1, v_1, \bar{m}_1) = (u_2, v_2, \bar{m}_2)$ and $c_2 \geq c_1$.*

Theorem 3 provides a behavioral measure of comparative ability to acquire information. In particular, Theorem 3 implies that the utility difference between a menu and the singleton equivalent of the menu is higher for a DM who is less able to acquire information. As such, the DM will be willing to pay a higher premium for the option to have the singleton menu, thereby avoiding higher information costs.

5 Special cases

Special cases of CBL preferences can be characterized in terms of the additional restrictions they impose on menu-choice data.

5.1 Rationally inattentive preferences

As indicated earlier, de Oliveira et al. [2017]'s RI preferences are a special case of our CBL preferences. We can establish this relation both by directly inspecting the representations or by establishing the underlying behaviors.

Structural relation. The RI model, which assigns a utility value to each menu of acts, can be given in our setting as

$$\hat{V}(F) = \max_{\hat{\pi} \in \Pi(\bar{\mu})} \left(E_{\hat{\pi}} \left[\max_{f \in F} \left(E_{f_\mu} w(x) \right) \right] - \hat{c}(\hat{\pi}) \right), \tag{5}$$

where $\bar{\mu}$ is a first-order prior, $\Pi(\bar{\mu}) = \{\pi \in \Delta(\Delta(\Omega)) : \bar{\mu} = \int \mu d\pi(\mu)\}$ is the set of viable information structures conveying information about the states, $w : Z \to \mathbb{R}$ is a utility function, and $\hat{c} : \Pi(\bar{\mu}) \to [0, \infty]$ is a proper information cost function.

Note that whenever the CBL model has an affine transformation function v, then the utility the DM obtains from each act f can be given by the first-order expected utility

$$E_{f_{\mu_{\bar{m}}}} v(u(x)) = \int_{\Omega} v[u(f(\omega)] d\mu_{\bar{m}}(\omega), \tag{6}$$

where $\mu_{\bar{m}} = \int_{\Delta(\Omega)} d\bar{m}(\mu)$ is the barycenter of the second-order probability measure $\bar{m}$. As such, according to the CBL model with an affine transformation function v, the value of a menu of acts F will be

$$V(F) = \max_{\pi \in \Pi(\bar{m})} \left(E_{\pi} \left[\max_{f \in F} E_m \left[v \left(E_{f_{\mu}} u(x) \right) \right] \right] - c(\pi) \right), \tag{7}$$

which is equivalent to

$$\max_{\pi \in \Pi(\bar{m})} \left(E_{\pi} \left[\max_{f \in F} E_{f_{\mu_m}} \left[v \left(u(x) \right) \right] \right] - c(\pi) \right), \tag{8}$$

where $\mu_m = \int_{\Delta(\Omega)} \mu \, m(d\mu)$ for all $m \in \Delta(\Delta(\Omega))$, which can be equivalently given as

$$\max_{\hat{\pi} \in \Pi(\mu_{\bar{m}})} \left(E_{\hat{\pi}} \left[\max_{f \in F} E_{f_{\mu_m}} \left[v \left(u(x) \right) \right] \right] - \hat{c}(\hat{\pi}) \right), \tag{9}$$

where $\Pi(\mu_{\bar{m}}) = \{\pi \in \Delta(\Delta(\Omega)) : \mu_{\bar{m}} = \int_{\Delta(\Omega)} \mu_m \pi(dm)\}$ is the set of viable information structures conveying information about the states and $\hat{c} : \Pi(\mu_{\bar{m}}) \to [0, \infty]$ is the suitably substituted for information cost function. Then, letting $\bar{\mu} = \mu_{\bar{m}}$ and $w(x) = v(u(x))$ for all $x \in Z$ above, we see that equations (5) and (9) coincide showing that the RI model can be embedded into our setting as a CBL model representing preferences over menus of acts. Moreover, these RI preferences can be immediately extended to our general setting of menus of act-lotteries where the value of a menu of act-lotteries F will be

$$V(F) = \max_{\hat{\pi} \in \Pi(\mu_{\bar{m}})} \left(E_{\hat{\pi}} \left[\max_{P \in F} E_P \left(E_{f_{\mu_m}} \left[v \left(u(x) \right) \right] \right) \right] - \hat{c}(\hat{\pi}) \right). \tag{10}$$

Axiomatic relation. The RI model can be established as a special case of our CBL model by imposing either one of the five axioms below.

First, according to the extended RI model given in equation (10), the DM can be seen as if she learns about the states, and not about the distribution of the states. This fact induces the following state-wise dominance condition, which is clearly a strengthening of our posterior-wise dominance axiom.

Axiom 9 (Statewise dominance). *For all menus F and act-lotteries Q, if there exists $P \in F$ such that $P_{\delta_\omega} \succsim Q_{\delta_\omega}$ for all $\omega \in \Omega$, then $F \sim F \cup \{Q\}$.*

Axiom 9 states that the DM acquires information only about the states in Ω. As a result, adding an act-lottery Q to a menu F can make her strictly better off only when there is some information about the true state that would lead the DM to choose Q from $F \cup \{Q\}$. Thus, if F already contains an act-lottery that is preferred to Q state by state, adding Q to her opportunity set can make her neither better off nor worse off.

Second, for the equivalence of equations (7) and (8) above, we moved the expectation by the first-order beliefs out of the transformation function and reduced the second-order expectation to a first-order expectation. This possibility of reduction implies the following three axioms.

Axiom 10 (Indifference to the timing of resolution). *For all act-lotteries P, we have $f_P \sim P$.*

Axiom 10 states that the DM's preferences are not sensitive to the timing of resolution of uncertainty about which act of the possible ones will be relevant for her payoffs.

Third, similar to the second observation, the DM must be indifferent between whether the mixture of acts is resolved before or after the state is determined.

Axiom 11 (Reversal of order). *For all acts f, g, and mixture weights $\alpha \in (0, 1)$, we have $\alpha\delta_f + (1 - \alpha)\delta_g \sim \alpha f + (1 - \alpha)g$.*

Axiom 11 states that the DM's preferences are not sensitive to the timing of resolution of uncertainty about which act of the possible two will be relevant for her payoffs.

Fourth, similar to the third observation, the DM must be indifferent between whether a second-order lottery is resolved at the first-stage or the second-stage.

Axiom 12 (Reduction of compound lotteries)**.** *For all prize-lotteries* p, q*, and* $\alpha \in (0,1)$*, we have* $\alpha\delta_p + (1-\alpha)\delta_q \sim \alpha p + (1-\alpha)q$*.*

Axiom 12 states that the DM's preferences are not sensitive to the timing of resolution of uncertainty about which constant act of the possible two will be relevant for her payoffs.

Finally, the value of a single act f will be given by the first-order expected utility given in equation (6) whenever the CBL model has an affine transformation. This implies the following independence of acts axiom.

Axiom 13 (Independence of acts)**.** *For all acts* f, g, h *and* $\alpha \in (0,1)$*,* $f \succsim g$ *if and only if* $\alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$*.*

Axiom 13 is a standard independence over acts axiom and typically imposed in order to obtain the subjective expected utility model in an Anscombe-Aumann choice setting.

Counterparts of Axioms 9, 11 and 12 can be found in Seo [2009] who uses them in the original Anscombe-Aumann act choice setting to show that the first-order (subjective) expected utility representation can be obtained from his second-order expected utility representation. A counterpart of Axiom 10 was first given by Kreps [1988, p.107] in an Anscombe-Aumann choice setting by way of implicitly describing the condition within a discussion about the order of acts (i.e., horse-race lotteries) and prize-lotteries (i.e., roulette-wheel lotteries). Clearly, Axiom 10 implies Axiom 11, which implies Axiom 12. In fact, the following results shows that for a given CBL preferences all of the five axioms stated above are equivalent to each other, and each one of them can characterize the extended RI preferences in our setting as a special case of the CBL preferences.

Proposition 1. *Let* $\succsim$ *be a costly Bayesian learning preference. Then the following are equivalent:*

(i) $\succsim$ *satisfies either one of Axioms 9, 10, 11, 12, or 13.*

(ii) $\succsim$ *is represented by the model given in equation (10).*

Discussion. Proposition 1 provides a way to separate models of information acquisition proposed in the literature that are about either learning the distribution of the state (like the CBL model) or the state itself (like the RI model). Both types of models are utilized for various choice problems in macroeconomics, finance, IO, and other disciplines; however, it is not clear what type of choice implications these models have and what distinguishes them from each other in a general choice framework. This is an issue that can be of critical importance to understand which type of model should be used for particular applications, for instance when deciding modeling rationally inattentive behavior. Proposition 1 shows that these models can be distinguished from each other by several equivalent ways on the basis of either a dominance axiom, or timing of resolution type axioms, or an independence of acts axiom.

First, the dominance axiom reveals directly what type of learning is more suitable for the information acquisition problem. For instance, if the DM satisfies the stronger axiom of dominance, Axiom 9, then RI as a model of learning about the state with information acquisition is more suitable to describe the DM's behavior; however, if the DM violates this stronger axiom of dominance, but satisfies the relatively weaker axiom of dominance, Axiom 7, then CBL as a general model of learning about the distribution of states with information acquisition is more suitable to describe the DM's behavior. Second, the timing of resolution type axioms reveal whether the DM is sensitive to the timing of resolution of randomizations or not. For the RI model the DM is indifferent, though in general there can be viable reasons to expect that the DM is not indifferent to the timing of resolutions; for instance, this is the case when the DM has multiple first-order priors, as in the case of general CBL preferences. And third, verification (or violations) of the independence of acts axiom reveals if the DM is a subjective expected utility maximizer or not. In general, the DM can violate the independence of acts axiom; for instance, this is the case when the DM exhibits uncertainty aversion, a phenomenon which is widely documented in the experimental literature.

What we have shown and argued above was that because of the reducibility of the CBL model whenever its transformation function v is affine, the DM can be seen as if she acts according to the RI model and acquires information about the states, and not the distributions. One can wonder, at this point, whether a hybrid model can be considered where information can be gathered both about the states and distributions at the same time. Although this type of information acquisition is a possibility in general, our Proposition 1 implies that such a model will violate some of the axioms listed above, most likely the axiom of preference for flexibility, Axiom 4. Thus, to allow for more general type of learning models, one might need to allow for the possibility that information can sometimes be harmful and induce the DM to prefer smaller menus.

5.2 Other special cases

In addition to the rationally inattentive preferences, there are other special cases of the CBL preferences.

Ambiguity averse preferences

We have discussed in section 5.1 that any of the Axioms 9-13 can be violated by the DM with CBL preferences, whenever she is not indifferent to the timing of resolution of randomizations. This is in particular a case for ambiguity averse decision-makers, who typically satisfy an uncertainty aversion axiom, first given by Schmeidler [1989] in an Anscombe-Aumann choice setting .

Axiom 14. *[Uncertainty aversion] For all acts f, g and $\alpha \in (0,1)$, if $f \sim g$, then $\alpha f + (1-\alpha) g \succsim f$.*

This axiom says that the DM can improve her payoff that she can obtain from singleton menus by way of mixing the acts in them with ex-post randomizations (i.e., hedging acts). As a result, such a DM with CBL preferences will have a non-affine transformation function, and so she will not be indifferent to the timing of resolution of randomizations.

Proposition 2. *Let $\succsim$ be a costly Bayesian learning preference with an increasing transformation v. Then the following are equivalent:*

(i) $\succsim$ satisfies Axiom 14.

(ii) v is concave.

Note that a similar result, where the DM will be ambiguity seeking, can be easily characterized with an uncertainty seeking axiom in which case the transformation function v will be convex.

Constrained Bayesian learning

A general constrained Bayesian learning (ConBL) model can be given in our setting such that the value of a menu of act-lotteries will be

$$V(F) = \max_{\pi \in \Gamma(\bar{m})} \left(E_\pi \left[\max_{P \in F} U_P^{u,v}(m) \right] \right), \tag{11}$$

where $\Gamma(\bar{m})$ is a closed convex subset of the set of information structures $\Pi(\bar{m})$. Clearly, the ConBL model is a special case of our CBL preferences where the information cost function c satisfies $c(\pi) \in \{0, \infty\}$ with $c(\pi) = 0$ if and only if $\pi \in \Gamma(\bar{m})$. This model can be characterized by an axiom allowing for weak indifference to contingent plans with singleton menus (de Oliveira et al. [2017]).

Axiom 15. *[Weak indifference to contingent planning] For all menus F and act-lotteries P, if $F \sim P$, then $F \sim \alpha F + (1-\alpha)P$ for all $\alpha \in (0,1)$.*

In the CBL model that we have characterized in Theorem 1, a DM is averse to mixing menus F and G unless there is a common information structure π that is optimal for both menus. For the constrained information acquisition problem in equation 11, any feasible information structure π in the set $\Gamma(\bar{m})$ is optimal for a singleton menu. Thus, the DM is indifferent towards mixtures with singleton menus. The following result shows that Axiom 15 is the only additional behavioral restriction of a ConBL model within the class of our general CBL model.

Proposition 3. *Let $\succsim$ be a costly Bayesian learning preference. Then the following are equivalent:*

(i) $\succsim$ satisfies Axiom 15,

(ii) $\succsim$ is represented by the model given in equation (11).

Passive Bayesian learning

There are many information acquisition models used in applied literature where the DM does not actively seek information, but rather is a passive recipient of it (see, e.g., Baaley and Veldkamp [2021]). A general passive Bayesian learning (PBL) model can be given in our setting such that the value of a menu of act-lotteries will be

$$V(F) = E_\pi \left[\max_{P \in F} U_P^{u,v}(m) \right] \tag{12}$$

where $\pi \in \Pi(\bar{m})$ is some fixed information structure that the DM expects to have for no cost. These types of models imply the following axiom in our choice setting, which de Oliveira et al. [2017] call indifference to contingent planning for arbitrary menus.

Axiom 16. *[Indifference to contingent planning]* For all menus F and G, if $F \sim G$, then $F \sim \alpha F + (1-\alpha)G$ for all $\alpha \in (0,1)$.

The following result shows that Axiom 16 is the only additional behavioral restriction of a PBL model within the class of our general CBL model.

Proposition 4. *Let $\succsim$ be a costly Bayesian learning preference. Then the following are equivalent:*

(i) $\succsim$ satisfies Axiom 16,

(ii) $\succsim$ is represented by the model given in equation (12).

By Proposition 1, it is immediate to see that the PBL model contains Dillenberger et al. [2014]'s subjective-learning preferences as a special case, and thus the subjective learning model can be characterized in our setting by adding one of the Axioms 9-13 to the PBL model.

6 Related literature

In this section, we provide a brief review of the related literature focusing on attitudes towards the timing of resolutions and on truthful elicitation of the CBL preferences even when they exhibit uncertainty aversion.

Attitudes towards the timing of resolutions

Our work is most related to de Oliveira et al. [2017] who study a costly information acquisition model (given in equation (9)), where learning is about the states. They give an axiomatic characterization of this model and call the induced preferences over menus of acts Rationally Inattentive Preferences. Due to the structure of their framework and the way they define mixed acts, the RI preferences exhibit neutrality towards the timing of resolution of randomizations between acts; that is, the DM is indifferent between learning which act is relevant for her payoff before or after the state is revealed. As such, the DM is a first-order expected utility maximizer. However, the RI preferences do exhibit desire for early resolution of randomizations between menus; that is, a DM with RI preferences would like to know which menu will eventually be relevant for her payoffs before she makes the choice of a costly information to acquire.

Our work extends de Oliveira et al. [2017]'s analysis by using the richer setting of menus of lotteries over acts. The costly Bayesian learning (CBL) model that we study formalizes the DM's acquisition of information as learning about the distribution of states by modeling the DM's uncertainty using a second-order prior belief. Therefore, the CBL model can permit differing attitudes towards the timing of resolution of uncertainty unlike the RI preferences. For instance, when the increasing transformation function v is concave, then the DM with CBL preferences finds mixing of acts with late resolution helpful by means of hedging against the uncertainty of the state, and therefore she exhibits a desire for late resolution of randomizations between acts. On the other hand, the CBL preferences do exhibit a desire for early resolution of randomizations between menus, just like the RI preferences; a DM with CBL preferences would like to know which menu will eventually be relevant for her payoffs before she makes the choice of a costly information to acquire.

There are other axiomatic studies that are related to de Oliveira et al. [2017] (and therefore our work) in view of attitudes towards the timing of resolutions. Ergin and Sarver [2010] consider a choice setting of menus of lotteries over prizes to study a model of costly contemplation. As de Oliveira et al. [2017] argues, the RI preferences can be seen (and similarly the CBL preferences) as a class of the costly contemplation preferences. As such, similar to the RI (and CBL) preferences, the costly contemplation preferences exhibit a desire for early resolution of randomizations between menus (i.e., Axiom 5). While there are such close connections between these preferences, in Ergin and Sarver [2010] there is no counterpart of having attitudes towards the timing of resolution of randomizations between within-menu-alternatives (e.g., lotteries in this case) since there is no objective state space in their setting unlike in de Oliveira et al. [2017] or in our work. In fact due to the lack of an objective state space in their setting, there is no counterpart of a Blackwell ordering in Ergin and Sarver [2010], which plays a key role for the unique elicitation of costs in de Oliveira et al. [2017]'s or in our study.

Ergin and Sarver [2015] extends the choice setting of Ergin and Sarver [2010] by considering lotteries over menus of lotteries. Within this framework, they provide a rationale for the DM's seemingly *intrinsic* preferences for early resolution of uncertainty (see, e.g., Kreps and Porteus [1978]) by the possibility of taking *hidden actions* at a later unmodeled stage (see, e.g., Kreps [1979]). In another work, Pennesi [2015] consider a choice setting with lotteries over menus of acts extending de Oliveira et al. [2017]'s framework, and provides an axiomatic characterization for the costly information acquisition model that de Oliveira et al. [2017] study. In particular, following Ergin and Sarver [2015], Pennesi [2015] defines more explicitly the notion of desire for the early resolution of uncertainty and shows that this attitude is a key feature of the RI model in addition to other standard behavior. Finally, when the DM is indifferent toward the timing of resolution of any uncertainty, then the DM has passive Bayesian learning (PBL) preferences, which extends Dillenberger et al. [2014]'s subjective-learning preferences into our menus of lotteries over acts choice setting.

Ambiguity aversion and Raiffa's critique

Since the seminal work of Ellsberg [1961], ambiguity aversion has been an active area of research, both theoretically and empirically. One of the well known arguments against Ellsberg [1961] style paradoxes is known as Raiffa's critique. Raiffa [1961] argued that by mixing acts, the DM can hedge against ambiguity and simply turn the uncertainty about the states into an objective risk. In fact, this is how

Schmeidler [1989] defines uncertainty aversion which we use as Axiom 14 in our setting.

A recent line research warns that just because of the very nature of uncertainty aversion, it might not be possible to truthfully elicit the DM's preferences under uncertainty in an incentivized experiment.[9] The reason is that simply the DM can recognize the fact that she can hedge against uncertainty by sometimes choosing possibly inferior acts over constant acts in a given random incentive system (RIS). The use of RIS is widespread in experimental economics, but this simple argument implies that it may have limitations for eliciting preferences under uncertainty.[10]

Our model of CBL can provide a new perspective on this debate. Notice that in a typical incentivized experiment, the DM is given a sequence of choice problems. Each choice problem is a menu of acts, typically consisting of an act and a certain outcome, from which the DM is asked to make a choice. After the DM makes all the choices, she is made a payment by randomly selecting one of the choice sets she was given. As such, since a DM with CBL preferences satisfies Axiom 5, she will prefer not randomizing over choice sets and will reveal her true preferences in each choice set as a result. The reason is that resolution of randomizations over her choice sets are typically made before the state is revealed to her and thus these mixtures practically do not provide hedging against the uncertainty she faces. This means that random payments do not create unintended implications for elicitation of CBL preferences.

While this line of argument works well for CBL preferences, it will not be helpful when the DM satisfies the dual of Axiom 5, who will be someone who prefers randomization over choice sets. In that case, truthful elicitation of the DM's preferences can still be a problem, but this time perhaps not due to uncertainty aversion, but due to a desire for contingent planning. We believe this is an interesting point of observation that should be considered when eliciting preferences using menu choice data in controlled experiments, not just for decision making under uncertainty in specific, but more in general.

[9] See, e.g., Bade [2015], Azrieli et al. [2018], Baillon et al. [2021] for some related wok on this.

[10] See, e.g., Azrieli et al. [2018] for a demonstration of shortcomings of RIS in an experiment when there is ambiguity aversion. Azrieli et al. [2018] also propose a modification of the RIS procedure to restore incentive compatibility.

7 Conclusion

In this paper, we show how menu choice data can be used to study models of costly information acquisition with learning about the distribution of states. These models, which we call costly Bayesian learning (CBL), have been widely applied in the literature to study important questions in finance, macroeconomics, IO and related disciplines. A challenging issue for applied work on costly information acquisition problems is that the costs can be subjective and therefore not directly observable by the analyst. Following de Oliveira et al. [2017], who work on rational inattention (RI) models, we show that behavior of individuals with CBL preferences can be directly tested, and their hidden information costs can be identified and elicited with observable choice data in a richer setting.

We also show that the RI model, which have recently gained prominence with the rational inattention literature (Sims [1998, 2003]), can be embedded into our setting as a special case of the CBL model. By imposing either (i) a stronger dominance axiom, or (ii) various forms of indifference to timing of resolution axioms, or (iii) an independence over acts axiom, the RI model can be characterized in our setting. Importantly this characterization shows that a DM seemingly following the RI model can in fact be someone who does follow the CBL model with multiple first-order prior beliefs, but due to the linear transformation function her behavior is seen as if she has a unique first-order prior belief. This means there is indeed a strong connection between the information acquisition models where learning is about the distribution of the states or learning about the actual state. We believe, therefore, our results in general help better understand the relation of these two seemingly different strands of the literature.

Our model of CBL preferences is flexible enough to accommodate non-linear attitudes towards the resolution of uncertainty, such as when the DM is uncertainty averse. As such, our analysis provides a plausible point of start for research on the behavioral foundations of dynamic models of costly information acquisition. Extending on our analysis in dynamic choice settings, future research can provide guidance for empirical analysis on the implications of costly information acquisition in dynamic choice environments.

A Appendix

In this section, we prove the results given in Sections 4 and 5.

A.1 Preliminaries

We first introduce some additional notation and preliminary results required for the proofs. To this end, let $\Delta^2(\Omega)$ denote the space $\Delta(\Delta(\Omega))$, and let $\Delta^3(\Omega)$ denote $\Delta(\Delta(\Delta(\Omega)))$ to simplify the exposition.

Niveloids. Denote by $C(\Delta^2(\Omega))$ the linear space of real-valued continuous functions defined on $\Delta^2(\Omega)$, and by $ca(\Delta^2(\Omega))$ the linear space of signed measures of bounded variation on $\Delta^2(\Omega)$ (Aliprantis and Border [2006, p. 399]). For each $\pi \in ca(\Delta^2(\Omega))$ and for each $\phi \in C(\Delta^2(\Omega))$, let

$$\langle \phi, \pi \rangle = \int_{\Delta^2(\Omega)} \phi(m)\, \pi(dm).$$

The linear space $C(\Delta^2(\Omega))$ is endowed with the supnorm and $ca(\Delta^2(\Omega))$ with the weak* topology. Therefore $ca(\Delta^2(\Omega))$ can be identified with the continuous dual space of $C(\Delta^2(\Omega))$ (Aliprantis and Border [2006, Corollary 14.15]), and $C(\Delta^2(\Omega))$ can be identified with the continuous dual space of $ca(\Delta^2(\Omega))$ (Aliprantis and Border [2006, Theorem 5.93]).

Let Ψ be a subset of $C(\Delta^2(\Omega))$, and consider a function $V : \Psi \to \mathbb{R}$. We say that V is *normalized* if $V(\alpha) = \alpha$ for each constant function $\alpha \in \Psi$; *monotone* if $V(\phi) \geq V(\psi)$ for all $\phi, \psi \in \Psi$ such that $\phi \geq \psi$; *translation invariant* if $V(\phi + \alpha) = V(\phi) + \alpha$ for each $\phi \in \Psi$ and $\alpha \in \mathbb{R}$ such that $\phi + \alpha \in \Psi$; and a *niveloid* if $V(\phi) - V(\psi) \leq \sup\{\phi(m) - \psi(m) : m \in \Delta^2(\Omega)\}$ for each $\phi, \psi \in \Psi$. If V is a niveloid, then it is monotone and translation invariant, while the converse is true whenever $\Psi = \Psi + \mathbb{R}$. Moreover, if V is a niveloid, then V is Lipschitz continuous. If Ψ is a convex set and V is a convex niveloid, then there is a convex niveloid that extends V to $C(\Delta^2(\Omega))$.[11]

[11] See Cerreia-Vioglio, Maccheroni, Marinacci, and Rustichini [2014] for the proofs of these results and a detailed analysis about niveloids in general.

Notation and Auxiliary Results. Let Φ be the set of convex functions belonging to $C(\Delta^2(\Omega))$: Φ is a closed convex cone such that $0 \in \Phi$. Denote by Φ^* the dual cone of Φ, that is,

$$\Phi^* = \{\pi \in ca(\Delta^2(\Omega)) : \langle \phi, \pi \rangle \geq 0 \text{ for all } \phi \in \Phi\}.$$

The set Φ^* is also a closed convex cone such that $0 \in \Phi^*$. Moreover $\Phi = \Phi^{**}$ (see Aliprantis and Border [2006, Theorem 5.103]), that is,

$$\Phi = \{\phi \in C(\Delta^2(\Omega)) : \langle \phi, \pi \rangle \geq 0 \text{ for all } \pi \in \Phi^*\}.$$

Let $u : Z \to \mathbb{R}$ be a function and $v : u(Z) \to \mathbb{R}$ be an increasing transformation. Denote by $\Phi_{\mathbb{F}}$ the set of functions $\phi_F : \Delta^2(\Omega) \to \mathbb{R}$ for each menu F where,

$$\phi_F(m) = \max_{P \in F} E_P[E_m[v(E_{f_\mu}[u(x)])]],$$

for all $m \in \Delta^2(\Omega)$. Similarly, let $\Phi_{\Delta(\mathcal{F})}$ denote the set of functions $\phi_P : \Delta^2(\Omega) \to \mathbb{R}$ for each act-lottery P where $\phi_P(m) = E_P[E_m[v(E_{f_\mu}[u(x)])]]$ for each $m \in \Delta^2(\Omega)$, and let $\Phi_{\Delta(Z)}$ denote the set of functions $\phi_p : \Delta^2(\Omega) \to \mathbb{R}$ for each prize-lottery p where $\phi_p(m) = v(u(x))$ for each $m \in \Delta^2(\Omega)$. Observe that we have $\Phi_{\Delta(Z)} \subset \Phi_{\Delta(\mathcal{F})} \subset \Phi_{\mathbb{F}} \subset \Phi$. Moreover, $\alpha\phi_F + (1-\alpha)\phi_G = \phi_{\alpha F + (1-\alpha)G}$ for each pair of menus F and G, and $\alpha \in [0,1]$. Hence, in particular, $\Phi_{\mathbb{F}}$ is convex.

For any given menu F, let co(F) denote its convex hull. For any $P \in \Delta(\mathcal{F})$, let $supp(P)$ denote its support. For any $P \in \Delta(\mathcal{F})$, let $v(u(P))$ denote the vector $\lambda_P \in \mathbb{R}^{\Delta(\Omega)}$ such that $\lambda_P(\mu) = E_P[v(E_{f_\mu}[u(x)])]$ for each $\mu \in \Delta(\Omega)$ and let $v(u(F)) = \{\lambda_P : P \in F\}$ for each $F \in \mathbb{F}$. For any $f \in \mathcal{F}$, let $u_f \in u(Z)^{\Delta(\Omega)}$ denote the vector such that $u_f(\mu) = E_{f_\mu}u(x)$ for each $\mu \in \Delta(\Omega)$. Let $u(\mathcal{F}) = \{u_f : f \in \mathcal{F}\}$.

A.2 Implications of Axioms 1–8

In this Section, we state and prove a lemma that provides a representation for a binary relation satisfying Axioms 1–8. We consider below utility functions which are unbounded above, while the case where they are unbounded below is analogous and therefore omitted.

Lemma 1. *Let $\succsim$ be a binary relation on $\mathbb{F}$ that satisfies Axioms 1–8. Then:*

(i) Every menu $F \in \mathbb{F}$ has a singleton equivalent $P_F \in \Delta(\mathcal{F})$ such that $F \sim P_F$.

(ii) There exist an unbounded utility function $u : Z \to \mathbb{R}$, an unbounded increasing transformation $v : u(Z) \to \mathbb{R}$, and a second-order prior $\bar{m} \in \Delta(\Delta(\Omega))$ such that the restriction of the preference order $\succsim$ over the set of singletons is represented by the second-order expected utility defined with parameters $(u, v, \bar{m})$.

(iii) The function c^* such that $c^*(\pi) = \sup_{F \in \mathbb{F}} \left[b_F^{u,v}(\pi) - U_{P_F}^{u,v}(\bar{m}) \right]$ for all $\pi \in \Pi(\bar{m})$ is proper.

(iv) The functional V defined by $V(F) = \max_{\pi \in \Pi(\bar{m})} \left[b_F^{u,v}(\pi) - c^*(\pi) \right]$ for all $F \in \mathbb{F}$ represents $\succsim$.

Proof. Let $\succsim$ be a binary relation on $\mathbb{F}$ that satisfies Axioms 1–8.

[**Part (i)**]: We establish this part in two claims.

Claim 1. *Let F and G be menus such that for each $Q \in G$ there is $P \in F$ such that $P_\mu \succsim Q_\mu$ for each $\mu \in \Delta(\Omega)$. Then $F \succsim G$.*

Proof. By Axiom 7, $F \sim F \cup \{Q_1\} \sim F \cup \{Q_1, Q_2\} \sim ... \sim F \cup G$. By Axiom 4, $F \cup G \succsim G$. Combining these, we obtain $F \succsim G$. □

Claim 2. *Every menu F has a singleton equivalent P_F such that $P_F \sim F$.*

Proof. Let $s(F) = \{f(w) : f \in supp(P),\, P \in F\}$ denote the set of prize-lotteries that are possible within menu F. Since F has finitely many act-lotteries, each act-lottery has finitely many acts in its support, and the state space has finitely many states, $s(F)$ must be finite. Therefore, there are some $p, q \in s(F)$ such that $p \succsim P_\mu \succsim q$ for all $P \in F$ and for all $\mu \in \Delta(\Omega)$. Notice that p, q and P_μ are viewed as singleton menus with corresponding act-lotteries in the previous binary comparisons. By Claim 1, we have $p \succsim F \succsim q$. By Axiom 2, the two sets

$$A = \{\alpha \in [0,1] : \alpha p + (1-\alpha) q \succsim F\} \quad \text{and} \quad B = \{\alpha \in [0,1] : F \succsim \alpha p + (1-\alpha) q\}.$$

are closed. Since $[0,1]$ is connected, there exists $\alpha \in A \cap B$ such that $\alpha p + (1-\alpha) q \sim F$. Let P_F be equal to the singleton $\alpha p + (1-\alpha) q$. $\square$

[**Part (ii)**]: Restriction of the DM's preferences $\succsim$ over the set of first-order prize-lotteries $\Delta(Z)$ satisfies the vNM axioms of expected utility, and so they can be represented by the expected utility model with a utility function $u : Z \to \mathbb{R}$ such that $p \succsim q$ if and only if $E_p u(x) \geq E_q u(x)$. Note that by Axiom 3, u is unbounded. Moreover, u can be normalized such that $E_{p'} u(x) = 1$ and $E_{q'} u(x) = 0$ for some $p' \succ q'$.

Restriction of the DM's preferences $\succsim$ over the set of act-lotteries $\Delta(\mathcal{F})$ satisfies the axioms of expected utility. In particular, to see that the independence axiom holds, take any $\alpha \in (0,1)$ and $P, Q \in \Delta(\mathcal{F})$ such that $P \sim Q$. By Axiom 5, $P = \alpha P + (1-\alpha) P \succsim \alpha Q + (1-\alpha) P$. By Axiom 6, $\alpha P + (1-\alpha) Q \succsim \alpha Q + (1-\alpha) Q = Q$. By Axiom 5, $Q = \alpha Q + (1-\alpha) Q \succsim \alpha P + (1-\alpha) Q$. So we conclude that $P \sim \alpha P + (1-\alpha) Q$ for any $\alpha \in (0,1)$ whenever $P \sim Q$. But then, by Ozbek [2021, Proposition 1], the preference order must satisfy independence over act-lotteries.

As a result, restriction of the DM's preferences $\succsim$ over the set of second-order prize-lotteries $\Delta(\Delta(Z))$ satisfies the axioms of expected utility. Thus, there exists a function $\bar{U} : \Delta(Z) \to \mathbb{R}$ such that $\bar{P} \succsim \bar{Q}$ if and only if $E_{\bar{P}} \bar{U}(r) \geq E_{\bar{Q}} \bar{U}(r)$ for all $\bar{P}, \bar{Q} \in \Delta(\Delta(Z))$. Moreover, $\bar{U}$ can be normalized such that $E_{\bar{P}'} \bar{U}(r) = 1$ and $E_{\bar{Q}'} \bar{U}(r) = 0$ for some $\bar{P}' \succ \bar{Q}'$. Without loss of generality, let $\bar{P}' = \delta_{p'}$ and $\bar{Q}' = \delta_{q'}$. Since $\Delta(Z)$ can be embedded into $\Delta(\Delta(Z))$, we have $E_{\delta_p} \bar{U}(r) \geq E_{\delta_q} \bar{U}(r)$ if and only if $E_p u(x) \geq E_q u(x)$. This means there exists an unbounded increasing transformation $v : u(Z) \to \mathbb{R}$ such that $\bar{U}(p) = v(E_p u(x))$ for all $p \in \Delta(Z)$.

Finally, since restriction of the DM's preferences $\succsim$ over the set of act-lotteries $\Delta(\mathcal{F})$ satisfies the vNM axioms of expected utility, there exists a function $U : \mathcal{F} \to \mathbb{R}$ such that $P \succsim Q$ if and only if $E_P[U(f)] \geq E_Q[U(f)]$ for all $P, Q \in \Delta(\mathcal{F})$. Note that we can let U such that $U(p) = \bar{U}(p) = v(E_p u(x))$ for all $p \in \Delta(Z)$. But then, since $\succsim$ satisfies Axiom 7, following Seo [2009, Lemma B.3-B10], there must exist a second-order belief $\bar{m} \in \Delta(\Delta(\Omega))$ such that $U(f) = E_{\bar{m}}[v(E_{f_\mu} u(x))]$ for all $f \in \mathcal{F}$. Combining all these, we conclude that the preference order $\succsim$ restricted to the set of singletons can be represented by the second-order expected utility model given in equation (2). $\square$

[**Part (iii)**]: We need to show that (i) $c^*(\pi) \geq 0$ for all $\pi \in \Pi(\bar{m})$, (ii) $c^*(\pi) < \infty$ for some $\pi \in \Pi(\bar{m})$, and (iii) c^* is lower semi-continuous. Note that since u and v are normalized, and so we have $0 \in \Phi_{\mathbb{F}}$, it follows that $c^*(\pi) \geq E_\pi 0 - 0 = 0$ for all $\pi \in \Pi(\bar{m})$, showing (i). By Axiom 4, we have $F \succsim P$ for any $P \in F$, and so $P_F \succsim P$ for all $P \in F$ by Part (i). This means $c^*(\delta_{\bar{m}}) = \sup_{F \in \mathbb{F}} \left[b_F^{u,v}(\delta_{\bar{m}}) - U_{P_F}^{u,v}(\bar{m}) \right] \leq 0$, and so $c^*(\delta_{\bar{m}}) = 0$, showing (ii). Finally, since c^* is a pointwise supremum of a family of continuous functions, it is lower semi-continuous. $\square$

[**Part (iv)**]: We establish this part in several claims. Without loss of generality assume that $u(x) \geq 0$ for each $x \in Z$ whenever $u(Z)$ is lower bounded and closed.

Define the functional $V : \Phi_{\mathbb{F}} \to \mathbb{R}$ such that $V(\phi_F) = E_{P_F} U(f)$ where P_F is a singleton equivalent of F.[12] If P_F and Q_F are two certainty equivalents of F, then $P_F \sim Q_F$ and so $E_{P_F} U(f) = E_{Q_F} U(f)$. Next we show in two claims that V is monotone (i.e., $\phi_F \geq \phi_G$ implies $V(\phi_F) \geq V(\phi_G)$), and so V is well-defined; that is, whenever $\phi_F = \phi_G$, then $F \sim G$ for each pair of menus F and G.

Claim 3. *Consider a pair of menus F and G. If $\phi_F \geq \phi_G$, then for each $Q \in G$ there exists $P \in \mathrm{co}(F)$ such that $P_\mu \succsim Q_\mu$ for each $\mu \in \Delta(\Omega)$.*

Proof. Assume, for contradiction, that that there is some $Q \in G$ such that for all $P \in \mathrm{co}(F)$ we have $Q_\mu \succ P_\mu$ for some $\mu \in \Delta(\Omega)$. Since the vector valued function $v(u(P))$ is linear in P, we have $\mathrm{co}(v(u(F))) = v(u(\mathrm{co}(F)))$, so that $v(u(\mathrm{co}(F)))$ is convex, closed and bounded. Let $A = \left\{ a \in \mathbb{R}^{\Delta(\Omega)} : a \geq v(u(Q)) \right\}$, then A is a closed convex cone. Clearly, $v(u(\mathrm{co}(F)))$ and A are disjoint sets. By a separating hyperplane theorem (Rockafellar [1970, Corollary 11.4.2]), there exists some $m \in \mathbb{R}^{\Delta(\Omega)}$ such that $E_P[E_m[v(E_{f_\mu} u(x))]] < E_m a(\mu)$ for all $a \in A$ and $P \in F$. Since $v(u(Q))$ belongs to A we have

$$\max_{P \in F} E_P[E_m[v(E_{f_\mu} u(x))]] < E_Q[E_m[v(E_{f_\mu} u(x))]].$$

Since A is a cone, we can let $m \in \Delta^2(\Omega)$ implying $\phi_F(m) < \phi_G(m)$, a contradiction. $\square$

[12] For convenience we use V to denote both the representation over menus and the induced representation over support functions.

Claim 4. *Consider a pair of menus F and G. If $G \subset \text{co}(F)$, then $F \succsim G$.*

Proof. Let $G = \{Q_1, \ldots, Q_n\} \subset \text{co}(F)$. For all $i = 1, \ldots, n$ we can write each $Q_i = \sum_{j=1}^{m_i} \alpha_j^i P_j^i$ for $\alpha_1^i, \ldots, \alpha_{m_i}^i \geq 0$ summing up to one, and $P_1^i, \ldots, P_{m_i}^i \in F$. Hence

$$G \subset \sum_{j=1}^{m_1} \cdots \sum_{j'=1}^{m_n} \alpha_j^1 \cdots \alpha_{j'}^n F = \sum_{k=1}^{l} \beta_k F.$$

By Axiom 4 we have that $\sum_{k=1}^{l} \beta_k F \succsim G$, so it is enough to check that $F \sim \sum_{k=1}^{l} \beta_k F$. We show this by induction on l. If $l = 1$, then $\sum_{k=1}^{l} \beta_k F = F \sim F$. Suppose now the claim is true for $l - 1$. Observe that

$$\sum_{k=1}^{l} \beta_k F = \beta_l F + (1 - \beta_l) \left(\sum_{k=1}^{l-1} \frac{\beta_k}{1 - \beta_l} F \right).$$

Moreover, by inductive assumption $F \sim \sum_{k=1}^{l-1} \frac{\beta_k}{1-\beta_l} F$. Therefore by Axiom 5 $F \succsim \sum_{k=1}^{l} \beta_k F$. Since $F \subset \sum_{k=1}^{l} \beta_k F$, by Axiom 4 we obtain $\sum_{k=1}^{l} \beta_k F \succsim F$. Therefore $F \sim \sum_{k=1}^{l} \beta_k F$, as desired. □

By Claim 3, if $\phi_F \geq \phi_G$, then there exists a subset $H \subset \text{co}(F)$ such that for each $Q \in G$ there exists $R \in H$ such that $R_\mu \succsim Q_\mu$ for all $\mu \in \Delta(\Omega)$. By Claim 4, F is preferred to H, which, by Claim 1, is preferred to G. This shows that V is monotone, and so well-defined. Moreover, since $F \succsim G$ if and only if $P_F \succsim P_G$ by definition, we deduce that V represents $\succsim$ in the sense that $F \succsim G$ if and only if $V(\phi_F) \geq V(\phi_G)$.

Claim 5. *The functional V is a monotone, normalized, convex niveloid.*

Proof. We have already established that V is monotone. To see that it is normalized, notice that the set of constant functions in $\Phi_{\mathbb{F}}$ is identified with the set $\Phi_{\Delta(Z)}$, and for every prize-lottery p we have $V(\phi_p) = v(E_p u(x)) = \phi_p$, so that V is normalized.

We now show that V is a convex niveloid in several steps.

Step 1 (V is translation invariant): Using Axiom 6, the obvious adaptation of the argument in Maccheroni, Marinacci, and Rustichini [2006, Proof of Lemma 28] provides that whenever k belongs to $v(u(Z))$ we have for any $\phi_F \in \Phi_{\mathbb{F}}$,

$$V(\beta\phi_F + (1-\beta)k) = V(\beta\phi_F) + (1-\beta)k \quad \forall \beta \in (0,1).$$

Pick $\gamma > 1$, so that $\gamma\phi_F \in \Phi_{\mathbb{F}}$. Then,

$$V\left(\frac{1}{\gamma}(\gamma\phi_F) + \frac{\gamma-1}{\gamma}\left(\frac{\gamma}{\gamma-1}k\right)\right) = V\left(\frac{1}{\gamma}(\gamma\phi_F)\right) + \frac{\gamma-1}{\gamma}\left(\frac{\gamma}{\gamma-1}k\right) \quad \forall \alpha > 0.$$

This implies that $V(\phi_F + k) = V(\phi_F) + k$ whenever $k > 0$. Notice that we have just shown $V(\phi_F + k - k) = V(\phi_F + k) - k$ implying that $V(\phi_G + t) = V(\phi_G) + t$ for any $t < 0$ such that $\phi_G + t \in \Phi_{\mathbb{F}}$. Thus, V is translation invariant on $\Phi_{\mathbb{F}}$.

Step 2 (V is convex): To show that V is convex, suppose $V(\phi_F) = V(\phi_G)$. Then $F \sim G$ and, by Axiom 5, $F \succsim \alpha F + (1-\alpha)G$. Hence,

$$\alpha V(\phi_F) + (1-\alpha)V(\phi_G) = V(\phi_F) \geq V(\phi_{\alpha F+(1-\alpha)G}) = V(\alpha\phi_F + (1-\alpha)\phi_G).$$

Now suppose $V(\phi_G) > V(\phi_F)$, and define $\beta = V(\phi_G) - V(\phi_F) > 0$. Since $\phi_F + \beta \in \Phi_{\mathbb{F}}$,

$$V(\phi_F + \beta) = V(\phi_F) + \beta = V(\phi_F) + V(\phi_G) - V(\phi_F) = V(\phi_G),$$

where the first equality holds by translation invariance. Therefore,

$$\begin{aligned} V(\phi_G) &\geq V(\alpha(\phi_F + \beta) + (1-\alpha)\phi_G) = V(\alpha\phi_F + (1-\alpha)\phi_G) + \alpha\beta \\ &= V(\alpha\phi_F + (1-\alpha)\phi_G) + \alpha(V(\phi_G) - V(\phi_F)), \end{aligned}$$

so that $V(\alpha\phi_F + (1-\alpha)\phi_G) \leq \alpha V(\phi_F) + (1-\alpha)V(\phi_G)$ showing that V is convex.

Step 3 (V is a niveloid): Since V is translation invariant on $\Phi_{\mathbb{F}}$, we can extend V uniquely to $\Phi_{\mathbb{F}} + \mathbb{R}$ by defining $V(\phi) = V(\phi + k) - k$ for any $\phi \in \Phi_{\mathbb{F}} + \mathbb{R}$ and $k \in \mathbb{R}$ such that $\phi + k \in \Phi_{\mathbb{F}}$. This extension preserves not only translation invariance, but also monotonicity and convexity. Hence the extension of V is a convex niveloid on $\Phi_{\mathbb{F}} + \mathbb{R}$, and therefore on $\Phi_{\mathbb{F}}$. □

To complete to proof we apply the well-known Fenchel-Moreau theorem adapted to our framework.

Claim 6. *There exist a proper cost function* $c : \Pi(\bar{m}) \to [0, \infty]$ *such that*

$$V(\phi_F) = \max_{\pi \in \Pi(\bar{m})} \left(\langle \phi_F, \pi \rangle - c(\pi) \right), \quad \forall F \in \mathbb{F}.$$

Proof. Since $\Phi_{\mathbb{F}}$ is convex and V is a convex niveloid, there is a real-valued functional W defined on $C(\Delta^2(\Omega))$ which is a convex niveloid extending V (see Section A.1). Since W is a niveloid, it is continuous. Since W is continuous, convex and real-valued, by Rockafellar [1974, Theorem 11] the subdifferential of W is nonempty at each $\phi \in C(\Delta^2(\Omega))$, that is, for each ϕ there is $\pi \in ca(\Delta^2(\Omega))$ such that

$$\langle \phi, \pi \rangle - W(\phi) \geq \langle \psi, \pi \rangle - W(\psi) \quad \forall \psi \in C(\Delta^2(\Omega)). \tag{13}$$

Moreover, since W is a niveloid, it is monotone and translation invariant, so by Ruszczyński and Shapiro [2006, Theorem 2.2] we can let π be in $\Delta^3(\Omega)$. Define $V^* : \Delta^3(\Omega) \to (-\infty, \infty]$ such that

$$V^*(\pi) = \sup_{F \in \mathbb{F}} \langle \phi_F, \pi \rangle - V(\phi_F) \quad \forall \pi \in \Delta^3(\Omega).$$

Thus, for all ϕ_F and π, $V^*(\pi) \geqslant \langle \phi_F, \pi \rangle - V(\phi_F)$ and hence $V(\phi_F) \geqslant \langle \phi_F, \pi \rangle - V^*(\pi)$. Moreover, for any ϕ_F there exists a $\pi \in \Delta^3(\Omega)$ such that $\langle \phi_F, \pi \rangle - V(\phi_F) = V^*(\pi)$ by (13). As a result,

$$V(\phi_F) = \max_{\pi \in \Delta^3(\Omega)} \langle \phi_F, \pi \rangle - V^*(\pi) \quad \forall F \in \mathbb{F}.$$

Finally, we want to show that c is the restriction of V^* to $\Pi(\bar{m})$, in which case c coincides with c^* showing that it is a proper cost function by Part (iii). For this, we need to show that $V^*(\pi) < \infty$ implies $\pi \in \Pi(\bar{m})$ whenever v is not affine, and $V^*(\pi) < \infty$ implies $\pi \in \Pi(m)$ where $E_m \mu = E_{\bar{m}} \mu$ whenever v is affine.

First, suppose that v is not affine. In this case, suppose, for contradiction, that there exists some $\pi \in \Delta^3(\Omega) \setminus \Pi(\bar{m})$ such that $V^*(\pi) < \infty$. Let $m_\pi = E_\pi m \in \Delta^2(\Omega)$. By definition, we have $m_\pi \neq \bar{m}$. This means we can find some f with $f(\omega) \in \{\delta_x, \delta_y\}$

for some $\delta_x \succ \delta_y$ such that

$$E_{m_\pi}[v(E_{f_\mu}u(x)] - E_{\bar{m}}[v(E_{f_\mu}u(x)] > 0. \tag{14}$$

But since u and v are unbounded, x and y above can be chosen such that the payoff difference in equation (14) becomes arbitrarily large. This means we have $V^*(\pi) \geq \sup_{f\in\mathcal{F}}\langle\phi_f, \pi\rangle - V(\phi_f) = \infty$, a contradiction.

Now suppose that v is affine. In this case, for each $n \in \mathbb{N}$, choose prizes x_n and y such that $v(u(x_n)) = n$ and $v(u(y)) = 0$. Fix some $\omega \in \Omega$ and consider an act f assuming prize x_n on ω and y otherwise. Then

$$\langle\phi_f, \pi\rangle - V^*(\pi) = nE_\pi[E_m[\mu(\omega)]] - V^*(\pi) \leq V(\phi_f) = nE_{\bar{m}}[\mu(\omega)].$$

Since the above inequality holds for each n, as long as $V^*(\pi) < \infty$, it follows that

$$E_\pi[E_m[\mu(\omega)]] \leq E_{\bar{m}}[\mu(\omega)] \quad \forall\omega \in \Omega,$$

and so, since $E_{\bar{m}}[\mu] \in \Delta(\Omega)$, it follows that $E_\pi[E_m[\mu(\omega)]] = E_{\bar{m}}[\mu(\omega)]$ for all $\omega \in \Omega$. Thus,

$$V(\phi_F) = \max_{\pi\in\Pi(m)}\langle\phi_F, \pi\rangle - V^*(\pi) \quad \forall F \in \mathbb{F},$$

where $m \in \Delta^2(\Omega)$ is such that $m = \bar{m}$ whenever v is not affine, and $E_m\mu = E_{\bar{m}}\mu$ whenever v is affine. □

With the demonstration of Part (iv), we complete the proof of Lemma 1. □

A.3 Proofs of the results in the text

Proof of Theorem 1

It is straightforward to show that a CBL preference satisfies Axioms 1–8. For the converse, let $\succsim$ be a binary relation that satisfies Axioms 1–8. Then by Lemma

1, the functional $V : \mathbb{F} \to \mathbb{R}$ defined by $V(F) = \max_{\pi \in \Pi(\bar{m})} [b_F^{u,v}(\pi) - c^*(\pi)]$ for all $F \in \mathbb{F}$ represents $\succsim$. □

For the following proofs of Propositions 1-4, let $\succsim$ be a CBL preference represented by $(u, v, \bar{m}, c)$ and suppose, without loss of generality, that c is canonical.

Proof of Proposition 1

It is clear that when $\succsim$ is represented by the RI model given in equation (10), then $\succsim$ satisfies Axioms 9-13.

For the converse, we first show that each one of Axioms 9-11, and 13 implies Axiom 12. Notice that for any given $P \in \Delta(\mathcal{F})$, by definition we have $P_{\delta_\omega} = \delta_{f_P(\omega)} = (\delta_{f_P})_{\delta_\omega}$ for all $\omega \in \Omega$. Thus by Axiom 4, we have $\{P, \delta_{f_P}\} \succsim P, \delta_{f_P}$. Hence, whenever Axiom 9 holds we have $P, \delta_{f_P} \succsim \{P, \delta_{f_P}\}$, and so $P \sim \delta_{f_P}$, showing that Axiom 10 holds. Clearly, Axiom 10 implies Axiom 11, which implies Axiom 12.

Note that by Axiom 6, for all $f, g, h \in \mathcal{F}$ and $\alpha \in (0, 1)$, we have $f \succsim g$ if and only if $\alpha\delta_f + (1-\alpha)\delta_h \succsim \alpha\delta_g + (1-\alpha)\delta_h$. Moreover, if Axiom 13 holds, then we have $f \succsim g$ if and only if $\alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$. In this case, for all $f, g, h \in \mathcal{F}$ and $\alpha \in (0, 1)$, we have $f \succsim g$ if and only if $V_{\Delta(\mathcal{F})}(\alpha\delta_f + (1-\alpha)\delta_h) \geq V_{\Delta(\mathcal{F})}(\alpha\delta_g + (1-\alpha)\delta_h)$ and $V_{\mathcal{F}}(\alpha f + (1-\alpha)h) \geq V_{\mathcal{F}}(\alpha g + (1-\alpha)h)$, where $V_{\Delta(\mathcal{F})}$ and $V_{\mathcal{F}}$ are restrictions of V over $\Delta(\mathcal{F})$ and $\mathcal{F}$, respectively. This means there exists an increasing transformation $\lambda : \mathbb{R} \to \mathbb{R}$ such that $V_{\Delta(\mathcal{F})}(\delta_f) = \lambda(V_{\mathcal{F}}(f))$ for all $f \in \mathcal{F}$. But since $\delta_f \sim f$ for all $f \in \mathcal{F}$, the transformation λ must be the identify map. Hence, when Axiom 13 holds, we have $\alpha\delta_f + (1-\alpha)\delta_h \sim \alpha f + (1-\alpha)h$ for all $f, h \in \mathcal{F}$ and $\alpha \in (0, 1)$ showing that Axiom 11 holds (which implies Axiom 12).

In sum, Axioms 9-11, and 13 all imply Axiom 12. Thus, if we show that Axiom 12 implies that v is affine, then we are done. To see this, let $p, q \in \Delta(Z)$ and $\alpha \in (0, 1)$. Since $\succsim$ is represented by the CBL model given in equation (3) with a canonical cost function, we have $V(\alpha\delta_p + (1-\alpha)\delta_q) = \alpha\, v(E_p u(x)) + (1-\alpha)\, v(E_q u(x))$ and $V(\alpha p + (1-\alpha)q) = v(\alpha E_p u(x) + (1-\alpha)E_q u(x))$. Thus, if Axiom 12 holds, then we must have $V(\alpha\delta_p + (1-\alpha)\delta_q) = V(\alpha p + (1-\alpha)q)$. But this can happen only when v is affine, as desired. □

Proof of Proposition 2

It is immediate to see that when $\succsim$ is represented by the CBL model with a concave transformation v, then $\succsim$ satisfies Axiom 14. For the converse, suppose that $\succsim$ satisfies Axiom 14. In this case, for any $f, g \in \mathcal{F}$ with $V(f) = V(g)$, we have $V(\alpha f + (1-\alpha)g) \geq V(f)$ for all $\alpha \in (0,1)$. Then, by a similar argument given in Claim 5, V is concave over $\mathcal{F}$; that is for all $f, g \in \mathcal{F}$, we have $V(\alpha f + (1-\alpha)g) \geq \alpha V(f) + (1-\alpha)V(g)$ for all $\alpha \in (0,1)$. Since $V(f) = E_{\bar{m}} v(E_{f_\mu} u(x))$ for all $f \in \mathcal{F}$, it follows that the function $E_{\bar{m}}[v] : u(\mathcal{F}) \to \mathbb{R}$ is concave, where for all $f \in \mathcal{F}$, we have $E_{\bar{m}}[v](u_f) = E_{\bar{m}} v(E_{f_\mu} u(x))$. Thus $v : u(Z) \to \mathbb{R}$ is concave. □

Proof of Proposition 3

If $\succsim$ is represented by the ConBL model given in equation (11), clearly $\succsim$ satisfies Axiom 15. The converse direction of the proof follows immediately from the proof of de Oliveira et al. [2017, Corollary 1] after making the obvious adaptations. □

Proof of Proposition 4

It is clear that if $\succsim$ is represented by the passive information acquisition model given in equation (12), then $\succsim$ satisfies Axiom 16. The converse direction of the proof follows immediately from the proof of de Oliveira et al. [2017, Corollary 2] after making the obvious adaptations. □

Proof of Theorem 2

Let $(u, v, \bar{m}, c)$ represents a CBL preference. By an obvious adaptation of the arguments in the proof of de Oliveira et al. [2017, Theorem 2], c can be taken as a cost function satisfying the canonical properties given in Definition 3. But then, the rest of the proof that c is the unique canonical cost function follows immediately from the proof of de Oliveira et al. [2017, Theorem 2]. □

Proof of Corollary 1

Assume that the given CBL preference $\succsim$ is represented both by $(u, v, \bar{m}, c)$ and $(u', v', \bar{m}', c')$, where c and c' are canonical. Since the restriction of $\succsim$ to the set of prize-lotteries has an expected utility representation, there exist some $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $u' = \alpha u + \beta$. Similarly, since the restriction of $\succsim$ to the set of second-order prize-lotteries has an expected utility representation, there exist some $\lambda > 0$ and $\gamma \in \mathbb{R}$ such that $v'(u') = \lambda v(u) + \gamma$. Since the restriction of $\succsim$ to the set of act-lotteries has a second-order expected utility representation, by Seo [2009, Lemma C1] for any $P \in \Delta(\mathcal{F})$, we have $E_P E_{\bar{m}'} v(E_{f_\mu} u(x)) = E_P E_{\bar{m}} v(E_{f_\mu} u(x))$ showing that $\bar{m}$ and $\bar{m}'$ are essentially equivalent. Finally, by Theorem 2, for all $\pi' \in \Pi(\bar{m}')$,

$$
\begin{aligned}
c'(\pi') &= \\
&= \sup_{F \in \mathbb{F}} \Big[E_\pi \max_{P \in F} \Big[E_P E_{m'} v'(E_{f_\mu} u'(x)) \Big] - E_P E_{\bar{m}'} v'(E_{f_\mu} u'(x)) \Big] \\
&= \sup_{F \in \mathbb{F}} \Big[E_\pi \max_{P \in F} \Big[E_P E_{m'} \lambda v(E_{f_\mu} \alpha u(x) + \beta) \Big] - E_P E_{\bar{m}'} \lambda v(E_{f_\mu} \alpha u(x) + \beta \Big] \\
&= \lambda \sup_{F \in \mathbb{F}} \Big[E_\pi \max_{P \in F} \Big[E_P E_{m'} v(E_{f_\mu} \alpha u(x) + \beta) \Big] - E_P E_{\bar{m}'} v(E_{f_\mu} \alpha u(x) + \beta) \Big] \\
&= \lambda \sup_{F \in \mathbb{F}} \Big[E_\pi \max_{P \in F} \Big[E_P E_{m'} v(E_{f_\mu} u(x)) \Big] - E_P E_{\bar{m}'} v(E_{f_\mu} u(x)) \Big] \\
&= \lambda \sup_{F \in \mathbb{F}} \Big[E_\pi \max_{P \in F} \Big[E_P E_{m} v(E_{f_\mu} u(x)) \Big] - E_P E_{\bar{m}} v(E_{f_\mu} u(x)) \Big] \\
&= \lambda\, c(\pi)
\end{aligned}
$$

where π' and π are induced by the same signal structure from second-order priors $\bar{m}'$ and $\bar{m}$, respectively. This completes the proof. □

Proof of Theorem 3

It is straightforward to show that whenever $\succsim_2$ has a stronger desire for singletons than $\succsim_1$, then the restriction of $\succsim_2$ and $\succsim_1$ over the set of act-lotteries coincide. Thus we can normalize the parameters such that $(u_1, v_1, \bar{m}_1) = (u_2, v_2, \bar{m}_2)$. In this case, we have $c_2 \geq c_1$ if and only if $V_1 \geq V_2$ by Theorem 2. Thus, given that we have $(u_1, v_1, \bar{m}_1) = (u_2, v_2, \bar{m}_2)$, to show the equivalence of the conditions in Theorem 3, we only need to show that (i) $P \succ_1 F$ implies $P \succ_2 F$ for all P and F if and only if (ii) $V_1(F) \geq V_2(F)$ for all F. First, suppose (i) holds. Take a menu F and an act-lottery P such that $F \sim_2 P$. By (i), we must have $F \succsim_1 P$. This means we have $V_1(F) \geq V_1(P) = V_2(P) = V_2(F)$, and so $V_1(F) \geq V_2(F)$ for all F showing that (ii) is satisfied. For the converse, suppose (ii) holds; that is, $V_1(G) \geq V_2(G)$ for all G. Take a menu F and an act-lottery P such that $V_2(F) \geq V_2(P)$. Since $V_1(F) \geq V_2(F)$ and $V_2(P) = V_1(P)$, we have $V_1(F) \geq V_1(P)$, as desired. □

References

Aliprantis, C. D. and K. C. Border (2006): *Infinite Dimensional Analysis*, Springer, third ed.

Azrieli, Y., C. P. Chambers, and P. J. Healy (2018): "Incentives in Experiments: a Theoretical Analysis," *Journal of Political Economy.*

Baaley, I. and L. Veldkamp (2021): "Bayesian Learning," Working paper.

Bade, S. (2015): "Randomization devices and the elicitation of ambiguity-averse preferences," *Journal of Economic Theory*, 159, 221–235.

Baillon, A., Y. Halevy, and C. Li (2021): "Experminetal Elicitation of Ambiguity Attitude using the Random Incentive System," Working paper.

Blackwell, D. (1951): "Comparison of Experiments," in *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 93–102.

Borovickova, K. (2016): "Job ows, worker ows and labor market policies," Tech. rep., New York University.

Caplin, A. and M. Dean (2015): "Revealed Preference, Rational Inattention, and Costly Information Acquisition," *American Economic Review*, 105, 2183–2203.

Cerreia-Vioglio, S., F. Maccheroni, M. Marinacci, and A. Rustichini (2014): "Niveloids and Their Extensions: Risk Measures on Small Domains," *Journal of Mathematical Analysis and Applications*, 413, 343–360.

Cogley, T. and T. J. Sargent (2005): "The conquest of us ination: Learning and robustnessto model uncertainty," *Review of Economic Dynamics*, 8, 528–563.

Cover, T. M. and J. A. Thomas (2006): *Elements of Information Theory*, Wiley-Interscience, second ed.

de Oliveira, H., T. Denti, M. Mihm, and K. Ozbek (2017): "Rationally inattentive preferences and hidden information costs," *Theoretical Economics*, 12, 621–654.

Dillenberger, D., J. S. Lleras, P. Sadowski, and N. Takeoka (2014): "A Theory of Subjective Learning," *Journal of Economic Theory*, 153, 287–312.

Ellis, A. (2018): "Foundations for optimal inattention," *Journal of Economic Theory*, 173, 56–94.

Ellsberg, D. (1961): "Risk, Ambiguity and the Savage Axioms," *Quarterly Journal of Economics*, 75, 643–669.

Ergin, H. and T. Sarver (2010): "A Unique Costly Contemplation Representation," *Econometrica*, 78, 1285–1339.

——— (2015): "Hidden Actions and Preferences for Timing of Resolution of Uncertainty," *Theoretical Economics*, 10, 489–541.

Hebert, B. M. and M. Woodford (2019): "Rational inattention when decisions take time," .

Jovanovic, B. and Y. Nyarko (1996): "Learning by doing and the choice of technology," *Econometrica*, 64, 1299–1310.

Kelly, D. L. and C. D. Kolstad (1999): "Bayesian learning, growth, and pollution," *Journal ofeconomic dynamics and control*, 23, 491–518.

Klibanoff, P., M. Marinacci, and S. Mukerji (2005): "A smooth model of decision making under ambiguity," *Econometrica*, 73, 1849–1892.

Kreps, D. M. (1979): "A Representation Theorem for Preference for Flexibility," *Econometrica*, 47, 565–577.

——— (1988): *Notes on the theory of choice*, Boulder, CO: Westview Press.

Kreps, D. M. and E. L. Porteus (1978): "Temporal Resolution of Uncertainty and Dynamic Choice Theory," *Econometrica*, 46, 185–200.

Maccheroni, F., M. Marinacci, and A. Rustichini (2006): "Ambiguity Aversion, Robustness, and the Variational Representation of Preferences," *Econometrica*, 74, 1447–1498.

Mackowiak, B., F. Matejka, and M. Wiederholt (2021): "Rational Inattention: A Review," Working paper.

Manzini, P. and M. Mariotti (2014): "Stochastic Choice and Consideration Sets," *Econometrica*, 82, 1153–1176.

Masatlioglu, Y., D. Nakajima, and E. Y. Ozbay (2012): "Revealed Attention," *American Economic Review*, 102, 2183–2205.

Moscarini, G. (2001): "Excess Worker Reallocation," *The Review of Economic Studies*, 68, 593–612.

Ortoleva, P. (2013): "The Price of Flexibility: Towards a Theory of Thinking Aversion," *Journal of Economic Theory*, 148, 903–934.

Ozbek, K. (2021): "Expected utility, independence, and continuity," Working paper.

Pastor, L. and P. Veronesi (2009): "Learning in nancial markets," *Annual Review of Financial Economics*, 1, 361–381.

Pennesi, D. (2015): "Costly information acquisition and the temporal resolution of uncertainty," *Journal of Mathematical Economics*, 60, 115–122.

Raiffa, H. (1961): "Risk, Ambiguity, and the Savage Axioms: Comment," *Quarterly Journal of Economics*, 75, 690–694.

Rockafellar, R. T. (1970): *Convex Analysis*, Princeton University Press.

——— (1974): *Conjugate Duality and Optimization*, Society for Industrial and Applied Mathematics.

Ruszczyński, A. and A. Shapiro (2006): "Optimization of Convex Risk Functions," *Mathematics of Operations Research*, 31, 433–452.

Schmeidler, D. (1989): "Subjective probability and expected utility without additivity," *Econometrica*, 57, 571–587.

Seo, K. (2009): "Ambiguity and second-order belief," *Econometrica*, 77, 1575–1605.

Sims, C. A. (1998): "Stickiness," *Carnegie-Rochester Conference Series on Public Policy*, 49, 317–356.

——— (2003): "Implications of Rational Inattention," *Journal of Monetary Economics*, 50, 665–690.