Networks, beliefs, and asset prices

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Abstract

We set out a model of the stock market in which investors with heterogeneous beliefs update their type based on the past performance of neighbours in an arbitrary social network. We study how the network structure and the degree of agents' attention to performance affect the coupled price-type dynamics. Types converge to a group-consensus characterised by network centrality if updating is purely social and to either the group's most fundamental or most chartist type if updating is purely performance-based, with the time to convergence being finite and proportional to the network diameter. For intermediate cases, we identify two key mechanisms which can make group-consensus non-monotonic with respect to investors' attention to performance. These results shed light on when performance-based updating from social networks is stabilising – or destabilising – for asset prices. As an application, our model can explain price bubbles and price oscillations by network-performance effects.

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> *[T]he time has come to move beyond behavioral finance to "social finance", which studies the structure of social interactions, how financial ideas spread and evolve, and how social processes affect financial outcomes.* (David Hirshleifer, 2015)

1. Introduction

A central question in the emerging field of social finance is: how do social networks influence beliefs, investment decisions, and asset prices? This question is motivated by empirical evidence pointing to a role of social networks in investment decisions. Shiller and Pound (1989) surveyed 131 investors and found that many stock purchases were influenced by interactions with personal contacts, such as friends and relatives. Hong et al. (2005) show that U.S. mutual fund managers' holdings are similar to those of other fund managers in the same city, while Ivković and Weisbenner (2007) show that households are more likely to purchase stocks from a particular industry if their neighbours did so. Shive (2010) finds that

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social interactions affect trading decisions and realised returns, and the same result has been found in asset pricing experiments (see Steiger and Pelster, 2020). Despite this evidence, relatively little is known about the mechanisms by which different social network structures affect investor beliefs and asset price dynamics.

In this paper, we therefore set out a social network model of asset prices. Belief updating in the model depends on a generic social network – such that agents imitate others – and also on a *performance component*. As a result, beliefs and prices evolve as a coupled dynamics. We contribute to the literature by highlighting the separate roles of the network and the performance component in shaping belief and price dynamics, and by relating these dynamics to concrete features of the network and market conditions. Our results therefore shed light on when performance-based updating from a social network is stabilising – or destabilising – for asset prices. Two numerical applications show how the model can explain price bubbles and belief-price oscillations by network-performance effects without any exogenous shocks.

In our model, heterogeneous agents with different initial beliefs form expectations about future asset prices, which feed into price determination. Similar to Brock and Hommes (1998) and several other influential papers we focus on fundamentalist and chartist beliefs, but we go beyond past work by endogenising the degree of trend-following of individuals.[1] Our agents are located on an exogenously given (possibly directed) network such that agents' observation of past investment decisions and returns of other agents is restricted to their network. We allow general network structures for which investors belong either to closed subgroups or are outside such groups; this is important in the context of financial markets as there are some well-known 'opinion leaders' who influence a large number of investors but may be little influenced by others (e.g. Warren Buffett), yet empirical evidence also indicates that investors are influenced by close contacts such as friends and relatives (e.g. Shiller and Pound (1989)), in which case clustering will be observed.

Based on observation of neighbours, agents revise their beliefs, weighing in their own and others' past success: belief types of strongly-performing agents receive higher weight than the beliefs of poorly-performing agents. Beliefs and stock prices thus evolve as a system of coupled dynamics, with network structure influencing beliefs and price determination, and prices feeding back on beliefs by determining which neighbours investors pay most attention to when updating (performance feedback). Such performance-based imitation has been shown to be important for price dynamics in experimental asset markets (Kroll and Levy (1992); Schoenberg and Haruvy (2012)) and estimated models with performance feedback help to explain booms / crashes and some empirical features of price returns such as heavy tails and volatility clustering (see Chiarella et al. (2014); Hommes et al. (2017)).

The long-run type distribution in our model depends on the strength of the performance feedback. On the one hand, if belief updates are independent of past performance, then the updating of belief types is equivalent to a model of opinion dynamics due to DeGroot (1974) where agents in strongly connected and closed groups reach a long-run consensus and each agent's influence on the consensus is determined by their network centrality. Price converges in this case if the long-run average type determined by network centrality is not too

[1]See, for example, Beja and Goldman (1980), Chiarella (1992) and Lux and Marchesi (1999).

strongly chartist. At the other extreme, if the performance feedback is so strong that only the very best past performers are imitated within each agent's network, then only the most fundamental or the most chartist initial type will be adopted in the long-run (depending on the initial average type) and price converges only if the adopted type is not too strongly chartist. Interestingly, adoption of extreme types is guaranteed only if the outside supply of shares is *zero*. If, instead, the outside supply of shares is positive, then the performance ranking of agents is reversed if the price breaches an endogenous threshold. In the latter case, adoption of extreme initial types can be guaranteed only if the initial price and types are restricted such that the threshold is not crossed prior to type convergence.

In the case of intermediate performance feedback, consensus types cannot be characterised analytically. However, we provide sufficient conditions for price convergence and show that, under these conditions, types are guaranteed to converge on a consensus in closed subgroups whereas heterogeneous types prevail (in general) in the other groups, mirroring the outcome in the two polar cases. Thus, in general, heterogeneous beliefs will co-exist alongside consensus beliefs (for those within closed groups). In line with this, some asset price experiments have documented convergence of beliefs on a common price predictor (e.g. Hommes et al. (2005), Bao et al. (2017)), while others find that heterogeneity persists.

With intermediate performance feedback, there are two key mechanisms – reversal of performance ranking and a lagged network structure effect – which make consensus non-monotonic, in general, with respect to investors' attention to performance. As a result, anything goes in the sense that all possible consensuses can be a long-run outcome for some parameter constellations. We therefore present two numerical applications. In the first, there are many strong chartists and one fundamental investor. The chartists update their beliefs by observing others, but the fundamentalist agent believes so strongly in the fundamentals of the asset that this agent's investment decision is not influenced by others. We show that price bubbles emerge and the size of the bubble increases with performance feedback since chartists initially outperform the fundamentalist investor. In the second application, we show that the combination of the network and a sufficiently strong performance feedback leads to permanent oscillations in types and asset prices. Both applications have a network structure in which one or more 'opinion leaders' are followed by the other agents, and the dynamics arise endogenously from initial conditions without any exogenous shocks.

We are not the first to study the implications of social interactions for asset prices. Kirman (1993) and Lux (1995) set out models in which herding is possible because investors are more likely to imitate the dominant type in the population, while Alfarano and Milaković (2009) enrich the Kirman-Lux herding model with local networks. In a similar vein, Cont and Bouchaud (2000) and Iori (2002) consider models of herding in a random networks setting, whereas Chang (2007) studies social interactions when utility exhibits an exogenous preference for social conformity as in Brock and Durlauf (2001). Relative to these papers, our model differs in allowing arbitrary network structures and endogenous weighting of neighbours based on past performance.

The closest paper to ours is Panchenko et al. (2013), in which a version of the Brock and Hommes (1998) model with an explicit social network is introduced, allowing them to relax the complete network assumption behind the baseline model. They study three particular local networks and find they have quite different implications for regions of stability and price volatility. Our framework differs from this literature in several respects. First, we do not assume agents to be either fundamentalist or chartist, but instead allow continuous types between these polar cases. Our continuous-types specification is motivated by survey evidence showing that investors use both fundamental and technical analysis but to differing degrees (Frankel and Froot (1990); Menkhoff (2010)).[2] Second, whereas the Brock and Hommes model has endogenous shares of investors adopting *fixed* types, our model endogenises both the population shares and the types, thus allowing for type consensus. Finally, we put minimal restrictions on the network structures, allowing for all possible directed (including undirected) networks. Our model thereby facilitates the study of consensus types and the implications for asset prices under arbitrary network structures.

We are able to relate price and type dynamics to aspects of the network and market conditions, such as distance between agents (diameter), network centrality, asset supply, and initial price and types. Some previous works have highlighted the impact of network topologies on price volatility and trading volume (Ozsoylev and Walden (2011); Han and Yang (2013)), but they study information transmission in rational expectations models, in contrast to our behavioural setting.[3] By comparison, we provide conditions such that asset prices converge, or not, on the rational expectations solution (fundamental price) when there is performance feedback; our paper thus sheds light on when performance-based updating from social networks is stabilising – or destabilising – for asset prices.

Our paper also contributes to the classical opinion dynamics literature on networks originated by DeGroot (1974) and extended to multiple settings (e.g. the wisdom of the crowds, Golub and Jackson, 2010) which studies the result of repeated weighted average updating of an initially exogenously given object (the individual opinions). A common assumption in this literature is that updating weights are independent of time and reflect only the network position. Some effort has been made to abstract from this assumption and allow updating weights to vary with time to account for e.g., the persuasion bias (DeMarzo et al., 2003), arrival of new information in every period (Jadbabaie et al., 2013), cultural traits (Buechel et al., 2014), conformity (Buechel et al., 2015), or to find general convergence conditions (Lorenz, 2005, 2007). In all these models the updating weights (and the way they change) are exogenously given. We present a model of type updating that resembles the opinion dynamics in the sense that types are updated according to repeated weighted averaging. However, the weights are proportional to each individual's performance on a market. We thus contribute to the opinion dynamics literature by allowing beliefs to influence market decisions and the resulting success on the market to influence the way that beliefs are formed.

In such a setting of coupled opinion dynamics and market interaction, we derive general convergence conditions for the type and price dynamics. Because of the stock market application, there is an interesting difference between weighing in performance and updating the actual types. The investment decision is made based on the type distribution from one

[2]To our knowledge, the only asset pricing model with mixed fundamentalist and trend-following beliefs is Barberis et al. (2018), in which there is no social interaction and individual types are purely exogenous.

[3]Both Ozsoylev and Walden (2011) and Han and Yang (2013) build on the seminal paper of Hellwig (1980), and hence the problem they study (information and its price/welfare implications) and the assumptions about agents' cognitive abilities (rational rather than behavioural) are very different to our paper.

time step before. So the updating weight depends on types with a two period lag while the update itself is from past types. We also find that classical convergence to consensus may not obtain if individuals only update from best-performing agents or if price diverges.

The paper proceeds as follows. In Section 2 we set out the model, and Section 3 presents our main results on long-run price and type dynamics. In Section 4 we explain two key mechanisms at work in the case of finite performance effect, and in Section 5 we provide two applications (price bubbles and price oscillations). Finally, we conclude in Section 6.

2. Model

Consider a finite set of risk-neutral investors $N = \{1, \ldots, n\}$ and let time be discrete, $t \in \mathbb{N}$. At each point in time $t \in \mathbb{N}$, agents choose holdings of a risky asset x_t^i with *unknown return* (in fixed supply $X \geq 0$) and a riskless bond (in flexible supply). Agents buy the risky asset at price p_t and sell it at price p_{t+1} having received (stochastic) dividends d_{t+1}; the riskless bond has a known return $r > 0$ and price of 1. Both price p_{t+1} and realisations of dividends d_{t+1} are unknown in period t such that the unknown (excess) return of the asset is given by $R_{t+1} = p_{t+1} + d_{t+1} - (1+r)p_t$. At each point in time $t \in \mathbb{N}$, agents $i \in N$ are characterised by their wealth w_t^i and their subjective expectation $\tilde{E}_t^i$ about the future asset price p_{t+1} and dividends d_{t+1}. We explain in Section 2.3 how these expectations are determined by an agent's type. Agents are myopic and choose holdings of the risky asset and the riskless bond to maximize expected next period wealth $\tilde{E}_t^i w_{t+1}^i$. Investors can take short positions, but investments into the risky asset are subject to a quadratic cost that depends on the parameter $\phi > 0$, such that total transaction costs in period t are given by $\frac{\phi}{2}(x_t^i)^2$.

Therefore, at any $t \in \mathbb{N}$, each investor $i \in N$ solves the problem:

$$\max_{x_t^i} \; \tilde{E}_t^i w_{t+1}^i$$

$$\text{s.t.} \quad w_{t+1}^i = (p_{t+1} + d_{t+1})x_t^i + (1+r)(w_t^i - p_t x_t^i) - \frac{\phi}{2}(x_t^i)^2 \tag{1}$$

where $w_t^i - p_t x_t^i$ denotes the holdings of the riskless asset. The first term in Equation (1) is the payoff on stocks (dividend plus resale price); the second term is the gross return on holdings of the riskless asset; and the third term is the portfolio transaction cost.

The first-order condition yields the following demand schedule:[4]

$$x_t^i = \delta \left(\tilde{E}_t^i \left[p_{t+1} + d_{t+1} \right] - (1+r)p_t \right) \tag{2}$$

where $\delta = 1/\phi$. Equation (2) shows the demand of agent $i \in N$ for the risky asset at any given price p_t. As is standard, demand is proportional to the expected excess return on the risky asset.

Dividends follow a stochastic process:

$$d_t = \bar{d} + \varepsilon_t \tag{3}$$

[4]The demand schedule (2) has the same form as under mean-variance utility (see e.g. Hommes (2021): p. 171); thus, assuming risk-neutral investors who face a transaction cost is convenient but not essential.

where $\bar{d} > 0$ and ε_t is chosen from some IID distribution with mean 0 and support in an interval $[d^-, d^+]$ such that $d^- < 0$ and $d^+ > 0$.[5] Agents know the dividend process, and hence their subjective expectations coincide with the objective (rational) expectation:

$$\tilde{E}_t^i[d_{t+1}] = E_t(d_{t+1}) = \bar{d}, \qquad \forall i. \tag{4}$$

where $E_t(.)$ denotes the conditional expectation operator with respect to past realisation of dividends d_t.

Next, we present the details of investor types. First, consider the two cases of fundamentalists and chartists before introducing the hybrid types.

2.1. Fundamentalists

Fundamentalists forecast future stock prices by calculating the expected stock market price in a world where *all investors are fundamentalists*.[6] Formally, the price expectation of a pure fundamentalist is

$$\tilde{E}_t^f[p_{t+1}] = p^f := \frac{\bar{d} - \frac{X}{n\delta}}{r} \tag{5}$$

where p^f is the fixed fundamental price.

The fundamental price p^f is a benchmark notion of 'fundamental solution'. In particular, the fundamental price is the price for which the aggregate demand for stocks equals the fixed supply X when all investors are fundamentalists with rational expectations $E_t(.)$. The fundamental price p_t^* must then satisfy

$$\sum_{i \in N} x_t^i = X \quad \Leftrightarrow \quad \delta\left(nE_t(p_{t+1}^*) + nE_t(d_{t+1}) - n(1+r)p_t^*\right) = X.$$

Given IID dividends with mean $\bar{d}$, we have $E_t(d_{t+1}) = \bar{d}$ and hence p_t^* is constant. Setting $p_t^* = p^f$ gives the expression in (5). Note that this is lower than the fundamental price of $\bar{d}/r$ in Brock and Hommes (1998) as we allow shares to be in positive net supply in our model.[7]

2.2. Chartists

Chartists base their expectations on the most recent observed price, p_{t-1}. In particular, the subjective price expectations of chartists are given by

$$\tilde{E}_t^c[p_{t+1}] = p_{t-1}. \tag{6}$$

This specification for chartist beliefs follows Brock and Hommes (1998), except that they allow chartists to place some weight on the fundamental belief, p^f. There is no loss of generality, however, since we allow investors to weight the two polar beliefs according to their type g in the general model that follows.

[5]Our assumption that dividends are drawn from a fixed interval is not restrictive since the interval can be chosen arbitrarily large. Note that assuming $d^- \geq -\bar{d}$ will ensure non-negative dividends.

[6]Note that such expectation formation is a *misspecified model* of actual market prices because it ignores the presence of chartists (see Section 2.2).

[7]Clearly, setting $X = 0$ as in Brock and Hommes (1998), the fundamental prices coincide in both models.

2.3. g-Traders

We consider more general types of investors who form their price expectation by following a rule of thumb that takes a weighted average between the price expectation of a fundamentalist and a price expectation of a chartist. Let $g_t^i \in \mathbb{R}_+$ denote the weight that such a trader $i \in N$ attaches to the chartist's price expectation at some point of time $t \in \mathbb{N}$. We call this trader a g_t^i-trader. The price expectation of an g_t^i-trader is, hence, given by:

$$\tilde{E}_t^i[p_{t+1}] = g_t^i \tilde{E}_t^c[p_{t+1}] + (1 - g_t^i)\tilde{E}_t^f[p_{t+1}] = g_t^i p_{t-1} + (1 - g_t^i)p^f. \tag{7}$$

The price expectation of a g_t^i trader nests the polar cases of fundamentalist, when $g_t^i = 0$, and chartist when $g_t^i = 1$. More generally, g_t^i traders arrive at a price expectation by taking a weighted average of the fundamentalist and chartist beliefs (if $g_t^i \leq 1$). Note that we also allow $g_t^i > 1$, in which case the agent expects the price to move further away from the fundamental price in the future. We call such agents strong chartists.

The g_t^i investors use fundamental and technical information as in Barberis et al. (2018), but we allow g_t^i to be endogenously determined by the network of agent i and the relative performance of different members in the network. Our setup also differs from that in Brock and Hommes (1998) because they consider fixed types, with the probability to switch type determined by the relative performance (fitness) of the different predictors, as agents observe performance of all others on a complete network.

By comparison, we allow investor beliefs to deviate from the polar cases of fundamentalist and chartist and place no restrictions on the network structure. As noted in the Introduction, there is considerable evidence that real-world investors combine fundamental and technical information when making forecasts and are influenced by social ties.

2.4. Market clearing

For the stock market to clear, we require that $\sum_{i \in N} x_t^i = X$. By Eqs. (2) and (4), market clearing implies

$$\sum_{i \in N} x_t^i = X \quad \Leftrightarrow \quad \delta \left(\sum_{i \in N} E_t^i(p_{t+1}) + n\bar{d} - n(1+r)p_t \right) = X.$$

Using (7) and rearranging the market clearing condition, the equilibrium stock price can be written in the form:

$$p_t = \frac{\bar{d} + \left(1 - \sum_{i \in N} \frac{g_t^i}{n}\right) \tilde{E}_t^f[p_{t+1}] + \left(\sum_{i \in N} \frac{g_t^i}{n}\right) \tilde{E}_t^c[p_{t+1}] - \frac{X}{n\delta}}{1+r}. \tag{8}$$

From (5) and (6) we can simplify (8) by using the notation of average type $\bar{g}_t := \sum_{i \in N} \frac{g_t^i}{n}$, and considering the deviation from the fundamental price, $\tilde{p}_t := p_t - p^f$ to get the law of motion of the price dynamics:

$$\tilde{p}_t = \frac{\bar{g}_t}{1+r}\tilde{p}_{t-1}. \tag{9}$$

From this law of motion, we can already conclude that asset price converges to the fundamental price if average belief type $\bar{g}_t$ converges to a value smaller than $1 + r$, while price diverges if average type is of too strong chartist nature in the long-run.

2.5. Return and Fitness

To derive excess returns per share, consider the demands x_t^i and investor types g_t^i for all agents $i \in N$ at some point in time $t \in \mathbb{N}$,

$$x_t^i = \delta\left(\tilde{E}_t^i[p_{t+1}] + \overline{d} - (1+r)p_t\right) = \delta\left(g_t^i \tilde{p}_{t-1} + \frac{X}{n\delta} - (1+r)\tilde{p}_t\right) \tag{10}$$

where $\overline{d} = rp^f + \frac{X}{n\delta}$ has been used (compare (5)). From the law of motion of the price (9), we then receive

$$x_t^i = \delta\left([g_t^i - \overline{g}_t]\tilde{p}_{t-1} + \frac{X}{n\delta}\right). \tag{11}$$

Equation (11) has a particularly nice interpretation when $X = 0$: agents less optimistic than the average will short-sell the asset and agents more optimistic than the average will buy the asset. If last period's price is below the fundamental price, then more fundamentalist type agents expect the price to increase more (i.e. are are more optimistic), while the opposite is true if last period's price is above the fundamental price.

Similar algebra can be used to show that the excess return per share is

$$\begin{aligned} R_t = p_t + d_t - (1+r)p_{t-1} &= \tilde{p}_t + \frac{X}{n\delta} - (1+r)\tilde{p}_{t-1} + \varepsilon_t \\ &= \left(\frac{\overline{g}_t}{1+r} - (1+r)\right)\tilde{p}_{t-1} + \frac{X}{n\delta} + \varepsilon_t. \end{aligned} \tag{12}$$

Combining the two results, the fitness measure or net profits at time t can be written as

$$u_t^i = R_t x_{t-1}^i = \left[\left(\frac{\overline{g}_t}{1+r} - (1+r)\right)\tilde{p}_{t-1} + \frac{X}{n\delta} + \varepsilon_t\right] \delta\left([g_{t-1}^i - \overline{g}_{t-1}]\tilde{p}_{t-2} + \frac{X}{n\delta}\right). \tag{13}$$

Note that if returns per share are positive, the agents with highest demand (the most optimistic agents) will have the highest fitness, whereas if returns per share are negative the best-performing agent is the least optimistic.

2.6. Performance Ranking and Critical Price

Who the most optimistic agents are, depends on the sign of the price deviation from the fundamental price. By (11), the more chartist an agent's type, the higher the demand if price deviation $\tilde{p}_t$ is positive while the inverse holds for negative price deviation, i.e.

$$\tilde{p}_{t-2} > 0 \quad \Rightarrow \quad g_{t-1}^i > g_{t-1}^j \Leftrightarrow x_{t-1}^i > x_{t-1}^j.$$

Hence, depending on the sign of the returns there will be a performance ranking. If $R_t > 0$, then the most extreme chartist type performs best, while if $R_t < 0$, then the most fundamentalist type performs best. There is a critical price level at which returns switch sign such that when the price crosses this critical price level a switch in performance ranking occurs. To see this we can rewrite (12) using (9) to obtain,

$$sgn(R_t) = sgn(p_t^{crit}) \cdot sgn(p_t^{crit} - \tilde{p}_t) \qquad \text{s.t. } p_t^{crit} := \frac{\bar{g}_t}{(1+r)^2 - \bar{g}_t} \cdot \left(\frac{X}{n\delta} + \varepsilon_t\right) \tag{14}$$

which holds for all $\bar{g}_t \neq (1+r)^2$ where sgn denotes the sign function.[8]

Clearly, the best-performing agents can then be found among the extreme types (the most chartist or the most fundamental type). We denote the set of best-performing agents from some subset of agents $S \subset N$ at time $t \in \mathbb{N}$ as $U_t^{\max}(S) := \{i \in S \mid u_t^i \geq u_t^j \; \forall j \in S\}$. It will turn out that these are either those with maximal or minimal type, which is denoted by $g_t^{\max}(S) := \max\{g_t^i \mid i \in S\}$, respectively $g_t^{\min}(S) := \min\{g_t^i \mid i \in S\}$. For $S = N$, we simply write $g_t^{\max} := g_t^{\max}(N)$, respectively $g_t^{\min} := g_t^{\min}(N)$. Analogously, the set of agents from $S \subset N$ with maximal or minimal type are denoted by $G_t^{\max}(S) := \{i \in S \mid g^i = g_t^{\max}(S)\}$, respectively $G_t^{\min}(S) := \{i \in S \mid g^i = g_t^{\min}(S)\}$. For $S = N$, we analogously drop the argument and write $G_t^{\max} := G_t^{\max}(N)$.

2.7. The Network, Type Updating

We consider a directed network given by a $n \times n$ matrix $\mathbf{A}$ with entries $a_{ij} \in \{0, 1\}$. If $a_{ij} = 0$, then investor i does not observe investor j. If, instead, $a_{ij} = 1$, then i observes j's type, and j's returns and fitness u_t^j. We assume that $a_{ii} = 1$ for all $i \in N$ such that each agent always observes their own type and returns. Denote by $N^i := \{j \in N : a_{ij} = 1\}$ the set of traders that i observes and by $M^i := \{j \in N : a_{ji} = 1\}$ the set of traders that observe i. By above assumption, we have $i \in N^i$ and $i \in M^i$. For a subset $S \subset N$ we denote by $M(S) := \{j \in N \mid \exists i \in S : a_{ji} = 1\}$ the set of agents who observe agents in S. Further, denote by $\bar{\mathbf{A}}$ the matrix with entries $\bar{a}_{ij} = \frac{1}{|N^i|} a_{ij}$ which is row stochastic.

A path in the network from node i to node j of length $k \in \mathbb{N}$ exists if there is a sequence of connected nodes $(i^1, \ldots i^k)$ such that $a_{i^l, i^{l+1}} = 1$ for all $1 \leq l \leq k-1$ and $i^1 = i$ and $i^k = j$. Note that a path of length k from i to j exists, if and only if we have $(\mathbf{A}^k)_{ij} > 0$ where $\mathbf{A}^k$ denotes the k-th power of the matrix $\mathbf{A}$. The set of nodes that lie on a path that starts in node i are defined as $P^i := \{j \in N \mid \exists k \in \mathbb{N} : (\mathbf{A}^k)_{ij} > 0\}$. Clearly $P^j \subseteq P^i$ for all $j \in P^i$. The distance between two nodes i and j in network $\mathbf{A}$ is defined as the minimal path length denoted by $d(i,j) := \min\{k \in \mathbb{N} : (\mathbf{A}^k)_{ij} > 0\}$. If two nodes are not connected by a path, we set $d(i,j) = \infty$. A network is called strongly connected if $d(i,j) < \infty$ for all $i, j \in N$. We further define the distance between two sets of nodes $B, C \subset N$ by $d(B,C) = \min_{i \in B, j \in C} d(i,j)$. The diameter of the network is given by $D(\mathbf{A}) = \max_{i,j \in N} d(i,j)$.

We assume that each trader is only influenced by those she observes in the network. Similarly to the Brock and Hommes model, traders evaluate the performance of other traders they observe and each investor updates according to the logit response model such that:

$$g_{t+1}^i = \left(\sum_{k \in N^i} \exp(\gamma u_t^k) \right)^{-1} \sum_{j \in N^i} \exp(\gamma u_t^j) g_t^j, \qquad \forall i \in N. \tag{15}$$

There are two important differences in (15) relative to the Brock and Hommes model. First, in our model agents $i \in N$ do not update from the entire set of agents N, but, as in

[8]The sign function is defined by $sgn(x) = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$

Panchenko et al. (2013), only update from their neighbours N^i. Second, in contrast to Brock and Hommes (1997, 1998), where relative fitness (the ratio $\exp(\gamma u_t^j)/\sum_{k\in N}\exp(\gamma u_t^k)$) determines the probability with which an agent adopts one of the polar types, an agent's type in period $t+1$ is a weighted average of those traders she observes in her network giving higher weight to more successful individuals. The parameter γ measures the performance feedback to beliefs. It is similar to the intensity of choice in the Brock and Hommes model measuring how fast agents switch between different prediction strategies.

Denoting the updating weights by $\tilde{a}_{ij}(t) = \left(\sum_{k\in N^i}\exp(\gamma u_t^k)\right)^{-1}\exp(\gamma u_t^j)$, and the (column) vector of types by $\mathbf{g}_t = (g_t^1, \ldots, g_t^n)'$, the matrix $\tilde{\mathbf{A}}(t) = \left(\tilde{a}_{ij}(t)\right)_{i,j\in N}$ presents the law of motion of the type dynamics in the sense that (15) can be expressed as

$$\mathbf{g}_{t+1} = \tilde{\mathbf{A}}(t)\mathbf{g}_t. \tag{16}$$

Note that $\tilde{\mathbf{A}}(t)$ is always row stochastic by (15) such that each iteration is a weighted average of the type vector of the previous period. Further, for any finite γ, we have $\tilde{a}_{ij}(t) = 0$ if and only if $a_{ij} = 0$ for any $t \in \mathbb{N}$. This means that in the course of repeated updating, agents can only influence each other if they are connected by a path in the network $\mathbf{A}$.

2.8. Closed and Strongly Connected Groups and the Rest of the World

Recall that agents can only influence each other if they are connected by a path in the network $\mathbf{A}$. This motivates the following definition (compare also Buechel et al., 2015).

Definition 1. *Let $\Pi(N, \mathbf{A}) = \{\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_K, \mathcal{R}\}$ be a partition of N into $K(\geq 1)$ groups and the (possibly empty) rest of the world $\mathcal{R}$ such that:*

- *Each group $\mathcal{C}_k$ is strongly connected, i.e. $j \in P^i$ for all $i, j \in \mathcal{C}_k$, for all $k = 1, \ldots, K$.*
- *Each group $\mathcal{C}_k$ is closed, i.e. $P^i \subseteq \mathcal{C}_k$ for all $i \in \mathcal{C}_k$, for all $k = 1, \ldots, K$.*
- *The (possibly empty) rest of the world $\mathcal{R}$ consists of the agents who do not belong to any closed and strongly connected set, i.e. $\mathcal{R} = N \setminus \bigcup_{k=1}^{K} \mathcal{C}_k$.*

A subgroup of agents is strongly connected if there exists a path from every i to every j within that group. If the entire player set is strongly connected, we call the network strongly connected. A subgroup of agents is closed if agents within that group only observe others from the same group (i.e. there is no influence from types outside this group). The only interaction across closed groups is through the market, but updating of types only occurs between members of the same group. The rest of the world are those agents whom none of the others listen to and who themselves have a path to some closed and strongly connected group. Obviously, every network has at least one strongly connected and closed group.

With a suitable renumeration, the matrix $\mathbf{A}$ can be organized into irreducible blocks $\mathbf{A}_{\mathcal{C}_k}$ for $k = 1, \ldots, K$ (which correspond to the closed and strongly groups of the partition $\Pi(N, \mathbf{A})$) and the possibly empty rest of the world. Clearly for any given (i.e. finite) γ, the matrices $\mathbf{A}$ and $\tilde{\mathbf{A}}(t)$ have the same block structure for all $t \in \mathbb{N}$ since $a_{ij} > 0 \Leftrightarrow \tilde{a}_{ij}(t) > 0$ such that

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{\mathcal{C}_1} & & & \mathbf{0} \\ & \ddots & & \\ \mathbf{0} & & \mathbf{A}_{\mathcal{C}_K} & \\ \mathbf{A}_{\mathcal{R}1} & \cdots & \mathbf{A}_{\mathcal{R}K} & \mathbf{A}_{\mathcal{R}\mathcal{R}} \end{pmatrix} \quad \Rightarrow \quad \tilde{\mathbf{A}}(t) = \begin{pmatrix} \tilde{\mathbf{A}}_{\mathcal{C}_1}(t) & & & \mathbf{0} \\ & \ddots & & \\ \mathbf{0} & & \tilde{\mathbf{A}}_{\mathcal{C}_K}(t) & \\ \tilde{\mathbf{A}}_{\mathcal{R}1}(t) & \cdots & \tilde{\mathbf{A}}_{\mathcal{R}K}(t) & \tilde{\mathbf{A}}_{\mathcal{R}\mathcal{R}}(t) \end{pmatrix}. \quad (17)$$

From this we can infer steady states of the type dynamics. A steady state vector of types $\mathbf{g} = (g^1, \ldots, g^n)'$ must be invariant to the law of motion $\tilde{\mathbf{A}}(t)$, i.e. $\mathbf{g} = \tilde{\mathbf{A}}(t)\mathbf{g}$. Since any $\tilde{\mathbf{A}}(t)$ is row stochastic and such that $\tilde{a}_{ij}(t) = 0$ if and only if $a_{ij} = 0$, the only potential steady state vectors of types are such that consensus in every closed and strongly connected group is obtained, i.e. for each $\mathcal{C}_k$ it must be that $g^i = g^j$ for all $i, j \in \mathcal{C}_k$. The consensus reached in each closed and strongly connected group may differ. On the other hand, every type vector such that some consensus is reached in each closed and strongly connected group (and some additional conditions on the types of the rest of the world are satisfied) is a steady state of the type dynamics. Hence, there is a continuum of steady states of the type dynamics.

Regarding the steady-state price, we see from (9) that there is a unique fundamental steady state price $\tilde{p} = 0$ if the steady state average type $\overline{g}$ differs from (1+r), whereas if average type equals (1+r) then there is a fundamental steady state price and a continuum of non-fundamental steady state prices $\tilde{p} \neq 0$.

2.9. Timing and Initial Conditions

From time period $t \geq 0$ on, the dynamics evolve according to what has been described above. In other words, at the beginning of each time period $t \geq 0$, investor's types are given by $g_t^i \in \mathbb{R}_+$ for all $i \in N$ with stock holdings of last period x_{t-1}^i and the last period's price deviation given by $\tilde{p}_{t-1} = p_{t-1} - p^f$. Investors then form their demands x_t^i according to (11) such that the price deviation $\tilde{p}_t$ can be derived from the price deviation $\tilde{p}_{t-1}$ according to the law of motion in (9). From this, returns R_t are realised and fitness u_t^i of each investor $i \in N$ is given by (13). Investors observe the fitness of others in their network and at the end of period t update their type according to (15).

To have a consistent model, we need assumptions for the period before type updating occurs for the first time, i.e. before $t = 0$. We assume that initially there is a price of the stock p_{-2} (and hence $\tilde{p}_{-2}$), which may be interpreted as the price at the emission of the stock, and there are investors types g_{-1}^i for all $i \in N$. We assume that $p_{-2} \neq p^f$.[9] Given g_{-1}^i and $\tilde{p}_{-2}$, demand x_{-1}^i can be computed according to (11) yielding equilibrium price p_{-1} such that $\tilde{p}_{-1}$ is determined by (9). At the end of period -1, we set $g_0^i = g_{-1}^i$ for all $i \in N$ (updating of the types can only occur once the agents realise differences in performance). In period 0, price $\tilde{p}_0$ and demand x_0^i are determined by (9) and (11), respectively, and performance u_0^i is

[9]If $p_{-2} = p^f$, then the price will always remain at the fundamental price, all types will predict the same price, and types will evolve according to network effects only, i.e. as if $\gamma = 0$, see Section 3.1.

evaluated according to (13), and at the end of period 0, the first type updating occurs such that g_1^i is determined by (15). We can therefore refer to $\tilde{p}_0$ as the initial price.[10]

For all periods $t \geq 1$, demand, price, fitness, and types are determined by (9) – (15).

3. Dynamics

Types and prices co-evolve over time: the price dynamics depend on the average type and the type dynamics depend on the performance of agents, determined by previous types and price. We thereby study a coupled dynamics of prices and types. The only exception is when there is no performance feedback effect, i.e. $\gamma = 0$. We start with this case to study the network effect in isolation. We then establish general conditions for convergence of price and types for arbitrary but finite performance feedback effect. Finally, we study the other polar case where the performance feedback becomes infinitely strong such that $\gamma \to \infty$.

3.1. $\gamma = 0$: Pure network-based updating

We first take a brief look at the dynamics when $\gamma = 0$. In this case, agents simply update their own type through their network independently of how others perform such that the updating rule (15) implies that the updating weights are given by $\tilde{a}_{ij}(t) = \bar{a}_{ij}$ for all $t \in \mathbb{N}$, or, equivalently,

$$(15) \quad \overset{\gamma=0}{\Rightarrow} \quad g_{t+1}^i = \frac{1}{|N^i|} \sum_{j \in N^i} g_t^j. \tag{18}$$

Hence, the dynamical system (16) can be written as

$$\mathbf{g}_{t+1} = \tilde{\mathbf{A}}(t)\mathbf{g}_t = \left(\bar{\mathbf{A}}\right)^{t+1} \mathbf{g}_0. \tag{19}$$

The dynamics of types do not depend on fitness, and are, therefore, completely independent of prices and stock holdings. Agents just observe their neighbours' types and adopt the average of these types in the next period independently of how each of their neighbours performs. Such a dynamic model is closely related to a model of opinion dynamics first formulated by DeGroot (1974). Since we assumed agents observe their own type and fitness such that the diagonal of $\mathbf{A}$ is strictly positive, the matrix $\bar{\mathbf{A}}$ is aperiodic. From standard results, the type dynamics converge and we can characterize the terminal types.

Proposition 1. *Suppose $\gamma = 0$. For each strongly connected and closed group $\mathcal{C}_i$, let $\mathbf{v}_i$ be the (unique) left-unit eigenvector of $\tilde{\mathbf{A}}_{C_i}$ such that entries of $\mathbf{v}_i$ sum to 1, and let $\mathbf{1}_i$ be the $\mid \mathcal{C}_i \mid \times 1$ vector with all entries equal to 1. Denoting $\bar{\mathbf{A}}^{\infty}_{\mathcal{R}k} := (\mathbf{I} - \bar{\mathbf{A}}_{\mathcal{R}\mathcal{R}})^{-1}\bar{\mathbf{A}}_{\mathcal{R}k}$, the types of traders converge to*

$$\lim_{t\to\infty} g_t = \lim_{t\to\infty} (\tilde{\mathbf{A}})^t g_0 = \begin{pmatrix} \mathbf{1}_1\mathbf{v}_1' & & & \mathbf{0} \\ & \ddots & & \\ \mathbf{0} & & \mathbf{1}_K\mathbf{v}_K' & \\ \bar{\mathbf{A}}^{\infty}_{\mathcal{R}1} & \cdots & \bar{\mathbf{A}}^{\infty}_{\mathcal{R}K} & \mathbf{0} \end{pmatrix} g_0.$$

[10]In all examples in the paper we specify only the initial price $\tilde{p}_0$ since the initial values at earlier dates can be inferred from the equation for the price dynamics, (9), in conjunction with the initial types.

for any realisation of the dividends. Price converges to the fundamental price if $\lim_{t\to\infty} \bar{g}_t < 1 + r$*, price converges to some price other than the fundamental price if* $\lim_{t\to\infty} \bar{g}_t = 1 + r$*, and price diverges to* $\pm\infty$ *if* $\lim_{t\to\infty} \bar{g}_t > 1 + r$*.*

Note that agents in different closed and strongly connected groups do not influence each other. To focus on what happens in one closed and strongly connected group, suppose first that the network is strongly connected, i.e. the partition from Definition 1 is such that $\Pi(N, \mathbf{A}) = \{N\}$ which is equivalent to $d(i, j) < n$ for all $i, j \in N$. From Proposition 1, we can conclude that in this case all players reach a consensus and

$$\lim_{t\to\infty} \bar{g}_t = \lim_{t\to\infty} g^i_t = \sum_{j\in N} v^j g^j_0 \quad \forall i \in N$$

where $\mathbf{v}$ is the left-hand unit eigenvector of $\tilde{\mathbf{A}}$ with entries summing to one, i.e. $\mathbf{v} = \mathbf{v}\tilde{\mathbf{A}}$ and $\sum_{j\in N} v^j = 1$ which is a measure of centrality of agents in the network, also called eigenvector-centrality. Hence, the social influence of an agent depends on their initial type weighted by their network centrality. In this case, price converges to the fundamental price if and only if the sum of initial types weighted by the eigenvector centrality sum to less than $1 + r$. Note that an agent with a type that exceeds unity believes that the price will move away from the fundamental price. Hence, if there are not many strong chartists or if the strong chartists are not central in the network, price will converge to the fundamental price.

The same considerations are true for each closed and strongly connected group $\mathcal{C}_k$ itself since there is no influence from other groups neither through the network nor through the market: a consensus is reached in $\mathcal{C}_k$ and the influence of each agent $i \in \mathcal{C}_k$ on the consensus depends on the agent's eigenvector centrality in their group (the entry of the left-hand unit eigenvector of the matrix $\mathbf{A}_{\mathcal{C}_k}$). The more central an agent in their group, the more influential is that agent for the type consensus in their group.

Each agent in the rest of the world then ends up with a weighted average of consensuses in the strongly connected and closed groups. Hence, the initial type of any agent from the rest of the world will not influence their or any other agents' terminal type. The weights with which agents from the rest of the world average over the consensuses in closed and strongly connected groups are according to the connections from the rest of the world to the strongly connected and closed groups weighted by the mutual connections within the rest of the world $(\mathbf{I} - \bar{\mathbf{A}}_{\mathcal{RR}})^{-1} = \sum_{k=0}^{\infty} (\bar{\mathbf{A}}_{\mathcal{RR}})^k$.[11] This also implies that the long-run types in the rest of the world are generically heterogeneous.

3.2. $\gamma \geq 0$*: Mixed network- and performance-based updating*

Now we turn on the performance effect. In such a setting, the dynamics of types and prices become highly complex and stochastic: asset prices depend on beliefs, resulting performance depends on stochastic dividends, and belief updating then depends on the network and performance. Nevertheless, we can make some concrete statements about price and type convergence even in this setting, as the following sections show.

[11] Note that $\bar{\mathbf{A}}_{\mathcal{RR}}$ has a spectral radius strictly smaller than 1 since, by definition of the rest of the world, each row of $\bar{\mathbf{A}}_{\mathcal{RR}}$ sums to a value strictly less than 1. Therefore, $(\mathbf{I} - \mathbf{A}_{\mathcal{RR}})$ is always invertible.

3.2.1. Price convergence for (finite) $\gamma \geq 0$

Consider first the price dynamics for positive performance feedback. We will show that the coupled dynamics are still ensured to converge under fairly general assumptions. We first show in Proposition 2 that price always converges to the fundamental price if no extreme chartist types exist initially. On the other hand, if only extreme chartists exist initially, then the price will diverge. This insight may seem trivial given the law of motion of the price dynamics (9) and the nature of type updating being a repeated weighted average, but the second part of the Proposition 2 shows that these conditions are quite tight.

We then establish in Proposition 3 that under price convergence, types will also converge such that in all closed and strongly connected groups a consensus is established. If the price converges to the fundamental price we can also characterize the types in the rest of the world as a function of the consensuses in the closed and strongly connected groups.

Proposition 2. *If $g_0^{\max} < 1 + r$, then price converges to the fundamental price (for all realisations of d_t). Further, if $g_0^{\min} > 1 + r$, then price diverges to $+\infty$ or $-\infty$.*

If $g_0^{\max} > 1+r$, then for any strongly connected network $\mathbf{A}$ and any distribution of types g, there exist initial conditions such that price diverges to $+\infty$ or $-\infty$, while if $g_0^{\min} < 1 + r$ then for any strongly connected network $\mathbf{A}$ and any distribution of types g, there exist initial conditions such that price converges to the fundamental price even when dividends are non-stochastic $d_t = \bar{d}$ for all $t \in \mathbb{N}$.

The first part of the result is straightforward. By the nature of weighted average updating of types, the convex hull of types never expands over time. Hence, at each point in time $t \in \mathbb{N}$, the average type $\bar{g}_t$ is always contained in the interval of initial extreme types $[g_0^{\min}, g_0^{\max}]$. Hence, if $g_0^{\max} < 1 + r$ then $\bar{g}_t < 1 + r$ for all $t \in \mathbb{N}$. By the law of motion of the price dynamics (9), price convergence to the fundamental price is implied. Note that convergence is smooth in the sense that price deviation $\tilde{p}_t$ does not change sign and its absolute value is strictly decreasing. Analogously, price divergence is straightforward if $g_0^{\min} > 1 + r$.

Although this may seem trivial, the second part of the result shows that these conditions are quite tight in the sense it is not possible to have any more general conditions on the initial type distribution without additional conditions on the performance feedback parameter γ, or other initial conditions. While for stochastic dividends this may not be surprising since there may exist realisations of dividends that lead to the adoption of one of the extreme types, this result also holds for non-stochastic dividends. We illustrate this in Example 1.

Example 1. *Consider the case where $n = 2$ such that the two agents observe each other, i.e. $a_{12} = a_{21} = 1$. Initially, let agent 2 be a fundamentalist $g_0^2 = 0$. If the initial type of agent 1 is not too extreme such that $g_0^1 < 1 + r$, then by Proposition 2, price will converge to the fundamental price. Instead, suppose that $g_0^1 = 1 + r + \varepsilon$ for some $0 < \varepsilon < (1 + r)^2 - (1 + r)$. Then, by the second part of Proposition 2, we can find initial conditions for price $\tilde{p}$, outside supply of shares X, and performance feedback parameter γ such that price diverges.*

To illustrate this, consider the deterministic skeleton of the dividends process, $d_t = \bar{d}$ for all $t \in \mathbb{N}$, and let initial price $\tilde{p}_0$ relate to X, δ, and r such that $0 < \tilde{p}_0 < \frac{1+r+\varepsilon}{2(1+r)^2+1+r+\varepsilon}\frac{X}{2\delta}$. For any $X > 0$, clearly such an initial price $\tilde{p}_0$ exists since $\delta, \varepsilon, r > 0$, and $\varepsilon < (1 + r)^2 - (1 + r)$. Above assumption on the initial price implies that $p_0 < p_0^{crit}$ and, hence, $R_0 > 0$ by (14).

Therefore agent 1 (the more optimistic agent which is the one with a more chartist type) obtains higher profit than agent 2, implying $u_0^1 > u_0^2$ since dividends are non-stochastic.

Note that for any $\gamma \in \mathbb{R}_+$, both agents obtain a consensus after one period since both use the same updating rule,

$$g_1^1 = g_1^2 = \frac{1}{\exp(\gamma u_0^1) + \exp(\gamma u_0^2)} \left(\exp(\gamma u_0^1) g_0^1 + \exp(\gamma u_0^2) g_0^2 \right).$$[12]

After reaching this consensus, types will not change henceforth, $g_t^i = g_1^1$ for all $i \in \{1, 2\}$ and $t \in \mathbb{N}$. For $\gamma \to \infty$, both agents will adopt the type with maximal fitness,[13] and hence $g_1^1 = \frac{1}{\exp(\gamma u_0^1)+\exp(\gamma u_0^2)} \left(\exp(\gamma u_0^1) g_0^1 + \exp(\gamma u_0^2) g_0^2 \right) \to g_0^1 = 1+r+\varepsilon$. Thus, there exists a $\bar{\gamma} \in \mathbb{R}_+$ such that $\frac{1}{\exp(\gamma u_0^1)+\exp(\gamma u_0^2)} \left(\exp(\gamma u_0^1) g_0^1 + \exp(\gamma u_0^2) g_0^2 \right) > 1 + r$ for all $\gamma > \bar{\gamma}$ since $\varepsilon > 0$. This, however, implies $\bar{g}_t > 1 + r$ for all $t \geq 1$. Hence, price will diverge by (9).

The second part of Proposition 2 usually requires large performance feedback parameter γ to ensure that small violations of the condition $g_0^{\max} < 1 + r$ actually lead to violations of the statement of the first part of Proposition 2. We will later show that for any network and small enough price deviations, the types will converge to $g_0^{\max}$ as $\gamma \to \infty$ (see Proposition 5, part 2b) providing a generalisation of the type convergence observed in Example 1.

Recall that in the DeGroot case the terminal types are characterised by the average of initial types weighted by the eigenvector centrality, and the terminal average type determines if price converges. In that case, for arbitrarily small violations of the condition $g_0^{\max} < 1 + r$ to lead to price divergence we would need an eigenvector centrality that is arbitrarily close to 1 of the agent(s) with initial type equal $g_0^{\max}$, which can only be achieved if that agent forms the only closed and strongly connected set (i.e. for particular network structures). Instead, Proposition 2 holds for arbitrary networks.

3.2.2. Type convergence for (finite) $\gamma \geq 0$

Having derived conditions for price convergence, we can now turn our attention to the dynamics of types. Recall that $\gamma \geq 0$ is a measure of how much agents weigh in performance of others in their updating. Performance itself depends on the stochastic dividends and the type distribution of two periods ago when the investment decision was made. All this implies that the weights with which each agent updates from their neighbours are highly time-dependent and stochastic.

Although the weights change over time, updating of types is still a weighted average at each time step. Moreover, from (17), we know that the block structure will be preserved. Hence, as long as some of the weights do not converge to 0 (or jump between positive and 0 values), agents will always find a consensus in each closed and strongly connected group due to an ergodicity property. The following result makes use of this.

Proposition 3. *Suppose that $\lim_{t\to\infty} \tilde{p}_t$ exists and let $\gamma \in \mathbb{R}_+$. Then, $g_\infty^i := \lim_{t\to\infty} g_t^i$ exists and is such that for all closed and strongly connected groups $\mathcal{C}_k$ there exists $g_\infty^{\mathcal{C}_k} \in$*

[12]Recall that $g_{-1}^i = g_0^i$ for all $i \in \mathbb{N}$ is assumed as an initial condition and used here.

[13]See also Equation (20) and the accompanying discussion

$[g_0^{\min}(\mathcal{C}_k), g_0^{\max}(\mathcal{C}_k)]$ such that $g_\infty^i = g_\infty^{\mathcal{C}_k}$ for all $i \in \mathcal{C}_k$. If $\lim_{t\to\infty} \tilde{p}_t = 0$, then the beliefs in the rest of the world converge to $g_\infty^l = \sum_{k=1}^K \mathbf{e}_l \left(\mathbf{I} - \bar{\mathbf{A}}_{\mathcal{R}\mathcal{R}}\right)^{-1} \bar{\mathbf{A}}_{\mathcal{R}k} \mathbf{1}_k g_\infty^{\mathcal{C}_k}$ for all $l \in \mathcal{R}$ where $\mathbf{e}_l$ denotes the $l-th$ unit vector of dimension $|\mathcal{R}|$.

The proof of Proposition 3 rests on the fact that for bounded price, given γ (which is, hence, finite), and bounded dividends being drawn from the interval $[\bar{d} + d^-, \bar{d} + d^+]$, the profits u_t^i are bounded.[14] Hence, if i observes j in the network, i.e. $a_{ij} > 0$, then i will also update from j such that the corresponding updating weights in the law of motion can be bounded away from 0, i.e. there exists a $\delta > 0$ such that $\tilde{a}_{ij}(t) \geq \delta$ for all $t \in \mathbb{N}$. The resulting ergodicity property ensures convergence to a consensus.

Together with Proposition 2, we can conclude that price converges to the fundamental price and types converge to a consensus in each closed and strongly connected group, if there are initially no extremely strong chartists in the population, i.e. if $g_0^{\max} < 1 + r$. Because of the weighted-average nature of updating, the consensus is such that it is from the interval from min type to max type in each strongly connected group. As an additional remark to Proposition 3, note that the consensus in each strongly connected and closed group $g_\infty^{\mathcal{C}_k}$ will be *strictly* contained in this interval, i.e. $g_\infty^{\mathcal{C}_k} \in (g_0^{\min}(\mathcal{C}_k), g_0^{\max}(\mathcal{C}_k))$ for any (finite) $\gamma \in \mathbb{R}_+$ if $g_0^{\min}(\mathcal{C}_k) \neq g_0^{\max}(\mathcal{C}_k)$.

If the price converges to the fundamental price, then in the long-run there are no differences in performance between types since all types will use more and more similar forecasts once the price gets closer and closer to the fundamental price. Hence, in the long-run agents from the rest of the world will use a DeGroot type updating, implying that each link is given equal weight. As a result, the way the terminal type of each agent in the rest of the world is formed is the same as in Proposition 1, which is a weighted average of the consensuses in the closed and strongly connected groups which generically differs from other agents within the rest of the world.

Note that while Proposition 3 provides conditions that guarantee the existence of a long-run consensus in each closed and strongly connected group, it provides little guidance on the consensuses themselves. In fact, the consensuses are analytically intractable for finite performance parameter, though not when $\gamma \to \infty$. We therefore turn next to infinite performance effect $\gamma \to \infty$ and characterise terminal types in this case. We return to the case of finite performance effect after these results have been presented to allow us to 'inspect the mechanisms' behind analytical intractability of this case in some detail and show that in this case all feasible consensuses can be reached by just varying outside supply of shares X and performance feedback parameter γ (see Section 4).

3.3. $\gamma \to \infty$: Pure performance-based updating

Consider the polar case where performance becomes infinitely important, $\gamma \to \infty$. In this case, agents update from their best-performing neighbours, i.e. only the performance of

[14] The assumption that dividend shocks are drawn from a bounded interval comes in handy here. If, instead, this interval is allowed to be unbounded, then Proposition 3 will still hold almost surely.

those neighbours with maximal fitness matter. In particular, we obtain

$$(15) \quad \overset{\gamma\to\infty}{\Rightarrow} \quad g^i_{t+1} = \lim_{\gamma\to\infty} \frac{\sum_{j\in N^i} \exp(\gamma u^j_t) g^j_t}{\sum_{k\in N^i} \exp(\gamma u^k_t)} = \frac{1}{|U^{\max}_t(N^i)|} \sum_{j\in U^{\max}_t(N^i)} g^j_t. \tag{20}$$

When $\gamma \to \infty$, the network plays only a minor role: agents are still restricted to update from those they observe, but they only update from the best-performing agents within that set. Hence, we only have that $a_{ij} = 0 \Rightarrow \tilde{a}_{ij}(t) = 0$ for any $t \in \mathbb{N}$, but the other direction does not hold anymore for $\gamma \to \infty$.

Note that the case $\gamma \to \infty$ was not covered by Proposition 3. The reason is that convergence to consensus may not obtain if $\gamma \to \infty$ (even if price converges) because the crossing of the critical price (14) can potentially lead to cycling behaviour of types. In this section we first provide an example of non-convergence of types when $\gamma \to \infty$ (Section 3.3.1), before providing conditions for which types will converge, such that a consensus is reached in each closed and strongly connected group (Sections 3.3.2 – 3.3.3). For the latter, we find conditions for which no switch in the performance ranking occurs, such that we have monotonic convergence of the type dynamics in (20) and obtain analytically tractable results.

We assume throughout Section 3.3 that dividends are non-stochastic, though we later relax this assumption in a numerical example (see Section 3.3.4).

3.3.1. Non-convergence of types when $\gamma \to \infty$

Consider the undirected wheel network where each agent is connected to its successor and its predecessor.[15] Suppose $n = 5$ for which the wheel network is illustrated in Figure 1 omitting own links. Let initial types be given by $g_0 = (0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1)$. Further, we consider

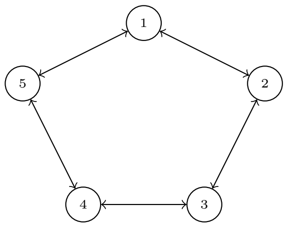

Figure 1: The wheel network for $n = 5$.

deterministic dividends $d_t = \bar{d}$ for all $t \in \mathbb{N}$. Finally, let $\tilde{p}_0$ be such that $\frac{1/2}{(1+r)^2-1/2}\frac{X}{n\delta} < \tilde{p}_0 < \frac{1+r}{(1+r)^2-1/6}\frac{X}{n\delta}$, which exists for any $r \geq 0$. Since $\bar{g}_0 = \frac{1}{2}$, the first inequality implies that $p_0 > p^{crit}_0$. Thus, by (14) initial return (relative to investment in the riskless asset) is negative, $R_0 < 0$, implying $u^i_0 > u^j_0$ if and only if $g^i_{-1} < g^j_{-1}$ by (13) such that more fundamental types (from period -1) are performing better. Since by assumption $g^i_{-1} = g^i_0$ for all $i \in N$, all agents adopt the minimal type from their neighbours in period 1 such that types become $g_1 = (0, 0, \frac{1}{3}, \frac{1}{2}, 0)$ with average type being given by $\bar{g}_1 = \frac{1}{6}$.

[15]The wheel network is defined by $a_{ij} = 1$ if and only if $|j - i| \in \{0, 1, n-1\}$.

Further, from (9) we get that $\tilde{p}_1 = \frac{1}{6(1+r)} p_0$. Above assumption $\tilde{p}_0 < \frac{1+r}{(1+r)^2-1/6}\frac{X}{n\delta}$ implies that $\tilde{p}_1 < \frac{1/6}{(1+r)^2-1/6}\frac{X}{n\delta} = p_1^{crit}$ which means that returns in period 1 are positive, i.e. $R_1 > 0$. Hence, $u_1^i > u_1^j$ if and only if $g_0^i > g_0^j$ by (13) such that more chartist types (from period 0) are performing better. Thus, all agents update by adopting the type (of period 1) from the neighbour with the maximal type in period 0. Thus, types become $g_2 = (0, \frac{1}{3}, \frac{1}{2}, 0, 0)$ with average type being unchanged at $\bar{g}_2 = \frac{1}{6}$.

Hence, $p_2^{crit} = p_1^{crit}$ and $\tilde{p}_2 = \frac{1}{6(1+r)}\tilde{p}_1 < p_2^{crit}$. Analogous arguments to above imply that $R_2 > 0$, hence, $u_2^i > u_2^j$ if and only if $g_1^i > g_1^j$. Agents, again, average over types of those neighbours with maximal type in period 1. The neighbours of agent 1 all have the same type in period 1, hence agent 1 takes the average over neighbour's types in period 2 such that $g_3^1 = (\frac{1}{3} + 0 + 0)\frac{1}{3}$. Agent 2 updates from agent 3's type in period 2 and all other agents update from agent 4's type in period 2 which results in the type vector $g_3 = ((\frac{1}{3})^2, \frac{1}{2}, 0, 0, 0)$. Inductive arguments lead to the types in any period t being given by:

$$g_3^i = \begin{cases} \frac{1}{3^{t-1}} & \text{if } i+t+1 \text{ is divisible by } 5 \\ \frac{1}{2} & \text{if } i+t \text{ is divisible by } 5 \\ 0 & \text{else} \end{cases}$$

Clearly, types do not converge to a consensus. Instead, oscillating dynamics emerge. The technical reason that Proposition 3 does not apply here is that for $\gamma \to \infty$ updating weights $\tilde{a}_{ij}(t)$ may converge to zero. As a result, the ergodicity property may fail to hold, and oscillating dynamics result from the switch in performance ranking.

We now provide conditions such that classical convergence to consensus is guaranteed when $\gamma \to \infty$ by ensuring that the critical price is not crossed.

3.3.2. Characterisation of Consensus Beliefs for Zero Outside Supply of Shares

For the case of zero outside supply of shares $X = 0$, characterising terminal types (and hence long-run price dynamics) turns out to be analytically tractable. Depending on the prevailing average type, either the more fundamental types are doing better in terms of fitness (if $\bar{g}_t < (1+r)^2$) or the more chartist types are (if $\bar{g}_t > (1+r)^2$). The technical reason for this is that when $X = 0$ the critical price is zero, such that no switch in performance ranking can occur in this case if average type stays below (above) the $(1+r)^2$ threshold; see (14). Intuitively, if the initially best-performing agents remain the best performers, then eventually their type should be adopted by all other agents along paths in the network. This result is shown for arbitrary network structures in Proposition 4.

Proposition 4. *Suppose $X = 0$. For $\gamma \to \infty$, we get the following*

1. *If $\bar{g}_0 < (1+r)^2$, then any agent adopts the most fundamental type on their path in finite time, i.e for all $i \in N$, $t \geq 2d(i, G_0^{\min}(P^i)) - 1$, we have $g_t^i \to g_0^{\min}(P^i)$.*

2. *If $\bar{g}_0 > (1+r)^2$, then any agent adopts the most chartist type on their path in finite time, i.e for all $i \in N$, $t \geq 2d(i, G_0^{\max}(P^i)) - 1$, we have $g_t^i \to g_0^{\max}(P^i)$.*

3. *Price converges to the fundamental price if $\bar{g}_0 < (1+r)^2$ and $\sum_{i \in N} g_0^{\min}(P^i) < n(1+r)$, price converges (to some finite limit) if $\bar{g}_0 < (1+r)^2$ and $\sum_{i \in N} g_0^{\min}(P^i) = n(1+r)$, and price diverges if $\bar{g}_0 < (1+r)^2$ and $\sum_{i \in N} g_0^{\min}(P^i) > n(1+r)$, or $\bar{g}_0 > (1+r)^2$.*

With Proposition 4 we can compare the case $\gamma \to \infty$ with the other polar case ($\gamma = 0$) that we already studied in Proposition 1 to isolate the network effect from the performance effect. Here, instead, fitness becomes infinitely important and agents only update from the best-performing agents in their network, i.e. to whom there exist a path.

Who the best-performing agents are, depends on the average type in the population. As explained above, if $\bar{g}_t < (1+r)^2$, then the best-performing agents are those with minimal type (with a one period lag), and vice versa for $\bar{g}_t > (1+r)^2$. Hence, if initial average type is above $(1+r)^2$, then higher types initially perform better, implying that the maximal types in each neighbourhood are adopted. This means that average type is not decreasing, such that $\bar{g}_t > (1+r)^2$ for all $t \in N$. Intuitively speaking, strong chartist beliefs become reinforcing in the sense that strong chartists expect the price to move away from the fundamental price which indeed happens if there are sufficiently many strong chartists, i.e. if the initial average type is large. As a result, types converge to the maximal type in each group and price diverges for any network structure.

On the other hand, if initial average type is small enough, then fundamental expectations are doing better and these beliefs become reinforcing. Note that price may still diverge in this case (which occurs for sure if initial average type is below $(1+r)^2$ and the initial minimal type is above $(1+r)$), but more fundamental expectations are still yielding a higher fitness in every period, ensuring convergence to the minimal type. For a strongly connected network, this implies that the globally minimal type will be adopted by all other agents in finite time since for every agent $i \in N$ there exists a path to any other agent, i.e. $P^i = N$. Hence, only a subset of initial types matter for the consensus and hence for whether price converges. One implication is that if investors focus strongly on performance, then one weak enough chartist (type $< 1+r$ is enough) is enough to stabilise asset prices, regardless of the network centrality of that agent.

Clearly, the same considerations outlined above hold within each closed and strongly connected group. These groups then converge to one of the extreme types from their group which is summarized in the following Corollary.

Corollary 1. *Suppose $X = 0$. For $\gamma \to \infty$, we get the following:*

- *If $\bar{g}_0 < (1+r)^2$, then any closed and strongly connected group $\mathcal{C}$ forms a consensus on the most fundamental type from their group in finite time, i.e. $g_t^i \to g_0^{\min}(\mathcal{C})$ for all $t \geq 2D(\mathbf{A}_{\mathcal{C}}) - 1$, $i \in \mathcal{C}$.*

- *If $\bar{g}_0 > (1+r)^2$, then any closed and strongly connected group $\mathcal{C}$ forms a consensus on the most chartist type from their group in finite time, i.e. $g_t^i \to g_0^{\max}(\mathcal{C})$ for all $t \geq 2D(\mathbf{A}_{\mathcal{C}}) - 1$, $i \in \mathcal{C}$.*

Moreover, each agent in the rest of the world $i \in \mathcal{R}$ will adopt a type that is given by the best-performing consensus out of the closed and strongly connected groups to which i

has a path. Clearly, agents from the rest of the world may obtain heterogeneous long-run types since they may be connected to different closed and strongly connected groups.

If the initial average type is small enough, then the terminal average type will depend on the minimum types reached within each strongly connected and closed group and the population shares. In this case, a single group will be pivotal for the terminal average type (and thus whether price converges) if the minimum type in the group is smaller than $(1+r)$ and the group has a sufficiently large population share to make the average terminal type also smaller than $(1+r)$.

In a strongly connected network, the best-performing type (which is either the globally minimal or maximal initial type) will be adopted by the entire population. Hence, contrary to the case of pure social updating, network centrality plays no role for terminal type if agents only pay attention to performance. The network structure does affect, however, time to convergence which is finite and does not exceed the threshold $2D(\mathbf{A})-1$. The reason is the following. If the best-performing type is adopted by some agent $i \in N$ at time $t \in \mathbb{N}$, then all agents j who directly observe i (such that $a_{ji}=1$) will have adopted this type themselves at latest by time step $t+2$ and will remain with this type forever. Since the first updating occurs in period 1, the maximal convergence time is given by the twice the length of the longest path in the network reduced by 1, i.e. $2D(\mathbf{A})-1$ where $D(\mathbf{A})$ is the diameter of the network. If all agents observe all other agents $a_{ij}=1$ for all $i,j \in N$ (i.e. a complete network), then convergence will obtain after 1 period.

Lastly, note that if we have a strongly connected network where some initial type is a pure fundamentalist and we are in Part 1 of Proposition 4, then we will have convergence on the pure fundamental type in at most $2D(\mathbf{A})-1$ periods, and hence also convergence to the fundamental price in at most $2D(\mathbf{A})-1$ periods, such that mispricing is eliminated in finite time. This result speaks to a common notion that stock markets are inefficient in the short run but efficient in the long-run – but with the added observation that the network influences how quickly mispricing is eliminated via the network diameter $D(\mathbf{A})$. If instead, there exists a strongly connected and closed group that does not have a pure fundamental investor, then convergence to the fundamental price is not possible in finite time.

3.3.3. Partial Characterisation of Consensus Beliefs for Positive Outside Supply

We were able to characterise the long-run properties and bound the time to convergence in the zero outside supply case for $\gamma \to \infty$ since no switch in performance ranking occurs. Similarly, we now derive conditions for the positive outside supply case such that the critical price is not crossed. If the initial price deviation has the opposite sign as the critical price, then critical price will never be crossed. Otherwise, assuming the initial price deviation to be small enough (in absolute value) or large enough also ensures that the critical price will not be crossed. Putting things together, we are in a position to derive a result for $\gamma \to \infty$.

Proposition 5. *Suppose $X > 0$. For $\gamma \to \infty$, we get the following:*

1. *A closed and strongly connected group $\mathcal{C}$ forms a consensus on the most fundamental type in finite time, i.e. $g_t^i \to g_0^{\min}(\mathcal{C})$ for all $t \geq 2D(\mathbf{A}_{\mathcal{C}})-1$, if one of the following conditions is satisfied:*

(a) $\bar{g}_0 < (1+r)^2$ and $\tilde{p}_0 < 0$, or

(b) $g_0^{\min} > (1+r)^2$ and $\frac{(1+r)^{2D(\mathbf{A}_\mathcal{C})-1}}{(g_0^{\max})^{2D(\mathbf{A}_\mathcal{C})-2}((1+r)^2-g_0^{\max})}\frac{X}{n\delta} \leq \tilde{p}_0 < 0$, or

(c) $g_0^{\max} < (1+r)^2$ and $\frac{(1+r)^{2D(\mathbf{A}_\mathcal{C})-1}}{(g_0^{\min^*})^{2D(\mathbf{A}_\mathcal{C})-2}((1+r)^2-g_0^{\max})}\frac{X}{n\delta} \leq \tilde{p}_0$ where $g_0^{\min^*} := \min\{1+r, g_0^{\min}\}$.

2. *A closed and strongly connected group $\mathcal{C}$ forms a consensus on the most chartist type in finite time, i.e. $g_t^i \to g_0^{\max}(\mathcal{C})$ for all $t \geq 2D(\mathbf{A}_\mathcal{C})-1$, if one of the following conditions is satisfied:*

   (a) $\bar{g}_0 > (1+r)^2$ and $\tilde{p}_0 > 0$, or

   (b) $g_0^{\max} < (1+r)^2$ and $0 < \tilde{p}_0 \leq \frac{1}{(1+r)^{2D(\mathbf{A}_\mathcal{C})-3}((1+r)^2-g_0^{\min})}\frac{X}{n\delta}$, or

   (c) $g_0^{\min} > (1+r)^2$ and $\tilde{p}_0 \leq -\frac{1+r}{g_0^{\min}-(1+r)^2}\frac{X}{n\delta}$.

3. *The rest of the world adopts the most fundamental type on their path, $g_t^j \to g_0^{\min}(P^j)$ for all $j \in \mathcal{R}$, $t \geq 2d(j, G_0^{\min}(P^j)) - 1$ if $\bar{g}_0 < (1+r)^2$ and $\tilde{p}_0 < 0$, while it adopts the most chartist type on their path, $g_t^j \to g_0^{\max}(P^j)$ for all $j \in \mathcal{R}$, $t \geq 2d(j, G_0^{\max}(P^j)) - 1$ if $\bar{g}_0 > (1+r)^2$ and $\tilde{p}_0 > 0$.*

The cases 1a and 2a of Proposition 5 are analogous to the case of zero outside supply of shares when $\gamma \to \infty$; see Proposition 4. If $\bar{g}_0 < (1+r)^2$ and $\tilde{p}_0 < 0$, then types will never increase and since the price deviation stays negative, no switch in performance ranking ever occurs. Similarly if $\bar{g}_0 > (1+r)^2$ and $\tilde{p}_0 > 0$, then types will never decrease, and again no switch in performance ranking occurs. In both cases we get convergence to consensus for closed and strongly connected groups in finite time where time to convergence must be smaller than twice the diameter of the group. In this case, the rest of the world adopts the most fundamental (resp. most chartist) type.

To ensure that no switch in performance ranking occurs in the remaining cases, we assume for small enough initial types ($g_0^{\max} < (1+r)^2$) and positive price deviation ($\tilde{p}_0 > 0$) that the latter is either small enough (case 2b) or large enough (case 1c) such that no switch in performance ranking occurs before types in a closed and strongly connected group form a consensus. If price deviation stays below the critical price for sufficiently many time periods, then agents will converge on the most chartist type (case 2b). Similarly, if price deviation stays above the critical price for sufficiently many time periods, then agents form a consensus on the most fundamental type (case 1c). The assumption of $g_0^{\max} < (1+r)^2$ is needed to ensure that no switch in performance ranking occurs by crossing the threshold $(1+r)^2$. Analogous considerations hold for $g_0^{\min} > (1+r)^2$ and $\tilde{p}_0 < 0$ (cases 1b and 2c).

The cases are not exhaustive because it is not analytically tractable to characterise the long term convergence properties if a switch in performance ranking occurs, even in the case of $\gamma \to \infty$. The rest of the world only adopts the minimum (resp. the maximum type) if we can ensure that performance ranking does not switch for any $t \in \mathbb{N}$. This is only true if we are in case 1a (resp. 2a) of Proposition 5.

Though the most fundamental types are the only survivors of evolutionary competition if the outside supply of shares is zero and price converges to the fundamental price, Proposition 5 shows that this need not be the case when shares are in positive outside supply. Thus, in general, price stability can coexist with convergence on the most chartist type. To put it differently, fundamental beliefs can be driven out of the market even if the price converges on the fundamental price as in Proposition 5, case 2b.

Whether price converges to the fundamental price depends on whether the terminal average type is smaller than $(1 + r)$, and in a strongly connected network the (terminal) consensus type will be reached in at most $2D(\mathbf{A})-1$ periods for all the cases in Proposition 5 since the only closed and strongly connected group is the entire set of agents, $\mathcal{C} = N$. Hence, for positive outside supply of shares, we still have that consensus on a pure fundamental type is possible in finite time (at most $2D(\mathbf{A}) - 1$ periods), and in such cases price is guaranteed to equal the rational expectations solution (fundamental price) for all $t \geq 2D(\mathbf{A}) - 1$.

It is worth pointing out that in all the cases in Proposition 5, we only get convergence on one of the extreme types. Clearly, for any finite $\gamma \geq 0$, the consensus of each group will lie between the extreme types (if these do not coincide), as shown by Proposition 3. Moreover, if either $\tilde{p}_0 < 0$ and $g_0^{\min} > (1+r)^2$ or $\tilde{p}_0 > 0$ and $g_0^{\max} < (1+r)^2$, then it is possible to move between the two extreme consensuses by just altering the outside supply of shares. Since the consensus is continuous in γ (as the limit of continuous linear mappings), this implies that by just varying performance parameter γ and the outside supply of shares X, we are able to generate all feasible consensuses, i.e. all consensuses which are included in convex hull of the initial types. In this sense, this can be interpreted as an ‘anything goes’ result.

3.3.4. The Impact of Stochastic Dividends for $\gamma \to \infty$

To characterize the terminal types in Propositions 4 and 5 we assumed dividends were non-stochastic. Clearly, for small enough variance of dividend shocks, both results should be robust. We demonstrate this and study the effect of stochastic dividends in this section.

Consider a wheel network as in Section 3.3.1 with $n = 10$ agents and the initial types given by $g_0 = \left(0, \frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}, \frac{9}{10}, 1\right)$. We let $\gamma \to \infty$ and consider the case of stochastic dividends, with the shocks ε_t drawn from a truncated-normal distribution with standard deviation σ_d. We set $X = 0.25$, $r = 0.04$, $\phi = 0.4$ and $\overline{d} = 0.5$. We truncate the normal distribution to the interval $[-\overline{d}, \overline{d}]$, which guarantees dividends are non-negative.

In Figure 2 we show the impact of increasing the shock variance, starting from a very low value. We plot both the consensus (left panels) and the period in which the consensus was reached (right panels) for 200 different sequences of dividend shocks; we consider both positive initial price deviations (blue) and negative initial price deviations (orange) because our analytical results in Proposition 5 suggest that whether the asset is initially overvalued or undervalued matters for the consensus type.

For sufficiently low variance (top panel), we see that the predictions of Proposition 5 hold: consensus is reached in 6 periods in all simulations and for positive initial price deviation the consensus is the pure chartist type (Proposition 5, 2a), while for negative initial price deviation the consensus is the minimal type (Proposition 5, 1a), i.e. the pure fundamentalist. This is intuitive since the *sign* of the realized return – which determines the fitness ranking of agents – will be unaffected for sufficiently small shocks (see (12)–(13)). As variance

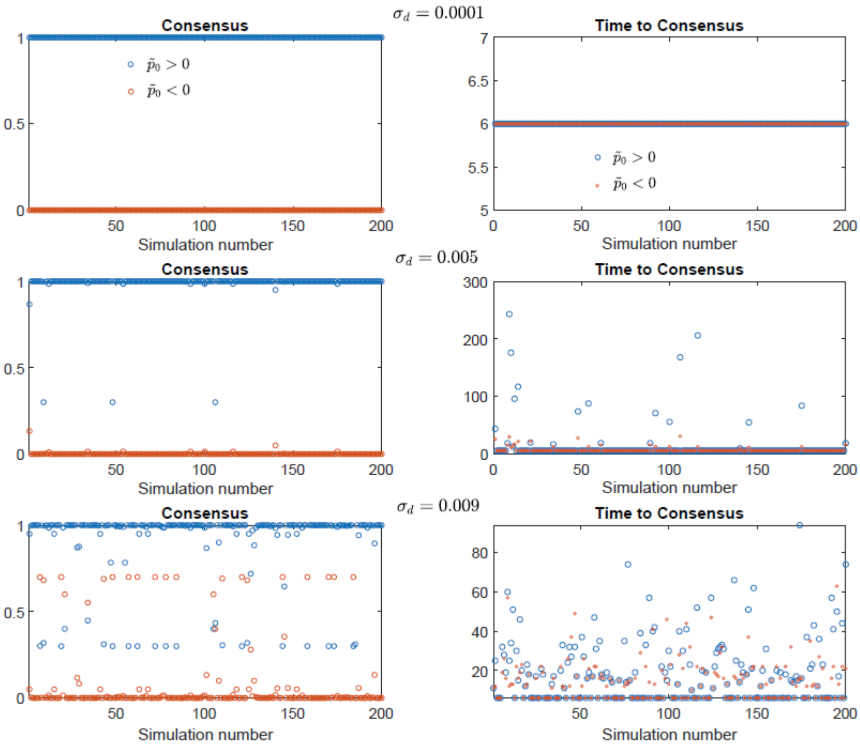

Figure 2: Consensus and speed of agreement in the wheel network when there are dividend shocks

is increased, the consensus and date of agreement deviate, in some simulations, from the predictions of Proposition 5 for the deterministic case. For instance, while the bulk of simulations have consensus at the polar types, there are several exceptions for which the consensus is a mixed chartist-fundamentalist type, and in some cases the consensus is closer to the opposite pole, showing a substantial shift in opinion (left, middle and bottom panels).

Time to consensus is much higher in some cases, and amounts to hundreds of periods for some simulations with intermediate variance (middle right). There does not appear to be any clear-cut relationship between time to consensus and shock variance, but the *maximum* time to consensus is highest in the intermediate variance case (cf. middle and bottom right). A potential explanation is as follows: if variance takes on intermediate values, then performance ranking will switch with low probability. Hence, with some probability this may happen in the first period and then not for some time, which may lead to alternating dynamics similar to Example 3, implying slow convergence rates. As variance is increased, the probability to update from different types increases, leading to less extreme convergence times.

For consensus the relationship is clearer: we see many more examples of consensus away from the poles in the highest variance case (bottom left), as well as substantial deviations of the consensus type from the deterministic case in Proposition 5. Even for small variances, the consensus becomes very difficult to predict. In short, both type and price dynamics may be disturbed materially by small shocks if investors are strongly focused on performance.

4. Positive performance effect $\gamma > 0$: Inspecting the mechanisms

In this section we explain two key mechanisms underlying the type dynamics and elaborate on the main obstacles for analytical tractability in the case of a finite but positive performance parameter γ. In this case, both the network effect and the performance effect are present when updating, i.e. higher weight will be given to better performing agents, but some weight is also given to any connected agent. We first explain how a *reversal of performance ranking* due to crossing of the critical price impacts on the dynamics and then we illustrate a surprising effect that occurs in some network structures, through a mechanism that we refer to as a *lagged network structure effect* (since it relates to the lagged updating of types). The combination of both these effects may lead to an anything goes result.

4.1. Reversal of performance ranking

The critical price in (14) generally plays a role in the type dynamics whenever $X > 0$ and $\gamma > 0$ – and it is also the reason that for $\gamma \to \infty$ it was necessary to bound the initial price in the case where supply is positive (see Section 3.3.3). To better understand the role of the critical price, consider the standard case of not excessively many strong chartists, such that $\bar{g}_t < (1+r)^2$, and take dividends as non-stochastic: $\varepsilon_t = 0$. Then, $p_t^{crit} > 0$ and, hence, returns are positive $R_t > 0$ if and only if $\tilde{p}_t < p_t^{crit}$. In this case, (13) implies that low types are doing better for $\tilde{p}_t > p_t^{crit}$ or $\tilde{p}_t < 0$ while high types are performing better if $0 < \tilde{p}_t < p_t^{crit}$. The other case where there are many strong chartists such that $\bar{g}_t > (1+r)^2$ is opposite: $p_t^{crit} < 0$ is implied and, hence, returns are positive $R_t > 0$ if and only if $\tilde{p}_t > p_t^{crit}$. Observing fitness given by (13) implies that low types are doing better for $\tilde{p}_t < p_t^{crit}$ or $\tilde{p}_t > 0$ while high types perform better if $p_t^{crit} < \tilde{p}_t < 0$.

To interpret this, suppose that the price deviation from the fundamental price $\tilde{p}_t$ is positive and $\bar{g}_t < (1+r)$, such that the asset is overvalued and there are not excessively many strong chartists. In this case, the price deviation from the fundamental price $\tilde{p}_t$ will decrease (see (9)) such that asset price p_t also decreases and approaches the fundamental price from above. Therefore, higher types are more optimistic than lower types at any point in time, meaning that more chartist types buy more of the risky asset. The intuition is that with decreasing asset price, the returns from investing in stocks are smaller than the returns from investing in the riskless asset, i.e. $\frac{p_t+\bar{d}}{p_{t-1}} - 1 < r$ and lower types are performing better. Note that when the outside supply of shares is zero, i.e. $X = 0$, we simply have that the returns from investing in the asset approach $\frac{\bar{d}}{p^f}$ from below where $\frac{\bar{d}}{p^f} = r$, implying that low types (i.e. more fundamental) always perform better.

Now consider the case of positive outside supply of shares. The returns from investing in the asset become $\frac{\bar{d}}{p^f} = \frac{\bar{d}}{\frac{\bar{d}}{r} - \frac{X}{n\delta}} > r$ when price approaches the fundamental price since the positive outside supply lowers the fundamental price. Hence, when the asset price approaches the fundamental price from above, it is actually the high types who perform better. The critical price where such a switch in performance occurs is given by (14). Note that if we had instead assumed a negative price deviation, then the critical price (which is positive) would not have been crossed. Therefore, if supply is positive, the price and type dynamics can be highly asymmetric to whether the asset is initially overvalued or undervalued.

The difference between critical price p_t^{crit} defined in (14) and the price deviation $\tilde{p}_t$ determines the sign of the returns and, hence, the performance ranking. That performance ranking can switch makes it very difficult to derive analytical results for the long-run consensus. A switch in performance ranking may even happen multiple times when the price is strictly decreasing, because the critical price changes over time with the average type; an example where price crossed the critical price multiple times is provided in Example 2.

Example 2. *To see that the price can cross the critical price multiple times, consider the following network being defined by* $\mathbf{A}$ *given in Figure 3 for large* n.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 & 0 \\ 0 & 1 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 1 & 0 \\ 0 & 1 & \dots & 1 & 1 \\ 0 & \dots & 0 & 1 & 1 \end{bmatrix}$$

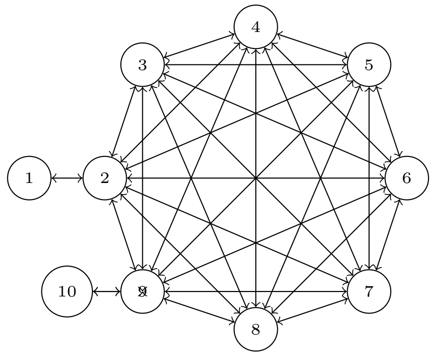

Figure 3: The network configuration for Appendix 2 defined by the matrix $\mathbf{A}$ for arbitrary n and illustrated as a graph for $n = 10$ omitting the self loops.

Let the initial types be $g_0 = (\varepsilon, 1, 1-\varepsilon, \dots, 1-\varepsilon)'$ *such that* $\bar{g}_0 \to 1-\varepsilon$ *for large* n. *Suppose that initial price satisfies* $\left(\max\left\{\frac{(1-\varepsilon)}{(1+r)^2-(1-\varepsilon)}, \frac{(1+r)^2}{((1+r)^2-\varepsilon)(1-\varepsilon)}\right\}\right)\frac{X}{\delta n} < \tilde{p}_0 < \frac{1+r}{(1+r)^2-(1-\varepsilon)}\frac{X}{\delta n}$. *For sufficiently small* $\varepsilon > 0$ *such a* $\tilde{p}_0$ *exists.*

Since by assumption $\tilde{p}_0 > \frac{(1-\varepsilon)}{(1+r)^2-(1-\varepsilon)}\frac{X}{\delta n}$, *we get that* $\tilde{p}_0 > p_0^{crit}$ *for large* n. *Hence,* $R_0 < 0$ *and, therefore, the performance ranking favours low types. The best-performing agent in period* 0 *is the one with minimal type, which is agent* 1. *For* $\gamma \to \infty$, *agent* 2 *will adopt agent* 1*'s type while all others are only connected to others with minimal type equal to* $1-\varepsilon$.

Hence, $g_1 = (\varepsilon, \varepsilon, 1-\varepsilon, \dots 1-\varepsilon)'$ *implying* $\bar{g}_1 \to 1-\varepsilon$ *for* n *large, and, therefore,* $p_1^{crit} \to p_0^{crit}$. *From* (9), *we get* $\tilde{p}_1 = \frac{\bar{g}_1}{1+r}\tilde{p}_0$, *implying that* $\tilde{p}_1 < p_1^{crit}$ *since* $\tilde{p}_0 < \frac{1+r}{(1+r)^2-(1-\varepsilon)}\frac{X}{\delta n}$ *by assumption. Hence,* $R_1 > 0$ *and, therefore, the performance ranking now favours high types. Thus, the best-performing agent in period 1 is the one with maximal type in period 0, which is agent* 2. *For* $\gamma \to \infty$, *all traders other than trader* n *will adopt agent* 2*'s type from period* 1 *in period* 2 *since all are connected to agent* 2 *except for agent* n.

Hence, $g_2 = (\varepsilon, \dots, \varepsilon, 1-\varepsilon)'$ *implying* $\bar{g}_2 \to \varepsilon$, *and, therefore,* $p_2^{crit} \to \frac{\varepsilon}{(1+r)^2-\varepsilon}\frac{X}{\delta n}$ *for* $n \to \infty$. *From* (9), *we get* $\tilde{p}_2 = \frac{\bar{g}_2\bar{g}_1}{(1+r)^2}\tilde{p}_0$, *implying that* $\tilde{p}_2 > p_2^{crit}$ *since* $\tilde{p}_0 > \frac{(1+r)^2}{((1+r)^2-\varepsilon)(1-\varepsilon)}\frac{X}{\delta n}$ *by assumption. The price crosses the critical value therefore at least three times in this example since it will eventually fall below the critical value again.*

4.2. Lagged network structure effect: non-monotonic consensus and 'anything goes'

An obvious problem for analytically characterising a consensus (if existing) is when the critical price is crossed since performance ranking is reversed; see (14). To rule this out, we may assume deterministic dividends $d_t = \bar{d}$ for all $t \in \mathbb{N}$ and zero outside supply of shares $X = 0$. Then by (14) the critical price is zero and fitness given by (13) thus simplifies to

$$X = 0 \;\Rightarrow\quad u_t^i = \delta\left(\bar{g}_t - (1+r)^2\right)\left(\frac{g_{t-1}^i}{\bar{g}_{t-1}} - 1\right)\left(\tilde{p}_{t-1}\right)^2 \tag{21}$$

which is obtained using (9).

Here, we have the very clear relation between types and performance that we exploited in Proposition 4: if average type $\bar{g}_t$ is below $(1+r)^2$, then those agents which had more fundamental (i.e. lower) types in period $t-1$ perform better, while the reverse holds for average type above $(1+r)^2$. Intuitively, one may expect to see that types then decrease with increasing performance feedback parameter γ. Indeed, this would be the case if fitness relates proportionally to types within the same period. However, because of the one-period lag between investment and realisation of profits, this relation may not hold in certain network structures.

To illustrate this point we employ a bipartite network with the following structure: there are $n = k+l$ traders, initial chartists $C = \{1, ..., k\}$ and fundamentalists $F = \{k+1, ..., k+l\}$, such that initial types are given by $g_0^i = 1$ for all $i \in C$ and $g_0^j = 0$ for all $j \in F$. Let the network be given by the complete bipartite network with loops,[16] such that each agent listens to themselves and those agents with a different initial type, as illustrated in Figure 4.[17]

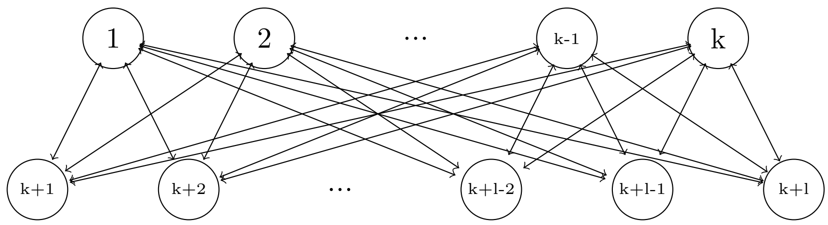

Figure 4: The complete bipartite network of the two sets $C = \{1, ..., k\}$ and $F = \{k+1, ..., k+l\}$.

Example 3. *First, consider the symmetric case of the complete bipartite network in Figure 4 with equal population shares of initial chartists and fundamentilists $|C| = k = |F| = l = 5$ (and hence $n = 10$). Let the exogenous initial conditions be $X = 0$, $\delta = 2$, $r = 0.04$, $d_t = \bar{d} = 0.1$ for all t, and $p_0 = 2.75$. These values imply a fundamental price $p^f = 2.5$, such that the initial price deviation is $\tilde{p}_0 = 0.25 > 0$. Since there are only two distinct initial types and all traders of the same type use the same updating rule, there are only two distinct types at any point in time (for any γ). The dynamics of these two distinct individual types g_t^i, average type $\bar{g}_t$, and price deviation $\tilde{p}_t$ are displayed in Figure 5 for different values of γ.*

In the case where $\gamma = 0$ (first column of Figure 5), fitness has no effect on updating. The alternation of the type dynamics is entirely caused by the network because each agent mainly

[16]Formally the complete bipartite network with loops between sets C and F is defined as $g_{ij} = 1$ if and only if $i \in C$ and $j \in F$ or $j \in C$ and $i \in F$ or $i = j$

[17]Note that for all networks drawn in this paper, we have omitted the self loops from each agent to themselves for the sake of simplicity.

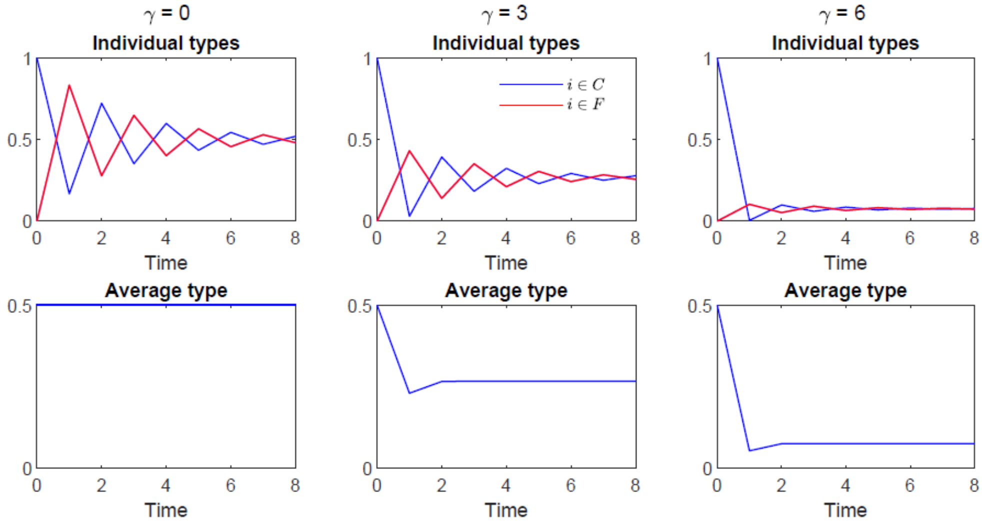

Figure 5: Dynamics of individual types, average type and price in the bipartite network for various γ

updates from the other type of agents (as well as herself). Since the alternation in individual types is symmetric, the average type remains constant at $\bar{g}_t = 0.5$ at any time $t \in \mathbb{N}$. From Proposition 1, we already know that the type dynamics converge to a consensus and because each agent is in a symmetric network position, the eigenvector centralities are the same for all agents. Hence, consensus is obtained on $\lim_{t\to\infty} g_t^i = 0.5$ for all $i \in N$.

For increasing values of γ, we observe several effects. First, there is an initial decrease in average type. Compared to the type dynamics for $\gamma = 0$, the low types increase by a smaller magnitude and the high types decrease by a greater magnitude for increasing values of γ in period 1. This is because more fundamental types receive higher fitness and this is adopted with higher weight for larger values of γ since types are the same in the initial two periods.

Second, from period 1 on, average type increases although it is still that the low types receive higher fitness. The reason can be found in the above described effect that individuals update with a one period lag. Agents who had low types in period $t-1$ now have high types in period t because of the bipartite nature of the network. Finally, we observe that the oscillation of type dynamics flattens for higher values of γ. This is because the initial decrease in average type is greater for higher values of γ such that the reverse effect from period 1 on becomes smaller and smaller since types are already very low by period 2.

The drift towards more chartist types from period 1 is due to the 'lagged network structure effect' mentioned at the start of this section. In particular, since types are updated with higher weights to neighbours with higher past fitness, higher weight is given to those investors who were more successful in period $t-1$. Figure 5 illustrates that there exist network structures such that the resulting updating weights are inconsistent with the performance ranking, such that average type drifts towards worse performing types for $t \geq 1$.

Given that direction of average type can be reversed, it is natural to ask whether reversals could 'overshoot', such that stronger focus on performance (higher γ) could imply a higher average type than when performance is ignored (i.e. $\gamma = 0$). We now show that this happens when we change the population shares in Example 3, such that the consensus becomes non-monotonic with respect to the performance feedback parameter.

Example 3 (continued). *Now consider the bipartite network in Figure 4 where the initial chartists have a higher population share such that $|C| = k = 16$ and $|F| = l = 4$ (and hence $n = 20$). Let the initial conditions be the same as in Example 3. The dynamics of the average type $\bar{g}_t$ for $\gamma = 0, 20$ and how the consensus varies with γ are displayed in Figure 6.*

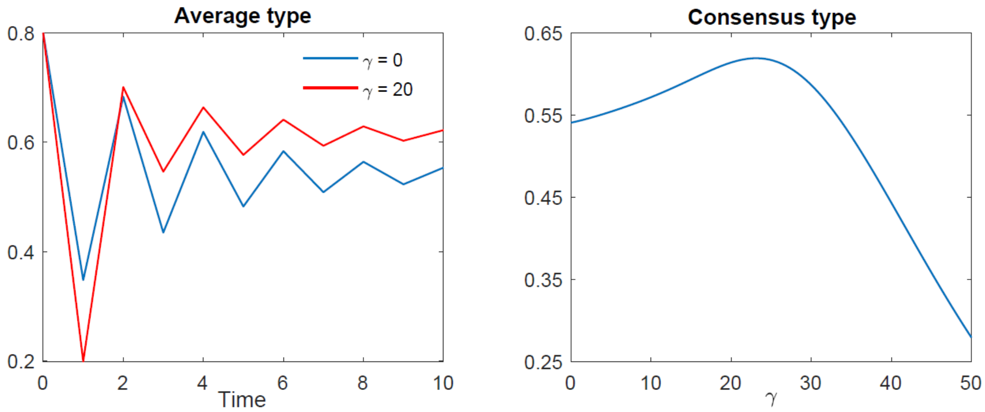

Figure 6: Left: Dynamics of average type for $\gamma = 0$ (blue) and $\gamma = 20$ (red). Right: Consensus versus γ.

In this example, average type alternates (contrary to the symmetric case in Figure 5). This is again due to the network structure since such alternating dynamics of the average type are observed for $\gamma = 0$ in the left hand panel of Figure 6. The initial drop in average type is amplified when the performance feedback parameter increases with the same explanation as above. From period 1 on, however, more weight is given to the worse performing agents for $\gamma = 20$ compared to $\gamma = 0$ (because of the lagged network effect described in above); as a result, average type drifts more towards chartist types when $\gamma = 20$, despite the fact that fundamentalists are outperforming chartists at any point in time.

The right hand panel of Figure 6 shows how consensus varies in this example with the performance feedback parameter γ. Clearly, the impact of the performance feedback parameter on consensus is non-monotonic, and suggests that behavioural agents concerned with performance may in fact settle on worse performing (higher) types. This counter-intuitive result arises because by combining repeated-average updating with a fitness measure (past profitability) that depends on previous types, our agents fail to internalise the channel that 'imitating those who imitate you' can be problematic (from a performance perspective) if there are substantive differences in type.

In Figure 6, we see that this problem is corrected for high enough values of γ (e.g. > 25) since the initial downwards kick outweighs the type reversals lagged network effect. We have seen in Proposition 4 that for $\gamma \to \infty$ consensus will be attained on best-performing type (i.e the fundamental type in this example), since the lagged network effect is absent.

The parameter setting of Example 3 producing Figure 6 makes clear that, with certain network structures, worse performing types may be adopted with higher weight when the performance feedback parameter γ increases but remains finite, such that the consensus may drift away from the best-performing type. Clearly, this result is not restricted to bipartite networks, though more clustered networks will be less prone to such type reversals. Real world networks usually do exhibit some clustering and will less likely observe such a theoretic effect. However, in light of these examples, general (analytical) results on the relation between consensus and performance feedback parameter γ are not possible.

To illustrate the variety of possible consensuses even more in the bipartite network, we now also allow the supply of shares to be positive, i.e. $X > 0$, such that crossing of the critical price (and hence a reversal of performance ranking) becomes possible, and we study the implications for the dynamics of the average type and the long-run consensus.

Example 3 (continued). *We now return to the symmetric bipartite network previously defined with* $k = l = 5$*, except that we allow supply* X *to be positive, such that there is a positive critical price:* $p_t^{crit} := \frac{\bar{g}_t}{(1+r)^2-\bar{g}_t} \cdot \frac{X}{\delta n} > 0$*; see* (14)*. All other parameters values are unchanged and we keep the initial price deviation fixed at* $\tilde{p}_0 = 0.25$ *as in Example 3. We investigate numerically how consensus is influenced by the performance feedback parameter* γ *at different values of the outside supply of shares* X*; see Figure 7.*

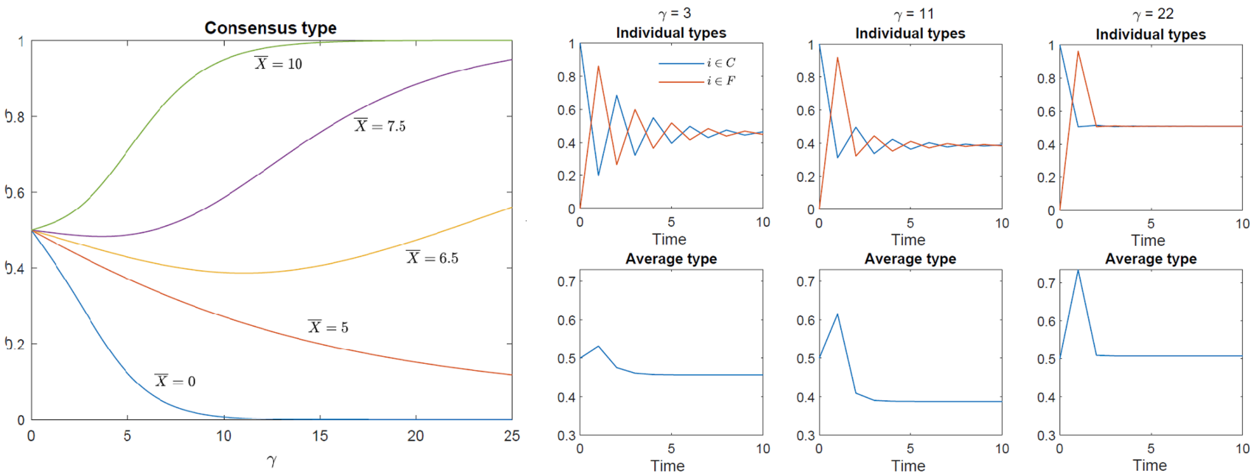

Figure 7: Left: Long-run consensus varying with X and γ. Right: Dynamics of individual and average types for $X = 6.5$ and different values of γ.

When $X = 0$*, increasing* γ *lowers the long-run consensus (Figure 7 left, blue line) as previously shown in Example 3. Hence, for high enough* γ*, we see a consensus at the pure fundamental type of 0. For sufficiently small* $X > 0$*, we see a similar pattern: consensus falls as the performance effect is strengthened and approaches 0 for sufficiently high* γ *(red line,* $X = 5$*); intuitively, this is because there is no reversal in performance ranking and hence higher values of* γ *lower the average type at date 1 as well as flattening the subsequent (alternating) type dynamics (analogous to Figure 5).*

However, as X *is increased further, the increase in net supply raises* p_0^{crit} *somewhat relative to the initial price deviation* $\tilde{p}_0 = 0.25$*. At* $X \approx 5.8$ *the initial return* R_0 *switches from negative to positive and hence for higher supply, the initial update instead favours the chartists* C *(see* (14) *and the dicussion that follows). Thus, if outside supply is sufficiently high, the type consensus increases as* γ *is increased and converges on the chartist type of 1 when the performance effect is sufficiently strong (Figure 7 left, green line).*

The most interesting cases occur at intermediate values of X such that there is a non-monotonic relationship between consensus and the performance feedback parameter. For $X = 7.5$ (Figure 7, purple line), consensus initially falls as γ is increased, but then increases once sufficiently large values of γ are reached. For $X = 6.5$ we instead see a U-shape because, even at intermediate values of γ, the initial update is not so central in determining the consensus, since it can be reversed or offset by the subsequent type dynamics.

We examine the case $X = 6.5$ in more detail in the right side of Figure 7. The U-shape relationship for the consensus arises because, although the initial update increases the average type (the initial return R_0 is positive and hence $u_0^i > u_0^j$ if and only if $g_0^i > g_0^j$), a modest increase in average type in period 1 can be more than reversed in subsequent periods such that average type eventually falls below its starting value of 0.5 (see $\gamma = 3$, left panel); we see a similar pattern for $\gamma = 11$ (middle panel), except that the changes in average type are exacerbated.

These results are related to the additional lag in updating weights, i.e. the network structure effect mentioned above. Hence, both this and the performance ranking effect can play an important role in shaping the consensus when both effects are present. Finally, note that as stated below Proposition 3, the consensus is strictly contained in the interval between maximal and minimal initial type, in contrast to the result when $\gamma \to \infty$ (see Proposition 5).

Example 3 illustrates the two effects at play here: even when the critical price is not crossed, the lagged updating may lead to type reversals for some networks structures implying that consensus is in general not monotonic with respect to performance feedback. When additionally the ranking of types is reversed due to crossing of the critical price, which is possible if supply X is positive, then Figure 7 shows that all achievable consensuses (i.e. that are included in the convex hull of initial types) can be achieved by just varying the outside supply of shares X and the performance feedback parameter γ. Thus, Example 3 illustrates the 'anything goes'-result discussed below Proposition 5, and there is little hope for analytically characterising long-run types for finite γ without restricting the network structures and other parameters.

5. Applications

We close by presenting two applications – asset price bubbles and price oscillations. These applications are motivated by evidence from experimental asset markets, as well as our desire to highlight some concrete implications for asset prices via numerical results for some interesting cases not settled by our analytical results. We consider deterministic dividends in this section to make clear that the results do not require exogenous shocks.

5.1. Price bubbles

As a first application, we consider an example with price bubbles in the sense of positive deviations from the fundamental price that initially grow to reach a peak before price collapses and returns to the fundamental value. Following the seminal paper of Smith et al. (1988), such 'bubbly' price dynamics have been documented in numerous studies of experimental asset markets, and it has been found that introducing communication between participants affects the incidence of asset price bubbles.[18] In our example, the bubble dynamics are generated by the network without any exogenous disturbance to asset prices or fundamental values. Besides highlighting the possibility of price bubbles, our example has the network effect and the performance effect either competing against each other, or reinforcing one other, depending on the value of the average type $\bar{g}_t$.

Consider a network with one pure fundamentalist, agent 1, who does not listen to any other agents (i.e. $a_{1j} = 1$ if $j = 1$ and 0 otherwise). The remaining agents, 2 to n, start out as strong chartists but update their types based on their network, with the weights depending upon performance when $\gamma > 0$. For convenience we set $g_0 = (0, 2, ..., 2)$ and $n = 10$, so that $\bar{g}_0 = 1.8$. Agents $i = 2, \ldots, 10$ listen to each of their nearest neighbours, i.e. $a_{ij} = 1$ if and only if $|j - i| \in \{0, 1, n-1\}$ and $i > 1$. Hence, the network matches the wheel network considered in Section 3.3.1, except that agent 1 does not listen to agent 2 or agent 10.

With the terminology used in this paper, agent 1 forms the only strongly connected and closed group while the rest of the world is composed of the agents 2,...,10. Note that this example can be interpreted as a world with a 'die hard' fundamentalist (agent 1) and many followers who either follow the fundamentalist directly (agents 2,10) or follow her indirectly by following her followers or her followers' followers (agents 3 to 9); intuitively, we may think of this example as 'one Warren Buffet and many sheep'.

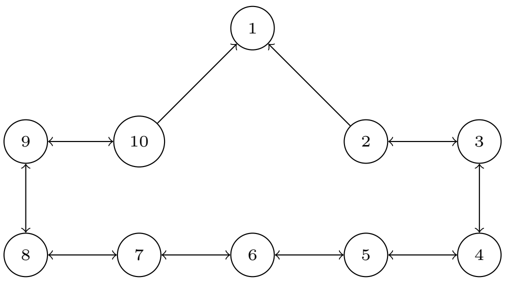

Figure 8: The network configuration of Section 5.1 for $n = 10$ omitting the self loops.

We set $r = 0.04$, $\overline{d} = 0.02$, $\phi = 1$ and choose a zero net supply of shares, $X = 0$, so that the fitness ranking among agents depends only on the average type $\bar{g}_t$ relative to $(1+r)^2$ (see (21)). Initially we keep n fixed at 10 so that chartists ('sheep') outnumber the fundamentalist by 9 to 1; however, we later consider the impact of varying n. Note that when $\gamma = 0$, the fitness ranking is irrelevant and hence the updating, of agents 2 to 10, depends on the (local) average computed from their own type and their neighbour on either side, whereas agent 1 always keeps $g_t^1 = 0$. As a result, the type dynamics for $\gamma = 0$ are guaranteed to converge smoothly to zero, giving us a benchmark to compare with the case when both a network effect and a performance effect are present ($\gamma > 0$).

The price follows a stylised 'bubble' dynamic, first increasing for several periods to reach a peak before collapsing and returning to the fundamental price (see Figure 9). The right panel shows the corresponding average type dynamics. Price initially increases because $\bar{g}_0 = 1.8 > 1 + r$, and it goes on increasing for several periods while $\bar{g}_t$ remains above $1 + r$.

[18]Noussair et al. (2001), Oechssler et al. (2011), Schoenberg and Haruvy (2012), Steiger and Pelster (2020).

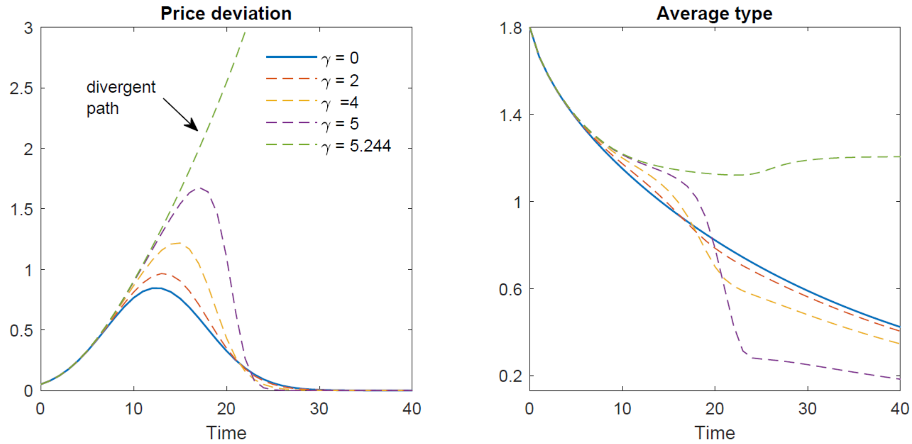

Figure 9: Asset price bubbles and type dynamics

Only after several rounds of updating has the fundamentalist type spread sufficiently through the population to lower the average type below $(1+r)$, so that price starts falling and the bubble collapses. Hence, the price 'bubble' here is generated by a network effect.

Once we turn on the performance effect ($\gamma > 0$), we see that the bubble is prolonged and peaks at a higher value. At the same time, the bubble becomes strongly asymmetric, with price collapsing rather quickly after reaching its peak (left panel). This dynamic occurs because while the average type satisfies $\bar{g}_t > (1+r)^2 \approx 1.082$, more optimistic agents earn higher returns and hence chartist beliefs outperform the pure fundamentalist; however, once $\bar{g}_t$ is below $(1+r)^2$, the performance ranking is reversed, and there is a shift toward more fundamental beliefs, which can be seen in the dramatic decline of $\bar{g}_t$ soon after period 10 (dashed lines, right panel). This decline is especially evident in the case of $\gamma = 5$ (purple dashed line). Note that price quickly collapses once $\bar{g}_t < (1+r)$ (left panel) because a falling price, which itself makes expectations more conservative, is reinforced by a fall in average type $\bar{g}_t$, so that price declines are exacerbated.

Note that while the network effect wins the 'battle' against the performance effect up to $\gamma = 5$, this result is reversed in the final case ($\gamma = 5.244$, green dashed line). That is, once γ is large enough, the performance effect is strong enough that $\bar{g}_t$ always exceeds $(1+r)^2$. In this case the reversal in performance ranking does not happen (see right panel) and since the terminal average type exceeds $(1+r)$, we have a perpetual bubble for which price diverges to $+\infty$.[19] We thus see that the importance placed on performance, as controlled by feedback parameter γ, has both quantitative and qualitative implications for the dynamics.

The findings here are consistent with the theoretical results earlier in the paper. In this example the agents 2,..,10 form the rest of the world and Proposition 1 implies that for $\gamma = 0$, the rest of the world will adopt a weighted average of the consensuses in the closed an strongly connected groups. The same is true for any finite γ if the price converges to the fundamental price by Proposition 3. Since the only closed and strongly connected group is

[19]Clearly, consensus does not obtain in this case since the 'die hard' fundamentalist has type of 0 while the remaining agents have types above the average (though arbitrarily close to it for sufficiently large n).

the singleton set composed of agent 1, Propositions 1 and 3 thereby imply that the types of the agents 2,...,10 converge to the fundamental type of agent 1 if price converges to the fundamental price. In contrast for $\gamma \to \infty$, since the initial average type is above $(1+r)^2$, Proposition 4 implies that the rest of the world will converge to the maximal type from their path which is clearly their initial type. Hence, for $\gamma \to \infty$, the initial types of agents $2, \ldots, 10$ will never change, leading to price divergence.

How sensitive are the results to the mass of chartists, as determined by n? Figure 10 varies the number of agents at three different values of gamma, including the case $\gamma = 0$ (no performance effect, left panel). We see that increasing n raises the magnitude and persistence

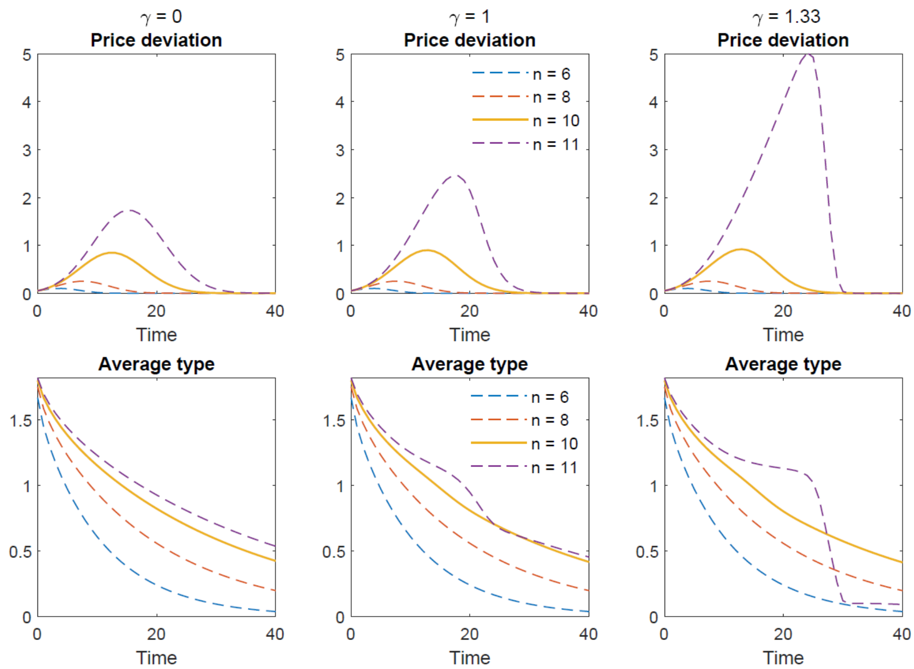

Figure 10: Sensitivity of bubble paths to increasing the mass of chartists

of the price bubble; intuitively, this is because a large population of chartists corresponds to greater initial optimism, which takes longer to die out as the fundamentalist belief spreads. At the same time, the diameter of the network increases for greater n which has a reinforcing effect, implying that it takes longer for all agents to adopt the fundamental belief. For the cases with positive γ (Fig. 9, middle and right panel), the bubble is amplified and more persistent when investors are more focused on performance, and even small increases in n have substantial effects on the bubble size and duration. Due to the switch in the performance ranking noted above, a strong asymmetry develops as n is increased: the bubbles build over many periods, but the price collapses very quickly after reaching its peak.

Although we only plot cases where price converges, the case $\gamma = 1.33, n = 11$ (right panel of Figure 10, dashed purple) is right on the border, and hence a small increase in γ will lead to price divergence as seen above. Note that there is some degree of substitutability of γ and n in this example, in the sense that increasing either γ or n while holding the other parameter fixed will increase the magnitude and persistence of the price bubble and, for sufficiently large values, will imply that there is a perpetual bubble so that price diverges to $+\infty$. There is an important distinction here since if $\gamma = 0$ (performance effect absent), then regardless of how large n is, types will converge on a consensus of 0 (pure fundamentalist) and hence the price bubble will always collapse, giving price convergence to the fundamental price. By contrast, if $\gamma > 0$ then for large enough n we may have the average type settle above $1+r$ so that the bubble never ends and price diverges. Again, we see that performance has both quantitative and qualitative implications for price and type dynamics.

Although in the above example the performance effect makes price divergence 'more likely', this is not a general result. With the parameters chosen in this example, the initial average type is greater than $(1+r)^2$, and hence Proposition 4, part 3 implies that the rest of the world converges to the maximal type on their path when performance becomes infinitely important. Clearly, this maximal type can be found in rest of the world itself, leading to price divergence with increasing performance feedback parameter. More generally, depending on the example at hand, the performance effect may lead to price divergence, have no impact on long-run price dynamics, or lead to price convergence when it would otherwise be absent.

5.2. Price oscillations

In this section we provide an example in which there can be never-ending price oscillations in the absence of external shocks. The presence of price oscillations – or 'recurrent bubbles' – has been seen in studies of experimental asset markets with multiple repeated rounds of the same group of participants (see e.g. Hommes et al. (2005) or Bao et al. (2017)).

We assume there are $n = 3$ agents. Agents 1 and 3 are "extreme agents" who listen only to themselves, i.e. for $i \in \{1, 3\}$, $a_{ij} = 1$ if $j = i$ and 0 otherwise. Agent 2 listens to both agents 1 and 3, so $a_{2,j} = 1$ for $j \in \{1, 2, 3\}$. Agent 1's initial type is $g_0^1 = 1$ and Agent 3's initial type is $g_0^3 = (1 + r + \varepsilon)^2$, where $\varepsilon = 0.001$. Since these agents update only from themselves, $g_t^1 = 1$ and $g_t^3 = (1 + r + \varepsilon)^2$ for all $t \in \mathbb{N}$. Agent 2's initial type is $g_0^2 = 1 + r$, and their subsequent types g_t^2, for all $t \geq 1$, will depend on their updates from past types (of all agents), which are weighted according to their relative performance. It follows that any changes in the average type $\overline{g}_t$ must come from changes in Agent 2's type.

The other parameters are set at $X = 3$, $r = 0.04$, $\overline{d} = 0.5$ and $\phi = 0.5$, giving $\delta = 1/\phi = 2$. Together these parameters imply a fundamental price of $p^f = \frac{1}{r}(\overline{d} - X(n\delta)^{-1}) = 0$. The initial deviation from the fundamental price is set at $\tilde{p}_0 = 10$.

Consider first the benchmark of DeGroot updating ($\gamma = 0$) where the performance effect is absent. In this case, Agent 2 updates from 1 and 3 with equal weight of 1/3 and hence their first update is $g_1^2 = \frac{1}{3}[1 + (1 + r) + (1 + r + \varepsilon)^2] \approx 1.0412 > 1 + r$. It follows that all of Agent 2's subsequent types $g_t^2 = \frac{1}{3}[1 + g_{t-1}^2 + (1 + r + \varepsilon)^2]$ are increasing, implying $g_t^2 \geq 1 + r$ for all $t \in \mathbb{N}$. The average type thus satisfies $\overline{g}_t > 1 + r$ and hence price will diverge to $+\infty$. We therefore focus in what follows on cases where $\gamma > 0$. It turns out that this is sufficient to prevent price divergence; as we shall see, the presence of permanent price fluctuations is closely linked to the size of the performance feedback parameter γ.

We start by plotting some time series for the price deviation $\tilde{p}_t$ and the average type $\overline{g}_t$ at select values $\gamma \in \{2.5, 5, 6, 7.5\}$ (see Figure 11). For sufficiently low values of γ, price is attracted to a non-fundamental steady state in which the price deviation is positive and average type settles on $1 + r$ (see left panel), implying that the terminal belief of Agent 2 is

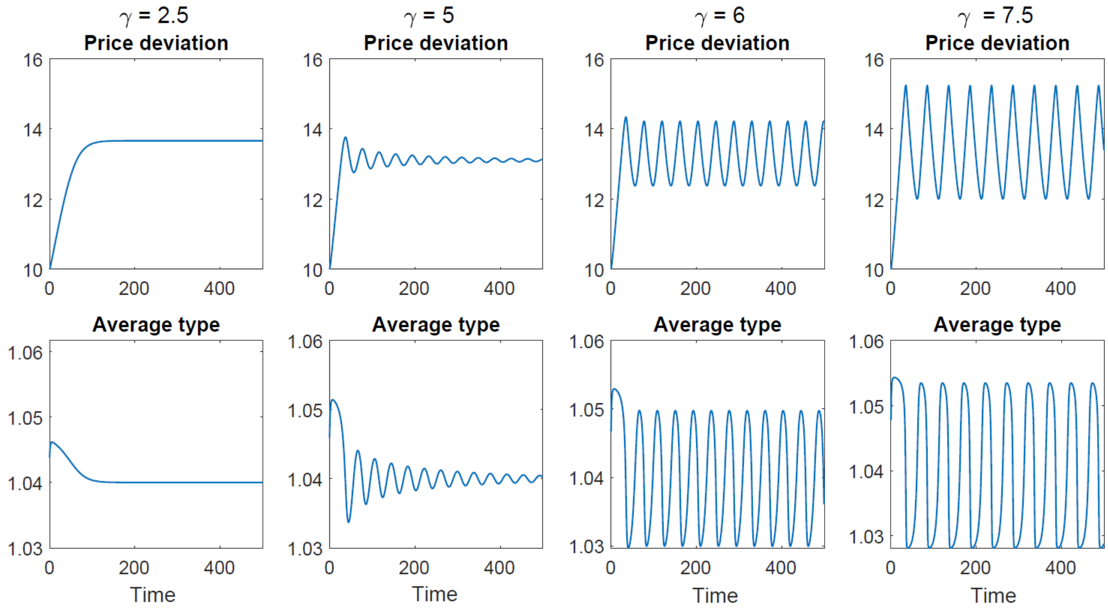

Figure 11: Time series for selected values of the performance parameter γ.

smaller than $(1+r)$.[20] For $\gamma = 5$ we see visible oscillations in the asset price and the average type that dampen over time. Once γ is large enough, however, the fluctuations in price and average type no longer die out (right panels). Both price and average type follow cycles of consistent amplitude and frequency, with the fluctuations larger for $\gamma = 7.5$ than for $\gamma = 6$. The latter cycles remain as the simulation horizon is increased, while for $\gamma = 5$ (middle left) the price deviation converges on a (non-fundamental) steady state with average type equal to $1 + r$ and the price deviation lower than for $\gamma = 2.5$ (left panel).

The intuition for the persistent price fluctuations is quite simple. When the average type $\overline{g}_t < 1+r$, iterations of the price equation imply that price is falling, while if $\overline{g}_t > 1+r$ price is increasing (see (9)). Therefore, for price to fluctuate continually, the average type must fluctuate between values $< 1+r$ and values $> 1+r$. This is exactly what we see is the lower right panels of Figure 11, and recall that these fluctuations come from the type updates of Agent 2, who changes their weighting of types according to performance. Agent 1 (moderate chartist) receives a higher weight once price exceeds a critical value, which leads price to start falling, at which point performance again favours Agent 3 (strong chartist) and price starts increasing again. The critical prices at which the switches in performance occur are given by (14) and although the critical price is unique, it depends on average type $\overline{g}_t$ – and hence there are both upper and lower price thresholds at which the switches occur.[21]

The simulations in Figure 11 leave open the question of whether the same pattern is

[20] Let $g_{2,\infty}$ be the terminal belief of Agent 2. Given a terminal average type of $\overline{g}_\infty = 1 + r$, we have $1+r = \frac{1}{3}[1+g_{2,\infty}+(1+r+\varepsilon)^2] \implies g_{2,\infty} = 3(1+r)-[1+(1+r+\varepsilon)^2] = (1+r)(1-2\varepsilon)-(r^2+\varepsilon^2) < 1+r$.

[21] Note that since the critical price is decreasing in $\overline{g}_t$, the upper (lower) price threshold 'kicks in' when average type is low (high). Therefore, the peaks and troughs in price and type are not coincident.

observed for a wide range of γ values. We therefore conducted a bifurcation analysis with respect to γ – and we found that the general pattern is preserved. In Figure 12 we plot the attractors for price deviation and average type for 7 different γ values in simulations of 15,000 periods in which the last 1,500 observations are plotted.[22]

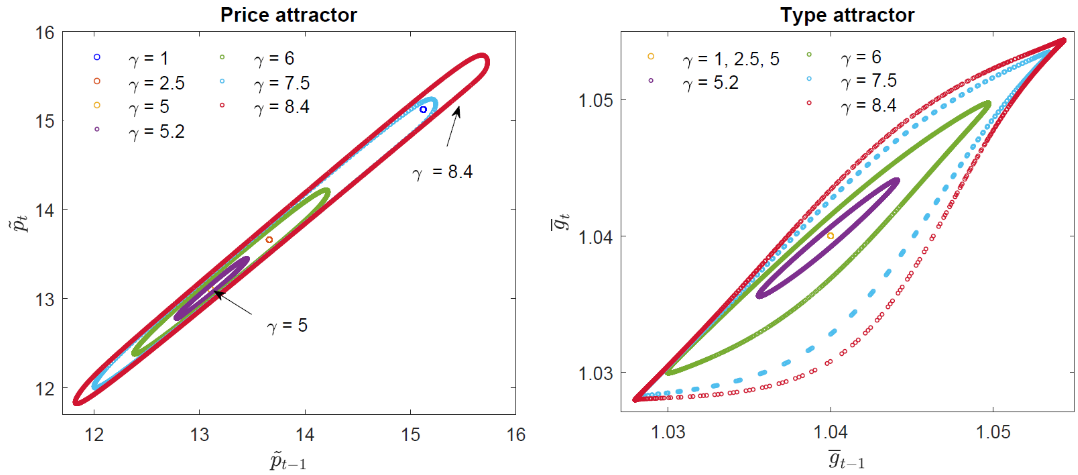

Figure 12: Price and average type attractors for selected values of γ.

For $\gamma > 0$ we find that the price dynamics converge. If γ is small enough, the attractors collapse to a single point – in particular, price and average type converge to a non-fundamental steady state in which price deviation is positive (left panel) and average type equals $1 + r$ (right panel), as seen in the above simulations. For high enough values of γ (≈ 5.1), we see a qualitative change in the dynamics: there are permanent price and type cycles, as visible in the attractors of Figure 12. The cyclical behaviour of the price and average type is indicated by the 'oval shape' attractors seen for values such as $\gamma = 5.2$ (purple) and $\gamma = 6$ (green); see left and right panel. Note that for moderate values of γ the attractors are 'solid'; however, as γ is increased further, some further qualitative changes emerge: the type attractor becomes strongly asymmetric and contains 'gaps', i.e. not all regions between the min and max are 'hit'. Note that the asymmetry of the type attractor also implies that price deviations get larger on the upside as γ is increased, so that the risky asset is often strongly overvalued (left panel).

6. Conclusion

There has been considerable interest in the influence of social ties on investment decisions, yet much formal work has relied on models in which social factors do not play an important role. As argued by Hirshleifer (2015), "the time has come to move beyond behavioral finance to social finance." In this paper we moved in this direction by studying the impact of network structure on belief formation and asset prices in a setting where prices and belief types evolve

[22]We picked a relatively small number of values to keep the attractors discernible from one another.

as a system of coupled dynamics. Our model builds on a benchmark asset pricing model by allowing investors to be connected via arbitrary social networks and to adopt continuous types on the spectrum from pure fundamentalist to arbitrarily strong chartist.

The influence of the social network on price and type dynamics depends on investors' attention to past performance when agents update their belief type. We obtained a sharp characterisation of the long-run type distribution for the polar cases where (i) updates are purely social, and (ii) agents only update from best-performing neighbours. For pure social updating, the long-run type distribution and price dynamics depend on network centrality as in the opinion dynamics literature. However, if only the best performer(s) within each agent's network are imitated, then (depending on initial price and average type), either the most fundamental type or the most chartist type in each closed subgroup is adopted in finite time, and price converges if these extreme types are not too strongly chartist.

In this case of updating from best-performers, the network only affects the time to consensus, and price can converge to the rational expectations solution (fundamental price) in finite time if there are one or more pure fundamentalists. For intermediate performance feedback, the network matters in shaping the consensus belief, and we provided conditions such that a long-run consensus is reached, although the consensus itself is analytically intractable. Particularly, we showed that anything goes in the sense that all achievable consensuses are achieved by just varying the outside supply of shares and the performance feedback parameter. Our two applications – price bubbles and price oscillations – illustrated some concrete implications of network-performance effects in the absence of exogenous shocks.

Relative to previous work, these results clarify how asset price and type dynamics depend on concrete features of networks and market conditions – such as distance between agents (diameter), network centrality, asset supply, and initial price and initial types – and when the most fundamental type or most chartist type will survive if investors are strongly focused on performance. Our results thereby provide an understanding of when performance-based updating from a social network is stabilising – or not – for asset prices. An important implication of our results is that policymakers concerned with financial market stability will require information not just on social networks, but also on the extent to which observed differences in performance affect investment decisions.

There are several promising avenues for future research. First, it would be of interest to investigate whether the model does a good job at replicating empirical stylised facts in stock markets when it is disciplined with networks that display a high degree of clustering as in the data. An investigation of price and type dynamics under some empirically plausible network structures – along the lines of the exercise in Panchenko et al. (2013) – would be a useful contribution. Second, estimation of the model could shed light on model performance, as well as helping to discipline model parameters in numerical analysis.

Last but not least, an important finding from several studies of experimental asset markets is that after a sufficiently large number of rounds with a group of participants, beliefs appears to coordinate on a common predictor (see Hommes et al. (2005), Hommes et al. (2008), Bao et al. (2017))). Since the standard framework of discrete types cannot, by definition, replicate such belief-type consensus (except as an extreme case), an open question is whether our model of continuous types would provide an improved fit to experimental asset market data while helping to explain this important stylised fact.

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Appendix A. Two Lemmas for the case of $\gamma \to \infty$

Lemma 1. *Let $\gamma \to \infty$ and suppose there exists $\bar{t} \in \mathbb{N}$ such that $sgn(R_t) = sgn(R_0)$ or $\bar{g}_t = 0$ for all $t \leq \bar{t}$. Then the following holds:*

1. *If $\tilde{p}_0 R_0 < 0$, then $g_t^i \geq g_{t-1}^{\min}(N^i) \geq g_{t+1}^i$ for all $i \in N$, for all $t \leq \bar{t}$.*
2. *If $\tilde{p}_0 R_0 > 0$, then $g_t^i \leq g_{t-1}^{\max}(N^i) \leq g_{t+1}^i$ for all $i \in N$, for all $t \leq \bar{t}$.*

Proof. First, note that $sgn(\tilde{p}_0) = sgn(\tilde{p}_t)$ holds for all $t \in \mathbb{N}$ such that $\bar{g}_t \neq 0$ by (9). With the assumption of the Lemma, this implies $sgn(R_0\tilde{p}_0) = sgn(R_t\tilde{p}_t)$ for all $t \leq \bar{t}$.

1. Consider the case $\tilde{p}_0 R_0 < 0$, implying $\tilde{p}_t R_t < 0$ or $\bar{g}_t \neq 0$ for all $t \leq \bar{t}$. Note that if there exists $t' \leq \bar{t}$ such that $\bar{g}_{t'} = 0$ then $g_{t'}^i = 0$ for all $i \in N$, and by weighted average updating (by row stochasticity of $\tilde{\mathbf{A}}$) there is nothing to show since then $g_{t'}^i = 0$ for all $i \in N$, $t' \geq t$ is implied.

   Therefore, suppose for the remainder that $\tilde{p}_t R_t < 0$ holds for all $t \leq \bar{t}$. Note that $sgn(R_t) = -sgn(\tilde{p}_t)$ implies $u_t^i > u_t^j$ if and only if $g_{t-1}^i < g_{t-1}^j$ by (13) (since $sgn(\tilde{p}_t) = sgn(\tilde{p}_{t-1}) = sgn(\tilde{p}_{t-2})$). Hence, more fundamental types perform better for any $t \leq \bar{t}$.

   First, consider $t = 0$ and let $\gamma \to \infty$. By (20), each agent $i \in N$ only updates from those with maximal fitness in their neighbourhood, i.e. from those with minimal type. Note, $g_{-1}^i = g_0^i$ for all $i \in N$ by assumption on initial conditions. Hence, $U_0^{\max}(N^i) = G_{-1}^{\min}(N^i) = G_0^{\min}(N^i)$ for all $i \in N$ recalling that $U_t^{\max}(N^i) := \arg\max_{j\in N^i}\{u_t^j\}$ and, similarly, $G_t^{\min}(N^i) := \arg\min_{j\in N^i}\{g_t^j\}$.

   We therefore get the following:

   $$g_1^i = \frac{1}{|U_0^{\max}(N^i)|} \sum_{j \in U_0^{\max}(N^i)} g_0^j = \frac{1}{|G_0^{\min}(N^i)|} \sum_{j \in G_0^{\min}(N^i)} g_0^j = \min_{j\in N^i} g_0^j \leq g_0^i \quad \forall i \in N,$$

   since $i \in N^i$ for all $i \in N$. Thus, $\tilde{p}_0 R_0 < 0$ implies $g_1^i \leq g_0^i$ for all $i \in N$.

   Further suppose for some $t \leq \bar{t}$ we have $g_t^i \leq g_{t-1}^i$ for all $i \in N$. We show that this implies $g_{t+1}^i \leq g_t^i$ for all $i \in N$. Since $R_t\tilde{p}_t < 0$, we get similar to above:

   $$g_{t+1}^i = \frac{1}{\left|G_{t-1}^{\min}(N^i)\right|} \sum_{j \in G_{t-1}^{\min}(N^i)} g_t^j \leq \frac{1}{\left|G_{t-1}^{\min}(N^i)\right|} \sum_{j \in G_{t-1}^{\min}(N^i)} g_{t-1}^j = \min_{j\in N^i} g_{t-1}^j \quad \forall i \in N$$

   By weighted average updating (by row stochasticity of $\tilde{\mathbf{A}}$) we have $g_t^i \geq \min_{j\in N^i} g_{t-1}^j$ for all $i \in N$. Hence, we have shown that if $\tilde{p}_0 R_0 < 0$, then $g_1^i \leq g_0^i$ and if $\tilde{p}_t R_t < 0$ and $g_t^i \leq g_{t-1}^i$, then

   $$g_{t+1}^i \leq \min_{j\in N^i} g_{t-1}^j \leq g_t^i. \quad \forall i \in N, \tag{A.1}$$

   Induction implies that (A.1) holds for all $t \leq \bar{t}$ which is what we had to show.

2. Now, consider the case $\tilde{p}_0 R_0 > 0$ which is completely analogous to above. $\tilde{p}_0 R_0 > 0$ implies $\tilde{p}_t R_t > 0$ for all $t \in \mathbb{N}$ for which there exists $i \in N$ such that $g_t^i \neq 0$. Note that we will show that for all such $t \in \mathbb{N}$, we have $g_t^i \leq g_{t+1}^i$. Hence, if there exists a $i \in N$ such that $g_0^i > 0$, then this will hold for all $t \leq \bar{t}$.

   Note that $sgn(R_t) = sgn(\tilde{p}_t)$ implies $u_t^i < u_t^j$ if and only if $g_{t-1}^i < g_{t-1}^j$ by (13). Hence more chartist types are always performing better at any point in time $t \leq \bar{t}$.

   Analogously to above for $t = 0$, we get because of the initial assumptions,

$$g_1^i = \frac{1}{|U_0^{\max}(N^i)|} \sum_{j \in U_0^{\max}(N^i)} g_0^j = \frac{1}{|G_0^{\max}(N^i)|} \sum_{j \in G_0^{\max}(N^i)} g_0^j = \max_{j \in N^i} g_0^j \geq g_0^i \quad \forall i \in N.$$

   Assuming $g_t^i \geq g_{t-1}^i$ for all $i \in N$, we get from $R_t \tilde{p}_t > 0$ that:

$$g_{t+1}^i = \frac{1}{\left|G_{t-1}^{\max}(N^i)\right|} \sum_{j \in G_{t-1}^{\max}(N^i)} g_t^j \geq \frac{1}{\left|G_{t-1}^{\max}(N^i)\right|} \sum_{j \in G_{t-1}^{\max}(N^i)} g_{t-1}^j = \max_{j \in N^i} g_{t-1}^j \quad \forall i \in N$$

   and, hence,

$$g_{t+1}^i \geq \max_{j \in N^i} g_{t-1}^j \geq g_t^i. \quad \forall i \in N, t \leq \bar{t}. \tag{A.2}$$

□

Lemma 2. *Let $\gamma \to \infty$ and and suppose there exists $\bar{t} \in \mathbb{N}$ such that $sgn(R_t) = sgn(R_0)$ or $\bar{g}_t = 0$ for all $t \leq \bar{t}$. Then the following holds:*

1. *If $\tilde{p}_0 R_0 < 0$, then $g_t^i \to g_0^{\min}(P^i)$ $\forall i \in N$, $t \in \mathbb{N} : 2d(i, G_0^{\min}(P^i)) - 1 \leq t \leq \bar{t}$.*

   If for a closed and strongly connected group $\mathcal{C}$ we have $\bar{t} \geq 2D(\mathbf{A}_{\mathcal{C}}) - 1$, then $g_t^i \to g_0^{\min}(\mathcal{C})$ for all $t \geq 2D(\mathbf{A}_{\mathcal{C}}) - 1$.

2. *If $\tilde{p}_0 R_0 > 0$, then $g_t^i \to g_0^{\max}(P^i)$ $\forall i \in N$, $t \in \mathbb{N} : 2d(i, G_0^{\max}(P^i)) - 1 \leq t \leq \bar{t}$.*

   If for a closed and strongly connected group $\mathcal{C}$ we have $\bar{t} \geq 2D(\mathbf{A}_{\mathcal{C}}) - 1$, then $g_t^i \to g_0^{\max}(\mathcal{C})$ for all $t \geq 2D(\mathbf{A}_{\mathcal{C}}) - 1$.

Proof. For some agent $i \in N$, recall that $P^i := \{j \in N | \exists k \in \mathbb{N} : (\mathbf{A}^k)_{ij} > 0\}$ denotes the set of agents to which there exists a path from i. Since $a_{jj} = 1$ for all $j \in N$, we have $j \in P^j$ for all $j \in N$. Further recall that for any $M \subseteq N$, we denote by $g_t^{\min}(M) := \min\{g_0^j | j \in M\}$ the minimal initial type of all agents in the set M and by $g_t^{\max}(M) := \max\{g_0^j | j \in M\}$ the maximal initial type of all agents in the set M for some point in time $t \in \mathbb{N}$. Clearly, $P^j \subseteq P^i$, and hence, $g_t^{\min}(P^i) \leq g_t^{\min}(P^j)$ while $g_t^{\max}(P^i) \geq g_t^{\max}(P^j)$ for all $j \in P^i$, $t \in \mathbb{N}$. Note that by weighted average updating (by row stochasticity of $\tilde{\mathbf{A}}$), we have $g_{t+1}^j \geq g_t^{\min}(P^j) \geq g_t^{\min}(P^i) \geq g_{t-1}^{\min}(P^i)$ and $g_{t+1}^j \leq g_t^{\max}(P^j) \leq g_t^{\max}(P^i) \leq g_{t-1}^{\max}(P^i)$ for all $j \in \mathcal{P}^i$, $t \in \mathbb{N}$.

1. Consider the case $\tilde{p}_0 R_0 < 0$ such that (A.1) holds for all $t \leq \bar{t}$ by Lemma 1. Hence, the first inequality in (A.1) must be satisfied with equality if $\min_{k \in N^j} g_{t-1}^k = g_{t-1}^{\min}(P^i)$

for some $j \in P^i$. Thus, all agents $j \in P^i$ who listen to agents within the set $G_{t-1}^{\min}(P^i)$ must adopt $g_{t-1}^{\min}(P^i)$ at latest by period $t+1$. Hence,

$$G_{t+1}^{\min}(P^i) \supseteq \bigcup_{j \in G_{t+1}^{\min}(P^i)} M^j(G_{t-1}^{\min}(P^i)) \quad \forall t \in \mathbb{N}. \tag{A.3}$$

Note that (A.1) also implies $G_{t+1}^{\min}(P^i) \supseteq G_t^{\min}(P^i)$ for all $t \in \mathbb{N}$. Since, further, by assumption on initial conditions, $G_{-1}^{\min}(P^i) = G_0^{\min}(P^i)$, (A.3) implies

$$G_{t+1}^{\min}(P^i) \supseteq \begin{cases} G_t^{\min}(P^i) & \text{if } t \text{ is even} \\ \bigcup_{j \in G_t^{\min}(P^i)} M^j(G_t^{\min}(P^i)) & \text{if } t \text{ is odd} \end{cases}$$

Since $j \in M^j$ for all $j \in N$, the set $G_t^{\min}(P^i)$ just expands over the path P^i at latest at every odd time-step by the neighbours of the previous set. Thus, each agent $j \in P^i$ within distance $d(j, G_0^{\min}(P^i))$ of the agents with initial minimal types of P^i has adopted this minimal type at latest at time step $2d(j, G_0^{\min}(P^i)) - 1$ and will keep it from there for all $t \geq 2d(j, G_0^{\min}(P^i)) - 1$ as long as $t \leq \bar{t}$.

Now, for a closed and strongly connected set $\mathcal{C}$, we have by definition $P^i = P^j$ for all $i, j \in \mathcal{C}$. Thus if $\bar{t} \geq 2D(\mathbf{A}_{\mathcal{C}}) - 1$, then all agents in $\mathcal{C}$ obtain a consensus on $g_0^{\min}(\mathcal{C})$ after at most $2D(\mathbf{A}_{\mathcal{C}}) - 1$ steps where, as before, $\mathbf{A}_{\mathcal{C}}$ is the matrix $\mathbf{A}$ restricted to the set $\mathcal{C}$ and $D(\mathbf{A}_{\mathcal{C}})$ is the length of the longest path within the network $\mathbf{A}_{\mathcal{C}}$. Since $g_{2D(\mathbf{A}_{\mathcal{C}})-1}^i = g_0^{\min}(\mathcal{C})$ for all $i \in \mathcal{C}$ if $\bar{t} \geq 2D(\mathbf{A}_{\mathcal{C}}) - 1$, these will not change henceforth by the nature of weighted average updating. Hence, $g_t^i = g_0^{\min}(\mathcal{C})$ for all $t \geq 2D(\mathbf{A}_{\mathcal{C}}) - 1$.

2. Now, consider the case $\tilde{p}_0 R_0 > 0$ such that (A.2) holds for all $t \leq \bar{t}$ by Lemma 1. Analogously to above, we conclude that for any $i \in N$,

$$G_{t+1}^{\max}(P^i) \supseteq \begin{cases} G_t^{\max}(P^i) & \text{if } t \text{ is even} \\ \bigcup_{j \in G_t^{\max}(P^i)} M^j(G_t^{\max}(P^i)) & \text{if } t \text{ is odd} \end{cases}$$

Thus, for a closed and strongly connected set $\mathcal{C}$, all agents in $\mathcal{C}$ obtain a consensus on $g_0^{\max}(\mathcal{C})$ after at most $2D(\mathbf{A}_{\mathcal{C}}) - 1$ steps and do not change types after that.

□

Appendix B. Proofs of the Main Results

Proof of Proposition 1. The type dynamics characterization follows from standard results of the DeGroot model (see e.g. Golub and Jackson, 2010; Buechel et al., 2015). If the limit average type $\bar{g}_\infty$ is below $1+r$, then there exists a $t' \in \mathbb{N}$ such that $\bar{g}_{t'} < 1+r$ for all $t \geq t'$ which implies by Eq (9) that $\lim_{t\to\infty} \tilde{p}_t = 0$, i.e. price converges to the fundamental price. If the limit average type is above $1+r$, then there exists a $t' \in \mathbb{N}$ such that $\bar{g}_{t'} > 1+r$ for all $t \geq t'$ which implies by Eq (9) that price diverges (to $+\infty$ if $\tilde{p}_t > 0$ and to $-\infty$ if $\tilde{p}_t < 0$). If $\bar{g}_\infty = 1+r$, then for any $\epsilon > 0$ there exists t_ϵ such that $|\bar{g}_t - (1+r)| < \epsilon$ for all $t \geq t_\epsilon$. By Eq (9), price will settle on some value. □

Proof of Proposition 2. Note $g_{t+1}^{\max} \leq g_t^{\max}$ and $g_{t+1}^{\min} \geq g_t^{\min}$ for all $t \in \mathbb{N}$ by row stochasticity of $\tilde{\mathbf{A}}$. Hence if $g_0^{\min} > 1+r$, then $\bar{g}_t > 1+r$ for all $t \in \mathbb{N}$, implying $\prod_{k=1}^{t} \frac{\bar{g}_k}{1+r} \overset{t\to\infty}{\longrightarrow} \infty$ and, hence, price diverges by (9). On the other hand, if $g_0^{\max} < 1+r$, then $\bar{g}_t < 1+r$ for all $t \in \mathbb{N}$ implying $\prod_{k=1}^{t} \frac{\bar{g}_k}{1+r} \overset{t\to\infty}{\longrightarrow} 0$ and, hence, price converges by (9). Further $0 \leq \frac{\bar{g}_t}{1+r} < 1$ for all $t \in \mathbb{N}$ implying smooth convergence as stated.

Now suppose some vector of types $\mathbf{g}_0 \in \mathbb{R}_+^n$ with $g_0^{\max} > 1+r$. Given any network $\mathbf{A}$, we show that there are initial conditions such that price diverges: let $X > 0$ such that we are in the case of positive outside supply of shares, consider the case of large feedback to beliefs, i.e. $\gamma \to \infty$, non-stochastic dividends, $d_t = \bar{d}$ for all $t \in \mathbb{N}$, and let the initial price deviation $\tilde{p}$ be positive, but small enough such that

$$0 < \tilde{p}_0 < \frac{(1+r)^{2D(\mathbf{A})-1}}{(g_0^{\max})^{2D(\mathbf{A})-1}} \frac{1+r}{n(1+r)^2-(1+r)} \frac{X}{\delta n}. \tag{B.1}$$

Note that the later exists since the right-hand side is positive. Since $\tilde{p}_t = \frac{1}{(1+r)^t}(\prod_{k=1}^{t} \bar{g}_k)\tilde{p}_0$ and $\bar{g}_t \leq g_0^{\max}$ for all $t \in \mathbb{N}$, we have $\tilde{p}_t \leq \frac{(g_0^{\max})^t}{(1+r)^t}\tilde{p}_0 \leq \frac{(g_0^{\max})^{2D(\mathbf{A})-1}}{(1+r)^{2D(\mathbf{A})-1}}\tilde{p}_0$ for all $0 \leq t \leq 2D(\mathbf{A})-1$. Together with (B.1), we get $\tilde{p}_t < \frac{1+r}{n(1+r)^2-(1+r)}\frac{X}{\delta n}$ for all $0 \leq t \leq 2D(\mathbf{A})-1$. Now, if $1+r < g_t^{\max}$, then $\frac{1+r}{n} < \bar{g}_t$, implying $\tilde{p}_t \leq \frac{1+r}{n(1+r)^2-(1+r)}\frac{X}{\delta n} < \frac{\bar{g}_t}{(1+r)^2-\bar{g}_t}\frac{X}{\delta n}$ if $\bar{g}_t < (1+r)^2$ and, hence by (14), $R_t > 0$. If, instead, $\bar{g}_t \geq (1+r)^2$, then trivially $R_t > 0$. Hence, $R_t\tilde{p}_t > 0$ (since $\bar{g}_t > \frac{1+r}{n}$ and hence $\tilde{p}_t > 0$) for all $0 \leq t \leq 2D(\mathbf{A})-1$.

By Lemma 2, this implies that each agent $i \in N$ within distance $d(i, G_0^{\max})$ of the agents with initial maximal types has adopted the maximal type at latest at time step $2d(i, N(g_0^{\max}))-1$ and will keep it from there forever, since the network is strongly connected and, hence, we have $P^i = N$ for all $i \in N$. Therefore, type-convergence to maximal type is obtained at latest at time step $2d\Big((N \setminus G_0^{\max}), G_0^{\max}\Big)-1 \leq 2D(\mathbf{A})-1$, i.e. for all $t \geq 2D(\mathbf{A})-1$: $g_t^i = g_0^{\max}$ for all $i \in N$. Further, since $g_0^{\max} > 1+r$, we get $\bar{g}_t > (1+r)$ for all $t \geq 2D(\mathbf{A})-1$ implying price divergence. Clearly, this holds also for finite but large enough γ, i.e. there exists $\bar{\gamma}$ such that for all $\gamma > \bar{\gamma}$ price diverges.

Finally suppose that $g_0^{\min} < 1+r$. Showing the possibility of price convergence in this case for some type vector g and some strongly connected network $\mathbf{A}$ works analogously to above by assuming the initial price to be large enough such that by Lemma 2 even after $2D(\mathbf{A})-1$ rounds of updating, we still have $R_t < 0$. Hence, $R_t\tilde{p}_t < 0$ for all $0 \leq t \leq 2D(\mathbf{A})-1$ such that by Lemma 2 types converge on the minimal type $g_0^{\min}$, i.e. for all $t \geq 2D(\mathbf{A})-1$: $g_t^i = g_0^{\min}$ for all $i \in N$. Since $g_0^{\min} < 1+r$, price converges. □

Proof of Proposition 3. By assumption $\tilde{p}_t$ is bounded for all $t \in \mathbb{N}$. Since dividends are are bounded, $d_t \in [\bar{d}-d^-, \bar{d}+d^+]$, and γ is some (finite) non-negative real number, we get that u_i^t is bounded for all $i \in N$ for all $t \in \mathbb{N}$. Hence, there exists $\zeta > 0$ such that

$$(\tilde{a}(t))_{ij} := \left(\sum_{k \in N^i} \exp(\gamma u_t^k)\right)^{-1} \exp(\gamma u_t^j) > \zeta \quad \Leftrightarrow \quad a_{ij} > 0 \quad \forall t \in \mathbb{N}$$

(otherwise as stated before, $(\tilde{a}(t))_{ij} = 0$).

Now, for all strongly connected and closed groups $\mathcal{C}_k$ (see Definition 1), denote by $\mathbf{A}_{\mathcal{C}_k}$ the restriction of $\mathbf{A}$ to $\mathcal{C}_k$. Hence, $\tilde{\mathbf{A}}_{\mathcal{C}_k}(t)$ is strongly connected with a positive diagonal and each entry is bounded below by ζ. Thus, for each $t \in N$, the matrix $\tilde{\mathbf{A}}_{\mathcal{C}_k}(t, t+n)$, defined by

$$\tilde{\mathbf{A}}_{\mathcal{C}_k}(t, t+n) := \tilde{\mathbf{A}}_{\mathcal{C}_k}(t+n) \cdot \tilde{\mathbf{A}}_{\mathcal{C}_k}(t+n-1) \cdot \ldots \cdot \tilde{\mathbf{A}}_{\mathcal{C}_k}(t),$$

is strictly positive with all entries bounded below by ζ^n for every closed and strongly connected group $\mathcal{C}_k$. Finally note that $\tilde{\mathbf{A}}_{\mathcal{C}_k}(t, t+n)$ is still row stochastic since the product of two row stochastic matrices is also row stochastic. If such a sequence of sub-accumulations $\tilde{\mathbf{A}}_{\mathcal{C}_k}(t, t+n)$ appears infinitely often, then by Lorenz (2007), Theorem 3.2.33 convergence to consensus in all closed and strongly connected groups is obtained. Hence, denoting by $\mathbf{g}_t^{\mathcal{C}_k}$ the type vector in period $t \in \mathbb{N}$ restricted to $\mathcal{C}_k$, we get that

$$\lim_{t\to\infty} \mathbf{g}_t^{\mathcal{C}_k} = \lim_{T\to\infty} \tilde{\mathbf{A}}_{\mathcal{C}_k}(0, T)\mathbf{g}_0^{\mathcal{C}_k}$$

exists, and is such that consensus is achieved, i.e. $g_\infty^i = g_\infty^j$ for all $i, j \in \mathcal{C}_k$. Since all $\tilde{\mathbf{A}}(t)$ are row stochastic, the consensus must be such that $g_\infty^{\mathcal{C}_k} \in [g_{\mathcal{C}_k}^{\min}, g_{\mathcal{C}_k}^{\max}]$ such that the consensus in the interior of the interval if $g_{\mathcal{C}_k}^{\min} \neq g_{\mathcal{C}_k}^{\max}$.

Finally, if $\lim_{t\to\infty} \tilde{p}_t = 0$, then expectations of all types converge to the fundamental price, $\lim_{t\to\infty} \tilde{E}_t^i[\tilde{p}_{t+1}] = 0$. Hence, $\lim_{t\to\infty} u_t^i = \lim_{t\to\infty} u_t^j$ for all $i, j \in N$. Thus, $\lim_{t\to\infty} \tilde{a}_{ij}(t) = \frac{1}{|N_i|} =: \bar{a}_{ij}$ for all $i \in R$. Thereby $\lim_{t\to\infty} \tilde{\mathbf{A}}_{\mathcal{R}\mathcal{R}}(t) = \lim_{t\to\infty} \bar{\mathbf{A}}_{\mathcal{R}\mathcal{R}}(t) = 0$, and hence $\lim_{t\to\infty} \tilde{\mathbf{A}}_{\mathcal{R}k}(0, t) = \left(I - \bar{\mathbf{A}}_{\mathcal{R}\mathcal{R}}\right)^{-1} \bar{\mathbf{A}}_{\mathcal{R}k}$. Thus, $\lim_{t\to\infty} \mathbf{g}_t^{\mathcal{R}} = \left(I - \bar{\mathbf{A}}_{\mathcal{R}\mathcal{R}}\right)^{-1} \bar{\mathbf{A}}_{\mathcal{R}k}\mathbf{g}^{\mathcal{C}}$. □

Proof of Proposition 4. Let $X = 0$. By (14), we get $sgn(R_t) = -sgn(\tilde{p}_t)$ if $0 < \bar{g}_t < (1+r)^2$ while $sgn(R_t) = sgn(\tilde{p}_t)$ if $\bar{g}_t > (1+r)^2$. If $0 = \bar{g}_t$ for some $t \in \mathbb{N}$ then $g_t^i = 0$ for all $i \in N$ and by weighted average updating convergence to the fundamental belief is obtained in which case there is nothing to show.

1. Suppose now $\bar{g}_0 < (1+r)^2$ then $sgn(R_0) = -sgn(\tilde{p}_0)$ and by part 1 of Lemma 1, we get that $g_0^i \geq g_1^i$ for all $i \in N$. Hence, $\bar{g}_1 < (1+r)^2$, implying $sgn(R_1) = -sgn(\tilde{p}_1)$. Repeatedly applying part 1 of Lemma 1 implies that $g_{t+1}^i \leq g_t^i < (1+r)^2$ for all $t \in \mathbb{N}$ and hence, $sgn(R_t) = -sgn(\tilde{p}_t)$ for all $t \in \mathbb{N}$.

   Lemma 2 then implies that for beliefs we have $g_t^i \to g_0^{\min}(P^i))$ for $\gamma \to \infty$ for all time steps $t \geq 2d(i, G_0^{\min}(P^i)) - 1$ for all $i \in \mathbb{N}$. In particular, each closed and strongly connected group $\mathcal{C}$ obtains a group consensus on $g_t^i \to g_0^{\min}(\mathcal{C})$ for all $i \in \mathcal{C}$ and each agent in the rest of the world $j \in \mathcal{R}$ does not change after obtaining the belief $g_0^{\min}(P^j)$ since $R_t\tilde{p}_t < 0$ for all $t \in \mathbb{N}$ such that $\exists i \in N$: $g_t^i \neq 0$ implying that Lemma 2 holds for all $t \in \mathbb{N}$.

2. If instead $\bar{g}_0 > (1+r)^2$ then $R_0\tilde{p}_0 > 0$ and by part 2 of Lemma 1, we get, completely analogously to above, that $g_0^i \leq g_1^i$ for all $i \in N$. Hence, $\bar{g}_1 > (1+r)^2$, implying $R_1\tilde{p}_1 > 0$. Repeatedly applying part 2 of Lemma 1 implies that $(1+r)^2 < g_t^i \leq g_{t+1}^i$ for all $t \in \mathbb{N}$ and hence, $R_t\tilde{p}_t > 0$ for all $t \in \mathbb{N}$.

Lemma 2 then implies that for beliefs we have $g_t^i \to g_0^{\max}(P^i)$ for $\gamma \to \infty$ for all time steps $t \geq 2d(i, G_0^{\min}(P^i)) - 1$. In particular, each closed and strongly connected group $\mathcal{C}$ obtains a group consensus on $g_t^i \to g_0^{\max}(\mathcal{C})$ for all $i \in \mathcal{C}$ and each agent in the rest of the world $j \in \mathcal{R}$ does not change does not change after obtaining the belief $g_0^{\max}(P^j)$ since $R_t\tilde{p}_t > 0$ for all $t \in \mathbb{N}$ such that $\exists i \in N$: $g_t^i \neq 0$ and hence Lemma 2 holds for all $t \in \mathbb{N}$.

3. Clearly, if $\bar{g}_0 > (1+r)^2$ price diverges since from Case 2 we have $\bar{g}_t > (1+r)^2$ for all $t \in \mathbb{N}$, implying price divergence by (9). Instead, suppose that $\bar{g}_0 > (1+r)^2 < 0$. By Case 1, $g_t^i = g_0^{\min}(P^i)$ for all $i \in N$, $t \geq 2D(\mathbf{A}) - 1$. Hence, $\bar{g}_t = \frac{1}{n}\sum_{i\in N} g_0^{\min}(P^i)$ for all $t \geq 2D(\mathbf{A}) - 1$ implying that price converges to the fundamental price, i.e. $\tilde{p}_t \to 0$, if $\frac{1}{n}\sum_{i\in N} g_0^{\min}(P^i) < 1 + r$, price converges to some finite limit price, if $\frac{1}{n}\sum_{i\in N} g_0^{\min}(P^i) = 1 + r$ and price diverges if $\frac{1}{n}\sum_{i\in N} g_0^{\min}(P^i) > 1 + r$, see (9).

□

Proof of Proposition 5. Let $\gamma \to \infty$ and $X > 0$.

1. Consider some closed and strongly connected group $\mathcal{C}$. We show that for each of the cases in Proposition 5, part 1, we have $R_t\tilde{p}_t < 0$ or $\bar{g}_t = 0$ for all $t \geq 2D(\mathbf{A}_\mathcal{C}) - 1$ such that the critical price is not crossed within the first $2D(\mathbf{A}_\mathcal{C}) - 1$ periods.

   (a) Suppose $\tilde{p}_0 < 0$ and $\bar{g}_0 < (1+r)^2$. Then $\tilde{p}_t < 0$ for all $t \in \mathbb{N}$ such that $\bar{g}_t > 0$ by (9). As before, in case $\bar{g}_t = 0$ there is nothing to show. Further, $p_0^{crit} > 0$ and, hence, $\tilde{p}_0 < 0 < p_0^{crit}$ implying $R_0 < 0$ by (14). Repeatedly applying part 1 of Lemma 1 implies that $g_t^i \geq g_{t+1}^i$ for all $t \in \mathbb{N}$. Hence, $\tilde{p}_t < 0 < p_t^{crit}$ for all $t \in \mathbb{N}$ such that $\bar{g}_t > 0$ implying $R_t\tilde{p}_t < 0$ for all $t \in \mathbb{N}$ such that $\bar{g}_t \neq 0$.

   (b) Next consider the case $g_0^{\min} > (1+r)^2$ and let $\frac{(1+r)^{2D(\mathbf{A}_\mathcal{C})-1}}{(g_0^{\max})^{2D(\mathbf{A}_\mathcal{C})-2}((1+r)^2 - g_0^{\max})}\frac{X}{n\delta} \leq \tilde{p}_0 < 0$. Note that $g_0^{\max} \geq \bar{g}_t > (1+r)^2$ for all $t \in \mathbb{N}$ by weighted average updating. Hence for all $t \leq 2D(\mathbf{A}_\mathcal{C}) - 1$ we get,

   $$\tilde{p}_t = \frac{\prod_{k=1}^t \bar{g}_k}{(1+r)^t}\tilde{p}_0 \geq \frac{(g_0^{\max})^{t-1}}{(1+r)^t}\bar{g}_t\tilde{p}_0 \geq -\frac{(1+r)^{2D(\mathbf{A}_\mathcal{C})-t-1}}{(g_0^{\max})^{2D(\mathbf{A}_\mathcal{C})-t-1}}\frac{\bar{g}_t}{g_0^{\max} - (1+r)^2}\frac{X}{n\delta} > p_t^{crit}$$

   since $p_t^{crit} = -\frac{\bar{g}_t}{\bar{g}_t - (1+r)^2}$ and $0 > \tilde{p}_0 \geq \frac{(1+r)^{2D(\mathbf{A}_\mathcal{C})-1}}{(g_0^{\max})^{2D(\mathbf{A}_\mathcal{C})-2}((1+r)^2 - g_0^{\max})}\frac{X}{n\delta}$ has been used. Note that (14) gives us $R_t > 0$ and, hence, $R_t\tilde{p}_t < 0$ for all $t \leq 2D(\mathbf{A}_\mathcal{C}) - 1$.

   (c) Finally, let $g_0^{\max} < (1+r)^2$ and $\frac{(1+r)^{2D(\mathbf{A}_\mathcal{C})-1}}{(g_0^{\min^*})^{2D(\mathbf{A}_\mathcal{C})-2}((1+r)^2 - g_0^{\max})}\frac{X}{n\delta} \leq \tilde{p}_0$. Note that $g_0^{\min^*} = \min\{1+r, g_0^{\min}\} > 0$ is implicitly assumed here, otherwise the condition on the initial price cannot be satisfied. Further, $g_0^{\min^*} \leq \bar{g}_t \leq g_0^{\max}$ for all $t \in \mathbb{N}$ by weighted average updating. Hence for all $t \leq 2D(\mathbf{A}_\mathcal{C}) - 1$ we get,

   $$\tilde{p}_t = \frac{\prod_{k=1}^t \bar{g}_k}{(1+r)^t}\tilde{p}_0 \geq \frac{\left(g_0^{\min^*}\right)^{t-1}}{(1+r)^t}\bar{g}_t\tilde{p}_0 \geq \frac{(1+r)^{2D(\mathbf{A}_\mathcal{C})-t-1}}{\left(g_0^{\min^*}\right)^{2D(\mathbf{A}_\mathcal{C})-t-1}}\frac{\bar{g}_t}{(1+r)^2 - g_0^{\max}} > p_t^{crit}$$

since $p_t^{crit} = \frac{\bar{g}_t}{(1+r)^2-\bar{g}_t}$ and $0 < \frac{(1+r)^{2D(\mathbf{A}_\mathcal{C})-1}}{(g_0^{\min *})^{2D(\mathbf{A}_\mathcal{C})-2}((1+r)^2-g_0^{\max})}\frac{X}{n\delta} \leq \tilde{p}_0$ has been used. Note that (14) gives us $R_t < 0$ and, hence, $R_t p_t > 0$ for all $t \leq 2D(\mathbf{A}_\mathcal{C}) - 1$.

Since we have shown that $R_t\tilde{p}_t < 0$ or $\bar{g}_t = 0$ for all $t \geq 2D(\mathbf{A}_\mathcal{C}) - 1$ is satisfied in all of the above cases, Part 1 of Lemma 2 implies that the strongly connected group $\mathcal{C}$ obtains a consensus on $g_0^{\min}$ which gives us the desired result.

2. Again, consider some closed and strongly connected group $\mathcal{C}$. We show that for the remaining cases in Proposition 5 we have $R_t\tilde{p}_t > 0$ for all $t \geq 2D(\mathbf{A}_\mathcal{C}) - 1$ which is works mostly analogously to part 1.

   (a) Suppose $\tilde{p}_0 > 0$ and $\bar{g}_0 > (1+r)^2$. Then $\tilde{p}_t > 0$ for all $t \in \mathbb{N}$ by (9). Further, $p_0^{crit} < 0$ and, hence, $\tilde{p}_0 > 0 > p_0^{crit}$ implying $R_0 > 0$ by (14). Repeatedly applying part 1 of Lemma 1 implies that $g_t^i \leq g_{t+1}^i$ for all $t \in \mathbb{N}$. Hence, $\tilde{p}_t > p_t^{crit}$ for all $t \in \mathbb{N}$ implying $R_t\tilde{p}_t > 0$ for all $t \in \mathbb{N}$.

   (b) Next consider the case $g_0^{\max} < (1+r)^2$ and let $0 < \tilde{p}_0 \leq \frac{1}{(1+r)^{2D(\mathbf{A}_\mathcal{C})-3}((1+r)^2-g_0^{\min})}\frac{X}{n\delta}$. By weighted average updating we get $\bar{g}_t < (1+r)^2$ for all $t \in \mathbb{N}$ since $g_{t+1}^{\max} \leq g_t^{\max}$ for all $t \in \mathbb{N}$. Hence for all $t \leq 2D(\mathbf{A}_\mathcal{C}) - 1$ we get,

   $$\tilde{p}_t = \frac{\prod_{k=1}^t \bar{g}_k}{(1+r)^t}\tilde{p}_0 \leq \frac{(1+r)^{2t-2}}{(1+r)^t}\bar{g}_t\tilde{p}_0 \leq \frac{1}{(1+r)^{2D(\mathbf{A})-t-1}}\frac{\bar{g}_t}{(1+r)^2 - g_0^{\min}}\frac{X}{n\delta} < p_t^{crit}$$

   since $p_t^{crit} = \frac{\bar{g}_t}{(1+r)^2-\bar{g}_t}$ and where $0 < \tilde{p}_0 < \frac{1}{(1+r)^{2D(\mathbf{A}_\mathcal{C})-3}((1+r)^2-g_0^{\min})}\frac{X}{n\delta}$ has been used. Note that (14) gives us $R_t > 0$ and, hence, $R_t\tilde{p}_t > 0$ or $\bar{g}_t = 0$ for all $t \leq 2D(\mathbf{A}_\mathcal{C}) - 1$.

   (c) Finally, let $g_0^{\min} > (1+r)^2$ and $\tilde{p}_0 \leq -\frac{1+r}{g_0^{\min}-(1+r)^2}\frac{X}{n\delta} < 0$. Note that $g_0^{\min} \leq \bar{g}_t \leq g_0^{\max}$ for all $t \in \mathbb{N}$ by weighted average updating. Hence for all $t \leq 2D(\mathbf{A}_\mathcal{C}) - 1$ we get,

   $$\tilde{p}_t = \frac{\prod_{k=1}^t \bar{g}_k}{(1+r)^t}\tilde{p}_0 \leq \frac{\left(g_0^{\min}\right)^{t-1}}{(1+r)^t}\bar{g}_t\tilde{p}_0 \leq -\frac{\left(g_0^{\min}\right)^{t-1}}{(1+r)^{t-1}}\frac{\bar{g}_t}{g_0^{\min} - (1+r)^2} < p_t^{crit}$$

   since $p_t^{crit} = -\frac{\bar{g}_t}{\bar{g}_t-(1+r)^2}$ and where $\tilde{p}_0 \leq -\frac{1+r}{g_0^{\min}-(1+r)^2}\frac{X}{n\delta} < 0$ has been used. Note that (14) gives us $R_t < 0$ and, hence, $R_t p_t > 0$ for all $t \leq 2D(\mathbf{A}_\mathcal{C}) - 1$.

   Applying Part 1 of Lemma 2 by setting $\bar{t} = 2D(\mathbf{A}) - 1$ implying $\bar{t} \geq 2D(\mathbf{A}_\mathcal{C}) - 1$ for all closed and strongly connected groups $\mathcal{C}$ gives us the desired result.

3. Finally note that for $\tilde{p}_0 < 0$ and $\bar{g}_0 < (1+r)^2$ we have shown $R_t\tilde{p}_t < 0$ or $\bar{g}_t^i$ for all $t \in \mathbb{N}$ in case 1a and for $\tilde{p}_0 > 0$ and $\bar{g}_0 > (1+r)^2$ we have shown $R_t\tilde{p}_t > 0$ or $\bar{g}_t^i$ for all $t \in \mathbb{N}$. Hence, by Lemma 2 we get the desired result for the rest of the world.

□