Version: November 13, 2022

Heterogeneous Attitudes toward Risk, Growth, and Redistribution

Aloisio Pessoa de Araujo
IMPA and EPGE/FGV

Juan Pablo Gama
Universidade Federal de Minas Gerais

Timothy J. Kehoe
University of Minnesota,
Federal Reserve Bank of Minneapolis,
and National Bureau of Economic Research

ABSTRACT ___

We develop an endogenous growth model of overlapping generations in which agents leave warm-glow bequests. There are dynasties of risk neutral investors and dynasties of risk averse investors. We start with a simple model in which risk averse investors can invest only in a safe asset while risk neutral investors, whom we often refer to as entrepreneurs, can invest in a risky asset with a higher expected return. This simple structure allows us to analytically calculate the invariant distributions of wealth holdings. We define a social welfare function for this model and calculate tax and transfer policies that maximize social welfare in the invariant distribution. We extend our results to models where (1) risk averse investors can invest in the risky asset, (2) a fraction of risk neutral parents have risk averse children and vice-versa, and (3) there is both labor and capital and endogenous wages and rental rates.

Keywords: Endogenous growth, Inequality, Redistribution, Overlapping generations, Invariant distribution, Social welfare function.

JEL Codes: C62, D51, H21, O4

* We thank participants at the seminar at IMPA, the Federal Reserve Bank of Minneapolis, University of Chicago, University of Minnesota for comments, and NYU especially Fabrizio Perri, Ellen Phelan, Manuel Amador, Jonathan Heathcoat, Varadarajan Chari, Lars Hansen, Jim Heckman, Nancy Stokey, Harold Uhlig, Alberto Bisin, Jess Benhabib, and Debraj Ray. The authors gratefully acknowledge financial support from CAPES. Gama thanks IMPA for generous financial support. The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

1. Introduction

The role of inequality and taxation in growth has been extensively studied in a variety of models. Kuznets (1955) was one of the first economists who analyzed economic growth and inequality. In his model, inequality tends to increase once the economic growth increases due to industrial revolution since low productivity labor with low inequality among workers is substituted with high productivity labor with high inequality among workers. Once the country achieves a middle-high level of industrialization, however, inequality decreases due to the development of a welfare state.

More recently, several authors analyzed theoretical models in which growth and inequality could be analyzed by general equilibrium models. Persson and Tabellini (1994) studied the impact of growth and inequality in democratic countries. In their model, redistributive taxes may affect the accumulation of capital implying a reduction in the long-term growth. Alesina and Rodrik (1994) developed a model with capitalists and workers in which the optimal taxes with the highest growth rates benefitted capitalists more.

For most of the classical literature in growth and inequality, taxes and redistribution have a negative impact on growth due to the loss of capital in the economy. More recently, some authors including Persson and Tabellini (1994) and Perotti (1996) argued that inequality has a negative impact on the amount of capital invested in the firms due to a possible increment in redistributive taxes caused by a high level of inequality in the economy. Most of the authors who argued that inequality and growth are inversely correlated, as the ones mentioned before, based their studies on empirical evidence using cross-country growth regressions. However, as Temple (1999) argued, most of these studies have been criticized due to the fragility of several of their results and their ad hoc specialization.

We consider three models, one without growth and two with endogenous growth rate and heterogeneous production technologies - one with segmentation and one with risk lovers - to analyze the dynamics of taxation in OLG economies with risk lovers. In the latter models, taxes and redistribution have a negative impact on growth if the more productive technologies involve larger amount of idiosyncratic risk. However, taxation on bequest and income have different effects on growth and inequality. Moreover, in absence of taxes, the most productive technologies will dominate the economy in the long run, and inequality in the long run will depend mainly on the risk

Comentado [TJK1]: Refer to later

Comentado [TJK2]: Refer to literature on redistributive taxation and capital accumulation. Put this later.

Comentado [TJK3]: Replace with extended version of abstract that the most productive one involves. In the presence of taxes and different expected technology returns, bequest or income taxes ensure the existence of an invariant distribution of wealth among the agents and an invariant growth rate of the economy. We also show that the invariant distribution with a single positive type of tax with constant marginal tax rate is also ergodic as in Piketty (1997), but only among the agents of the same type. Therefore, there is no poverty trap among the agents with the most productive and most risky technologies. Among the agents that do not have access to the most productive technologies, their wealth might not reach the top in any future date.

The study of the models are closely related. The distribution and convergence to the invariant distribution of the former model is strongly related to the latter ones. Additionally, the second one can be seen as a restricted case of the latter if the risky technology is more productive than the safe one as we assume in most part of the article, allowing us to have a better understanding of the latter model.

We study optimal taxation by introducing a central planner who chooses taxes to maximize the social welfare of the economy. We show that the social welfare function of the social planner can be written as the sum of three independent functions. The first one depends only on growth, the second one depends only on inequality, and third one depends only on the difference of the discount factors of the agent and the social planner. The first one is directly correlated with the growth rate of the economy, implying that the presence of low taxes might be optimal in some cases. The second function is inversely correlated with the inequality of the invariant distribution which implies that high taxes might be optimal in some cases. We prove that bequest taxes are always worse than income taxes for discount rate of the agents and the social planner.

We also find that, for a fixed discount rate of the consumers[1], the optimal taxation is strictly decreasing on how the social planner discounts the future. Moreover, the optimal tax is such that the invariant wealth distribution tends to a completely equal one if the social planner strongly discounts the future. On the other hand, the optimal tax is zero if the social planner does not discount the future at all. The intuition behind these results is that, if a social planner discounts the future strongly, the weight of distant dates and the growth rate become almost irrelevant. Therefore, the social welfare function is dominated by the inequality effect. However, if a social planner

[1] We consider the discount rate of the agent as the bequest rate since, as we will show latter on, the bequest is a form of how the agent is concerned about his/her descendants' future consumptions.

discounts the future very weakly, the weight of future consumption will dominate the inequality effect even when both effects are considerably large. Note that, in the presence of a social planner that almost does not discount the future, inequality affects strongly the social welfare function due to the presence of a very unequal distribution of wealth that causes a high impact in the social welfare function.

We developed an overlapping generations model with bequest as in Galor and Zeira (1993). To analyze the impact of different production technologies on the accumulation of wealth, we use a model with bequest and idiosyncratic uncertainty on the technologies as in Piketty (1997). However, we consider frictions on the use of the technologies separating the agents in two groups: skilled and unskilled, and risk lovers and risk averters. As it was mentioned before, we also consider redistributive taxes as in Alesina and Rodrik (1994). The impact of different technologies with different levels of idiosyncratic uncertainty can also be related to models with different attitudes toward risk as in Araujo, Gama and Kehoe (2017) and Araujo, Chateauneuf, Gama and Novinski (2018).

The idea that high taxes should be imposed due to the existing spread between asset returns and real returns has been explored by Piketty and Saez (2014). They argued that changes in growth rate in western economies, mainly in the US, has a strong impact on inequality due to the gap among the interest rate and the real GDP growth rate. Other authors as Lindert and Williamson (2016) studied long term data to analyze the inequality in the US economy since colonial times. Bargain et al. (2015) made a deeper analysis of the tax policy and inequality from 1979-2007 in which changes in the tax policy increased income inequality causing more accumulation among the top one percent. Weide and Milanovic (2014) showed with a study on micro data of the US economy from 1960 to 2010 that high levels of inequality reduce the economic growth of the poor, but it might enhance the economic growth of the rich. Benhabib, Bisin, and Zhu (2011, 2015, 2016), Jones (2015), and Acemoglu and Robinson (2015) analyze income or wealth distribution with taxation. Gabaix, Lasry, Lions, and Moll (2016) (analytically), and Aoki and Nirei (2017) (numerically) analyze the dynamics of income distribution with taxes. And Garcia-Peñalosa and Wen (2008) analyze the effect of taxation on growth with risk averse agents. Most of works mentioned above support the idea analyzes Pareto distribution of wealth or income. Moreover, Benhabib, Bisin, and Zhu (2011, 2015, 2016) found conditions to generate fat tails for transformation processes induced by investment risk. On the other hand, Beare and Toda (2018) show that tails of wealth distribution decay exponentially in a heterogeneous-agent dynamic general equilibrium model with idiosyncratic endowment risk. Our model has larger similarities with the former than with the latter. However, we analyze the effect of different marginal taxation rate on growth, showing that there is a trade off between growth and taxation for middle and high marginal taxation rates from which there is no empirical evidence against it.

Our results support some of the ideas mentioned above since low taxes imply higher growth rates and high levels of inequality. At the same time, a reduction of income taxes in our model will change the invariant distribution to a more unequal one, then it will cause a gradual increment of the inequality supporting several empirical studies mentioned above. Moreover, our model suggest that changes in the tax policy may be based on changes on how the social planner discounts the future compared with the other agents do so.

There is an important exception to our result that taxes reduce growth: We identify parameter values for the model where agents are restricted to invest in only one type of assets in which high enough taxes and transfers insure risk averters and induce poor risk averters to invest in the risky asset. Those risk averters who are lucky and accumulate a large enough level of wealth choose to switch to investing in the safe asset. In this case, increasing taxes and increases growth and the welfare of risk averters although it decreases the welfare of risk lovers.

The paper is organized as follows. In section 2, we define the basic model including the notion of equilibrium. In subsection 2.2, we define the basic properties of the model including the relationship between growth and inequality without taxes, and, in subsection 2.3, we analyze the basic properties with taxes. In section 2.4, we show the existence of an invariant growth rate, an invariant distribution of wealth among the agents and their basic properties. In section 2.5, we analyze the existence of optimal taxes by a social welfare function, and we also prove the basic properties of this function and of the optimal taxes. In subsection 2.6, we give some numerical examples that help us to the analysis made in section 2.5. In section 3, we analyze the case with effort cost and the extension to a model with capital and labor. Finally, in section 4, we give some concluding remarks.

2. Model with production and segmentation and taxation

Let us consider an overlapping generation economy with a continuum of two-period agents (young and old). There are two different types of agents *unskilled* or type a, and unskilled or type l. The former can use only one of the available technologies in the economy. On the other hand, the latter can invest in the two types of technologies available. These technologies are linear and represented by $R_S: \{1,2\} \to \mathbb{R}_+$ and $R_R: \{1,2\} \to \mathbb{R}_+$ where R_S is the safe one, the technology available for both types of agents, and R_R is the risky one, available for the *skilled* agents only. Then, the returns of the safe technology are represented by a constant value $R_S > 0$, and the returns of the risky technology are represented by the vector $(\overline{R}_R, \underline{R}_R)$ where $\overline{R}_R > 0$ if event 1 occurs, and $\underline{R}_R \geq 0$ if event 2 occurs. Note that the probability of each event is equal to $1/2$ and independent among the agents. Due to the risk involved in the technologies used by the agents, each agent is exposed to idiosyncratic risk caused by the uncertainty of the technologies used. Then, the uncertainty in our model is independent among the agents, which implies that, in the aggregate economy, there is no aggregate uncertainty. Note also that the investment in one of the technologies by an agent is a *one-period* investment.

For any date $t \geq 1$, there is a single consumption good at every state s with date $t \geq 1$ that the young agents will use it to invest in the technologies, and the old agents will use it to consume, c_s^i, and to give a bequest, b_s^i, to his successor. All the agents give a bequest that is a proportion of the agent's total wealth. In $t = 0$, there is no consumption since there is no old generation, and every young agent has an initial endowment w_0^i, to be invested in the technologies.

2.1.Taxes

At each state s of length $t \geq 1$, there is an income tax ${\tau^I}_s^+(\cdot)$ and bequest tax ${\tau^B}_s^+(\cdot)$ imposed for any agent if her level of income, consumption and bequest is above some threshold $\overline{W}_s^i$ and $\overline{B}_s$, respectively. Additionally, there is also an income subsidy ${\tau^I}_s^-(\cdot)$ and bequest subsidy ${\tau^B}_s^-(\cdot)$ given to any agent with an income, consumption, and bequest below some threshold $\underline{W}_s^i$ and $\underline{\mathrm{B}}_s$, respectively. For simplicity, each type of tax will be summarized by ${\tau^I}_s(\cdot) = {\tau^I}_s^+(\cdot) + {\tau^I}_s^-(\cdot)$ and ${\tau^B}_s(\cdot) = {\tau^B}_s^+(\cdot) + {\tau^B}_s^-(\cdot)$. From now on, each type of tax is defined by a constant marginal tax rate above the upper threshold and a constant marginal subsidy rate below the lower threshold. ${\tau^I}_s^+, {\tau^I}_s^- \in [0,1]$ are the marginal rates related to the income policy, the ones related with consumption, and ${\tau^B}_s^+, {\tau^B}_s^- \in [0,1]$ the ones related with bequests. Therefore, the income tax mentioned above can be written as ${\tau^I}_s(\mathrm{x}) = {\tau^I}_s^+(\mathrm{x} - \overline{W}_s)^+ + {\tau^I}_s^-\left(\underline{W}_s - \mathrm{x}\right)^+$ and the bequest tax can be written as ${\tau^B}_s(\mathrm{b}) = \frac{{\tau^B}_s^+}{1-{\tau^B}_s^+}(\mathrm{b} - \overline{B}_s)^{\pm} \frac{{\tau^B}_s^-}{1-{\tau^B}_s^-}\left(\underline{B}_s - \mathrm{b}\right)^+$. Note that the marginal tax rates τ_s^{I+}

and τ_s^{B+} as well as the thresholds are exogenously defined by the central planner. On the other hand, the marginal subsidy rates τ_s^{I-} and τ_s^{B-} are endogenously determined in equilibrium to ensure a balanced government budget.

Therefore, the government can define the tax policy by choosing the marginal tax rates τ_s^{I+}, and τ_s^{B+} and the thresholds $\underline{W}_s$, $\overline{W}_s$, $\underline{B}_s$ and $\overline{B}_s$. For simplicity, $\underline{W}_s = \overline{W}_s = \overline{w}_s$ where $\overline{w}_s$ is the average income of the economy in state s, and $\underline{B}_s = \overline{B}_s = \overline{b}_s$ where $\overline{b}_s$ is the average bequest of the economy in state s. Therefore, $\tau_s^{I+} = \tau_s^{I-} = \overline{\tau}_s^I$, and $\tau_s^{B+} = \tau_s^{B-} = \overline{\tau}_s^B$.

Additionally, we will suppose that

$$\overline{\mathrm{R}}_\mathrm{R} > \overline{\mathrm{R}}_\mathrm{S} \geq \underline{\mathrm{R}}_\mathrm{S} > \underline{\mathrm{R}}_\mathrm{R} \geq 0 \tag{3.1}$$

which implies that the R-technology involves higher levels of risk compared to the S-technology such that

$$E[R_R] \geq \frac{1}{\delta} \geq E[R_S] \tag{3.2}$$

where $\delta \in (0,1)$ is the natural bequest rate that will be explained properly latter on.

As it was mentioned above, the initial endowment is given by an initial amount of the available good, $\{w_0^i\}_i$, and then, the problem of the agent i in the first date is defined by

$$\begin{aligned}
\max_{(c,b,\theta)} \quad & \frac{1}{2}u^i(c_1,b_1) + \frac{1}{2}u^i(c_2,b_2) \\
s.t. \quad & \theta_R + \theta_S \leq w_0^i, \\
& 0 \leq c_1 + b_1 + \frac{\tau_s^B}{1-\tau_s^B}(b_1) \leq \overline{R}_R\theta_R + R_S\theta_S - \tau_s^I(\overline{R}_R\theta_R + R_S\theta_S), \\
& 0 \leq c_2 + b_2 + \frac{\tau_s^B}{1-\tau_s^B}(b_2) \leq \overline{R}_R\theta_R + R_S\theta_S - \tau_s^I\left(\underline{R}_R\theta_R + R_S\theta_S\right).
\end{aligned}$$

where $u^i(c,b)$ is the utility index of the agent i. For $t \geq 1$, w_0^i is substituted by b_s^i the bequest that the agent i receives from her predecessor at state s. Each agent has a utility index given by $u^i(c,b) = c^{1-\delta}b^{\delta}$. Using the form of the tax policy, the budget constraint when the agent i is old can be written as

$$0 \leq c_1 + +b_1 + \frac{\overline{\tau}_s^B}{1-\overline{\tau}_s^B}\left(b_1 - \overline{b}_s\right) \leq \overline{R}_R\theta_R + R_S\theta_S + \overline{\tau}_s^I\left(\overline{w}_s - \overline{R}_R\theta_R - R_S\theta_S\right),$$
$$0 \leq c_2 + b_2 + \frac{\overline{\tau}_s^B}{1-\overline{\tau}_s^B}\left(b_2 - \overline{b}_s\right) \leq \underline{R}_R\theta_R + R_S\theta_S + \overline{\tau}_s^I\left(\overline{w}_s - \underline{R}_R\theta_R - R_S\theta_S\right).$$

Then, each agent receives three types of transfers that depends on the average income, consumption, and bequest. Additionally, income tax reduces the agent income by a proportion of $1 - \overline{\tau}_s^I$, and consumption and bequest taxes make more expensive to consume and to give part of her income as bequest, respectively. Therefore, this constraint can be written as

$$0 \leq c_1 + \left(1 + \frac{\overline{\tau}_s^B}{1 - \overline{\tau}_s^B}\right) b_1 \leq \left(1 - \overline{\tau}_s^I\right)\left(\overline{R}_R\theta_R + R_S\theta_S\right) + \overline{\tau}_s^I\overline{w}_s + \frac{\overline{\tau}_s^B}{1 - \overline{\tau}_s^B}\overline{b}_s,$$
$$0 \leq c_2 + \left(1 + \frac{\overline{\tau}_s^B}{1 - \overline{\tau}_s^B}\right) b_2 \leq \left(1 - \overline{\tau}_s^I\right)\left(\underline{R}_R\theta_R + R_S\theta_S\right) + \overline{\tau}_s^I\overline{w}_s + \frac{\overline{\tau}_s^B}{1 - \overline{\tau}_s^B}\overline{b}_s.$$

Note that, in our model, the existence of income taxes can also be seen as wealth taxes since the capital is completely transformed in each state s. Therefore, we will focus mainly on interpret τ^I as income taxes. However, we understand the clear difference between both concepts, that coincide due to the capital properties of our model.

2.2. Equilibrium

Now, let us define the equilibrium for the economy as $\left(\left(c^i, b^i, \theta^i\right)_i, (\tau_s^I, \tau_s^B)_s\right)$ such that $\left(c^i, b^i, \theta^i\right)$ maximizes the consumer problem mentioned above for any $t \geq 0$, and for each state s of length $t \geq 1$, we have

$$\int_i \tau_s^B\left(b_s^i\right) = 0,$$
$$\int_i \tau_s^I\left(R_{R,s}\theta_R^i + R_S\theta_S^I\right) = 0.$$

Due to the FOC, we know that

$$b_s^i = \frac{\delta\left(1 - \overline{\tau}_s^B\right)}{(1-\delta)} c_s^i$$

for all agent i, then in absence of income or bequest taxes, each agent will bequest a proportion δ of her income and consume the other part. However, in the presence of consumption or bequest taxes, the agent will deviate from this proportion since the cost of consuming or requesting increasing due to taxation.

Based on the effects of bequest taxes on consumer's problem, the average consumption and bequest can be written as

$$\bar{c}_s = \frac{(1-\delta)\overline{w}_s}{1-\delta\bar{\tau}_s^B},$$
$$\bar{b}_s = \frac{\delta\left(1-\bar{\tau}_s^B\right)\overline{w}_s}{1-\delta\bar{\tau}_s^B}.$$

Due to the form of the utility index and Equation (3.2) a skilled agent l_i will never invest in the safe technology, that is, $\theta_t^{l_i} = 0$ for all $i \in [0,1]$. Therefore, all agents invest only in one technology at the same time. The unskilled ones invest in the safe one, the less productive, and the skilled ones invest in the risky, the most productive one.

Since bequest taxes affect the incentives that each agent has for consumption and bequest, the average consumption and the average bequest also depends on these marginal taxation rates. Moreover, higher bequest taxes imply a larger proportion of consumption by all agents and a lower proportion of bequest, which decreases the descendant income. More specifically, we have that if $\overline{w}_s^a$ is the after taxes mean income of the *unskilled* at the node s, and $\overline{w}_s^l$ is the after taxes mean income of the *skilled* at the node s, the average income at a node s' an immediate successor of s is given by

$$\overline{w}_{s'} = \frac{E[R_R]\delta\left(1-\bar{\tau}_s^B\right)}{1-\delta\bar{\tau}^B}\overline{w}_s^l + \frac{R_S\delta\left(1-\bar{\tau}_s^B\right)}{1-\delta\bar{\tau}^B}\overline{w}_s^a, \tag{3.3}$$

which implies the following result.

Proposition 1. For any fiscal policy plan with marginal tax rates given by $\left(\bar{\tau}_s^I, \bar{\tau}_s^B\right)_s$, such that $\bar{\tau}_s^I, \bar{\tau}_s^B < 1$, we have that any increment on the income tax rate at state s (from $\bar{\tau}_s^I$ to $\bar{\tau}_s^I + \epsilon$) induces a higher growth rate than an increment on the bequest tax rate (from $\bar{\tau}_s^B$ to $\bar{\tau}_s^B + \epsilon$) at state s' the immediate successor of s.

2.3. Dynamic properties of the equilibrium

From now on, we will analyze the dynamic properties of the equilibrium and the existence of an invariant distribution of income. From now on, let us suppose that the fiscal policy plans satisfy that the marginal tax rates are constant over time, that is, $\overline{\tau}_s^I = \overline{\tau}_{s'}^I$, and $\overline{\tau}_s^B = \overline{\tau}_{s'}^B$ for all s, s'. Then, we will simply denote the fiscal plans as $\left(\overline{\tau}^I, \overline{\tau}^B\right) \in [0,1]^2$.

Note that there is no aggregate uncertainty. It is a consequence of the continuum numbers of agent and the idiosyncratic risk that each agent has once they invest in the available technologies for them. Therefore, from now on, we will denote each state s at date t simply as t, and for the successors of s at date t we will denote as $t+1$ for aggregate variables. However, for individual variables like optimal consumption, bequest, and income of an agent i, we will denote a successor of a node s as s, k with $k = 1, 2$.

2.4. Invariant distribution

From now on, let us assume that $\underline{R}_R = 0$ for simplicity, and, in case that this condition does not hold, we will inform you. Let us now analyze the existence of an invariant distribution of relative wealth, that is, the distribution of wealth of the agents divided by the aggregate wealth in each state. From now on, we will focus our attention in positive fiscal policies, that is, $\left(\overline{\tau}^I, \overline{\tau}^B\right)$ since in the absence of taxes there is no invariant distribution if $\underline{R}_R = 0$. Moreover, if $\underline{R}_R > 0$, the wealth of the skilled agents will have all the wealth in the economy in the long run in even when the wealth is not completely in hands of the agents who get lucky all the time.

Proposition 2. There is no invariant concentration of wealth if $\overline{\tau}^I = \overline{\tau}^B = 0$ unless $\underline{R}_R = \overline{R}_R = R_S$. In this case, any initial endowment distribution such that $x^{l_i} = 0$ for all i $\in [0,1]$ is an invariant distribution.

We have that any initial distribution converges to an invariant distribution. The following Theorem shows that any tax policy with constant and nonnegative marginal taxation rates implies that the distribution of income converges to an invariant distribution in the long run.

Theorem 1. Given a fixed marginal tax rate $\left(\bar{\tau}_s^I, \bar{\tau}_s^B\right)_s \in [0,1]^2$ for any initial distribution of endowment $\{w_0^i\}_i$, there is an invariant distribution of the proportion of wealth among the agents.

2.5.Social welfare function and optimal tax rate

We know that any increment of the marginal tax rate causes a change in the invariant distribution and the long-run growth rate of the economy. The invariant distribution will tend to be more equal among the agents, and the long-run growth rate might also decrease if a social planner taxes more income.

Cases like $\bar{\tau}^{\mathrm{I}}, \bar{\tau}^{\mathrm{B}} = 0$, or $\bar{\tau}^I = 1$, or $\bar{\tau}^B = 1$ are too extreme in this framework. The first one will imply the survival of a small amount of agents with arbitrarily large amount of wealth only. The second one will imply a considerably lower growth rate of the economy and in some cases even negative growth rates. The third one, consumption taxes equal to 1, will imply that any agent cannot consume any amount of good. And the fourth one, bequest taxes equal to 1, imply that any agent cannot bequest to the next generation. These two latter conditions imply that the agents will not survive in one way or another. Then, we will focus on less extreme type of taxation plans.

Therefore, the analysis of an optimal tax rate should consider inequality and growth rate. Since all agents has linear utility index, the whole dynasty will be only worried about the expected return of their investments in the long run than to have levels of wealth that are bounded from below in a positive measure set of paths. Therefore, a more suitable welfare function for a taxation plan $\hat{\tau} = (\overline{\tau}^I, \overline{\tau}^B) \in [0,1]^2 \setminus \{(0,0)\} = \mathcal{T}$ could be defined by

$$W\left(\left(U^i\right)_i, \left(c^i_{\hat{\tau},t}\right)_i, \left(b^i_{\hat{\tau},t}\right)\right) := \sum_{t=1}^{\infty} \mathrm{d}^t \int log U^i\left(c^i_{\hat{\tau},t}\ b^i_{\hat{\tau},t}\right) di \qquad (3.8)$$

where $d \in (0,1)$ is the discounted factor used by the social planner. In this case, the social welfare function does not have problems related with the convergence of the series when the economy has positive growth rate in the long run since $\log \mathrm{U}^{\mathrm{i}}\left(\mathrm{c}^{\mathrm{i}}_{\hat{\tau},\mathrm{t}}, \mathrm{b}^{\mathrm{i}}_{\hat{\tau},\mathrm{t}}\right)$ is at most linear.

Note that for any fixed marginal tax rate plan $\widehat{\tau} \in [0,1]^2$, the consumption and bequest in equilibrium will depend strongly on the marginal taxation rates $\hat{\tau}$. Therefore, to avoid any confusion, we will denote the consumption and bequest in equilibrium will be denoted using the tax rate used by the social planner, that is,$\left(\left(c^i_{\hat{\tau},t}\right)_i, \left(c^i_{\hat{\tau},t}\right)_i\right)$.

Let us analyze the equilibrium with an initial endowment consistent with the invariant distribution, that is, $\left(\left((1-\delta)\mathrm{x}^{\mathrm{i}}_{\tau}(1+\mathrm{g}_{\tau})^{\mathrm{t}}\right)_{\mathrm{i}}, \left(\delta \mathrm{x}^{\mathrm{i}}_{\tau}(1+\mathrm{g}_{\tau})^{\mathrm{t}}\right)_{\mathrm{i}}\right) = \left(\left(\frac{(1-\delta)\left(1-\overline{\tau}^C\right)}{1-\delta\overline{\tau}^B}\right) x^i_{\hat{\tau}}\left(1+g_{\hat{\tau}}\right)^t, \left(\frac{\delta\left(1-\overline{\tau}^B\right)}{1-\delta\overline{\tau}^B}\right) x^i_{\hat{\tau}}\left(1+g_{\hat{\tau}}\right)^t\right)$, the welfare function can be re written as

$$W\left(\left(U^i\right)_{i'}\left(c^{\underset{\frown}{i}}_{\tau,t}\right)_i,\left(b^{\underset{\frown}{i}}_{\tau,t}\right)\right): \quad = \sum_{t=1}^{\infty} \mathrm{d}^t \int log U^i \left(c^{\underset{\frown}{i}}_{\tau,t}\ b^{\underset{\frown}{i}}_{\tau,t}\right) di$$

$$= \sum_{t=1}^{\infty} \mathrm{d}^t \int \delta log\left(\left(\frac{\delta\left(1-\overline{\tau}^B\right)}{1-\delta\overline{\tau}^B}\right)\left(x^{\underset{\frown}{i}}_{\tau} + x^{[[i+1/2]]}_{\underset{\frown}{\tau}}\right)\left(1+g_{\underset{\frown}{\tau}}\right)^t\right) di$$

$$+\sum_{t=1}^{\infty} \mathrm{d}^t \int (1-\delta) log\left(\left(\frac{(1-\delta)\left(1-\overline{\tau}^C\right)}{1-\delta\overline{\tau}^B}\right)\left(x^{\underset{\frown}{i}}_{\tau} + x^{[[i+1/2]]}_{\underset{\frown}{\tau}}\right)\left(1+g_{\underset{\frown}{\tau}}\right)^t\right) di$$

$$= \sum_{t=1}^{\infty} \mathrm{d}^t \int log\left(\left(x^{\underset{\frown}{i}}_{\tau} + x^{[[i+1/2]]}_{\underset{\frown}{\tau}}\right)\left(1+g_{\underset{\frown}{\tau}}\right)^t\right) di$$

$$+\sum_{t=1}^{\infty} d^t \int log\left(\left(\frac{\delta^{\delta}(1-\delta)^{1-\delta}\left(1-\overline{\tau}^B\right)}{1-\delta\overline{\tau}^B}\right)\right) di \quad (3.9)$$

where $[[\cdot]]: \mathbb{R} \to [0,1)$ is the function that considers the non-integral part of a number, that is, $[[10.45]] = 0.45$.

Using the properties of the logarithm, we have that we can separate the social welfare function W in three different parts, one that only depends on the invariant distribution implying that it is strictly increasing with τ, another component that only depends on the growth rate of the economy which means that is strictly decreasing with τ, and a component that depends on the bequest rate of the agents. This separation is given by

$$W\left(\left(U^i\right)_{i'}\left(c^{\underset{\frown}{i}}_{\tau,t}\right)_i,\left(b^{\underset{\frown}{i}}_{\tau,t}\right)\right) = \frac{\mathrm{d}}{1-d}\int log\left(x^{\underset{\frown}{i}}_{\tau} + x^{[[i+1/2]]}_{\underset{\frown}{\tau}}\right) + \sum_{t=1}^{\infty} d^t \int log\left(\left(1+g_{\underset{\frown}{\tau}}\right)^t\right) di +$$
$$\sum_{t=1}^{\infty} \mathrm{d}^{\mathrm{t}} \int log\left(\left(\frac{\delta^{\delta}(1-\delta)^{1-\delta}\left(1-\overline{\tau}^B\right)}{1-\delta\overline{\tau}^B}\right)\right) di \quad (3.10)$$

Therefore, we have the following result.

Proposition 3. Under the hypotheses mentioned above, if the initial endowment distribution is consistent with the invariant concentration of wealth for the marginal tax rate $\widehat{\tau} \in \mathcal{T}$ chosen by the social planner, the social welfare function, W, in the equilibrium allocation can be written as $W\left(\left(U^i\right)_{i'}\left(c^{\underset{\frown}{i}}_{\tau,t}\right)_i,\left(b^{\underset{\frown}{i}}_{\tau,t}\right)\right) = X\left(\mathrm{d},\widehat{\tau}\right) + G\left(\mathrm{d},\widehat{\tau}\right) + D(\mathrm{d})$, where $X: [0,1) \times \mathcal{T} \to \mathbb{R}$,

$G: [0,1) \times \mathcal{T} \to \mathbb{R}$, and $D: [0,1) \to \mathbb{R}$ are differentiable functions in $(0,1) \times (0,1)^3$, strictly increasing in the first component, and, in the second component, G is strictly decreasing.

As it was mentioned above, there are two different things that are affecting the social welfare function, the invariant concentration of wealth (X) and the growth of the economy (G). Therefore, for a fixed discount rate for the social planner $d \in (0,1)$ and the bequest rate $\delta \in (0,1)$ there is a trade off between growth and inequality since growth rate tends to increase and inequality tends to decrease when taxes are diminished. However, each type of taxation has different implications on growth and inequality. In general, income taxes are the ones that reduce more inequality, and bequest taxes are the ones that generate higher consumption in the first dates. Therefore, it is not completely natural to determine each combination of taxes are better.

The characterization of the social welfare function it is extremely useful to understand the phenomena behind the marginal taxation rate, the inequality, the growth rate of the economy and the relationship between the bequest rate and how the social planner discounts the future. As it can observed, inequality and growth are in opposite direction in the social welfare function. When you increase a marginal tax rate, growth and inequality will always reduce. However, their impact in the social welfare function is not comonotonic since the growth term of the social welfare function is comonotic with the growth of the economy due to the monotonicity of the logarithmic function. On the other hand, the social welfare function is anticomonotonic with respect to inequality since the logarithmic function is a strictly concave function that always decreases if you consider a more diverse type of distribution or variable[2]. Consequently, the social planner must find a balance between low taxes to have large economic growth and high taxes to reduce inequality.

[2] This is the case of martingales in probability, that is, a process that has the same expected conditional value based on previous information, but it diversifies it values over time. In this case, if $(M_t, F_t)_t$ with $F_t \subset F_s$ for all $t \leq s$ is a martingale, that is, M_t is a F_t-measurable function such that $E[M_{t+s}|F_t] = M_t$ almost certainly, then,

Since G and X are logarithm functions related to the growth and inequality, we conjecture that both functions are strictly concave functions.

Conjecture 1. $G(d,\cdot)$ and $X(d,\cdot)$ are strictly concave functions for every $d \in (0,1)$.

Therefore, we have that the social welfare function is a strictly concave function respect to τ.

Conjecture 2. The function $W\left(\left(U^i\right)_i, \left(c^i_{\hat{\tau},t}\right)_i, \left(b^i_{\hat{\tau},t}\right)_i\right)$ is strictly concave respect to τ for every $d \in (0,1)$, and, then, there is only one marginal tax rate, τ_d, that maximizes the social welfare function.

Intuitively, if a social planner is more worried about distant consumptions, it will give more attention to growth than inequality since the latter is maintained over time since we analyze the inequality of the invariant distribution, and the former is strongly related with distant consumption since by definition, an increment in the discount facto of the social planner, increases the weight of the future events and the only thing that changes over time in this equilibrium allocation is the growth rate of the economy. Therefore, it is natural to think that a social planner who decides to be more concerned about the future will choose a lower marginal tax rate than a social planner who do not.

$E[g(M_{t+s})|F_t] \leq g(M_t)$ almost certainly if g is a concave function. Note that in this case, the invariant concentration of wealth is not a martingale. It is more as a analogy of what happens.

Conjecture 3. In absence of bequest taxes, let be τ_{d} the optimal tax for a social planner given by $\mathrm{d} \in (0,1)$. If $\mathrm{d}_1 < \mathrm{d}_2$ then $\tau_{\mathrm{d}_1} > \tau_{d_2}$.

Mathematically, if the social planner is concerned more about distant consumptions, that is, he moves from d_1 to d_2 with $\mathrm{d}_1 < \mathrm{d}_2$. the value of X and X' increase only by small fraction (of the order of $\frac{1}{1-\mathrm{d}}$). If the marginal tax rate is maintained. However, the value of G increases a lot and, more importantly, its derivative due to the existence of a linear factor in the sum (of the order of at least of $\sum_t t^2 \mathrm{d}^{t-1}$). Therefore, we have that the optimal tax seems to be sensible for changes in the discount rate of the social planner.

The sensibility of the optimal taxes with the discount factor of the social planner does not imply that the optimal marginal tax rate of the economy must be such that the economy must have a positive growth rate at every possible discount factor $\mathrm{d} \in (0,1)$. In the following subsection, we compute some numerical examples in which the optimal tax has a negative growth rate for a large class of discount factors. Nevertheless, it does not imply that, under our conditions, the optimal tax for every discount factor $\mathrm{d} \in (0,1)$ is such that the growth rate of the economy is negative. Moreover, we have the following result.

Proposition 4. Under the conditions mentioned above, there exist d_1 and d_2 in (0,1) such that $d_1 < \mathrm{d}_2$, and

1. for every $\mathrm{d} < \mathrm{d}_2$, $g_{\tau_d} < 0$, and
2. for every $\mathrm{d} > \mathrm{d}_2$, $g_{\tau_d} > 0$.

Moreover, $\lim_{d \to 0^+} g_{\tau_{\mathrm{d}}} = \delta \mathrm{E}[R_R] - 1 > 0$, $\lim_{d \to 0^+} x_{\tau_{\mathrm{d}}} = 0$ almost everywhere, and $\lim_{d \to 1^-} g_{\tau_{\mathrm{d}}} = \frac{\delta}{2}(E[R_R] + R_S) - 1 < 0$, $\lim_{d \to 1^-} x_{\tau_{\mathrm{d}}} = 1$ almost everywhere.

All the previous conjectures mentioned above have their economic and mathematical intuition. However, due to specific form of the invariant distribution for every possible marginal tax rate, it is not possible to analyze these properties analytically as it was made before. Moreover, we also found numerical evidence that support the conjectures and these results mentioned above. These numerical examples will be explain in the following subsection in which we will analyze more deeply the properties of the optimal marginal tax and the robustness of the model to analyze the trade off between inequality and growth.

The following result shows that positive bequest taxes induce a lower welfare compared with only income taxes unless the bequest rate is large.

Proposition 5. For any marginal taxation plan $\hat{\tau} \in \mathcal{T}$ with positive bequest taxes, any taxation plan $\hat{\tau}' \in \mathcal{T}$ such that $\bar{\tau}'^{I} > \bar{\tau}^{I} + \frac{\delta\bar{\tau}^{B}}{1-\delta\bar{\tau}^{B}}$ induces a strictly higher welfare compared with the welfare obtained by the taxation plan $\hat{\tau}$.

Note that in for low levels of bequest rate, almost all levels of bequest taxes are worst than allocations with only income taxes. This will imply that if we want to look for conditions in which bequest taxes are positive, we must look for situations in which the bequest rate is higher. In the following subsection, we analyze that the optimal bequest tax is positive only if the bequest rate is quite large.

2.6. Numerical Example

If we analyze the social welfare function in Example 1 with an initial distribution of wealth such that the aggregate wealth of each group is maintained over time, we found that the optimal tax rate for a discounted factor for the social planner $d = \delta$ is 80.3% that is a little bit more than critical tax in which the economy has a stationary aggregate wealth. Therefore, if the social planner discounts the future similarly as the agents do, the economy will collapse in the long run. The explanation for this phenomenon is that the social planner is not concerned about very distant low consumptions making that low rates of contraction of the economy could be optimal because reduces the unequally in the first dates. A mathematical explanation of this is that the discount factor of the social planner converges to zero faster than the collapse of the economy. Nevertheless, if the social planner discounts less the future, that is $d > \delta$, the optimal tax rate is lower than the critical rate. More precisely, if $d = \frac{1+\delta}{2}$, the optimal marginal tax rate will be around 52.4%. Moreover, if $d = \frac{2+\delta}{3}$, the optimal marginal tax rate will be approximately 36%. Figure 1 shows how the welfare function changes for all the possible tax rates for the three different discount factors mentioned above.

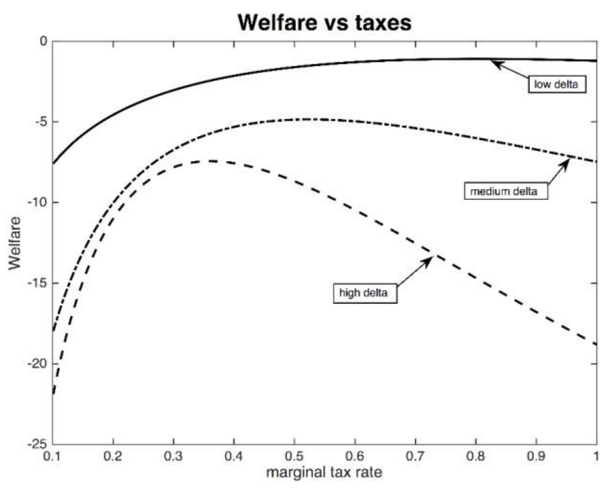

Figure 1: Social welfare function vs income taxes for different values of $d = \delta, \frac{\delta+1}{2}, \frac{\delta+2}{3}$.

From the numerical examples showed above, we know that an increment in the discount factor of the social planner implies an increment in growth rate of the economy and a more unequal invariant distribution of wealth. However, in any case, the inequality does not necessarily increase over time since it will depend strongly on the initial distribution of wealth. Therefore, if we start from a more unequal distribution than the invariant distribution for the chosen marginal tax rate, the inequality will decrease over time.

As we mentioned before, the social welfare function has a strictly concave behavior in the discount factor of the social planner. Moreover, the set of discount factors in which the economy will have a negative growth rate is considerably larger than the one in which the economy has a positive growth rate. In fact, for every $d \in (0, \alpha)$, the growth rate of the economy, g_{τ_δ} is negative, and for every $d \in (\alpha, 1)$, the growth rate of the economy, g_{τ_δ} is positive with $\alpha \sim 0.6$.

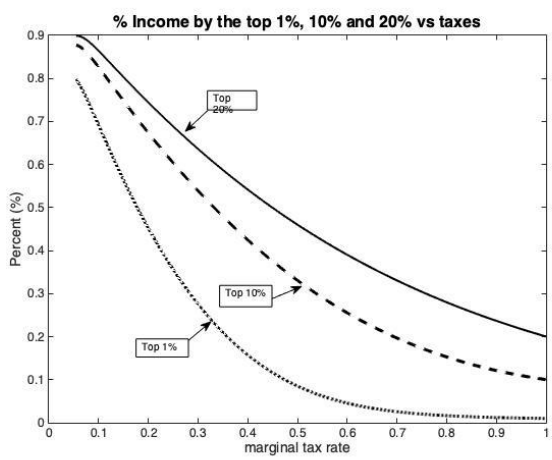

Figure 2: Income inequality for different income taxation rates.

As can be observed in Figure 1 and Figure 2, the level of inequality implemented by a social planner with a discount factor equal to the agent is quite low (2.85% of the total income for the top 1%, 20.7% for the top 10%, and 34.15% for the top 20%) if it is compared to very equal countries as Japan where the top 1% earns around 10% of the national income. For a social planner that with a discount factor equal to $d = \frac{1+\delta}{2}$, the inequality is clearly larger with 9.69% for the top 1%, 34.85% for the top 10%, and 47.6% for the top 20% which seems to be like Japan. Finally, for a social planner with a discount factor equal to $d = \frac{2+\delta}{3}$, the inequality is clearly larger than the other two cases with 21.44% for the top 1%, 48.37% for the top 10%, and 59.09% for the top 20% which seems to be like the US where the top 1% earns around 20% of the national income. All these results support the idea that the social planner should be more patience than the agents and that most of the governments are indeed more patience than their population since most of them are generally quite worried about increasing the growth path of the economy than to almost eliminate any inequality.

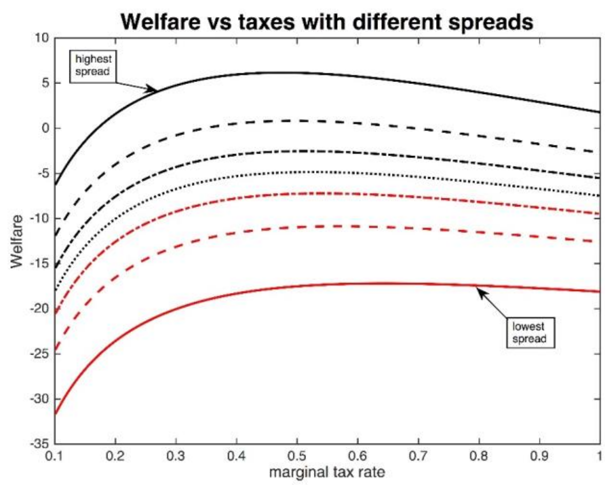

Figure 3: Welfare vs changes on productivity of R_R.

In Figure 3, we can notice that changes in productivity of the risky technology, that in our case implies changes in the spread of the risky one, cause a change in the optimal taxation. In this case, a more productive risky technology leads to lower optimal taxation rate. An explanation to this is that a more productive economy due to its risky technology needs lower taxation rates compared to economies less productive to achieve its maximum welfare. Therefore, in this case, for the social planner is optimal to increase the inequality due to the increment of productivity. This is also observed in a large variety of economies around the world, one of these examples are the US and the largest economies in Europe, such as Germany, France, and England. The former has a larger productivity than the latter and it has also a considerably larger income inequality.

	$\delta = 0.25, d = 0.05$	$\delta = 0.25, d = 0.25$	$\delta = 0.25, d = 0.375$	$\delta = 0.25, d = 0.5$
Optimal beq. tax	0	0	0	0
Optimal inc. tax	0,99	0,75	0,805	0,61
Welfare	4,650257	1,979259	0,69723137	-1,77627379
Growth	0,81371224	0,83835788	0,84042284	0,88011891
Top 1%	0,01018153	0,02141054	0,01667438	0,04566337
	$\delta = 0.375, d = 0.075$	$\delta = 0.375, d = 0.375$	$\delta = 0.375, d = 0.5$	$\delta = 0.375, d = 0.75$
Optimal beq. tax	0	0	0	0
Optimal inc. tax	0,99	0,805	0,515	0,26
Welfare	7,597522180	12	27,44189082	108,9278328
Growth	1,220568360	1,260634260	1,35775239	1,493556190
Top 1%	0,01018153	0,01667438	0,08296093	0,35163233
	$\delta = 0.5, d = 0.1$	$\delta = 0.5, d = 0.5$	$\delta = 0.5, d = 0.75$	$\delta = 0.5, d = 0.9$
Optimal beq. tax	0	0	0	0
Optimal inc. tax	0,82	0,575	0,265	0,1
Welfare	9,87818850	35,07289614	192,40648409	1.478,44045608
Growth	1,65141381	1,76591892	1,98683324	2,13368991
Top 1%	0,01563683	0,05592454	0,35591541	0,69395095

Table 1: Optimal taxation plans for low bequest rates, δ, and discount factor of the social planner, d, for $R_S = 2, \overline{R}_R = 9$, $\underline{R}_R = 0$.

We observe in Table 1 that if the social planner has a discount factor close to zero, the optimal income tax is high to reduce inequality. Moreover, once the social planner discounts less the future, the social planner is less concerned about inequality and more about growth which implies a lower optimal tax rate. Note that the in these cases, the bequest rate has a role on the optimal taxes, but it also seems important the relationship with the discount factor. In some ways, the bequest rate of the investors can be seen as a form of discounting the income of future generations. Therefore, the growth rate is positively related with the bequest rate also with the discount factor of the social planner.

As it was observed before, inequality is negatively correlated with income taxes. Then, under conditions in which income taxes are higher, inequality tends to reduce. We observe this clearly by the top 1% of income of the investors.

Note that the bequest rates are always equal to zero due to low values of the discount factor. Under these conditions, the social planner does not need to decrease the economy saving rate by reducing the incentives to leave bequests to the next generation.

	$\delta = 0.8, d = 0.08$	$\delta = 0.8, d = 0.4$	$\delta = 0.8, d = 0.6$	$\delta = 0.8, d = 0.8$
Optimal beq. tax	0	0	0	0
Optimal inc. tax	0,99	0,765	0,46	0,205
Welfare	13,77058490	37,9308063	106,56725262	554,032556
Growth	2,60387916	2,7118590	2,94941965	3,264103
Top 1%	0,01018153	0,0196542	0,1167149	0,45446689
	$\delta = 0.9, d = 0.09$	$\delta = 0.9, d = 0.45$	$\delta = 0.9, d = 0.675$	$\delta = 0.9, d = 0.9$
Optimal beq. tax	0	0,1950000	0,08	0
Optimal inc. tax	0,99	0,6200000	0,310000	0,100000
Welfare	15,05243647	52,4617563	196,263123	270,945738
Growth	2,92936405	3,0309488	3,412300	3,855993
Top 1%	0,01018153	0,0289359	0,209761	0,700383
	$\delta = 0.95, d = 0.095$	$\delta = 0.95, d = 0.475$	$\delta = 0.95, d = 0.7125$	$\delta = 0.95, d = 0.95$
Optimal beq. tax	0	0,62	0,375000	0
Optimal inc. tax	0,99	0,14	0,050000	0,065000
Welfare	15,68928389	64,0820324	278,868501	9014,584138
Growth	3,09210650	3,0587289	3,524039	4,139790
Top 1%	0,01018153	0,03622528	0,207739	0,785182

Table 2: Optimal taxation plans for high bequest rates, δ, and discount factor of the social planner, d, for $R_S = 2, \overline{R}_R = 9$, $\underline{R}_R = 0$.

From Table 1 and Table 2, we see that the optimal income taxes are always extremely high independently of the bequest rate when the discount factor of the social planner is close to zero. As it was mentioned above, this is due to the social planner discounts strongly the future making that his almost completely worried about inequality and not in the growth rate. When the social planner discounts less the future, optimal income taxes decreases, and the top 1% income share increases.

Differently from Table 1, in Table 2 the optimal bequest taxes are positive when the bequest rate is quite large, $\delta \geq 0.9$, and it also tends to increase as the bequest rate increases. However, it only occurs when the discount factor of the social planner is lower than the bequest rate. Moreover, it is observed that optimal bequest taxes are positively correlated with the bequest rate. This is due to the social planner's intention to decrease the proportion of the wealth that is given to the next generation. However, this concern does not occur in all cases, for discount factors close to zero, inequality tends to dominate social planner optimal tax policy, and for discount factor close to one, the social planner is more concerned about growth. The former case implies extremely high optimal income taxes, and the latter case implies extremely low optimal income taxes. Then, for intermediate levels of the discount factor, the importance of inequality is not particularly dominant, and the social planner's optimal "levels of savings" is quite low compared to the agent actual bequest rate. Therefore, it is observed numerically that the optimal bequest taxes are positive if $\frac{d}{\delta}$ is in an interval around 0.5 with a length that is positively correlated with δ. Note that this interval does not include the value 1. The reason for always excluding the value 1 is that the social planner's concern about the future and the investor's concern about his/her successors must be different enough as it was mentioned above.

3. Extensions

3.1.Model without segmentation

Let us define a model based on Section 2 in which both agents have access to both technologies.

Hypothesis I2: Risk-averse investors and entrepreneurs have access to the safe technology and to risky one, but not simultaneously.

Then, the investors cannot have access to both at the same time. That is, if an agent decides to invest in one of the technologies, she cannot invest in the other technology. This can be justified by the fact that each agent has a limited capacity to manage investments with quite different type of properties at the same time.

Under this hypothesis, entrepreneurs will continue investing only in the risky one $E[R_R] \geq R_S$. Therefore, they will have an unequal distribution of wealth if $\delta \mathrm{R_R} < \frac{1-\delta\bar{\tau}_S^B}{\left(1-\bar{\tau}_S^B\right)(1-\bar{\tau}^I)}$. For the risk lovers, no matter how much the agent is being taxed, if the risky return is at least as productive as the safe one, she will invest in the risky one for any marginal taxation.

From now on, we will assume that $E[R_R] > \frac{1}{\delta} > \frac{E[R_R]+R_S}{2} > R_S$, $\bar{\tau}^B = 0$, and $\bar{\tau}^I = \tau \in (0,1]$.

For the risk averters, the marginal taxation rate and the level of wealth of the agent will affect her optimal solution. More precisely, we have that:

Proposition 7. Given a marginal taxation rate $\tau \in (0,1)$, there is a constant

$$\alpha^*_{\pm,\tau} = \frac{\tau\left(R_R \pm \sqrt{{R_R}^2 - 4R_S^2}\right)}{R_S^2(1-\tau)} \tag{4.1}$$

such that:

1. if $w_t^{a_i} > \alpha^*_{+,\tau}\overline{w}_{t+1}$ or $w_t^{a_i} < \alpha^*_{+,\tau}\overline{w}_{t+1}$, the agent a_i invests in the safe technology at date $t+1$,

2. if $w_t^{a_i} \in \left(\alpha^*_{-,\tau}\overline{w}_{t+1}, \alpha^*_{+,\tau}\overline{w}_{t+1}\right)$, the agent a_i invests in the risky technology at date $t+1$, and

3. if $w_t^{a_i} = \alpha^*_{\pm,\tau}\overline{w}_{t+1}$, the agent a_i is indifferent between both type of investments at date $t+1$.

Therefore, taxation might have a positive impact on growth since it makes that a proportion of the risk-averse investors invest in the risky technology. Moreover, low levels of taxes will induce that the wealth invested in the risky technology by the risk-averse investors is quite low inducing a lower growth rate. However, if taxes increase, the interval mentioned in Proposition 7 becomes larger, and the proportion of the wealth invested in the risky technology increases. Then, the growth rate increases. More specifically, we have the following result.

Proposition 8.

1. If $\tau \geq \frac{R_R}{R_S} + 1 - \frac{R_R^2}{2R_S^2}$, any increment of the marginal tax rate increases the growth rate of the economy. The proportion of risk averters that invest in the safe technology is a decreasing positive function of the marginal tax that converges to 0 when τ converges to one.

2. If $\tau < \frac{R_R}{R_S} + 1 - \frac{R_R^2}{2R_S^2}$, all the risk averters invest in the safe technology. Therefore, the economy converges to an invariant distribution as Section 2.

Note that if $E[R_R] > R_S$, $\frac{R_R}{R_S} + 1 - \frac{R_R^2}{2R_S^2} < 1$ implying that for any economy, there is a positive marginal tax rate $\tau < 1$ such that, at any give date t, a proportion of the risk averters invest in the risky technology. However, if

$$R_R \geq \left(1 + \sqrt{3}\right) R_S \tag{4.3}$$

$\frac{R_R}{R_S} + 1 - \frac{R_R^2}{2R_S^2} \leq 0$ implying that there are risk averters investing in the risky technology at any period t.

3.2.Model with changes of type of investors (call skilled/unskilled investors in which skilled are entrepreneurs)

Let us suppose that a proportion $p \in [0,0.5]$ of agents switch from entrepreneurs to risk averse, and vice versa. A risk averse investor who has a son that is an entrepreneur gives a proportion δ of his wealth as bequest. But an entrepreneur who has a son that is a risk averse investor decides to leave to his descendant the average bequest that risk-averse investors receive from his predecessors. Additionally, we will assume that only sons of entrepreneurs who received the high return R_R might become “risk averse”. Therefore, the poorest agents will continue investing in the more productive technologies to overcome their adverse situation.

Hypothesis T1: A successor of an entrepreneur that received the highest return has a probability of $2p \in [0,1]$ of becoming a risk-averse investor, and the bequest received is equal to the average bequest of the risk-averse investors receive, $\bar{b}_t^a$.

Hypothesis T2: A successor of a risk-averse investor has a probability of $p \in [0,1]$ of becoming an entrepreneur.

If entrepreneurs are quite poor, they will tend to make a larger effort to overcome that adverse situation. On the other hand, entrepreneurs that have been successful in the past are less concerned about keeping their skilled capabilities to continue being entrepreneurs, having a positive probability of becoming “risk averse” investors.

Under these conditions, the arguments used above to ensure the convergence of the income distribution to an invariant distribution.

Proposition 11. Under the additional Hypotheses T1 and T2, for any positive tax policy, $\left(\overline{\tau}_s^I, \overline{\tau}_s^B\right)_s \in [0,1]^2$, and initial distribution of wealth, w_0, the wealth distribution converges to an invariant distribution.

The proof can be found in Appendix C.

Remark 16. Other possible transitions are: 1) a successor of an entrepreneur that received 0 as a return has a probability of $2p \in [0,1]$ of becoming a risk-averse investor, or 2) the probability of an entrepreneur of having a successor that is a risk-averse investor is p independently of the wealth.

In both cases, the constructive argument used above to ensure the existence of an invariant distribution can be applied.

3.3.Extension to a model with capital, labor, and innovation

Using the model exposed before, we can extend it to a capital, labor, and model as follows. $K_t^i = \left(R_R\theta_R^i + R_S\theta_S^i\right)\delta b_t^i$ for $t \geq 1$ and $K_0^i = 1$ is the amount of capital of the agent i's firm that depreciates completely, $\Theta_t^i = \Theta_{t-1}^i\left(\frac{K_t^i}{K_{t-1}^i}\right)^{1-\alpha}$ with $\alpha \in (0,1)$ for $t \geq 1$ and $\Theta_0^i = K_0^i$ is the innovation factor, $L_t^i \in [0,1]$ without any utility for leisure which implies that $L_t^i = 1$ for all t, r_t is the price of the capital at date t, and w_t is the salary. The technology of the firm i at t is given by

$$y_t^i = \Theta_t^i\left(K_t^i\right)^{\alpha}\left(L_t^i\right)^{1-\alpha}$$

The consumer constraint is given by

$$c_{t+1} + b_{t+1} \leq w_t^i L_t^i + r_t^i K_t^i$$

In equilibrium, since the firm has constant returns to scale, $w_t^i = (1-\alpha)\Theta_t^i\left(K_t^i\right)^{\alpha}\left(L_t^i\right)^{-\alpha}$, $w_t^i = \alpha\Theta_t^i\left(K_t^i\right)^{\alpha-1}\left(L_t^i\right)^{1-\alpha}$. Therefore, in equilibrium, the consumer problem of each agent is defined as before, and all the results related to the dynamics of the wealth are still valid.

4. Conclusions

We developed an overlapping generation model with endogenous growth rate and heterogeneous technology productions. In this model, taxes and redistribution has a negative impact on growth if the more productive technologies involve larger amount of idiosyncratic risk. Moreover, in absence of taxes, the most productive technologies will dominate the economy in the long run and the long run inequality will depend mainly in the risk that it involves. In the presence of taxes,

taxes ensure the existence of an invariant distribution of wealth among the agents and an invariant growth rate of the economy. We also showed that there is no poverty trap among the agents with the most productive. Among the agent that do not have access to the most productive technologies, their wealth may not reach the top in any future date.

Redistribute taxes has a negative effect in growth rate and inequality. To establish an optimal taxation, we introduced a central planner that considers the consumption of the agents at equilibrium. We showed that the social welfare function can be written as the sum of three independent functions, one depending on growth, one depending on inequality, and one depending on the difference of the discount factors of the agent and the social planner. The first function is comonotonic with the growth rate of the economy, implying that this function might be an increasing function on taxes. The second one is anticomonotonic with the inequality of the invariant distribution which implies that this function might be a decreasing function on taxes.

We also found that, for a fixed discount rate for every agent in the economy, the optimal taxation is strictly decreasing on how the social planner discounts the future. Moreover, the optimal tax will be such that the invariant wealth distribution tends to an equal one if the social planner strongly discounts the future, and, on the other hand, the optimal tax is zero when the social planner does not discount the future at all. The intuition behind these results is that, if a social planner discounts the future strongly, the weight of distant dates becomes almost irrelevant, and analogously with the growth rate of the economy. Therefore, the social welfare function will be dominated by the inequality effect. However, if a social planner almost does not discount the future, the weight of future consumption will dominate the inequality effect even when both effects are increasing. Additionally, our model suggests that changes in the tax policy may be based on changes on the form of the social planner discounts the future compared to how the other agents do so.

The optimal bequest taxes are positive when the bequest rate is quite large, and the ratio of the discount factor of the social planner with the bequest rate is lower than 1. This is due to the social planner's intention to decrease the proportion of the wealth that is given to the next generation. However, this concern does not occur in all cases, for discount factors close to zero, inequality tends to dominate social planner optimal tax policy, and, for discount factor close to one, the social planner is more concerned about growth. Then, for intermediate levels of the discount factor, the importance of inequality is not particularly dominant, and the social planner's optimal "levels of savings" is quite low compared to the agent actual bequest rate.

For economies where extremely wealthy families stop being as *entrepreneurs* as their ancestors, they can switch from investing in the more productive to less productive technologies. Additionally, if their bequest rate is also larger than the discount factor of the social planner, our model suggests that a positive bequest tax for them is socially optimal due to a reduction of the growth rate and the extremely high levels of saving that this type of investors do.

5. Bibliography

Acemoglu, D. and **Robinson, J. A.** 2015. The rise and decline of general laws of capitalism. Journal of Economic Perspectives, 29(1), 3-28.

Alesina, Alberto, and **Dani Rodrik**. 1994. "Distributive politics and economic growth." The quarterly journal of economics 109 (2), 465-490.

Aoki, S., and **Nirei, M.** 2017. Zipf's Law, Pareto's Law, and the Evolution of Top Incomes in the United States. American Economic Journal: Macroeconomics, 9(3), 36-71.

Araujo, Aloisio, **Alain Chateauneuf**, **Juan P. Gama**, and **Rodrigo Novinski**. 2018. "General Equilibrium with uncertainty seeking preferences." Econometrica, 86(5), 1859-1871.

Araujo, Aloisio, **Juan P. Gama**, and **Tim Kehoe**. 2017. "Dynamics with uncertainty loving preferences.” working paper.

Bargain, O., **M. Dolls**, **H. Immervoll**, **D. Neumann**, **A. Peichl**, **N. Pestel**, and **S. Siegloch.** 2015. “Tax policy and income inequality in the United States, 1979-2007.” Economic Inquiry, 53(2), 1061-1085. Beare, Brendan K., and Alexis Akira Toda. 2018. “Geometrically stopped Markovian random growth processes and Pareto tails.” working paper.

Benhabib, J., **Bisin, A.**, and **Zhu, S**. 2011. “The distribution of wealth and fiscal policy in economies with finitely lived agents. ” Econometrica, 79 (1), 123-157.

Benhabib, J., **Bisin, A.**, and **Zhu, S.** 2015. “The wealth distribution in Bewley economies with capital income risk." Journal of Economic Theory, 159, 489-515.

Benhabib, J., **Bisin, A.**, and **Zhu, S**. 2016. “The distribution of wealth in the Blanchard-Yaari model.” Macroeconomic Dynamics, 20 (2), 466-481.

De Gregorio, Jose. 1992. “Economic growth in Latin-america.” Journal of development economics 39(1), 59-84.

Gabaix, X., **Lasry, J. M.**, **Lions, P. L.**, and **Moll, B**. 2016. “The dynamics of inequality.”Econometrica, 84(6), 2071-2111.

Galor, Oded, and **Joseph Zeira.** 1993. “Income distribution and macroeconomics.” The review of economic studies 60 (1), 35-52.

Garca-Peñalosa, Cecilia and **Jean-Francois Wen**. 2008. “Redistribution and entrepreneurship with Schumpeterian growth.” Journal of Economic Growth, 13 57-80.

Jappelli, Tullio, and **Marco Pagano**. 1994. “Saving, growth, and liquidity constraints.”The Quarterly Journal of Economics 109 (1), 83-109.

Jones, C. I. 2015. “Pareto and Piketty: The macroeconomics of top income and wealth inequality.” Journal of Economic Perspectives, 29(1), 29-46.

Kehoe, Timothy J., and **David K. Levine.** 1984. “Intertemporal separability in overlapping-generations models.” Journal of Economic Theory, 34(2), 216-226.

Kuznets, Simon. 1955. “Economic growth and income inequality..” The American economic review 45 (1), 1-28.

Lindert, Peter H., and **Jeffrey G. Williamson.** 2016. “Unequal gains:American growth and inequality since 1700.” Juncture 22 (4), 276-283.

Papanek, Gustav F. 1973. “Aid, foreign private investment, savings, and growth in less developed countries.” Journal of political Economy 81(1), 120-130.

Perotti, Roberto. 1996. “Growth, income distribution, and democracy: What the data say.” Journal of Economic growth 1 (2), 149-187.

Persson, Torsten, and **Guido Tabellini.** 1994. “Is inequality harmful for growth?.” The American Economic Review 600-621.

Piketty, Thomas. 1997. “The dynamics of the wealth distribution and the interest rate with credit rationing.” The Review of Economic Studies 64 (2), 173-189.

Piketty, Thomas, and **Emmanuel Saez**. 2014. “Inequality in the long run.” Science 344 (6186), 838-843.

Saltz, Ira S. 1999. “An examination of the causal relationship between savings and growth in the third world.” Journal of Economics and Finance 23(1), 90-98.

Temple, Jonathan. 1999. “The new growth evidence.” Journal of economic Literature 37 (1), 112-156.

van der Weide, Roy, and **Branko Milanovic.** 2014. "Inequality is Bad for Growth of the Poor (But Not for that of the Rich)". World Bank Policy Research Working Paper No. 6963. Washington, DC.

Appendix A. Proofs

A.1. Proof of Proposition 2

Proof. The case in which $\underline{R}_R = 0$ is a direct consequence of the fact that, with probability one, all skilled agents will have zero consumption wealth in the long run, and, at the same time, the aggregate endowment of the economy is always positive.

To prove the case in which $\underline{R}_R > 0$, notice that since the production of the skilled agents is at least as good as the unskilled agents, 1) the initial income of the invariant distribution of the skilled agents can only be zero, or 2) equal to the total aggregate initial wealth if $E[R_R] > R_S$, or 3) it can be any possible value between these two extremes if $E[R_R] = R_S$. If we analyze the case in which x^l is not equal to zero almost everywhere (case 2 or 3), the aggregate production over the aggregate production of the skilled ones converges to a positive constant when t goes to infinity, and, since $\frac{\frac{\overline{\mathrm{R}}_R}{\overline{R}_R}}{E[R_R]} < 1$, $\left(\frac{\frac{\overline{\mathrm{R}}_R}{\overline{R}_R}}{E[R_R]}\right)^n$ converges to zero when n goes to infinity, and $\frac{\frac{\overline{\mathrm{R}}_R^n}{\overline{R}_R^n}}{E[R_R]^{(k+1)n}}$ also converge to zero when n goes to infinity for every $k \in \mathbb{N}$, part of the income will be concentrated in hands of a zero-measure set of skilled agents in the long run. Therefore, the only possible case is that $x^l \equiv 0$ almost everywhere.

A.1. Proof of Theorem 1

Let us prove a preliminary result that ensures that for any initial distribution $(w_0^i) \gg 0$, the aggregate income in hands of the l agents over the aggregate wealth in hands of the a agents, $\frac{\overline{w}_t^l}{\overline{w}_t^a}$, converge to a positive constant even if $\underline{R}_R \neq 0$ or $\underline{R}_S \neq \overline{R}_S$.

Lemma 1. For taxes defined by nonnegative marginal tax rates $(\overline{\tau}^l, \overline{\tau}^B) > 0$ with technology returns such that satisfy Equations 3.1 and 3.2, $\lim_{t\to\infty} \frac{\overline{w}_t^l}{\overline{w}_t^a} = \gamma_{(\overline{\tau}^l,\overline{\tau}^B)}$ where $\gamma_{(\overline{\tau}^l,\overline{\tau}^B)} \in [1,\infty)$.

Proof. To simplify the proof, we will assume that $\overline{\tau}^l = \tau > 0, \overline{\tau}^B = 0$. The idea of the proof is to show that the function $f: [0,\infty) \to [0,\infty)$ defined by $z_\tau^{t+1} = \frac{E[R_R]\overline{b}_t^l}{E[R_S]\overline{b}_t^a} = f(z_\tau^t) = \left(\frac{E[R_R]}{E[R_S]}\right)\left(\frac{(1-\frac{\tau}{2})z_\tau^t + \frac{\tau}{2}}{\frac{\tau}{2}z_\tau^t + (1-\frac{\tau}{2})}\right)$ satisfies that $f(0) > 0$, $\lim_{z\to\infty} f(z) = \left(\frac{E[R_R]}{E[R_S]}\right)\frac{2-\tau}{\tau}$, $f'(z) > 0 \forall z \in [0,\infty)$, $f'(\infty) = 0$ and f' is a decreasing function. Under these conditions, f has only one fixed point γ_τ defined by

$$z_\tau = -\left(\frac{1}{2}-\frac{1}{\tau}\right)\left(\frac{E[R_R]}{E[R_S]}-1\right) + \sqrt{\left(\frac{1}{2}-\frac{1}{\tau}\right)^2\left(\frac{E[R_R]}{E[R_S]}-1\right)^2 + \left(\frac{E[R_R]}{E[R_S]}\right)} \tag{3.5}$$

and, for each $z^0 \in (0,\infty)$, z_τ^t converge to z_τ. Since z_τ^t is the proportion of the aggregate production of the l agents and the a agents, the sequence $\left\{\frac{\overline{w}_t^l}{\overline{w}_t^a}\right\}_t$ also converge, and since $z_\tau^{t+1} \geq 1$, $\gamma_\tau^t = \frac{\overline{w}_t^l}{\overline{w}_t^a} \geq 1$ implying that $\gamma_\tau = \frac{z_\tau(1-\frac{\tau}{2})+\frac{\tau}{2}}{z_\tau(\frac{\tau}{2})+1-\frac{\tau}{2}} \in [1,\infty)$. The proof is analogous to the other two types of taxation.

From the Proof of Lemma 1, we can interpret z_τ as the ratio of the aggregate production of the skilled agents and the aggregate production of the unskilled ones. Therefore, note that γ can be seen as monotonic function of z_τ, implying that $\gamma_\cdot$ is a C^1 function for $\tau \in [0,1]$ that decreases when τ and $E[R_S]$ increases, and that increases when $E[R_R]$ increases.

Since the aggregate production depends on aggregate wealth of each of the groups, the convergence of the ratio of the skilled and unskilled aggregate wealth ensures the convergence of the growth path.

Corollary 1. For any fixed tax rate $(\overline{\tau}^I, \overline{\tau}^B) > 0$, the growth rate of the economy, $g_{(\overline{\tau}^I, \overline{\tau}^B),t}$ converges when t goes to infinity.

Due to the convergence of how each group invest in each technology, the growth rate of the economy will also converge. Then, the proportion of income of the poorest skilled agent converges, which implies that the proportion of income of a skilled agent that has received at least once the lower return $\underline{R}_R = 0$ also converges.

Proof of Theorem 1. For simplicity, let us consider a positive marginal income tax rate $\tau^I = \tau$ only. The convergence of the proportion of wealth of the l agents to an invariant distribution is a direct consequence of the conditions mentioned above. In fact, the invariant distribution of the proportion of the l agents is

$$\sum_{k=0}^{n-1} \frac{R_R^k \frac{\tau}{2}(1-\tau)^k \delta^k}{(1+g_\tau)^k}$$

for the n^{th} poorest group of l agents with weight $\left(\frac{1}{2}\right)^{n+1}$ for $n \in \mathbb{N}$.

To conclude the proof, we must ensure that Equation 3.7 converges for any initial $w_0^{a_i}$.

Since $R_S < E[R_R]$, $\frac{R_s^t \delta^{t-1}}{\bar{w}_t} \to 0$ when $t \to \infty$, implying that $\delta^{t-1} R_S^t (1-\tau)^t \frac{w_0^{a_i}}{\bar{w}_t} \to 0$ when $t \to \infty$.

$\sum_{k=0}^{t} \left(\frac{\delta^k (1-\tau)^k \frac{\tau}{2} R_S^k}{\prod_{l=1}^{k} (1+g_{\tau,l})} \right)$ converges when t goes to infinity since $0 < \frac{\delta (1-\frac{\tau}{2}) R_S}{(1+g_{\tau,l})} < 1 - \frac{\tau}{2}$ for all $l \in \mathbb{N}$. Therefore, the proportion of the wealth of the a agents in the limit is

$$\frac{1}{1-\frac{\delta(1-\tau)R_S}{1+g_\tau}} \frac{\tau}{2} = \frac{\frac{\tau}{2}(1+g_\tau)}{1+g_\tau - \delta(1-\tau)R_S} = \frac{1}{\gamma_\tau + 1}$$

The proof is analogous to the other two types of taxation.

Based on the proof of Theorem 1, with $\overline{\tau}^I = \tau > 0$, and $\overline{\tau}^B = 0$, the proportion of the income of the poorest l agents is $\tau/2$, and the weight of this group is 1/2. The income of the second poorest group of l agents only depends on the average income and the income of the poorest l agents in the previous period. Therefore, the proportion of the second poorest group of l agents is $\frac{R_R\left(\frac{\tau}{2}\right)\delta}{1+g_{\tau,t-1}} + \tau\left(\frac{1}{2} - \frac{R_R\left(\frac{\tau}{2}\right)\delta}{1+g_{\tau,t-1}}\right) = \frac{\tau}{2}\left(\frac{R_R(1-\tau)\delta}{1+g_{\tau,t-1}} + 1\right)$, and its weight is 1/4. If we continue this process, we obtained that proportion of the n^{th} poorest group of l agents is

$$\sum_{k=0}^{n-1} \frac{R_R^k \frac{\tau}{2} (1-\tau)^k \delta^k}{\prod_{j=1}^{k} (1+g_{\tau,t-k})}, \tag{3.6}$$

and the weight of this group is $\frac{1}{2^{n+1}}$ if $t \geq n$.

For the a agents, since $\underline{R}_S = \overline{R}_S$, the proportion of the income of an a_i agent is given by

$$\sum_{k=0}^{t-1}\left(\frac{\delta^k \tau^k \frac{\tau}{2} R_S^k}{\prod_{l=1}^{k}(1+g_{\tau,l})}\right) + \delta^{t-1} R_S^t (1-\tau)^t \frac{w_0^{a_i}}{\overline{w}_t} \tag{3.7}$$

for each node s of length $t \geq 0$. Since $\gamma^{\tau,t} \to \gamma_\tau \in [0,\infty)$; when $t \to \infty$, $g_{\tau,t} \to g_\tau \in [R_S\delta - 1, E[R_R]\delta - 1]$ when $t \to \infty$, which concludes the proof. The proof with positive income and consumption taxes is analogous.

A.2. Proof of Proposition 3

Proof of Proposition 3. Due to Equation 3.10, we can define $X(\cdot,\cdot)$ as $X\left(\mathrm{d}, \hat{\tau}\right) = \frac{\mathrm{d}}{1-\mathrm{d}} \int log\left(x_{\hat{\tau}}^{i} + x_{\hat{\tau}}^{[[i+1/2]]}\right)$, $G(\cdot,\cdot)$ as $G\left(\mathrm{d}, \hat{\tau}\right) = \sum_{t=1}^{\infty} \mathrm{d}^t \int log\left(\left(1 + g_{\hat{\tau}}\right)^t\right) di$ which is clearly a decreasing function in $\hat{\tau}$, and $D(\cdot,\cdot)$ as $\mathrm{D}(d) = \sum_{t=1}^{\infty} d \int l\,og\left(\left(\frac{\delta^{\delta}(1-\delta)^{1-\delta}\left(1-\overline{\tau}^{B}\right)}{1-\delta\overline{\tau}^{B}}\right)\right) di$. In this case, each function depends directly on indirectly on δ since the bequest rate affects the distribution of the invariant distribution and the growth rate of the economy by increasing inequality and the growth rate when δ increases.

The properties of X are consequence of the dominated convergence theorem and the properties of the invariant concentration of wealth (in the aggregate, is constant, and it is more unequal every time that you decrease the marginal tax rate). The properties of G are consequence of the invariant growth rate and the fact that the series that defines this function is absolutely convergent. Finally, the properties of D can be easily obtained due to its functional form.

A.3. Proof of Proposition 4

Proof of Proposition 4. To prove the second part, it is enough to analyze asymptotic behavior of X and G when d goes to zero and goes to one. When d goes to zero, G becomes null and X does not, then, the W only depends on X for d small enough which implies our result. When d goes to one, X becomes null and G does not. However, in this case X is unbounded from below. Therefore, the optimal taxes are small but no zero. Nevertheless, when the discount factor of the social planner goes to zero, we can find taxes that converge to zero that keep constant X and increases the value of G implying our result.

Finally, the first part is a consequence of the second one.

A.4. Proof of Proposition 5

Proof of Proposition 5. Consider a taxation plan $\hat{\tau}' \in \mathcal{T}$ such that $1 > \overline{\tau}'^{I} > \overline{\tau}^{I} + \frac{\delta\overline{\tau}^{B}}{1-\delta\overline{\tau}^{B}}$. Since we analyze the welfare function in the invariant distribution, we can assume that $\overline{w}_0 = 1$ in both cases.

For the poorest *skilled agents*, we have that their level of after tax nominal income with the taxation plan $(\bar{\tau}^{I}, \bar{\tau}^{B})$ is given by $\bar{\tau}^{I} = \overline{w}_0\bar{\tau}^{I} + \overline{w}_0\frac{\bar{\tau}^{B}}{1-\bar{\tau}^{B}}\left(\frac{\delta(1-\bar{\tau}^{B})}{1-\delta\bar{\tau}^{B}}\right) = \overline{w}_0\left(\bar{\tau}^{I} + \frac{\delta\bar{\tau}^{B}}{1-\delta\bar{\tau}^{B}}\right) < \overline{w}_0\bar{\tau}'^{I} = \bar{\tau}'^{I}$ which is the after tax income with the new taxation plan.

Note that that since the function $f_k(x) = \frac{x}{(1-x)^k}$ is an increasing function for $x \in [0,1)$ for all $k, l \in \mathbb{N}$ and $\bar{\tau}^{I} < \bar{\tau}'^{I}$, we have that $\bar{\tau}^{I}(1-\bar{\tau}^{I})^k(1-\bar{\tau}^{B})^l < \bar{\tau}^{I}(1-\bar{\tau}^{I})^k < \bar{\tau}'^{I}(1-\bar{\tau}'^{I})^k$. The second poorest group of *skilled agents*, we have that their level of after tax nominal income with the taxation plan $(\bar{\tau}^{I}, \bar{\tau}^{B})$ is given by $\left(R_R\bar{\tau}^{I}\overline{w}_0\delta(1-\bar{\tau}^{B})\right)(1-\bar{\tau}^{I}) + \overline{w}_0\bar{\tau}^{I} + \overline{w}_0\frac{\bar{\tau}^{B}}{1-\bar{\tau}^{B}}\left(\frac{\delta(1-\bar{\tau}^{B})}{1-\delta\bar{\tau}^{B}}\right) = \overline{w}_0\left(R_R\delta\bar{\tau}^{I}(1-\bar{\tau}^{B})(1-\bar{\tau}^{I}) + \bar{\tau}^{I} + \frac{\bar{\tau}^{B}\delta}{1-\delta\bar{\tau}^{B}}\right) < \overline{w}_0(R_R\delta\bar{\tau}'^{I}(1-\bar{\tau}'^{I}) + \bar{\tau}'^{I})$.

For the *n*-poorest group of *skilled agents*, we have

$$\sum_{k=0}^{n-1}\left(R_R^k\delta^k(1-\bar{\tau}^{B})^k\right)(1-\bar{\tau}^{I})^k\left(\bar{\tau}^{I}\overline{w}_0 + \frac{\bar{\tau}^{B}\delta}{1-\delta\bar{\tau}^{B}}\overline{w}_0\right) = \overline{w}_0\left(\sum_{k=0}^{n-1}\left(R_R^k\delta^k(1-\bar{\tau}^{B})^k(1-\bar{\tau}^{I})^k\left(\bar{\tau}^{I} + \frac{\bar{\tau}^{B}\delta}{1-\delta\bar{\tau}^{B}}\right)\right)\right) < \overline{w}_0\left(\sum_{k=0}^{n-1}\left(R_R^k\delta^k(1-\bar{\tau}^{I})^k\left(\bar{\tau}^{I} + \frac{\bar{\tau}^{B}\delta}{1-\delta\bar{\tau}^{B}}\right)\right)\right) < \overline{w}_0\left(\sum_{k=0}^{n-1}\left(R_R^k\delta^k(1-\bar{\tau}^{I})^k\left(\bar{\tau}^{I} + \frac{\bar{\tau}^{B}\delta}{1-\delta\bar{\tau}^{B}}\right)\right)\right) < \overline{w}_0\left(\sum_{k=0}^{n-1}\left(R_R^k\delta^k(1-\bar{\tau}'^{I})^k\bar{\tau}'^{I}\right)\right).$$

Then, the income of the invariant distribution in each period is always lower with the taxation plan $(\bar{\tau}^I, \bar{\tau}^B)$ than with $(\bar{\tau}'^I, 0)$. For the *unskilled ones*, the result is also true because of the convergence of an analogous series as the one described above. To conclude the proof, notice that the utility of the agent i when the after taxes nominal incomes in each state are w_1^i and w_2^i is

$$\begin{aligned}
\frac{1}{2}u^i(c_1^i, b_1^i) + \frac{1}{2}u^i(c_2^i, b_2^i) &= \frac{1}{2}(c_1^i)^{1-\delta}(b_1^i)^{\delta} + \frac{1}{2}(c_2^i)^{1-\delta}(b_2^i)^{\delta} \\
&= \frac{1}{2}\left((1-\delta)w_1^i\right)^{1-\delta}\left(\delta(1-\bar{\tau}^B)w_1^i\right)^{\delta} + \frac{1}{2}\left((1-\delta)w_2^i\right)^{1-\delta}\left(\delta(1-\bar{\tau}^B)w_2^i\right)^{\delta} \\
&= \frac{1}{2}\left((1-\delta)\right)^{1-\delta}\delta^{\delta}(1-\bar{\tau}^B)^{\delta}w_1^i + \frac{1}{2}\left((1-\delta)\right)^{1-\delta}\delta^{\delta}(1-\bar{\tau}^B)^{\delta}w_2^i \\
&< \frac{1}{2}\left((1-\delta)\right)^{1-\delta}\delta^{\delta}w_1^i + \frac{1}{2}\left((1-\delta)\right)^{1-\delta}\delta^{\delta}w_2^i.
\end{aligned}$$

Appendix B. Model without segmentation and an effort cost

Let us define a model based on Section ??? only with income taxes in which both agents have access to both technologies, but each agent that decides to invest in the risky technology will have an effort cost, $L \geq 0$, a fixed effort cost that the agent must take to have a positive probability of wining the highest return. In absence of this cost, the agent will have a null return in the next period. Therefore, if an agent decides to invest in the risky one, it is always optimal to pay the effort cost.

Hypothesis E1: Investments of the risk technology have constant effort costs L_α for the risk-averse investors and L_β for the entrepreneurs. This effort cost is paid if R_R occurs.

B.1. Effort cost for risk-averse investors

Note that these costs will reduce the return of the risky technology. If L_α is small, some risk-averse investors will continue investing in the risky one, but the proportion of agent willing to invest will decrease. If L_β is small, entrepreneurs will continue investing as before. However, if L_β or the income taxes are large, the poorest entrepreneurs do not have incentives to invest in the risky technology. The following propositions analyze these cases.

Proposition 12. Given a marginal taxation rate $\tau \in (0,1)$, there is a constant

$$\alpha^*_{\pm,\tau} = \frac{\tau\left((R_R - 2R_S\, exp(L_\alpha)) \pm \sqrt{(R_R - 2R_S\, exp(L_\alpha))^2 - 4(exp(L) - 1)R_S^2}\right)}{R_S^2(1-\tau)} \qquad (4.1)$$

such that:

1. if $w_t^{a_i} > \alpha^*_{+,\tau}\overline{w}_{t+1}$ or $w_t^{a_i} < \alpha^*_{+,\tau}\overline{w}_{t+1}$, the agent a_i invests in the safe technology at date $t+1$,

2. if $w_t^{a_i} \in \left(\alpha^*_{-,\tau}\overline{w}_{t+1}, \alpha^*_{+,\tau}\overline{w}_{t+1}\right)$, the agent a_i invests in the risky technology at date $t+1$, and

3. if $w_t^{a_i} = \alpha^*_{\pm,\tau}\overline{w}_{t+1}$, the agent a_i is indifferent between both type of investments at date $t+1$.

B.2. Effort cost for entrepreneurs

For entrepreneurs, we have the following result.

Proposition 13. Given a marginal taxation rate $\tau \in (0,1)$ and the aggregate wealth in $t+1$, $\overline{w}_{t+1}$, there is a constant $\beta^*_{t+1} = \frac{L_\beta}{(R_R - 2R_S)\overline{w}_{t+1}}$ such that:

1. if $w_t^{l_i} < \beta_{t+1}^* \overline{w}_{t+1} = \frac{L_\beta}{(R_R - 2R_S)}$, the agent l_i invests in the safe technology at date $t+1$,

2. if $w_t^{l_i} > \beta_{t+1}^* \overline{w}_{t+1}$, the agent l_i invests in the risky technology at date $t+1$, and

3. if $w_t^{l_i} = \beta_{t+1}^* \overline{w}_{t+1}$, the agent l_i is indifferent between both type of investments at date $t+1$.

Note that, if the economy has a negative growth rate for some periods in a row, it might cause that the economy collapses in the long run since all entrepreneurs invest in the safe investment eventually due to β_t^* going to infinity. On the other hand, the risk-averse investors might not invest in the risky technology due to Inada condition. In Subsection B.2, we will show this numerically.

If the entrepreneurs are risk lovers with a utility index given by $u(c,b) = \left(c^{1-\delta} b^{\delta}\right)^2$, entrepreneurs will consider not only the return of the technologies and the effort cost, but also consider taxes since they can reduce the amount of risk that they are taking.

Proposition 14. Given a marginal taxation rate $\tau \in (0,1)$ and the aggregate wealth in $t+1$, $\overline{w}_{t+1}$, there is a constant

$$\beta_{t+1,\tau}^* = \frac{-(R_R - 2R_S)\tau + \sqrt{(R_R - 2R_S)^2 \tau^2 + \frac{L_\beta}{\overline{w}_{t+1}^2}\left(R_R^2 - 2R_S^2\right)}}{\left(R_R^2 - 2R_S^2\right)(1-\tau)} \tag{4.2}$$

such that:

1. if $w_t^{l_i} < \beta_{t+1,\tau}^* \overline{w}_{t+1}$, the agent l_i invests in the safe technology at date $t+1$,

2. if $w_t^{l_i} > \beta_{t+1,\tau}^* \overline{w}_{t+1}$, the agent l_i invests in the risky technology at date $t+1$, and

3. if $w_t^{l_i} = \beta^*_{t+1,\tau}\overline{w}_{t+1}$, the agent l_i is indifferent between both type of investments at date $t + 1$.

B.3. Invariant Distribution

Due to Proposition 12, Proposition 13, and Proposition 14, we can ensure that a process similar to the one made in the other models can be done in this case since the wealth of each risk averter (more precisely, almost every risk averter) can be computed by a recursive.

Numerically, it requires only to compute a preliminary proportion of agents that invest in each technology in each period t by using an increasing function $\alpha^*_\cdot$ defined above. Then, we compute $\alpha^*_\tau \frac{\overline{w}_{t+1}}{\overline{w}_t}$ and the distribution of wealth step by step. Now, we restart the process with the proportion of investment induced by the distribution that we have just found.

By doing this, we can find the invariant distribution of wealth invested in each technology, and, therefore, the invariant growth rate of the economy, which are the only things that we need to know the invariant concentration of wealth among the agents.

Lemma 2. Under a fixed and positive marginal tax rates and a wealth dynamic process such that the growth rate $g_{\tau,t}$ satisfies that $g_{\tau,t} \to g_\tau$ when $t \to \infty$, the wealth concentration of wealth converges to an invariant distribution.

This result implies the existence of the invariant concentration of wealth and it also suggests that if the concentration is considerably close to the invariant one, it converges in the long run to the invariant one which is what we found in the numerical examples that will be afterwards.

Proposition 15. Under a fixed and positive marginal tax rate, there is an invariant concentration of wealth.

B.2. Numerical examples with risk lovers' entrepreneurs

Based on the numerical examples defined before, we consider $R_R = 4.5$, $R_S = 1.6$, $\delta = 0.5$, $L = 0.04879$. We start with a distribution constant distribution of wealth among the agents.

In this case, increments on taxation generate different effects on growth depending on the marginal tax rate that we start on. For very low levels of marginal taxation rate, increment in taxation rates might decrease the growth rate due to the transfers from the risk lovers to risk averters. The former are investing completely in the risky and more productive type of investment, and the latter are investing part in the risky and part in the safe investment. Therefore, the increment in the marginal taxation rate implies less investments in the risky and more investments in the safe one.

For marginal tax rates between 0.3 and 0.5, the growth rate increases when the marginal tax rate increases. In this case, the risk averters are investing considerably more in the risky than in the safe one since they need a larger number of successful periods investing in the risky investment to reach the indifference threshold, $\alpha^*_{+,\tau}$. Intuitively, the threshold being attainable for the risk averters means that the insurance effect caused by taxes is observed, that is, some risk averters decide to invest in a risky and more productive type of investment because the government ensures that he/she will receive minimum level of wealth if his/her investment does not give any return. In this case, given an increment of the marginal tax rate, the proportion of agents that decide to invest in the risky one compensates the transfers of wealth from the risk lovers to the risk averters who decide to invest in the safe one.

For marginal taxation rate slightly above 0.5, there is no fat tails in the economy. Moreover, there is always an upper bound depending on the aggregate wealth that is unattainable for any agent, including the ones who always invest in the risky investment and succeed. Therefore, there is $\hat{\tau} \in (0.5,0.6)$ the lowest positive marginal taxation rate such that the growth rate in the long run is maximum, that is, $g_{\tau,t} \to \frac{R_R \delta}{2} - 1 = 0.125 = 12.5$ when $t \to \infty$.

When the marginal taxation rate is larger than 0.75, the poorest risk averter satisfies $w_t^{a_i} = \tau \overline{w}_t \sim \alpha^*_{-,\tau} \overline{w}_{t+1}$ therefore, an increment on the marginal taxation rate will imply that these agents decide to invest in the safe and less productive investment implying reductions in the growth rate. This phenomenon continues until the growth rate is positive. Once the marginal tax rate is such that the growth rate in the long run is slightly negative, we are under the conditions in which the poorest risk lovers start to invest in the safe one reducing even more the growth rate which makes that more agents (risk averters and lovers) witch to the safe one. Then, in the long run, almost all agents invest in the safe one instead of the risky one implying that the economy has the lowest growth rate possible, that is, $g_{\tau,t} \to R_S \delta - 1 = -0.2 = -20\%$ when $t \to \infty$.

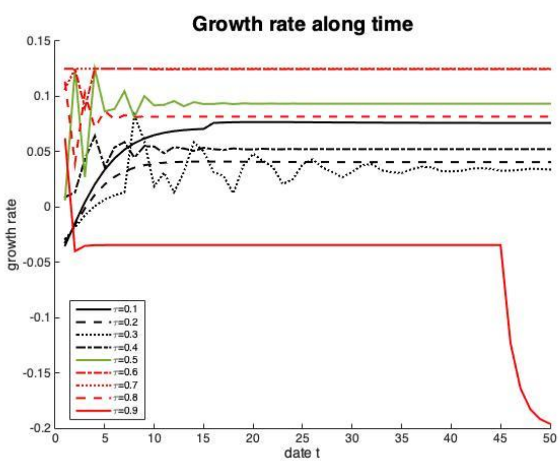

Figure 6: Growth rate over time for different income tax rates.

Note that, based on Figure 6, the economy converges to a constant growth rate in the long run for all the marginal tax rates analyzed. Moreover, we observed that for almost all the marginal tax rates analyzed, the convergence of the growth rate holds. However, it might be a problem for marginal tax rates around 0.83 since the induced growth rate oscillates between positive and negatives growth rates. We note that these problems might be convergence problems or numerical problems induced by computational errors.

From Figure 6, we noticed that for low marginal tax rate, all risk averters invest in the safe one implying a constant distribution. When $\tau = 0.3$, 0.4, and 0.5 the risk averters invest in the risky one when they are poor and in the safe one when they are above the threshold $\alpha^{*}_{+,\tau}$. The biggest difference between these two distributions is that the threshold is considerably higher when $\tau = 0.4$ (moreover when $\tau = 0.5$) implying that a larger proportion of wealth is being invested in the risky in this case which causes the phenomenon mentioned before, an increment on the growth rate. This happens because this effect overcome the transfers made from the risk lovers to the risk averters that invest in the safe one- a proportion of agents that is small in this case and decrease every time that you increase the marginal tax rate.

When $\tau = 0.6$ and 0.7, all risk averters invest in the risky one all the time since there is no over-accumulation of wealth (fat tails) in this case. Therefore, a risk averter that is infinitely successful by investing in the risky one has a wealth in period t bounded by $2.6\overline{w}_{t+1}$ for $\tau = 0.6$, and $2.6\overline{w}_{t+1}$ for $\tau = 0.7$.

When $\tau = 0.8$, the poorest risk averters, the ones that their parents only received the transfers in the previous period, the invest in the safe one. However, once they invest in the safe one, the transfers made by taxes increases the wealth of their successors in such a way that they decide to invest in the risky one generating the invariant distribution observed. This phenomenon continues making that a larger proportion of risk averters decide to invest in the safe one instead of the risky one. However, once the growth rate is negative for a long number of periods, the risk lovers will switch too implying that the invariant distribution is even lower. In this case ($\tau = 0.9$), all risk lovers and all risk averters will eventually invest only in the safe one implying that the invariant distribution is constant.

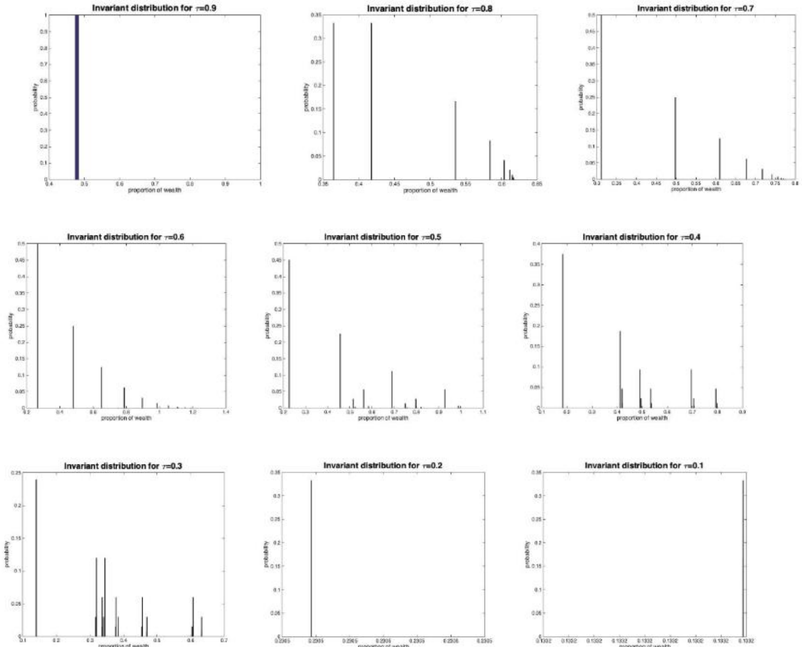

Figure 7: Invariant concentration of wealth for the risk averters for different income tax rates.

Appendix C. Properties of the model with changes of type of investors

Note that in every period, the proportion of each type of agent keeps constant over time since a proportion p of the agents changes from entrepreneurs to risk-averse investors, and vice versa. Then, the average wealth in date $t+1$, $\overline{w}_{t+1}$, is given by

$$\overline{w}_{t+1} = \frac{E[R_R]\delta\left(1-\overline{\tau}^B\right)}{2\left(1-\delta\overline{\tau}^B\right)}\left(\frac{\overline{\tau}^I\overline{w}_t}{2} + (1-p)\left(\overline{w}_t^l - \frac{\overline{\tau}^I\overline{w}_t}{2}\right) + p\overline{w}_t^a\right) + \frac{R_S\delta\left(1-\overline{\tau}^B\right)}{2\left(1-\delta\overline{\tau}^B\right)}\overline{w}_t^a.$$

FIX IT IN THE ORIGINAL VERSION WHEN IT IS READY!!1

Where the left term $\frac{\overline{\tau}^I\overline{w}_t}{2}$ is the aggregate wealth of the entrepreneurs that received 0 at the date t, $(1-p)\left(\overline{w}_t^l - \frac{\overline{\tau}^I\overline{w}_t}{2}\right)$ is the aggregate wealth of the entrepreneurs that received R_R as a return at the date t and have a successor that are entrepreneurs, and $p\overline{w}_t^a$ is the aggregate wealth of the predecessors of the entrepreneurs who were risk-averse investors. Then, the average wealth in date $t+1$, $\overline{w}_{t+1}^l$ and $\overline{w}_{t+1}^a$, are given by

$$\overline{w}_{t+1}^l = \frac{E[R_R]\delta\left(1-\overline{\tau}^B\right)}{1-\delta\overline{\tau}^B}\left(\left(1-\frac{\overline{\tau}^I}{2}\right)\left(\frac{\overline{\tau}^I\overline{w}_t}{2} + (1-p)\left(\overline{w}_t^l - \frac{\overline{\tau}^I\overline{w}_t}{2}\right) + p\overline{w}_t^a\right)\right)$$

$$+\,\overline{\tau}^I\,\frac{R_S\delta\left(1-\overline{\tau}^B\right)}{2\left(1-\delta\overline{\tau}^B\right)}\overline{w}_t^a,$$

$$\overline{w}_{t+1}^a = \frac{E[R_R]\delta\left(1-\overline{\tau}^B\right)}{1-\delta\overline{\tau}^B}\left(\frac{\overline{\tau}^I}{2}\right)\left(\frac{\overline{\tau}^I\overline{w}_t}{2} + (1-p)\left(\overline{w}_t^l - \frac{\overline{\tau}^I\overline{w}_t}{2}\right) + p\overline{w}_t^a\right)$$

$$+\frac{R_S\delta\left(1-\overline{\tau}^B\right)}{1-\delta\overline{\tau}^B}\left(1-\frac{\overline{\tau}^I}{2}\right)\overline{w}_t^a.$$

Substituting $\overline{w}_t = \frac{1}{2}\left(\overline{w}_t^l + \overline{w}_t^a\right)$, we have that

$$\overline{w}_{t+1}^{l} = \frac{E[R_R]\delta\left(1-\overline{\tau}^{B}\right)}{1-\delta\overline{\tau}^{B}}\left(\left(\left(1-\frac{\overline{\tau}^{I}}{2}\right)\frac{\overline{\tau}^{I}}{4}+(1-p)\left(1-\frac{\overline{\tau}^{I}}{4}\right)\right)\overline{w}_{t}^{l}\right.$$

$$\left.+\left(\left(1-\frac{\overline{\tau}^{I}}{2}\right)\frac{\overline{\tau}^{I}}{4}-(1-p)\left(\frac{\overline{\tau}^{I}}{4}\right)+p\right)\overline{w}_{t}^{a}\right)+\overline{\tau}^{I}\frac{R_S\delta\left(1-\overline{\tau}^{B}\right)}{2\left(1-\delta\overline{\tau}^{B}\right)}\overline{w}_{t}^{a},$$

$$\overline{w}_{t+1}^{a} = \frac{E[R_R]\delta\left(1-\overline{\tau}^{B}\right)}{1-\delta\overline{\tau}^{B}}\left(\frac{\overline{\tau}^{I}}{2}\right)\left(\left(\frac{\overline{\tau}^{I}}{4}+(1-p)\left(1-\frac{\overline{\tau}^{I}}{4}\right)\right)\overline{w}_{t}^{l}+\left(\frac{\overline{\tau}^{I}}{4}-(1-p)\left(\frac{\overline{\tau}^{I}}{4}\right)+p\right)\overline{w}_{t}^{a}\right)$$

$$+\frac{R_S\delta\left(1-\overline{\tau}^{B}\right)}{1-\delta\overline{\tau}^{B}}\left(1-\frac{\overline{\tau}^{I}}{2}\right)\overline{w}_{t}^{a}.$$

Then,

$$\frac{\overline{w}_{t+1}^{l}}{\overline{w}_{t+1}^{a}} = \frac{\left(\left(1-\frac{\overline{\tau}^{I}}{2}\right)\frac{\overline{\tau}^{I}}{4}+(1-p)\left(1-\frac{\overline{\tau}^{I}}{4}\right)\right)\frac{\overline{w}_{t}^{l}}{\overline{w}_{t}^{a}}+\left(\left(1-\frac{\overline{\tau}^{I}}{2}\right)\frac{\overline{\tau}^{I}}{4}-(1-p)\left(\frac{\overline{\tau}^{I}}{4}\right)+p\right)+\frac{\overline{\tau}^{I}}{2}\frac{R_S}{E[R_R]}}{\left(\frac{\overline{\tau}^{I}}{2}\right)\left(\frac{\overline{\tau}^{I}}{4}+(1-p)\left(1-\frac{\overline{\tau}^{I}}{4}\right)\right)\frac{\overline{w}_{t}^{l}}{\overline{w}_{t}^{a}}+\left(\frac{\overline{\tau}^{I}}{2}\right)\left(\frac{\overline{\tau}^{I}}{4}-(1-p)\left(\frac{\overline{\tau}^{I}}{4}\right)+p\right)+\frac{R_S}{E[R_R]}\left(1-\frac{\overline{\tau}^{I}}{2}\right)}$$

Similar to the proof of Theorem 1, it can be proved that the function $f\colon [0,\infty) \to [1,\infty)$ defined by $f\left(\frac{\overline{w}_{t}^{l}}{\overline{w}_{t}^{a}}\right) = \frac{\overline{w}_{t+1}^{l}}{\overline{w}_{t+1}^{a}}$ satisfies that $f(0) > 1$, $f'(z) > 0 \forall z \in [0,\infty)$, $f'(\infty) = 0$ and f' is a decreasing function. Under these conditions, f has only one fixed point $\gamma_{\overline{\tau}}$ and that $\frac{\overline{w}_{t}^{l}}{\overline{w}_{t}^{a}}$ converges to $\gamma_{\overline{\tau}}$. Which implies that the growth rate and the wealth of the risk averse investors also converges. These results also ensure that the proportion of wealth that entrepreneurs leave as a bequest to the next generation of entrepreneurs does not exceed δ.

To finish the proof, we can construct recursively the invariant distribution as in Theorem 1. Note that the big difference is that there are two levels of wealth of the entrepreneurs at the bottom of the distribution, the ones who invest and got only the transfer which represent half of them, and the ones whose immediate predecessor was a risk-averse investor which represent a proportion p of them. Note that this concludes the proof.