Foreign Reserves and Exchange Rates

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Abstract

In a traditional open macro model, we show that when we account for the common exchange rate puzzles, we also generate general equilibrium levels for foreign reserves and exchange rate volatility consistent with common emerging markets values. In such an environment, the country finds it optimal to issue debt in domestic currency to finance assets in foreign currency. In a model without such correction, we show that the optimal portfolio on foreign currency consists of a short position, *e.g.* a debt in foreign currency. This correction is done through financial friction and key calibration. We show this result using both a reduced form and a micro foundation for financial friction. To outline the intuition, all results are obtained with closed-form solutions.

Keywords: key1, key2, key3

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1 Introduction

The reasons why countries accumulate reserves are not yet so clear and are difficult to measure. Some reasons to retain foreign exchange include engendering confidence in the national currency, countering disorderly market conditions, supporting the conduction of monetary policy, build assets for intergenerational purposes or influencing the exchange rate. Although there is a consensus on the cost of retaining reserves as the spread of domestic and foreign rates, there is not much consensus on the benefits, especially on the quantitative scope. This difficulty arises from the complexity of quantifying external risks, making it harder to argue about the role of reserves on risk premia maintenance.

We contribute to the literature by showing how reserves can be a useful asset even when there is no debt crisis, occasionally binding constraints, or large shocks such as disasters. By adjusting a typical open macro model to reproduce realistic features of exchange rates through the inclusion of a financial shock, foreign assets are desirable because of the exchange rate pricing structure that emerges in general equilibrium. As shown by **Oleg and Dmitri (2021)**, proper calibration of this shock can solve many of the exchange rate puzzles that appear in standard open macro models. Without the financial shock, we show that a standard open macro model can't generate this structure and the country wishes to issue non-defaultable debt rather than buy international reserves. The mechanism that generates this feature is the standard productivity shock and/or financial market completeness because they will imply in both contemporaneous exchange rate depreciation and consumption increases[1].

This is a typical correlation that appears in open macro models. If domestic endowment (or productivity) is higher relative to the foreign endowment, and there is home-bias[2], then

[1]A positive high correlation, near one in the traditional models. So-called in the exchange rate literature by Backus-Smith Puzzle.

[2]Home-bias occurs when more than half of the domestic consumption bundle is produced domestically. This is consistent with countries that present imports-GDP rates lower than 50%.

it's possible to assemble more domestic consumption baskets relative to foreign consumption baskets. Because of home bias, these consumption baskets are different since the first is more concentrated on home goods, and the second on foreign goods. Therefore, after a positive domestic endowment shock, the relative price of the foreign consumption basket should be higher, meaning by definition a real exchange rate depreciation. But in this case, we also have an increase in domestic consumption. In this environment, the foreign bond is an asset that pays excess returns only in good states. Because agents are risk-averse, they desire an asset that pays excess returns in bad states, that is they wish for a short position in the foreign bond, *e.g.* a (non-defaultable) debt in foreign currency. In other words, in this environment, the domestic country issues debt in foreign currency because the service of debt is lower in bad states. Abstracting the option of default for tractability, we show this result with closed form solutions.

Including a persistent financial shock can soften or even change this feature. A risk-premium shock increases the return on foreign bonds, and the endogenous response is both an increase in the domestic real rates and an expected real exchange rate appreciation for the future, to maintain the uncovered interest rate parity[3]. The expected real exchange appreciation is done through a high contemporaneous depreciation, which is expected to slowly appreciate again in the future as the shock is mitigated. Both movements are consistent with a drop in domestic consumption, caused both by an increase in the price of consuming today rather than tomorrow, and a crowd-out of domestic consumption by foreign demand, since the price of the domestic endowment is lower. Therefore the risk-premium shock provides an opposite correlation for real exchange rates and consumption, and its inclusion in the model can even fixes many exchange rate puzzles[4], bringing exchange rate dynamics closer to the data. To obtain intuition, we obtain closed-form solutions for the portfolios, allowing us to see the transmission channels involved. Adding this shock to the model, the domestic

[3]Adjusted by the risk-premium shock

[4]As formally shown by **OLEG and ITSKHOKI (2021)**

country now wishes to hold international reserves rather than debt for almost any positive value of the financial shock volatility. As we correct the model to reproduce more realistic features of exchange rates, we also start to observe reserve accumulation.

The risk-premium shock is a reduced form of friction in the financial markets of the economies. Another advantage of this reduced form is that there is many microfoundations, and a large field of the macro-finance attempts to endogenize such frictions[5]. We also provide a typical microfoundation from the financial frictions literature, that will endogenize such reduced form. The model embeds in general equilibrium the noise trader and limits-to-arbitrage model of **De Long, Shleifer, Summers, and Waldmann (1990)** and its adaptation to the exchange rate market by **Jeanne and Rose (2002)**. It captures the spirit of the idea that emerging countries are still developing their financial markets, through a limited amount of financial intermediaries and their risk-aversion. The microfoundation consists of a positive mass of noisy traders that exogenously demands foreign bonds, and this demand must be intermediated by a relatively small mass of risk-averse financial intermediaries, that will require a risk premium for the transaction. We show that this microfoundation can generate in equilibrium the empirical feature that higher reserves imply lower equilibrium volatility of exchange rates. The key parameters that will shape this relation are the mass of noisy traders, and the mass and risk aversion of financial intermediaries. Though such parameters are difficult to calibrate using data, they set an important intuition for the maturity of financial markets, especially for developing economies. We can calibrate these parameters so that the resulting general equilibrium implies foreign reserves consistent with emerging markets values, and equilibrium exchange rate volatility half the size. To obtain intuition, we obtain results with closed-form solutions.

[5]See: exogenous preferences for foreign assets **Dekle, Jeong, and Kiyotaki (2014)**; Shock to the net worth of financial intermediaries **Hau and Rey 2006, Brunnermeier, Nagel, and Pedersen 2009, Gabaix and Maggiori 2015, Adrian, Etula, and Shin 2015**; Incomplete information, heterogeneous beliefs and expectational errors **Evans and Lyons (2002), Gourinchas and Tornell (2004) and Bacchetta and van Wincoop (2006)**

The traditional literature that approaches the reserves problem usually addresses it in a sovereign debt crisis or sudden stops model, making international reserves a natural emergency saving for large crises. Laura **Alfaro and Fabio Kanczuk (2009) ?** builds a sovereign default model, similar to **Cristina Arellano (2008)**, that takes a quantitative look into the joint accumulation of both reserves and debt in an environment where a country has the option to default on external debt, but they do not get numerical results consistent with data. From there, many other papers surge in a similar environment to try to improve numerical results, using additional assumptions such as debt maturity and default haircuts. For example, **Bianchi, Hatchondo, and Martinez (2018) ?** works in a similar model, but the external debt is a long-term debt instead of a one-period debt. In this way, defaultable debt and reserves are less similar assets, and reserves actively contribute to reducing the rollover risk, because this risk concerns the ability of payment in longer horizons. Authors also include risk-averse lenders, which helps to improve results. **Ricardo Sabbadini (2019) ?** designs a similar model, also with risk-averse lenders but with debt haircut, also called as partial default. These extensions can be calibrated to reproduce observable values of international reserves for the common emerging markets. More recently, **Alfaro and Fabio Kanczuk (2018) ?** extend their previous model with a non-tradeable sector, trying to capture some of the exchange rate behavior in the model, and indeed find a calibration that can improve their numerical results, although exchange rate modeling presents many distortions when compared to data[6]

On a different approach, **Hur and Kondo (2016) ?** model a short-term international debt as a contract with international investors, where the sovereign uses these resources to finance long-term investment. The structure of the model is similar to **Diamond and Dybvig (1983) ?**, where there is a liquidity shock and international investors suddenly decide not to roll the debt to the extent needed for the investment maturity. In this environment,

[6] In their model, real exchange rate equals the relative price of non-tradeable, while the non-tradeable prices in data is much less volatile and much more predictable than exchange rates. See

it is optimal for the country to save part of the resources raised by debt as reserves, in case such liquidity runs happen in the future before investment maturity. **Bianchi (2011)** approaches sudden stop as a binding constraint on debt accumulation. His model brings the idea of endogenizing a sudden stop with the non-linearity of policy functions near the endogenous constraints bind. These constraints bind when debt is high and endowment is low, so with the binding constraint, the only possible policy is to drastically reduce the amount of debt raised, which is interpreted as a sudden stop. **Arce, Bengui, and Bianchi (2022) ?** extend this model allowing for reserves accumulations as a macroprudential policy and also introducing a financial shock as flexibility in the binding constraint for debt, which was originally measured as a parameter.

What all this literature has in common is that the role of international reserves is to guarantee some consumption smoothing in large crises. That's a valuable incentive since financial markets are sensitive and are in constant threat of quick corrosion. We differ from the traditional literature showing that reserves are also an attractive asset even when these large crises are not present, but only due to the fact that international reserves are denominated in foreign currency rather than domestic currency. In our model, it is optimal to issue debt in domestic currency to finance assets in foreign currency, just because of the hedging incentive of exchange rates. Therefore, this paper aims to complement the literature with one more reason to retain international reserves, which is a hedge through exchange rates. Note that, for that to make sense, the exchange rate dynamics in the model must be minimally realistic. As it may be a surprise, this is a challenging task and there is a whole literature documenting the exchange rate puzzles generated by open macro models. We rely on a recent paper of **Oleg and Itskhoki (2021) ?** that accounts for a correction of the main exchange rate puzzles in traditional open macro models, with simple changes.

2 Baseline Model

In this section we outline what is defined as a *baseline* model. The baseline model is only a traditional open macro model, which consists of a model with no friction and only endowment fluctuation, and an open economy. Then, in this baseline model, we include financial friction through an exogenous shock in the return of foreign bonds.

Home and Foreign Sector

The economy is populated by two countries, each containing a representative agent. There are two goods in the economy, good H, and good F. We denote with a subscript the origin of the good/bond being demanded, and with a star superscript to denote that is the foreign household demanding such good/bond[7]. Both households combine both goods into baskets and gain utility over the consumption of such baskets:

$$C_t = \left[(1-\gamma)^{\frac{1}{\theta}} C_{H,t}^{\frac{\theta-1}{\theta}} + \gamma^{\frac{1}{\theta}} C_{F,t}^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} \tag{1}$$

$$C_t^* = \left[(1-\gamma)^{\frac{1}{\theta}} C_{F,t}^{*\,\frac{\theta-1}{\theta}} + \gamma^{\frac{1}{\theta}} C_{H,t}^{*\,\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} \tag{2}$$

Where $\theta > 1$ is the elasticity of substitution between good H and F in each basket. Higher θ means that, when one of the goods becomes more expensive, the household can easily substitute this good with small amounts of the other good to compensate for the drop in utility level. Therefore, higher θ will be associated with low levels of terms of trade and real exchange rate volatility[8].

[7] For example, $C_{H,t}$ is the domestic household demand for the H good, and $C_{H,t}^*$ is the foreign household demand also for the H good.

[8] This intuition explains why, on most calibrations and estimation of open macro models, $\theta \sim 1.5$. A value much lower than the common elasticity of substitution in New-Keynesian models (around $9 \sim 11$)

Another important parameter is γ, which measures the degree of home bias. Lower values of γ mean that there is much more good H than good F in the domestic consumption bundle. Therefore, when there are more H goods available than F goods, agents can build many more C_t baskets than C_t^* baskets. In general equilibrium, the price of the C_t basket will be lower than the price of the C_t^* basket, which is by a definition a real exchange rate depreciation. Therefore, lower γ is associated with higher volatility of the real exchange rate. From the cost minimization problem of a typical household allocating H and F goods in a basket, given prices, we have:

$$C_{H,t} = (1-\gamma)P_{H,t}^{-\theta}C_t, \quad C_{F,t} = \gamma P_{F,t}^{-\theta}C_t \tag{3}$$

$$C_{F,t}^* = (1-\gamma)P_{F,t}^{*-\theta}C_t^*, \quad C_{H,t}^* = \gamma P_{H,t}^{*-\theta}C_t^* \tag{4}$$

Where $P_{H,t}$, $P_{H,t}^*$ is the price of good H in domestic and foreign currency, respectively. Similarly for $P_{F,t}, P_{F,t}^*$. For simplicity, we assume that monetary policy fixes the nominal price levels, which are the prices of the domestic basket in local currency $P_t = 1$, and the foreign basket in foreign currency, $P_t^* = 1$. Therefore, by definition, real exchange rate and nominal exchange rate are equal[9]. From the price index definition, such that total expenditure for goods equals total bundle quantity:

$$P_t = 1 = \left[(1-\gamma)P_{H,t}^{1-\theta} + \gamma P_{F,t}^{1-\theta}\right]^{\frac{1}{1-\theta}} \tag{5}$$

$$P_t^* = 1 = \left[(1-\gamma){P_{F,t}^*}^{1-\theta} + \gamma {P_{H,t}^*}^{1-\theta}\right]^{\frac{1}{1-\theta}} \tag{6}$$

[9]Let Q_t and $\mathcal{E}_t$ be the real and nominal exchange rate, respectively. From the definition of real exchange rate $Q_t = P_t^*\mathcal{E}_t/P_t = \mathcal{E}_t$

Terms of Trade and Real Exchange Rate

The baseline model allows the law of one price for each good. Let Q_t be the real exchange rate, which represents how many domestic consumption baskets C_t can be exchanged for one foreign consumption basket, C_t^*. The law of one price implies that[10] $P_{H,t} = Q_t P_{H,t}^*$ and $P_{F,t} = Q_t P_{F,t}^*$. Terms of trade are defined by:

$$S_t = \frac{P_{F,t}}{P_{H,t}} \tag{7}$$

Terms of trade (or relative price) are defined such that an increase in S_t means that the domestic country has a consumption increase relative to the foreign country. This is because if S_t increases, then $P_{F,t}$ is higher than $P_{H,t}$, which, in this endowment economy, means that there are more goods H than F available in the economy, therefore more C_t can be built over C_t^*. To simplify model equations, define the following functions of the terms of trade:

$$\frac{1}{P_{H,t}} = \left[(1-\gamma) + \gamma S_t^{1-\theta}\right]^{\frac{1}{1-\theta}} \equiv g(S_t) \tag{8}$$

$$\frac{1}{P_{F,t}} = \left[(1-\gamma) S_t^{-(1-\theta)} + \gamma\right]^{\frac{1}{1-\theta}} \equiv h(S_t) \tag{9}$$

$$\frac{1}{P_{H,t}^*} = \left[\gamma + (1-\gamma) S_t^{1-\theta}\right]^{\frac{1}{1-\theta}} \equiv g^*(S_t) \tag{10}$$

$$\frac{1}{P_{F,t}^*} = \left[\gamma S_t^{-(1-\theta)} + (1-\gamma)\right]^{\frac{1}{1-\theta}} \equiv h^*(S_t) \tag{11}$$

To obtain these functions just divide the price index definitions by each individual price. Using the law of one price, we have that $Q_t = g^*(S_t)/g(S_t)$. Inverting this function we have

[10] It may seem quite obvious, but actually this contributes to a Purchase Power Parity puzzle. In this specification, terms of trade are more volatile than the real exchange rates. There are alternatives such as local currency pricing, or pricing to market, that make the price faced by foreign households for the good H actually different from the price faced by domestic households, corrected by the nominal exchange rate.

the non-linear relation between terms of trade and real exchange rate:

$$S_t = \left[\frac{Q_t^{1-\theta}(1-\gamma)-\gamma}{1-\gamma(1+Q_t^{1-\theta})}\right]^{\frac{1}{1-\theta}} \tag{12}$$

Observe that if $\gamma = 1/2$, then $S_t = 1, \forall t$. In such case, from $Q_t = g^*(S_t)/g(S_t)$, it follows that $Q_t = 1, \forall t$. The source of real exchange rate fluctuation in the model is the home bias[11] $\gamma < 1/2$.

Market-Clear and Budget Constraints

Market clear occurs in good markets and bond markets. Every period domestic economy receives a stochastic endowment of Y_t, denominated in H goods. Similarly, the foreign economy receives Y_t^* denominated in F goods. Market-clear implies that aggregate demand toward each good must be equal to its aggregate supply:

$$Y_t = C_{H,t} + C_{H,t}^*, \quad Y_t^* = C_{F,t} + C_{F,t}^* \tag{13}$$

Using equations 3 and 4 we can write this conditions as function of S_t, C_t and C_t^*. We could rule out S_t using equation 7 and only work with real exchange rates Q_t, but we carry S_t to avoid larger expressions:

$$Y_t = (1-\gamma)g(S_t)^{\theta}C_t + \gamma g^*(S_t)C_t^* \tag{14}$$

$$Y_t^* = (1-\gamma)h^*(S_t)^{\theta}C_t^* + \gamma h(S_t)C_t \tag{15}$$

[11]Clearly $\gamma > 1/2$ would also provide real exchange rate fluctuation. But $\gamma > 1/2$ implies more imported goods than domestic goods in the aggregate domestic consumption, a not consistent behavior with data. An additional force of real exchange rate fluctuation is pricing to market behavior, which is later included for numerical enhancement.

Let $B_{H,t}$ be the domestic demand for home bonds, and $B_{F,t}$ be the domestic demand for foreign bonds. Both are denominated in domestic consumption units, C_t. Domestic bonds pay a return in units of domestic currency in the next period, and foreign bonds pay a return in units of foreign currency in the next period. Since we consider a constant price level for each economy for simplicity, bond payments are the respective consumption baskets themselves.

We measure the size of all portfolios in units of C_t, therefore currency variations are incorporated through returns. The real return of a domestic bond purchased in period $t-1$ is simply R_{t-1}, and its price is normalized as one. The real return of a foreign bond purchased in period $t-1$ is composed of the real return in the foreign currency R^*_{t-1} and the currency variation in the period, measured by Q_t/Q_{t-1}, and its price is also normalized to one. The price normalization comes from writing both budget constraints in units of the domestic basket C_t. The point of such is to be able to compare portfolios without adjusting for exchange rates.

Summarizing, real returns are endogenously determined in equilibrium, but each bond consists of a risk-free asset, one paying in C_t units, and the other paying in C^*_t units. The stochastic component comes from the fact that a domestic household can only infer utility over C_t units, so when she receives C^*_t payments from a foreign bond, she has to trade it for C_t units in the market, which will depend on the stochastic exchange rate. Domestic household budget constraint is given by:

$$C_t + B_{H,t} + B_{F,t} = \frac{P_{H,t}}{P_t} Y_t + R_{t-1} B_{H,t-1} + e^{\psi_t} \frac{Q_t}{Q_{t-1}} R^*_{t-1} B_{F,t-1} \tag{16}$$

The ψ_t term is an exogenous risk-premium shock, as a reduced form of financial friction. In the next section, we provide a microfoundation for the shock. A persistent increase in the risk-premium, due to the resulting adjusted interest-rate parity, will decrease the expected return on the exchange rate. In equilibrium, this is achieved through a strong current depreciation, with expected slower appreciation in the future. The intuition as a risk-premium shock comes from the fact that, after a surprise of such shock, real exchange rates depreciate and real interest rates rise, decreasing domestic consumption. Foreign households do not face risk-premium risk and their budget constraint, denominated in units of domestic baskets C_t, is given by:

$$Q_t C_t^* + B_{H,t}^* + B_{F,t}^* = Q_t \frac{P_{F,t}^*}{P_t^*} Y_t + R_{t-1} B_{H,t-1}^* + \frac{Q_t}{Q_{t-1}} R_{t-1}^* B_{F,t-1}^*$$

We will omit this equation of the system using Walras law. Combine the household budget constraint 16, the H good market clear 13 and the property of the price index $C_t = P_{H,t} C_{H,t} + P_{F,t} C_{F,t}$ to rewrite budget constraint as:

$$B_{H,t} + B_{F,t} = R_{t-1} B_{H,t-1} + e^{\psi_t} \frac{Q_t}{Q_{t-1}} R_{t-1}^* B_{F,t-1} + \left[P_{H,t} C_{H,t}^* - P_{F,t} C_{F,t} \right]$$

Where the last term on the right-hand side is the current account, that is, exports minus imports[12]. The financial account would be the liquid returns over previous bond positions. We can rewrite the current account in terms of C_t, C_t^*, S_t, and Q_t. Then, it is possible to write equilibrium conditions with fewer variables.

$$B_{H,t} + B_{F,t} = R_{t-1} B_{H,t-1} + e^{\psi_t} \frac{Q_t}{Q_{t-1}} R_{t-1}^* B_{F,t-1} + \gamma \left[Q_t g^*(S_t)^{\theta-1} C_t^* - h(S_t)^{\theta-1} C_t \right] \qquad (17)$$

Where we substituted $C_{F,t}$, $C_{H,t}^*$ using equations 3 and 4, applied the law of one price, and then used equations 9, 10 to write in terms of the terms of trade. We assume a zero-net supply for both bonds. Bonds market clear are:

[12] In this model, current account equals commercial balance. In an endogenous production setup, if firms import labor, we would additionally have wage payments sent to the foreign country, breaking this equality.

$$B_{H,t} + B^*_{H,t} = 0 \tag{18}$$

$$B_{F,t} + B^*_{F,t} = 0 \tag{19}$$

By Walras Law, we can omit the budget constraint for foreign households. If the domestic budget is satisfied, and the market clears of bonds e goods markets are satisfied, the foreign budget constraint must be satisfied from an excess demand condition, and prices will be such that support this allocation.

Households

As previously mentioned, each country is populated by a representative infinitely lived household. Domestic household maximizes the expected discounted instantaneous utility:

$$\max_{\{C_t, B_{H,t}, B_{F,t}\}} \quad E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma}, \quad \text{such that}$$

$$C_t + B_{H,t} + B_{F,t} = P_{H,t} Y_t + R_{t-1} B_{H,t-1} + e^{\psi_t} \frac{Q_t}{Q_{t-1}} R^*_{t-1} B_{F,t-1}$$

Foreign households face a similar problem, but we write their budget constraint in terms of the domestic consumption bundle, C_t. They maximize:

$$\max_{\{C^*_t, B^*_{H,t}, B^*_{F,t}\}} E_0 \sum_{j=0}^{\infty} \beta^t \frac{C_t^{*1-\sigma}}{1-\sigma} \quad \text{subject to}$$

$$Q_t C^*_t + B^*_{H,t} + B^*_{F,t} = Q_t P^*_{F,t} Y_t + R_{t-1} B^*_{H,t-1} + \frac{Q_t}{Q_{t-1}} R^*_{t-1} B^*_{F,t-1}$$

First-order conditions are conventional euler equations for both households:

$$E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}R_t\right]=1 \tag{20}$$

$$E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}R_t^*\frac{Q_{t+1}}{Q_t}e^{\psi_{t+1}}\right]=1 \tag{21}$$

$$E_t\left[\beta\left(\frac{C_{t+1}^*}{C_t^*}\right)^{-\sigma}R_t^*\right]=1 \tag{22}$$

$$E_t\left[\beta\left(\frac{C_{t+1}^*}{C_t^*}\right)^{-\sigma}\left(\frac{Q_t}{Q_{t+1}}\right)R_t\right]=1 \tag{23}$$

2.1 Equilibrium

Definition 1 *An equilibrium are functions* $\{C_t, C_t^*, Q_t, S_t, B_{H,t}, B_{H,t}^*, B_{F,t}, B_{F,t}^*, R_t, R_t^*\}$ *defined over the states* $(Y_t, Y_t^*, \psi_t, B_{H,t-1}, B_{F,t-1})$ *such that satisfies equations 12, 14, 15, 17, 18, 19, 20-23. Those equations correspond to 1 terms of trade equation, 2 goods market-clear, 2 bonds market-clear, 4 Euler equations, and 1 budget constraint.*

To provide intuition on the role of the ψ_t shock through closed-form solutions, we will work with an approximation of such equilibrium. To identify portfolios in this approximation, we use the technique of **DEVEREUX and SUTHERLAND (2014)**. They show how to obtain the *zero-order* term of a Taylor series expansion of portfolios policy function. Therefore, in the first-order approximated solution, this value will be the ergotic mean of portfolios. If the first-order approximation is good enough, the *zero-order* portfolio should be closer to the true ergotic mean of portfolios ergotic distribution.

Usually, we choose a point at which to make an approximation around, and is usually the steady state. To obtain the steady state we must analyze a version of the system that is neither dynamic nor stochastic. In this system, asset positions are not identifiable. Any position is actually a consistent solution. Using their method, we find such a point that is consistent with first-order consumption and returns dynamics and a second-order risk aversion arising from preferences. Their method consists of noticing that, in a second-order approximation of Euler equations[13], we get an equation that depends on a combination of only first-order terms:

$$E_t\left(\sigma c^R_{t+1} - q_{t+1}\right) r^X_{t+1} = 0 \tag{24}$$

Where $c^R_{t+1} \equiv c_{t+1} - c^*_{t+1}$, $r^X_{t+1} \equiv q_{t+1} - q_t + r^*_t - r_t + \psi_t$, and all variables are written as log-deviation of their steady-state. To solve for the *zero-order* portfolio, we need a first-order solution for such variables. These first-order solutions will depend on such *zero-order* portfolio. To see how, define the domestic country's net wealth as $W_t \equiv B_{H,t} + B_{F,t}$. Write the domestic budget constraint as:

$$W_t = R_t W_{t-1} + R^X_t B_{F,t-1} + \gamma\left[Q_t g^*(S_t)^{\theta-1} C^*_t - h(S_t)^{\theta-1} C_t\right] \tag{25}$$

Where $R^X_{t+1} \equiv e^{\psi_t}\frac{Q_{t+1}}{Q_t}R^*_t - R_t$ is the excess return on foreign bonds. Perform a first-order linear approximation around the steady state, using the fact that, due to symmetry $W_{ss} = 0$, and the fact that, at first-order, $q_t = (1-2\gamma)s_t$. Also recall that the steady-state for foreign bonds $B_{F,t-1}$ is the *zero-order* portfolio that we wish to endogenously determine, say B_F. Divide the result both sides by Y_{ss}:

$$w_t = \frac{1}{\beta}w_{t-1} + r^X_t b_F + \gamma\left(\frac{(2\theta(1-\gamma)-1)}{1-2\gamma}q_t - c^R_t\right) \tag{26}$$

Where $w_t \equiv (W_t - W_{ss})/Y_{ss}$, $b_F = B_F/Y_{ss}$, $r^x_t = q_{t+1} - q_t + r^*_t - r_t + \psi_t$, $y^R_t = \log(Y_t) - \log(Y^*_t)$. Observe that now the relevant state variable is only the total wealth position[14],

[13]Under CRRA preferences. See **DEVEREUX and SUTHERLAND (2014)**

[14]This is not a property of the first-order approximation of equilibrium. In the non-linear version, we can define total wealth as the unique endogenous state variable, but policy functions must be consistent with such wealth for each state, increasing the number of equations in the system by the number of states.

w_{t-1}. A positive position on foreign bonds ($b_F > 0$) implies that the country will be wealthier when the excess returns on foreign bonds is positive, increasing its consumption in the next periods.

2.2 Solution

Observe that the time variation of the portfolio does not matter for an approximation of first-order solution. The only relevant term is the *zero-order* term in a Taylor series approximation of the true equilibrium portfolio function. Therefore, this method delivers a solution that exhausts all the macroeconomic implications of portfolio choice at this level of approximation.

The *zero-order* solution serves as an approximation of the mean of the ergotic distribution of portfolios when the volatilities of the shocks are small and the model is stationary because, under such conditions, first-order solutions are reasonable approximations. Thus, we can look at the *zero-order* portfolio as an approximation of the *long-run* portfolio position, or as **DEVEREUX and SUTHERLAND (2014)** refers, an (endogenous) *steady-state* portfolios. We begin reducing the linearized model as most as possible:

Lemma 1 *Let $\xi_t \equiv b_F r_t^X$ be a zero-mean shock, and assume that both countries' endowment processes are equal, but with different innovations. We can reduce the linearized model into a system of two equations and two variables:*

$$E_t q_{t+1} = q_t - \omega_1 y_t^R - \omega_2 \psi_t \tag{27}$$

$$w_t = \frac{1}{\beta} w_{t-1} + \mu q_t - \frac{\gamma}{1-2\gamma} y_t^R + \xi_t, \quad \textit{where} \tag{28}$$

$$\omega_1 \equiv \frac{(1-2\gamma)(1-\rho)\sigma}{4\sigma\theta\gamma(1-\gamma)+(1-2\gamma)^2} > 0, \quad \omega_2 \equiv \frac{(1-2\gamma)^2}{4\theta\sigma\gamma(1-\gamma)+(1-2\gamma)^2} > 0, \quad \textit{and}$$

$$\mu \equiv \frac{2\gamma\theta(1-\gamma)-\gamma(1-2\gamma)}{(1-2\gamma)^2} > 0$$

Proof. Shown in the appendix

The spirit of this *lemma* is to arrange the model such that we would if $b_F r_t^X$ were an *i.i.d zero mean* shock. Actually, $E_t r_{t+1}^X = 0$ is an equilibrium condition, but the covariance structure of $b_F r_t^X$ is endogenous. However, we do not need covariance information for a first-order solution. Therefore, this *lemma* is an intermediate step to solve for first-order dynamics of consumption and exchange rate given a *zero-mean* surprise every period.

Proposition 1 *Assume that endowment shocks follow an AR(1) process with the same lag coefficient. The solution of real exchange rate consistent with the appropriated transversality condition, given the exogenous zero-mean shock ξ_t, is:*

$$q_t = \lambda_y y_t^R + \lambda_\psi \psi_t - \frac{1-\beta}{\mu}\left(\frac{1}{\beta} w_{t-1} + \xi_t\right), \quad \textit{where} \tag{29}$$

$$\lambda_y \equiv \left(\frac{\beta\sigma(1-2\gamma)(1-\rho^y)}{4\theta\sigma\gamma(1-\gamma)+(1-2\gamma)^2} + \frac{(1-2\gamma)(1-\beta)}{2\theta(1-\gamma)-(1-2\gamma)}\right)\frac{1}{1-\beta\rho^y} > 0$$

$$\lambda_\psi \equiv \frac{(1-2\gamma)^2}{4\theta\sigma\gamma(1-\gamma)+(1-2\gamma)^2}\frac{\beta}{1-\beta\rho^\psi} > 0$$

Proof. Shown in the appendix

The coefficient $\lambda_y > 0$ implies that an increase in relative output depreciates the real exchange rate. This occurs because higher relative output means a higher quantity of goods H relative to good F available in the economy. Therefore, the relative price of good H falls. Due to home bias ($\gamma < 1/2$), domestic consumption basket relative price also falls, or, in another word, the real exchange rate depreciates.

The coefficient λ_ψ is also a positive number. As shown in **ITSKHOKI and DMITRI, JPE 2021**, the calibration that corrects the exchange rate puzzle in the model includes $\beta\rho^\psi \to 1$, which implies $\lambda_\psi \to \infty$. This implies that even low fluctuations of ψ_t will cause high real exchange rate fluctuations. This accounts for a solution for some puzzles such as high exchange rate volatility, low predictability, and UIP break.

Corollary 1 *Relative consumption $c_t^R \equiv c_t - c_t^*$ solution that is consistent with the appropriate transversality condition, given the exogenous zero-mean shock ξ_t, is given by:*

$$c_t^R = \Theta_1 y_t^R + \Theta_2 \left(\frac{1}{\beta} w_{t-1} + \xi_t \right) - \Theta_3 \psi_t \quad \textit{where} \tag{30}$$

$$\Theta_1 \equiv \frac{1 - 2\gamma - 4\theta\gamma(1-\gamma)\lambda_y}{(1-2\gamma)^2} > 0, \quad \textit{and}$$

$$\Theta_2 = \frac{4\theta\gamma(1-\gamma)(1-\beta)}{2\gamma\theta(1-\gamma) - \gamma(1-2\gamma)} > 0, \quad \textit{and}$$

$$\Theta_3 = \frac{4\theta\gamma\beta(1-\gamma)}{(1-\beta\rho^\psi)(4\theta\sigma\gamma(1-\gamma) + (1-2\gamma)^2)} > 0$$

Proof. Shown in the appendix

From the consumption policy function, we see that a positive endowment shock is associated with higher consumption and a higher exchange rate. But a risk-premium shock is associated with lower consumption and higher exchange rate[15]. Shocks provide different incentives for the role of the foreign asset in the portfolio. Under the endowment shock, the foreign asset provides good remuneration when consumption increases. Under the risk-premium shock, the foreign asset provides excess returns when consumption falls.

The reason that a risk-premium shock depreciates the exchange rate and drops consumption is through the UIP parity. Higher risk-premium shocks increase the return on foreign bonds, which endogenously causes a decrease in both $E_t \Delta q_{t+1}$ and an increase in r_t to maintain such a condition. The decrease in $E_t \Delta q_{t+1}$ is achieved through a high increase in q_t,

[15] Again, if $\beta\rho^\psi \to 1$, Θ_3 may be a higher coefficient. But as shown by **ITSKHOKI and DMITRI, JPE 2021**, a necessary calibration that solves the exchange rate disconnect puzzle is $\gamma \to 0$. But in that case $\Theta_3 \to 0$. Therefore, a lower γ will compensate for a high $\beta\rho^\psi$.

which is expected to decrease in the future. The increase in r_t is responsible to draw consumption down, through the elasticity of substitution in time. The increase in q_t also crowds out domestic consumption, because foreign households will switch expenditure toward the H good, which is cheaper, reinforcing the drop in domestic consumption.

Due to risk-aversion that arises from the second-order approximation of Euler equations, agents want assets that provide excess returns when consumption falls. In the stylized open macro model ($\psi_t = 0$), the foreign asset provides a hedge to consumption if the domestic country holds a short position. Intuitively, the country prefers to issue debt denominated in foreign currency rather than buy the asset[16], because when consumption falls, the cost of such debt also falls.

When only the financial friction is included ($\psi_t \neq 0$), even a low volatility will affect consumption if $\beta\rho^{\psi} \to 1$. When this shock happens, a risk-averse agent will want to have a long position on the foreign asset, because exchange rate increases will provide excess returns when consumption drops. Clearly, when both shocks are present we will have some combination of both effects, and volatilities will be relevant information for portfolio composition.

We already characterize the law of motion for real exchange rates. We can proceed to characterize excesses returns as a function of the foreign position b_F:

Lemma 2 *Excesses returns on the foreign asset, given the (endogenous) position long-run position b_F, is given by:*

$$r^X_{t+1} = \frac{1}{1 + \frac{(1-\beta)}{\mu} b_F} \left[\lambda_y \xi^{y^R}_{t+1} + \lambda_\psi \psi_{t+1} \right]. \tag{31}$$

Proof. Shown in the appendix

[16]Recall that here we have perfect enforcement in financial markets. Therefore, a one-period short or long position are perfect substitute asset.

If $|b_F|$ is not too large, we can use the heuristic $1-\beta \approx 0$ to interpret the excess return equation. In general equilibrium, both shocks cause an excessive return on the foreign asset, but the endowment shock increases consumption and the risk-premium shock decreases consumption. In the next result, we formalize the intuition obtained through policy functions.

Proposition 2 *In the complete markets stylized version of the model with $\psi_t = 0$, optimal (long-run) portfolio on foreign currency is:*

$$b_F^0 = \frac{1-2\gamma+\sigma(1-2\theta(1-\gamma))}{\sigma(1-2\gamma)} \frac{\gamma}{1-\beta\rho^y} \tag{32}$$

If there is home-bias ($\gamma < 1/2$), the domestic country takes a short position on the foreign asset, $b_F^0 < 0$.

Proof. Shown in the appendix

The intuition for the result arises from the correlation between the endowment shock and consumption. When the (relative) endowment shock is positive, (relative) consumption increases and the real exchange rate depreciates. The real exchange rate depreciates due to the fact that there are more resources to build additional domestic bundles C_t than foreign bundles C_t^*, therefore the price of the latter will be higher. This effect occurs due to the presence of home-bias in consumption. With real exchange rates rising, excess returns on foreign assets are positive. Also, observe that endowment volatilities do not appear in the solution. This occurs due to the complete market structure without the risk-premium shock. Since households can allocate consumption in each linearly independent state of nature, volatility does not matter for portfolio position.

Proposition 3 *In the incomplete markets version of the stylized model with $\psi_t \neq 0$, the optimal (long-run) portfolio on foreign currency is:*

$$b_F = b_F^0 + \Omega \left(\frac{\beta}{1-\beta\rho^\psi}\right)^2 \frac{\sigma_\psi^2}{\sigma_y^2}, \quad \textit{where} \tag{33}$$

$$\Omega = \frac{(2\theta\gamma(1-\gamma) - \gamma(1-2\gamma))(1-2\gamma)}{\sigma\lambda_y(1-\beta)(4\sigma\theta\gamma(1-\gamma) + (1-2\gamma)^2)} > 0$$

Under the calibration $\beta\rho^{\psi} \to 1$, the domestic country takes a long position on the foreign asset, $b_F > 0$.

Proof. Shown in the appendix.

When $\psi_t \neq 0$, foreign bonds pay excess returns that correlate negatively with the risk-premium shock, raising demand for the asset from domestic agents that wish to hedge against negative risk-premium shock. The optimal long-run portfolio for the domestic agent now can be decomposed in two terms: the (negative) zero-volatility portfolio b_F^0 that is optimal when the agent only faces endowment risk, and a positive term that is higher whenever the risk-premium shock is more persistent ($\beta\rho^{\psi} \to 1$) or more volatile (higher σ_{ψ}). If the risk-premium shock is persistent enough, then even low risk-premium shocks will induce large, and almost permanent, changes in the real exchange rate. This induces a persistent decrease in consumption and a large increase in the excess return of the foreign asset. Since the domestic country is risk-averse, a positive position on such a bond will guarantee a large one-time increase in wealth, which will soften the impact of the shock on consumption across time.

3 Microfoundation of the Financial Friction

Until now we've shown that the baseline open macro model with only endowment shocks generates *long-run* negative position on foreign currency. Countries wish to hold *non-defaultable* debt denominated in foreign currency, instead of reserves. The reason is that higher consumption is associated with more depreciated real exchange rates[17]. This relation

[17]This is actually a general feature of complete market models. As it is known that, in the model with full *Arrow-Debreu* securities, the linearized condition $\sigma c_t^R = q_t$ emerges. Under such an equation, consumption will always be positively correlated with the real exchange rate, no matter the shock. Therefore, an incomplete market is a necessary condition to reproduce more realistic real exchange rates and, consequently, reserves.

comes from the general equilibrium pricing of goods H and F. Due to home bias, higher domestic endowment means that there are more resources to build C_t baskets than C_t^*, decreasing the price of the first, *e.g.* a real exchange rate depreciation, although consumption also rises. Therefore, when consumption is low, real exchange rates are lower and the service of debt is also lower. The negative position on foreign bonds softens consumption drops in bad states.

We further extended the baseline model with a reduced form of financial friction and showed that, under some calibration, this financial friction can account for a positive *long-run* position on foreign currency. The reason is that such financial friction is a source of a negative correlation between consumption and real exchange rates, a correlation much more consistent with data, especially for emerging markets. The *risk-premium* shock induces higher returns on the foreign bond, demanding higher levels of expected appreciation on foreign currency or higher domestic interest rates. This is achieved with a contemporaneous one-time increase of real exchange and domestic interest rates, which tends to decrease in expectation in the future. The increase in domestic interest rates is consistent with a decrease in domestic consumption. Households spot higher domestic rates, meaning that consuming today is expensive. This shock causes, through general equilibrium, contemporaneous exchange rate depreciation, higher interest rates, and lower consumption, a convenient interpretation for a *risk-premium* shock.

In this section we provide a microfoundation of such shock. The microfoundation is a version of **ITSKHOKI and DMITRI, (2021)**, which is based on the noise trader and limits-to-arbitrage model of **De Long, Shleifer, Summers, and Waldmann (1990)** and its adaptation to the exchange rate market by **Jeanne and Rose (2002)**. The extension allows the domestic country to have access to foreign bonds, but only accommodated by households and chosen by the central bank through a rule. The rule is the same second-order approximation of the Euler equation 24, so it is an optimal rule in the light of Euler equations, respecting household preferences and risk-aversion. In this microfoundation, there are *noise-traders* that exogenously demand foreign bonds. The word *noise* comes from the fact that such demand does not depend directly on the country fundamentals, captured by c_t^R, w_t, or q_t. Therefore, their demands (short or long) are purely viewed as shocks in the light of the model[18]. Domestic country euler equation now is:

$$E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}R^*_{t+1}\frac{Q_{t+1}}{Q_t}\right]=1 \tag{34}$$

There are four types of agents operating in financial markets for each bond. Two of them were already included in the baseline model, which represents each country's demand for each bond. Those are captured by the bond market clear equations 18- 19. Now we include a *zero*-capital noise trade of mass $0 < n < 1$, demanding N_t of the domestic bond and N_t^* of the foreign bond. The *zero*-capital means that any short position on one bond must be financed by a long position on the other bond:

$$N_t = -N_t^* \tag{35}$$

Where, as usual, both quantities are denominated in terms of domestic consumption bundle, C_t. As mentioned, we assume that noise traders demand for foreign bonds is exogenous and given by a shock:

$$N_t^* = n\left(e^{\psi_t}-1\right) \tag{36}$$

The last type of agents are financial intermediaries of mass $0 < m < 1$. They are responsible to intermediate both countries and noise trader demand for bonds. Let D_t and D_t^* be the amount of domestic bond and foreign bond held by the intermediary, respectively.

[18] An alternative intuition is of a *noisy country*. Imagine that policymakers keep making public statements that are not consistent with welfare-improving behavior. Even though these statements are not made in practice, through breaks of euler equations or imposing additional frictions, traders become early anxious and demand foreign bonds to zero the exposition on the domestic country.

Both quantities are denominated in terms of domestic consumption bundle C_t. Financial intermediaries are also *zero*-capital based:

$$D_t = -D_t^* \tag{37}$$

Since they intermediate the demand for bonds, the total demand for each bond must be equal to the total bond supplied by the financial intermediary. Already imposing the zero capital position assumption, bond market clear now becomes:

$$B_{H,t} + B_{H.t}^* + N_t = -D_t \tag{38}$$

$$B_{F,t} + B_{F,t}^* - N_t = D_t \tag{39}$$

Each financial intermediary in the mass $[0, m]$ chooses the amount of foreign bonds to intermediate, d_t^*. Because of the *zero*-capital position, each foreign bond intermediated yields a return of $R_{t+1}^X = \frac{Q_{t+1}}{Q_t} R_t^* - R_t$. The position d_t^* is chosen in order to maximize the following mean-variance utility function:

$$\max_{d_t^*} E_t R_{t+1}^X d_t^* - \frac{\omega}{2} \text{var}_t \left(R_{t+1}^X \right) d_t^{*2}$$

,

Where ω is a risk-aversion parameter of the mean-variance agent. Since the financial intermediary is infinitely small in the continuum, it does not internalize her position impact on R_{t+1}^X, and it takes as a given process. Aggregating the individual solution we have:

$$D_t^* = m \frac{E_t R_{t+1}^X}{\omega \text{var}_t \left(R_{t+1}^X \right)} \tag{40}$$

3.1 Equilibrium

Definition 2 *An equilibrium are functions* $\{C_t, C_t^*, Q_t, S_t, B_{H,t}, B_{H,t}^*, B_{F,t}, B_{F,t}^*, R_t, R_t^*, D_t, N_t\}$ *defined over the states* $(Y_t, Y_t^*, \psi_t, B_{H,t-1}, B_{F,t-1})$ *such that satifies equations 12, 14, 15, 17, 38, 39, 20, 34, 22, 23 and now, additionally, 36 and 40. Those equations correspond to 1 terms of trade equation, 2 goods market-clear, 2 bonds market-clear, 4 Euler equations, 1 budget constraint, 1 noise trader demand, and 1 intermediary demand.*

Equilibrium definition now consists of two additional variables, N_t, D_t and two additional equations. The noise trader variable and equation are actually trivial and can be omitted from the system. The intermediation quantity D_t is important since it will affect household budget constraints through bond demands.

A positive noise trader shock will increase the amount of intermediation required from the financial intermediaries. This can be achieved through two channels. The first is the simple increase of the bond supply from the intermediary. It must be accompanied by high expected excess returns to compensate for the larger position. The second is a *crowd out* of private demand for bonds. The total demand for the bond after the noise trader shock may be too high which would induce large drops in consumption, through the necessary exchange rates to accommodate the necessary excess return asked by financial intermediaries. Households may wish to reduce the demand for such bonds to accommodate some noise trader demand and avoid a larger impact on exchange rates and consumption.

3.2 Solution

Solving this model implies finding policy functions defined over the state variables that are consistent with the model equations, for any point in the state space. This is not a trivial task due to the high nonlinearity of the system. Therefore, to acquire a *pen and paper* solution, we make some simplifications of the model equations, so we can linearize an obtain a closed-form solution.

Although it is a simplification of the model, it brings much intuition into it. We suppose that households do not choose the foreign bonds portfolio, but just accommodate a decision made by the central bank. The central bank chooses a portfolio that is consistent with a second order approximation of euler equation, given by 24:

$$E_t\left(\sigma c^R_{t+1} - q_{t+1}\right) r^X_{t+1} = 0$$

Household takes the position as given, and the decision impacts their consumption and wealth but is taken as a zero-mean exogenous shock received by households. This is consistent with the first-step solution portfolios as in 1. An additional simplification is that all reserves must be financed by domestic debt, and not by financial intermediaries. This is consistent with the idea that a central bank is choosing the reserves and households are just internalizing them because reserves may be a large pool of resources, and, at least in the *long-run*, it must be financed by other agents that are not mere financial intermediaries.

The necessity of this simplification is technical, due to the non-existence of first-order approximated equilibria. To see this, note that the financial shock impacts the system through the bonds market clearing, and not through a UIP condition.

$$B_{H,t} + B^*_{H,t} + n(e^{\psi_t} - 1) + m\frac{E_t(R^X_{t+1})}{\omega \mathrm{var}_t(R^X_{t+1})} = 0. \tag{41}$$

The modified UIP appears after linearization of such equation, using the intermediation position and noise trader process[19]:

$$r_{t+1} - r^*_{t+1} - \mathbb{E}_t(\Delta q_{t+1}) = \chi_1\psi_t - \chi_2 b_t \tag{42}$$

19

Where $\chi_1 = \frac{n\omega\sigma_q^2}{m}$, and $\chi_2 = \frac{\omega\sigma_q^2}{m}$, and b_t is the total demand for domestic assets $B_t \equiv B_{H,t} + B^*_{H,t}$, over steady-state GDP. The parameter $\sigma_q \equiv \text{var}_t \Delta q_{t+1}$ is endogenous but taken as given by the financial intermediary. We show later that $\text{var}_t \Delta q_{t+1}$ is constant in time, due a general property of linear processes.

A positive noise trader demand shock ψ_t must be accommodated by increasing expected returns from the intermediary position, or by crowding out domestic-denominated debt through a reduction in b_t. Now, combine the Euler equations from the domestic and foreign household utility maximization problem to obtain:

$$E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma} R^X_{t+1}\right] = 0 \tag{43}$$

$$E_t\left[\beta\left(\frac{C^*_{t+1}}{C^*_t}\right)^{-\sigma} \frac{Q_{t+1}}{Q_t} R^X_{t+1}\right] = 0 \tag{44}$$

These are pricing equations for the excess return on bonds and are a necessary condition for an existence of an interior solution for debt quantities $B_{H,t}$, $B_{F,t}$, $B^*_{H,t}$ and $B^*_{F,t}$. A violation of any of these conditions will generate an infinite demand for one bond, which will be financed by a corresponding infinite short position on the other, and no equilibrium will be possible. Note that equations (41), (43) and (44) are consistent: the former constrains the path for the expected excess returns $E_t[R^X_{t+1}]$ while the latter prices the excess returns path according to the agents pricing kernel.

However, linearizing equations (43) and (44) and combining results in a risk-neutral version of the pricing equation, which corresponds to the conventional uncovered interest parity (UIP) condition:

$$r_t - r^*_t - \mathbb{E}_t(\Delta q_{t+1}) = 0. \tag{45}$$

While the non-linear versions (41), (43) and (44) are consistent, its linear counterparts (42) and (45) are no longer consistent, since one equation implies an UIP deviation while the other does not. In the non-linear analog of equation (45), we have $E_t R^X_{t+1} = 0$, which is inconsistent with an equilibrium since both intermediaries and households would not demand a non-zero leveraged position on bonds that would be necessary to finance noise traders, due to risk-aversion. Linearization removes the risk-aversion component of the portfolio selection, effectively muting a crucial dynamic for determining the position of agents on assets, which is out goal.

To solve this issue, we remove the linearized versions of the Euler equations that emerge from the maximization problem over foreign bonds for the domestic and foreign households, so that equation (45) is no longer part of the model. Their remotion can be interpreted as households not choosing portfolios on foreign bonds, but just internalizing some given amount in their budget constraint. The quantity chosen will be set by the Central Bank that follows a rule, which is a non-linear version of Euler equations, given by equation 24. Since we have two fewer equations on the model, we need to impose additional constraints on the linearized system so that it can be solved. This is done by restricting $b_t = b^*_t = 0$, which states that one country should finance the position of the other. It has a straightforward interpretation: a country only holds bonds that were written by the other country, and not by financial intermediaries. This assumption is natural, since in the long run, as noise trader shocks dissipate, equation (41) (or its linear analog (42)) imply that $B_t \to 0$, and if our goal is to determine the steady-state position on foreign bonds by the domestic household, this steady-state position must be consistent with a zero total domestic debt $B = 0$.

Since the remaining equations of the system are unchanged, it is straightforward to establish the following Lemma, analogous to Lemma 1:

Lemma 3 *Let $\xi_t \equiv b_F r^X_t$ be a zero-mean shock, and assume that both countries endowment process are equal, but with different innovations. We can reduce the linearized model approx-*

imation with the previous assumptions of $b_t = b_t^* = 0$ into a system of two equations and two variables:

$$E_t q_{t+1} = q_t - \omega_1 y_t^R - \hat{\omega}_2 \psi_t \tag{46}$$

$$w_t = \frac{1}{\beta} w_{t-1} + \mu q_t - \frac{\gamma}{1-2\gamma} y_t^R + \xi_t, \quad \textit{where} \tag{47}$$

$$\omega_1 \equiv \frac{(1-2\gamma)(1-\rho^y)\sigma}{4\sigma\theta\gamma(1-\gamma) + (1-2\gamma)^2} > 0, \quad \hat{\omega}_2 \equiv \frac{n\omega\sigma_q^2}{m}\omega_2, \quad \textit{and}$$

$$\mu \equiv \frac{2\gamma\theta(1-\gamma) - \gamma(1-2\gamma)}{(1-2\gamma)^2} > 0$$

Proof. Shown in the appendix

The linear system is nearly identical to the previous model, except that the parameter $\hat{\omega}_2$ that multiplies the noise trader shock is now dependent on the ratio of the measure of noise traders and of financial intermediaries, n/m, the intermediaries risk aversion level ω and the (endogenous) volatility of the exchange rate, σ_q^2[20]. As a consequence, we can establish the main result:

Proposition 4 *In the incomplete markets version of the model with microfoundation for the financial shock, the optimal (long-run) portfolio on foreign currency is:*

$$\hat{b}_F = b_F^0 + \Omega \left(\frac{\beta}{1-\beta\rho^\psi} \right)^2 \left(\frac{n\omega}{m} \right)^2 \frac{\sigma_\psi^2}{\sigma_y^2} \sigma_q^4. \tag{48}$$

Under the calibration $\beta\rho^\psi \to 1$, the domestic country takes a long position on the foreign asset, $\hat{b}_F > 0$.

Proof. Shown in the appendix.

Persistent risk-premium shocks that are associated with an almost permanent drop in consumption induce agents to hold reserves, which pay a greater real return in such events due to real exchange rate depreciation. The intuition now is analogous: noise trader persistent

20

demand for a long position on foreign bonds causes a depreciation of domestic currency which is associated with a permanent drop in consumption. This movement induces agents to hold reserves, to hedge against the reduction on consumption. Demand for reserves will be higher the larger the ratio of noise traders to financial intermediaries n/m grows, and as the risk aversion coefficient of financial intermediaries ω becomes larger.

While the above intuition is valid, the model must be closed by determining the equilibrium conditional variance for the real exchange rate $\sigma_q^2 = Var_t(\Delta q_{t+1})$, which we now show that is constant at any point in time[21]. Since the equilibrium level for σ_q^2 will also depend on the level of reserves on the steady-state, we obtain a system of equations that determine an unique pair for $\hat{b}_F$ and σ_q^2 that is consistent with the equilibrium dynamical system.

Proposition 5 *The ex-post solution for the real exchange rate growth in the microfounded linearized system is given by:*

$$\Delta q_{t+1} = -\Sigma_1 y_t^R - \hat{\omega}_2 \psi_t + \lambda_y \xi_{t+1}^y + \lambda_\psi \xi_{t+1}^\psi - \frac{1-\beta}{\mu} \xi_{t+1}$$

$$where \quad \Sigma_1 \equiv \frac{\sigma(1-2\gamma)(1-\rho^y)}{4\theta\sigma\gamma(1-\gamma)+(1-2\gamma)^2} > 0.$$

This solution implies a constant conditional variance $Var_t(\Delta q_{t+1}) = \sigma_q^2$ with a unique stable solution given by:

$$\sigma_q^2 = \frac{1}{2\left(\frac{n\omega}{m}\right)^2 \lambda_\psi^2 \sigma_\psi^2} \left(1 + \frac{1-\beta}{\mu}\hat{b}_F\right)^2 \left[1 - \sqrt{1 - 4\lambda_y^2\sigma_y^2 \left(\frac{n\omega}{m}\right)^2 \lambda_\psi^2 \sigma_\psi^2 \left(1 + \frac{1-\beta}{\mu}\hat{b}_F\right)^{-4}}\right] \tag{49}$$

The system made of equations (48) and (49) for σ_q^2 and $\hat{b}_F$ contains an unique solution, such that $\sigma_q^2 > 0$.

[21]But is different from the unconditional variance of exchange rates depreciation. This is a property of linear stochastic processes. To see this, let $x_t \sim AR(1)$ stationary, then $var_t x_{t+1} = \sigma_\epsilon^2$, while $var x_{t+1} = \frac{1}{1-\rho}\sigma_\epsilon^2$, where σ_ϵ^2 is the variance of the $i.i.d$ innovation. Both values are constant in time, but different while conditioning or not.

Proof. Shown in the appendix.

Definition 3 *Let $b_F(\sigma_q^2)$ from equation 48 be the optimal demand of Central Bank for foreign assets given the volatility of exchange rate perceived by financial intermediaries, and $\sigma_q^2(b_F)$ from equation 5 the equilibrium exchange rate volatility given the Central Bank portfolio on the foreign asset. A general equilibrium consists of the pair (b_F^*, σ_q^{2*}) such that $b_F(\sigma_q^{2*}) = b_F^*$ and $\sigma_q^2(b_F^*) = \sigma_q^{2*}$.*

One can see from equation 5 that, depending on the calibration and on the value of foreign assets portfolio b_F, the square root of such number may yield a complex value. In such case, for such portfolio b_F, a general equilibrium does not exist for such approximation. This is especially the case when the parameters of the financial friction are high enough.

4 Numerical Analysis

4.1 Baseline Model

Intertemporal discount $\beta = 0.99$ is set to a quarterly frequency. The risk-aversion $\sigma = 2$ is standard. The degree of substitution between home and foreign goods is set to $\theta = 1.5$. This is the most contested calibration, it follows the estimates of **Feenstra, Luck, Obstfeld, and Russ (2014**. The persistence and volatility of the endowment fluctuations are set accordingly in Brazil's real GDP growth[22]: $\rho^Y = 0.87$, $\sigma_Y = 0.063$. Although calibrated using a specific country, these values are consistent with most quarterly endowment or productivity shock calibrations. Concerning home-bias, we set $\gamma = 0.05$. This parameter is typically calibrated using the average imports over GDP. Under this argument, this may be an unusual calibration, which was expected to be around 15% or 20% however, as shown by **Oleg and Dmitri (2021)**, when $\gamma \to 0$ and $\rho^\psi \to 1$ we observe a solution of a common exchange

[22]Here we take monthly Brazil's GDP, use the inflation index IPCA to deflate the series, and transform it to quarters, summing up every three months. After, we remove the trend using a quadratic regression in time. Finally, we estimate an AR(1) without constant and obtain the calibrated values.

rate puzzle called exchange rate disconnect. This puzzle is referred to as the similar level of fluctuations between exchange rate and other macro fundamentals that occurs in traditional models, but in the data, exchange rates appear to be much more volatile and accompanied by much smaller movements in consumption and income. Therefore, the calibration is set to reproduce the most possible realistic exchange rate in the model. We still can calibrate $\gamma = 0.05$ and have higher imports over GDP if we consider an endogenous production model, where firms use imported units as input, as it is considered in the extensions section. We follow **Oleg and Dmitri (2021)** and set $\rho^{\psi} = 0.97$. Table 1 resumes the calibration, and figure 1 illustrates the results for different calibrations of the financial shock volatility, σ_{ψ}.

Description	**Parameter**	**Calibration**
Intertemporal Discount Factor	β	0.99
Risk-aversion	σ	2
Substitution Degree between H and F goods	θ	1.5
Endowment Persistence	ρ^Y	0.87
Endowment Volatility	σ_Y	0.063
Home-Bias	γ	0.05
Financial Shock Persistence	ρ^{ψ}	0.97

Table 1: Calibration Baseline Model

Without the risk-premium shock ($\sigma_{\psi} = 0$), the economy holds on average 25% of annual GDP of debt denominated in foreign currency. The country chooses to hold debt instead of assets because in bad times (low y_t^R) real exchange rates appreciate, and the service of debt decreases because foreign currency becomes cheaper. This decrease in the bad state is much appreciated by the home economy because consumption falls in this bad state. So a risk-averse country wishes to hold debt in foreign currency when exchange rates are mainly driven by endowment or productivity shocks.

Now, as the volatility of the risk-premium shock increases, the risk of a bad risk-premium state increases, and such a bad state is expected to last long a time due to the high calibration of $\rho^{\psi} = 0.97$. In such a state, consumption falls and the exchange rate, therefore a risk-averse

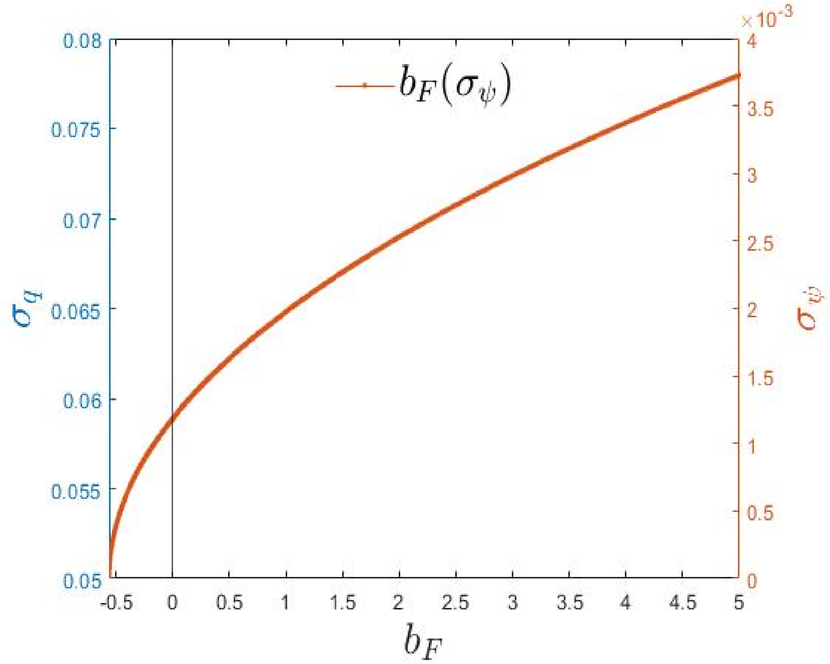

Figure 1: Optimal Portfolios and Exchange Rate Volatility

country wishes to hold a positive position of such assets as a hedge for such a state. The previous intuition for endowment shocks is still there, so the volatility of the risk-premium shock must be high enough to overcompensate for the correlation caused by such shock.

Positive levels of reserves already appear with significantly low values of risk-premium volatility. With $\sigma_\psi = 0.3\%$, the country wishes to hold 75% of annual GDP on reserves. One can see that the model can reproduce the reserves and exchange rate volatility consistent with the empirical evidence. For example, if we set $\sigma_\psi = 0.2\%$, the resulting equilibrium will be reserves at 24% of annual GDP and exchange rate volatility around 6.5%, which is similar to Brazil's values. The reason why such small volatilities already induce these high values is that the financial risk-premium shocks matter a lot for the linearized system. The microfoundation of the risk-premium shock will induce a much smaller coefficient on the linearized system, allowing more considerable volatilities.

4.2 Microfoundation of the Risk-Premium Shock

Here we provide a calibration for the microfounded model. This version of the model can allow more realistic risk-premium shocks without assigning high values to the portfolios. We can see why this happens when we look at the linearized system. Recall that ω_2 is the coefficient that multiplies the risk-premium shock in the baseline model:

$$\omega_2 = \frac{(1-2\gamma)^2}{4\theta\sigma\gamma(1-\gamma)+(1-2\gamma)^2}$$

Now, when we look at the linearized system with the microfoundation of the risk-premium shock, we see that $\hat{\omega}_2 \equiv \frac{n\omega\sigma_q^2}{m}\omega_2$. Since, in equilibrium, real exchange rate volatility is a low value, then $\hat{\omega}_2 << \omega_2$, meaning that the risk-premium shock is quantitatively less relevant to the system. This allows us to tune much higher volatilities to the ψ_t shock.

We set $\sigma_\psi = 0.05$. In this calibration, around 65% of noise trader's demands increase are around 5%. We still have three important parameters to be calibrated. The noise traders mass n, financial intermediaries mass m and their aversion ω. What actually matters is the value of their combination $n\omega/m$. We set $m = 1$ for normalization. We choose $\omega = 5$, a value that is considered high risk-aversion in typical efficient frontier problems. The intuition is that we're analyzing what was to be considered emerging markets, therefore financial intermediaries are still very risk-averse. Finally, we set $n = 1.3$ to reproduce an equilibrium level of foreign reserves of 24% of annual GDP, consistent with Brazil's current holdings. Table 2 summarizes the calibration and figure 2 shows the results. We can mixture more values between σ_ψ and $n\omega/m$ and still get similar results. The remaining parameters are the same.

The orange line illustrates the resulting equilibrium of exchange rate volatility, given a portfolio of foreign assets, given by equation 49. What this curve states is that, under the financial friction structure, higher foreign assets decrease exchange rate volatility. That is,

Description	Parameter	Calibration
Noise traders volatility	σ_ψ	0.05
Mass of Financial Intermediary	m	1
Risk-aversion of Financial Intermediary	ω	5
Mass of noise traders	n	1.3

Table 2: Calibration Microfounded Model

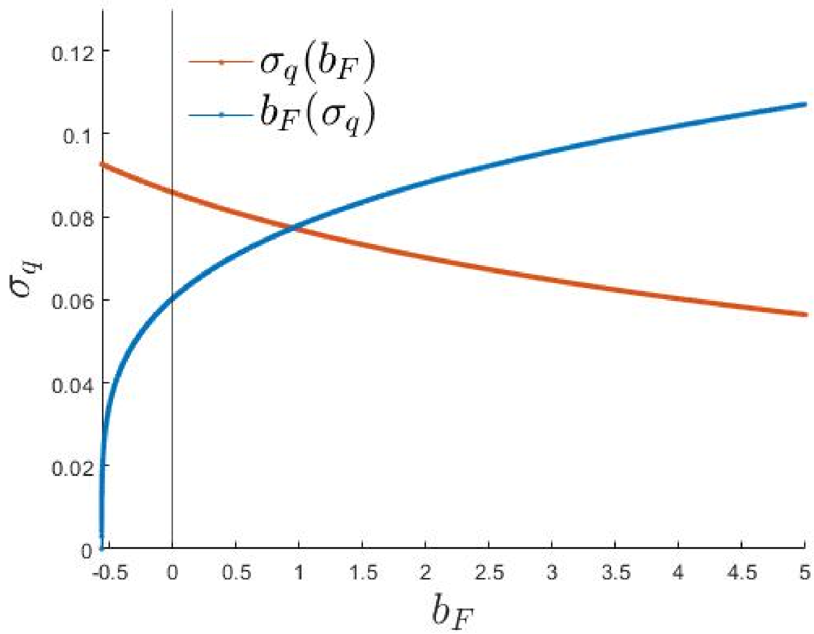

Figure 2: Equilibrium Foreign Assets and Exchange Rate Volatility

countries with higher reserves will present lower exchange rate volatility, a feature consistent with empirical evidence and endogenously generated in the model[23]. The blue line states the Central Bank optimal demand for foreign assets, given an exchange rate volatility perceived by financial intermediaries. Higher exchange rate volatilities perceived by financial intermediaries cause higher demand for reserves. This occurs because the financial shock becomes more relevant to the model, increasing the desire for hedge through foreign assets.

[23]Although this effect may be stronger in the empirical evidence due to the fact that Central Bank actively interferes in exchange rate markets for such purpose. A fact not captured in the model, but highlighted that such intervention is not necessary for long-run volatilities.

The resulting general equilibrium is the pair (σ_q^*, b_F^*) such that $b_F(\sigma_q^*) = b_F^*$ and $\sigma_q(b_F^*) = \sigma_q^*$, which is the point where these curves crosses. This can be viewed as a Nash equilibrium, where one player is a central bank choosing reserves, and the other player is the financial market delivering exchange rate volatilities, and the curves are each player's best response given the other player's strategy. In the calibration, the equilibrium is given by $(b_F^*, \sigma_q^*) = (0.95, 0.077)$, that is, an amount of 24% of annual GDP of reserves and 7.7% of exchange rate volatilities, both values consistent with Brazil's data.

It is also possible to see the stability of such equilibrium if we imagine a phase diagram in this figure. Say that we are currently out of equilibrium with the current exchange rate volatility actually higher, such as 10%. Then, the optimal policy for reserves is to increase, but as reserves increase, the resulting exchange rate volatility in equilibrium decreases. As the volatility decreases, reserves demand decreases, until the point where the resulting exchange rate volatility is consistent with the demand for reserves[24].

We also compare equilibrium portfolios and exchange rate volatility when changing some important parameters calibration. This is often called a sensitivity analysis, which consists of checking the results changes when we change the calibration. Figure 4 shows the results increasing both financial friction parameters and other more traditional parameters.

When we increase any of the parameters related to the size of the financial friction, we observe the same movement of both curves and an increase in foreign assets holdings with slightly the same exchange rate volatility. This occurs due to two movements. The first is that with higher relevance of the financial friction shock in the dynamics of the variables, agents find it optimal to hold more international reserves given the level of exchange rate

[24]We have to remember that this is an analysis of the stability of such steady-state values, and not a transition dynamics analysis. We can compare different steady-states, but in this approximation, we can't obtain transition dynamics.

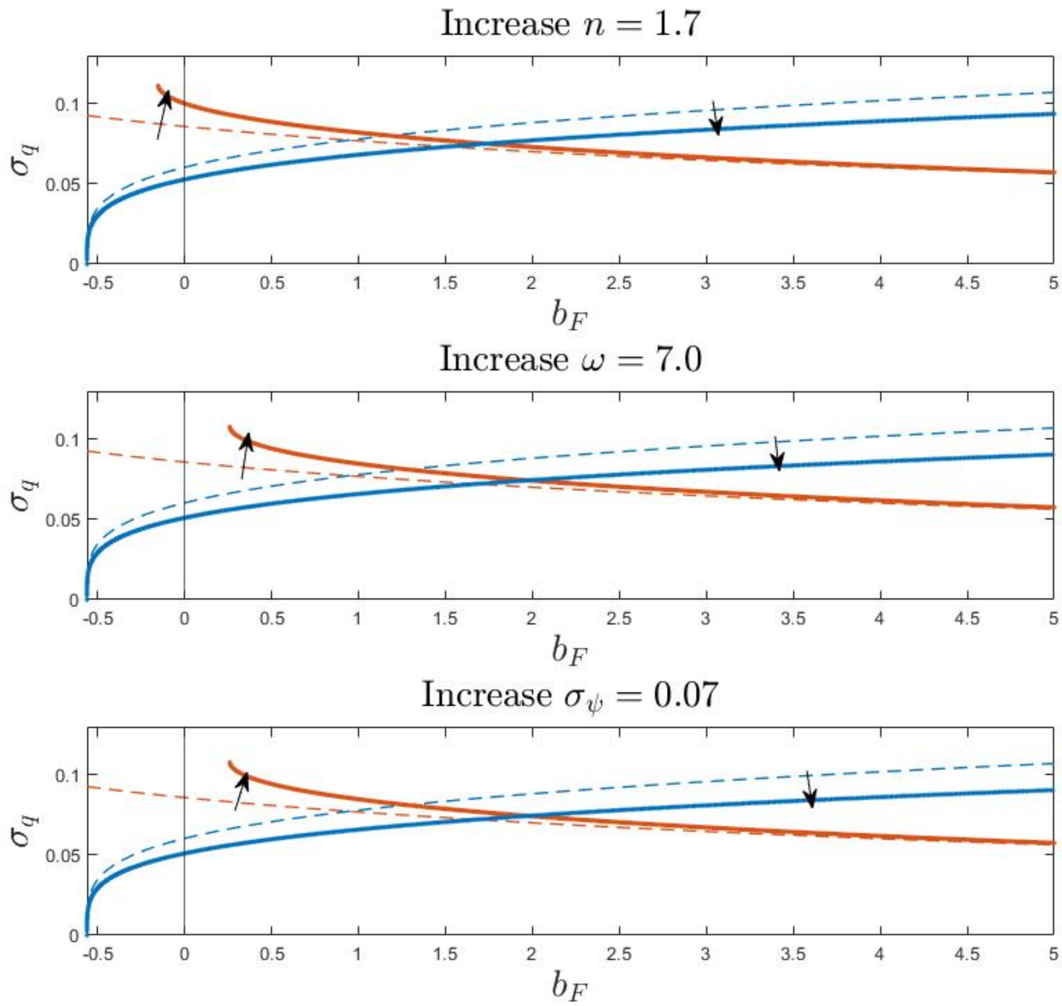

Figure 3: Sensitivity Analysis - Financial Friction

volatility perceived by the financial intermediary. This corresponds to a shift to the right of foreign asset demand. The other movement is that, given the same level of foreign assets, the equilibrium volatility of the exchange rate increases, due to the higher relevance of the financial shock. We can interpret both of these movements as the monetary authority buying more foreign assets to contain the increase in foreign exchange rate volatility.

Although the equilibrium results are not so different for these more intense calibrations of the financial shock, note that the orange line starts to appear only in some parts of the state space. This corresponds to the nonexistence of equilibrium for exchange rate volatility at negative (or positively small) levels for foreign assets. This can be interpreted as the Central Bank being unable to maintain exchange rate expectations anchored by the financial intermediaries. That is, the level of reserves is so small or the level of debt is so high that financial intermediaries will always increase their expected volatility for the real exchange rate, such that the equilibrium exchange rate volatility will always be higher than their expectation. But when foreign assets increase, Central Bank can achieve stability. If we set the financial friction to be strong enough, we may even not have the existence of an equilibrium[25]. The remaining parameters are more structural and deeper, hiding a lot of complexity behind them, therefore is natural to expect more relative importance for the general equilibrium.

When we increase σ, we increase both risk-aversion but also decrease the elasticity of substitution of consumption in time. The first effect corresponds to lower exposition to risk, and the last corresponds to lower incentive in saving consumption in favor of future consumption. The risk-aversion effect is consistent with the upward shift of the blue curve, which is less exposition to the risky asset and is also consistent with less consumption smoothing through savings. The shift upward of the orange curve corresponds to the lesser elasticity of substitution in time. That is, with the same amount of assets, agents are willing to consume more today, giving more volatility to exchange rates. The achieved equilibrium is a similar level of reserves but with more volatile exchange rates. The reason for increased equilibrium volatility is mainly due to the decrease in elasticity of consumption in time, but asset holdings remain the same due to higher risk aversion.

[25]This corresponds to the case where the downward slope line starts to appear only after the upward slope line.

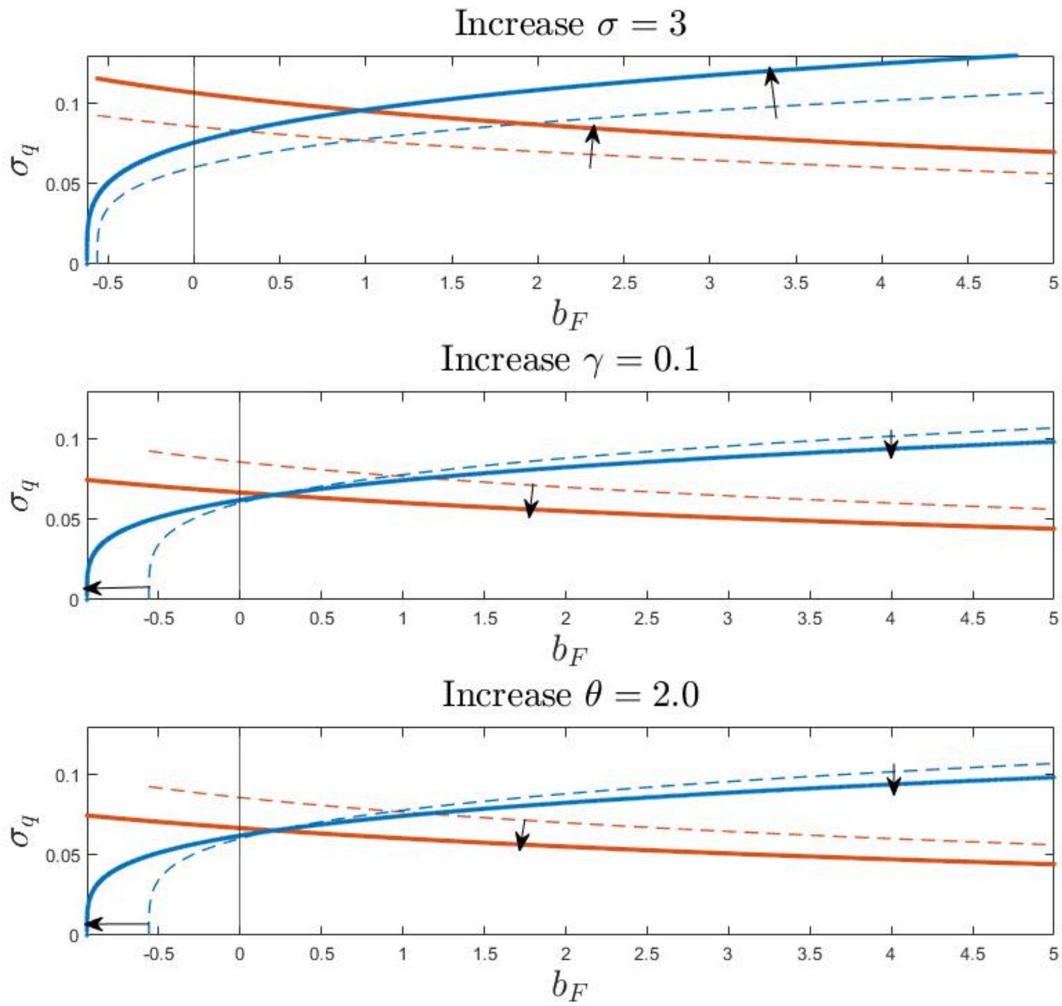

Figure 4: Sensitivity Analysis - Deep Parameters

When we increase γ, as shown by **Oleg and Dmitri (2021)**, the exchange rate disconnect puzzle starts to appear again in the model. Exchange rate fluctuations now are more associated with consumption fluctuations, meaning that when the exchange rate changes, consumption changes by more similar proportions. The optimal behavior for Central Bank is to increase the exposition to the asset because large movements in the asset price will be associated with larger movements in consumption. In the frictionless world, the demand for external debt is higher, and, when the friction is sufficiently relevant to hold reserves, the amount of reserves demanded by the country is higher for each unit of exchange rate volatility, explaining the movement of the blue curve. The orange also relates to such a puzzle. Because exchange rate volatility is more connected to macro fundamentals, the central bank loses the ability to control exchange rate volatility with foreign asset holdings, and the orange curve becomes flatter. The reason is that the driver for exchange rates is now more attributed to endowment shocks rather than financial shocks, weakening the hedging power of portfolios. Therefore, the orange curve is shifted down because endowment shock has a present value effect way lower than the financial shock, decreasing exchange rate volatility.

A very similar feature can be observed when θ increases. When θ increases, goods H and F are closer to substitutes, meaning that households can change one for another when the prices change. If the price of the good F increases, a household can rapidly substitute some amount of good F with good H, which is cheaper, making the cost of building the C_t basket only slightly increase. Therefore, higher θ is associated with less volatile exchange rates, shifting the orange curve down. Because this mechanism implies less volatile exchange rates[26], Central Bank wishes to expand their exposition to the asset to maintain the amount of hedge for consumption. This corresponds to an increase in the slope of the blue curve when b_F, and a decrease when $b_F < 0$.

Because the financial friction theory should be considered both in emerging and developed economies, the γ, θ parameter gives an important source of connection between these worlds. Lower levels of γ, θ will imply more intense exchange rate disconnection and volatility, which is consistent with emerging markets. Here we show that it will also imply higher levels of foreign assets holdings, also consistent with emerging markets. While higher values of γ, θ may still preserve some amount of exchange rate features, this exercise shows that they are consistent with lower exchange rate volatilities and lower, or none at all, reserves, a salient feature of developed economies.

[26]Not the exchange rate perceived by financial intermediaries, but the actual exchange rate would emerge in the model solution given some value for σ_q^2.

5 Conclusion

This paper contributes to the literature on foreign reserves with a reason not much considered yet, which is exchange rates. Foreign reserves may be a useful asset even when we do not consider large risks, a sovereign default, or sudden stop events. This can occur due to the general equilibrium stochastic behavior of exchange rates, which is outlined in the empirical evidence as a negative correlation between consumption or income and exchange rates, also bringing the common intuition that in bad states of nature, or when the country goes bad, the exchange rate depreciates.

When designing a model that is capable of endogenously reproducing these exchange rate features, we show that we can account for most of the reserves observed in the empirical evidence. Such design is no easy task. There is a large literature documenting many exchange rate puzzles that appear in conventional open macro models, and several proposals of correction of such models to solve some puzzles. Using one solution that can account for many of them, we show that it also endogenously generates levels of foreign assets and exchange rate volatility consistent with emerging markets values.

In general equilibrium, the country wishes to issue debt denominated in domestic currency just to finance assets denominated in foreign currency, trading domestic debt for reserves. The reason arises from the fear of a bad state that decreases consumption while increasing exchange rates. A financial friction shock is capable of generating such a mechanism, and to specify the intuition behind we show the resulting portfolio with closed-form solutions, using both a reduced form and a micro foundation for the shock. When the financial friction becomes stronger, the demand for foreign assets increases in general equilibrium.

We highlight that the standard endowment or productivity shock can't generate this comovement and we outline the intuition for it with closed-form solutions. We maintain the shock in the model to keep the connection with more traditional models. Nevertheless, endowment or productivity is still an important source of business cycle fluctuations. We show that with only endowment shocks, the country wishes to hold non-defaultable debt instead of reserves, *e.g.* a short position on foreign assets. With the two shocks present, the country switches to a long position only when the financial friction becomes relevant in the calibration. But under the same calibration that accounts for the solution of exchange rate puzzles, the general equilibrium implies a long position on foreign assets, *e.g.* positive reserves.