# The Ramsey Steady-State Conundrum in Heterogeneous-Agent Economies*

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**Abstract**

In infinite horizon, heterogeneous-agent and incomplete-market models, the existence of an interior Ramsey steady state is often assumed instead of proven. This paper makes two fundamental contributions: (i) We prove that the interior Ramsey steady state assumed by Aiyagari (1995) does not exist in the standard Aiyagari model. Specifically, a steady state featuring the modified golden rule and a positive capital tax is feasible but not optimal. (ii) We design a modified, analytically tractable version of the standard Aiyagari model to unveil the necessary and/or sufficient conditions for the existence of a Ramsey steady state. These conditions are shown to be quite demanding and sensitive to structural parameter values pertaining to the economy's fiscal space for providing full self-insurance, such as the government's capacity to finance public debt, the degree of intertemporal elasticity of substitution, and the extent of history dependence of individual wealth on idiosyncratic shocks. In addition, we characterize the basic properties of both interior and non-interior Ramsey steady states and show that researchers may draw fundamentally misleading conclusions on optimal fiscal policies (such as the optimal capital tax rate) from their analysis when an interior Ramsey steady state is erroneously assumed to exist.

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Key Words: Optimal Fiscal Policy, Ramsey Problem, Incomplete Markets, Heterogeneous Agents

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# 1 Introduction

When solving the Ramsey taxation problem in a standard Aiyagari model with heterogeneous agents and incomplete insurance markets, the existence of a Ramsey steady state is often assumed instead of proven (see, e.g., Aiyagari (1995) and the literature that follows), because proving the existence of the Ramsey steady state is a daunting challenge in such models due to their intractability. For example, Aiyagari (1995) openly acknowledges that "It seems quite difficult to guarantee that a solution to the optimal tax problem converges to a steady state." (See Aiyagari (1995) Footnote 14). Yet without such an "existence assumption," the Ramsey allocation is difficult to analyze and not even numerically solvable. But, optimal tax policies drawn from the analyses hinge critically on the validity of such an "existence assumption." In this paper we refer to this difficult situation as the Ramsey steady-state conundrum.[1]

We tackle the Ramsey steady-state conundrum by proving first that an interior Ramsey steady state with the properties proposed by Aiyagari (1995) or commonly assumed in the literature does not exist in the standard Aiyagari (1994) model.[2] Because of this, erroneously assuming the existence of such an interior Ramsey steady state have led researchers to draw fundamentally misleading conclusions about optimal fiscal policies, such as the conclusion that the optimal capital tax is positive while it in fact could be zero or negative.

The intractability of the standard Aiyagari model is the root cause of the Ramsey steady-state conundrum and it originates from the infinite history dependence of individual wealth on past idiosyncratic shocks. This property implies that the wealth distribution may become an infinite dimensional object in the Aiyagari model, hence making it difficult to obtain the full set of the Ramsey first-order conditions (FOCs). Yet without obtaining the full set of the Ramsey optimal conditions, it is impossible to fully analyze the Ramsey allocation or prove the existence of a Ramsey steady state, let alone to characterize optimal fiscal policies. Nonetheless, this daunting challenge does not prevent us from falsifying Aiyagari's result, because we can derive a few more Ramsey FOCs and show inconsistency between the existence assumption made by Aiyagari (1995) and these additional FOCs that Aiyagari (1995) omitted from his analysis.

In this paper we also go far beyond such a negative result for the conundrum. To unveil

[1]This issue was first raised by Chen, Chien, and Yang (2019).

[2]There may exist two types of Ramsey steady states in general: an interior one and a non-interior one. If all quantity variables converge to finitely positive values, it is called an interior steady state. Otherwise it is called a non-interior steady state if one or more quantity variables (such as aggregate consumption) converge to zero.

the mechanism behind our alarming finding, we design a modified, analytically tractable version of the standard Aiyagari model to explain why the interior Ramsey steady state assumed by Aiyagari (1995) does not exist. Working with the power utility function over consumption, $u(c) = c^{1-\sigma}/(1-\sigma)$, we then characterize the properties of a non-interior Ramsey steady state—which is shown to be the only possible Ramsey steady state in the standard Aiyagari model if the intertemporal elasticity of substitution (IES) is less than or equal to 1 (namely, $\sigma \geq 1$). A nice feature of our modified model is that it converges to the standard Aiyagari model in the limit when a key parameter in our model changes its value. We find that the conditions for the existence of an interior Ramsey steady state are quite demanding and sensitive to the government's fiscal space for providing full self-insurance (FSI) to borrowing-constrained individuals, which critically depends on the persistence of idiosyncratic shocks, the degree of risk aversion, and the extent of history-dependence of individual wealth on idiosyncratic shocks.

Specifically, we use our modified Aiyagari model to show analytically that under the normal parameter condition of $\sigma \geq 1$, the following results must be true: (i) If the fiscal space permits an FSI allocation, then an FSI interior Ramsey steady state exists and it is the only possible Ramsey steady state; furthermore, in such an interior Ramsey steady state the modified golden rule (MGR) holds and the optimal capital tax is zero.[3] (ii) When an interior Ramsey steady state does not exist but is erroneously assumed to exist, the "optimal" steady-state capital tax always appears positive in order to be consistent with the MGR; yet the only possible Ramsey steady state in this case is non-interior, where aggregate consumption approaches zero, the optimal labor tax goes to 100%, and the capital tax is indeterminate. (iii) An interior Ramsey steady state exists in our modified Aiyagari model, but it rapidly converges to the non-interior Ramsey steady state when our modified model approaches the standard Aiyagari model by changing a key parameter in our model.

On the other hand, when the IES parameter $\sigma < 1$ such that the degree of risk aversion is low or the utility function is sufficiently linear, we show that the following results must hold: If the fiscal space does not permit an FSI allocation, then the only possible Ramsey steady state is an interior steady state where the Ramsey Lagrangian multiplier associated with the aggregate resource constraint diverges, the MGR fails to hold, the interest rate lies below the time discount rate, and most importantly, the optimal capital tax is non-positive.

[3] A full self-insurance allocation is defined as a competitive equilibrium allocation where no individual's borrowing constraint is strictly binding.

In addition, we show that solving Aiyagari-type models by assuming that the Ramsey planner maximizes only the steady-state welfare of a competitive equilibrium can trivially ensure the existence of an interior Ramsey steady state, but the result distorts the picture of the Ramsey allocation in a dynamic setting that maximizes the time-zero expected welfare of a competitive equilibrium. This distortion occurs because the steady-state welfare approach ignores the transitional dynamics of the Ramsey problem. Although it is well known in the literature that optimal policies may look dramatically different between steady-state welfare analysis and time-zero dynamic welfare analysis (see, e.g., Domeij and Heathcote (2004), Heathcote (2005), and, Rohrs and Winter (2017)), our analytical approach makes a further contribution to the literature by showing the underlying mechanism driving the sharp differences between these two approaches. The culprit for obtaining different results for optimal fiscal policies between steady-state welfare analysis and dynamic welfare analysis in heterogeneous-agents models is the arbitrage opportunity arising from the gap between the market interest rate and the time discount rate; this gap does not matter when maximizing the steady-state welfare, whereas it does matter greatly when maximizing the time-zero expected welfare. This is so because the Ramsey planner opts to take advantage of the cheap interest rate for debt financing in a dynamic setting by frontloading consumption; however, such a frontloading incentive disappears in the static welfare analysis.

**Brief Literature Review.** First, our work is motivated by Straub and Werning (2020), who questioned the classical zero-capital-taxation result of Judd (1985) by considering the possibility of an non-interior Ramsey steady state where aggregate consumption approaches zero. This paper questions instead the positive-capital-taxation result of Aiyagari (1995) by showing that an non-interior Ramsey steady state is the only possible Ramsey steady state in the standard Aiyagari model if the IES parameter $\sigma \geq 1$. We obtain our result analytically despite the fact that the mechanism underlying our non-interior Ramsey steady state is totally different from that of Straub and Werning (2020).

Second, our analysis relates to the work of Bassetto and Cui (2020), who also argue that when the government's fiscal capacity is insufficient to support an FSI allocation, the optimal Ramsey allocation could converge to a non-FSI interior steady state where the Lagrangian multiplier diverges. However, in their model, government debt has a "crowding in" effect, whereas in our model it has a "crowding out" effect on capital accumulation. More importantly, we are able to prove with certainty that this type of interior Ramsey steady state can emerge only under a high degree of IES or low degree of risk aversion (such as under linear utility in consumption) and that the optimal capital tax is unambiguously non-positive in such a Ramsey steady state.

Third, our work includes our previous work in Chien and Wen (2021a) as a special case, which utilizes a tractable heterogeneous-agents model with quasi-linear preferences and a regular IES parameter value to show that the optimal capital tax must be zero in an FSI Ramsey steady state that can be proven to exist. The intuition provided by Chien and Wen (2021a) indicates that the desire of the Ramsey planner to frontload consumption by issuing an increasing amount of debt never goes away unless the market discount rate is equal to the time discount rate, which can only be achieved in an FSI allocation. We show in this paper that this mechanism is a special property of structural parameter values pertaining to fiscal capacity and individual risk, and this result carries through to our more general model. Most importantly, our modified framework can approximate the standard Aiyagari model arbitrarily well when a key parameter in our model changes. We also prove that the never-ending purist of an FSI allocation by the Ramsey planner leads to a non-interior Ramsey steady state because of government's limited fiscal space to fulfill the strong precautionary saving motives on the consumer side.

Our work also relates to a large literature in studying the optimal responses of fiscal policies to aggregate shocks, which originates from the works by Barro (1979) and Lucas and Stokey (1983) in the representative-agent framework. There is a strong tradition and renewed interest in extending this literature into a heterogeneous-agents framework, such as Bassetto (2014) and Bhandari, Evans, Golosov, and Sargent (2021). However, the existence of an interior stationary Ramsey allocation in a heterogeneous-agents model with both aggregate and idiosyncratic uncertainty is often assumed instead of proven. We think our modified Aiyagari model can be extended to include aggregate risks and hence complement this literature by offering a more transparent analysis.

The rest of the paper is organized as follows: Section 2 sets up the standard Aiyagari model and defines the competitive equilibrium. Section 3 shows that the interior Ramsey steady state described by Aiyagari (1995) cannot possibly exist. Section 4 builds a modified Aiyagari model to unveil the mechanism behind the Ramsey steady-state conundrum and provide necessary and/or sufficient conditions for the existence of various types of Ramsey steady states. Section 5 considers two modifications of the modified model to further explore the underlying mechanism of our results. Finally, Section 6 concludes.

# 2 A Standard Aiyagari Model

**Firms.** A representative firm produces output according to the constant-returns-to-scale Cobb-Douglas technology, $Y_t = F(K_t, N_t) = K_t^\alpha N_t^{1-\alpha}$, where $Y$, $K$, and $N$ denote aggregate output, capital, and labor, respectively. The firm rents capital and hires labor by paying a competitive rental rate and real wage, denoted by $q_t$ and $w_t$, respectively. The firm's optimal conditions for profit maximization at time $t$ satisfy

$$w_t = \frac{\partial F(K_t, N_t)}{\partial N_t} \equiv MPN_t, \tag{1}$$

$$q_t = \frac{\partial F(K_t, N_t)}{\partial K_t} \equiv MPK_t. \tag{2}$$

**Government.** In each period $t$, the government can issue bonds, $B_{t+1}$, and levy both a flat-rate labor tax $\tau_{n,t}$ and a flat-rate capital tax $\tau_{k,t}$. Denote $Q_{t+1}$ as the price of the risk-free government bonds in period $t$, which pay one unit of consumption goods in period $t+1$; then, the risk-free interest rate is given by $r_{t+1} \equiv Q_{t+1}^{-1}$. The flow government budget constraint in period $t$ is

$$\tau_{n,t} w_t N_t + \tau_{k,t} q_t K_t + Q_{t+1} B_{t+1} \geq B_t, \tag{3}$$

where the initial level of government bonds $B_0$ is exogenously given. For simplicity, government spending is assumed to be zero. Later in this paper, Section 5 considers the case where the government can impose unconditional lump-sum transfers (or taxes if negative), which is denoted by $T_t$ .

**Individuals.** There is a unit measure of ex-ante identical individuals with initial wealth $a_0 > 0$. Ex post, each individual is subject to an idiosyncratic labor productivity shock in every period. The shock process follows a finite-state first-order Markov process $\theta_t \in Z$. We denote $\theta^t \equiv \{\theta_0, \theta_1, ..., \theta_t\}$ as an individual's shock history up to period $t$; $\pi(\theta^t)$ as the unconditional probability of the realization of state $\theta^t$; and $\pi(\theta^{t+1}|\theta^t)$ as the transition probability of event $\theta^t$ to $\theta^{t+1}$, which is equal to $\pi(\theta_{t+1}|\theta_t)$ given that the shock process is first-order Markov.

Denote $\widehat{w}_t \equiv (1 - \tau_{n,t})\, w_t$ as the after-tax wage rate. In period $t$ given the shock history $\theta^t$, let $a_{t+1}(\theta^t)$, $n_t(\theta^t)$, $c_t(\theta^t)$, and $z_t(\theta^t)$ be an individual's asset holding, labor supply, consumption and labor productivity level, respectively. The budget constraint for an individual with history $\theta^t$ is given by

$$a_t(\theta^{t-1}) + \widehat{w}_t z_t(\theta^t) n_t(\theta^t) - c_t(\theta^t) - Q_{t+1} a_{t+1}(\theta^t) \geq 0, \text{ for all } t \geq 0, \tag{4}$$

where $a_0$ is the exogenously given initial wealth. All individuals are subject to the following borrowing constraint in all periods $t \geq 0$ regardless of their history $\theta^t$:

$$a_{t+1}(\theta^t) \geq 0. \tag{5}$$

The individual's welfare criterion is given by

$$U = \sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \left[ u(c_t(\theta^t)) - v\left(n_t(\theta^t)\right) \right] \pi(\theta^t), \tag{6}$$

where $\beta \in (0, 1)$ is the time-discounting factor and the utility function takes the standard power form:

$$u(c) = \frac{1}{1-\sigma} c^{1-\sigma} \text{ and } v(n) = \frac{1}{1+\gamma} n^{1+\gamma},$$

where the IES parameter $\sigma \in (0, \infty)$ and the Frisch elasticity parameter $\gamma > 0$.

Given the market prices $\{Q_{t+1}, \widehat{w}_t\}_{t=0}^{\infty}$, the government policies $\{\tau_{n,t}, \tau_{k,t}, B_{t+1}\}_{t=0}^{\infty}$, and the initial asset holdings $a_0$, each individual chooses a plan $\{c_t(\theta^t), n_t(\theta^t), a_{t+1}(\theta^t)\}_{t=0}^{\infty}$ to maximize (6) subject to (4) and (5). Let $\beta^t \xi_t(\theta^t) \pi(\theta^t)$ and $\beta^t \psi_t(\theta^t) \pi(\theta^t)$ denote the Lagrangian multipliers associated with constraints (4) and (5), respectively; the FOCs with respect to $c_t(\theta^t)$, $n_t(\theta^t)$, and $a_{t+1}(\theta^t)$ are given, respectively, by

$$u_{c,t}(\theta^t) = \xi_t(\theta^t), \tag{7}$$

$$v_{n,t}(\theta^t) = \xi_t(\theta^t) \widehat{w}_t z_t(\theta^t), \tag{8}$$

$$Q_{t+1} \xi_t(\theta^t) = \beta \sum_{\theta^{t+1}} \xi_{t+1}(\theta^{t+1}) \pi(\theta^{t+1} | \theta^t) + \psi_t(\theta^t); \tag{9}$$

where $u_{c,t}(\theta^t)$ and $v_{n,t}(\theta^t)$ denote, respectively, the marginal utility of consumption and leisure in period $t$.

There is no aggregate uncertainty. Government bonds and capital are perfect substitutes as a store of value for individuals. As a result, the after-tax gross rate of return to capital must equal the gross risk-free rate (no-arbitrage condition):

$$\frac{1}{Q_t} \equiv r_t = 1 + (1-\tau_{k,t})q_t - \delta, \text{ for all } t \geq 0. \tag{10}$$

## 2.1 Competitive Equilibrium

**Definition 1.** Denote $A_{t+1}$ and $C_t$ as the aggregate asset holdings and aggregate consumption in the end of period $t$, respectively. Given the initial asset holdings $a_0$, the initial government bond supply $B_0$, the initial capital stock $K_0$, and the sequence of policies $\{\tau_{n,t}, \tau_{k,t}, B_{t+1}\}_{t=0}^{\infty}$, a competitive equilibrium is defined as the sequences of prices $\{w_t, Q_{t+1}\}_{t=0}^{\infty}$, aggregate allocations $\{C_t, N_t, K_{t+1}, A_{t+1}\}_{t=0}^{\infty}$, and individual allocations $\{c_t(\theta^t), n_t(\theta^t), a_{t+1}(\theta^t)\}_{t=0}^{\infty}$, such that

1. given $\{Q_{t+1}, w_t, \tau_{n,t}\}_{t=0}^{\infty}$, the allocations $\{c_t(\theta^t), n_t(\theta^t), a_{t+1}(\theta^t)\}_{t=0}^{\infty}$ solves the individual's problem;
2. the no-arbitrage condition holds: $1/Q_t = 1 + (1-\tau_{k,t})\, q_t - \delta$ for all $t \geq 0$;
3. given $\{w_t, q_t\}_{t=0}^{\infty}$, the path of aggregate quantity $\{N_t, K_t\}_{t=0}^{\infty}$ solves the representative firm's problem;
4. all markets clear:

$$\begin{aligned}
F(K_t, N_t) + (1-\delta)K_t &= C_t + K_{t+1}, \\
N_t &= \sum_{\theta^t} n_t(\theta^t) z_t(\theta^t) \pi(\theta^t), \\
C_t &= \sum_{\theta^t} c_t(\theta^t) \pi(\theta^t), \\
B_{t+1} + \frac{K_{t+1}}{Q_{t+1}} &= \sum_{\theta^t} a_{t+1}(\theta^t) \pi(\theta^t) \equiv A_{t+1}
\end{aligned} \tag{11}$$

for $t \geq 0$, and the government flow budget constraint holds:

$$\tau_{n,t} w_t N_t + \tau_{k,t} q_t K_t + Q_{t+1} B_{t+1} \geq B_t$$

for $t \geq 0$.

**Proposition 2.** *Define $\boldsymbol{\theta}_h^t$ as the shock history of the lucky individuals who receive the highest productivity shock in every period from* 0 *to* $t$*. Then, the competitive equilibrium has the following properties:*

1. *For all $t \geq 0$, it must be true that $c_t(\boldsymbol{\theta}_h^t) \geq c_t(\theta^t)$ and $a_{t+1}(\boldsymbol{\theta}_h^t) > a_{t+1}(\theta^t) \geq 0$ for all $\theta^t \neq \boldsymbol{\theta}_h^t$. That is, the borrowing constraints of the lucky individuals with history $\boldsymbol{\theta}_h^t$ are always slack: $a_{t+1}(\boldsymbol{\theta}_h^t) > 0$, which implies that the associated Lagrangian multiplier must be zero: $\psi_t(\boldsymbol{\theta}_h^t) = 0$.*

2. *Since $\psi_t(\boldsymbol{\theta}_h^t) = 0$ and $\psi_t(\theta^t) \geq 0$ for all $\theta^t \neq \boldsymbol{\theta}_h^t$, the intertemporal price $Q_{t+1}$ must satisfy the following equations with equality and inequality for all $t \geq 0$:*

$$Q_{t+1} = \beta \sum_{\theta^{t+1}} \frac{u_{c,t+1}(\theta^{t+1})}{u_{c,t}(\boldsymbol{\theta}_h^t)} \pi(\theta^{t+1}|\boldsymbol{\theta}_h^t) \geq \beta \sum_{\theta^{t+1}} \frac{u_{c,t+1}(\theta^{t+1})}{u_{c,t}(\theta^t)} \pi(\theta^{t+1}|\theta^t). \tag{12}$$

3. *Most importantly, in the competitive equilibrium's steady state a liquidity premium exists such that $Q > \beta$ (or $r < \beta^{-1}$).*[4]

*Proof.* See Appendix A.1. □

Define an FSI allocation as the allocation where all individual borrowing constraints are slack regardless of their shock history. The above proposition shows that an FSI steady state is impossible to achieve in the standard Aiyagari model—because it requires $Q = \beta$. More specifically, when $Q = \beta$ every individual's marginal utility of consumption would follow a supermartingale and individual's asset demand would diverge to infinity, which cannot be a steady-state competitive equilibrium.

In other words, a positive measure of individuals' borrowing constraints must be strictly binding such that $Q > \beta$, and there must exist aggregate allocative inefficiency due to overaccumulation of capital in a laizzes-faire competitive equilibrium. For this reason, Aiyagari (1995) argues that the best outcome that the Ramsey planner can achieve is an allocation where the MGR holds by taxing the capital stock in the steady state so that the aggregate allocative efficiency can be restored in the long run.

[4] Throughout this paper, a variable without subscript $t$ represents its steady-state value.

However, Aiyagari makes his argument by relying only on one of the many Ramsey FOCs and more importantly under the assumption that an interior Ramsey steady state exists with convergent Ramsey Lagrangian multiplier(s). In this paper we will go beyond the Aiyagari's approach by deriving more than one Ramsey FOC and showing that the assumption of the existence of an interior Ramsey steady state leads to contradictions or is inconsistent with additional Ramsey FOCs.

# 3 Ramsey Outcome in a Standard Aiyagari Model

To solve the Ramsey problem, we adopt the primal approach by first substituting out all market prices and policy variables by using a subset of the competitive equilibrium's conditions, and then choosing the allocation to maximize social welfare subject to the rest of the equilibrium conditions. The solution is called a Ramsey allocation or a Ramsey outcome.

## 3.1 Conditions to Support a Competitive Equilibrium

To facilitate our analysis, define $\boldsymbol{c}_t^s(\theta^t)$, $\boldsymbol{n}_t^s(\theta^t)$, and $\boldsymbol{a}_{t+1}^s(\theta^t)$ as the consumption share, labor share, and asset share, respectively, of each individual with history $\theta^t$ in the population:

$$\boldsymbol{c}_t^s(\theta^t) \equiv \frac{c_t(\theta^t)}{C_t},\ \boldsymbol{n}_t^s(\theta^t) \equiv \frac{n_t(\theta^t)z_t(\theta^t)}{N_t}, \text{ and } \boldsymbol{a}_{t+1}^s(\theta^t) \equiv \frac{a_{t+1}(\theta^t)}{A_{t+1}}. \tag{13}$$

To ensure that a Ramsey outcome constitutes a competitive equilibrium, we must show first that all possible allocations in the choice set of the Ramsey planner constitute a competitive equilibrium. The choice set includes the individual share variables $\{\boldsymbol{c}_t^s(\theta^t), \boldsymbol{n}_t^s(\theta^t), \boldsymbol{a}_{t+1}^s(\theta^t)\}_{t=0}^{\infty}$ and the aggregate allocation $\{C_t, N_t, K_{t+1}, A_{t+1}\}_{t=0}^{\infty}$. The following proposition states the conditions that any constructed Ramsey allocation must satisfy to constitute a competitive equilibrium[5]:

**Proposition 3.** *Given the initial asset holdings $a_0$, the initial capital tax $\tau_{k,0}$, the initial government bond $B_0$, and the initial capital stock $K_0$, the individual share allocation $\{\boldsymbol{c}_t^s(\theta^t), \boldsymbol{n}_t^s(\theta^t), \boldsymbol{a}_{t+1}^s(\theta^t)\}_{t=0}^{\infty}$ and aggregate allocation $\{C_t, N_t, K_{t+1}, A_{t+1}\}_{t=0}^{\infty}$ can be supported as a competitive equilibrium if and only if they satisfy the following conditions:*

[5] In addition, the initial capital tax rate, $\tau_{k,0}$, should be a choice variable for the Ramsey planner. However, given that the initial capital is pre-installed, taxing the initial capital is essentially the same as allowing a lump-sum tax. As is standard in the literature, we restrict the planner's ability to choose $\tau_{k,0}$ in the Ramsey problem.

1. *the aggregate resource constraint:*

$$F(N_t, K_t) + (1-\delta)K_t - C_t - K_{t+1} \geq 0, \ \forall t \geq 0; \tag{14}$$

2. *the implementability condition:*

$$\boldsymbol{c}_t^s(\theta^t)C_t^{1-\sigma} - \frac{(\boldsymbol{n}_t^s(\boldsymbol{\theta}_h^t))^\gamma}{(\boldsymbol{c}_t^s(\boldsymbol{\theta}_h^t))^{-\sigma}} \frac{N_t^{\gamma+1}}{(z_t(\boldsymbol{\theta}_h^t))^{\gamma+1}} \boldsymbol{n}_t^s(\theta^t) + Q_{t+1}C_t^{-\sigma}A_{t+1}\boldsymbol{a}_{t+1}^s(\theta^t) - C_t^{-\sigma}A_t\boldsymbol{a}_t^s(\theta^{t-1}) = 0 \tag{15}$$

*for all* $t \geq 0$ *and* $\theta^t$, *where*

$$Q_{t+1}C_t^{-\sigma} = \beta C_{t+1}^{-\sigma} \sum_{\theta_{t+1}} (\frac{\boldsymbol{c}_{t+1}^s(\theta^{t+1})}{\boldsymbol{c}_t^s(\boldsymbol{\theta}_h^t)})^{-\sigma} \pi(\theta_{t+1}|\boldsymbol{\theta}_h^t);$$

3. *the initial-period asset market-clearing condition:*

$$\frac{K_0}{Q_0} + B_0 = a_0, \tag{16}$$

*where the initial bond price satisfies*

$$\frac{1}{Q_0} = 1 + (1-\tau_{k,0})MPK_0 - \delta;$$

4. *the individual marginal rate of substitution conditions:*

$$\frac{(\boldsymbol{n}_t^s(\boldsymbol{\theta}_h^t))^\gamma}{(\boldsymbol{c}_t^s(\boldsymbol{\theta}_h^t))^{-\sigma}} \frac{1}{(z_t(\boldsymbol{\theta}_h^t))^{\gamma+1}} - \frac{(\boldsymbol{n}_t^s(\theta^t))^\gamma}{(\boldsymbol{c}_t^s(\theta^t))^{-\sigma}} \frac{1}{(z_t(\theta^t))^{\gamma+1}} = 0 \tag{17}$$

*for all* $t \geq 0$ *and* $\theta^t \neq \boldsymbol{\theta}_h^t$;

5. *the borrowing constraints and their associated complementary slackness conditions:*

$$\boldsymbol{a}_{t+1}^s(\theta^t) \geq 0, \boldsymbol{g}_t^s(\theta^t) \geq 0, \ \boldsymbol{g}_t^s(\theta^t)\boldsymbol{a}_{t+1}^s(\theta^t) = 0 \tag{18}$$

*for all* $t \geq 0$ *and all* $\theta^t \neq \boldsymbol{\theta}_h^t$, *where the function* $\boldsymbol{g}_t^s(\theta^t)$ *is defined as (based on equation (12))*

$$\boldsymbol{g}_t^s(\theta^t) \equiv \frac{\boldsymbol{c}_t^s(\theta^t)^{-\sigma}}{\sum_{\theta_{t+1}} (\boldsymbol{c}_{t+1}^s(\theta_{t+1})^{-\sigma}\pi(\theta_{t+1}|\theta^t)} - \frac{\boldsymbol{c}_t^s(\boldsymbol{\theta}_h^t)^{-\sigma}}{\sum_{\theta_{t+1}} \boldsymbol{c}_{t+1}^s(\theta_{t+1})^{-\sigma}\pi(\theta_{t+1}|\boldsymbol{\theta}_h^t)};$$

*6. and finally the aggregation conditions for shares:*

$$\sum_{\theta^t} \boldsymbol{c}_t^s(\theta^t)\pi(\theta^t) - 1 \;=\; 0, \tag{19}$$

$$\sum_{\theta^t} \boldsymbol{n}_t^s(\theta^t)\pi(\theta^t) - 1 \;=\; 0, \tag{20}$$

$$\sum_{\theta^t} \boldsymbol{a}_{t+1}^s(\theta^t)\pi(\theta^t) - 1 \;=\; 0. \tag{21}$$

*Proof.* See Appendix A.2. □

## 3.2 Ramsey Outcome

Armed with the above definitions of the share variables as well as Proposition 3, the Ramsey problem can be written as

$$\max_{\{\boldsymbol{c}_t^s(\theta^t),\boldsymbol{n}_t^s(\theta^t),\boldsymbol{a}_{t+1}^s(\theta^t),C_t,N_t,A_{t+1},K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[ \begin{array}{c} \frac{1}{1-\sigma} C_t^{1-\sigma} \sum_{\theta^t} (\boldsymbol{c}_t^s)^{1-\sigma} \pi(\theta^t) \\ -\frac{1}{1+\gamma} N_t^{1+\gamma} \sum_{\theta^t} \left( \frac{\boldsymbol{n}_t^s(\theta^t)}{z_t(\theta^t)} \right)^{1+\gamma} \pi(\theta^t) \end{array} \right]$$

subject to constraints (14) to (21). But before solving the Ramsey problem, we define first the Ramsey steady state in our economy:

**Definition 4.** Given $\{K_0, B_0, a_0\}$, a Ramsey steady state is a long-run Ramsey allocation where the aggregate variables $\{N_t, C_t, K_{t+1}, A_{t+1}\}$ all converge to constant values and the share variables $\{\boldsymbol{c}_t^s(\theta^t), \boldsymbol{n}_t^s(\theta^t), \boldsymbol{a}_{t+1}^s(\theta^t)\}$ converge to stationary distributions. In addition, a Ramsey steady state is called "interior" if all of the aggregate variables are strictly positive; otherwise, if one or more of these aggregate variables (such as consumption $C_t$) converge to zero, the Ramsey steady state is called "non-interior."[6]

Define $Q^t$ as the compounded consumption price between time zero and time $t : Q^t \equiv \prod_{j=0}^{t} Q_j$. Denote $\beta^t \mu_t$ and $Q^t \lambda_t(\theta^t)\pi(\theta^t)$ as the Ramsey Lagrangian multipliers for constraints (14) and (15), respectively. The Ramsey FOC with respect to $K_{t+1}$ is given by

$$\mu_t = \beta \mu_{t+1} \left( MPK_{t+1} + 1 - \delta \right), \tag{22}$$

[6]For example, using the two-class model of Judd (1985), Straub and Werning (2020) show that the Ramsey outcome could converge to a non-interior Ramsey steady state.

which is identical to that in Aiyagari (1995).

According to equation (22), Aiyagari (1995) then obtains his famous result of $\tau_k > 0$ in the steady state based on the following critical assumptions: (i) an interior Ramsey steady state exists and (ii) the Ramsey Lagrangian multiplier associated with the aggregate resource constraint $\mu_t$ converges to a positive constant. More specifically, under the assumption that $\mu_t$ converges, equation (22) implies that the MGR holds in the steady state: $1 = \beta\left(MPK + 1 - \delta\right)$. The optimal steady-state capital tax is chosen such that the no-arbitrage condition, $1 = Q\left((1-\tau_k)MPK + 1 - \delta\right),$ is consistent with the MGR:

$$\tau_k = 1 - \frac{\frac{1}{Q} - (1-\delta)}{\frac{1}{\beta} - (1-\delta)} > 0,$$

which is strictly positive because $Q > \beta$ in any competitive equilibrium. This implies that the Ramsey planner "opts" to levy a (permanent) capital tax to select a long-run competitive equilibrium consistent with the MGR.[7]

However, we can derive a few more Ramsey FOCs and show that Aiyagari's assumptions lead to contradictions with these additional Ramsey FOCs. In particular, we can derive at least three more Ramsey FOCs with respect to $A_{t+1}$, $N_t$, and $C_t$. Among them is a very important Ramsey FOC in establishing a key result in Proposition 5 with respect to aggregate labor:

$$\frac{\beta^t}{Q^t} N_t^{\gamma} \sum_{\theta^t} \left(\frac{\boldsymbol{n}_t^s(\theta^t)}{z_t(\theta^t)}\right)^{1+\gamma} \pi(\theta^t) + (1+\gamma) N_t^{\gamma} \frac{(\boldsymbol{n}_t^s(\boldsymbol{\theta}_h^t))^{\gamma}}{(\boldsymbol{c}_t^s(\boldsymbol{\theta}_h^t))^{-\sigma} z_t(\boldsymbol{\theta}_h^t)^{1+\gamma}} \sum_{\theta^t} \lambda_t(\theta^t) \boldsymbol{n}_t^s(\theta^t) \pi(\theta^t) = \frac{\beta^t}{Q^t} \mu_t MPN_t, \tag{23}$$

where $Q^t \equiv \prod_{j=0}^{t} Q_j$. Recall that $Q_t$ is the market discounting factor for household intertemporal budget constraints, which the Ramsey planner must respect, and that $\beta$ is the Ramsey planner's discounting factor for the social welfare function. The wedge between $Q$ and $\beta$ is the hallmark feature of Aiyagari-type models and can be exploited to establish our first main result in this paper. The key step of the proof is that if an interior Ramsey steady state exists where $Q > \beta$ and the multiplier $\mu_t$ converges, then since $\lim_{t\to\infty} \frac{\beta^t}{Q^t} = 0$, the above Ramsey FOC with respect

[7] It is well acknowledged that Aiyagari (1995) also introduces an endogenous government spending in household utilities. However, the introduction of endogenous government spending does not contribute to the different results found in this paper. The critical assumptions to uphold the result of Aiyagari (1995) are the interior Ramsey steady state and the convergence of Ramsey Lagrangian multiplier. It is straightforward to show that even with endogenous government spending, our results remain unchanged.

to $N_t$ in the limit becomes

$$\lim_{t\to\infty}(1+\gamma)N_t^{\gamma}\frac{(\boldsymbol{n}^s(\boldsymbol{\theta}_h^t))^{\gamma}}{(\boldsymbol{c}^s(\boldsymbol{\theta}_h^t))^{-\sigma}z(\boldsymbol{\theta}_h^t)^{1+\gamma}}\sum_{\theta^t}\lambda_t(\theta^t)\boldsymbol{n}_t^s(\theta^t)\pi(\theta^t)=0, \tag{24}$$

which cannot be true because the left-hand side of the above equation is strictly positive in an interior Ramsey steady state (see the proof in Appendix A.3). Hence, an interior Ramsey steady state with convergent multiplier $\mu_t$ cannot be the long-run Ramsey outcome. Similarly, we can use all three Ramsey FOCs in addition to the one with respect to $K_{t+1}$ (the only one used by Aiyagari (1995)) to consider the case where the multiplier $\mu_t$ does not converge. Hence, we have the following proposition:

**Proposition 5.** *A Ramsey allocation in the standard Aiyagari model has the following properties:*

1. *Under the parameter condition $\sigma \geq 1$, there does not exist an interior Ramsey steady state regardless of the convergence property of the Lagrangian multiplier $\mu_t$. The only possible Ramsey steady state under $\sigma \geq 1$ must be non-interior with $C_t \to 0$.*

2. *If $\sigma < 1$, an interior Ramsey steady state may exist; however, if it exists it must feature the following characteristics:*

   *(a) a divergent Ramsey Lagrangian multiplier $\mu_t$,*

   *(b) failure of the MGR,*

   *(c) a non-positive capital tax $\tau_k \leq 0$ .*

*Proof.* See Appendix A.3. □

Proposition 5 shows that the interior Ramsey steady state assumed by Aiyagari (1995) cannot possibly exist. As shown in the proof of Proposition 5, such an interior Ramsey steady state is inconsistent with the other Ramsey FOCs, which are omitted in the analysis of Aiyagari (1995). In other words, the common practice of assuming the existence of an interior Ramsey steady state in the literature might not be innocuous.

In addition, this proposition shows that the result is also sensitive to the utility curvature parameter $\sigma$, which determines the consumer saving behaviors, as it measures both the degree of risk aversion and the inverse degree of IES. A higher risk aversion implies a stronger incentive for precautionary saving to avoid consumption fluctuations, and at the same time a lower IES implies a lower substitutability between current and future consumption (or a stronger income effect than substitution effect). Since these two aspects are captured by the same parameter, we can use these two terms interchangeably in this paper.

Proposition 5 also shows that if $\sigma < 1$ such that the degree of risk aversion is low or IES is sufficiently high, then there may exist an interior Ramsey steady state only if the Lagrangian multiplier $\mu_t$ diverges such that the MGR fails. This result is consistent with the finding of Bassetto and Cui (2020), who in a model featuring firms' borrowing constraints constructed an interior Ramsey steady state under the condition that $\sigma$ is zero or sufficiently close to zero. Nonetheless, we prove more generally in a standard Aiyagari model that if such an interior steady state exists, the optimal capital tax must be non-positive—this result is still inconsistent with the message of Aiyagari (1995) regarding the rationale of capital taxation based on the heterogeneous-agents and incomplete-markets argument.

# 4 A Modified Aiyagari Model

It may be surprising and even puzzling that an interior Ramsey steady state commonly assumed in the literature does not exist in a standard Aiyagari model. Note that a steady state featuring the MGR and positive capital tax as described by Aiyagari (1995) is certainly feasible, but not optimal to the Ramsey planner. The question is, why does the Ramsey allocation not converge to such a feasible steady state to uphold the MGR by taxing capital. Also, what are the optimal level of public debt and the optimal tax rates in the Aiyagari model?

In order to answer these questions, in this section we design a modified and tractable version of the standard Aiyagari model so as to show clearly the conditions under which an interior Ramsey steady state with convergent multiplier(s) may or may not exist. This modified Aiyagari model can also help us understand the properties of both the interior and the non-interior Ramsey steady states whenever they exist, as well as determine the optimal tax rates for capital and labor in a corresponding Ramsey steady state.

Our intuition tells us that the key to destroying an interior Ramsey steady state in the standard Aiyagari model is the government's inability to provide full self-insurance to individuals due to limited fiscal space. For example, in a special version of the Aiyagari model with a completely degenerate wealth distribution under log-linear preferences, Chien and Wen (2021a) argue that since precautionary saving due to borrowing constraints is the root cause and the only friction in heterogeneous-agents models that generates any allocative inefficiency, the Ramsey planner may opt to flood the economy with a sufficient amount of government bonds to eliminate borrowing constraints rather than impose a steady-state capital tax to correct the capital-overaccumulation problem caused by such borrowing constraints. However, in a standard Aiyagari model, an FSI allocation is infeasible to the Ramsey planner because the household asset demand (and the required bond supply) would approach infinity when the asset price $Q_t$ approaches $\beta$.

Hence, to construct a model that can unveil the fundamental mechanisms behind the Ramsey steady-state conundrum in the standard Aiyagari model, we need a model that can analytically reveal the important role of the government's fiscal space in determining the existence/non-existence of a Ramsey steady state. In other words, we need a model that not only guarantees the existence of an interior Ramsey steady state when the fiscal space is sufficient, but also converges to the standard Aiyagari model in the limit when the fiscal space becomes sufficiently tight or insufficient.

For this purpose, we introduce an ad hoc wealth-pooling technology into an otherwise standard Aiyagari model to permit partial risk sharing across individuals. Our wealth-pooling technology follows the spirit of Lucas (1990), Heathcote and Perri (2018), and Bilbiie and Ragot (2021). This wealth-pooling technology allows individuals with identical idiosyncratic-shock histories in the last $\kappa$-periods ($\infty > \kappa \geq 0$) to share risk by pooling their wealth together in the beginning of each period after the idiosyncratic shock is realized. As a result, individuals with the same truncated $\kappa$-period shock history make the same consumption and saving decisions, leading to a partially degenerate wealth distribution. Moreover, our modified model can become arbitrarily close to the standard Aiyagari model as $\kappa$ increases to infinity so that the effect of the wealth-pooling technology becomes ineffective or non-existent—in which case the probability of any two individuals having identical histories goes to zero as $\kappa \to \infty$. On the other extreme where $\kappa = 0$, individuals can pool their wealth in the beginning of every period $t$ as long as their current idiosyncratic-shock status are the same, leading to an almost complete degenerate wealth distribution. Our model thus includes both the standard Aiyagari model and the model of Chien and Wen (2021b) as special limiting cases without the need to appeal to log-linear preferences (as in Chien and Wen (2021a)) to gain model tractability. The technical details of the risk-sharing technology is described below.

**Individuals and Families.** We introduce a unit measure of representative families and assume that there is a unit measure of individuals within each representative family. Within each representative family, there is a family head who is equipped with a wealth-pooling technology to permit partial risk sharing according to each individual's truncated history of $\kappa \geq 0$ periods. Denote the $\kappa$-period truncated history in period $t$ as $h_t^\kappa = \{\theta_{t-\kappa}, \theta_{t-\kappa+1}, .., \theta_t\} \in Z^\kappa$. Specifically, the head of a family can reshuffle asset holdings among individual family members with the same truncated history $h^\kappa$. However, the family head cannot reshuffle resources across individual members across different $h^\kappa$. Hence, the family head can provide a limited amount of risk sharing among certain family members while not completely eliminating the idiosyncratic risk faced by individuals. Without such a limited wealth-pooling technology, or as $\kappa \to \infty$, our model becomes identical or converges to the standard Aiyagari model.

The transition probability from type-$h^\kappa$ individuals to type-$h^{\kappa\prime}$ individuals is then denoted by $\pi(h^{\kappa\prime}|h^\kappa)$, which is determined by transition probability of the first-order Markov process of $\theta$. Moreover, the invariant probability of each group $h^\kappa$ is denoted by $\pi(h^\kappa)$. Also, we label the group of individuals experiencing the highest shock and the lowest shock in every period during the entire period of truncated history as $\boldsymbol{h}_h^\kappa$ and $\boldsymbol{h}_l^\kappa$, respectively.

For simplicity and without loss of generality, we assume that $\pi(h^\kappa)$ also represents the initial period's share of individuals in time 0. The utilitarian welfare criterion of a family head is then given by

$$U = \sum_{t=0}^{\infty} \beta^t \sum_{h^\kappa} \left[u(c_t(h^\kappa)) - v\left(n_t(h^\kappa)\right)\right] \pi(h^\kappa). \tag{25}$$

Denote $z_t(h^k)$ as the period-$t$ (current period) labor productivity shock for group $h^\kappa$. The budget constraints for type-$h^\kappa$ individuals in period 0 are given by

$$a_0(h^\kappa) + \widehat{w}_0 z_0(h^\kappa) n_0(h^\kappa) - c_0(h^\kappa) - Q_1 a_1(h^\kappa) \geq 0. \tag{26}$$

Under the wealth-pooling technology, the total assets available for type-$h^\kappa$ individuals in the beginning of period $t \geq 1$ is given by $\sum_{h_{-1}^\kappa} a_t(h_{-1}^\kappa)\pi(h_{-1}^\kappa)\pi(h^\kappa|h_{-1}^\kappa)$. Therefore, for all $t \geq 1$, the budget constraints for type-$h^\kappa$ individuals can be written as

$$\sum_{h_{-1}^\kappa} \frac{a_t(h_{-1}^\kappa)\pi(h_{-1}^\kappa)\pi(h^\kappa|h_{-1}^\kappa)}{\pi(h^\kappa)} + \widehat{w}_t z_t(h^\kappa) n_t(h^\kappa) - c_t(h^\kappa) - Q_{t+1} a_{t+1}(h^\kappa) \geq 0. \tag{27}$$

The individual borrowing constraint is still given by

$$a_{t+1}(h^\kappa) \geq 0 \text{ for all } t \geq 0 \text{ and } h^\kappa. \tag{28}$$

Finally, each family head chooses a plan of $\{c_t(h^\kappa), n_t(h^\kappa), a_{t+1}(h^\kappa)\}_{t=0}^\infty$ to maximize (25) subject to (26), (27) and (28).

To make the ad hoc wealth-pooling technology meaningful in analytically addressing our problems at hand, we assume that the idiosyncratic shock process is non-negatively autocorrelated such that in each period $t \geq 1$ the initial wealth of the type-$\boldsymbol{h}_h^\kappa$ individuals is no less than that of the other individuals in the population. This assumption rules out the uninteresting case where any individual may become wealthier than type-$\boldsymbol{h}_h^\kappa$ individuals from time to time. More specifically, if $a_t(\boldsymbol{h}_h^\kappa) > a_t(h^\kappa)$ for all $h^\kappa$, then under the assumption that the idiosyncratic shock process is non-negatively autocorrelated, the following inequality must hold:

$$\sum_{h_{-1}^\kappa} \frac{a_t(h_{-1}^\kappa)\pi(h_{-1}^\kappa)\pi(\boldsymbol{h}_h^\kappa | h_{-1}^\kappa)}{\pi(\boldsymbol{h}_h^\kappa)} \geq \sum_{h_{-1}^\kappa} \frac{a_t(h_{-1}^\kappa)\pi(h_{-1}^\kappa)\pi(h^\kappa | h_{-1}^\kappa)}{\pi(h^\kappa)} \text{ for all } h^\kappa \neq \boldsymbol{h}_h^\kappa,$$

which says that the period-$t$ initial asset holdings of $\boldsymbol{h}_h^\kappa$ individuals are no less than that of any other type of individuals.

**Proposition 6.** *Assume for simplicity that in period 0 the initial asset holdings satisfy $a_0(\boldsymbol{h}_h^\kappa) > a_0(h^\kappa) \geq 0$ for all $h^\kappa \neq \boldsymbol{h}_h^\kappa$. The competitive equilibrium of the modified Aiyagari model must have the following properties:*

1. *For all $t \geq 0$, it must be true that $c_t(\boldsymbol{h}_h^\kappa) \geq c_t(h^\kappa)$ and $a_{t+1}(\boldsymbol{h}_h^\kappa) > a_{t+1}(h^\kappa) \geq 0$ for all $h^\kappa \neq \boldsymbol{h}_h^\kappa$. That is, the borrowing constraints of type-$\boldsymbol{h}_h^\kappa$ individuals are always slack: $a_{t+1}(\boldsymbol{h}_h^\kappa) > 0$, which implies that the Lagrangian multiplier associated with constraint (28) is $\psi_t(\boldsymbol{h}_h^\kappa) = 0$. Also, depending on the level of $B_t$, the borrowing constraints of the currently unemployed individuals may or may not be binding: $a_{t+1}(h^\kappa) \geq 0$.*

2. *Because $\psi_t(\boldsymbol{h}_h^\kappa) = 0$, the intertemporal price $Q_{t+1}$ can be expressed as*

$$Q_{t+1} = \beta \sum_{h^{\kappa\prime}} \frac{u_{c,t+1}(h^{\kappa\prime})}{u_{c,t}(\boldsymbol{h}_h^\kappa)} \pi(h^{\kappa\prime} | \boldsymbol{h}_h^\kappa) \geq \beta \sum_{h^{\kappa\prime}} \frac{u_{c,t+1}(h^{\kappa\prime})}{u_{c,t}(h^\kappa)} \pi(h^{\kappa\prime} | h^\kappa) \quad \textit{for all } t \textit{ and } h^\kappa \neq \boldsymbol{h}_h^\kappa. \tag{29}$$

3. *In the steady state, if the individual asset holdings are sufficiently large such that the corresponding multiplier $\psi_t(h^\kappa) = 0$ for all individuals regardless of their truncated history $h^\kappa$, then the competitive equilibrium in the modified Aiyagari model features FSI with two properties: (i) consumption equality $c(\boldsymbol{h}_h^\kappa) = c(h^\kappa)$ for all $h^\kappa$ and (ii) a zero liquidity premium with $Q = \beta$ (or $r = \beta^{-1}$). Otherwise, in the case of only partial self-insurance (absence of FSI) it must be true that $Q > \beta$ (or $r < \beta^{-1}$).*

*Proof.* See Appendix A.4. □

Proposition 6 states that if the asset holdings $a_{t+1}(h^\kappa)$ are sufficiently large for all individuals regardless of their truncated history $h^\kappa$, such that every one's borrowing constraint is slack, then they can obtain the same level of steady-state consumption regardless of their truncated idiosyncratic history. In this FSI competitive equilibrium, the steady-state market interest rate equals the time discount rate.

It is well known that a competitive equilibrium featuring FSI is not possible in the standard Aiyagari model. In our modified Aiyagari model, however, the FSI steady state can be achieved with only a finite level of asset holdings because the optimal asset demand does not go to infinity even when $Q = \beta$, thanks to the wealth-pooling technology that permits partial risk sharing across individuals with different histories. Proposition 6 also implies that even if the laissez-faire competitive equilibrium does not feature FSI because of an insufficient initial asset supply, the Ramsey planner can potentially achieve the FSI allocation by issuing enough public debt if desired.

Hence, to make our Ramsey problem interesting in the modified Aiyagari model, we assume that the initial capital $K_0$ and bond supply $B_0$, as well as the initial distribution of household wealth $a_0(h^\kappa)$, are such that the laissez-faire competitive equilibrium (without further policy intervention) *does not* feature FSI. Namely, in the absence of further government intervention ($B_t = B_0$ for all $t > 0$), the steady-state competitive equilibrium features consumption inequality $c(\boldsymbol{h}_h^\kappa) > c(h^\kappa)$ and precautionary saving behaviors with a positive liquidity premium: $r < \beta^{-1}$.

## 4.1 Ramsey Outcome in the Modified Aiyagari Model

To facilitate the analysis below, we define $A$ as the minimum asset level required to achieve an FSI allocation in the Ramsey steady state, and define $\phi \equiv \frac{A}{C}$ as the ratio between $A$ and aggregate consumption $C$ in an FSI allocation. Note that the equilibrium value of $\phi$ depends on the persistence of idiosyncratic shocks and the extent of risk-sharing technology parameter $\kappa$; and $\phi$ essentially captures the tightness of the required fiscal space to achieve an FSI interior Ramsey steady state in our modified Aiyagari model.

**Proposition 7.** *If the fiscal space is sufficient such that $\phi(1-\beta) < 1$, then under $\sigma \geq 1$ there exists an interior Ramsey steady state with the following properties:*

1. *The allocation features FSI where all individuals have the same steady-state consumption and non-binding borrowing constraints, and the Lagrangian multiplier associated with the aggregate resource constraint, $\mu_t$, converges to finite positive value.*

2. *The risk-free rate satisfies $r = 1/\beta$, the MGR holds, and the steady-state capital tax is zero: $\tau_k = 0$.*

3. *The optimal labor tax rate and optimal debt-to-output ratio depend on $\phi$ and are given, respectively, by*
$$\tau_n = (1-\beta)\frac{\phi(1-\beta+\delta\beta(1-\alpha)) - \alpha}{(1-\alpha)(1-\beta+\delta\beta)} \in (0,1),$$
$$\frac{B}{Y} = \frac{(1-\beta+\delta(1-\alpha)\beta)\,\phi - \alpha}{(1-\beta+\delta\beta)};$$
*where $\tau_n < 1$ if and only if the fiscal-space condition $\phi(1-\beta) < 1$ holds.*

4. *This is the only possible type of Ramsey steady state.*

*On the other hand, if the fiscal space is insufficient such that $\phi(1-\beta) \geq 1$, then the following properties hold:*

1. *Under the parameter condition $\sigma \geq 1$, the only possible Ramsey steady state (if it exists) is a non-interior allocation with zero aggregate consumption.*

2. *Under the parameter condition $\sigma < 1$, the only possible Ramsey steady state (if it exists) is an interior allocation with*

   (a) *partial self-insurance and a divergent Lagrangian multiplier $\mu_t$,*

   (b) *a non-positive capital tax rate $\tau_k \leq 0$.*

*Proof.* See Appendix A.5. ☐

One of the key insights from Proposition 7 is that, under the parameter space $\sigma \geq 1$ and the fiscal space condition $\phi(1-\beta) < 1$, the only possible Ramsey steady state is an FSI interior Ramsey steady state and it necessarily exists; and more importantly the Ramsey planner opts to achieve it even at the cost of a possibly very high steady-state labor tax rate while still setting the optimal long-run capital tax rate to zero. Although in such an interior Ramsey steady state the optimal labor tax rate $\tau_n$ is bounded above from 1—only because of the restriction $(1-\beta)\phi < 1$—but $\tau_n$ could be arbitrarily close to 100% depending on the parameter values that influence the value of $\phi$. In particular, the value of $\phi$ depends critically on the length of the truncated history $\kappa$ for risk sharing. A close-to-100% labor tax rate then implies close-to-zero steady-state aggregate output and consumption. Namely, the Ramsey planner may opt to achieve consumption equality at "all costs."

In addition, in this interior steady state the optimal capital tax is zero, suggesting that the Ramsey planner will never levy a steady-state capital tax to achieve the MGR even if the labor tax is close to 100%; the MGR is achieved instead by having a sufficiently high public debt-to-GDP ratio such that the borrowing constraints of all individuals are slack, despite that it is feasible to use a capital tax to achieve the MGR. In other words, the Ramsey planner never uses capital taxation to alleviate the burden of labor taxation despite the fact that a close-to-100% labor tax rate implies close-to-zero labor income and consumption (a similar result also holds in representative-agent models when exogenous government spending is sufficiently high).[8]

How can such a long-run FSI allocation be optimal? Common sense seems to tell us that the marginal benefit of reducing the consumption inequality to achieve FSI must decline with an increasing stock of public debt (or debt-to-consumption ratio), while the marginal cost of financing the debt under distortionary labor taxes must also increase rapidly. Therefore, there should be a trade-off between consumption equality and absolute steady-state consumption such that at some point taxing capital may become optimal. But this is not the case, surprisingly and counter-intuitively.

The fundamental reason for such a counter-intuitive result is as follows: Given that the market discount (interest) rate is lower than the time discount rate ($r < \beta^{-1}$ or $Q > \beta$)—a hallmark feature of Aiyagari-type models—the Ramsey planner opts to frontload household consumption by borrowing more cheaply in the short run as a trade-off for low consumption in the long run,

[8] For a literature survey on optimal capital taxation in representative-agent models, see Atkeson, Chari, and Kehoe (1999) or more recent work by Chari, Nicolini, and Teles (2020)

because the future cost of debt financing is heavily discounted by a higher time-discounting factor $\beta^{-1}$ compared with the market interest rate. Also, consumption frontloading must be supported by a higher labor supply to increase production, which requires a lower or even negative labor tax in the transition to incentivize hard working. However, a rapidly growing debt and a low labor tax rate in the short run must imply a heavy tax burden in the long run to finance the sky-rocketing government debt.

Therefore, when $\sigma \geq 1$, the consumer's strong precautionary saving motive and low IES are consistent with the Ramsey planner's intention to increase the debt supply and bond growth in the short run to support consumption frontloading, because the anticipated high labor tax in the future to finance the burden of government debt leads to higher current saving when the income effect dominates the substitution effect.[9] Namely, individuals are willing to hold more government debt in the short run in anticipation of a high labor tax rate in the long run if $\sigma \geq 1$, which facilitates the government's consumption-frontloading strategy since the Ramsey planner is then able to issue debt or increase the bond supply more rapidly. This consumption-frontloading incentive never disappears unless the equilibrium interest rate becomes equal to the time discount rate, which can only be achieved with an excessive government bond supply (relative to output) in the FSI allocation where $Q = \beta$ and the optimal capital tax $\tau_k = 0$; consequently, the Ramsey planner must finance the "sky-rocketing" debt by increasing the steady-state labor tax rate, even if this implies a low (or close to zero) consumption in the remote future (long run). Therefore, when the fiscal space is not sufficient to support an FSI allocation, under the parameter condition $\sigma \geq 1$ the only possible Ramsey steady state must be non-interior. Such a dynamic implication for the Ramsey allocation is consistent with the finding of Albanesi and Armenter (2012), who argue that frontloading intertemporal distortions induces a first-order welfare gain in a broad class of second-best economies. Also, the zero-capital-tax result echoes that in a representative-agent model.

However, without taking into account the transitional dynamics, as in the case of maximizing only the steady-state welfare, an FSI steady state may not be optimal, because the asymmetric discounting between $Q$ and $\beta$ is no longer a relevant issue in this setup (see Section 5 below for analytical details and more discussions).

[9]Interestingly, the result found by Straub and Werning (2020) also depends on the IES parameter. The intuition also hinges on the response of households' current saving to the change of future tax rate, which is called the "anticipatory savings effects" in their model.

The above discussions on the case of $\sigma \geq 1$ also indicate that when $\sigma < 1$, since the substitution effect dominates the income effect and the individual precautionary saving motive is weak, the anticipated future increase in the labor tax will lead to a reduction instead of an increase in individuals' current saving. As a result, the lack of saving motives can significantly limit the government's fiscal space to increase debt growth in spite of the planner's intention to frontload consumption, thus producing a counterforce against the planner's pursuit of FSI and consumption frontloading. In other words, the government is unable to issue as much debt as required to support consumption frontloading and FSI, resulting in a lower future tax burden in the steady state. Therefore, the non-interior Ramsey steady state with zero consumption is deterred or detoured into an interior steady state where the interest rate is lower than the time discount rate ($Q > \beta$). In such a case with a sufficiently tight fiscal space ($\phi(1-\beta) \geq 1$), the Ramsey planner opts to reduce the extent of consumption frontloading and even subsidize capital to encourage individual saving so as to support higher future consumption, leading to an interior Ramsey steady state where the Lagrangian multiplier diverges, the optimal capital tax is non-positive, and the MGR fails.

This result is reminiscent of that in Bassetto and Cui (2020), who also show that when the government's fiscal capacity is insufficient to support an FSI allocation, the optimal Ramsey allocation converges to a non-FSI interior steady state where the Lagrangian multiplier diverges. Nevertheless, we are able to prove rigorously that this type of interior Ramsey steady state can emerge only under a sufficiently high IES or low risk aversion ($\sigma < 1$) and that the optimal capital tax is unambiguously non-positive. In addition, our explanation and intuition for this type of interior Ramsey steady state are based on the different responses of current saving to the rise of future tax rate under the planner's consumption-frontloading incentive (thanks to the wedge between $Q$ and $\beta$), which is controlled by the risk-aversion or IES parameter $\sigma$. The different saving behaviors then lead to different debt-supply policies of the Ramsey planner and eventually different Ramsey steady states; in contrast, the explanation of Bassetto and Cui (2020) is based on the Laffer curve and their interior Ramsey steady state is constructed numerically by choosing different growth rates of government spending and the Lagrangian multiplier.

Notice that the Ramsey planner's incentive for frontloading consumption is always present regardless of the IES parameter $\sigma \in (0, \infty)$. Also notice that the equivalence between $\tau_n < 1$ and $\phi(1-\beta) < 1$ holds in the FSI Ramsey steady state, where the multiplier $\mu_t$ converges. This equivalence no longer applies if the multiplier diverges. Therefore, when $\sigma < 1$ the fiscal-space condition $\phi(1-\beta) \geq 1$ does not imply a more than 100% labor tax rate; it only implies that the required asset-to-consumption ratio to achieve the FSI allocation is too high.

A nice property of this modified Aiyagari model is that as $\kappa$ increases (or as the role of the wealth-pooling technology diminishes), the model will converge to the standard Aiyagari model; in this case the asset demand (or the asset-to-consumption ratio $\phi$) will rise with $\kappa$ to reflect the increasing demand of self-insurance under $\sigma \geq 1$. As a result, the condition $\phi(1-\beta) < 1$ becomes harder and harder to satisfy and the fiscal space of the Ramsey planner becomes tighter and tighter. Eventually, when the value of $\kappa$ is large enough, the condition $\phi(1-\beta) < 1$ will be violated and, consequently, the interior Ramsey steady state featuring the MGR disappears and becomes (turns into) the non-interior Ramsey steady state—because under the condition $(1-\beta)\phi \geq 1$ and $\sigma \geq 1$ the only possible Ramsey steady state is non-interior regardless of the convergence property of the multiplier(s), in which case the optimal labor tax goes to 100% asymptotically.

In what follows, we use numerical simulations of the modified Aiyagari model to confirm that the interior Ramsey steady state featuring the MGR will indeed converge to the non-interior Ramsey steady state as $\kappa$ increases.

## 4.2 Numerical Approximation of the Standard Aiyagari Model

The above theoretical analysis can be confirmed by numerical simulations. In particular, we can numerically solve the modified Aiyagari model's FSI Ramsey steady state by ensuring that all Ramsey FOCs are satisfied under proper parameter values and a given set of values for $\kappa$, or as the length of the truncation history $\kappa$ extends. To demonstrate, we set the preference parameters to $\gamma = \sigma = 2$, the capital deprecation rate to $\delta = 0.1$, and the capital's share to $\alpha = 0.35$, which are all standard in the macroeconomic literature. For simplicity, we consider a two-state Markov process where $Z = \{e, u\}$, $z(e) = 1$ and $z(u) = 0$. In other words, an individual can work and receive labor income if $\theta = e$; otherwise, if $\theta = u$, the individual cannot work and has no labor income. In addition, to demonstrate the mechanism more sharply, the time-discounting factor is deliberately set to a low value of $\beta = 0.65$, which allows less space to raise the value of $\kappa$ (otherwise the changes in the Ramsey allocation in our numerical simulations would be less visible before $\kappa$ becomes extremely large). In this way, when $\kappa = 0$, the Ramsey steady-state labor tax rate would be very low because the interest-cost burden in an FSI Ramsey steady state is low; however, as $\kappa$

increases from 0 to 10, the optimal rate of steady-state labor tax can rise rapidly to near 100%. Finally, the transition probability matrix of the $\theta$ shock is given by

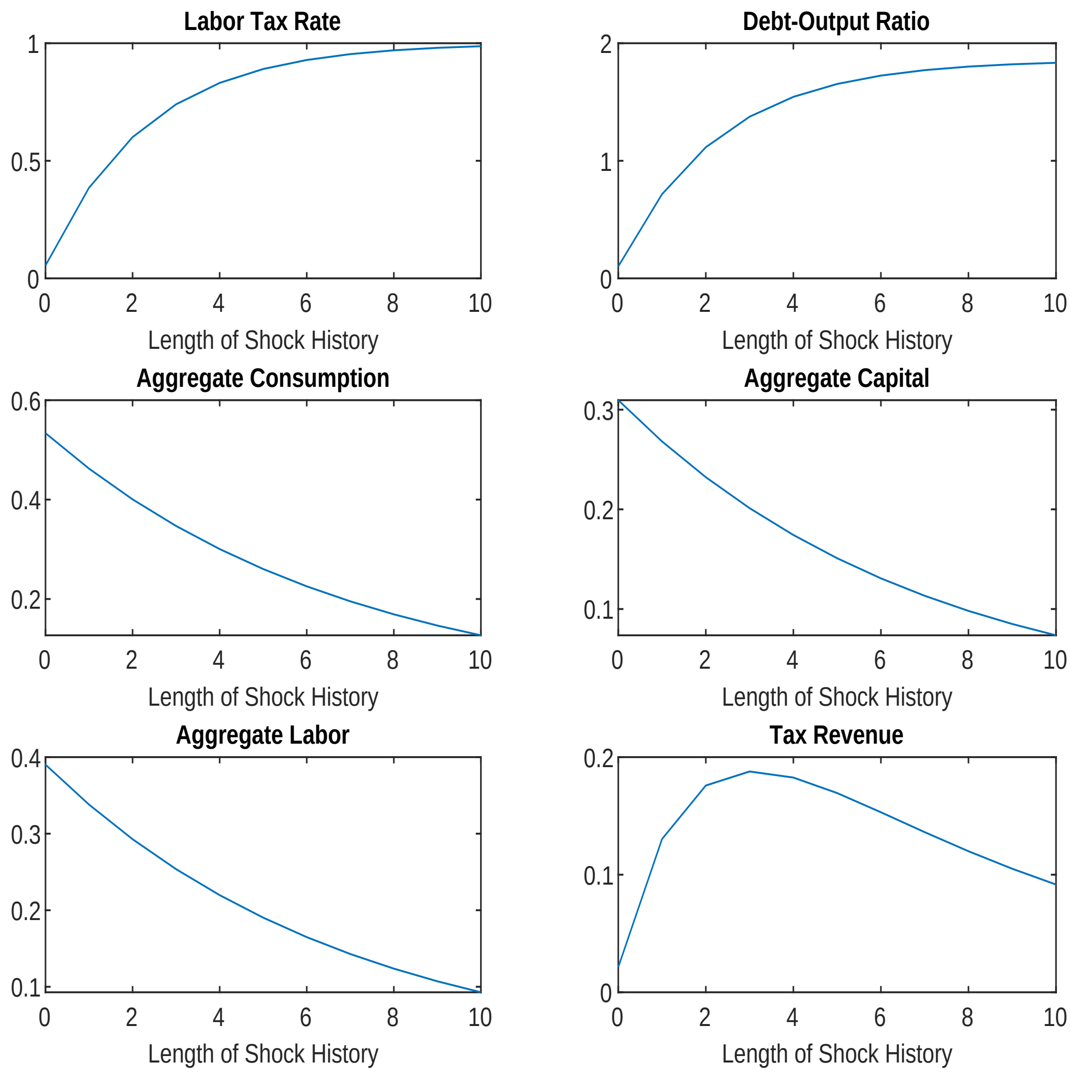
$$\pi = \begin{bmatrix} \pi(u|u) & \pi(e|u) \\ \pi(u|e) & \pi(e|e) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}. \tag{30}$$

We find that as $\kappa$ increases from 0 to 10, the implied optimal debt-to-output ratio required for an FSI interior Ramsey steady state grows rapidly, the steady-steady labor tax $\tau_n$ approaches 100%, and the steady-state aggregate consumption approaches zero. Consequently, the allocation in the interior Ramsey steady state approaches a non-interior Ramsey steady state, suggesting that the interior Ramsey steady state featuring the MGR will eventually disappear even within a finite value of $\kappa$. Such a numerical analysis is valid because the interior Ramsey steady state has been proven to exist in Proposition 7 and can also be proven to exist numerically by solving all of the Ramsey FOCs.

Figure 1 shows the Ramsey policies and other endogenous variables in the interior Ramsey steady state as we extend $\kappa$ from 0 to 10. It shows that as $\kappa$ increases, namely, as the risk-sharing capacity brought in by the wealth-pooling technology becomes less and less effective, the implied optimal steady-state labor tax rate $\tau_n$ (top-left panel or panel[1,1]) rises from 5.5% to nearly 100%. This is so because the optimal debt-to-output ratio required to support an FSI Ramsey allocation (top-right panel or panel [1,2]) increases rapidly as the degree of risk embodied in wealth increases. Consequently, the levels of aggregate consumption, aggregate capital, and aggregate labor (panel [2,2], panel [2,2] and panel [3,1], respectively) decline toward zero. That is, as $\kappa$ increases, the interior Ramsey steady state moves toward the non-interior Ramsey steady state. During this process of prolonging the past-shock history $\kappa$, a gradually rising labor tax rate and declining labor supply also imply that the total tax revenue (fiscal capacity) will rise first (e.g., for $\kappa < 4$) but eventually decline toward zero, as indicated by the bottom-right panel.

Since our model converges to the standard Aiyagari model (that has no risk-sharing technology) when $\kappa$ approaches infinity, our numerical finding thus confirms our theoretical result that under parameter value $\sigma \geq 1$, an interior Ramsey steady state does not exist in the standard Aiyagari model. The intuition is as follows: Because the Ramsey planner opts to pursue an FSI allocation to completely eliminate borrowing constraints—driven by the arbitrage opportunity when the market

Figure 1: The FSI Ramsey Steady-State with Extended Truncated History




interest rate lies below the time discount rate—the asset demand in the standard Aiyagari model will approach infinity when the interest rate approaches the time discount rate (i.e., as $Q \to \beta$). Thus, once our modified Aiyagari model approaches the standard Aiyagari model by reducing the risk-sharing efficacy of the wealth-pooling technology (implied by increasing $\kappa$), the optimal debt level required for sustaining an FSI allocation will rise to infinity accordingly. This rise to infinity makes the interior FSI Ramsey steady state infeasible as a competitive equilibrium. As a result, the FSI interior Ramsey steady state eventually disappears and converges to a non-interior Ramsey steady state.

Therefore, the numerical exercise indicates that if the IES parameter satisfies $\sigma \geq 1$, then the only possible Ramsey steady state in the standard Aiyagari model is the non-interior steady state where the aggregate consumption, aggregate capital stock, aggregate labor, and aggregate output are all zero, and the optimal labor tax rate is 100%.

### 4.3 Characterization of a Ramsey Steady State with $\kappa = 0$

This subsection considers a further simplified version of our model by setting $\kappa = 0$ so as to illuminate how the persistence of idiosyncratic shocks affects the Ramsey allocation in our modified Aiyagari model. Continue to assume a two-state Markov process for the idiosyncratic shock process as in Section 4.2, which means there are only two types of individuals in every period, denoted by $e$ and $u$ types. In this further simplified case we can go one step further in analytically displaying the role of shock persistence in determining the fiscal-space condition and proving the existence of the Ramsey steady state, whether that be interior or non-interior.

When $\kappa = 0$, the implementability condition in the FSI Ramsey steady state can be simplified to

$$\frac{a^e \pi(e)\pi(u|e)}{\pi(u)} = c^u = c^e,$$

and the asset-to-consumption ratio $\phi$ in the fiscal-space condition can be simplified to $\phi \equiv \frac{A}{C} = \frac{a^e \pi(e)}{c^e} = \frac{\pi(u)}{\pi(u|e)}$. According to Proposition 7, under the parameter conditions $(1-\beta)\frac{\pi(u)}{\pi(u|e)} < 1$ and $\sigma \geq 1$, an FSI interior Ramsey steady state necessarily and uniquely exists where the optimal labor tax rate and the debt-to-output ratio are given by

$$\tau_n = \frac{(1-\beta+\delta(1-\alpha)\beta)\,\frac{\pi(u)}{\pi(u|e)}(1-\beta) - \alpha(1-\beta)}{(1-\alpha)\,(1-\beta+\delta\beta)} \in (0,1),$$

and

$$\frac{B}{Y} = \frac{(1-\beta+\delta(1-\alpha)\beta)\,\frac{\pi(u)}{\pi(u|e)} - \alpha}{(1-\beta+\delta\beta)},$$

respectively. Clearly, the optimal $B/Y$ ratio required to support FSI is proportional to $\frac{\pi(u)}{\pi(u|e)}$, which can be rewritten as $\frac{\pi(u)}{\pi(u|e)} = \frac{1}{2-(\pi(u|u)+\pi(e|e))}$. Under a simple two-state Markov process, $\pi(u|u) + \pi(e|e)$ represents the persistence of idiosyncratic shocks.

If the idiosyncratic shock becomes permanent, for example, then $\pi(u|u) + \pi(e|e)$ becomes 2 and the value of $\frac{\pi(u)}{\pi(u|e)}$ goes to infinity. In other words, the required $B/Y$ ratio to support FSI can be arbitrarily close to infinity even in the case of a quite effective risk-sharing technology under $\kappa = 0$. This finding suggests that once the idiosyncratic shock process becomes highly persistent or permanent, an interior FSI Ramsey steady state will fail to exist because the condition $(1-\beta)\frac{\pi(u)}{\pi(u|e)} < 1$ will be violated and the steady-state labor tax rate will converge to or exceed 1, ruling out the FSI interior Ramsey steady state.

In addition, the previous analysis under general $\kappa \geq 0$ shows that there may exist another type of interior Ramsey steady state where the multiplier $\mu_t$ diverges, the interest rate lies below the time discount rate ($Q > \beta$), and the optimal capital tax is non-positive ($\tau_k \leq 0$). By using all the Ramsey FOCs and constraints in the case $\kappa = 0$, the existence of such types of Ramsey steady states can be guaranteed, as shown in the following proposition:

**Proposition 8.** *If $(1-\beta)\frac{\pi(u)}{\pi(u|e)} \geq 1$ and $\sigma \geq 1$, there is no interior Ramsey steady state but there uniquely exists a non-interior steady state where (i) aggregate consumption $C_t$, aggregate capital $K_t$, and aggregate labor $N_t$, all converge to zero; (ii) the optimal labor tax rate $\tau_{n,t}$ converges to 100%; (iii) the optimal capital tax is undetermined; and (iv) the multiplier $\mu_t$ diverges to infinity.*

*On the other hand, if $(1-\beta)\frac{\pi(u)}{\pi(u|e)} \geq 1$ and $\sigma < 1$, a non-interior Ramsey steady state does not exist but there uniquely exists an interior Ramsey steady state featuring (i) a divergent multiplier $\mu_t$, (ii) partial self-insurance with $Q > \beta$, and (iii) a non-positive capital tax $\tau_k \leq 0$.*

*Proof.* See Appendix A.6. □

In other words, even under a very effective risk-sharing technology $\kappa = 0$, if the idiosyncratic shock is persistent enough (such that the fiscal-space condition is violated, $(1-\beta)\frac{\pi(u)}{\pi(u|e)} \geq 1$), then the non-interior Ramsey steady state is the only possible Ramsey steady state and it necessarily exists when $\sigma \geq 1$. This finding indicates that as risk sharing goes to zero ($\kappa \to \infty$), the demand for self-insurance must rise steadily; hence, the condition $(1-\beta)\phi \geq 1$ becomes much easier to meet even with less persistent idiosyncratic shocks. This outcome further reinforces the message that a non-interior Ramsey steady state exists and is the only possible steady state in the standard Aiyagari model under the commonly accepted parameter value $\sigma \geq 1$.

Finally, the above proposition also shows that with a highly persistent idiosyncratic shock process (such that the fiscal-space condition is violated, $(1-\beta)\frac{\pi(u)}{\pi(u|e)} \geq 1$), if individuals' willingness to save or hold assets is too weak (under a sufficiently high IES or low risk aversion, $\sigma < 1$), then an interior Ramsey steady state featuring a divergent multiplier can exist; and more importantly, this is the only possible Ramsey steady state. In addition, this interior Ramsey steady state must feature a non-positive capital tax and the failure of the MGR with $Q > \beta$. The reason is that under a sufficiently high IES or low degree of risk aversion, individuals' saving incentives are reduced significantly in response to an anticipated rise of the future labor tax under the government's consumption-frontloading strategy, making it difficult for the government to issue plenty of debt during the transition, which forces the Ramsey planner to restraint from excessive consumption frontloading and debt accumulation. To trade off the insufficient consumption frontloading, the Ramsey planner then opts to increase the steady-state consumption by reducing the steady-state labor tax and even subsidizing capital; but the high capital stock in the steady state takes the allocation further away from the MGR.

# 5 Additional Analyses

This section considers two additional analyses by utilizing the modified Aiyagari model (with $\kappa \geq 0$) for two scenarios. The first scenario allows the Ramsey planner to have an additional fiscal tool—an unconditional lump-sum tax or transfer. In the second scenario, the Ramsey planner maximizes only the steady-state welfare of the competitive equilibrium. These analyses shed further light on the Ramsey steady-state conundrum and the insight provided above.

## 5.1 Scenario 1: Unconditional Lump-Sum Tax/Transfer

The previous sections have revealed the critical importance of public debt in providing self-insurance and achieving consumption equality in incomplete-market economies. But the analysis is conducted in the absence of lump-sum transfers. As argued by Werning (2007), there may be good reasons to avoid lump-sum taxes as a source of government revenue, but it may not be reasonable to assume away lump-sum transfers as an alternative fiscal tool to public debt in mitigating consumption risk and inequality. This is so because a lump-sum transfer is not only realistic but can also substitute for public debt in income/wealth redistribution, especially in heterogeneous-agent models with inequality.

Yet the distinctive role of lump-sum transfers in the presence of government debt is not well understood in the heterogeneous-agent literature. In this subsection we explore the tractability of our model to show analytically that a lump-sum transfer is not an effective tool to improve consumption inequality in the presence of public debt. In addition, if a lump-sum tax $T_t$ (negative transfer) is allowed, then the Ramsey planner can even achieve the first-best allocation starting from the initial period $t = 0$ and all the way to $t \to \infty$. Specifically, the Ramsey planner opts to issue plenty of public debt to relax the borrowing constraints of all households and use lump-sum taxes ($T_t < 0$) as the only tax instrument to finance public debt (i.e., $\tau_{n,t} = \tau_{k,t+1} = 0$ for all $t \geq 1$). This result also logically implies that if lump-sum taxes are not allowed but lump-sum transfers are still available (i.e., under the constraint $T \geq 0$), then the Ramsey allocation must feature $T = 0$ in the steady state, which is identical to our previous result in Proposition 7 where $T = 0$ by assumption.

With lump-sum transfers, the individual budget constraints in the modified model become

$$a_0(h^\kappa) + \widehat{w}_0 z_0(h^\kappa) n_0(h^\kappa) + T_0 - c_0(h^\kappa) - Q_1 a_1(h^\kappa) \geq 0$$

for $t = 0$ and

$$\sum_{h_{-1}^\kappa} \frac{a_t(h_{-1}^\kappa)\pi(h_{-1}^\kappa)\pi(h^\kappa|h_{-1}^\kappa)}{\pi(h^\kappa)} + T_t + \widehat{w}_t z_t(h^\kappa) n_t(h^\kappa) - c_t(h^\kappa) - Q_{t+1} a_{t+1}(h^\kappa) \geq 0$$

for $t \geq 1$. The government budget constraint is rewritten as

$$\tau_{n,t} w_t N_t + \tau_{k,t} q_t K_t + Q_{t+1} B_{t+1} \geq B_t + T_t.$$

**Definition 9.** The first-best allocation is defined as the optimal allocation chosen by a social planner that maximizes the welfare function (25) subject only to the aggregate resource constraint.

It is straightforward to see that, in the first-best allocation, all individuals have the same consumption regardless of their shock history. Denote $\{c_t^{FB}, K_{t+1}^{FB}\}_{t=0}^\infty$ as the individual consumption and aggregate capital sequence of the first-best allocation. The following proposition characterize the Ramsey allocation when lump-sum taxes or transfers are available.

**Proposition 10.** *Set $\tau_{k,0} = 0$ in period 0. Given the initial values of $K_0 > 0$, $\frac{K_0}{Q_0} + B_0 > 0$, the Ramsey outcome in the modified model with lump-sum taxes/transfers and with a finite $\kappa$ achieves the first-best allocation that features $c_t(h^\kappa) = c_t^{FB}$ and FSI in every period $t \geq 0$. This allocation can be implemented by the following policy mix:*

1. *The distortionary labor tax and capital tax are zero for all $t \geq 0$: $\tau_{n,t} = 0$ and $\tau_{k,t+1} = 0$.*

2. *The sequence of government debt is chosen to satisfy the asset market-clearing condition for all $t \geq 0$:*

$$B_{t+1} = A_{t+1} - \frac{\beta u_{c,t+1}^{FB}}{u_{c,t}^{FB}} K_{t+1}^{FB},$$

   *where $A_{t+1}$ is aggregate saving.*

3. *The steady-state lump-sum transfer is strictly negative and given by $T = (\beta - 1)B < 0$.*

*Proof.* See Appendix A.7. □

Several subtle implications of Proposition 10 are worth mentioning: First, Proposition 10 indicates that a lump-sum tax is a very powerful tool in our modified model to sustain government debt; it permits the first-best allocation for the entire dynamic path (from $t = 0$ to $\infty$) without the need to levy distortionary taxes or have state-contingent fiscal tools. Second, the government prefers using bonds instead of a lump-sum transfer to achieve FSI and consumption equality. Intuitively, in the Aiyagari-type models with ex-post heterogeneous agents, government bonds are more suitable than lump-sum transfers to address the lack of self-insurance problem caused by incomplete insurance markets; hence, the Ramsey planner opts to use debt exclusively instead of lump-sum transfers. Finally, this also suggests that if lump-sum taxes are not allowed and only lump-sum transfers are available, then the Ramsey planner will not use lump-sum transfers at all, which means that the constraint $T_t \geq 0$ must be strictly binding at least in the Ramsey steady state. In other words, using lump-sum transfers financed by a labor/capital tax in the steady state is never optimal.

Proposition 10 provides an explanation for why an FSI Ramsey steady state tends to prevail in our modified Aiyagari model. That is, Proposition 10 indicates that the Ramsey allocation in the absence of lump-sum taxes becomes the first-best allocation with lump-sum taxes and that this result holds for any finite value of $\kappa$. However, as $\kappa$ increases to infinity, then the first-best allocation surely becomes infeasible because the required aggregate asset demand to support FSI approaches infinity. This suggests that in the standard Aiyagari model (where $\kappa = \infty$), even if lump-sum transfers/taxes are available, an interior first-best steady state is never feasible to the benevolent planer. However, the theoretical property of the Ramsey outcome with lump-sum taxes/transfers in a standard Aiyagari model remains an open question (when $\sigma \in (0, \infty)$), which is left for future research.

### 5.2 Scenario 2: Maximizing Steady-State Welfare

In Section 4, we mention that when $Q > \beta$, the Ramsey planner has incentives to increase the bond supply to pursue FSI allocation and frontload consumption even if this implies a close-to-100% labor tax rate in the long run to finance the sky-rocketing public debt-to-GDP ratio. To further support our argument and the intuition behind it, here we conduct a different kind of analysis by supposing that the Ramsey planner maximizes only the steady-state welfare of the competitive equilibrium (as in the works of Aiyagari and McGrattan (1998) and Floden (2001)) instead of the time-zero present value of the dynamic path of social welfare. We will see that the Ramsey planner's design of debt and tax policies in this situation differs fundamentally from that discussed above when the incentive for frontloading consumption is no longer present in a static optimization problem.

To maximize the steady-state welfare of the economy, the Ramsey problem becomes

$$\max_{\{c(h^\kappa), n(h^\kappa), a(h^\kappa), \overline{w}, Q, K\}_{t=0}^{\infty}} \sum_{h^\kappa} \left[ u(c_t(h^\kappa)) - v\left(n_t(h^\kappa)\right) \right] \pi(h^\kappa)$$

subject to

$$F(\sum_{h^\kappa} n(h^\kappa) z(h^\kappa) \pi(h^\kappa), K) - \delta K - \sum_{h^\kappa} c(h^\kappa)\pi(h^\kappa) \geq 0, \tag{31}$$

$$c(h^\kappa) - \overline{w} z(h^\kappa) n(h^\kappa) + Q a(h^\kappa) - \sum_{h^\kappa_{-1}} \frac{a(h^\kappa_{-1})\pi(h^\kappa_{-1})\pi(h^\kappa|h^\kappa_{-1})}{\pi(h^\kappa)} = 0, \forall h_\kappa, \tag{32}$$

$$\chi(h^\kappa) n(h_\kappa) \pi(h_\kappa) : \overline{w} u_c(h^\kappa) z(h^\kappa) - v_n(h^\kappa) = 0, \forall h^\kappa$$

$$\mu^q : \beta \sum_{h^{\kappa\prime}} u_c(h^{\kappa\prime}) \pi(h^{\kappa\prime}|\boldsymbol{h}^\kappa_h) - Q u_c(\boldsymbol{h}^\kappa_h),$$

and for all $h^\kappa \neq \boldsymbol{h}^\kappa_h$,

$$a(h^\kappa) \geq 0, \tag{33}$$

$$g(h^\kappa) \equiv \frac{u_c(h^\kappa)}{\sum_{h^{\kappa\prime}} u_c(h^{\kappa\prime})\pi(h^{\kappa\prime}|h^\kappa)} - \frac{u_c(\boldsymbol{h}^\kappa_h)}{\sum_{h^{\kappa\prime}} u_c(h)\pi(h^{\kappa\prime}|\boldsymbol{h}^\kappa_h)} \geq 0, \tag{34}$$

$$g(h^\kappa) a(h^\kappa) = 0 = 0. \tag{35}$$

Note that this Ramsey problem features no dynamics, as there is no dynamic consideration for the Ramsey planner. In other words, the "future" is no longer "discounted" compared with the "present." We prove that an FSI allocation is *no longer optimal*, as shown in the following proposition:

**Proposition 11.** *If the Ramsey planner cares only about the steady-state welfare of the competitive equilibrium, then an FSI allocation is not optimal regardless of IES, even if FSI is feasible. Instead, it is optimal to set* $c(\boldsymbol{h}_h^{\kappa}) > c(h^{\kappa}) > c(\boldsymbol{h}_l^{\kappa})$ *and have the borrowing constraints of low-income individuals strictly binding.*

*Proof.* See Appendix A.8. □

The result in Proposition 11 holds for all values of $\kappa$ and is thus applicable also to the standard Aiyagari model. This result is intuitive. By maximizing only the steady-state welfare of the competitive economy, the Ramsey planner no longer has the incentive to exploit the difference between the interest rate and the time discount rate, since the time discount rate is no longer relevant in maximizing the steady-state welfare. Consequently, without transitional dynamics the issue of frontloading consumption also becomes irrelevant. In such a case, the Ramsey planner opts not to pursue an FSI allocation by equalizing consumption across employed and unemployed individuals, because the cost of doing so in terms of levying distortionary taxes and issuing too much debt is too high at the margin, where there is no time discounting. In other words, from the viewpoint of the competitive equilibrium's steady-state welfare, the marginal benefit of achieving FSI by increasing public debt is at some point dominated by the marginal cost of distortionary taxation, such that the Ramsey planner will stop issuing bonds at a certain level before FSI is achieved.

# 6 Conclusion

Capital taxation has been a vital source of government revenues in history and is often viewed as a critical means for reducing income/wealth inequality. Yet macroeconomic models had been unable to rationalize this popular practice using representative-agent models until the seminal work of Aiyagari (1995) that broke the ice. Aiyagari (1995) argued that in heterogeneous-agents and incomplete-markets models it is optimal for the government to tax capital—because capital is overaccumulated under precautionary saving motives. Aiyagari's analysis, however, is based on the critical assumption that a Ramsey steady state exists without proof.

In this paper we analyze the Ramsey steady-state conundrum in a class of Aiyagari-type models. We prove that in the standard Aiyagari model the assumption of the existence of an interior Ramsey steady state with convergent Lagrangian multiplier(s) (commonly made in both the theoretical literature and the numerical literature) is incorrect and not innocuous for policy implications. We show instead that if a Ramsey steady state exists at all in the standard Aiyagari model, it must be non-interior if $\sigma \geq 1$; alternatively, if $\sigma < 1$, an interior Ramsey steady state (if it exists) must feature a divergent Ramsey Lagrangian multiplier and the optimal capital tax must be zero or negative.

We then design a tractable version of the Aiyagari model to unveil the mechanisms and conditions behind the various types of possible Ramsey steady states. We find that the conditions for the existence of an interior Ramsey steady state are quite demanding and sensitive to structural parameter values pertaining to the economy's ability to sustain public debt and mitigate idiosyncratic risk. In particular, we prove that an interior Ramsey steady state can exist under certain fiscal-space conditions, but the steady state either features FSI and a zero capital tax (under $\sigma \geq 1$) or the failure of the MGR and a non-positive capital tax (under $\sigma < 1$); both are in sharp contrast to the rationale of using heterogeneous-agents and incomplete-markets models to justify positive capital taxes in the real world. We also prove that if the fiscal-space condition is violated (as in the standard Aiyagari model), the only possible Ramsey steady state is non-interior under normal parameter values for IES or risk aversion (i.e., $\sigma \geq 1$).

The key reasons behind our unconventional results are the following: Because of the arbitrage opportunity created by the gap between the interest rate and the time discount rate in Aiyagari-type models, the Ramsey planner opts to issue a sufficiently large amount of debt to achieve FSI, even at the cost of an extremely high labor tax to finance public debt. If the fiscal space is sufficient and individual saving motives are not too weak ($\sigma \geq 1$), the Ramsey outcome features an interior FSI steady state where no one is borrowing constrained and the optimal capital tax is zero. The optimal capital tax is zero in the steady state because the root cause of any allocative inefficiency (due to incomplete insurance markets) is fully addressed in the FSI allocation by a sufficient supply of public debt—which can be financed fully by a labor tax (as is also the case in representative-agent models). This outcome is in sharp contrast to the interior Ramsey steady state imagined by Aiyagari (1995). On the other hand, if the fiscal space is insufficient, the dominant motive of the Ramsey planner to pursue FSI may lead to an unsustainable amount of government debt, rendering an interior Ramsey steady state non-existent. Hence, the only possible Ramsey steady state in a standard Aiyagari model with a normal IES parameter is non-interior with zero aggregate consumption and a 100% labor tax rate in the limit. However, when individual saving motives are too weak (under $\sigma < 1$), the Ramsey planner's intention to frontload consumption cannot be supported by a rapidly rising debt level, due to individuals' weak asset demand for government debt; it is then optimal for the Ramsey planner to encourage consumption in the long run, leading to an interior Ramsey steady state; but in such a case the optimal capital tax must be non-positive and the MGR must fail—still in sharp contrast to the interior Ramsey steady state imagined by Aiyagari.

Therefore, our analysis not only suggests that Aiyagari-type models cannot rationalize positive capital taxation in the real world but also that any result obtained in the heterogeneous-agents literature under the popular practice of assuming the existence of an interior Ramsey steady state without proof could be dubious and needs to be interpreted with caution.

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# A Appendix

## A.1 Proof of Proposition 2

Given that individuals have identical initial wealth and that $\boldsymbol{\theta}_h^t$ is the best possible path of idiosyncratic shock, then it must be the case that,

$$a_{t+1}(\boldsymbol{\theta}_h^t) > a_{t+1}(\theta^t) \geq 0 \text{ for all } t \geq 0 \text{ and } \theta^t \neq \boldsymbol{\theta}_h^t,$$

which implies that the associated multipliers satisfy $\psi_t(\boldsymbol{\theta}_h^t) = 0$ and $\psi_t(\theta^t) \geq 0$. This result together with equations (9) and (7) lead to equation (12).

Also, it is well know that in any steady state of a competitive equilibrium in the standard Aiyagari model, it must be the case that $Q > \beta$. Otherwise, the individual's asset demand goes to infinity, which cannot constitute a competitive equilibrium. For more details, see Aiyagari (1994) and Ljungqvist and Sargent (2012).

## A.2 Proof of Proposition 3

### A.2.1 The "If" Part:

Given the initial values of $(B_0, K_0, a_0, \tau_{k,0})$, the individual share's allocation $\{\boldsymbol{c}_t^s(\theta^t), \boldsymbol{n}_t^s(\theta^t), \boldsymbol{a}_{t+1}^s(\theta^t)\}_{t=0}^{\infty}$, and aggregate allocation $\{K_{t+1}, N_t, C_t, A_{t+1}\}_{t=0}^{\infty}$, a competitive equilibrium can be constructed by using the conditions in Proposition 3 and by following the steps below to uniquely back up the sequences of the other prices and tax variables:

1. $w_t$ and $q_t$ are determined by $w_t = MPN_t$ and $q_t = MPK_t$, respectively.

2. $Q_{t+1} \equiv \frac{1}{r_{t+1}}$ is determined by

$$\frac{1}{r_{t+1}} = Q_{t+1} = \beta \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \sum_{\theta_{t+1}} \left(\frac{\boldsymbol{c}_{t+1}^s(\theta^{t+1})}{\boldsymbol{c}_t^s(\boldsymbol{\theta}_h^t)}\right)^{-\sigma} \pi(\theta_{t+1}|\boldsymbol{\theta}_h^t) \text{ for all } t > 0. \tag{36}$$

3. $\tau_{n,t}$ is determined by

$$\widehat{w}_t = (1 - \tau_{n,t}) MPN_t = \frac{v_{n,t}(\boldsymbol{\theta}_h^t)}{u_{c,t}(\boldsymbol{\theta}_h^t) z_t(\boldsymbol{\theta}_h^t)} = \frac{N_t^{\gamma} (\boldsymbol{n}_t^s(\boldsymbol{\theta}_h^t))^{\gamma}}{C_t^{-\sigma} (\boldsymbol{c}_t^s(\boldsymbol{\theta}_h^t))^{-\sigma}} \frac{1}{(z_t(\boldsymbol{\theta}_h^t))^{\gamma+1}}. \tag{37}$$

4. $\tau_{k,t}$ is determined by

$$\frac{1}{Q_t} = 1 + (1 - \tau_{k,t})MPK_t - \delta$$

for all $t \geq 0$.

5. $B_{t+1}$ is pinned down by the asset market-clearing condition

$$B_{t+1} + \frac{K_{t+1}}{Q_{t+1}} = A_{t+1}, \text{ for all } t \geq 0.$$

6. The following constraints are satisfied:

   (a) The implementability conditions, displayed in equation (15), can be derived by plugging equations (37) and (36) into the household budget constraints.

   (b) The resource constraint is listed in equation (14).

   (c) The individual FOCs (8) are listed in equation (17).

   (d) The asset constraint in period 0 is as shown in equation (16).

   (e) The individual FOCs (9) and borrowing constraints, which are listed in equation (18) as constraints in Proposition 3.

   (f) The aggregation conditions are as listed in equations (19) to (21).

7. Finally, it is straightforward to verify that the sum of all implementability conditions together with the aggregate resource constraint imply the government budget constraint.

### A.2.2 The "Only If" Part:

The constraints listed in Proposition 3 are trivially satisfied because they are part of the competitive-equilibrium conditions.

## A.3 Proof of Proposition 5

In what follows, we prove that under the parameter condition $\sigma \geq 1$,there is no interior Ramsey steady state with $1 > Q > \beta$. The proof is done by contradiction. Assume there exists an interior Ramsey steady state with $1 > Q > \beta$, and consider two possible cases depending on the convergence of $\mu_t$:

1. Suppose $\mu_t$ converges.

   (a) The FOC with respect to $N_t$ in equation (23) can be rewritten as

$$\begin{aligned}
& \frac{\beta^t}{Q^t} N_t^\gamma \sum_{\theta^t} \left( \frac{\boldsymbol{n}_t^s(\theta^t)}{z_t^s(\theta^t)} \right)^{1+\gamma} \pi(\theta^t) + (1+\gamma) N_t^\gamma \frac{(\boldsymbol{n}_t^s(\boldsymbol{\theta}_h^t))^\gamma}{(\boldsymbol{c}_t^s(\boldsymbol{\theta}_h^t))^{-\sigma} z_t(\boldsymbol{\theta}_h^t)^{1+\gamma}} \sum_{\theta^t} \lambda_t(\theta^t) \boldsymbol{n}_t^s(\theta^t) \pi(\theta^t) \\
= \; & \frac{\beta^t}{Q^t} \mu_t MPN_t.
\end{aligned}$$

   Under $Q > \beta$, it must be true that $\lim_{t\to\infty} \frac{\beta^t}{Q^t} = 0$, so the above equation in the limit becomes

$$(1+\gamma) N^\gamma \lim_{t\to\infty} \frac{(\boldsymbol{n}_t^s(\boldsymbol{\theta}_h^t))^\gamma}{(\boldsymbol{c}_t^s(\boldsymbol{\theta}_h^t))^{-\sigma} z_t(\boldsymbol{\theta}_h^t)^{1+\gamma}} \sum_{\theta^t} \lambda_t(\theta^t) \boldsymbol{n}_t^s(\theta^t) \pi(\theta^t) = 0,$$

   which must imply $\lim_{t\to\infty} \sum_{\theta^t} \lambda_t(\theta^t) \boldsymbol{n}_t^s(\theta^t) \pi(\theta^t) = 0$. Also, the same FOC with respect to $N_t$ can be written in the following way:

$$\begin{aligned}
& \lim_{t\to\infty} N_t^\gamma \sum_{\theta^t} \left( \frac{\boldsymbol{n}_t^s(\theta^t)}{z_t^s(\theta^t)} \right)^{1+\gamma} \pi(\theta^t) \\
& + (1+\gamma) \lim_{t\to\infty} \frac{Q^t}{\beta^t} N_t^\gamma \frac{(\boldsymbol{n}_t^s(\boldsymbol{\theta}_h^t))^\gamma}{(\boldsymbol{c}_t^s(\boldsymbol{\theta}_h^t))^{-\sigma} z_t(\boldsymbol{\theta}_h^t)^{1+\gamma}} \sum_{\theta^t} \lambda_t(\theta^t) \boldsymbol{n}_t^s(\theta^t) \pi(\theta^t) \\
= \; & \lim_{t\to\infty} \mu_t MPN_t,
\end{aligned}$$

   which implies that not only $\sum_{\theta^t} \lambda_t(\theta^t) \boldsymbol{n}_t^s(\theta^t) \pi(\theta^t)$ converges to zero but also the speed at which $\sum_{\theta^t} \lambda_t(\theta^t) \boldsymbol{n}_t^s(\theta^t) \pi(\theta^t)$ converges to zero in the limit must be greater than $\frac{Q}{\beta} > 1$ (or its convergent rate be lower than $\frac{\beta}{Q} < 1$ in limit) to overcome the explosive growth in $\frac{Q^t}{\beta^t}$, otherwise the second term on the left-hand side of the above equation would approach infinity.

   (b) The Ramsey FOC with respect to $A_{t+1}$ is given by

$$C_t^{-\sigma} \sum_{\theta^t} \lambda_t(\theta^t) \boldsymbol{a}_{t+1}^s(\theta^t) \pi(\theta^t) - C_{t+1}^{-\sigma} \sum_{\theta^{t+1}} \lambda_{t+1}(\theta^{t+1}) \boldsymbol{a}_{t+1}^s(\theta^t) \pi(\theta^{t+1}) = 0.$$

In the limit, since $\lim_{t\to\infty} C_t = C > 0$, the above equation can be written as

$$\lim_{t\to\infty} \frac{\sum_{\theta^{t+1}} \lambda_{t+1}(\theta^{t+1})\boldsymbol{a}^s_{t+1}(\theta^t)\pi(\theta^{t+1})}{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}^s_{t+1}(\theta^t)\pi(\theta^t)} = \lim_{t\to\infty} \left(\frac{C_t}{C_{t+1}}\right)^{-\sigma} = 1. \tag{38}$$

Given this, we must have

$$\lim_{t\to\infty} \sum_{\theta^{t+1}} \lambda_{t+1}(\theta^{t+1})\boldsymbol{a}^s_{t+1}(\theta^t)\pi(\theta^{t+1}) = \lim_{t\to\infty} \sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}^s_t(\theta^{t-1})\pi(\theta^t),$$

where the right-hand side is simply the left-hand side lagged by one period. Therefore, the above two equations imply

$$\lim_{t\to\infty} \sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}^s_{t+1}(\theta^t)\pi(\theta^t) = \lim_{t\to\infty} \sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}^s_t(\theta^{t-1})\pi(\theta^t),$$

which together with equation (38) gives

$$\lim_{t\to\infty} \frac{\sum_{\theta^{t+1}} \lambda_{t+1}(\theta^{t+1})\boldsymbol{a}^s_{t+1}(\theta^t)\pi(\theta^{t+1})}{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}^s_{t+1}(\theta^t)\pi(\theta^t)} = \lim_{t\to\infty} \frac{\sum_{\theta^{t+1}} \lambda_{t+1}(\theta^{t+1})\boldsymbol{a}^s_{t+1}(\theta^t)\pi(\theta^{t+1})}{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}^s_t(\theta^{t-1})\pi(\theta^t)} = 1.$$

Namely, the convergent rate of $\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}^s_t(\theta^{t-1})\pi(\theta^t)$ approaches 1 as $t \to \infty$.

(c) The Ramsey FOC with respect to $C_t$ can be written (after combining with the limiting Ramsey FOC with respect to $A_{t+1}$) as

$$\lim_{t\to\infty} C_t^{-\sigma} \sum_{\theta^t} (\boldsymbol{c}^s_t)^{1-\sigma}\pi(\theta^t) + (1-\sigma) \lim_{t\to\infty} \frac{Q^t}{\beta^t} C_t^{-\sigma} \sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{c}^s_t(\theta^t)\pi(\theta^t) = \lim_{t\to\infty} \mu_t,$$

which implies that for $\sigma \neq 1$, not only $\lim_{t\to\infty} \sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{c}^s_t(\theta^t)\pi(\theta^t)$ converges to zero but also the speed at which $\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{c}^s_t(\theta^t)\pi(\theta^t)$ approaches zero must be greater than $\frac{Q}{\beta} > 1$ (or its growth rate be lower than $\frac{\beta}{Q} < 1$), otherwise the second term on the left-hand side of the above equation will explode to infinity.

(d) Consider the household budget constraint for type-$\theta^t$ individual:

$$\boldsymbol{c}^s_t(\theta^t)C_t - \widehat{w}_t N_t \boldsymbol{n}^s_t(\theta^t) = A_t \boldsymbol{a}^s_t(\theta^{t-1}) - Q_{t+1} A_{t+1} \boldsymbol{a}^s_{t+1}(\theta^t).$$

Multiplying all terms in the above equation by $\lambda_t(\theta^t)\pi(\theta^t)$ for each type-$\theta^t$ individual and integrating over $\theta^t$ give

$$
\begin{aligned}
& C_t \sum_{\theta^t} \lambda_t(\theta^t) \boldsymbol{c}_t^s(\theta^t)\pi(\theta^t) - \widehat{w}_t N_t \sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{n}_t^s(\theta^t)\pi(\theta^t) \\
= \;& A_t \sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t) - Q_{t+1}A_{t+1}\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_{t+1}^s(\theta^t)\pi(\theta^t),
\end{aligned} \tag{39}
$$

which (after dividing each term on both sides by $\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t)$) leads to

$$
\begin{aligned}
& C_t \frac{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{c}_t^s(\theta^t)\pi(\theta^t)}{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t)} - \widehat{w}_t N_t \frac{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{n}_t^s(\theta^t)\pi(\theta^t)}{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t)} \\
= \;& A_t \left[ 1 - \frac{Q_{t+1}A_{t+1}\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}}}{A_t} \frac{\sum_{\theta^{t+1}} \lambda_t(\theta^{t+1})\boldsymbol{a}_{t+1}^s(\theta^t)\pi(\theta^{t+1})}{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t)} \right].
\end{aligned} \tag{40}
$$

Notice that $\lim_{t\to\infty} \frac{\sum_{\theta^{t+1}} \lambda_t(\theta^{t+1})\boldsymbol{a}_{t+1}^s(\theta^t)\pi(\theta^{t+1})}{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t)} = \lim_{t\to\infty} \frac{\sum_{\theta^{t+1}} \lambda_{t+1}(\theta^{t+1})\boldsymbol{a}_{t+1}^s(\theta^t)\pi(\theta^{t+1})}{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t)} = 1$ by the FOC with respect to $A_{t+1}$, and that $\lim_{t\to\infty} \frac{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{n}_t^s(\theta^t)\pi(\theta^t)}{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t)} = 0$ since the convergent rate of $\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{n}_t^s(\theta^t)\pi(\theta^t)$ to zero is faster than that of $\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t)$. Hence, assuming an interior Ramsey steady state in the limit, the equation above simplifies to

$$
\lim_{t\to\infty} \frac{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{c}_t^s(\theta^t)\pi(\theta^t)}{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t)} = \frac{A}{C}(1-Q). \tag{41}
$$

That is, the numerator and the denominator must have the same convergence rate. Now consider the following cases:

i. If $\sigma \neq 1$, we know that $\lim_{t\to\infty} \frac{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{c}_t^s(\theta^t)\pi(\theta^t)}{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t)} = 0$ since the convergent rate of $\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{c}_t^s(\theta^t)\pi(\theta^t)$ to zero is faster than that of $\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t)$, then equation (41) implies that $Q = 1$, a contradiction.

ii. If $\sigma = 1$ and suppose $\lim_{t\to\infty} \frac{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{c}_t^s(\theta^t)\pi(\theta^t)}{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t)} = 0$, then equation (41) implies that $Q = 1$, a contradiction.

iii. If $\sigma = 1$ and suppose $\lim_{t\to\infty} \frac{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{c}_t^s(\theta^t)\pi(\theta^t)}{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t)} = +\infty$ or $-\infty$, then equation (41) suggests that $(1-Q) = +\infty$ or $-\infty$, also a contradiction.

iv. If $\sigma = 1$ and suppose $\frac{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{c}_t^s(\theta^t)\pi(\theta^t)}{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t)}$ converges to a finite value, then it must be the case that the convergent rate of $\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{c}_t^s(\theta^t)\pi(\theta^t)$ is the same as that of $\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t)$, which is slower than the convergence rate of

$\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{n}_t^s(\theta^t)\pi(\theta^t)$; namely, we must have the following inequalities in the limiting growth rates (or relationship in the convergence rates):

$$\begin{aligned}
& \lim_{t\to\infty} \frac{\sum_{\theta^{t+1}} \lambda_{t+1}(\theta^{t+1})\boldsymbol{n}_{t+1}^s(\theta^t)\pi(\theta^{t+1})}{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{n}_t^s(\theta^{t-1})\pi(\theta^t)} \leq \frac{\beta}{Q} < 1 \\
= & \lim_{t\to\infty} \frac{\sum_{\theta^{t+1}} \lambda_{t+1}(\theta^{t+1})\boldsymbol{a}_{t+1}^s(\theta^t)\pi(\theta^{t+1})}{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t)} = \lim_{t\to\infty} \frac{\sum_{\theta^t} \lambda_{t+1}(\theta^t)\boldsymbol{c}_{t+1}^s(\theta^{t+1})\pi(\theta^{t+1})}{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{c}_t^s(\theta^t)\pi(\theta^t)}.
\end{aligned}$$

We prove below that this inequality contains a contradiction. Equation (40) implies

$$\begin{aligned}
& \widehat{w}_t N_t \sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{n}_t^s(\theta^t)\pi(\theta^t) \qquad (42) \\
= & \; C_t \sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{c}_t^s(\theta^t)\pi(\theta^t) - \left[ \begin{array}{c} A_t \sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t) \\ -Q_{t+1}A_{t+1} \sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_{t+1}^s(\theta^t)\pi(\theta^t) \end{array} \right],
\end{aligned}$$

which can be used to form the following limiting relationship in growth rates after dividing both sides of the equation by its own corresponding one-period lag:

$$\begin{aligned}
& \lim_{t\to\infty} \frac{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{n}_t^s(\theta^t)\pi(\theta^t)}{\sum_{\theta^{t-1}} \lambda_{t-1}(\theta^{t-1})\boldsymbol{n}_{t-1}^s(\theta^{t-1})\pi(\theta^{t-1})} \\
= & \frac{\begin{array}{c} C \lim_{t\to\infty} \sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{c}_t^s(\theta^t)\pi(\theta^t) \\ -A(1-Q) \lim_{t\to\infty} \sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t) \end{array}}{\begin{array}{c} C \lim_{t\to\infty} \sum_{\theta^{t-1}} \lambda_{t-1}(\theta^t)\boldsymbol{c}_{t-1}^s(\theta^{t-1})\pi(\theta^{t-1}) \\ -A(1-Q) \lim_{t\to\infty} \sum_{\theta^{t-1}} \lambda_{t-1}(\theta^{t-1})\boldsymbol{a}_{t-1}^s(\theta^{t-2})\pi(\theta^{t-1}) \end{array}}.
\end{aligned}$$

Clearly, as shown before the LHS of the above equation cannot be greater than $\frac{\beta}{Q}$; however, the RHS of the above equation approaches 1, thus leading to a contradiction. To see this, rewrite the RHS (after factoring out $A\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t)$ in the numerator and its one-period lag in the denominator) as

$$\frac{\lim_{t\to\infty} A\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t) \left( \frac{C\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{c}_t^s(\theta^t)\pi(\theta^t)}{A\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t)} - [1-Q] \right)}{\lim_{t\to\infty} A\sum_{\theta^{t-1}} \lambda_{t-1}(\theta^{t-1})\boldsymbol{a}_{t-1}^s(\theta^{t-2})\pi(\theta^{t-1}) \left( \frac{C\sum_{\theta^{t-1}} \lambda_{t-1}(\theta^{t-1})\boldsymbol{c}_{t-1}^s(\theta^{t-1})\pi(\theta^{t-1})}{A\sum_{\theta^{t-1}} \lambda_{t-1}(\theta^{t-1})\boldsymbol{a}_{t-1}^s(\theta^{t-2})\pi(\theta^{t-1})} - [1-Q] \right)},$$

which equals to 1 in the limit because $\lim_{t\to\infty} \frac{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{c}_t^s(\theta^t)\pi(\theta^t)}{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t)}$ converges to a finite value and $\lim_{t\to\infty} \frac{\sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_t^s(\theta^{t-1})\pi(\theta^t)}{\sum_{\theta^{t-1}} \lambda_{t-1}(\theta^{t-1})\boldsymbol{a}_{t-1}^s(\theta^{t-2})\pi(\theta^{t-1})} = 1$.

2. Suppose $\mu_t$ diverges to infinity at a certain rate such that $\frac{\beta^t}{Q^t}\mu_t$ does not converge to zero. Then the following occur:

   (a) The Ramsey FOC with respect to $N_t$ in the limit becomes

$$(1+\gamma)N_t^\gamma \lim_{t\to\infty} \frac{(\boldsymbol{n}_t^s(\boldsymbol{\theta}_h^t))^\gamma}{(\boldsymbol{c}_t^s(\boldsymbol{\theta}_h^t))^{-\sigma} z_t(\boldsymbol{\theta}_h^t)^{1+\gamma}} \sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{n}_t^s(\theta^t)\pi(\theta^t) = MPN \lim_{t\to\infty} \frac{\beta^t}{Q^t}\mu_t > 0.$$

   (b) The Ramsey FOC with respect to $A_{t+1}$ becomes

$$C_t^{-\sigma} \sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_{t+1}^s(\theta^t)\pi(\theta^t) - C_{t+1}^{-\sigma} \sum_{\theta^{t+1}} \lambda_{t+1}(\theta^{t+1})\boldsymbol{a}_{t+1}^s(\theta^t)\pi(\theta^{t+1}) = 0,$$

   and hence in the limit

$$\lim_{t\to\infty} \sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{a}_{t+1}^s(\theta^t)\pi(\theta^t) - \lim_{t\to\infty} \sum_{\theta^{t+1}} \lambda_{t+1}(\theta^{t+1})\boldsymbol{a}_{t+1}^s(\theta^t)\pi(\theta^{t+1}) = 0.$$

   Plugging the above equation into the steady-state version of the FOC with respect to $C_t$ leads to

$$\lim_{t\to\infty} \frac{\beta^t}{Q^t} \sum_{\theta^t} (\boldsymbol{c}_t^s)^{1-\sigma}\pi(\theta^t) + \lim_{t\to\infty}(1-\sigma) \sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{c}_t^s(\theta^t)\pi(\theta^t) = \lim_{t\to\infty} \frac{\beta^t}{Q^t}\mu_t C_t^\sigma,$$

   Under the assumption $Q > \beta$, the equation above becomes

$$(1-\sigma)\lim_{t\to\infty} \sum_{\theta^t} \lambda_t(\theta^t)\boldsymbol{c}_t^s(\theta^t)\pi(\theta^t) = \lim_{t\to\infty} \frac{\beta^t}{Q^t}\mu_t C_t^\sigma > 0,$$

   which leads to a contradiction if $\sigma \geq 1$ because the left-hand side of the above equation is non-positive.

3. The above proof also indicates that if $\sigma < 1$, then it may be possible to have an interior Ramsey steady state provided that the multiplier $\mu_t$ diverges fast enough. In particular, denote the steady-state growth rate of $\mu_t$ as $g_\mu$. For an interior Ramsey steady state to exist under $\sigma < 1$, a necessary condition is that $g_\mu \geq \frac{Q}{\beta} > 1$. Under this necessary condition, the

steady-state FOC with respect to $K_{t+1}$ can be written as

$$1 = \beta g_\mu \left( MPK + 1 - \delta \right),$$

which implies that the MGR fails to hold. In addition, comparing the above equation to equation (10) suggests that the steady-state capital tax is zero if $g_\mu = \frac{Q}{\beta}$ and negative if $g_\mu > \frac{Q}{\beta}$. Hence, it must be true that $\tau_k \leq 0$ if an interior Ramsey steady state exists under $\sigma < 1$. However, without studying all of the Ramsey FOCs, we cannot prove that such an interior Ramsey steady state with a divergent multiplier under $\sigma < 1$ necessarily exists.

4. Finally, the above proof also indicates that under $\sigma \geq 1$, the only possible Ramsey steady state (if it exists) must be a non-interior one, which (if it exists) must feature $C_t \to 0$. Again, without studying all of the Ramsey FOCs we cannot prove that such a non-interior Ramsey steady state necessarily exists.

## A.4 Proof of Proposition 6

Given that $a_0(\boldsymbol{h}_h^\kappa) > a_0(h^\kappa)$ and the assumption that the autocorrelation of the shock process is non-negative, then it must be the case that

$$a_{t+1}(\boldsymbol{h}_h^\kappa) > a_{t+1}(h^\kappa) \geq 0 \text{ for all } t \geq 0 \text{ and } h^\kappa \neq \boldsymbol{h}_h^\kappa,$$

which implies that the associated multipliers on the borrowing constraints satisfy $\psi_t(\boldsymbol{h}_h^\kappa) = 0$ and $\psi_t(h^\kappa) \geq 0$. This result together with the household FOCs with respect to $a_{t+1}(h^\kappa)$ and $c_t(h^\kappa)$ leads to equation (29).

Given such a wealth-pooling technology, the individual steady-state asset demand may remain finite even if $Q = \beta$. This fact opens the possibility that a FSI steady state may exist in such an economy if the level of aggregate assets is sufficiently large. In such a FSI steady state where $\psi(h^\kappa) = 0$ for all $h^\kappa$, the steady-state version of equation (29) must hold with equality for all $h^\kappa$, which implies that (i) $Q = \beta$ and (ii) $c(\boldsymbol{h}_h^\kappa) = c(h^\kappa)$ for all $h^\kappa$.

## A.5 Proof of Proposition 7

### A.5.1 Ramsey Problem

Denote $\beta^t \mu_t$, $\lambda_0(h^0)$, $\beta^t \lambda_t(h^\kappa)$, $\beta^t \chi_t(h^\kappa)$, $\zeta^0$, $\beta^t \mu_t^q$, $\beta^{t+1}\zeta_t^1(h^\kappa)$, $\beta^{t+1}\zeta_t^2(h^\kappa)$, and $\beta^{t+1}\zeta_t^3(h^\kappa)$ as the Lagrangian multipliers for the associated constraints listed below. It is straightforward to verify that the Ramsey planner's problem can be written as

$$\max_{\{c_t(h^\kappa), n_t(h^\kappa), a_{t+1}(h^\kappa), \overline{w}_t, Q_{t+1}, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \sum_{h^\kappa} \left[ u(c_t(h^\kappa)) - v\left(n_t(h^\kappa)\right) \right] \pi(h^\kappa)$$

subject to

$$\beta^t \mu_t : F(\sum_{h^\kappa} n_t(h^\kappa) z_t(h^\kappa) \pi(h^\kappa), K_t) + (1-\delta)K_t - \sum_{h^\kappa} c_t(h^\kappa)\pi(h^\kappa) - K_{t+1} \geq 0 \ \forall t \geq 0,$$

$$\lambda_0(h^0)\pi(h^0) : c_0(h^\kappa) - \overline{w}_0 z_0(h^\kappa) n_0(h^\kappa) + Q_1 a_1(h^\kappa) - a_0(h^\kappa) = 0 \ \forall h^\kappa,$$

$$\begin{aligned} \beta^t \lambda_t(h^\kappa)\pi(h^\kappa) \quad &: \quad c_t(h^\kappa) - \overline{w}_t z_t(h^\kappa) n_t(h^\kappa) + Q_{t+1} a_{t+1}(h^\kappa) - \sum_{h_{-1}^\kappa} \frac{a_t(h_{-1}^\kappa)\pi(h_{-1}^\kappa)\pi(h^\kappa | h_{-1}^\kappa)}{\pi(h^\kappa)} \\ &= \quad 0 \ \forall h_\kappa \text{ and } t \geq 1, \end{aligned}$$

$$\beta^t \chi_t(h^\kappa) n_t(h_\kappa)\pi(h_\kappa) : \overline{w}_t u_{c,t}(h^\kappa) z_t(h^\kappa) - v_{n,t}(h^\kappa) = 0 \ \forall h^\kappa \text{ and } t \geq 0,$$

$$\zeta^0 : (1 + (1-\tau_{k,0})MPK_0 - \delta)K_0 + B_0 - \sum_{h_\kappa} a_0(h_\kappa) = 0,$$

$$\beta^t \mu_t^q : \beta \sum_{h^{\kappa'}} u_{c,t+1}(h^{\kappa'})\pi(h^{\kappa'} | \boldsymbol{h}_h^\kappa) - Q_{t+1} u_{c,t}(\boldsymbol{h}_h^\kappa) \ \forall t \geq 0$$

and for all $h^\kappa \neq \boldsymbol{h}_h^\kappa$,

$$\beta^{t+1}\zeta_t^1(h^\kappa) : a_{t+1}(h^\kappa) \geq 0,$$

$$\beta^{t+1}\zeta_t^2(h^\kappa) : g_t(h^\kappa) \geq 0,$$

$$\beta^{t+1}\zeta_t^3(h^\kappa) : g_t(h^\kappa) a_{t+1}(h^\kappa) = 0,$$

where $g_t(h^\kappa)$ is defined as

$$g_t(h^\kappa) \equiv \frac{u_{c,t}(h^\kappa)}{\sum_{h^{\kappa'}} u_{c,t+1}(h^{\kappa'})\pi(h^{\kappa'}|h^\kappa)} - \frac{u_{c,t}(\boldsymbol{h}_h^\kappa)}{\sum_{h^{\kappa'}} u_{c,t+1}(h)\pi(h^{\kappa'}|\boldsymbol{h}_h^\kappa)}.$$

### A.5.2 Ramsey FOCs

For all $t \geq 0$, the FOCs of the Ramsey problem with respect to $K_{t+1}, \overline{w}_t, Q_{t+1}, a_{t+1}(\boldsymbol{h}_h^\kappa)$, and $a_{t+1}(h^\kappa)$ are given, respectively, by

$$\mu_t = \beta\mu_{t+1}\left(MP_{K,t+1} + 1 - \delta\right),$$

$$\sum_{h_\kappa} \lambda_t(h^\kappa) z_t(h^\kappa) n_t(h^\kappa)\pi(h^\kappa) = \sum_{h_\kappa} \chi_t(h^\kappa) u_{c,t}(h^\kappa) z_t(h^\kappa) n_t(h_\kappa)\pi(h_\kappa), \tag{43}$$

$$\sum_{h^\kappa} \lambda_t(h^\kappa) a_{t+1}(h^\kappa)\pi(h^\kappa) = \mu_t^q u_{c,t}(\boldsymbol{h}_h^\kappa), \tag{44}$$

$$\lambda_t(\boldsymbol{h}_h^\kappa) Q_{t+1} = \beta \sum_{h^{\kappa'}} \lambda_{t+1}(h^{\kappa'})\pi(h^{\kappa'}|\boldsymbol{h}_h^\kappa), \tag{45}$$

and

$$\lambda_t(h_\kappa) Q_{t+1} = \beta \sum_{h^{\kappa'}} \lambda_{t+1}(h^{\kappa'})\pi(h^{\kappa'}|h^\kappa) + \zeta_t^1(h^\kappa) + \zeta_t^3(h^\kappa) g_t(h^\kappa). \tag{46}$$

For all $t \geq 1$, the FOCs of the Ramsey problem with respect to $n_t(h^\kappa)$, $c_t(\boldsymbol{h}_h^\kappa)$, and $c_t(h^\kappa)$ are given, respectively, by

$$v_{n,t}(h^\kappa) + \lambda_t(h^\kappa)\overline{w}_t z_t(h^\kappa) + \chi_t(h^\kappa) n_t(h_\kappa) v_{nn,t}(h^\kappa) = \mu_t MPN_t z_t(h^\kappa) \tag{47}$$

$$\begin{aligned} &u_{c,t}(\boldsymbol{h}_h^\kappa) + \lambda_t(\boldsymbol{h}_h^\kappa) + \chi_t(\boldsymbol{h}_h^\kappa) n_t(\boldsymbol{h}_h^\kappa)\overline{w}_t u_{cc,t}(\boldsymbol{h}_h^\kappa) z_t(\boldsymbol{h}_h^\kappa) - Q_{t+1}\mu_t^q u_{cc,t}(\boldsymbol{h}_h^\kappa) \\ &+\mu_{t-1}^q u_{cc,t}(\boldsymbol{h}_h^\kappa)\pi(\boldsymbol{h}_h^\kappa|\boldsymbol{h}_h^\kappa) + \beta \sum_{h^\kappa \neq \boldsymbol{h}_h^\kappa} \zeta_t^2(h^\kappa)\frac{\partial g_t(h^\kappa)}{\partial c_t(\boldsymbol{h}_h^\kappa)} + \sum_{h^\kappa \neq \boldsymbol{h}_h^\kappa} \zeta_{t-1}^2(h^\kappa)\frac{\partial g_{t-1}(h^\kappa)}{\partial c_t(\boldsymbol{h}_h^\kappa)} \\ &+\beta \sum_{h^\kappa \neq \boldsymbol{h}_h^\kappa} a_{t+1}(h^\kappa)\zeta_t^3(h^\kappa)\frac{\partial g_t(h^\kappa)}{\partial c_t(\boldsymbol{h}_h^\kappa)} + \sum_{h^\kappa \neq \boldsymbol{h}_h^\kappa} a_t(h^\kappa)\zeta_{t-1}^3(h^\kappa)\frac{\partial g_{t-1}(h^\kappa)}{\partial c_t(\boldsymbol{h}_h^\kappa)} = \mu_t, \end{aligned} \tag{48}$$

$$
\begin{aligned}
& u_{c,t}(h^\kappa) + \lambda_t(h^\kappa) + \chi_t(h^\kappa) n_t(h_\kappa)\overline{w}_t u_{cc,t}(h^\kappa) z_t(h^\kappa) + \mu^q_{t-1} u_{cc,t}(h^\kappa)\pi(h^\kappa|\boldsymbol{h}^\kappa_h) \\
& +\beta \sum_{h^\kappa \neq \boldsymbol{h}^\kappa_h} \zeta^2_t(h^\kappa)\frac{\partial g_t(h^\kappa)}{\partial c_t(h^\kappa)} + \sum_{h^\kappa \neq \boldsymbol{h}^\kappa_h} \zeta^2_{t-1}(h^\kappa)\frac{\partial g_{t-1}(h^\kappa)}{\partial c_t(h^\kappa)} \\
& +\beta \sum_{h^\kappa \neq \boldsymbol{h}^\kappa_h} a_{t+1}(h^\kappa)\zeta^3_t(h^\kappa)\frac{\partial g_t(h^\kappa)}{\partial c_t(h^\kappa)} + \sum_{h^\kappa \neq \boldsymbol{h}^\kappa_h} a_t(h^\kappa)\zeta^3_{t-1}(h^\kappa)\frac{\partial g_{t-1}(h^\kappa)}{\partial c_t(h^\kappa)} = \mu_t.
\end{aligned} \tag{49}
$$

### A.5.3 Existence of an FSI Interior Ramsey Steady State

By the following steps, we conjecture and verify that there exists an FSI interior Ramsey steady state featuring (i) $c(\boldsymbol{h}^\kappa_h) = c(h^\kappa) > 0$ for all $h^\kappa$, (ii) $a(\boldsymbol{h}^\kappa_h) > a(h^\kappa) > a(\boldsymbol{h}^\kappa_l) = 0$ for all $h^\kappa \neq \boldsymbol{h}^\kappa_h$, (iii) $\zeta^1(h^\kappa) = 0$ for all $h^\kappa$, (v) $Q = \beta$, and (vi) $0 < \mu < \infty$. In the following steps, we show that an interior Ramsey steady state allocation can satisfy all of the Ramsey FOCs and the constraints in the Ramsey problem.

1. Consider the steady-state Ramsey allocation with $a(\boldsymbol{h}^\kappa_l) = 0$. The steady-state allocation $(c(\boldsymbol{h}^\kappa_h), n(h^\kappa), \{a(h^\kappa)\}_{h^\kappa \neq \boldsymbol{h}^\kappa_l}, K)$ can be solved by the following steady-state equations. Note that the number of unknowns is equal to the number of equations, which is $2^{\kappa+1} + 1$.

   (a) The FOC with respect to $K_{t+1}$ in the steady state is (1 equation)

   $$1 = \beta\left(MPK + 1 - \delta\right).$$

   (b) The steady state resource constraint is (1 equation)

   $$F\Big(\sum_{h^\kappa} n(h^\kappa) z(h^\kappa)\pi(h^\kappa), K\Big) - \delta K = c(\boldsymbol{h}^\kappa_h),$$

   (c) Given $c(\boldsymbol{h}^\kappa_h) = c(h^\kappa)$, the steady-state household FOCs with respect to labor are given by ($2^{\kappa+1} - 1$ equations)

   $$v_n(\boldsymbol{h}^\kappa_h)\frac{z(h^\kappa)}{z(\boldsymbol{h}^\kappa_h)} = v_n(h^\kappa)$$

   (d) The implementability conditions of each type-$h_\kappa$agent ($2^{\kappa+1}$ equations) are given by

   $$c(h_\kappa) - \frac{v_n(\boldsymbol{h}^\kappa_h)}{u_c(\boldsymbol{h}^\kappa_h)}\frac{z(h^\kappa)}{z(\boldsymbol{h}^\kappa_h)} n(h^\kappa) + \beta a(h_\kappa) = \sum_{h^\kappa_{-1}} \frac{a(h^\kappa_{-1})\pi(h^\kappa_{-1})\pi(h^\kappa|h^\kappa_{-1})}{\pi(h^\kappa)}.$$

2. Note that this Ramsey steady-state allocation satisfies all constraints of the Ramsey problem by construction. We then show that the Lagrangian multipliers of the Ramsey problem can be correctly solved such that all Ramsey FOCs are satisfied:

   (a) Set $\zeta^1(h^\kappa) = 0$ for all $h^\kappa$.

   (b) Given $Q = \beta$, $\zeta^1(h^\kappa) = 0$, and $g(h^\kappa) = 0$, the steady-state version of the Ramsey FOCs with respect to $a_{t+1}(h_\kappa)$, equations (45) and (46), can be rewritten as

$$\lambda(h^\kappa) = \sum_{h^{\kappa'}} \lambda(h^{\kappa'})\pi(h^{\kappa'}|h^\kappa),$$

   which can be satisfied only if $\lambda(\boldsymbol{h}_h^\kappa) = \lambda(h^\kappa) = \lambda$ for all $h^\kappa$.

   (c) Given the power utility assumption and $\frac{z(h^\kappa)}{z(\boldsymbol{h}_h^\kappa)} = \frac{v_n(h^\kappa)}{v_n(\boldsymbol{h}_h^\kappa)}$, we know that $\frac{v_n(\boldsymbol{h}_h^\kappa)}{z(\boldsymbol{h}_h^\kappa)} = \frac{v_n(h^\kappa)}{z(h^\kappa)} = \frac{n_t(h_\kappa)}{\gamma}\frac{v_{nn}(h^\kappa)}{z_t(h^\kappa)}$ for all $h^\kappa$. The FSI steady-state Ramsey FOCs with respect to $n_t(h^\kappa)$, equation (47), leads to

$$\frac{v_n(h^\kappa)}{z(h^\kappa)} + \lambda(h^\kappa)\overline{w} + \chi(h^\kappa)\gamma\frac{v_n(h^\kappa)}{z(h^\kappa)} = \mu MPN,$$

   which together with $\lambda(h_\kappa) = \lambda$ implies that $\chi(h_\kappa) = \chi$ for all $h_\kappa$. The equation above then can be rewritten as

$$\frac{v_n(h^\kappa)}{z(h^\kappa)} + \lambda\overline{w} + \chi\gamma\frac{v_n(h^\kappa)}{z(h^\kappa)} = \mu MPN. \tag{50}$$

   (d) The steady-state FOC with respect to $\overline{w}_t$ is then

$$\lambda = \chi u_c(\boldsymbol{h}_h^\kappa). \tag{51}$$

   (e) The steady-state version of the Ramsey FOC with respect to $Q_{t+1}$ is

$$\lambda \sum_{h^\kappa} a(h^\kappa)\pi(h^\kappa) = \mu^q u_c(\boldsymbol{h}_h^\kappa). \tag{52}$$

   (f) Set $\zeta^2(\boldsymbol{h}_l^\kappa) = 0$. The steady-state Ramsey FOC with respect to $c_t(\boldsymbol{h}_l^\kappa)$ together with

$a(\boldsymbol{h}_h^\kappa) = 0$ gives

$$u_{c,t}(\boldsymbol{h}_h^\kappa) + \lambda + \chi n_t(\boldsymbol{h}_l^\kappa)\overline{w}_t u_{cc,t}(\boldsymbol{h}_h^\kappa) z_t(\boldsymbol{h}_l^\kappa) + \mu^q u_{cc,t}(\boldsymbol{h}_h^\kappa)\pi(h^\kappa|\boldsymbol{h}_h^\kappa) = \mu_t. \tag{53}$$

(g) Hence, the multipliers $\lambda, \chi, \mu^q$,and $\mu$ can be solved by the four FOCs above, namely, equations (50) to (53).

(h) By properly choosing $\zeta^2(h^\kappa)$ and $\zeta^3(h^\kappa)$ for all $h^\kappa \neq \boldsymbol{h}_l^\kappa$, the remaining FOCs with respect to $c_t(h^\kappa)$ where $h^\kappa \neq \boldsymbol{h}_l^\kappa$ can be satisfied.

3. The optimal long-run policies, $\{B,\ \tau_n,\tau_k\}$, are pinned down by the following steps:

   (a) $Q = \beta$ by equation (12). Given $Q = \beta$, the MGR implies a zero steady-state capital tax:
   $$\tau_k = 1 - \frac{\frac{1}{Q} - (1-\delta)}{\frac{1}{\beta} - (1-\delta)} = 0.$$

   (b) The government debt can be solved by using the asset market-clearing condition: $B = \sum_{h^\kappa} a(h^\kappa)\pi(h^\kappa) - \frac{K}{Q} = A - \frac{K}{Q}$.

   (c) By plugging (i) $\sum_{h^\kappa} a(h^\kappa)\pi(h^\kappa) = \phi c(\boldsymbol{h}_h^\kappa)$, (ii) the asset market-clearing condition, $B = \sum_{h^\kappa} a(h^\kappa)\pi(h^\kappa) - \frac{K}{Q}$, (iii) the resource constraint in the steady state, $\frac{c^e}{K} = \frac{MP_K}{\alpha} - \delta = \frac{1-\beta+\delta\beta}{\alpha\beta} - \delta$, (iv) $\phi = \frac{A}{C} = \frac{\sum_{h^\kappa} a(h^\kappa)\pi(h^\kappa)}{c(\boldsymbol{h}_h^\kappa)}$, and (v) the MGR implied by the FOC with respect to $K$, into the steady-state government budget constraint, the optimal long-run labor tax rate can be expressed as

$$\begin{aligned}
\tau_n &= (1-\beta)\frac{B}{MPN \times N} = (1-\beta)\frac{A - \frac{K}{\beta}}{MPN \times N} \\
&= (1-\beta)\frac{(\phi\frac{C}{K} - \frac{1}{\beta})}{\frac{MPN \times N}{Y}}\frac{K}{Y} = (1-\beta)\frac{\phi(\frac{1-\beta+\delta\beta}{\alpha\beta} - \delta) - \frac{1}{\beta}}{(1-\alpha)}\frac{K}{Y} \\
&= (1-\beta)\frac{\phi(1-\beta+\delta\beta(1-\alpha)) - \alpha}{(1-\alpha)(1-\beta+\delta\beta)}.
\end{aligned}$$

   In addition, the debt-to-GDP ratio can be expressed as

$$(1-\beta)\frac{B}{Y} = \tau_n\frac{MPN \times N}{Y} = \tau_n(1-\alpha),$$

and hence

$$\frac{B}{Y}=\frac{\left(1-\beta+\delta(1-\alpha)\beta\right)\phi-\alpha}{\left(1-\beta+\delta\beta\right)},$$

which is an increasing function of $\phi$.

4. Notice that this interior steady state is feasible only if $\tau_n < 1$; otherwise, it violates the FOC of employed individuals. The following two steps verify that $\tau_n < 1$ if and only if $\phi(1-\beta) < 1$:

   (a) If $\phi(1-\beta) < 1$, then

   $$\tau_n=\frac{\left(1-\beta+\delta(1-\alpha)\beta\right)\phi(1-\beta)-\alpha(1-\beta)}{(1-\alpha)\left(1-\beta+\delta\beta\right)}<\frac{\left(1-\beta+\delta(1-\alpha)\beta\right)-\alpha(1-\beta)}{(1-\alpha)\left(1-\beta+\delta\beta\right)}=1.$$

   (b) If $\tau_n < 1$, then

   $$\left(1-\beta+\delta(1-\alpha)\beta\right)\phi(1-\beta)-\alpha(1-\beta)<(1-\alpha)\left(1-\beta+\delta\beta\right),$$

   which can be simplified as

   $$\left(1-\beta+\delta(1-\alpha)\beta\right)\phi(1-\beta)<(1-\alpha)\delta\beta+(1-\beta),$$

   and further simplified as $\phi(1-\beta) < 1$.

### A.5.4 Uniqueness of the FSI Interior Ramsey Steady State under $\sigma \geq 1$

This proof follows closely to that in Appendix A.7. To facilitate our proof, define $\boldsymbol{c}_t^s(h_\kappa)$, $\boldsymbol{n}_t^s(h_\kappa)$, and $\boldsymbol{a}_{t+1}^s(h_\kappa)$ as the consumption share, labor share, and asset share, respectively, for individuals with truncated history $h_\kappa$ in period $t$. More specifically,

$$\boldsymbol{c}_t^s(h^\kappa)\equiv\frac{c_t^s(h^\kappa)}{C_t},\ \ \boldsymbol{n}_t^s(h^\kappa)\equiv\frac{n_t^s(h^\kappa)z_t(h^\kappa)}{N_t},\text{ and }\boldsymbol{a}_{t+1}^s(h^\kappa)\equiv\frac{a_{t+1}(h^\kappa)}{A_{t+1}}. \tag{54}$$

Hence, it must be true that

$$\sum_{h^\kappa}\boldsymbol{c}_t^s(h^\kappa)\pi(h^\kappa)=1,\ \ \sum_{h^\kappa}\boldsymbol{n}_t^s(h^\kappa)\pi(h^\kappa)=1,\ \text{and}\ \sum_{h^\kappa}\boldsymbol{a}_{t+1}^s(h^\kappa)\pi(h^\kappa)=1. \tag{55}$$

### A.5.5 Rewriting the Ramsey Problem

By utilizing the share variables in equation (54), we can rewrite the Ramsey Problem in the following way:

$$\max_{\{\boldsymbol{c}_t^s(h^\kappa),\boldsymbol{n}_t^s(h^\kappa),\boldsymbol{a}_{t+1}^s(h^\kappa),C_t,N_t,A_{t+1},K_{t+1}\}_{t=0}^\infty} \sum_{t=0}^{\infty}\beta^t \begin{bmatrix} \frac{1}{1-\sigma}C_t^{1-\sigma}\sum_{\theta^t}(\boldsymbol{c}_t^s(h^\kappa))^{1-\sigma}\pi(h^\kappa) \\ -\frac{1}{1+\gamma}N_t^{1+\gamma}\sum_{\theta^t}(\boldsymbol{n}_t^s(h^\kappa))^{1+\gamma}\pi(h^\kappa) \end{bmatrix}$$

subject to

$$\beta^t\mu_t : F(N_t,K_t)+(1-\delta)K_t-C_t-K_{t+1}\geq 0,\ \forall t\geq 0,$$

$$\lambda_0(h_\kappa)\pi(h_\kappa) : \boldsymbol{c}_0^s(h_\kappa)C_0^{1-\sigma}-\frac{(\boldsymbol{n}_0^s(\boldsymbol{h}_h^\kappa))^\gamma}{(\boldsymbol{c}_0^s(\boldsymbol{h}_h^\kappa))^{-\sigma}z_0(\boldsymbol{h}_h^\kappa)^{1+\gamma}}N_t^{\gamma+1}n_0^s(h^\kappa)+Q_1C_0^{-\sigma}a_1(h_\kappa)-C_0^{-\sigma}a_0(h_\kappa)=0,\forall h^\kappa,$$

$$\begin{aligned} Q^t\lambda_t(h^\kappa)\pi(h^\kappa) \quad &: \quad c_t^s(h^\kappa)C_t^{1-\sigma}-\frac{(\boldsymbol{n}_t^s(\boldsymbol{h}_h^\kappa))^\gamma}{(\boldsymbol{c}_t^s(\boldsymbol{h}_h^\kappa))^{-\sigma}z_t(\boldsymbol{h}_h^\kappa)^{1+\gamma}}N_t^{\gamma+1}n_t^s(h^\kappa)+Q_{t+1}C_t^{-\sigma}A_{t+1}^s\boldsymbol{a}_{t+1}^s(h_\kappa) \\ &\quad -C_t^{-\sigma}A_t\sum_{h_{-1}^\kappa}\boldsymbol{a}_t^s(h_{\kappa,-1})\frac{\pi(h_{-1}^\kappa)}{\pi(h^\kappa)}\pi(h_\kappa|h_{-1}^\kappa) \\ &= \quad 0,\forall h_\kappa \text{ and } t\geq 1, \end{aligned}$$

$$\zeta^0 : (1+(1-\tau_{k,0})MPK_0-\delta)K_0+B_0-\sum_{h_\kappa}a_0(h^\kappa)\pi(h^\kappa)=0,$$

$$\eta_t^s : \sum_{h_\kappa}\boldsymbol{c}_t^s(h^\kappa)\pi(h^\kappa)-1=0,\ \forall t\geq 0,$$

$$\eta_t^n : \sum_{h_\kappa}\boldsymbol{n}_t^s(h^\kappa)\pi(h^\kappa)-1=0,\forall t\geq 0,$$

$$\eta_t^a : \sum_{h_\kappa}\boldsymbol{a}_{t+1}^s(h^\kappa)\pi(h^\kappa)-1=0,\forall t\geq 0,$$

$$\beta^{t+1}\zeta_t^1(h^\kappa) : \boldsymbol{a}_{t+1}^s(h^\kappa)\geq 0,\ \forall h_\kappa\neq h_\kappa^e,$$

$$\beta^{t+1}\zeta_t^2(h^\kappa) : \boldsymbol{g}_t^s(h^\kappa)\geq 0,\ \forall h_\kappa\neq h_\kappa^e,$$

and

$$\beta^{t+1}\zeta_t^3(h^\kappa) : \boldsymbol{g}_t^s(h^\kappa)\boldsymbol{a}_{t+1}^s(h^\kappa)=0,\ \forall h_\kappa\neq h_\kappa^e,$$

where

$$Q_{t+1}C_t^{-\sigma} = \beta C_{t+1}^{-\sigma} \sum_{h^{\kappa'}} \left(\frac{\boldsymbol{c}^s_{t+1}(h^{\kappa'})}{\boldsymbol{c}^s_t(\boldsymbol{h}^\kappa_h)}\right)^{-\sigma} \pi(h^{\kappa'}|\boldsymbol{h}^\kappa_h),$$

where $Q^t$ is defined as the compounded consumption price between time zero and time $t$:

$$Q^t \equiv \prod_{s=0}^{t} Q_s,$$

and the function $\boldsymbol{g}^s_t$ is defined as

$$\boldsymbol{g}^s_t(h^\kappa) \equiv \frac{\boldsymbol{c}^s_t(h^\kappa)^{-\sigma}}{\sum_{h^{\kappa'}}(\boldsymbol{c}^s_{t+1}(h^{\kappa'})^{-\sigma}\pi(h^{\kappa'}|h^\kappa)} - \frac{\boldsymbol{c}^s_t(\boldsymbol{h}^\kappa_h)^{-\sigma}}{\sum_{h^{\kappa'}} \boldsymbol{c}^s_{t+1}(h^{\kappa'})^{-\sigma}\pi(h^{\kappa'}|\boldsymbol{h}^\kappa_h)}.$$

### A.5.6 Ramsey FOCs

The Ramsey FOCs with respect to $A_{t+1}$, $N_t$, and $C_t$ are given, respectively, by

$$C_t^{-\sigma} \sum_{h^\kappa} \lambda_t(h^\kappa)\boldsymbol{a}^s_{t+1}(h^\kappa)\pi(h^\kappa) - C_{t+1}^{-\sigma} \sum_{h^\kappa} \lambda_{t+1}(h^\kappa) \sum_{h^\kappa_{-1}} \boldsymbol{a}^s_{t+1}(h^\kappa_{-1})\pi(h^\kappa_{-1})\pi(h^\kappa|h^\kappa_{-1}) = 0, \tag{56}$$

$$\beta^t N_t^\gamma \sum_{h^\kappa} \left(\frac{\boldsymbol{n}^s_t(h^\kappa)}{z^s_t(h^\kappa)}\right)^{1+\gamma} \pi(h^\kappa) + (1+\gamma)Q^t N_t^\gamma \frac{(\boldsymbol{n}^s_t(\boldsymbol{h}^\kappa_h))^\gamma}{(\boldsymbol{c}^s_t(\boldsymbol{h}^\kappa_h))^{-\sigma} z_t(\boldsymbol{h}^\kappa_h)^{1+\gamma}} \sum_{h^\kappa} \lambda_t(h^\kappa)\boldsymbol{n}^s_t(h^\kappa)\pi(h^\kappa) = \beta^t \mu_t MP_{N,t}, \tag{57}$$

$$\begin{aligned}
&\beta^t C_t^{-\sigma} \sum_{h_\kappa} (\boldsymbol{c}^s_t(h^\kappa))^{1-\sigma}\pi(h^\kappa) + (1-\sigma)C_t^{-\sigma}Q^t \sum_{h^\kappa} \lambda_t(h^\kappa)\boldsymbol{c}^s_t(h^\kappa)\pi(h^\kappa) \\
&-\sigma Q^{t-1}\beta C_t^{-\sigma-1} \frac{Q_t C_{t-1}^{-\sigma}}{\beta C_t^{-\sigma}} A_t \sum_{h_\kappa} \lambda_{t-1}(h^\kappa)\boldsymbol{a}^s_t(h^\kappa)\pi(h^\kappa) \\
&+\sigma Q^t C_t^{-\sigma-1} \sum_{h_\kappa} \lambda_t(h^\kappa) \sum_{h^\kappa_{-1}} \boldsymbol{a}^s_{t+1}(h^\kappa_{-1})\pi(h^\kappa_{-1})\pi(h^\kappa|h^\kappa_{-1}) = \beta^t \mu_t.
\end{aligned} \tag{58}$$

Note that the Ramsey FOC with respect to $K_{t+1}$ remains the same as before:

$$\mu_t = \beta\mu_{t+1}\left(MPK_{t+1} + 1 - \delta\right).$$

### A.5.7 No Interior Ramsey Steady State Featuring $Q > \beta$

We now show that under the parameter condition $\sigma \geq 1$, there cannot possibly exist any interior Ramsey steady state featuring $Q > \beta$. The proof is done by contradiction. Consider two possible cases depending on the convergence of $\mu_t$:

1. Suppose $\mu_t$ converges. The FOC with respect to $N_t$ in equation (23) can be rewritten as

$$\begin{aligned}&\frac{\beta^t}{Q^t}N_t^\gamma \sum_{h^\kappa}\left(\frac{\boldsymbol{n}_t^s(h^\kappa)}{z_t^s(h^\kappa)}\right)^{1+\gamma}\pi(h^\kappa) + (1+\gamma)N_t^\gamma \frac{(\boldsymbol{n}_t^s(\boldsymbol{h}_h^\kappa))^\gamma}{(\boldsymbol{c}_t^s(\boldsymbol{h}_h^\kappa))^{-\sigma} z_t(\boldsymbol{h}_h^\kappa)^{1+\gamma}}\sum_{h^\kappa}\lambda_t(h^\kappa)\boldsymbol{n}_t^s(h^\kappa)\pi(h^\kappa)\\ =\ &\frac{\beta^t}{Q^t}\mu_t MPN_t.\end{aligned}$$

   Under the assumptions of $Q > \beta$ and convergent multipliers $\mu_t$, it must be true that $\lim_{t\to\infty}\frac{\beta^t}{Q^t} = 0$, so the above equation becomes

$$\lim_{t\to\infty}(1+\gamma)N_t^\gamma \frac{(\boldsymbol{n}_t^s(\boldsymbol{h}_h^\kappa))^\gamma}{(\boldsymbol{c}_t^s(\boldsymbol{h}_h^\kappa))^{-\sigma} z_t(\boldsymbol{h}_h^\kappa)^{1+\gamma}}\sum_{h^\kappa}\lambda_t(h^\kappa)\boldsymbol{n}_t^s(h^\kappa)\pi(h^\kappa) = 0,$$

   which leads to a contradiction with an interior steady state because all terms on the left-hand side of the above equation are positive.

2. Suppose $\mu_t$ diverges to infinity at a certain rate such that $\lim_{t\to\infty}\frac{\beta^t}{Q^t}\mu_t$ converges to a positive constant. Then in the limit we have the following:

   (a) The Ramsey FOC with respect to $N_t$ in the limit is given by

$$(1+\gamma)N^\gamma \lim_{t\to\infty}\frac{(\boldsymbol{n}_t^s(\boldsymbol{h}_h^\kappa))^\gamma}{(\boldsymbol{c}_t^s(\boldsymbol{h}_h^\kappa))^{-\sigma} z_t(\boldsymbol{h}_h^\kappa)^{1+\gamma}}\sum_{h^\kappa}\lambda_t(h^\kappa)\boldsymbol{n}_t^s(h^\kappa)\pi(h^\kappa) = MPN\lim_{t\to\infty}\frac{\beta^t}{Q^t}\mu_t > 0.$$

   (b) The Ramsey FOC with respect to $A_{t+1}$ is given by

$$\lim_{t\to\infty}\sum_{h^\kappa}\lambda_t(h^\kappa)\boldsymbol{a}_{t+1}^s(h^\kappa)\pi(h^\kappa) - \lim_{t\to\infty}\sum_{h^\kappa}\lambda_{t+1}(h^\kappa)\sum_{h_{-1}^\kappa}\boldsymbol{a}_{t+1}^s(h_{-1}^\kappa)\pi(h_{-1}^\kappa)\pi(h^\kappa|h_{-1}^\kappa) = 0.$$

   Plugging the above equation into the FOC with respect to $C_t$ leads to

$$\lim_{t\to\infty}\frac{\beta^t}{Q^t}C_t^{-\sigma}\sum_{h^\kappa}(\boldsymbol{c}_t^s(h^\kappa))^{1-\sigma}\pi(h^\kappa) + \lim_{t\to\infty}(1-\sigma)C_t^{-\sigma}\sum_{h^\kappa}\lambda_t(h^\kappa)\boldsymbol{c}_t^s(h^\kappa)\pi(h^\kappa) = \lim_{t\to\infty}\frac{\beta^t}{Q^t}\mu_t.$$

Under the assumptions that $Q > \beta$ and that $\mu_t$ diverges at a certain rate such that $\lim_{t\to\infty} \frac{\beta^t}{Q^t}\mu_t > 0$, the equation above then becomes

$$(1-\sigma)C^{-\sigma} \lim_{t\to\infty} \sum_{h^\kappa} \lambda_t(h^\kappa)\mathbf{c}_t^s(h^\kappa)\pi(h^\kappa) = \lim_{t\to\infty} \frac{\beta^t}{Q^t}\mu_t > 0,$$

which leads to a contradiction because the two sides of the equation have opposite signs if $\sigma \geq 1$.

3. To recap, the above proof shows that under $\phi(1-\beta) < 1$, we have the following: (i) If $\sigma \geq 1$, the FSI interior steady state (with $Q = \beta$) exists and is unique; it is impossible to have an interior Ramsey steady state featuring $Q > \beta$. (ii) If $\sigma < 1$, it is possible to have an interior Ramsey steady state with $Q > \beta$ provided that the multiplier $\mu_t$ diverges at the proper rate $g_\mu \geq \frac{Q}{\beta} > 1$;in such a case the optimal capital tax must be non-positive, as shown in Appendix 5. Furthermore, under $\phi(1-\beta) \geq 1$ and $\sigma \geq 1$, it is impossible for any form of an interior Ramsey steady state to exist and the only possible Ramsey steady state is non-interior with $C_t \to 0$.

## A.6 The Existence of a Ramsey Steady State When $\kappa = 0$ and $Z = \{e, u\}$

### A.6.1 Ramsey Problem

With $\kappa = 0$ and $Z = \{e, u\}$, there are only two groups of individuals, which are denoted by $e$ and $u$ groups. Any variable denoted with superscript $e$ or $u$ then represents its value for the $e$ or $u$ group, respectively. Denote $\beta^t\mu_t$, $\lambda_0^e$, $\lambda_0^u$, $\beta^t\lambda_t^e\pi(e)$, $\beta^t\lambda_t^u\pi(u)$, $\zeta^0$, $\beta^{t+1}\zeta_t^1$, $\beta^{t+1}\zeta_t^2$, and $\beta^{t+1}\zeta_t^3$ as the Lagrangian multipliers for the conditions listed below. It is straight forward to verify that the Ramsey planner's problem can be written as

$$\max_{\{c_t^e, c_t^u, n_t^e, a_{t+1}^e, a_{t+1}^u, K_{t+1}\}_{t=0}^\infty} \sum_{t=0}^{\infty} \beta^t \left\{ \left[u(c_t^e) - v\left(n_t^e\right)\right]\pi(e) + u(c_t^u)\pi(u) \right\}$$

subject to

$$\beta^t\mu_t : F(n_t^e\pi(e), K_t) + (1-\delta)K_t - c_t^e\pi(e) - c_t^u\pi(u) - K_{t+1} \geq 0 \ \forall t \geq 0,$$

$$\lambda_0^e : u_{c,0}^e c_0^e \pi(e) - v_{n,0}^e n_0^e \pi(e) + Q_1 u_{c,0}^e a_1^e - u_{c,0}^e a_0^e = 0,$$

$$\lambda_0^u : u_{c,0}^e c_0^u \pi(u) + Q_1 u_{c,0}^e a_1^u - u_{c,0}^e a_t^u = 0,$$

$$\beta^t \lambda_t^e \pi(e) : u_{c,t}^e c_t^e - v_{n,t}^e n_t^e + Q_{t+1} u_{c,t}^e a_{t+1}^e - u_{c,t}^e \left[ \frac{a_t^e \pi(e)\pi(e|e) + a_t^u \pi(u)\pi(e|u)}{\pi(e)} \right] = 0,$$

$$\beta^t \lambda_t^u \pi(u) : u_{c,t}^e c_t^u + Q_{t+1} u_{c,t}^e a_{t+1}^u - u_{c,t}^e \left[ \frac{a_t^e \pi(e)\pi(u|e) + a_t^u \pi(u)\pi(u|u)}{\pi(u)} \right] = 0,$$

$$\zeta^0 : (1 + (1 - \tau_{k,0}) MP_{K,0} - \delta) K_0 + B_0 - a_0^e - a_{0^u} = 0,$$

$$\beta^{t+1} \zeta_t^1 : a_{t+1}^u \geq 0,$$

$$\beta^{t+1} \zeta_t^2 : g_t^u \geq 0,$$

$$\beta^{t+1} \zeta_t^3 : g_t^u a_{t+1}^u = 0,$$

where $Q_{t+1} u_{c,t}^e$ is given by

$$Q_{t+1} u_{c,t}^e = \beta \left[ u_{c,t+1}^e \pi(e|e) + u_{c,t+1}^u \pi(u|e) \right],$$

and the function $g_t^u$ is defined as

$$g_t^u \equiv \frac{u_{c,t}^u}{u_{c,t+1}^e \pi(e|u) + u_{c,t+1}^u \pi(u|u)} - \frac{u_{c,t}^e}{u_{c,t+1}^e \pi(e|e) + u_{c,t+1}^u \pi(u|e)}.$$

### A.6.2 Ramsey FOCs

We first state the Ramsey FOCs. For all $t \geq 0$, the FOCs of the Ramsey problem with respect to $K_{t+1}, a_{t+1}^e$, and $a_{t+1}^u$ are given, respectively, by

$$\mu_t = \beta \mu_{t+1} \left( MPK_{t+1} + 1 - \delta \right), \tag{59}$$

$$\lambda_t^e Q_{t+1} u_{c,t}^e = \beta u_{c,t+1}^e \left( \lambda_{t+1}^e \pi(e|e) + \lambda_{t+1}^u \pi(u|e) \right) \text{ for } t \geq 0, \tag{60}$$

and

$$\begin{aligned} \lambda_t^u Q_{t+1} u_{c,t}^e &= \beta u_{c,t+1}^e (\lambda_{t+1}^u \pi(u|u) + \lambda_{t+1}^e \pi(e|u)) \\ &\quad + \zeta_t^1 + \zeta_t^3 g(c_t^e, c_t^u, c_{t+1}^e, c_{t+1}^u). \end{aligned} \tag{61}$$

For all $t \geq 1$, the FOCs of the Ramsey problem with respect to $n_t^e, c_t^e$, and $c_t^u$ are given, respectively, by

$$v_{n,t}^e + \lambda_t^e(v_{n,t}^e + v_{nn,t}^e n_t^e) = \mu_t MPN_t, \tag{62}$$

$$\begin{aligned} & (u_{c,t}^e - \mu_t)\pi(e) + \lambda_t^e(u_{c,t}^e + u_{cc,t}^e c_t^e)\pi(e) - \lambda_t^e u_{cc,t}^e(a_t^e \pi(e)\pi(e|e) + a_t^u \pi(u)\pi(e|u)) \\ & +\lambda_{t-1}^e a_t^e \pi(e) u_{cc,t}^e \pi(e|e) + \lambda_t^u u_{cc,t}^e c_t^u \pi(u) - \lambda_t^u u_{cc,t}^e(a_t^e \pi(e)\pi(u|e) + a_t^u \pi(u)\pi(u|u)) \\ & +\lambda_{t-1}^u a_t^u \pi(u) u_{cc,t}^e \pi(e|e) + \beta\zeta_t^2 \frac{\partial g_t}{\partial c_t^e} + \zeta_{t-1}^2 \frac{\partial g_{t-1}}{\partial c_t^e} + \beta a_{t+1}^u \zeta_t^3 \frac{\partial g_t}{\partial c_t^e} + a_t^u \zeta_{t-1}^3 \frac{\partial g_{t-1}}{\partial c_t^e} \\ = \; & 0, \end{aligned} \tag{63}$$

and

$$\begin{aligned} & (u_{c,t}^u - \mu_t)\pi(u) + \lambda_t^u u_{c,t}^e \pi(u) + \lambda_{t-1}^u a_t^u \pi(u) u_{cc,t}^u \pi(u|e) \\ & +\lambda_{t-1}^e a_t^e \pi(e) u_{cc,t}^u \pi(u|e) + \beta\zeta_t^2 \frac{\partial g_t}{\partial c_t^u} + \zeta_{t-1}^2 \frac{\partial g_{t-1}}{\partial c_t^u} + \beta a_{t+1}^u \zeta_t^3 \frac{\partial g_t}{\partial c_t^u} + a_t^u \zeta_{t-1}^3 \frac{\partial g_{t-1}}{\partial c_t^u} \\ = \; & 0. \end{aligned} \tag{64}$$

For $t = 0$, the FOCs of the Ramsey problem with respect to $n_0^e$, $c_0^e$, and $c_0^u$ are given, respectively, by

$$v_{n,0}^e + \lambda_0^e(v_{n,0}^e + v_{nn,0}^e n_0^e) = \mu_0 MP_{N,0} + \zeta^0(1 - \tau_{k,0}) MP_{KN,0} K_0,$$

$$\begin{aligned} & (u_{c,0}^e - \mu_0)\pi(e) + \lambda_0^e(u_{c,0}^e + u_{cc,0}^e c_0^e)\pi(e) - \lambda_0^e u_{cc,0}^e a_0^e \\ & +\lambda_0^u u_{cc,0}^e c_0^u \pi(u) - \lambda_0^u u_{cc,0}^e a_0^u + \beta\zeta_0^2 \frac{\partial g_0}{\partial c_0^e} + \beta a_1^u \zeta_0^3 \frac{\partial g_0}{\partial c_0^e} \\ = \; & 0, \end{aligned}$$

and

$$(u_{c,0}^u - \mu_0)\pi(u) + \lambda_0^u u_{c,0}^e \pi(u) + \beta\zeta_0^2 \frac{\partial g_0}{\partial c_0^u} + \beta a_1^u \zeta_0^3 \frac{\partial g_0}{\partial c_0^u} = 0.$$

Note that

$$\begin{aligned}
\frac{\partial g_t}{\partial c_t^u} &= \frac{u_{cc,t}^u}{u_{c,t+1}^e \pi(e|u) + u_{c,t+1}^u \pi(u|u)}, \\
\frac{\partial g_t}{\partial c_t^e} &= -\frac{u_{cc,t}^e}{u_{c,t+1}^e \pi(e|e) + u_{c,t+1}^u \pi(u|e)},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial g_{t-1}}{\partial c_t^u} &= -\frac{u_{c,t-1}^u u_{cc,t}^u \pi(u|u)}{\left(u_{c,t}^e \pi(e|u) + u_{c,t}^u \pi(u|u)\right)^2} + \frac{u_{c,t-1}^e u_{cc,t}^u \pi(u|e)}{\left(u_{c,t}^e \pi(e|e) + u_{c,t}^u \pi(u|e)\right)^2}, \\
\frac{\partial g_{t-1}}{\partial c_t^e} &= -\frac{u_{c,t-1}^u u_{cc,t}^e \pi(e|u)}{\left(u_{c,t}^e \pi(e|u) + u_{c,t}^u \pi(u|u)\right)^2} + \frac{u_{c,t-1}^e u_{cc,t}^e \pi(e|e)}{\left(u_{c,t}^e \pi(e|e) + u_{c,t}^u \pi(u|e)\right)^2}.
\end{aligned}$$

### A.6.3 Existence of a Non-Interior Ramsey Steady State under $\kappa = 0$ and $\sigma \geq 1$

By Proposition 7, we know that if $(1-\beta)\frac{\pi(u)}{\pi(u|e)} \geq 1$ and $\sigma \geq 1$, there is no interior Ramsey steady state. Here we further prove that there exists a non-interior Ramsey steady state under these parameter conditions.

For this non-interior Ramsey steady state to exist, it must be the case that $C_t \to 0$. So the proof proceeds by considering an allocation path where (i) $c_t^e > c_t^u > 0$ for all $t < \infty$ and (ii) $\lim_{t\to\infty} c_t^e = \lim_{t\to\infty} c_t^u = 0$.

1. We first show that this non-interior steady-state allocation can satisfy all constraints and the Ramsey FOCs:

   (a) The resource constraint is satisfied if $K_t \to 0$ since $\lim_{t\to\infty} c_t^e = \lim_{t\to\infty} c_t^u = 0$.

   (b) Given that $c_t^e > c_t^u > 0$ for $t < \infty$ and $a_t^u = 0$, the implementability condition of the unemployed agents becomes $c_t^u = a_t^e \frac{\pi(e)\pi(u|e)}{\pi(u)}$, which can be satisfied in the limit by letting $a_t^e \to 0$. In addition, the implementability condition of the employed agents becomes

   $$c_t^e - \frac{v_{n,t}^e}{u_{c,t}^e} n_t^e + Q_{t+1} a_{t+1}^e - a_t^e \pi(e|e) = 0,$$

   which is satisfied in the limit given the condition $\lim_{t\to\infty} Q_{t+1} a_{t+1}^e = \lim_{t\to\infty} \frac{v_{n,t}^e}{u_{c,t}^e} n_t^e \pi(e) \geq 0$.

   (c) The borrowing constraints and complementary slackness conditions of the Ramsey problem are trivially satisfied.

2. We then show that this allocation satisfies all Ramsey FOCs by properly choosing the convergence properties of the Ramsey multipliers:

   (a) The FOC with respect to $K_{t+1}$ can be satisfied if the following condition holds

   $$\lim_{t\to\infty} MPK_{t+1} = \frac{1}{\lim_{t\to\infty}\frac{\mu_{t+1}}{\mu_t}}\frac{1}{\beta} - 1 + \delta.$$

   Notice that the equation above also implies the following:

      i. $\lim_{t\to\infty}\frac{\mu_{t+1}}{\mu_t} < \infty$ since $\lim_{t\to\infty} MPK_{t+1} \geq 0$. We also know that $\lim_{t\to\infty}\frac{\mu_{t+1}}{\mu_t} \geq 1$. A convergence of $\frac{\mu_{t+1}}{\mu_t}$ then implies that the capital-to-labor ratio $\frac{K_t}{N_t}$ contained in $MPK_t$ and $MPN_t$ must also converge to a finite positive value despite the fact that $\lim_{t\to\infty} K_t = \lim_{t\to\infty} N_t = 0$.

      ii. $n_t^e \to 0$ given that $\lim_{t\to\infty} MPK_t = \lim_{t\to\infty}\alpha(\frac{n_t^e\pi(e)}{K_t})^{1-\alpha} < \infty$ and $K_t \to 0$.

   (b) Let $\lambda_t^e \to \infty$ and $\frac{\mu_t}{\lambda_t^e} \to 0$. The Ramsey FOC with respect to $n_t^e$ in equation (62) is satisfied in the limit as $t \to \infty$ :

   $$MPN \lim_{t\to\infty}\frac{\mu_t}{\lambda_t^e} = (1+\gamma)\lim_{t\to\infty} v_{n,t}^e = 0.$$

   (c) Given $\zeta_t^1 > 0$, $\zeta_t^2 = 0$, $a_t^u = 0$, and $c_t^u = a_t^e\frac{\pi(e)\pi(u|e)}{\pi(u)}$, the Ramsey FOC with respect to $c_t^e$ can be rewritten as

   $$u_{c,t}^e\left(1+\lambda_t^e(1-\sigma)\right) + \left(\lambda_{t-1}^e - \lambda_t^e\right) u_{cc,t}^e c_t^u \frac{\pi(u)}{\pi(u|e)}\pi(e|e) = \mu_t,$$

   which can be further transformed to

   $$\frac{1}{\sigma\lambda_t^e} + \left(1 - \frac{\lambda_{t-1}^e}{\lambda_t^e}\right)\frac{c_t^u}{c_t^e}\frac{\pi(u)}{\pi(u|e)}\pi(e|e) = \frac{\mu_t}{\lambda_t^e}\frac{1}{u_{c,t}^e}\frac{1}{\sigma} + 1 - \frac{1}{\sigma}.$$

   As $t \to \infty$, the equation above becomes

   $$0 < \left(1 - \lim_{t\to\infty}\frac{\lambda_{t-1}^e}{\lambda_t^e}\right)\lim_{t\to\infty}\frac{c_t^u}{c_t^e} = \frac{\pi(u|e)}{\pi(e|e)\pi(u)}(1-\frac{1}{\sigma}). \tag{65}$$

   Hence, given $\sigma \geq 1$ (a necessary condition), the above FOC can be satisfied and does not lead to contradictions.

(d) Under the conditions that $\zeta_t^1 > 0$, $\zeta_t^2 = 0$, $a_t^u = 0$, and $c_t^u = a_t^e \frac{\pi(e)\pi(u|e)}{\pi(u)}$, the Ramsey FOC with respect to $c_t^u$ can be simplified to

$$u_{c,t}^u + \lambda_t^u u_{c,t}^e + \lambda_{t-1}^e c_t^u u_{cc,t}^u = \mu_t,$$

which can be rewritten as

$$\frac{1}{\lambda_{t-1}^e} + \frac{\lambda_t^u}{\lambda_{t-1}^e}\frac{u_{c,t}^e}{u_{c,t}^u} - \sigma = \frac{\mu_t}{\lambda_t^e}\frac{\lambda_t^e}{\lambda_{t-1}^e}\frac{1}{u_{c,t}^u}.$$

Since $\lim_{t\to\infty} \frac{\mu_t}{\lambda_t^e}\frac{\lambda_t^e}{\lambda_{t-1}^e}\frac{1}{u_{c,t}^u} = 0$, in the limit the equation above becomes

$$\lim_{t\to\infty} \frac{\lambda_t^u}{\lambda_{t-1}^e}\frac{u_{c,t}^e}{u_{c,t}^u} = \sigma. \tag{66}$$

We then have two subcases to consider:

i. $\sigma > 1$. Equation (66) can be satisfied if $\frac{\lambda_t^u}{\lambda_{t-1}^e}$ converges to a finite positive constant. From 2(c), we know that $\frac{u_{c,t}^e}{u_{c,t}^u}$ also converges to a finite positive constant given the convergence of $\frac{c_t^u}{c_t^e}$.

ii. $\sigma = 1$. Equation (66) can be satisfied if both $\frac{\lambda_t^u}{\lambda_{t-1}^e}$ and $\frac{c_t^u}{c_t^e}$ converge to finite positive values.

Hence, both sub-cases above are possible and do not lead to contradictions.

(e) The FOCs of $a_{t+1}^e$ and $a_{t+1}^u$ in equations (60) and (61) can be rewritten, respectively, as

$$\pi(e|e) + \frac{u_{c,t+1}^u}{u_{c,t+1}^e}\pi(u|e) = \frac{\lambda_{t+1}^e}{\lambda_t^e}\pi(e|e) + \frac{\lambda_{t+1}^u}{\lambda_t^e}\pi(u|e), \tag{67}$$

$$\pi(e|e) + \frac{u_{c,t+1}^u}{u_{c,t+1}^e}\pi(u|e) = \frac{\lambda_{t+1}^u}{\lambda_t^u}\pi(u|u) + \frac{\lambda_{t+1}^e}{\lambda_t^u}\pi(e|u) + \frac{\zeta_t^1 + \zeta_t^3 g(c_t^e, c_t^u, c_{t+1}^e, c_{t+1}^u)}{\lambda_t^u u_{c,t+1}^e}. \tag{68}$$

Given $\sigma \geq 1$ and from 2(c) and 2(d), we know that $\frac{u_{c,t}^u}{u_{c,t}^e}$, $\frac{\lambda_{t+1}^e}{\lambda_t^e}$, $\frac{\lambda_{t+1}^u}{\lambda_t^u}$, $\frac{\lambda_{t+1}^e}{\lambda_t^u}$, and $\frac{\lambda_{t+1}^u}{\lambda_t^e}$ all converge to finite positive constants. Hence, equation (67) is satisfied. In addition, $g_t(c_t^e, c_t^u, c_{t+1}^e, c_{t+1}^u)$ also converges to a finite positive constant according to its definition, so equation (68) can be satisfied if $\lim_{t\to\infty} \frac{\zeta_t^1 + \zeta_t^3 g(c_t^e, c_t^u, c_{t+1}^e, c_{t+1}^u)}{\lambda_t^u u_{c,t+1}^e}$ is chosen to be a finite constant, which is possible and does not lead to contradictions.

In short, we have shown that a non-interior Ramsey steady state exists and it must feature divergent multipliers. Also notice that the last equation in (c) in step 2 is the only critical condition involving the value of $\sigma$:

$$0<\left(1-\lim_{t\to\infty}\frac{\lambda^e_{t-1}}{\lambda^e_t}\right)\lim_{t\to\infty}\frac{c^u_t}{c^e_t}=\frac{\pi(u|e)}{\pi(e|e)\pi(u)}\left(1-\frac{1}{\sigma}\right), \tag{69}$$

which suggests that if $\sigma<1$, the right-hand side of the above inequality is negative; hence, there cannot exist a non-interior Ramsey steady state in the simplified model with $\kappa=0$.

3. Policy implication

   (a) Given $c^e_t\to 0$, $n^e_t\to 0$, and $MPN_t\to MPN>0$, it must be true that $\tau_{n,t}\to 1$ by equation (37).

   (b) The intertemporal price is

   $$Q_{t+1}=\beta\left[\frac{u^e_{c,t+1}}{u^e_{c,t}}\pi(e|e)+\frac{u^u_{c,t+1}}{u^u_{c,t}}\frac{u^u_{c,t}}{u^e_{c,t}}\pi(u|e)\right].$$

   We know that $\frac{u^e_{c,t+1}}{u^e_{c,t}}$, $\frac{u^u_{c,t+1}}{u^u_{c,t}}$, and $\frac{u^u_{c,t}}{u^e_{c,t}}$ all converge to finite values larger than or equal to 1 and hence $\infty>\lim_{t\to\infty}Q_{t+1}\geq\beta$.

   (c) The capital tax is determined by

   $$\tau_{k,t+1}=1-\frac{\frac{1}{Q_{t+1}}-(1-\delta)}{\frac{\mu_t}{\beta\mu_{t+1}}-(1-\delta)}.$$

   Hence, the sign of the capital tax in the limit ($\lim_{t\to\infty}\tau_{k,t+1}$) depends on the growth rate of $\mu_t$ relative to $\frac{Q_{t+1}}{\beta}$ in the limit. Given that $K_t\to 0$, the optimal capital tax is irrelevant.

   (d) By the asset market clearing condition, we have $B_t\to 0$.

### A.6.4 Existence of an Interior Partial-Insurance Ramsey Steady State under $\sigma<1$ and $\mu_t\to\infty$

By Proposition 7, we know that if $(1-\beta)\frac{\pi(u)}{\pi(u|e)}\geq 1$, there is no interior FSI steady state. In addition, from section A.6.3, we know that the existence of a non-interior Ramsey steady state requires condition $\sigma \geq 1$. We now prove that there exists an interior non-FSI Ramsey steady state under $\sigma < 1$. This Ramsey steady state is then the only possible Ramsey steady state if $(1-\beta)\frac{\pi(u)}{\pi(u|e)} \geq 1$.

Consider an interior non-FSI Ramsey steady state where (i) $c^e > c^u > 0$, (ii) the borrowing constraint for the unemployed individuals must be strictly binding with $a^u = 0$ and hence $\zeta^1 > 0$ and $\zeta^2 = 0$.

Let $g_\lambda^u$, $g_\lambda^e$, and $g_\mu$ denote the steady-state growth rate of $\lambda_t^u$, $\lambda_t^e$, and $\mu$, respectively. We first show that this Ramsey steady state must feature $g_\lambda^u = g_\lambda^e = g_\mu$.

1. From the Ramsey FOC with respect to $n^e$, we know that for an interior Ramsey steady state to exist, the growth rate of $\lambda_t^e$ and $\mu_t$ have to be the same: $g_\lambda^e = g_\mu$.

2. Moreover, we can show that $g_\lambda^e = g_\lambda^\mu$ by the following steps:

   (a) The FOC with respect to $a^e$ in the steady state is given by

   $$1 < \frac{Q}{\beta} = \pi(e|e) + \frac{u_c^u}{u_c^e}\pi(u|e) = g_\lambda^e \pi(e|e) + g_\lambda^u \frac{\lambda_t^u}{\lambda_t^e}\pi(u|e). \tag{70}$$

   (b) Suppose $g_\lambda^e < g_\lambda^u$, then $\frac{\lambda_t^u}{\lambda_t^e} \to \infty$. Equation (70) becomes $\infty > \pi(e|e) + \frac{u_c^u}{u_c^e}\pi(u|e) = \infty$, which is impossible.

   (c) Suppose $g_\lambda^e > g_\lambda^u$, then $\frac{\lambda_t^u}{\lambda_t^e} \to 0$. The FOC with respect to $c^u$ (under $\zeta_t^2 = 0$, $a^u = 0$, and $a^e\pi(e)\pi(u|e) = c^u\pi(u)$) becomes

   $$u_c^u + g_\lambda^u \lambda_{t-1}^u u_c^e + \lambda_{t-1}^e u_{cc}^u c^u = g\mu_{t-1}\mu_t,$$

   which implies
   $$\frac{u_c^u}{\lambda_{t-1}^e} + g_\lambda^u \frac{\lambda_{t-1}^u}{\lambda_{t-1}^e} u_c^e + u_{cc}^u c^u = g_\mu \frac{\mu_{t-1}}{\lambda_{t-1}^e}.$$

   As $t \to \infty$, the left-hand side is negative and the right-hand side is positive, which is a contradiction.

   (d) So, it must be true that $g_\lambda^e = g_\lambda^u = g_\mu$.

We show that such an interior steady state cannot exist under the condition that $\sigma \geq 1$ or $\mu_t \to \mu < \infty$. Namely, this Ramsey steady state exists only if $\sigma < 1$ and the following hold:

1. Given $g^e_\lambda = g^u_\lambda$, equation (70) implies that

$$\left(\frac{u^u_c}{u^e_c} - g^u_\lambda \frac{\lambda^u_t}{\lambda^e_t}\right)\pi(u|e) = (g^e_\lambda - 1)\pi(e|e). \tag{71}$$

2. Under $\zeta^2_t = 0$, $a^u = 0$, and $a^e\pi(e)\pi(u|e) = c^u\pi(u)$, the FOC with respect to $c^e$ can be rewritten as

$$u^e_c + \lambda^e_t u^e_c(1-\sigma) = \mu_t + u^e_{cc}c^u \frac{\pi(u)}{\pi(u|e)}\frac{\pi(e|e)}{\pi(e)}(g^e_\lambda - 1)\lambda^e_{t-1}, \tag{72}$$

and the FOC with respect to $c^u$ can be rewritten as

$$u^u_c + \lambda^e_{t-1}u^u_c(1-\sigma) = \mu_t - \lambda^u_t u^e_c + \lambda^e_{t-1}u^u_c. \tag{73}$$

Now, with the above two equations, consider the following cases:

(a) The growth rates satisfy $g^e_\lambda = g^u_\lambda = 1$. Without growth, $\lambda^e$ must converge. Equation (71) then implies $\lambda^u u^e_c = \lambda^e u^u_c$. The difference between equation (72) and equation (73) gives

$$(u^e_c - u^u_c) + \lambda^e(1-\sigma)(u^e_c - u^u_c) = 0,$$

which implies $u^e_c = u^u_c$ and contradicts the assumption $c^e > c^u$.

(b) The growth rates satisfy $g^e_\lambda = g^u_\lambda > 1$. The sum of the FOCs with respect to $c^e$ and $c^u$ can be written as

$$\begin{aligned} & \frac{u^e_c\pi(e) + u^u_c\pi(u)}{\lambda^e_{t-1}} + \frac{\lambda^e_t}{\lambda^e_{t-1}}u^e_c(1-\sigma)\pi(e) + u^u_c(1-\sigma)\pi(u) \\ = \; & \frac{\mu_t}{\lambda^e_{t-1}} - \frac{\lambda^u_t}{\lambda^e_{t-1}}u^e_c\pi(u) + u^u_c\pi(u) + u^e_{cc}c^u\frac{\pi(u)}{\pi(u|e)}\pi(e|e)(g^e_\lambda - 1). \end{aligned}$$

Since under positive growth $\lambda^e_{t-1} \to \infty$, the above equation becomes

$$\begin{aligned} & g^e_\lambda u^e_c(1-\sigma)\pi(e) + u^u_c(1-\sigma)\pi(u) \\ = \; & g_\mu \frac{\mu_{t-1}}{\lambda^e_{t-1}} - g^u_\lambda \frac{\lambda^u_{t-1}}{\lambda^e_{t-1}}u^e_c\pi(u) + u^u_c\pi(u) + u^e_{cc}c^u\frac{\pi(u)}{\pi(u|e)}\pi(e|e)(g^e_\lambda - 1), \end{aligned}$$

which together with (71) implies

$$\begin{aligned}
& g_\lambda^e u_c^e(1-\sigma)\pi(e) + u_c^u(1-\sigma)\pi(u) \\
= \; & g_\mu \frac{\mu_{t-1}}{\lambda_{t-1}^e} + \pi(u)u_c^e\left(\frac{u_c^u}{u_c^e} - g_\lambda^u \frac{\lambda_{t-1}^u}{\lambda_{t-1}^e}\right) + u_{cc}^e c^u \pi(u)\left(\frac{u_c^u}{u_c^e} - g_\lambda^u \frac{\lambda_{t-1}^u}{\lambda_{t-1}^e}\right) \\
= \; & g_\mu \frac{\mu_{t-1}}{\lambda_{t-1}^e} + \pi(u)\left(\frac{u_c^u}{u_c^e} - g_\lambda^u \frac{\lambda_{t-1}^u}{\lambda_{t-1}^e}\right)(u_c^e + u_{cc}^e c^u) \\
> \; & \pi(u)\left(\frac{u_c^u}{u_c^e} - g_\lambda^u \frac{\lambda_{t-1}^u}{\lambda_{t-1}^e}\right) u_c^e \left(1 - \sigma\frac{c^u}{c^e}\right) \\
> \; & \pi(u)\left(u_c^u - g_\lambda^u \frac{\lambda_{t-1}^u}{\lambda_{t-1}^e} u_c^e\right)(1-\sigma),
\end{aligned}$$

where the last two inequalities utilize the fact that (i) $g_\mu \frac{\mu_{t-1}}{\lambda_{t-1}^e} > 0$ and (ii) $\frac{c^u}{c^e} < 1$. Now, considering the parameter value $\sigma \geq 1$, the above inequality can be simplified to the following two possible relationships:

$$\begin{aligned}
0 &< g_\lambda^e u_c^e \pi(e) < -g_\lambda^u \frac{\lambda_t^u}{\lambda_t^e} u_c^e \pi(u) < 0, \text{ if } \sigma > 1, \\
\text{or } 0 &< 0 \text{ if } \sigma > 1,
\end{aligned}$$

and

$$\begin{aligned}
0 &> \left(\frac{u_c^u}{u_c^e} - g_\lambda^u \frac{\lambda_{t-1}^u}{\lambda_{t-1}^e}\right)(1-\sigma) = 0, \text{ if } \sigma = 1, \\
\text{or } 0 &> 0 \text{ if } \sigma = 1;
\end{aligned}$$

both of which are self-contradictory.

Third, we show that there indeed exists such an interior Ramsey steady state by showing that it satisfies all optimal Ramsey FOCs:

1. In this interior Ramsey steady state, there are eight variables to solve, which include $c^e, c^u, n^e, K, a^e, g_\lambda^e, \lim_{t\to\infty}\frac{\mu_t}{\lambda_t^e}$, and $\lim_{t\to\infty}\frac{\mu_t}{\lambda_t^u}$. Notice that $g_\mu = g_\lambda^u = g_\lambda^e$ and $\lim_{t\to\infty}\frac{\lambda_t^e}{\lambda_t^u}$ is known once we know $\lim_{t\to\infty}\frac{\mu_t}{\lambda_t^u}$ and $\lim_{t\to\infty}\frac{\mu_t}{\lambda_t^e}$.

2. There are eight Ramsey FOCs and constraints that can be used to solve these unknown variables in the limit:

(a) In the limit, the Ramsey FOCs with respect to $K_{t+1}, n_t^e, a_{t+1}^e, c_t^e$, and $c_t^u$ are given, respectively, by

$$1 = \beta g_\lambda^e \left(MPK + 1 - \delta\right),$$

$$v_n^e(1+\gamma) = MPN \lim_{t\to\infty} \frac{\mu_t}{\lambda_t^e},$$

$$\left(\frac{u_c^u}{u_c^e} - g_\lambda^u \lim_{t\to\infty} \frac{\lambda_t^u}{\lambda_t^e}\right) \pi(u|e) = (g_\lambda^e - 1)\pi(e|e),$$

$$u_c^e(1-\sigma) = \lim_{t\to\infty} \frac{\mu_t}{\lambda_t^e} + u_{cc}^e c^u \frac{\pi(u)}{\pi(u|e)} \frac{\pi(e|e)}{\pi(e)} (g_\lambda^e - 1) \frac{1}{g_\lambda^e}, \tag{74}$$

and

$$u_c^e \lim_{t\to\infty} \frac{\lambda_t^u}{\lambda_t^e} + \frac{1}{g_\lambda^e} c^u u_{cc}^u = \lim_{t\to\infty} \frac{\mu_t}{\lambda_t^e}. \tag{75}$$

(b) The resource constraint and the implementability conditions for type-$e$ and type-$u$ individuals are given, respectively, by

$$F(n^e\pi(e), K) - \delta K - c^e\pi(e) - c^u\pi(u) = 0,$$

$$c_t^e - \frac{v_{n,t}^e}{u_{c,t}^e} n_t^e + \beta \left(\pi(e|e) + \frac{u_c^u}{u_c^e}\pi(u|e)\right) a^e - \left[\frac{a^e\pi(e)\pi(e|e)}{\pi(e)}\right] = 0,$$

and

$$c^u = \frac{a^e\pi(e)\pi(u|e)}{\pi(u)}.$$

Finally, recall that in Proposition 7 we have shown $\tau_\kappa$ is non-positive. Therefore, we have proved the existence of such an interior Ramsey steady state under $\sigma < 1$.

## A.7 Proof of Proposition 10

The proof is done by construction. Conjecture that the Ramsey planner can achieve the first-best allocation starting from period 0. In other words, all constraints except the resource constraint do not bind: $\lambda_t(h^\kappa) = 0$ and $\zeta_t^1(h^\kappa) = \zeta_t^2(h^\kappa) = \zeta_t^3(h^\kappa) = 0$ for all $t$ and $h^\kappa$. It is then straightforward to see that the Ramsey planner's problem becomes the social planner's problem. In addition, it is straight forward to verify that the Ramsey FOCs become identical to the optimal conditions of the first-best allocation.

Given this first-best allocation $\{c_t^{FB}, n_t^{FB}(h^\kappa), K_{t+1}^{FB}\}$, the following steps show that the Ramsey planner can choose the corresponding policy in order to achieve the first-best allocation.

1. The optimal conditions of the first-best allocation implies that $\frac{v_{n,t}^{FB}(h^\kappa)}{u_{c,t}^{FB} z_t(h^\kappa)} = MPN_t^{FB}$ and $Q_{t+1} = \beta \frac{u_{c,\,t+1}^{FB}}{u_{c,\,t}^{FB}}$, which together with equation (59) further imply that the optimal labor tax and capital tax are both zero for all $t \geq 0$:

$$\begin{aligned}\tau_{n,t} &= 1 - \frac{v_{n,t}^{FB}}{u_{c,t}^{FB}} = 0, \\ \tau_{k,t+1} &= 1 - \frac{\frac{1}{Q_{t+1}} - (1-\delta)}{\frac{\mu_t}{\beta \mu_{t+1}} - (1-\delta)} = 0.\end{aligned}$$

   As a result, the government budget constraint is then reduced to

$$\beta \frac{u_{ct+1}^{FB}}{u_{ct}^{FB}} B_{t+1} - B_t = T_t. \tag{76}$$

2. Consider the case where $a_{t+1}(\boldsymbol{h}_l^\kappa) = 0$ for all $t \geq 0$. By the asset market-clearing condition, $B_{t+1} = A_{t+1} - \beta \frac{u_{c,\,t+1}^{FB}}{u_{c,\,t}^{FB}} K_{t+1}^{FB}$ for all $t \geq 0$, we show the sequences of $T_t$ and $a_{t+1}(h^\kappa)$ for $h^\kappa \neq \boldsymbol{h}_l^\kappa$ can be chosen to satisfy the implementability conditions in the following steps:

3. $T_0$ is chosen to satisfy the implementability condition of an unemployed agent at period 0 :

$$T_0 = c_0^{FB} - a_0(\boldsymbol{h}_l^\kappa) - MPN_0^{FB} z_0(\boldsymbol{h}_l^\kappa) n_0(\boldsymbol{h}_l^\kappa);$$

   for $h^\kappa \neq \boldsymbol{h}_l^\kappa$, $a_1(h^\kappa)$ is chosen such that the implementability condition of $h^\kappa$ individuals is satisfied by

$$\beta \frac{u_{c1}^{FB}}{u_{c0}^{FB}} a_1(h^\kappa) = a_0(h^\kappa) + T_0 + MPN_0^{FB} z_0(h^\kappa) n_0(h^\kappa) - c_0^{FB} > 0,$$

   which is strictly positive given $a_0(h^\kappa) > a_0(\boldsymbol{h}_l^\kappa)$ and $z_0(h^\kappa) > z_0(\boldsymbol{h}_l^\kappa)$.

4. For any $t \geq 1$, given $a_t(h^\kappa)$, $T_t$ and $a_{t+1}(h^\kappa)$ are chosen to satisfy the implementability condition of type-$\boldsymbol{h}_l^\kappa$ and type-$h^\kappa$ individuals:

$$T_t = c_t^{FB} - \sum_{h_{-1}^\kappa} \frac{a_t(h_{-1}^\kappa)\pi(h_{-1}^\kappa)\pi(\boldsymbol{h}_l^\kappa | h_{-1}^\kappa)}{\pi(h^\kappa)} - MPN_t^{FB} z_t(\boldsymbol{h}_l^\kappa) n_t(\boldsymbol{h}_l^\kappa),$$

and for $h^\kappa \neq \boldsymbol{h}_l^\kappa$, $a_1(h^\kappa)$ is chosen such that the implementability condition of type-$h^\kappa$ individuals is satisfied by

$$\beta \frac{u_{ct+1}^{FB}}{u_{ct}^{FB}} a_{t+1}(h^\kappa) = \sum_{h_{-1}^\kappa} \frac{a_t(h_{-1}^\kappa)\pi(h_{-1}^\kappa)\pi(h^\kappa|h_{-1}^\kappa)}{\pi(h^\kappa)} + T_t + MPN_t^{FB} z_t(h^\kappa) n_t(h^\kappa) - c_t^{FB} > 0,$$

which is strictly positive given $a_t(h^\kappa) > a_t(\boldsymbol{h}_l^\kappa)$ and $z_t(h^\kappa) > z_t(\boldsymbol{h}_l^\kappa)$.

5. Therefore, as shown by the steps above, the first-best allocation can be implemented by the Ramsey planner. Finally, the steady-state government budget constraint is simply given by $T = B(\beta - 1)$, which implies $T$ is negative (lump-sum tax) if $B > 0$.

## A.8 Proof of Proposition 11

Denote $\mu$, $\lambda(h^\kappa)\pi(h^\kappa)$, $\zeta^1(h^\kappa)$, and $\zeta^3(h^\kappa)$ as the Lagrangian multipliers for constraints (31), (32), (33) and (35), respectively. The Ramsey FOCs with respect to $c(h^\kappa)$ and $a(h^\kappa)$ are given, respectivley, by

$$u_c(h^\kappa) = \mu + \lambda(h^\kappa), \tag{77}$$

and

$$\lambda(h^\kappa)\pi(h^\kappa)Q = \sum_{h_{-1}^\kappa} \lambda(h_{-1}^\kappa)\pi(h^\kappa)\pi(h_{-1}^\kappa|h^\kappa) - \zeta^1(h^\kappa) - \zeta^3(h^\kappa)g(h^\kappa). \tag{78}$$

The proof is done by contradiction. Consider a Ramsey steady state featuring FSI: $c(h^\kappa) = c(\boldsymbol{h}_h^\kappa) > 0$, $a(h^\kappa) \geq 0$, $\zeta^1(h^\kappa) = 0$, $Q = \beta$, and $g(h^\kappa) = 0$ for all $h^\kappa$. Then, the Ramsey FOC with recpet to consumption in equation (78) implies $\lambda(h^\kappa) = \lambda(\boldsymbol{h}_h^\kappa)$ for all $h^\kappa$. As a result, equation (78) can be simplified as

$$\pi(h^\kappa)Q = \sum_{h_{-1}^\kappa} \pi(h^\kappa)\pi(h_{-1}^\kappa|h^\kappa),$$

which together with $Q = \beta$ gives $\beta = 1$, a contradiction. Thus, an FSI allocation where $c(h^\kappa) = c$ for all $h^\kappa$ cannot be the outcome of the static Ramsey problem that maximizes the steady-state welfare of the competitive equilibrium.