Do taxspots matter?*

A. Citanna† M. Tvede‡
NYU AD U Sheffield

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Abstract

Should the government run an uncertain fiscal policy to finance its liabilities? We call the resulting extrinsic uncertainty taxspots, and study under what conditions taxspots are optimal, and persistent, in standard Ramsey problems.

Keywords: Ramsey taxation, sunspots, lottery equilibrium.

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†New York University Abu Dhabi, UAE, ac6609@nyu.edu
‡University of Sheffield, UK, m.tvede@sheffield.ac.uk

1 Introduction

A standard principle in public finance (see, e.g., Barro (1979)) is that, under the usual assumption of risk aversion, taxation should be designed so as to smooth out lifetime consumption, aiming at fiscal certainty over time. Yet, tax uncertainty is recognized as a feature of many tax systems, in and outside the group of OECD countries, usually with a negative view.[1] We study when and how efficient fiscal policies must instead create fiscal uncertainty, or 'taxspots'. Taxspots equilibria are competitive equilibria where uncertainty stems (only) from variation in taxes. Taxspot equilibria then give new insights on the properties of optimal taxes. We focus on Ramsey problems, where taxes are linear in income and the planner is benevolent.

To illustrate the emergence of taxspots, we first examine a version of Lucas and Stokey's (1983) model of linear labor income taxation in a market economy where the government can issue a full set of state-contingent bonds. Here taxspots affect labor income taxes, and create stochastic variations in the workers' labor supply. Taxspots occur at the optimal fiscal policy in a large set of economies –dense in a sense we make precise, and under usual regularity conditions. The intuitive reason is that, under sufficient prudence, uncertain future disposable income spurs consumers to work more. The additional income can increase average consumption, compensating the individuals for the additional risk faced. However, without additional frictions taxspots have limited impact on the economy: as the government can issue state- (that is, taxspot-) contingent bonds, resulting in market completeness, taxspot uncertainty essentially dies out in finite time.

Next, we move to an economy with only capital. When capital is considered, the mechanism through which taxspots are beneficial to the economy becomes more transparent. The Ramsey problem requires the planner to accumulate capital with a 'demand constraint', that is, capital investment must be consistent with the individual's perceived optimal intertemporal consumption trade-off. If the government starts off with insufficient capital or large enough liabilities, as is assumed in standard analyses of the problem, the Ramsey planner needs to increase capital investment, that is, savings. As it is well known, this can obtained with either a lower or a higher interest rate, depending on the intertemporal elasticity of substitution —when the elasticity is lower than one, a lower interest rate (i.e., a positive capital tax) spurs higher investment. However, due to prudence, an increase in uncertainty also may spur higher investment. If prudence is large enough to offset the negative effect of additional consumption uncertainty (an adverse effect for risk averse individuals), then taxspots can be used to raise the demand for capital.

Again in a model where the government may issue a full set of state-contingent debt (as in, e.g., Chamley (1986) and Chari, Christiano and Kehoe (1994)), we confirm the above intuition: enough prudence induces taxspots, that is, a taxspot-free fiscal policy is suboptimal under enough prudence. Slight perturbations of the utility and of government initial liabilities parameters, away from the stan-

[1] See, e.g., Tax Certainty, the IMF/OECD 2017 Report for the G20 Finance Ministers.

dard isoelastic utility with elasticity coefficient less than one, also lead to the Pareto superiority of taxspots. When optimal taxspots are considered, almost surely zero average capital taxes obtain in stationary, ergodic solutions. However, as in the pure labor economy, since the government can issue a complete set of bonds, taxspots again essentially die off in finite time.

We then look at economies where the government cannot issue intertemporal bonds. Simultaneously, we consider a technology with both capital and labor, and (as in Judd (1985)) add population heterogeneity, via the introduction of hand-to-mouth workers and capitalists. If there is a desire for redistribution from capitalists to workers, then optimal taxspots arise persistently. Taxspots are no longer constrained to vanish in finite time. Thus, taxation dynamics can break with the serial correlation of government expenditures, and may display history dependence.

When workers supply labor elastically, we prove that persistent taxspots are optimal under sufficient workers' prudence and if government expenditure is large enough.[2] However, taxspots are coupled with the issuance of 'employment insurance bonds', i.e., subsidies or lump-sum taxes to workers that are contingent on taxspots.

We finally consider economies where the hand-to-mouth workers supply labor inelastically (as in Straub and Werning (2020)). Here taxspots affect capital taxes, that is, the capitalists' consumption and investment decisions. In the same environments considered in Straub and Werning, we claim the suboptimality of a taxspot-free long-run capital tax, possibly under a slight utility perturbation. Taxspots are welfare dominating with a strong enough precautionary savings motive for the capitalists, and occur infinitely often.[3] Straub and Werning (2020) have argued that, with low elasticity of intertemporal substitution, the optimal long-run capital tax is deterministic, and positive. While a 'front-loading' taxspot policy Pareto improves over any purely deterministic capital tax, generally optimal taxspots are not necessarily a substitute for a positive average capital tax. Optimal taxspots do lower the average tax relative to a path-equivalent deterministic tax policy (i.e., 'ceteris paribus') if workers are imprudent.[4]

The general observation of the uncertain nature of optimal fiscal policy all the more applies if there are other frictions that fiscal policy must face —strategic, as when the government has limited ability to commit or there are politico-economical consideration to fiscal spending (as in, e.g., Acemoglu,

[2]A sufficient condition is that public sector expenditure must exceed the capital share of output. This is not inconsistent with casual observation on the public finances of many countries (e.g., on average Denmark's capital share is 35% and public sector's expenditure is 50% of GDP; in France, public sector's expenditure is 60% of GDP, and the capital share is around 30. Ancillary assumptions on the elasticities of intertemporal substitution for consumption guarantee that regularity conditions are satisfied (for example, the workers' elasticity of intertemporal substitution is less than one). See Section 4.1.

[3]Our result does not cover the log utility case, aligning with Lansing's (1999) findings that log utility is a knife-edge case.

[4]Prudence has been correlated with wealth (see e.g., Noussair, Traumann and van de Kuilen, 2014) so it is not at all improbable that workers, whose wealth is low, display imprudence. Straub and Werning (2020) allow workers' utility to display imprudence.

Golosov, Tsyvisnki (2011)); informational, as when firms have noisy signals when maximizing profits (as in, e.g., Angeletos and La'O (2020)); or technological, such as when trade occurs through bilateral matching in labor market models a la Pissarides (2000)— introducing an infinite sequence of constraints. We conjecture taxspots to be a robust feature of optimal fiscal policy even if these more realistic constraints are placed on the planner. In this sense, our choice of a simple, if not so realistic, framework for fiscal policy helps understanding the origins of fiscal uncertainty. In fact, our taxspots are implementable by making fiscal policy depend on 'sentiments', or higher-order beliefs circulating in the market, along the interpretation provided in Angeletos and La'O (2020) and related literature. However, the reason why here fiscal policy should be made dependent on such extrinsic uncertainty is not informational, but resides in consumers' preferences, i.e., prudence and risk aversion.

Two studies have offered a positive view of legislative tax uncertainty, Bizer and Judd (1989) and Hasset and Metcalf (1999). Both make specific parametric assumptions on the tax process. The former evaluates welfare improvements on the basis of a comparison of direct efficiency loss and tax revenue increase. The latter measures improvements due to tax uncertainty in terms of reduced delays in the time to invest, and attributes the positive effects of tax uncertainty to a jump diffusion assumption of the tax process.

Our results confirm Hagedorn's (2010), who also challenged the common belief (see, e.g., Chari et al. (1994)) that efficient taxes are supposed to smooth out the dynamics of government expenditures and future liabilities. In fact, our sufficient conditions for taxspots resemble his. As a difference, Hagedorn pointed to the possibility of 2-period deterministic cycles, and his conditions are intertemporal.[5] We show that fiscal policy can be uncertain, within time periods. We further interpret the condition as related to prudence, and establish a link between a random Ramsey problem and taxspot equilibria.

Cole and Kubler (2012) study the computational problems arising from the use of public randomizations in the Marcet and Marimon's (2019) recursive formulation of a planning problem with incentive constraints. They give general conditions to use the saddlepoint solution method, but do not study conditions under which lotteries are Pareto improving.

We tie the planner's problem to competitive equilibria with 'extrinsic' uncertainty, as in Shell and Wright (1993) and related literature on sunspots and lotteries in static economies. We dub the equilibria corresponding to optimal random taxes 'taxspots' because, and unlike the case of sunspots, from the viewpoint of the private economy uncertainty here is on taxes —over and beyond the possibly uncertain flow of government expenditure. Absent tax uncertainty, as individuals are risk averse, there would be no effect of extrinsic uncertainty on equilibria when markets are complete.[6] Here, some tax randomness survives even if markets are complete. Thus, our results are similar in spirit

[5] Hassler, Krusell, Storesletten and Zilibotti (2008) also found optimal oscillatory taxes, but due to time-varying, non-geometric capital depreciation rates.

[6] With inelastic labor supply, equilibria are sunspot-free when taxes are deterministic even with totally incomplete markets, provided a mild condition on the production function is satisfied.

to Garratt, Keister, Cheng-Zhong and Shell (2002), where there are nonconvexities stemming from indivisibilities (see also Rogerson (1988)), and to Kehoe, Levine and Prescott (2002), where they are generated by incentive problems. As a difference, the nonconvexity here shows up through the lack of concavity –not of quasi-concavity–of the constraints. Also, here nonconvexity is a necessary but not sufficient condition for lotteries to matter.[7]

Section 2 presents the issues in the context of a Lucas-Stokey economy. Section 3 considers an economy with capital only, and government bonds. Section 4 finally expands into a workers-capitalists economy without government bonds. The Appendices contain proofs, organized by part of the arguments across the different economies studied: existence proofs (Appendix A), suboptimality proofs (Appendix B), characterization of optimal taxspots (Appendix C), and a perturbation argument (Appendix D).

2 A pure labor economy

It is instructive to start with the simplest Ramsey taxation model, following Lucas-Stokey (1983). An infinite horizon economy faces an uncertain and exogenous stream of government expenditures $g_t, t \geq 0$, with values $g_t \geq 0$ in a finite set G and with transition probability $\pi : G \to \Delta(G)$. A history is $g^t = (g_0, ..., g_t)$. Hereafter, if not otherwise stated all processes are adapted to the tree generated by the Markov chain on G. The set of bounded such processes with values in $\mathbb{R}$ is denoted $L(\mathbb{R})$, and the set of bounded sequences is $\ell(\mathbb{R})$.

The economy is populated by representative individuals with discounted expected utility preferences over consumption $\hat{c}_t \geq 0$ and leisure $x_t \in [0,1]$, with felicity index v and discount factor $\beta \in (0,1)$,

$$\mathbb{E}_0 \sum_{t \geq 0} \beta^t v(\hat{c}_t, x_t).$$

For every $t \geq 0$, material balance of resources requires

$$\hat{c}_t + g_t \ \leq \ 1 - x_t.$$

The government also receives state-contingent repayment obligations $\hat{b}_{0,t}, t \geq 0$, from the past: $\hat{b}_{0,t}(g^t)$ is the amount of payments the government promises to make at date t if history g^t has realized, with $\sup_t |\hat{b}_{0,t}| < \infty$.

There are competitive markets for labor, the consumption good and government debt. Given g_t and $\hat{b}_{0,t}$, $t \geq 0$, the government chooses a process of taxes (labor income linear tax rates) $\tau_t, t \geq 0$ and

[7] Goenka and Prechac (2006) found that, with enough prudence, sunspots could be better for some individuals in a two-period incomplete markets economy. We show in Section 3 that tax-induced uncertainty can be better even in a representative agent infinite horizon economy with complete markets.

a process $\hat{b}_t, t > 0$, with $\hat{b}_t = (\hat{b}_{t,s}, s \geq t)$ of future debt restructuring plans to satisfy the government sequential budget

$$\mathbb{E}_t \sum_{s \geq t} p_s \hat{b}_{t,s} + p_t g_t \; = \; p_t \tau_t (1 - x_t) + \mathbb{E}_t \sum_{s \geq t+1} p_s \hat{b}_{t+1,s},$$

where $p_s > 0$, for $s \geq t$, is the price of consumption at time s in time-0 dollars. It is assumed that at any history g^t there is a complete set of debt plans, one for each future contingency.

Individuals at time t, having invested in government liabilities $\hat{b}_t$ at time $t-1$ and given taxes τ_t and prices p_s for $s \geq t$, choose consumption $\hat{c}_t$ and leisure x_t as well as their holdings of government liabilities $\hat{b}_{t+1}$ with respect to all future contingencies $s > t$, facing the budget

$$p_t \hat{c}_t + \mathbb{E}_t \sum_{s \geq t+1} p_s \hat{b}_{t+1,s} \; \leq \; p_t (1 - \tau_t)(1 - x_t) + \mathbb{E}_t \sum_{s \geq t} p_s \hat{b}_{t,s}.$$

At any date $s > t$ the difference $\hat{b}_{t+1,s} - \hat{b}_{t,s}$ for individuals is a rebalancing in their asset portfolio; for the government, it is a restructuring of its liabilities. Using the present value definition, and since markets are dynamically complete,

$$b_t \; \equiv \; \mathbb{E}_t \sum_{s \geq t} \frac{p_s}{p_t} \hat{b}_{t,s}$$

we can write the budget equivalently as

$$p_t \hat{c}_t + \mathbb{E}_t p_{t+1} b_{t+1} \; \leq \; p_t (1 - \tau_t)(1 - x_t) + p_t b_t,$$

where b_0 (i.e., $\hat{b}_{0,t}, t \geq 0$) is given. Individuals only care about the present value of all future positions at any date t, and adjust their positions at date t in order to carry the proper amount of purchasing power in the successor states. Additionally, $b_t \geq \underline{b}_t$ for every $t > 0$, where $\underline{b}_t$ is a 'natural' debt limit, essentially necessary for the individual optimum to exist with sequential budgets.

An *equilibrium with taxes* τ_t *and liabilities* $\hat{b}_{0,t}, g_t, t \geq 0$, is a vector of consumption-leisure allocations, portfolios and price processes $(\hat{c}_t, x_t, b_t, p_t), t \geq 0$, such that the individual solves

$$\begin{aligned} &\max_{(\hat{c},x,b)} \quad \mathbb{E}_0 \sum_{t \geq 0} \beta^t v(\hat{c}_t, x_t) \\ &\text{s.t.} \quad \begin{cases} p_t \hat{c}_t + \mathbb{E}_t p_{t+1} b_{t+1} & \leq \; p_t (1 - \tau_t)(1 - x_t) + p_t b_t, \; t \geq 0 \\ b_{t+1} & \geq \; -\underline{b}_{t+1}, \; t \geq 0 \end{cases} \end{aligned}$$

with b_0 (i.e., $\hat{b}_0$) given; and markets clear.

Assumptions on utilities. Following the relevant literature, we assume that utilities v are continuous functions $v : \mathbb{R}^2_+ \to \mathbb{R} \cup \{-\infty\}$, at least $\mathscr{C}^3$ on $\mathbb{R}^2_{++}$, strictly increasing and concave in the arguments. We define shorthands $v_{\hat{c}t} \equiv v_{\hat{c}}(\hat{c}_t, x_t)$ and $v_{xt} \equiv v_x(\hat{c}_t, x_t)$ for the first derivatives, similarly $v_{\hat{c}\hat{c}t}$, $v_{x\hat{c}t} = v_{\hat{c}xt}$ and v_{xxt} for the second derivatives, and so on.

2.1 The standard Ramsey problem

In this context, the Ramsey problem is to find the best (linear) tax and debt combination constrained by (competitive) market equilibrium, and given the government expenditure process and the initial liabilities b_0. As standard, the Ramsey problem is expressed in 'primal form', i.e., in the allocation space $(\hat{c}, x)$. To this end, using the fact that markets are complete and a natural debt limit implies a transversality condition at the optimum,

$$\lim_{t\to\infty} \mathbb{E}_0\, p_{t+1} b_{t+1} \;=\; 0,$$

the individual budget can be cast equivalently in Debreu form. Thus, given τ_t and $\hat{b}_{0,t}$, $t \geq 0$, individuals solve the consumption-leisure planning problem

$$\begin{aligned} \max_{(\hat{c},x)} \quad & \mathbb{E}_0 \sum_{t\geq 0} \beta^t v(\hat{c}_t, x_t) \\ \text{s.t.} \quad & \mathbb{E}_0 \sum_{t\geq 0} p_t[\hat{c}_t - \hat{b}_{0,t} - (1-\tau_t)(1-x_t)] \;\leq\; 0. \end{aligned}$$

Note that $\hat{c}_t - \hat{b}_{0,t} - (1-\tau_t)(1-x_t)$ is equal to the government surplus at time t, or $\tau_t(1-x_t) - (g_t + \hat{b}_{0,t})$.

The corresponding first order conditions (FOC) are necessary and sufficient,[8] and using an Inada condition (see A1.b below) they admit an interior solution, i.e., $\hat{c}_t > 0$ (then, via market clearing, $x_t < 1$), and $x_t > 0$, all $t \geq 0$. They are

$$\beta^t v_{\hat{c}t} = v_{\hat{c}0} \text{ and } p_t v_{xt} = (1-\tau_t) v_{\hat{c}t}.$$

After substitution of the FOC into the budget constraint, and using market clearing to eliminate x_t, the *standard Ramsey problem* at liabilities $(\hat{b}_{0,t}, g_t), t \geq 0$, is

$$\begin{aligned} \max_{\hat{c}\in L(\mathbb{R}_+) : \hat{c}_t \leq 1-g_t \,\mathrm{all}\, t\geq 0} \quad & U(\hat{c}) \;\equiv\; \mathbb{E}_0 \sum_{t\geq 0} \beta^t U_t(\hat{c}_t) \\ \text{s. to} \quad & F(\hat{c}) \;\equiv\; \mathbb{E}_0 \sum_{t\geq 0} \beta^t a_t(\hat{c}_t) = 0, \end{aligned}$$

where

$$\begin{aligned} U_t(\hat{c}_t) &\equiv v(\hat{c}_t, 1-g_t-\hat{c}_t) \\ a_t(\hat{c}_t) &\equiv v_{\hat{c}t}(\hat{c}_t - \hat{b}_{0,t}) - v_{xt}(\hat{c}_t + g_t). \end{aligned}$$

Dependence of a_t, thus of F, on $\hat{b}_{0,t}, g_t$ is made explicit solely when necessary.

[8] Variables $(\hat{c}, x)$ are in $L(\mathbb{R}^2_+)$. When endowed with the sup-norm topology, $L(\mathbb{R}^2_+)$ is naturally isomorphic to $\ell^+_\infty \times \ell^+_\infty$. This space has a positive orthant with nonempty interior. The individual problem is convex, and at positive summable prices p the constraint set also has nonempty interior. Thus, Luenberger (1969, Thm 1, p. 217, and its Corollary) applies.

Equation $F(\hat{c}) = 0$ is an 'intertemporal Euler equation'. The choice of fiscal financing instruments τ and $\hat{b}_s, s > 0$ is implicit via the choice of $\hat{c}$, thus of x via market clearing. The allocation affects marginal utilities and then identifies taxes via $\tau_t = 1 - v_{xt}/v_{\hat{c}t}$, prices via the FOC, and finally government debt $\hat{b}_s$ using the sequential (individual) budget, as markets are complete.

The standard Ramsey problem: preliminaries

We follow the usual approach (see, e.g., Ljundqvist and Sargent (2012) and, implicitly, Messner et al. (2018)) of characterizing the Ramsey problem via the auxiliary problem

$$\begin{array}{cc} \max\limits_{\hat{c} \in L(\mathbb{R}_+): \hat{c}_t \leq 1 - g_t, \, \text{all}\, t \geq 0} & U(\hat{c}) \\ \text{s.t.} & F(\hat{c}) \geq 0. \end{array} \tag{M-RP}$$

In problem M-RP the planner is allowed to run a surplus in its intertemporal budget –and correspondingly, individuals spend more than their after-tax lifetime income. However, problem M-RP is shown to be a relaxed version of problem RP. Properties of problem M-RP are known, but it is useful to spell out the assumptions and state the ensuing results.

First, let $\tilde{c}_t$ be the unconstrained maximum of the per period utility $U_t(\hat{c})$, all $\hat{c} \in [0, 1 - g_t]$, and let $a_{\hat{c}t}$ be the first and $a_{\hat{c}\hat{c}t}$ the second derivative of $a_t(\hat{c}_t)$, and $U_{\hat{c}t}$ and $U_{\hat{c}\hat{c}t}$ the first and second derivative of $U_t(\hat{c})$. We use a superscript * when derivatives are computed at a solution $\hat{c}^*$ to problem M-RP. Consider the following assumptions:[9]

A1.a There exists $\hat{c}^o \in \times_{t,g^t}(0, 1 - g_t)$ with $\infty > F(\hat{c}^o) > 0$ and $U(\hat{c}^o) > -\infty$.

A1.b For every $t \geq 0$ and g^t there exists $\bar{a}_t(g^t)$ such that $a_t(\hat{c}) < \bar{a}_t(g^t)$ for all $\hat{c} \in [0, 1 - g_t]$, and $\mathbb{E}_0 \sum_{t \geq 0} \beta^t \bar{a}_t < \infty$, and (i) $v(0, x_t) = v(\hat{c}_t, 0) = -\infty$ or (ii) $\lim_{\hat{c} \to 0} a_{\hat{c}t}(\hat{c}) > 0$ and $\lim_{\hat{c} \to 1 - g_t} a_{\hat{c}t}(\hat{c}) < 0$.

A1.c $\mathbb{E}_0 \sum_{t \geq 0} \beta^t v_{\hat{c}}(\tilde{c}_t, 1 - g_t - \tilde{c}_t)(g_t + \hat{b}_{0,t}) \geq 0$.

[9] Here as in later sections, we give *sufficient* conditions for existence of a solution, to cover the most interesting cases. Some cases may be excluded -like here the separable quadratic case, which may violate A1.b. However, in such case existence is straightforward as all functions are continuous and bounded. Assumption A1.a guarantees nonemptiness of the constraint set. It holds for an open set of government liabilities $g, \hat{b}_0$ including a neighborhood of zero, via existence of standard competitive equilibrium. Indeed, at an interior competitive equilibrium $\hat{c}^e$ a feasible change in the consumption sequence at a single date event has nonzero impact $DF(\hat{c}^e, 0, 0)$ on the constraint by concavity of v. By continuity of F in the parameters $(\hat{b}_0, g)$, the impact will be nonzero in a neighborhood of zero. Assumption A1.b suffices to have interiority of a solution. It is consistent with standard Inada conditions on v for zero consumption and leisure. It is satisfied by separable utilities of the isoelastic family with (RRA) coefficient for consumption σ greater than 1 (recall that $\sigma(\hat{c}) \equiv -v_{\hat{c}\hat{c}}(\hat{c})\hat{c}/v_{\hat{c}}(\hat{c})$) only if $\hat{b}_{0,t} > 0$, all $t > 0$ –however small. Assumption A1.c implies that the initial liabilities $\hat{b}_0, g$ are asking for resources to be saved from the economy. This condition is satisfied when $\hat{b}_{0,t} \geq 0$, i.e., when indeed $\hat{b}_0$ is a process of government liabilities.

A1.d $a_{\hat{c}\hat{c}}(\hat{c},0,0) < 0$ all scalars $\hat{c} \in (0,1]$ and every $t \geq 0$.

Assumption A1.c is standardly used to show that problem M-RP relaxes problem RP, and also to eliminate trivial cases where the 'unconstrained optimum' is a solution. Assumption A1.d is a sufficient condition implying convexity of the constraint set, thus the validity of the Kuhn-Tucker conditions: a Slater (or 'positivity') condition delivers the saddlepoint property of the Lagrangian as a necessary condition for any solution.[10] The set of economies satisfying this concavity condition is nonempty, as it includes the case (used, e.g., in Lucas and Stokey (1983)) of separable quadratic v.

Lemma 1 *Under A1.a,b,c:* [11]

(i) A solution $\hat{c}^*$ to the modified Ramsey problem M-RP exists, and all solutions are interior.

(ii) Any solution to problem M-RP is also solution to problem RP.

Under A1.a-d:

(iii) There exist a neighborhood O of zero government liabilities and a scalar $\lambda \geq 0$ such that $U^*_{\hat{c}t} + \lambda a^*_{\hat{c}t} = 0$, every $t \geq 0$.

(iv) If $\hat{b}_{0,t}(g^t) = \hat{b}_{0,t}(g_t)$, then $\hat{c}^*_t(g^t) = \hat{c}^*(g_t)$.

The proof of Lemma 1.i-ii follows from the proof of Lemma 2, which is provided in Appendix A. Lemma 1.iii-iv are standard results following from the convexity of the problem under A1.d, and their proof is omitted. Lemma 1.iv is the main feature of the solution to the Ramsey problem under A1.d: when $\hat{b}_{0,t}$ only depends on current expenditure, consumption and leisure at time t also only depend on current expenditure. This feature shows that optimal taxes (and debt) are used to smooth out the expenditure shocks. Put it differently, serial correlation in taxes and government debt is inherited from the serial correlation in government expenditures.

The question remains on what happens when A1.d is not satisfied. Observe that, even without A1.d, problem M-RP still has solutions and represents the Ramsey problem. However, F is not guaranteed to be concave. Not only this entails well-known issues both with the necessity and sufficiency of the FOC, as per se they may not identify a saddlepoint, but the lack of concavity of F may also imply that the standard Ramsey problem does not deliver a welfare-maximizing tax solution. This we explore next.

[10] Marcet and Marimont (2019) also offer a solution technique based on the use of a regularity, or invertibility, constraint qualification condition. This difference is immaterial for our main point of discussion.

[11] Existence under A1 may be restricted to a neighborhood of zero government liabilities, as customary, since a competitive equilibrium may not exist, even without expenditures and taxes, unless endowments are restricted. The size of this neighborhood can be quite large, allowing for significantly nonzero government expenditures or initial debt.

2.2 The Ramsey problem is not a social optimum

We show that, outside the cases covered by the usual solution method (in particular, by A1.d), a planner can do better by using lotteries over processes $\hat{c}_t, t \geq 0$.

A *2-point lottery* $(\hat{c}_{1t}, \hat{c}_{2t}, \mu)$ over consumption at g^t is a pair of consumptions and a probability with $\hat{c}_{it} \in [0, 1-g_t]$ for both i and $\mu \in]0,1[$ where $\hat{c}_{1t}$ occurs with probability $\mu > 0$ and $\hat{c}_{2t}$ with probability $1-\mu$. The lottery $(\hat{c}_{1t}, \hat{c}_{2t}, \mu)$ *Pareto improves (at g^t) upon $\hat{c}^*$* provided $\mathbb{E}_\mu[U_t(\hat{c}_t)] > U_t(\hat{c}_t^*(g^t))$ and $\mathbb{E}_\mu[a_t(\hat{c}_t)] \geq a_t(\hat{c}_t^*(g^t))$ so the expected utility is higher and the constraint in problem M-RP is satisfied. Indeed, for $\hat{\beta}^t \equiv \pi(g^t)\beta^t$, $\mathbb{E}_\mu[a_t(\hat{c}_t)] \geq a_t(\hat{c}_t^*(g^t))$ implies

$$\mathbb{E}_0 \sum_{t' \neq t} \beta^{t'} a_{t'}(\hat{c}_{t'}^*) + \sum_{\hat{g}^t \neq g^t} \pi(\hat{g}^t)\beta^t a_t(\hat{c}_t^*(\hat{g}^t); \hat{b}_{0,t}(\hat{g}^t), \hat{g}_t) + \hat{\beta}^t \mathbb{E}_\mu[a_t(\hat{c}_t)] \ \geq \ F(\hat{c}^*) \ \geq \ 0.$$

Thus, lottery μ does not run a government deficit, and yet dominates $\hat{c}^*$. If such a dominating lottery exists, it leads us to consider a relaxed problem in lotteries where surpluses are possible – but are not optimal.

The following is shown in Appendix B.

Proposition 1 *For an RP optimum $\hat{c}^*$ assume there is a date-event g^t for which $a_{\hat{c}t}^* \neq 0$*[12] *and $\tau_t^* > 0$. Suppose*

$$-\frac{a_{\hat{c}\hat{c}t}^*}{a_{\hat{c}t}^*} > -\frac{U_{\hat{c}\hat{c}t}^*}{U_{\hat{c}t}^*}. \tag{D}$$

Then there is a 2-point lottery with $\mathbb{E}_\mu \hat{c}_t > \hat{c}_t^*$ that Pareto improves upon $\hat{c}^*$.

As $\tau_t(\hat{c}_t) = U_{\hat{c}t}/v_{\hat{c}t}$, under $\tau_t^* > 0$, if $a_{\hat{c}t}^* \neq 0$, then by optimality $a_{\hat{c}t}^* < 0$. Hence, condition D requires that the curvature of a_t and U_t be of the opposite sign. When a is strictly concave, i.e., under A1.d, condition D is never satisfied. If $a_{\hat{c}\hat{c}t}^* < 0$ and $a_t^* = a_{\hat{c}t}^* = 0$, then there is only one point satisfying $a_t \geq 0$, and again there is no room for uncertain taxes. Similarly, when $(\hat{b}_{0,t}, g_t) = (0,0)$, all $t \geq 0$, condition D cannot be met, because in such case $\hat{c}^*$ is a competitive equilibrium without taxes, i.e., $U_{\hat{c}t}^* = 0$, all t, and indeed any solution to RP is already the unconstrained global maximum for the function $U(\hat{c})$.

As a corollary to Proposition 1, a little algebra suffices to list sufficient conditions for D to obtain.

Corollary 1 *Suppose that $U_{\hat{c}t}^* > 0$ and $a_{\hat{c}t}^* \neq 0$ at some date-event g^t, $t \geq 0$. A 2-point lottery Pareto improves over $\hat{c}^*$ at g^t if:*

[12] Condition $a_{\hat{c}t}^* \neq 0$ is a regularity property, which in the infinite horizon optimal taxation problems is often assumed, and we follow the literature on this. The case when $T < +\infty$ leads to $a_{\hat{c}t}^* \neq 0$ generically. What if $a_{\hat{c}t}^* = 0$ whenever $U_{\hat{c}t}^* > 0$? Then,

$$\hat{c}_t^* = -\frac{(v_{\hat{c}\hat{c}t}^* - v_{\hat{c}xt}^*)\hat{b}_{0,t} - (v_{xxt}^* - v_{\hat{c}xt}^*)g_t - U_{\hat{c}t}^*}{U_{\hat{c}\hat{c}t}^*} \tag{KE}$$

We note that when $\hat{b}_{0,t} = 0$, it must be $U_{\hat{c}t}^* \geq (v_{\hat{c}xt}^* - v_{xxt}^*)g_t$.

(i) $\hat{c}_t^* > \hat{b}_{0,t}$, *and* $v^*_{\hat{c}\hat{c}\hat{c}t}$ *is positive and large enough;*

(ii) $\hat{b}_{0,t} \geq \hat{c}_t^*$, *and either* $v^*_{\hat{c}\hat{c}xt}$ *or* $v^*_{\hat{c}xxt}$ *is large enough.*

We notice that the conditions implying D never violate standard assumptions of risk aversion and prudence (the third partials $v_{\hat{c}\hat{c}\hat{c}t}$ and v_{xxxt} are positive, i.e., both marginal utilities for consumption and leisure are convex, implying a precautionary savings and precautionary working behavior). In fact, they show that *when prudence is sufficiently high relative to risk aversion, and if there is room for taxes in the standard Ramsey problem, random taxes can be Pareto improving*.

Thus, condition D depends on the strength of prudence, i.e., of the third derivatives of v, as explicitly stated in Corollary 1. As such, it has economic appeal: when prudence is strong enough to overcome risk aversion, i.e., the concavity of U_t, uncertainty in the form of spreads over consumption can be Pareto improving.

One can also regard condition D as quite pervasive if differences in preferences are judged minimal when they result in differences in utilities only around an arbitrary small neighborhood of the RP solution. In Appendix D we formally explore this notion of minimality via utility perturbations. We now provide an example in which, after a small perturbation making $v_{\hat{c}\hat{c}\hat{c}} > 0$ sufficiently large, lotteries do strictly better.

An example. We look at a finite economy with a single period, $t = 0$. Let $g_0 = g = 0$ and $\hat{b}_{0,0} = b = 0.13$. Then, the RP problem is the static problem $\max v(\hat{c}, 1-\hat{c})$ s. to $a(\hat{c}; 0.13, 0) = 0$. Assume that

$$v(\hat{c}, x) = \left(0.002\hat{c}^{-4} + x^{-4}\right)^{-1/4}.$$

Then, $v_{\hat{c}} = \left(0.002\hat{c}^{-4} + x^{-4}\right)^{-5/4} \hat{c}^{-5}$ and $v_x = \left(0.002\hat{c}^{-4} + x^{-4}\right)^{-5/4} x^{-5}$, while

$$a(\hat{c}; 0.13, 0) = \left(0.002\hat{c}^{-4} + x^{-4}\right)^{-5/4} [0.002\hat{c}^{-5}(\hat{c} - b) - (1-\hat{c})^{-5}\hat{c}]$$

Let $\hat{c}^*$ (i.e., $\hat{c}^*, x^*$ with $x^* = 1 - \hat{c}^*$) be the solution to the static problem. It is easily seen that $\hat{c}^* = 0.184337 \neq \tilde{c} = 0.223928$, the unconstrained maximum, and $U^*_{\hat{c}} > 0$, while $a^*_{\hat{c}} < 0$ (a achieves its maximum at 0.159502). Thus, we are in case (i) of the Corollary, and a perturbation of the third derivative of u, making $v_{\hat{c}\hat{c}\hat{c}}$ sufficiently large implies that condition D in Proposition 1 applies. None of the maintained assumptions on utility (except, of course, A1.d) has been violated.

From a mathematical viewpoint, it is not difficult to see that the issue of suboptimality of the deterministic fiscal policy stems not so much from the lack of convexity of the set of consumptions $\hat{c}$ where $a(\hat{c}; b, g) \geq 0$, rather from the lack of concavity of a. Even if a is quasi-concave, there is still a possibility of Pareto improving lotteries, provided condition D is met. In the example, this is seen by truncating the domain of a to the left of its maximum –the function is now monotone, but suboptimality still obtains. It is the nonconvexity of the hypograph of F, not of its upper contour sets, which matters here. We will come back to this later.

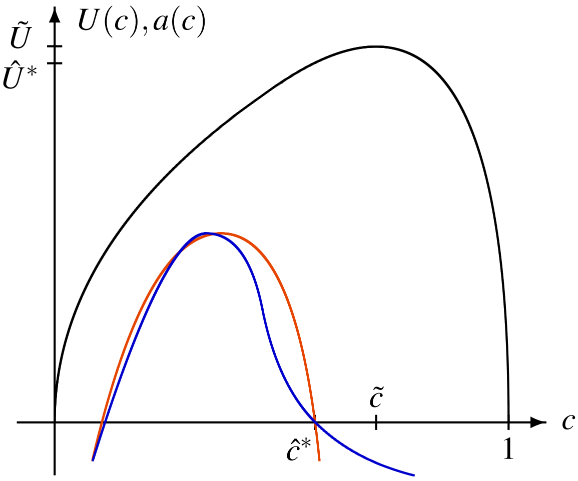

Figure 1: *No taxspots and taxspots.*

Finally, we comment on the average tax induced by the Pareto-improving lottery. Imposing some standard 'normality' assumptions for consumption and leisure,

$$v_{xxt} - v_{\hat{c}xt} \leq 0 \text{ and } v_{\hat{c}\hat{c}t} - v_{\hat{c}xt} \leq 0,$$

the impact of a consumption increase on the tax is negative:

$$\frac{d\tau_t}{d\hat{c}_t} = \frac{v_{xt}(v_{\hat{c}\hat{c}t} - v_{\hat{c}xt}) - v_{\hat{c}t}(v_{\hat{c}xt} - v_{xxt})}{(v_{\hat{c}t})^2} < 0.$$

Hence, the Pareto-improving 2-point lottery lowers taxes in one state. Its overall effect on the average tax is however ambiguous. When the optimal tax function is concave, the average tax induced by the Pareto-improving lottery may actually decrease.

2.3 Optimal lotteries, and taxspots

Given the limitations of the standard Ramsey problem, we are then lead to consider lotteries directly as policy instruments. We recast the planning problem in the space of lotteries over consumption plans.

A consumption plan is a process adapted to the filtration $\{\mathscr{G}_t\}_{t\geq 0}$, i.e., an element of $C = \times_{t\geq 0} \times_{g^t \in \mathscr{G}_t} [0, 1-g_t]$. To avoid decentralization issues with $t = 0$ randomizations, we restrict randomizations to periods $t > 0$, and with some abuse of notation we denote by $\Delta(C)$ the set of (Borel) probability measures over C so restricted.

Then, given liabilities $\hat{b}_{0,t}, g_t$, $t \geq 0$, the planner solves

$$\begin{aligned} \max_{\mu \in \Delta(C)} \quad & \int_C U(\hat{c}) d\mu \\ \text{s.t.} \quad & \int_C F(\hat{c}) d\mu = 0. \end{aligned} \tag{E-RP}$$

A solution μ^* to E-RP is interior if $\mu^*(C \backslash \overset{o}{C}) = 0$ where $\overset{o}{C}$ is the interior of C, i.e., where $0 < \hat{c}_t < 1 - g_t$, all $t \geq 0$.

Lemma 2 *Under A1.a,b,c a solution μ^* to problem E-RP exists, and all solutions are interior.*

For a proof, see Appendix A. We can easily give a 'taxspot' interpretation to the optimal lottery solving E-RP. Let length-t histories be pairs $\hat{\omega}^t = (g^t, s^t)$, where $s^t = (s_0, s_1, ..., s_t)$ and $s_\tau \in S = [0,1]$, all $\tau < t$. Processes are sequences of random variables adapted to the filtration generated by histories $\hat{\omega}^t$.

Let $\nu_t, t \geq 0$ be a process of distributions over S. A *competitive equilibrium with taxspots at liabilities* $\hat{b}_{0,t}, g_t, t \geq 0$, is a process $(\hat{c}_t, x_t, b_t, p_t, \tau_t), t \geq 0$, of consumption, leisure, bond holdings, prices and taxes, and a distribution process $\nu_t, t \geq 0$ such that

$$\max_{\hat{c},x,b} \quad \mathbb{E}_{0,\nu} \textstyle\sum_{t \geq 0} \beta^t v(\hat{c}_t, x_t)$$
$$\text{s.t.} \quad \begin{cases} p_t \hat{c}_t + \mathbb{E}_{t,\nu} p_{t+1} b_{t+1} & \leq & p_t(1-\tau_t)(1-x_t) + p_t b_t, \; t \geq 0 \\ b_{t+1} & \geq & -\underline{b}_{t+1}, \; t \geq 0, \end{cases}$$

with b_0 (i.e., $\hat{b}_0$) given, and markets clear, $\hat{c}_t + g_t = 1 - x_t$. Here, expectations $\mathbb{E}_{t,\nu}$, $t \geq 0$, are taken relative to the expenditure and the taxspot uncertainty, s^t, and we stress the use of the taxspot distribution process $\nu_t, t \geq 0$ by adding a corresponding subscript.[13] If the taxspot is a continuous variable, this market presumes the existence of a continuum of Arrow securities, or of linearly independent assets.

The taxspot variable distribution process ν_t is going to be derived from optimal lottery μ^*. The existence of an optimal taxspot equilibrium does not generally imply persistent policy uncertainty. In fact, market completeness implies that optimal taxspot uncertainty essentially resolves in finite time. That is, any optimal lottery is payoff equivalent to a lottery that translates into taxspots only at finitely many dates. This is stated in the following proposition, proved in Appendix C.

Proposition 2 *If μ^* is a lottery over consumption processes solving problem E-RP at fiscal liabilities $(\hat{b}_{0,t}, g_t), t \geq 0$, then there exists process $(\hat{c}_t, x_t, b_t, p_t, \tau_t), t \geq 0$, and process distribution $\nu_t, t \geq 0$ such that together they are a competitive equilibrium with taxspots at the given liabilities. Moreover, every*

[13] That is, let ν_π^t be the probability measure over histories $\hat{\omega}^t$ induced by π and $\nu_\tau, \tau \leq t$: for any cylinder $B = \times_{\tau \leq t} B_\tau$ where each $B_\tau \in \mathscr{B}[0,1]$, the Borel σ-algebra on $[0,1]$, it is

$$\nu_\pi^t(\{g^t\} \times B; \hat{\omega}^0) = \sum_{\tau \leq t} \pi(g^t) \int_{\times_{\tau \leq t} B_\tau} d\nu_t(s_t|\hat{\omega}^t) d\nu_{t-1}(s_{t-1}|\hat{\omega}^{t-1}) ... d\nu_1(s_1|\hat{\omega}^1) d\nu_0(s_0|\hat{\omega}^0),$$

and

$$\mathbb{E}_{0,\nu} \textstyle\sum_{t=0}^{+\infty} \beta^t v(\hat{c}_t, x_t) = \int \mathbb{E}_0 \textstyle\sum_{t=0}^{+\infty} \beta^t v(\hat{c}_t, x_t) d\nu = \displaystyle\sum_{t \geq 0} \beta^t \int v(\hat{c}_t, 1 - g_t - \hat{c}_t) d\nu_\pi^t.$$

optimal taxspot lottery is payoff equivalent to a taxspot equilibrium where uncertainty dies off in finite time.

Hence, and when $\hat{b}_{0,t}$ is measurable with respect to current expenditures, if A1.d holds, by Lemma 1.iv the optimal fiscal policy is only a function of current expenditures. When A1.d instead does not hold, and if condition D is met, optimal taxes are random and are no longer a deterministic function of current expenditure. Optimal fiscal policy may create randomness which conflicts with the smoothing role of taxes. In other words, serial correlation on government debt does not necessarily follow the serial correlation of government expenditures. However, any equilibrium with taxspot uncertainty can be recast as an equilibrium with uncertainty dying off in finite time. Thereafter, either $\tau_t^* = 0$, or condition D fails, and the Lucas and Stokey's (1983) solution does not hold anymore.

3 A pure capital economy

The previous section showed an economy where taxspot uncertainty generates fluctuations within a single time period, as opposed to via intertemporal cycles. We now move to the case of an economy with capital, where taxspots will affect the intertemporal marginal rate of substitution via random capital taxation. The economy's output is given by a standard concave constant return to scale technology $f(K_t, N_t)$ where K_t is aggregate capital and N_t aggregate labor. To focus on capital taxation, we simplify and assume that: 1) $f(K_t, N_t) = AK_t + BN_t$ where $A > 0$ and $B > 0$; and 2) households do not derive utility from leisure, but inelastically supply their unitary endowment of leisure, and $N_t = 1$, all t –the population size is normalized to one. Capital is normalized to $k_t \geq 0$, and $f(k_t)$ now denotes the production function, while the depreciation factor is $\delta \in [0,1]$. We write $\hat{f}(k_t) = f(k_t) + (1-\delta)k_t$. Initial capital, $k_0 > 0$ is given. Firms rent capital from households at profit-maximizing rental rates, and we let $\hat{R}_t = \hat{f}'(k_t)$ be the gross return on capital. Here and below, for enhanced readability we use the general notation for marginal productivity even though with this production function it is constant. Government initial liabilities $\hat{b}_0$ and expenses g are given as before. Material balance is

$$c_t + k_{t+1} = \hat{f}_t(k_t) \equiv \hat{f}(k_t) - g_t. \tag{M}$$

We continue to assume that the government (and the economy) has access to a complete set of assets (government bonds are state-contingent). Markets for the single physical good and government bonds are competitive. The government taxes away wages —as labor is provided inelastically. It then chooses a sequence of capital linear tax rates $\tau_t, t > 0$ and a sequence $b_t, t > 0$ of debt levels to satisfy the government budget

$$b_t + g_t = \tau_t \hat{R}_t k_t + q_{t+1} \cdot b_{t+1} + B.$$

where q_{t+1} is the bond price vector and p_t, the price of consumption at time t, is normalized to one.

To avoid trivial tax solutions, we apply here a standard restriction on fiscal instruments: only future capital income taxes $\tau_t, t > 0$ can be selected. This must be imposed to avoid trivial front-loading –financing any future expenditure by amassing (present value) of all needed resources at time zero with a non-distortive capital tax. Thus, $\tau_0 = 0$ is also given. Also, throughout we focus on taxes τ_t on capital wealth $\hat{R}_t k_t$ as opposed to on capital income, $R_t k_t$, but the distinction is largely irrelevant for our analysis.

Individuals have utility $u(c_t)$ from consumption $c_t \geq 0$ at time t. Having invested in capital k_t and government bonds b_t at time $t-1$ and given taxes τ_t and prices q_{t+1}, individuals choose consumption c_t, capital $k_{t+1} \geq 0$, as well as their holdings of government bonds $b_{t+1} \geq -\underline{b}_{t+1}$, where $\underline{b}_{t+1}$ is a natural debt limit, facing the budget constraint

$$c_t + k_{t+1} + q_{t+1} \cdot b_{t+1} \leq (1-\tau_t)\hat{R}_t k_t + b_t$$

where b_0 and k_0 are given.

We maintain standard assumptions on utility: $u : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ is continuous, strictly increasing and concave, and $\mathscr{C}^3$ on $\mathbb{R}_{++}$.

Along the lines used for the individual problem without capital, under our Inada's conditions (precisely stated below, as A2.c) it can be shown that a solution to the individual problem is interior ($c_t > 0$ and $k_{t+1} > 0$, all $t \geq 0$) and must satisfy the budgets as equalities, and

$$\begin{aligned} u_{ct} &= \beta \mathbb{E}_t u_{ct+1}(1-\tau_{t+1})\hat{R}_{t+1} \\ q_{t+1} &= \beta \pi_{t+1|t} \tfrac{u_{ct+1}}{u_{ct}} \\ 0 &= \lim_{t\to\infty} \mathbb{E}_0 \beta^t u_{ct} k_{t+1} = \lim_{t\to\infty} \mathbb{E}_0 \beta^t u_{ct+1} b_{t+1}. \end{aligned}$$

Since markets are complete, as in Section 2.1 we can equivalently compress the sequential budgets into a single budget constraint, after multiplying the budget at $t > 0$ by $p_t = \beta^t u_{ct}/u_{c0}$ for $t > 0$, and normalizing $p_0 = 1$. Letting

$$\hat{a}_t(c_t) \equiv u_{ct}(c_t - \hat{b}_{0,t}),$$

and after substitution of market clearing, the primal Ramsey problem is, for given k_0 and $\hat{R}_0$,

$$\begin{array}{cc} \max\limits_{k \in L(\mathbb{R}): 0 \leq k_{t+1} \leq \hat{f}_t(k_t)} & \hat{U}(k) \equiv \mathbb{E}_0 \sum_{t \geq 0} \beta^t u(\hat{f}_t(k_t) - k_{t+1}) \\ \text{s.t.} & \hat{F}(k) \equiv \mathbb{E}_0 \sum_{t \geq 0} \beta^t \hat{a}_t(\hat{f}_t(k_t) - k_{t+1}) = \hat{R}_0 k_0 u_{c0}. \end{array} \tag{RP-K}$$

Once the allocation process (c, k) is found, tax rates are derived implicitly as before, using the FOC for capital and the definition of $\hat{R}_t$.

Just like in the previous section, we consider economies where the taxation problem is nontrivial. To this end, let $\tilde{k}(k_0)$ be the 'adjusted optimal growth' solution, i.e., a solution to RP-K without the constraint $\hat{F}(k) = \hat{R}_0 k_0 u_{c0}$, and let $\tilde{u}_{c0}(k_0) \equiv u_c(\hat{f}_0(k_0) - \tilde{k}_1(k_0))$. Now $\hat{F}(\tilde{k})$ can be greater than, equal or less than $\hat{R}_0 k_0 \tilde{u}_(k_0)$ depending on the values of $\hat{b}_0$. These possibilities affect how we relax the RP-K problem. Hereafter, we focus on the case where $\hat{F}(\tilde{k}(k_0)) < \hat{R}_0 k_0 \tilde{u}_{c0}(k_0)$. Then, we consider the auxiliary problem[14]

$$\begin{aligned} \max_k \quad & \hat{U}(k) \\ \text{s.t.} \quad & \hat{F}(k) \geq \hat{R}_0 k_0 u_{c0}. \end{aligned} \tag{M-RP-K}$$

As in Section 2.1, problem M-RP-K allows the planner to run a primary surplus in the intertemporal budget of the government.

We give sufficient conditions to establish existence of an interior solution to M-RP-K. Let $K = \{k \in L(\mathbb{R}) \mid k_{t+1} \in [0, \hat{f}_t(k_t)], t \geq 0, k_0 \text{ given}\}$ and $\overset{o}{K}$ be its interior, where $k_{t+1} \in (0, \hat{f}_t(k_t))$, all t. Then, we make the following assumptions:

A2.a There exists an interior k^o such that $\infty > \hat{F}(k^o) > \hat{R}_0 k_0 u^o_{c0}$ and $\hat{U}(k^o) > -\infty$.

A2.b There exists a positive maximum accumulation level $\bar{k} : \bar{k} = \hat{f}(\bar{k}) - \max g_t$.

A2.c For all $t \geq 0$ and g^t there exists $\bar{a}_t(g^t)$ such that $\hat{a}_t(\hat{f}_t(k_t) - k_{t+1}) \leq \bar{a}_t(g^t)$ for all interior k, $\mathbb{E}_0 \sum_{t\geq 0} \beta^t \bar{a}_t < \infty$, and (i) $u(0) = -\infty$ or (ii) $\lim_{c\to 0} \hat{a}_t(c) = -\infty$.

A2.d $\hat{F}(\tilde{k}(k_0)) < \hat{R}_0 k_0 \tilde{u}_{c0}(k_0)$.

A2.a is used to determine nonemptiness of the constraint set. In requiring a uniform upper bound to the period utility, A2.b can be weakened to simply ask that utility is bounded from above on the feasible set, but the stronger form is enough for our purposes here. It is implied here by $A > \delta$. A2.c plays the role of A1.b in Section 2.1. A2.d mimics A1.c: the unconstrained (zero tax) optimum growth cannot be a solution to RP-K. It holds for sufficiently high initial liabilities $g_t, \hat{b}_{0,t}$, for at least some $t \geq 0$, and when labor taxes cannot alone always finance g_t. The latter is satisfied if, e.g., $B < \underline{g} = \min g_t$. Under A2, problem M-RP-K is a relaxation of problem RP-K, and it has an interior solution. The proof follows the logic of Lemma 2, and it is left to the reader.[15]

[14] A mirror image situation occurs when $\hat{F}(\tilde{k}(k_0)) > \hat{R}_0 k_0 u_c(\tilde{c}_0(k_0))$. Then, the relaxed problem takes the form of utility maximization under the constraint $\hat{F}(k) \leq \hat{R}_0 k_0 u_{c0}$.

[15] The case when u is isoelastic must be treated separately, as A2.c may fail –for instance, when $\hat{b}_{0,t} = 0$ for all $t \geq 0$. However, in the latter case and when $\sigma \neq 1$, then $\hat{F}(k) = \hat{U}(k)(1-\sigma)$, and the constraint in problem RP can be written as $\hat{U}(k)(1-\sigma) = \hat{R}_0 k_0 u_{c0}$, so product continuity of the constraint over a product compact, connected subset of K is readily established (using A2.b, and $u(0) = -\infty$ when $\sigma > 1$, when this product compact set is contained in $\overset{o}{K}$). When A2.a,d hold, a highest utility feasible k satisfying the previous equation then exists by the Intermediate Value Theorem. When $\sigma > 1$, A2.a,d hold immediately. When $\sigma < 1$, and if A2.d is violated, then A.2a holds for $k^o = \tilde{k}$, and A2.d holds for the feasible process with $c_t = 0$ all $t \geq 0$. If instead A2.d holds, then additional parametric conditions must guarantee A2.a. When $\sigma = 1$, the constraint is substituted with an initial condition on k_1, pinning down its value, so clearly the RP-K problem has a solution –optimal growth starting from k_1 as initial value.

Lemma 3 *Under A2:*

(i) A solution k^* to the modified Ramsey problem M-RP-K exists, and it is always interior.

(ii) At any solution k^* to M-RP-K it is $\hat{F}(k^*) = \hat{R}_0 k_0 u_{c0}^*$.

Average taxes. Before we address the optimal tax, we need to introduce a notion of average tax. Indeed, uncertainty of g_t already implies indeterminacy of ex-post tax rates —a well-known result found in Zhu (1990). Here, uncertainty may stem directly from τ_t. Thus, hereafter we focus on the long-run behavior of average taxes. Following the literature, we define an average capital tax via the marginal returns kernel as follows. Let $\psi_{t+1} \equiv u_{ct+1}\hat{R}_{t+1}/\mathbb{E}_t u_{ct+1}\hat{R}_{t+1}$, and note that $\psi_{t+1} > 0$ and $\mathbb{E}_t \psi_{t+1} = 1$. Let Ψ be the measure over histories defined using $\psi_{t+1}\pi$ as the density, and let $\mathbb{E}_t^{\Psi}$ be the expectation taken with respect to this probability measure. The Ψ-average capital tax is

$$\mathbb{E}_t^{\Psi} \tau_{t+1} \equiv 1 - \frac{u_{ct}}{\beta \mathbb{E}_t u_{ct+1}\hat{R}_{t+1}}.$$

3.1 Pareto improving lotteries

With no government expenditure uncertainty, and in fact with $g_t = g$, all t, and thus with a deterministic (interior, via Lemma 3) solution to problem M-RP-K, the first order conditions (FOC) for capital and no arbitrage then lead to[16]

$$(1-\tau_{t+1})\hat{R}_{t+1} = \frac{1}{q_{t+1}}.$$

Suppose that $(A+1-\delta)\beta = 1$. If the solution to M-RP-K is a steady state, combining no arbitrage and the FOC, it is

$$(1-\tau)\hat{f}'(k) = \frac{1}{q} = \frac{1}{\beta} = \hat{f}'(k),$$

implying $\tau = 0$, the celebrated Chamley-Judd result. However, it is not at all obvious that, absent concavity of $\hat{a}_t$, problem M-RP-K represents the normative standard for capital taxes, or that average taxes should then be zero. We explore this point next.

We start off by reproducing in this setting condition D, and the Pareto suboptimality of the solution to the standard Ramsey problem. Let

$$\begin{aligned} U_{t,t+1}(k_{t+1}) &\equiv u(\hat{f}_t(k_t) - k_{t+1}) + \beta \mathbb{E}_t u(\hat{f}_t(k_{t+1}) - k_{t+2}), \\ a_{t,t+1}(k_{t+1}) &\equiv \hat{a}_t(\hat{f}_t(k_t) - k_{t+1}) + \beta \mathbb{E}_t \hat{a}_{t+1}(\hat{f}_t(k_{t+1}) - k_{t+2}). \end{aligned}$$

The following is proved in Appendix B.

[16]As in the pure labor economy, problem M-RP-K is convex if $\hat{a}_{cct} < 0$ all t. Together with condition A2.a, this means that the Kuhn-Tucker conditions can be used to characterize its solutions. Alternatively a regularity condition, i.e., that $D\hat{F}(k^*)$ be onto, can lead to stationarity of the Lagrangian for problem M-RP-K, i.e., to FOC, as necessary for a solution. A similar proof is provided for Lemma 5.

Proposition 3 *Let k^* be a solution to problem M-RP-K and let $\mathbb{E}_t^\Psi \tau_{t+1}^* > 0$ and $a_{t,t+1}'^* \neq 0$ at some $t > 0$. Then, if*

$$-\frac{a_{t,t+1}''^*}{a_{t,t+1}'^*} > -\frac{U_{t,t+1}''^*}{U_{t,t+1}'^*} \tag{D-K}$$

a 2-point lottery at t Pareto improves over the M-RP-K solution. Further, $\mathbb{E}_\mu k_{t+1} > k_{t+1}^*$.

The statement obviously applies to ex-post tax rates when there is no government expenditure uncertainty to start with.

We note that when the Ψ-average tax is positive, $U_{t,t+1}'^* > 0$. This and optimality imply that $a_{t,t+1}'^* \leq 0$. If $a_{t,t+1}'^* \neq 0$, a weak regularity condition, it is then $a_{t,t+1}'^* < 0$, and the condition of the statement implies $a_{t,t+1}''^* > 0$. Simple algebra shows that the following sufficient conditions lead to D-K.

Corollary 2 *Suppose that $\mathbb{E}_t^\Psi \tau_{t+1}^* > 0$ and $\hat{a}_{ct}^* \neq 0$ at some date-event g^t, $t \geq 0$. A 2-point lottery Pareto improves over c^* at g^t if $c_t^* > \hat{b}_{0,t}$ and u_{ccct}^* is positive and large enough.*

When u is isoelastic, it is

$$\hat{a}_{cct} = u_{cct}(c_t)(1 - \sigma + \sigma \hat{b}_{0,t}/c_t) - u_{ct}(c_t)\sigma \hat{b}_{0,t}/c_t^2.$$

When $\hat{b}_{0,t} = 0$ for all $t > 0$ and $\sigma > 1$, it is $\hat{a}_{ct} = u_{ct}(c_t)(1-\sigma) < 0$ and $\hat{a}_{cct} = u_{cct}(c_t)(1-\sigma) > 0$, so function $\hat{a}_t$ is decreasing and convex, not concave, but condition D-K does not obtain, as

$$U_{t,t+1}' = a_{t,t+1}'/(1-\sigma) \text{ and } U_{t,t+1}'' = a_{t,t+1}''/(1-\sigma).$$

However, an arbitrarily small increase in the third-order derivative coefficient is going to deliver condition D-K, and if $\mathbb{E}_t^\Psi \tau_{t+1}^* > 0$, lotteries can do better than the solution to M-RP-K. Thus, condition D-K can be seen to hold pervasively around the isoelastic case, in the sense made formal in Appendix D.

In a world with no government expenditure uncertainty to start with, the claim that optimal taxes are deterministic and positive in a steady state (or in the long run) implies that $\mathbb{E}_t^\Psi \tau_{t+1}^* = \tau_{t+1}^* > 0$ for some large t. Hence, the previous proposition and corollary show that such a claim cannot be robust, especially around isoelastic utilities. Lotteries over capital, and thus taxspots, would 'typically' be optimal in that case.

Risk aversion and prudence. In the capital economy, when positive (average) capital taxes are obtained, and $U_{t,t+1}'^* > 0$, it is $\beta \mathbb{E}_t u_{ct+1} \hat{R}_{t+1} - u_{ct} > 0$, which means that the planner wants to trade off consumption today for capital (i.e., consumption tomorrow), but this trade-off is cut short by the Euler equation condition: the planner needs to convince the individual to demand more capital at t.

Inducing a higher demand for capital is tantamount to inducing higher savings. In problem M-RP-K, the planner does so by controlling the implicit (average) rate of return $\beta\mathbb{E}_t u_{ct+1}(1-\tau_{t+1})\hat{R}_{t+1}$. However, a second method exists to raise savings, namely by leveraging on the individual's prudence via mean-preserving spreads in returns. Condition D-K states that, for any additional uncertainty to pay off, prudence should be large enough to overcome the negative effect of the additional uncertainty due to risk aversion. Under isoelastic utilities this is not possible, because the two properties are linked by the same parameter, σ; but for arbitrarily close utilities, and $\sigma \neq 1$, taxspot uncertainty pays off. In the particular case where $\sigma > 1$, it is well-known that lowering the (average) return rate is going to increase the demand for capital. Thus, increasing taxes corresponds to raising the demand for capital. Whether or not injecting spreads is enough to eventually substitute out a lower average return, or an increase in taxes, relative to the first best is discussed further below.

3.2 Optimal taxspots

With lotteries we modify the Ramsey problem as we did for the labor-only economy. A capital process k adapted to the filtration $\{\mathscr{G}_t\}_{t\geq 0}$ generated by histories g^t is an element of K. For every $t \geq 0$ and every g^t, consumption is derived using feasibility. To focus on decentralizable lotteries, we impose no randomization over k_1, and let $\Delta(K)$ be the set of (Borel) probability measures over K satisfying this restriction. The extended Ramsey problem now is, given initial liabilities $\hat{b}_{0,t}, g_t$, $t \geq 0$, and initial capital k_0,

$$\begin{array}{c} \max_{\mu\in\Delta(K)} \int \hat{U}(k)d\mu \qquad \text{s. to} \\ \int[\hat{F}(k) - \hat{R}_0 k_0 u_{c0}]d\mu = 0 \end{array} \tag{E-RP-K}$$

A solution μ^* to E-RP-K is interior if $\mu^*[K \backslash \overset{o}{K}] = 0$. Existence of an interior solution is established, via the use of a 'relaxed' problem, using Lemma 3 and following the logic of Lemma 2. We state the result without proof.

Lemma 4 *Under A2, a solution μ^* to problem E-RP-K exists, and all solutions are interior.*

To go from optimal lotteries to taxspot equilibria we can also mimic what we did in Section 2.3. To define a taxspot equilibrium, we let $\{\mathscr{F}_t\}_{t\geq 0}$ be the filtration generated by histories $\hat{\omega}^t = (g^t, s^t)$, $t \geq 0$, where $s_t \in S = [0,1]$, and consider processes adapted to $\{\mathscr{F}_t\}_{t\geq 0}$. A *competitive taxspot equilibrium at liabilities* $\hat{b}_{0,t}, g_t, t \geq 0$, is a process $\left(c_t, k_t, b_t, q_t, \hat{R}_t, \tau_t\right), t \geq 0$, and a distribution process $\nu_t, t \geq 0$ over histories s^t, $t \geq 0$ with $s_t \in [0,1]$, such that

$$\begin{array}{c} \max_{c,k,b} \mathbb{E}_{0,\nu} \sum_{t\geq 0} \beta^t u(c_t) \qquad \text{s. to} \\ c_t + k_{t+1} + q_{t+1} \cdot b_{t+1} \leq (1-\tau_t)\hat{R}_t k_t + b_t \quad t \geq 0 \\ b_t \geq -\underline{b}_t, t > 0, \end{array}$$

given b_0 and $k_0 > 0$, and $\underline{b}_t$ a natural debt limit, with $\hat{R}_t = \hat{f}'(k_t)$ and market clearing

$$c_t + k_{t+1} \leq \hat{f}_t(k_t).$$

Here, again expectations $\mathbb{E}_{0,\nu}$ are taken relative to the expenditure and to the taxspot uncertainty, s^t.[17] An argument in all similar to the one offered in Section 2.3 delivers taxspots equilibria from optimal lotteries solving E-RP-K, with the property that randomness essentially disappears in finite time. The proof of this proposition is also left to the reader.

Proposition 4 *If μ^* is a lottery over capital processes solving E-RP-K at fiscal liabilities $(\hat{b}_{0,t}, g_t)$, $t \geq 0$, then there exists process $(c_t, k_t, b_t, q_t, \hat{R}_t, \tau_t), t \geq 0$, and distribution process $\nu_t, t \geq 0$ such that they constitute a competitive equilibrium with taxspots at those liabilities. Moreover, every optimal taxspot lottery is payoff equivalent to a taxspot equilibrium where uncertainty dies off in finite time.*

As in Section 2.3, Proposition 3 can be used to establish that at the optimal taxspot equilibrium condition D-K fails because either average taxes are zero or a regularity condition fails. Like for the economies without capital, market completeness implies that there is always an equilibrium where all taxspot uncertainty will be resolved in finite time, and will not occur over more than three sample paths of capital growth, thus at no more than two dates. We now revisit the question of whether optimal taxspot (average) tax rates are positive 'in the long run'. Now the average must take into account the optimal taxspot distribution $\nu_t^*, t > 0$. Let $\psi_{t+1,\nu^*}(\omega^{t+1}) \equiv u_{ct+1}\hat{R}_{t+1}/\mathbb{E}_{t,\nu^*}u_{ct+1}\hat{R}_{t+1}$ and let $\Psi(\nu^*)$ be the measure over histories defined using $\psi_{t+1,\nu^*}d\nu^{*t}_{\pi,t-1}$ as the density, and let $\mathbb{E}_t^{\Psi(\nu^*)}$ be the expectation taken with respect to this probability measure.

Using a form of regularity, we start by giving a martingale condition that characterizes nonnegative average taxspots. Let IE denote the intertemporal Euler constraint in M-RP-K, $\hat{F}(k) \geq \hat{R}_0 k_0 u_{c0}$; let MKT denote the market clearing constraint. We say that a constraint is *binding* if relaxing the constraint is improving the value function. We say that a taxspot equilibrium satisfies *regularity* if

(a) constraint IE is binding;

(b) constraint MKT is binding at all $t \geq 0$.

Next, for any pair $\omega^t = (g^t, k^t)$, let

$$M_t(\omega^t) \equiv \frac{\hat{a}_{ct}(\omega^t)}{u_{ct}(\omega^t)},$$

or in short, $M_t = \frac{\hat{a}_{ct}}{u_{ct}}$. We prove the next statement in Appendix C.

[17] That is, letting ν_π^t be the probability measure over histories (g^t, s^t) induced by π and $\nu_\tau, \tau \leq t$, it is

$$\mathbb{E}_{0,\nu} \sum_{t\geq 0} \beta^t u(c_t) = \int \mathbb{E}_0 \sum_{t\geq 0} \beta^t u(c_t) d\nu = \sum_{t\geq 0} \beta^t \int u(c_t) d\nu_\pi^t.$$

Lemma 5 *Under regularity, at an optimal taxspot equilibrium it is* $\mathbb{E}_t^{\Psi(\nu^*)}\tau_{t+1} \geq 0$ *if and only if* M_t *is a supermartingale under* $\Psi(\nu^*)$*, and equals zero if and only if* M_t *is a martingale at* $t \geq 0$.

With isoelastic utility with coefficient $\sigma \neq 1$ and with $\hat{b}_{0,t} = 0$, by the definition of M_t the latter is always a martingale, and $\bar{\tau}_{t+1} = 0$, all $t \geq 0$. Process $\Psi_t(\nu^*), t \geq 0$ defined via

$$\Psi_t(\nu^*) \equiv \Pi_{t'=0}^{t} \frac{u_{ct'+1}\hat{R}_{t'+1}}{\mathbb{E}_{t',\nu^*}u_{ct'+1}\hat{R}_{t'+1}}$$

is itself a nonnegative martingale, and the equivalent martingale condition can be written equivalently as

$$\Psi_t(\nu^*)M_t \geq \mathbb{E}_t \Psi_{t+1}(\nu^*)M_{t+1},$$

stating that $\Psi_t(\nu^*)M_t, t \geq 0$ is a bounded (super)martingale at equilibrium, and by Doob's Martingale Convergence Theorem both M_t and $\Psi_t(\nu^*)M_t$ converge almost surely to a finite random variable.

Next, we specialize to environments where the E-RP-K problem is stationary with a regular, stationary, ergodic solution over a compact state space. Specifically, let $G \times [0,1]$ be the compact state space with element $z = (g,s)$. We assume that all policy functions at the taxspot equilibrium are continuous time-invariant functions of the states, i.e., in particular,

$$c_t = c(z_t),\ k_{t+1} = k(z_t), \text{ and } b_{t+1} = b(z_t).$$

Thus, $M_t = M(z_t)$ and $\mathbb{E}_t^{\Psi(\nu^*)}\tau_{t+1} = \bar{\tau}(z_t)$. Let Pr be the transition probability over z derived from ν_π and the policy functions. Then, Pr is continuous as a function of z_t. For any Borel set $B \subset G \times [0,1]$ let

$$\mu^\infty(B) \equiv \lim_{j\to\infty} \Pr[z_{t+j} \in B | z_t]$$

all $t \geq 0$, be the corresponding stationary, ergodic distribution.

Proposition 5 *Under regularity, at an optimal stationary, ergodic taxspot equilibrium either* $\mathbb{E}_t^{\Psi(\nu^*)}\tau_{t+1} = 0$, μ^∞*-a.s., or if* $\mu^\infty\{z : \bar{\tau}(z) > 0\} > 0$*, then* $\mu^\infty\{z : \bar{\tau}(z) < 0\} > 0$.

The proof follows Zhu (1990) closely, and is therefore left to the reader. As we found out in Section 3.1, in the 'convex $\hat{a}$' case, except for the knife-edge case with isoelastic utility and $\hat{b}_{0,t} = 0$, all $t > 0$, taxspots are Pareto dominating. We just showed that at a (regular) stationary taxspot equilibrium (average) capital taxes are then zero. In particular, when $\sigma > 1$ and $\hat{b}_{0,t} = \hat{b}_0 \neq 0$, all $t > 0$, long-term average capital taxes will be zero.

4 Capital, labor, and no government bonds

So far we have underlined the economic intuition for the optimality of taxspots, but we have yet to show an environment where taxspots significantly matter –i.e., they occur more than finitely many times at every equilibrium. To this end, we now move to study an economy where there are no government bonds. Thus, there are no financial markets. To focus on purely endogenous uncertainty, and to simplify notation, hereafter we assume no initial uncertainty (thus, g_t is deterministic).

In fact, it is easy to cast our study in a Judd (1985) - inspired variant of the economy above where there is also heterogeneity in the population: in addition to our existing individuals owning capital k_0 and the production technology, and dubbed 'capitalists', there are hand-to-mouth 'workers' who supply labor. That is, we add an individual with per-period utility $v(\hat{c}_t, x_t)$, where $\hat{c}_t \geq 0$ is the worker's consumption and $x_t \in [0,1]$ is leisure. Workers also are endowed with some nontaxable wealth $e_t \geq 0$. The production function is $f(k_t, 1-x_t)$, where k_t is per capita capital –we borrow the notation from the previous sections when possible. The sizes of the populations of capitalists and workers are normalized to one.

In the competitive market, the government taxes both labor and capital income. With τ_t^x and τ_t^k the labor and capital tax rates, respectively, the government budget then is

$$\hat{b}_{0,t} + g_t = \tau_t^x w_t (1-x_t) + \tau_t^k \hat{R}_t k_t,$$

where w_t is the wage, and the workers' budget is

$$\hat{c}_t \leq (1-\tau_t^x) w_t (1-x_t) + e_t,$$

while the capitalists' budget is

$$c_t + k_{t+1} \leq \hat{b}_{0,t} + (1-\tau_t^k)\hat{R}_t k_t,$$

where k_0 (and thus, $\hat{R}_0$) is given. Market clearing is modified to

$$\hat{c}_t + c_t + k_{t+1} \leq \hat{f}_t(k_t, 1-x_t),$$

all $t \geq 0$, where $\hat{f}_t(k_t, 1-x_t) = f(k_t, 1-x_t) + (1-\delta)k_t + e_t - g_t$.

Beyond the maintained assumptions on u and v, we also impose that:

- For $\hat{c} > 0$ and $0 < x < 1$,

$$\lim_{x \to 0} \frac{v_x(\hat{c}, x)}{v_{\hat{c}}(\hat{c}, x)} = \infty \text{ and } \lim_{x \to 1} \frac{v_x(\hat{c}, x)}{v_{\hat{c}}(\hat{c}, x)} = 0.$$

- $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ is continuous, $\mathscr{C}^2$ on $\mathbb{R}_{++}^2$, differentially strictly increasing and concave, exhibiting constant returns to scale.

At a competitive equilibrium, profit-maximizing firms rent capital and labor, setting their demand for capital and labor so that $\hat{R}_t = f_1(k_t, 1-x_t) + 1 - \delta$ and $w_t = f_2(k_t, 1-x_t)$. Further, first order conditions for labor and capital mimic those found in previous sections, namely,

$$\frac{v_{xt}}{v_{\hat{c}t}} = (1-\tau_t^x) w_t \text{ and } \beta u_{ct+1}(1-\tau_{t+1}^k)\hat{R}_{t+1} = u_{ct}.$$

Using the primal approach, equilibrium constraints in the Ramsey problem, now dubbed RP-WK, are

$$\begin{aligned}
F_{1,t}(z) &\equiv \hat{f}_t(k_t,1-x_t)-\hat{c}_t-c_t-k_{t+1} \;=\; 0, \text{ all } t\geq 0\\
F_{2,t}(z) &\equiv \beta u_{ct+1}(c_{t+1}+k_{t+2}-\hat{b}_{0,t+1})-u_{ct}k_{t+1} \;=\; 0, \text{ all } t\geq 0\\
F_{3,t}(z) &\equiv v_{\hat{c}t}(\hat{c}_t-e_t)-v_{xt}(1-x_t) \;=\; 0, \text{ all } t\geq 0\\
\lim_{t\to\infty}\beta^t u_{ct}k_{t+1} &= 0
\end{aligned}$$

where $z\equiv(\hat{c},x,c,k)$ are the sequences of workers' consumption and leisure, and capitalists' consumption and investment, and k_0 is given. Here $F_{2,t}=0$ and $F_{3,t}=0$ are the first order condition for capital and leisure, respectively, after substitution of the agents' budget constraints –used as definitions of $(1-\tau_t^k)\hat{R}_t$ and $(1-\tau_t^x)w_t$. Letting $U_t(z,\theta)=(1-\theta)v(\hat{c}_t,x_t)+\theta u(c_t)$ for $0\leq\theta<1$, and $F_t=(F_{1,t},F_{2,t},F_{3,t})$, consider the modified Ramsey problem

$$\begin{aligned}
&\max_{z\in\ell(\mathbb{R}^4_+)} \quad U(z,\theta)\equiv\sum_{t\geq 0}\beta^t U_t(z,\theta)\\
&\text{s.t.}\quad \begin{cases} F_t(z) \;\geq\; 0, \text{ all } t\geq 0,\\ \lim_{t\to\infty}\beta^t u_{ct}k_{t+1} \;=\; 0, \text{ and } k_0 \text{ given.}\end{cases}
\end{aligned} \tag{M-RP-WK}$$

In problem M-RP-WK, the planner is allowed to run a primary surplus at any date $t\geq 0$, but never a deficit.[18] Surpluses cannot be carried over to the future, or borrowed against in the past, though.

To establish existence of an interior solution to M-RP-WK, we introduce the following notation and assumptions. Let $Z=C\times X\times C\times K\subset\ell(\mathbb{R}^4_+)$ the set of sequences of workers' consumption and leisure, capitalists' consumption and capital $z=(z_t,t\geq 0)$ satisfying

$$z_t\in Z_t=\{(\hat{c}_t,x_t,c_t,k_{t+1})\in\mathbb{R}^4_+ : 0\leq\hat{c}_t,c_t,k_{t+1}\leq\hat{f}_t(k_t,1-x_t),0\leq x_t\leq 1\};$$

and let $\overset{o}{Z}$ be its interior —where in particular $x_t\in(0,1)$ all $t\geq 0$. Sequence z is interior if it is in $\overset{o}{Z}$. Let $\sigma_u(c)$ for any scalar $c>0$ be the capitalist's RRA coefficient. We assume that

A3.a There exists an interior $z^o : F_t(z^o)\geq 0$, all $t\geq 0$, and $\lim_{t\to\infty}\beta^t u^o_{ct}k^o_{t+1}=0$, and $U(z^o,\theta)>-\infty$.

A3.b There exists a maximum accumulation level $\bar{k}:\bar{k}=\hat{f}(\bar{k},1)$, and $\min g_t=\underline{g}>\lim_{KL\to 0}\hat{f}(K,L)$.

A3.c Either (i) $\hat{b}_{0,t}\geq\underline{b}>0$ for all $t\geq 0$, and $\lim_{c\to 0}u_c(c)=\infty$; or (ii) $\hat{b}_{0,t}=0$ all $t\geq 0$, and $\sigma_u(c)>1$ all $c>0$ and $\lim_{y\to\infty}\sum_{t'\geq t}u_c^{-1}(y/(\beta^{t'-t}))=0$, all $t\geq 0$.

[18] To see this, observe that, as $\beta u_{ct+1}(1-\tau^k_{t+1})\hat{R}_{t+1}=u_{ct}$ and $v_{xt}=v_{\hat{c}t}(1-\tau^x_t)w_t$ at any corresponding market equilibrium, the inequalities $F_{2,t}(z)\geq 0$ and $F_{3,t+1}(z)\geq 0$ imply $c_{t+1}+k_{t+2}\geq\hat{b}_{0,t+1}+(1-\tau^k_{t+1})\hat{R}_{t+1}k_{t+1}$ and $\hat{c}_{t+1}\geq e_{t+1}+(1-\tau^x_{t+1})w_{t+1}(1-x_{t+1})$. From feasibility, and using the equilibrium values of $\hat{R}_{t+1}$ and w_{t+1}, it is seen that $\hat{b}_{0,t+1}+g_{t+1}\leq\tau^x_{t+1}w_{t+1}(1-x_{t+1})+\tau^k_{t+1}\hat{R}_{t+1}k_{t+1}$.

A3.d Either (i) $v(0,x) = v(\hat{c},0) = -\infty$; or (ii) $F_{3,t}(z) < 0$ when $\hat{c}_t x_t = 0$, $x_t < 1$.

The assumptions on v imply that a vanishing disposable salary yields a vanishing supply of labor, which contradicts feasibility with positive government expenditures.

Our assumptions are more specific than in previous sections, but guarantee existence in the important benchmark cases we focus on below.

Lemma 6 *Under our maintained assumptions on u, v and f, and A3, a solution z^* to problem M-RP-WK exists, and solutions are interior. Further, $\hat{c}_t^* > e_t$, all $t \geq 0$, and* $\liminf_t c_t^* > 0$.

Next, we impose conditions that imply that M-RP-WK is a relaxation of RP-WK, i.e., that at a solution the constraints hold as equalities. These conditions mimic A2.d in ruling out that the adjusted optimal growth solution is still a solution in problem M-RP-WK. They are more complex because here there are multiple trade-offs that need not to vanish. For all $t \geq 0$, let $\sigma_{ut} \equiv \sigma_u(c_t)$, and consider the matrix

$$A_t = \begin{bmatrix} -f_2(k_t, 1-x_t) & -1 \\ D_{x_t} F_{3,t} & D_{\hat{c}_t} F_{3,t} \end{bmatrix}.$$

Consider the following conditions: for all $t \geq 0$,

$$f_2(k_t, 1-x_t) = \frac{v_x(\hat{c}_t, x_t)}{v_{\hat{c}}(\hat{c}_t, x_t)} \Rightarrow \hat{c}_t \leq e_t + f_2(k_t, 1-x_t)(1-x_t) \tag{R1}$$

$$(1-\theta)\beta \frac{f_{1t}}{\det A_t}(v_{\hat{c}t} D_{x_t} F_{3,t} - v_{xt} D_{\hat{c}_t} F_{3,t}) - \theta u_{ct-1} > 0. \tag{R2}$$

Condition (R1) states that for any capital level, the unconstrained maximum for workers is attainable without subsidies. Condition (R2) is a priori compatible with any labor income tax value. It states that the economy's constrained best leaves room for cross-time transfers from capitalists to workers. Condition (R2) requires $\det A_t \neq 0$, a regularity condition. The latter is non-vacuous: economies where v is separable in $\hat{c}$ and x, the RRA coefficient satisfies $\sigma_{\hat{c}vt} \equiv \sigma_{\hat{c}v}(\hat{c}_t) > 1$ and where $e_t = 0$, display this property at every t.

Lemma 7 *(i) If at z^* (R1) holds, $F_{j,t}(z^*) = 0$, $j = 1, 3$, all $t \geq 0$.*

(ii) If in addition (R2) holds, then it is $F_{2,t}(z^*) = 0$, all $t \geq 0$.

Lemma 7 shows that, under (R), the planner will never optimally run primary surpluses. Hereafter, we confine our attention to environments where condition (R) holds. We obtain first order conditions. First, write the sequence of inequality constraints in problem M-RP-WK as $F(z) \geq 0$, where $F = F_t, t \geq 0$. A solution z^* to problem M-RP-WK is *regular* if the linear map $DF(z^*)$ is onto. Without regularity or other constraint qualifications, the Kuhn-Tucker conditions are not useful to characterize a solution to the optimization problem. Under regularity, the Kuhn-Tucker theorem can be applied to claim the usual dual characterization.[19] To further guarantee the existence of summable nonnegative multipliers, we follow Rustichini (1998). Summability is satisfied in problem M-RP-WK at any regular solution. Let $f_{2t} \equiv f_2(k_t, 1-x_t)$, and $\kappa_{t+1} \equiv k_{t+1}/c_t$.

Lemma 8 *Let a solution z^* to problem M-RP-WK satisfy condition (R). Then,*

(i) z^* is regular, and there is a process of nonnegative multipliers $\nu_t, \lambda_t, \phi_t, t > 0$ such that z^* satisfies

$$\begin{cases} \nu_t u_{ct} &= (1-\theta)v_{\hat{c}t} + \phi_t[v_{\hat{c}t} + v_{\hat{c}\hat{c}t}(\hat{c}_t - e_t) - v_{\hat{c}xt}(1-x_t)] \\ \nu_t u_{ct} f_{2t} &= (1-\theta)v_{xt} + \phi_t[v_{xt} - v_{xxt}(1-x_t) + v_{\hat{c}xt}(\hat{c}_t - e_t)] \\ \lambda_t &= \lambda_{t-1}(\frac{\sigma_{ut}-1}{\sigma_{ut}\kappa_{t+1}} + 1 - \sigma_{ut}\frac{\hat{b}_{0,t}}{\kappa_{t+1}}) + \frac{1}{\sigma_{ut}\kappa_{t+1}}(\nu_t - \theta) \\ \beta\nu_{t+1}u_{ct+1}\hat{R}_{1t+1} - \nu_t u_{ct} &= (\lambda_t - \lambda_{t-1})u_{ct}. \end{cases}$$

(ii) $\phi_t \nu_t > 0$ if and only if $\tau_t^{x*} > 0$, all $t \geq 0$.

Note that if $\hat{b}_{0,t} = e_t = 0$, $\sigma_{ut} > 1$ and v is separable and $\sigma_{\hat{c}vt} > 1$, all $t \geq 0$, and $\theta = 0$, then a solution exists, condition (R2) holds, all $t > 0$, all solutions are regular and the first order conditions hold.

4.1 Pareto improving lotteries

It is easy to see that in M-RP-WK the Euler equation function F_2 is not generally concave. In particular, suppose that u is isoelastic with coefficient $\sigma_u > 1$. The left-hand side of $F_{2,t}$ is convex, not concave.[20] Similarly, $F_{3,t}$ is not generally concave. If v is separable, it is

$$\begin{aligned}(\hat{c}_t - e_t, 1-x_t)\cdot D^2F_{3,t}\cdot(\hat{c}_t - e_t, 1-x_t) = {} & 2v_{\hat{c}\hat{c}t}(\hat{c}_t - e_t)^2 + 2v_{xxt}(1-x_t)^2 \\ & - v_{xxxt}(1-x_t)^3 + v_{\hat{c}\hat{c}\hat{c}t}(\hat{c}_t - e_t)^3\end{aligned}$$

[19] See Luenberger, either Thm 1, p. 243, or Thm 1, p. 249, which signs the dual linear map.

[20] Putting probability μ on $(\hat{c}^1_{t+1}, c^1_{t+1})$ and $1-\mu$ on $(\hat{c}^2_{t+1}, c^2_{t+1})$ such that each satisfies market clearing at initial capital k_{t+1} and investment k_{t+2}, and also, say,

$$\beta u^1_{ct+1}(c^1_{t+1} + k_{t+2} - \hat{b}_{0,t+1}) - u_{ct}k_{t+1} > 0 > \beta u^2_{ct+1}(c^2_{t+1} + k_{t+2} - \hat{b}_{0,t+1}) - u_{ct}k_{t+1},$$

by appropriate choice of μ we obtain

$$\begin{aligned} 0 &= \mu\beta u^1_{ct+1}(c^1_{t+1} + k_{t+2} - \hat{b}_{0,t+1}) + (1-\mu)\beta u^2_{ct+1}(c^2_{t+1} + k_{t+2} - \hat{b}_{0,t+1}) - u_{ct}k_{t+1} \\ &> \beta u(c^\mu_{t+1})(1-\sigma_u) + \beta u_{ct+1}(c^\mu_{t+1})(k_{t+2} - \hat{b}_{0,t+1}) - u_{ct}k_{t+1} \\ &= \beta u_{ct+1}(c^\mu_{t+1})(c^\mu_{t+1} + k_{t+2} - \hat{b}_{0,t+1}) - u_{ct}k_{t+1} \end{aligned}$$

by convexity of $u(c)(1-\sigma_u)$ and of u_c. Thus $(\hat{c}^\mu_{t+1}, c^\mu_{t+1}, k_{t+2})$ with $c^\mu_{t+1} = \mu c^1_{t+1} + (1-\mu)c^2_{t+1}$ and $\hat{c}^\mu_{t+1} + c^\mu_{t+1} + k_{t+2} = \hat{f}_{t+1}(k_{t+1})$ is not in the constraint set without lotteries.

and this can be positive if $v_{\hat{c}\hat{c}\hat{c}t} > 0$ and large enough –an insight we encountered in Section 2.2.

As before, these observations are not themselves sufficient to claim suboptimality of the standard Ramsey policy. However, using the property that at a solution $\hat{c}_t > e_t$ and $x_t \in (0,1)$, and that labor taxes are positive at some t, we now construct a 2-point lottery that Pareto improves over the Ramsey solution using the lack of concavity of $F_{3,t}$.

Proposition 6 *Suppose that at a regular M-RP-WK optimum z^* at some $t \geq 0$ it is $\tau_t^{x*} > 0$ and*

$$-\frac{dz_t \cdot D^2 F_{3,t}^* \cdot dz_t}{DF_{3,t}^* \cdot dz_t} > -\frac{dz_t \cdot D^2 U_t^* \cdot dz_t}{DU_t^* \cdot dz_t} \tag{D-WK}$$

for some $dz \in \ell(\mathbb{R}^4)$ with $DF_{\backslash(3,t)}^* \cdot dz = 0$. Then, there is a 2-point lottery that Pareto improves over z^*. Further, the Pareto improving policy lottery has $\mathbb{E}_\mu \hat{c}_t > \hat{c}_t^*$, and $f_{2t}^* dx_t = -d\hat{c}_t$. When v is separable, condition D-WK is satisfied provided $v_{\hat{c}\hat{c}\hat{c}t}^*$ is positive and large enough.

The Pareto improving lottery of Proposition 6 yields $\mathbb{E}_{t,\mu} F_{3,t}(z) \geq 0$, and only perturbs workers' consumption and their labor supply –not capitalists' consumption or investment. An optimal lottery then involves lotteries at least over $(\hat{c}_t, x_t)$ at every t. It solves

$$\begin{array}{ccc} \max_{\mu \in \Delta(Z)} \int U(z,\theta) d\mu & \text{s. to} & \\ F_{1,t}(z) = 0 & \text{all } t \geq 0, & \\ \mathbb{E}_{t,\mu} F_{2,t}(z) = 0 & \text{and} & \\ \mathbb{E}_{t,\mu} F_{3,t}(z) = 0 & \mu\text{-a.e., all } t \geq 0, & \\ \lim_{t' \to \infty} \beta^{t'} \mathbb{E}_{t,\mu} u_{ct+t'} k_{t+t'+1} = 0 & \text{all } t \geq 0. & \end{array} \tag{E-RP-WK}$$

As $\mathbb{E}_{t,\mu} F_{3,t}(z) \geq 0$ both in this lottery problem and for the Pareto-improving lottery of Proposition 6, these lotterizations need an interpretation as decentralized equilibria. At any corresponding taxspot equilibrium, and for every $t \geq 0$, the workers budget becomes

$$\begin{aligned} q_t \cdot b_t &= 0 \\ \hat{c}_{s,t} &= e_t + (1 - \tau_{s,t}^x) w_{s,t} (1 - x_{s,t}) + b_{s,t}, \ \mu - \text{a.e.} \end{aligned}$$

for some (complete) set of one-period assets b_t, where $b_{s,t}$ is an Arrow security paying in taxspot state s, and the government budget is

$$\hat{b}_{0,t} + g_t + b_{s,t} = \tau_{s,t}^x w_{s,t} (1 - x_{s,t}) + \tau_{s,t}^k \hat{R}_{s,t} k_t,$$

where $\hat{R}_{s,t} = f_1(k_t, 1 - x_{s,t}) + 1 - \delta$. If a lottery does not affect capitalists' consumption and investment, just as it happened in our construction for Proposition 6, then $c_t + k_{t+1} = \hat{b}_{0,t} + (1 - \tau_{st}^k) \hat{R}_{s,t} k_t$ for all s, i.e., $(1 - \tau_{s,t}^k) \hat{R}_{s,t}$ is s-invariant (though capital taxes vary with s if there is labor supply uncertainty, i.e., labor income taxspots).

Although bonds b_t do not allow for intertemporal government budget deficits, they add financial instruments to the government policy toolkit: a $b_{s,t} < 0$ contributes to financing government expenditure over and beyond labor and capital income tax revenues.

When optimal taxspots are persistent. Importantly, our previous remarks on the vanishing nature of taxspot uncertainty do not apply here, and taxspot uncertainty may not vanish in finite time.

Proposition 7 *Suppose that A.3 holds. Then, there exists a solution μ^* to E-RP-WK, and all solutions are interior. If at any solution to M-RP-WK condition (R) holds and workers are taxed at all $t \geq 0$, then modulo a small utility perturbation, at the corresponding optimal taxspot equilibrium of every solution μ^* there are taxspots infinitely often, μ^*-almost surely. If $\min e_t > 0$, then $\lim_{t_n \to \infty} \mu^*_{t_n}$ is not a Dirac measure.*

Thus, taxspots are persistent, i.e., they occur infinitely often, in all economies of Proposition 7. To see that the set of these economies is nonempty and contains economically interesting cases, suppose that $g_t = g > \hat{f}_1(\bar{k}_1, 1)\bar{k}_1 \geq \hat{f}_1(k, 1)k$ for all $k : \underline{k} \leq k \leq \bar{k}$, and that $\theta = 0$. Further, let $e_t = e > 0$. If v is separable with

$$\sigma_{\hat{c}v} > \frac{e^{1-\sigma \hat{c}v}}{\min v_x(1-x)}, \tag{1}$$

then it is verified that condition (R) holds and $\tau_t^{x*} > 0$ for all t at any solution to M-RP-WK, and Proposition 7 applies.

4.2 Inelastic labor supply

In this subsection we consider the special case studied by Straub and Werning (2020), where workers supply labor inelastically, i.e., $x_t = 1$ all $t \geq 0$, and $\hat{b}_{0,t} = 0$, $g_t = g$ and $e_t = 0$ for all $t \geq 0$. With some slight abuse of notation we let $v(\hat{c}_t, 0) = v(\hat{c}_t)$ and $f_t(k_t, 1) = f_t(k_t)$. Workers can still be taxed or subsidized, but in a lump-sum fashion. Letting T_t be this transfer, the government budget becomes

$$g + T_t = \tau_t \hat{R}_t k_t,$$

the worker's budget is $\hat{c}_t = \hat{f}(k_t) - \hat{f}'(k_t)k_t + T_t$, and the capitalists' budget and market clearing are unchanged. The equilibrium constraints in the Ramsey problem now are only the transversality condition and $(F_{1,t}, F_{2,t})(\hat{c}, c, k) = 0$ —here the sequences $(\hat{c}, x, c, k)$ with $x_t = 1$ all $t \geq 0$ are denoted $(\hat{c}, c, k)$. We let T_0, τ_0 also to be derived from $\hat{c}_0, c_0, k_0$ via the workers' and the government budgets at $t = 0$. These equations, together with $F_{1,0}(\hat{c}, c, k) = 0$ imply the remaining capitalist's budget constraint at $t = 0$. The modified Ramsey problem in this special case is dubbed M-RP-WKSW.

As we did in the previous section, under A3 existence of a solution to M-RP-WKSW can be preliminarily established —a proof now left to the reader. Problem M-RP-WKSW is a relaxation of the Ramsey problem provided condition (R) holds. Here, condition (R) actually reduces to a simplified version of (R2), or

$$(1-\theta)\beta \hat{f}_{1t} v_{\hat{c}t} - \theta u_{ct-1} > 0, \tag{R2'}$$

which holds for all small enough θ. We then write first order conditions as in Straub and Werning (2020), using our regularity result —in fact, it can be shown that regularity always holds here. Using

Lemma 8, we establish the existence of a sequence of multipliers $\lambda_t \geq 0$ such that optimal consumption implies

$$\lambda_t = \lambda_{t-1}(\frac{\sigma_{ut}-1}{\sigma_{ut}\kappa_{t+1}}+1)+\frac{1}{\sigma_{ut}\kappa_{t+1}}(\frac{(1-\theta)v_{\hat{c}t}-\theta u_{ct}}{u_{ct}})$$

and the FOC for capital is

$$(1-\theta)[\beta v_{\hat{c}t+1}\hat{R}_{t+1}-v_{\hat{c}t}] = (\lambda_t-\lambda_{t-1})u_{ct}.$$

It is immediately verified that an interior steady state with converging multipliers cannot exist when u is isoelastic with $\sigma_u > 1$ and for all θ sufficiently small, and that capitalists' consumption cannot converge to zero at any limit. As Straub and Werning (2020) note, paraphrasing Lansing (1999), at an interior steady state when $(1-\theta)v_{\hat{c}} > \theta u_c$ the r.h.s. of the FOC for capital is positive and so, from the l.h.s., $\hat{R} > 1/\beta$, and a positive capital tax emerges. Under these conditions, however, namely $\sigma_u > 1$, it may be possible to use lotteries to increase welfare, as we show next.

When labor supply was elastic, we perturbed it to obtain a Pareto improvement, exploiting the non-concavity of $F_{3,t}$. Here, we are going to exploit the potential lack of concavity of $F_{2,t}$, and perturb instead the capitalist's consumption and investment in a way consistent with the government inability to run deficits —and without introducing any bond.

We need the following notation. For an interior optimal solution $z^* = (\hat{c}^*, c^*, k^*)$ to M-RP-WKSW, let $\mathfrak{Z} = \{dz \in \ell(\mathbb{R}^3) : z^* \pm dz > 0\}$ be the set of admissible changes, and

$$\Delta U = \{dz \in \mathfrak{Z} :\ DU^* \cdot dz > 0\}$$

be the set of Pareto-improving changes. Letting $F^*_{2\setminus t} = F_{2,t'}(z^*)$ all $t' \neq t$,

$$TF^*_{2\setminus t} = \{dz \in \mathfrak{Z} : DF^*_{1,t'} \cdot dz = 0 \text{ all } t' > 0 \text{ and } DF^*_{2\setminus t} \cdot dz = 0\}$$

is the tangent space to the Euler equations except for at some time $t > 0$.

Changes dz in $TF^*_{2\setminus t}$ are feasible at every t', and budget balanced at every $t' \neq t+1$. If a solution to M-RP-WKSW is given, then every time we consider a Pareto-improving deviation $dz \in \Delta U \cap TF^*_{2\setminus t}$, by optimality and regularity it must be that $DF^*_{2,t} \cdot dz < 0$. That is, and interpreting the direction dz as a policy change, if we find a policy that increases the planner's objective while being budget balanced in every period but at date $t+1$, it must run a primary deficit at that date. We are then looking for a lottery randomizing between policy $dz \in \Delta U \cap TF^*_{2\setminus t}$ and its opposite, $-dz$, such that the average effect on the Euler equation at t is nonnegative (and it is zero at every other date). Indeed, and reverting to the corresponding competitive equilibrium, since with uncertainty the Euler equation must hold only in expectations, this requirement is coherent with obtaining budget-balance at both realizations dz and $-dz$: even if $F_{2,t}(z^*+dz) = DF^*_{2,t} \cdot dz + o(||dz||) < 0$, this is consistent with requiring feasibility and the capitalist's budget $c_{t+1}+k_{t+2} = R_{t+1}k_{t+1}$ to hold in both lottery realizations, while $\beta u_{ct+1}(c_{t+1} + k_{t+2}) < u_{ct}k_{t+1}$, or $\beta u_{ct+1}R_{t+1}/u_{ct} < 1$.

We then arrive at the following result.

Proposition 8 *Let z^* be a solution to problem M-RP-WKSW where, at some $t > 0$,*

$$-\frac{dz \cdot D^2 F^*_{2,t} \cdot dz}{DF^*_{2,t} \cdot dz} > -\frac{dz \cdot D^2 U^* \cdot dz}{DU^* \cdot dz} \tag{D-WKSW}$$

for some $dz \in \Delta U \cap TF^*_{2\backslash t}$. Then, a 2-point lottery at t Pareto improves over z^*.

As in Section 3, the intuition for Proposition 8 is that prudence can be exploited to increase the capitalists' savings with taxspot uncertainty.

Condition D-WKSW states that, since by concavity the r.h.s. is positive, it must be that $dz \cdot D^2 F^*_{2,t} \cdot dz > 0$, and large enough. The size of this second-order or curvature effect of $F^*_{2,t}$ generally depends on u^*_{ccct+1}, and must be larger than the curvature of the utility, which depends on $v^*_{\hat{c}\hat{c}t+1}$ and, if $\theta > 0$, on u^*_{cct+1}. This again opens the door to third-order derivative perturbations that do not change the slope or curvature of the utility.

Condition D-WKSW presumes that $\Delta U \cap TF^*_{2\backslash t}$ is nonempty for some $t > 0$: that a Pareto-improving budget balanced policy running a deficit at $t + 1 > 0$ exists. In the following, we establish the existence of such a direction $dz \in \Delta U \cap TF^*_{2\backslash t}$ for some $t > 0$ when the elasticity of substitution is less than one and the solution to problem M-RP-WKSW features a perennial desirable redistribution from capitalists to workers, as in Straub and Werning's economies, and their limiting optimal behavior.

Lemma 9 *Suppose $\sigma_{ut} = \sigma_u > 1$ and the solution z^* to problem M-RP-WKSW approaches a limit as $t \to \infty$ where $\tau^{k*} > 0$. Then, for any θ sufficiently close to zero, there is date $t > 0$ and a change $dz \in \Delta U \cap TF^*_{\backslash 2,t}$ such that $DF^*_{2,t} \cdot dz < 0$. Further,*

$$\mathbb{E}_\mu F_{2,t}(z^* + dz) = \beta \mathbb{E}_{t,\mu} u_{ct+1}(c^*_{t+1} + dc_{t+1})(c^*_{t+1} + dc_{t+1} + k^*_{t+2} + dk_{t+2}) - u^*_{ct} k^*_{t+1} \geq 0. \tag{DEC}$$

Combining Proposition 8 and Lemma 9, the suboptimality of a deterministic fiscal policy arises in economies arbitrarily close to those with isoelastic utility with elasticity less than one considered by Straub and Werning (2020).

The policy change dz we construct in Lemma 9 is built by arbitrarily picking dc_{t+1} and dk_{t+2}, and then deriving the consequent dc_{t+2} and dk_{t+3}. Since we have some freedom in the choice of the first elements, it is of interest to see if the corresponding policy results in '*front loading*': $d\tau_{t+1} > 0$ and $d\tau_{t+3} < 0$. The following proposition offers insights into the Pareto improving lottery.

Proposition 9 *Suppose $\sigma_{ut} = \sigma_u > 1$ and that the solution z^* to problem M-RP-WKSW approaches a limit as $t \to \infty$ where $\tau^{k*} > 0$, and θ is sufficiently close to zero. If $f(k) = k^\alpha$ for some positive $\alpha < 1$, then the Pareto improving policy of Lemma 9 can be front-loading, and $\mathbb{E}_\mu \tau_{t+1} \geq \tau^*_{t+1}$ and $\mathbb{E}_\mu \tau_{t+3} < \tau^*_{t+3}$.*

For other production functions, e.g., $f(k) = \alpha - \alpha/(1+k)$ for some $\alpha > 0$, the Pareto-improving policy is not guaranteed to be front loading.

Introducing lotteries, the extended Ramsey problem becomes

$$\begin{array}{lr} \max_{\mu\in\Delta(Z)} \int U(z,\theta) d\mu \qquad \text{s. to} & \\ F_{1,t}(z) = 0, & \\ \beta \mathbb{E}_{t,\mu} u_{ct+1}(c_{t+1} + k_{t+2}) - u_{ct} k_{t+1} = 0 \quad \text{all } t \geq 0, \mu\text{-a.e.}, & \text{(E-RP-WKSW)} \\ \lim_{T\to\infty} \beta^T \mathbb{E}_{t-1,\mu} u_{ct+T} k_{t+1+T} = 0, \qquad \text{all } t \geq 0 & \end{array}$$

where μ is a probability measure over sequences $(\hat{c}_t, c_t, k_{t+1}), t \geq 0$. Again, we note that μ does not allow averaging $F_{2,t}$ through the initial value $u_{ct} k_{t+1}$ at t, as it would if we required $\mathbb{E}_{t,\mu}[\beta u_{ct+1}(c_{t+1} + k_{t+2}) - u_{ct} k_{t+1}] = 0$. Decentralization of this second type of randomization would require insurance bonds. Instead, decentralization of a solution to problem E-RP-WKSW through taxspots can be established without introducing any bonds. The proof of this, together with existence, follows what we have seen in previous sections, and therefore it is omitted.

Again, due to market incompleteness, taxspot uncertainty may not vanish in finite time.[21] Indeed, the policy change dz we construct in Lemma 9 and use in Proposition 8 has the property that, at the resulting time t constraint, the associated Pareto-improving lottery μ does not randomize on c_t, k_{t+1}, as per (DEC), and also leads to $F_{2,t'}(z^* + dz) = 0$ at all $t' \neq t$. This is a key property to decentralizing the lottery via taxspots in this setting, as it too does not require issuing any bonds. Further, such a policy change is admissible, and can be used to move away from any deterministic long-run candidate solution, in the relaxed version of (E-RP-WKSW) —thus, of problem (E-RP-WKSW) itself when the two problems have the same solutions.

Proposition 10 *Let $\sigma_{ut} = \sigma_u > 1$ and θ be close to zero. Then, a solution μ^* to problem E-RP-WKSW exists, and all solutions are interior. Further, modulo a small utility perturbation, at the corresponding optimal taxspot equilibrium of every solution μ^* there are taxspots infinitely often, μ^*-almost surely.*

Convergence to a deterministic steady state, with positive capital taxation, may still happen if the taxspot uncertainty gets smaller as t grows arbitrarily large. A martingale characterization of the optimal average tax is possible, along the lines of what introduced in Section 3.2, but weaker than Lemma 5. There is no obvious way to compare the lottery and no lottery solution paths. However, if the two paths were to overlap, workers' imprudence would be enough to lower the average tax. Recall that imprudence obtains when the marginal utility is concave —a condition not excluded in Straub and Werning (2020).

Proposition 11 *Let $\sigma_{ut} = \sigma_u > 1$ and θ be close to zero. At the optimal taxspot equilibrium, if $v_{\hat{c}t}/u_{ct}$ is a supermartingale under $\Psi(\nu^*)$ at $t \geq 0$, then the $\Psi(\nu^*)$-average capital tax is nonnegative at*

[21] Note that without fiscal uncertainty there are no sunspots at equilibrium provided $f_t''/f_t' + 1$ is nonzero.

$t \geq 0$. *If workers are imprudent,* $\tau^{nl}_{t+1} > \mathbb{E}_{t,\nu^*} \tau_{t+1}$ *where* τ^{nl}_{t+1} *is the optimal deterministic tax at a path-equivalent history.*

One should observe that the no government liabilities restriction and the presence of taxspots effectively creates market incompleteness, which in itself could be a reason to support positive taxes in the long run. In fact, it is at this point straightforward to see that if the government were able to issue a full set of state-contingent bonds, the results of Section 3 would apply also to this capital-labor heterogeneous agents economy. The government budget would be expressed as in that section, allowing for deficits to be run at any date t, and the capitalists' sequential budgets could be equivalently reduced to a single intertemporal constraint. Zero average taxes would result at any stationary Ramsey solution, and this would involve taxspots only at finitely many dates.

5 Conclusions

Taxspots are optimal when prudence is large enough relative to risk aversion, even with a representative agent and complete markets. Taxspots matter in the long run when the government ability to transfer money across periods is limited (incomplete markets), and the government would want to tax (workers or capitalists). When the labor supply is inelastic, and the capitalists' intertemporal elasticity of substitution is less than one, the optimal capital tax is not deterministic (as instead assumed in, e.g., Straub and Werning 2020). Effects on the sign of the average tax are ambiguous, but there are situations –for example, if workers are imprudent– where the average tax declines thanks to taxspots. The quantitative assessment of the impact of taxspots on welfare remains an open question for future research.

6 Appendix A: Existence, basic properties of solutions

Proof of Lemma 2: We are going to show that the modified problem

$$\begin{array}{c} \max_{\mu \in \Delta(C)} \int_C U(\hat{c}) d\mu \quad \text{s. to} \\ \int F(\hat{c}) d\mu \geq 0 \end{array} \tag{E-M-RP}$$

has a solution. The set C is endowed with the product topology and the product Borel σ-algebra. Let $\overset{o}{C} = \times_{t \geq 0} \times_{g^t \in \mathcal{G}_t} (0, 1 - g_t)$, and

$$\bar{U} = \sup \left\{ \int U(\hat{c}) d\mu \mid \mu \in \Delta(\overset{o}{C}) \text{and} \int F(\hat{c}) d\mu \geq 0 \right\}.$$

For $(\bar{u}_t)_{t \geq 0}$ with $\bar{u}_t = U_t(\tilde{c}_t)$, $\bar{u}_t \leq v(1,1) = \bar{u}$. Then, the partial sums $U_T(\hat{c}) = \mathbb{E}_0 \sum_{t=0}^{T} \beta^t [U_t(\hat{c}_t) - \bar{u}] \leq 0$ are nonincreasing functions of T for any $\hat{c}$ in C, and thus the limit exists, implying the existence of $U(\hat{c})$, although the limit could be $-\infty$.

Notice that $\bar{U} > -\infty$, because of the existence of $\hat{c}^o$ as per A1.a. Also, $\bar{U} \leq \mathbb{E}_0 \sum \beta^t \bar{u}_t < +\infty$. With the weak*-topology $\Delta(\overset{o}{C})$ is a metrizable space. By definition of sup, there is a sequence $\{\mu^n\}_{n\geq 0}$ with $\mu^n \in \Delta(\overset{o}{C})$ and $\int F(\hat{c}) d\mu^n \geq 0$ for every n, such that

$$\lim_{n\to\infty} \int U(\hat{c}) d\mu^n = \bar{U}.$$

Since each $[0, 1-g_t]$ is compact in the Euclidean space, C is compact in the product topology via Tychonoff's Theorem, $\Delta(C)$ is a compact space in the weak* topology, and there is a weak* convergent subsequence μ^n. Define μ to be its limit. Further, since U is product continuous on C, it is $\int U(\hat{c}) d\mu = \bar{U}$.

It is shown in turn that: $\mu[C \backslash \overset{o}{C}] = 0$; $\int F(\hat{c}) d\mu \geq 0$. This implies that μ is a solution to problem E-M-RP.

To see that $\mu[C \backslash \overset{o}{C}] = 0$, suppose otherwise, and assume that (A1.b.i) holds. Then,

$$\begin{aligned} \bar{U} &= \int U(\hat{c}) d\mu = \int_{C \backslash \overset{o}{C}} U(\hat{c}) d\mu + \int_{\overset{o}{C}} U(\hat{c}) d\mu \\ &\leq \int_{C \backslash \overset{o}{C}} U(\hat{c}) d\mu + \bar{u} < \bar{U} \end{aligned}$$

since the first integral equals $-\infty$, a contradiction.

If (A1.b.ii) holds, then for every t there is $\underline{c}_t > 0$ small enough such that $a_{\hat{c}t}(\hat{c}) > 0$ all $\hat{c} \leq \underline{c}_t$ (and $a_{\hat{c}t}(\hat{c}) < 0$ all $\hat{c} \geq 1 - g_t - \underline{c}_t$), all g_t, all t. Suppose, contrary to the statement, that $\mu(B') > 0$ for

$$B' = \{\hat{c} : \exists t, g^t \text{ with } \hat{c}_t(g^t) < \underline{c}_t - \varepsilon\}$$

and for some $\varepsilon > 0$. Again, $\liminf \mu^n(B') > 0$. Then, let $r : C \to C$ be the (continuous) function defined as

$$r(\hat{c}) = \begin{cases} \hat{c} & \hat{c} \in B'^{\complement} \\ \hat{c}_{\underline{c}_t} & \text{if } \hat{c} \in B', \end{cases}$$

where

$$\hat{c}_{\underline{c}_t, t}(g^t) = \begin{cases} \underline{c}_t & \text{if } \hat{c}_t(g^t) \leq \underline{c}_t - 2\varepsilon \\ 2(\underline{c}_t - \varepsilon) - \hat{c}_t(g^t) & \text{if } \underline{c}_t - 2\varepsilon < \hat{c}_t(g^t) \leq \underline{c}_t - \varepsilon \\ \hat{c}_t(g^t) & \text{otherwise} \end{cases}$$

Then, let $\mu^n_{\underline{c}_t}$ be the measure defined via

$$\int U(\hat{c}) d\mu^n_{\underline{c}_t} = \int U(r(\hat{c})) d\mu^n.$$

As $\underline{c}_t < \bar{c}_t(g^t)$, from concavity and monotonicity of U_t it is

$$\int U(\hat{c}) d\mu^n_{\underline{c}_t} > \int_{B'} [U'(\underline{c}_t)\varepsilon + U(\hat{c})] d\mu^n + \int_{B'^c} U(\hat{c}) d\mu^n > U'(\underline{c}_t)\varepsilon \mu^n(B') + \bar{U} - 1/n,$$

Further, as $a_{\hat{c}_t}(\hat{c}_t(g^t)) > 0$ for any $t, g^t : \hat{c}_t(g^t) \leq \underline{c}_t$, it is

$$\int \mathbb{E}_0 \sum \beta^t a_t(\hat{c}) d\mu^n_{\underline{c}_t} > \mathbb{E}_0 \sum \beta^t a_t(\hat{c}_t) d\mu^n \geq 0,$$

so that $\mu^n_{\underline{c}_t} \in \Delta(\overset{o}{C})$and $\int F(\hat{c}) d\mu^n_{\underline{c}_t} \geq 0$. However,

$$\lim \int U(\hat{c}) d\mu^n_{\underline{c}_t} \geq U'(\underline{c}_t) \varepsilon \liminf \mu^n(B') + \bar{U} > \bar{U},$$

contradicting the definition of $\bar{U}$.

Suppose now $\liminf_T \int \mathbb{E}_0 \sum_{t=0}^{T} \beta^t a_t(\hat{c}_t) d\mu < 0$. Then there is $T > 0$ such that

$$\int \mathbb{E}_0 \sum_{t=0}^{T-1} \beta^t a_t(\hat{c}_t) d\mu + \mathbb{E}_0 \sum_{t \geq T} \beta^t \bar{a}_t < 0.$$

As for every n,

$$\int F(\hat{c}) d\mu^n \leq \int \mathbb{E}_0 \sum_{t=0}^{T-1} \beta^t a_t(\hat{c}_t) d\mu^n + \mathbb{E}_0 \sum_{t \geq T} \beta^t \bar{a}_t,$$

and by continuity of a_t on $\overset{o}{C}$,

$$\lim_{n \to \infty} \int \mathbb{E}_0 \sum_{t=0}^{T-1} \beta^t a_t(\hat{c}_t) d\mu^n = \int \mathbb{E}_0 \sum_{t=0}^{T-1} \beta^t a_t(\hat{c}_t) d\mu,$$

it is

$$\begin{aligned} 0 \leq \liminf_{n \to \infty} \int F(\hat{c}) d\mu^n &\leq \lim_{n \to \infty} \int \mathbb{E}_0 \textstyle\sum_{t=0}^{T-1} \beta^t a_t(\hat{c}_t) d\mu^n + \mathbb{E}_0 \sum_{t \geq T} \beta^t \bar{a}_t \\ &= \int \mathbb{E}_0 \textstyle\sum_{t=0}^{T-1} \beta^t a_t(\hat{c}_t) d\mu + \mathbb{E}_0 \sum_{t \geq T} \beta^t \bar{a}_t < 0, \end{aligned}$$

a contradiction. Therefore $\liminf_T \int \mathbb{E}_0 \sum^T \beta^t a_t(\hat{c}_t) d\mu \geq 0$.

Next, let $\varphi_T(\hat{c}) = \mathbb{E}_0 \sum_{t=0}^{T} \beta^t (a_t(\hat{c}_t) - \bar{a}_t)$ and $\Phi(T) = \int \varphi_T(\hat{c}) d\mu$, for all T. From $a_t(\hat{c}_t) - \bar{a}_t(g^t) < 0$ for all $\hat{c}_t \in [0, 1 - g_t]$ and all $g^t, t \geq 0$, we get that $\varphi_T(\hat{c})$ is a nonincreasing sequence of (integrable) nonpositive functions. Thus, $\Phi(T)$ is a nonincreasing sequence of real numbers, with

$$\Phi(T) \geq \Phi(T+1) > -\varepsilon - \mathbb{E}_0 \sum_{t=0}^{T+1} \beta^t \bar{a}_t$$

for all $T > \bar{T}(\varepsilon)$ and all $\varepsilon > 0$, where the last inequality comes from the $\liminf_T$ result. We conclude, by definition of $F(\hat{c})$ and the Monotone Convergence Theorem, that

$$\int [F(\hat{c}) - \mathbb{E}_0 \sum_{t \geq 0} \beta^t \bar{a}_t] d\mu = \lim_T \int \varphi_T(\hat{c}) d\mu = \lim_T \Phi(T) \geq -\mathbb{E}_0 \sum_{t \geq 0} \beta^t \bar{a}_t$$

implying $\int F(\hat{c}) d\mu \geq 0$, as wanted. Thus, μ is a solution to problem E-M-RP.

The proof is concluded by showing that, under A1.c, at any solution to E-M-RP the constraint is binding. If not, and $\int F(\hat{c})d\mu > 0$, then for $\varepsilon > 0$ consider the lottery $\mu_\varepsilon = (1-\varepsilon)\mu + \varepsilon\delta_{\tilde{c}}$, where $\delta_{\tilde{c}}$ is the Dirac on the feasible, maximum per period utility process $\tilde{c}_t, t \geq 0$. By A1.c, for ε small enough $\int F(\hat{c})d\mu_\varepsilon > 0$, while by definition of $\tilde{c}$, $\int U(\hat{c})d\mu_\varepsilon > \int U(\hat{c})d\mu$, a contradiction ends the proof. ■

Proof of Lemma 6: Write the sequence of inequality constraints in problem M-RP-WK as $F(z) \geq 0$, where $F = F_t, t \geq 0$. Define

$$\bar{U} = \sup\left\{U(z,\theta) \mid z = (\hat{c},x,c,k) \in \overset{o}{C} \times \overset{o}{X} \times \overset{o}{C} \times \overset{o}{K} \text{ and } F(z) \geq 0,\ \lim \beta^{t'} u_{ct+t'} k_{t+t'+1} = 0\right\}.$$

By A3.b, we can assume without loss of generality that $U_t(z,\theta) \leq 0$ all $t \geq 0$, so that $U(z,\theta) = \lim_T \sum_{t=0}^{T} U_t(z,\theta)$ exists and is finite for all $z \in \overset{o}{Z}$. By A3.a it is $\bar{U} > -\infty$. By definition of sup, there is $\{z^n\}_{n\geq 0} \subset \overset{o}{Z}$ with $F(z^n) \geq 0$ and such that

$$\lim_{n\to\infty} U(z^n,\theta) = \bar{U}.$$

As $\{z^n\}_{n\geq 0} \subset Z$, and Z is compact in the product topology by Tychonoff's Theorem, there is a subsequence converging to z.

At this point the proof proceeds along the lines of the proof of Lemma 3. It is verified in turn that $z \in \overset{o}{Z}$; that $U(z,\theta) \geq \bar{U}$; and that $F(z) \geq 0$ and $\lim_t \beta^t u_{ct} k_{t+1} = 0$, so z is a solution to problem M-RP-WK.

Since $F_{3,t}(z^n) \geq 0$, it is $v^n_{\hat{c}t}(\hat{c}^n_t - e_t) \geq \min_{x\in[0,1]} v_x(1-x) \geq 0$, thus $\hat{c}_t \geq e_t$. Then, as $\lim_{KL\to 0} f(K,L) < \underline{g}$, it is $k_{t+1} \geq \underline{k} > 0$ and $x_t \leq \bar{x} < 1$, all $t \geq 0$. If $e_t = 0$, under A3.d.i, as $\lim_{\hat{c}_t x_t \to 0} v_t(\hat{c}_t, x_t) = -\infty$, or if under A3.d.ii $\lim_{\hat{c}_t x_t\to 0} F_{3,t}(z) < 0$ for every $t \geq 0$, it must be $\hat{c}_t > 0$ and $x_t > 0$, all $t \geq 0$. When $\min e_t > 0$, $\hat{c}_t \geq \underline{c}$ for some $\underline{c} > 0$, all $t \geq 0$.

As the utility function $U_t(.,\theta)$ is continuous and bounded from above, $U(.,\theta)$ is product upper semicontinuous, and $U(z,\theta) = \lim_n U(z^n,0) \geq \bar{U}$.

Clearly, by continuity of all functions involved, $0 \leq \lim_n F_{1,t}(z^n) = F_{1,t}(z)$ and $0 \leq \lim_n F_{3,t}(z^n) = F_{3,t}(z)$, all $t \geq 0$.

Under condition A2.c.i, the strong Inada condition $\lim_{c\to 0} u_c(c) = \infty$ and $\hat{b}_{0,t} > \underline{b} > 0$ all $t \geq 0$ imply interiority of c_t. To see this, observe that by the uniform boundedness of $\hat{b}_{0,t}$ there exist a finite positive $\bar{a}$ such that $\bar{a} \geq u_{ct}(c_t - \hat{b}_{0,t})$ all $t \geq 0$.

Then, by contradiction suppose that $c_t = 0$ for some $t \geq 0$, and then $c^n_t < \varepsilon$ for some $\varepsilon > 0$ and all n large enough. The transversality condition and $F_{2,t'}(z^n) \geq 0$, all t', at any fixed n imply

$$\sum_{t'\geq t} \beta^{t'-t} u^n_{ct'}(c^n_{t'} - \hat{b}_{0,t'}) \geq u^n_{ct-1} k^n_{t+1}/\beta$$

for all $t \geq 0$. Then, as $\lim_{n\to\infty} u^n_{ct} = \infty$, for some $t'' < t$ and all n large enough

$$\begin{aligned}
0 &> \sum_{t'\geq t'', t'\neq t} \beta^{t'-t''}\bar{a} + \beta^{t-t''} u^n_{ct}(c^n_t - \hat{b}_{0,t}) \\
&\geq \sum_{t'\geq t'', t'\neq t} \beta^{t'-t''} u^n_{ct'}(c^n_{t'} - \hat{b}_{0,t'}) + \beta^{t-t''} u^n_{ct}(c^n_t - \hat{b}_{0,t}) \\
&= \sum_{t'\geq t''} \beta^{t'-t''} u^n_{ct'}(c^n_{t'} - \hat{b}_{0,t'}) \geq u^n_{ct''-1} k^n_{t''+1}/\beta \; > \; 0
\end{aligned} \tag{2}$$

a contradiction.

Then, for every t the function $F_{2,t}(.)$ is continuous at z, and we obtain $\lim_n F_{2,t}(z^n) = F_{2,t}(z)$, all $t \geq 0$.

Next, suppose $\lim_t \beta^t u_{ct} k_{t+1} > 0$, a violation of the transversality condition at z. Then, for any $\varepsilon > 0$ it must be that there exists a sequence t_n with $\lim_{n\to\infty} t_n = \infty$ such that $c^n_{t_n} < \varepsilon$ for all n large enough. However, for any such n, using the transversality condition at n and $F_{2,t}(z^n) \geq 0$, all t, again we obtain

$$\begin{aligned}
0 &> \sum_{s\geq 1} \beta^s \bar{a} + u^n_{ct_n}(c^n_{t_n} - \hat{b}_{0,t_n}) \\
&\geq \sum_{s\geq 0} \beta^s u^n_{ct_n+s}(c^n_{t_n+s} - \hat{b}_{0,t_n+s}) \geq u^n_{ct_n-1} k^n_{t_n+1}/\beta > 0
\end{aligned}$$

a contradiction.

If instead A3.c.ii holds, we show that $c_t > \underline{c} > 0$, all $t \geq 0$ holds true.

Let $R_{t,t'} = R_{t+1} \cdot \ldots \cdot R_{t'}$, all $t \geq 0, t' > t$, and $R_{t,t} = 1$. Recall that there are $\bar{k}, \underline{k} > 0$ such that for every $t \geq 0$ and $n \geq 1$, $\bar{k} \geq k^n_t \geq \underline{k}$. Assume by contradiction that for every $t \geq 0$, $\liminf_{n\to\infty} c^n_t = 0$.

First, for every $t \geq 0$ and every $n \geq 1$, using the capitalist's budget constraints,

$$\sum_{t'\geq t} \frac{1}{R^n_{t,t'}} c_{t'} \;\geq\; R^n_t k^n_t$$

because of transversality, i.e., $\lim_{t\to\infty} \beta^t u_c(c^n_t) k^n_{t+1} = 0$. Second, for every $t \geq 0$ and $n \geq 1$,

$$u_c(c^n_t) \;\geq\; \frac{u_c(c^n_{t-1})}{\beta R^n_{t'}}$$

so for every $t' \geq t$ and $n \geq 1$

$$u_c(c^n_{t'}) \;\geq\; \frac{u_c(c^n_t)}{\beta^{t'-t} R^n_{t,t'}}.$$

Therefore, for every $t' \geq t$ and $n \geq 1$,

$$c^n_{t'} \;\leq\; u_c^{-1}\left(\frac{u_c(c^n_t)}{\beta^{t'-t} R^n_{t,t'}}\right)$$

and the right-hand side divided by $R_{t,t'}$ is decreasing in $R_{t,t'}$ because $\sigma_u(c) > 1$ for all $c > 0$. Third, for every $t, t' \geq 0$ with $t' \geq t$ and $n \geq 1$, $R^n_{t,t'} \geq \underline{k}/\bar{k}$ so $1/R^n_{t,t'} \leq \bar{k}/\underline{k}$. Hence, for every $t, t' \geq 0$ with $t' \geq t$ and $n \geq 1$,

$$\underline{k} \;\leq\; k^n_t \;\leq\; \frac{1}{R^n_t}\sum_{t'\geq t}\frac{1}{R^n_{t,t'}}u_c^{-1}\left(\frac{u_c(c^n_t)}{\beta^{t'-t}R^n_{t,t'}}\right) \;\leq\; \left(\frac{\bar{k}}{\underline{k}}\right)^2 \sum_{s\geq 0} u_c^{-1}\left(\frac{\bar{k}}{\underline{k}}\frac{u_c(c^n_t)}{\beta^s}\right).$$

Since $\lim_{x\to\infty}\sum_{s\geq 0} u_c^{-1}(x/\beta^s) = 0$, but $\underline{k} > 0$, while the function $u_c^{-1}(.)$ is independent of t and n, for every $t \geq 0$ and $n \geq 1$ there is $\underline{c} > 0$ such that $c^n_t \geq \underline{c}$.

Then, $\liminf c_t k_{t+1} > 0$. As a result, $F_{2,t}(z)$ is continuous in z, and $0 \leq \lim_n F_{2,t}(z^n) = F_{2,t}(z)$, all $t \geq 0$. Finally, it is checked that $\lim_n \lim_t \beta^t u^n_{ct} k^n_{t+1} = \lim_t \lim_n \beta^t u^n_{ct} k^n_{t+1} = 0$.

As we already know that $U(z,\theta) \geq \bar{U}$, we have established existence. ■

Proof of Lemma 7: i) Suppose that $F_{3,t}(z^*) > 0$ at some $t \geq 0$. As $f^*_{2t} v^*_{\hat{c}t} \neq v^*_{xt}$, consider the change $d\hat{c}_t, dx_t$ such that $dx_t = -d\hat{c}_t/f_{2t}$ (all other variables are unchanged), and $v^*_{\hat{c}t} d\hat{c}_t + v^*_{xt} dx_t = (v^*_{\hat{c}t} - v^*_{xt}/f^*_{2t})d\hat{c}_t > 0$ for some $d\hat{c}_t$. Then, all constraints are satisfied and the objective function increases, a contradiction.

Next, suppose that $F_{1,t}(z^*) > 0$. Since v is concave, using $F_{3,t}(z^*) = 0$, it is verified that either $D_{x_t} F^*_{3,t} \neq 0$ or $D_{\hat{c}_t} F^*_{3,t} \neq 0$. Assuming at no loss of generality that the former is nonzero, then $v^*_{\hat{c}t} \neq v^*_{xt}\frac{D_{\hat{c}_t} F^*_{3,t}}{D_{x_t} F^*_{3,t}}$. Consider $d\hat{c}_t, dx_t$ such that

$$dx_t = -\frac{D_{\hat{c}_t} F^*_{3,t}}{D_{x_t} F^*_{3,t}} d\hat{c}_t$$

(all other variables are unchanged). Then,

$$v^*_{\hat{c}t} d\hat{c}_t + v^*_{xt} dx_t = (v^*_{\hat{c}t} - v^*_{xt}\frac{D_{\hat{c}_t} F^*_{3,t}}{D_{x_t} F^*_{3,t}}) d\hat{c}_t > 0$$

for some $d\hat{c}_t$, again a contradiction. □

ii) Suppose $F_{2,t} > 0$ for some $t \geq 0$, and assume (R) holds. We consider changes $dc_{t-1}, d\hat{c}_t, dx_t, dk_t$ at some $t > 0$ such that $dc_{t-1} = -dk_t$ and

$$d\hat{c}_t = f_{1t}\frac{D_{x_t} F_{3,t}}{\det A_t} dk_t \text{ and } dx_t = -f_{1t}\frac{D_{\hat{c}_t} F_{3,t}}{\det A_t} dk_t.$$

Then, feasibility holds at $t-1, t$, and we have $F_{3,t} = 0$ as well. For small enough changes, $F_{2,t} > 0$ as well. The impact on the objective is

$$[(1-\theta)\beta\frac{f^*_{1t}}{\det A_t}(v^*_{\hat{c}t} D_{x_t} F^*_{3,t} - v_{xt} D_{\hat{c}_t} F^*_{3,t}) - \theta u_{ct-1}] dk_t.$$

If $t = 1$, then the impact on the objective can be made positive for some $dk_1 \neq 0$, a contradiction.

Now consider $t > 1$, and suppose that condition (R2) holds, all $t > 0$. Then, we transfer consumption from capitalists at $t-1$ to workers at t: $dk_t > 0$. The impact on the objective again is positive. As for equation $F_{2,t-1}$, it is

$$\begin{aligned} dk_t - (\sigma_{ut-1} - 1 + \sigma_{ut-1}\kappa_t)dc_{t-1} &\geq 0, \text{ or} \\ (\sigma_{ut-1} + \sigma_{ut-1}\kappa_t)dk_t &> 0, \end{aligned}$$

as $dk_t > 0$, a contradiction. ■

Proof of Lemma 8: Under (R), we can write problem M-RP-WK in its equivalent formulation where the constraint set is defined by the map $\tilde{F} : \ell(\mathbb{R}^4_+) \to \ell(\mathbb{R}^3)$ whose coordinates $\tilde{F}_t(\hat{c},x,c,k)$ are, for $t \geq 0$, $\tilde{F}_{j,t}(\hat{c},x,c,k) = F_{j,t}(\hat{c},x,c,k)$ for $j = 1,3$, and

$$\tilde{F}_{2,t}(\hat{c},x,c,k) = \beta \sum_{j\geq 0} \beta^j u_{ct+1+j}(c_{t+1+j} - \hat{b}_{0,t+1+j}) - u_{ct}k_{t+1}.$$

By Lemma 6, we consider an interior solution. For any $d\xi \in \ell(\mathbb{R}^4)$ and for each $t \geq 0$, $\tilde{F}_{2,t}$ is perturbed using capital k_{t+1}, and $\tilde{F}_{1,t}$ and $\tilde{F}_{3,t}$ are perturbed using workers' consumption and leisure. More precisely, let $dz = (d\hat{c}, dx, dc, dk)$ be defined recursively by, for all $t \geq 0$, given the τ-th coordinates $dz_\tau = (d\hat{c}_\tau, dx_\tau, dc_\tau, dk_{\tau+1})$ with $\tau < t$ and $dk_0 = 0$: $dc_t = 0$; $dk_{t+1} = -\frac{1}{u_{ct}}d\xi_{2,t}$; and $d\hat{c}_t, dx_t$ are chosen so that

$$\begin{bmatrix} d\xi_{1,t} \\ d\xi_{3,t} \end{bmatrix} - \begin{bmatrix} D\tilde{F}^*_{1,t} \\ D\tilde{F}^*_{3,t} \end{bmatrix} \cdot (d\hat{c}_{\setminus t}, dx_{\setminus t}, dc_{\setminus t}, dk_{\setminus t+1}) = \begin{bmatrix} D\tilde{F}^*_{1,t} \\ D\tilde{F}^*_{3,t} \end{bmatrix} \cdot (d\hat{c}_t, dx_t, 0, dk_{t+1}), \tag{*}$$

and $(d\hat{c}_{\setminus t}, dx_{\setminus t}, dc_{\setminus t}, dk_{\setminus t+1})$ is the vector dz where we have zeroed the τ-th coordinates dz_τ, $\tau \geq t$. Note that $D\tilde{F}^*_{3,t} \cdot (d\hat{c}_t, dx_t, 0, dk_{t+1}) = 0$, all $t \geq 0$. Further,

$$\begin{bmatrix} D\tilde{F}^*_{1,t} \\ D\tilde{F}^*_{3,t} \end{bmatrix} \cdot dz = \begin{bmatrix} D\tilde{F}^*_{1,t} \\ D\tilde{F}^*_{3,t} \end{bmatrix} \cdot (d\hat{c}_{\setminus t+1}, dx_{\setminus t+1}, dc_{\setminus t+1}, dk_{\setminus t+2})$$

and

$$\begin{bmatrix} D\tilde{F}^*_{1,t} \\ D\tilde{F}^*_{3,t} \end{bmatrix} \cdot (d\hat{c}_{\setminus t}, dx_{\setminus t}, dc_{\setminus t}, dk_{\setminus t+1}) = \begin{bmatrix} D\tilde{F}^*_{1,t} \\ D\tilde{F}^*_{3,t} \end{bmatrix} \cdot (0,0,0,dk_t) = \begin{pmatrix} f^*_{1t}dk_t \\ 0 \end{pmatrix},$$

so that dz is well-defined. Also, as $\det A_t \neq 0$, (*) can be solved for $(d\hat{c}_t, dx_t)$, all $t \geq 0$. Thus, $d\xi = D\tilde{F}(z^*) \cdot dz$. Further, as $d\xi_2 \in \ell(\mathbb{R})$ and at an interior solution $\inf_{t\geq 0} u_{ct} > 0$, and $\{dk_t\}_{t\geq 0}$ is bounded, $k_t \in [\underline{k}, \overline{k}]$ with $\underline{k} > 0$, then $d\xi_1 \in \ell(\mathbb{R})$. Finally, by construction $z^* + dz > 0$ for small enough changes dz. Thus, the linear map $D\tilde{F}^*$ is onto –and so is DF^*. Existence of a positive Lagrange vector follows from Luenberger (1969, Thm 1, p. 249).

Next, $D\tilde{F}^*$ is a matrix-representable linear operator, as $D\tilde{F}^*_t$ is summable at every t. Finally, $D\tilde{F}^*$ is upper triangular, i.e.,

$$D_{\hat{c}_{t'}}\tilde{F}^*_{1,t} = D_{x_{t'}}\tilde{F}^*_{1,t} = D_{c_{t'}}\tilde{F}^*_{1,t} = 0 \text{ all } t' < t,$$

and $D_{k_{t'}}\tilde{F}^*_{1,t} = 0$ all $t' < t$; $D_{\hat{c}_{t'}}\tilde{F}^*_{3,t} = D_{x_{t'}}\tilde{F}^*_{3,t} = 0$ all $t' < t$, and $D_{c_{t'}}\tilde{F}^*_{3,t} = D_{k_{t'}}\tilde{F}^*_{3,t} = 0$ all t'; while $D_{c_{t'}}\tilde{F}^*_{2,t} = D_{k_{t'}}\tilde{F}^*_{2,t} = 0$ all $t' < t$, and $D_{\hat{c}_{t'}}\tilde{F}^*_{2,t} = D_{x_{t'}}\tilde{F}^*_{2,t} = 0$ all t'.

Thus, Theorem 5.5 and its Corollary in Rustichini (1998) apply and deliver summable nonnegative multipliers $\hat{v}_t, \hat{\lambda}_t, \hat{\phi}_t, t > 0$. It is now just a matter of straightforward calculations to see that the first order conditions are as stated, where we rearrange the multipliers $\hat{\lambda}_{2,t}$ so that $\lambda_t\beta = \sum_{t'\leq t}\hat{\lambda}_{2,\tau}\beta^{-t'}$. Clearly, they are also the first order conditions for the recursive formulation which uses F_t. □

ii) Multiplying the first FOC equation by $\hat{c}_t$, the second by $(1-x_t)$, and subtracting the latter from the former, and using $F^*_{3,t} = 0$, we get

$$\nu_t u_{ct}(f_{2t}(1-x_t) - \hat{c}_t) + \phi_t(\hat{c}_t, 1-x_t)D^2 v_t(\hat{c}_t, 1-x_t) = 0,$$

and by concavity of v, $f_{2t}(1-x_t) > \hat{c}_t$ as $\phi_t \nu_t > 0$, hence $\tau^x_t > 0$, as wanted. For the other direction, let $\tau^x_t > 0$. Observe that from the previous equation, if $\phi_t = 0$, then $\nu_t = 0$, but equating the first two FOC yields $(1-\theta)(v_{\hat{c}t} f_{2t} - v_{xt}) = 0$, thus via $F^*_{3,t} = 0$ it is $\tau^x_t = 0$, a contradiction. Then, $\phi_t > 0$, and the previous equation and concavity of v imply $\nu_t > 0$. ■

7 Appendix B: Pareto improving lotteries.

Proof of Proposition 1: Suppose that $\hat{c}^*$ is a solution to problem M-RP

$$\begin{array}{l} \max \mathbb{E}_0 \sum_t \beta^t U_t(\hat{c}_t) \quad s.to \\ \mathbb{E}_0 \sum_t \beta^t a_t(\hat{c}_t) \geq 0 \end{array}$$

with $\tau^*_t > 0$ at some g^t and $t \geq 0$. As $\tau^*_t = U^*_{\hat{c}t}/v_{\hat{c}t} > 0$ and $v_{\hat{c}t} > 0$, it is $U^*_{\hat{c}t} > 0$. If $a^*_{\hat{c}t} \neq 0$ and since $U^*_{\hat{c}t} > 0$, by optimality of $\hat{c}^*$ it is $a^*_{\hat{c}t} < 0$, and $\mathbb{E}_0 \sum_t \beta^t a_t(\hat{c}^*_t) = 0$.

Hereafter we drop reference to history g^t from $\hat{c}^*_t(g^t)$ and $\hat{b}_{0,t}(g^t)$, denoting them $\hat{c}^*_t$ and $\hat{b}_{0,t}$, respectively. Let $P : \mathbb{R} \to \mathbb{R}$ be the polynomial in $d\hat{c}$ of derivatives of U_t up to the second order, where all derivatives are evaluated at $\hat{c}^*_t$ (and $x^*_t = 1 - g_t - \hat{c}^*_t$). If $d\hat{c} \in B_\varepsilon(0)$ a small enough neighborhood of zero, P is a second-order Taylor approximation of U_t at $\hat{c}^*_t$. Similarly, let $\phi : \mathbb{R} \to \mathbb{R}$ be the polynomial in $d\hat{c}$ of derivatives of a_t up to the second order evaluated at $\hat{c}^*_t$. That is,

$$\begin{aligned} P(d\hat{c}) &= U_t(\hat{c}^*_t) + U^*_{\hat{c}t} d\hat{c} + \frac{1}{2} U^*_{\hat{c}\hat{c}t}(d\hat{c})^2, \\ \phi(d\hat{c}) &= a_t(\hat{c}^*_t) + a^*_{\hat{c}t} d\hat{c} + \frac{1}{2} a^*_{\hat{c}\hat{c}t}(d\hat{c})^2, \end{aligned}$$

Consider the 2-point lottery μ over consumption where $\hat{c}_{1t} = \hat{c}^*_t + d\hat{c}$ and $\hat{c}_{2t} = \hat{c}^*_t - d\hat{c}$. Using the second-order approximations, let

$$\begin{array}{l} \mathbb{E}_\mu[P(d\hat{c})] = \mu P(d\hat{c}) + (1-\mu)P(-d\hat{c}) \\ \mathbb{E}_\mu[\phi(d\hat{c})] = \mu\phi(d\hat{c}) + (1-\mu)\phi(-d\hat{c}) \end{array}$$

and observe that at $d\hat{c} = 0$ we obtain back $\mathbb{E}_\mu[P(0)] = U_t(\hat{c}_t^*)$ and $\mathbb{E}_\mu[\phi(0)] = a_t(\hat{c}_t^*)$.

Suppose we find $d\hat{c} \in B_\varepsilon(0)$ such that

$$\begin{aligned} \mathbb{E}_\mu[P(d\hat{c})] > U_t(\hat{c}_t^*), \text{ and} \\ \mathbb{E}_\mu[\phi(d\hat{c})] > a_t(\hat{c}_t^*). \end{aligned} \tag{LP}$$

Then, we have found a 2-point lottery at g^t better than the M-RP solution. After substitution of the relevant derivatives, a 2-point lottery at g^t is better than the M-RP solution if we can find a nonzero $d\hat{c} \in B_\varepsilon(0)$ such that

$$\begin{aligned} (2\mu-1)U_{\hat{c}t}^* d\hat{c} + \frac{1}{2}U_{\hat{c}\hat{c}t}^*(d\hat{c})^2 > 0, \\ (2\mu-1)a_{\hat{c}t}^* d\hat{c} + \tfrac{1}{2}a_{\hat{c}\hat{c}t}^*(d\hat{c})^2 > 0. \end{aligned} \tag{LP-in}$$

As $a_{\hat{c}t}^* = 0$ is excluded here, and $d\hat{c} \neq 0$, then let μ and $d\hat{c}$ solve

$$1 - 2\mu = \frac{1}{2}\frac{a_{\hat{c}\hat{c}t}^*}{a_{\hat{c}t}^*} d\hat{c}. \tag{3}$$

Equation (3) is consistent with μ being a probability provided

$$0 < \frac{1}{2} - \frac{a_{\hat{c}\hat{c}t}^*}{4a_{\hat{c}t}^*} d\hat{c} < \frac{1}{2},$$

which occurs for $d\hat{c}$ (i.e., ε) small enough. Substituting equation (3) into the first inequality of (LP-in), we obtain equivalently

$$\frac{1}{2}\frac{U_{\hat{c}\hat{c}t}^*}{U_{\hat{c}t}^*}(d\hat{c})^2 > \frac{1}{2}\frac{a_{\hat{c}\hat{c}t}^*}{a_{\hat{c}t}^*}(d\hat{c})^2$$

which is implied by condition D for $d\hat{c} \neq 0$. To assure that the second inequality in (LP-in) is satisfied, we choose $d\hat{c} > 0$. By optimality of $\hat{c}^*$, $a_{\hat{c}t}^* < 0$, and we can lower μ slightly from its (3) value so that the last inequality in (LP-in) still holds.

Finally, from $\frac{1}{2}U_{\hat{c}\hat{c}t}^*(d\hat{c})^2 > (1-2\mu)U_{\hat{c}t}^* d\hat{c}$ and $U_{\hat{c}t}^* > 0 > U_{\hat{c}\hat{c}t}^*$, we obtain $(1-2\mu)d\hat{c} = -\mathbb{E}_\mu d\hat{c} < 0$, thus $\mathbb{E}_\mu \hat{c}_t > \hat{c}_t^*$. ■

Proof of Proposition 3: We follow the logic of the proof of Proposition 1. Suppose that k^* is a solution to problem M-RP-K where $\bar{\tau}_{t+1}^* \equiv \mathbb{E}_t^\Psi \tau_{t+1}^* > 0$ at some g^t and $t \geq 0$ (the average capital tax is positive). By optimality of k^* it is $\hat{F}(k^*) = \hat{R}_0 k_0 u_{c0}^*$. Since

$$U_{t,t+1}'(k_{t+1}) = \beta \mathbb{E}_t u_{ct+1} \hat{R}_{t+1} - u_{ct},$$

and $\bar{\tau}_{t+1}^* > 0$, it is $U_{t,t+1}'^* > 0$.

We consider changes in consumption in the two subsequent periods t and $t+1$ so that market clearing holds. Thus, the changes are adjusted by, and equivalent to, changes in capital stock k_{t+1}.

Hereafter we drop reference to histories from $k_{t+1}^*(g^t)$, denoting it simply k_{t+1}^*. With some abuse of notation, let $P : \mathbb{R} \to \mathbb{R}$ be the second-order polynomial in dk of derivatives of $U_{t,t+1}$ with respect to

k_{t+1} up to the second order, where all derivatives are evaluated at k^*_{t+1}, i.e., P is a second-order Taylor approximation of $U_{t,t+1}$ at k^*_{t+1}, and similarly $\phi : \mathbb{R} \to \mathbb{R}$ is a second-order approximation of $a_{t,t+1}$ at k^*_{t+1}. Consider the 2-point lottery μ over capital where $k_{1t+1} = k^*_{t+1} + dk$ and $k_{2t+1} = k^*_{t+1} - dk$, with $dk \in B_\varepsilon(0)$ a small enough neighborhood of zero.

Using the second-order approximations P and ϕ, after substitution of the relevant derivatives, and further using $a'^*_{t,t+1} \neq 0$, we conclude that under (D-K) a 2-point lottery at g^t Pareto improves over k^* since it implies the existence of a nonzero solution dk, μ to

$$(2\mu-1)U'^*_{t,t+1}dk + \frac{1}{2}U''^*_{t,t+1}(dk)^2 \quad > \quad 0 \tag{4}$$

$$(2\mu-1)a'^*_{t,t+1}dk + \frac{1}{2}a''^*_{t,t+1}(dk)^2 \quad > \quad 0. \tag{5}$$

and

$$1-2\mu = \frac{1}{2}\frac{a''^*_{t,t+1}}{a'^*_{t,t+1}}dk. \tag{6}$$

Finally, as

$$U''_{t,t+1}(k_{t+1}) = u_{cct} + \beta \mathbb{E}_t u_{cct+1}\hat{R}^2_{t+1} + \beta \mathbb{E}_t u_{ct+1} f''(k_{t+1}),$$

it is $U''^*_{t,t+1} < 0$ by concavity of u and f, and $U'^*_{t,t+1} > 0$ and (D-K) further imply $a''^*_{t,t+1}/a'^*_{t,t+1} < 0$, while from (4) it is $(1-2\mu)\,dk < 0$. Thus, either $k_{1t+1} > k^*_{t+1}$ with probability $\mu > 1/2$, or $k_{1t+1} < k^*_{t+1}$ with probability $\mu < 1/2$, and $\mathbb{E}_\mu k_{t+1} > k^*_{t+1}$, as wanted. ■

Proof of Proposition 6: We consider $dz_t = (d\hat{c}_t, dx_t, dc_t, dk_{t+1})$ where $(dc_t, dk_{t+1}) = 0$ and $f_{2t}dx_t = -d\hat{c}_t$. Let z^*_t be the initial optimal solution component at $t > 0$, and consider assigning probability μ to $z_{1t} = z^*_t + dz_t$ and $1-\mu$ to $z_{2t} = z^*_t - dz_t$. Let dz be a change in z which is zero at every $t' \neq t$. The effect of dz on utility U is of the order

$$P(dz) = U^* + DU^*_t \cdot dz_t + \frac{1}{2}dz_t \cdot D^2U^*_t \cdot dz_t$$

and it is

$$\phi(dz) = F^*_{3t} + DF^*_{3t} \cdot dz_t + \frac{1}{2}dz_t \cdot D^2F^*_{3t} \cdot dz_t$$

on F_{3t}, while by construction $F_{3t'}(z^*+dz) = 0$ all $t' \neq t$, and $F_{2t'}(z^*+dz) = 0 = F_{1t'}(z^*+dz) = 0$ all $t' \geq 0$. Hence, by regularity, if

$$\mathbb{E}_\mu P(dz) > U^* \text{ and } \mathbb{E}_\mu \phi(dz) > F^*_{3t} = 0,$$

then we will have found a Pareto improving lottery. Zooming in on $\mathbb{E}_\mu \phi(dz)$, it is

$$\mathbb{E}_\mu \phi(dz) - F^*_{3t} = (2\mu-1)DF^*_{3t} \cdot dz_t + \frac{1}{2}dz_t \cdot D^2F^*_{3t} \cdot dz_t.$$

Suppose $d\hat{c}_t > 0$. Then $DU^*_t \cdot dz_t = (v_{\hat{c}t} - \frac{v_{xt}}{f_{2t}})d\hat{c}_t > 0$, and under our regularity assumptions it is $DF^*_{3t} \cdot dz_t < 0$. Thus, if μ satisfies

$$2\mu - 1 = \frac{-\frac{1}{2}dz_t \cdot D^2F^*_{3t} \cdot dz_t}{DF^*_{3t} \cdot dz_t}$$

then $\mathbb{E}_\mu \phi(dz) = F_{3t}^* = 0$. Plugging into the expression for $\mathbb{E}_\mu P(dz)$, we obtain $\mathbb{E}_\mu P(dz) > U^*$ if and only if

$$-\frac{1}{2}\frac{dz_t \cdot D^2 F_{3t}^* \cdot dz_t}{DF_{3t}^* \cdot dz_t} > -\frac{1}{2}\frac{dz_t \cdot D^2 U_t^* \cdot dz_t}{DU_t^* \cdot dz_t}.$$

By continuity, for a closeby μ, a 2-point lottery Pareto improves over the standard solution. Since $2\mu - 1 > 0$ and $d\hat{c}_t > 0$, it is $\mathbb{E}_\mu \hat{c}_t > \hat{c}_t^*$, as wanted. The last statement follows from direct computation of $D^2 F_{3,t}$. ∎

Proof of Proposition 8: We follow the logic of the proof of Proposition 1. Suppose that $z^* = (\hat{c}^*, c^*, k^*)$ is an interior solution to problem M-RP-WKSW. By optimality and Lemma 7, it is $F_2^* = 0$ (and $F_1^* = 0$).

Consider a 2-point lottery μ over processes $(\hat{c}, c, k)$ where $(\hat{c}_i, c_i, k_i) = z^* + dz_i, i = 1, 2$, and $dz_1 = -dz_2$ and $dz_2 = dz$ as stated in the assumption. Again with some abuse of notation, let $P : Z \to \mathbb{R}$ be the polynomial in $dz \in Z$ of derivatives of U up to the second order, where all derivatives are evaluated at z^*. For dz sufficiently small, P is a second-order Taylor approximation of U at z^*; and similarly let $\phi : Z \to \mathbb{R}$ be a second-order approximation of $F_{2,t}$ computed at z^*. Further, letting $\mathbb{E}_\mu[P(dz)] = \sum_i \mu_i P(dz_i)$, for $\mu_i \geq 0$, $i = 1, 2$, and $\sum_i \mu_i = 1$; similarly defining $\mathbb{E}_\mu[\phi(dz)]$, we see that $\mathbb{E}_\mu[P(0)] = U^*$ and $\mathbb{E}_\mu[\phi(0)] = F_{2,t}^*$. Thus, if at nonzero $dz_2 = dz \in \Delta U \cap TF_{2\backslash t}^*$, and for probability $\mu_1 = \mu > 0$ it is

$$\begin{array}{ll} (1) \quad \sum_i \mu_i P(dz_i) > U^* & \text{and} \\ (2) \quad \sum_i \mu_i \phi(dz_i) > F_{2,t}^*, & \end{array} \tag{LP}$$

then we have found a 2-point lottery Pareto improving over the M-RP-WKSW solution.

Indeed, suppose such dz_i exist. By $(DF_{1,t'}^*, t' \geq 0, DF_{2\backslash t}^*)$ onto, for every $\varepsilon > 0$ we can find $dz_{i,\varepsilon} \in Z$ such that $DF_{1,t'}^* \cdot dz_{i,\varepsilon} = \varepsilon \mathbf{1}$ all $t' \geq 0$ and $DF_{2\backslash t}^* \cdot dz_{i,\varepsilon} = \varepsilon \mathbf{1} \gg 0$. Then, $DF_{1,t'}^* \cdot \alpha dz_{i,\varepsilon} \gg 0$ all $t' \geq 0$ and $DF_{2\backslash t}^* \cdot \alpha dz_{i,\varepsilon} \gg 0$ for every $\alpha > 0$. Now, letting $dz_i' = dz_i + \alpha dz_{i,\varepsilon}$, $i = 1, 2$, we have

$$\begin{aligned} \mathbb{E}_\mu[P(dz')] &= U^* + \sum_i \mu_i [DU^* \cdot dz_i + \frac{1}{2} dz_i \cdot D^2 U^* \cdot dz_i] \\ &\quad + \alpha \sum_i \mu_i [DU^* \cdot dz_{i,\varepsilon} + \frac{1}{2}[2 dz_i \cdot D^2 U^* \cdot dz_{i,\varepsilon} + \alpha dz_{i,\varepsilon} \cdot D^2 U^* \cdot dz_{i,\varepsilon}] \end{aligned}$$

and $\mathbb{E}_\mu[P(dz')] > U^*$ as α can be taken arbitrarily small. A similar argument can be used to obtain $\mathbb{E}_\mu[\phi(dz')] > F_{2,t}^*$, $F_{2\backslash t}(z^* + dz_i') \gg 0$, and $F_{1,t'}(z^* + dz_i') > 0$, all $t' \geq 0$. Hence, the processes $(\hat{c}_i', c_i', k_i') = z^* + dz_i'$ are feasible and satisfy all the constraints, and the 2-point lottery over $(\hat{c}_i', c_i', k_i')$, $i = 1, 2$ Pareto-improves over z^*.

To verify that LP has a solution, write LP-2 as equality by choosingμ and dz_2 so that

$$1 - 2\mu = -\frac{1}{2}\frac{dz_2 \cdot D^2 F_{2,t}^* \cdot dz_2}{DF_{2,t}^* \cdot dz_2},$$

which is consistent with μ being a probability provided dz_2 is small enough. Using the μ, substituting into LP-1, and further using the assumption $DU^* \cdot dz_2 > 0$, we obtain

$$-\frac{dz_2 \cdot D^2 F^*_{2,t} \cdot dz_2}{DF^*_{2,t} \cdot dz_2} > -\frac{dz_2 \cdot D^2 U^* \cdot dz_2}{DU^* \cdot dz_2}$$

or D-WKSW. ■

Proof of Lemma 9: Hereafter, we write σ_t for σ_{ut}. Pick a $t > 0$ and let $dz_{t'} = 0$ for all $t' < t+1$, and $dz_{t'} = 0$ for $t' > t+2$. Then, $dz \in \mathfrak{Z}$, and only three Euler equations, at dates t, $t+1$ and $t+2$, are affected by dz. If we neglect the Euler equation at t, then $(dc_{t+1}, dk_{t+2}) \in \mathbb{R}^2$ determines $(dc_{t+2}, dk_{t+3}) \in \mathbb{R}^2$ via the Euler equations at dates $t+1$ and $t+2$ with $(dc_{t+3}, dk_{t+4}) = 0$:

$$\begin{bmatrix} \beta(1-\sigma_{t+2}(1+\kappa_{t+3})) & \beta \\ \sigma_{t+2}\kappa_{t+3} & -1 \end{bmatrix} \begin{bmatrix} dc_{t+2} \\ dk_{t+3} \end{bmatrix} = \begin{bmatrix} m_{t+2}(-\sigma_{t+1}\kappa_{t+2}dc_{t+1}+dk_{t+2}) \\ 0 \end{bmatrix}. \tag{*}$$

for $m_t = u_{ct-1}/u_{ct}$. The remaining elements $d\hat{c}_{t'}$ are determined via the feasibility equations at dates $t+1$, $t+2$ and $t+3$. Thus, $DF^*_{2,t+1} \cdot dz = 0$ and $DF^*_{2,t+2} \cdot dz = 0$, and $DF^*_{2,t'} \cdot dz = 0$ for all $t' < t$ and all $t' > t+2$, while $DF^*_{1,t} \cdot dz = 0$ at all $t \geq 0$.

Since the determinant of the matrix on the l.h.s. is $-\beta(1-\sigma_{t+2})$, for $\sigma_{t+2} \neq 1$ the matrix is invertible and

$$\begin{bmatrix} dc_{t+2} \\ dk_{t+3} \end{bmatrix} = \frac{1}{\beta(1-\sigma_{t+2})} \begin{bmatrix} m_{t+2}(-\sigma_{t+1}\kappa_{t+2}dc_{t+1}+dk_{t+2}) \\ m_{t+2}\sigma_{t+2}\kappa_{t+3}(-\sigma_{t+1}\kappa_{t+2}dc_{t+1}+dk_{t+2}) \end{bmatrix}.$$

Let $\triangle_{t+1} \subset \mathbb{R}^4$ be the set of changes in $(c_{t+1}, k_{t+2}) \in \mathbb{R}^2$ and $(c_{t+2}, k_{t+3}) \in \mathbb{R}^2$, where the change in (c_{t+1}, k_{t+2}) is arbitrary and the change in (c_{t+2}, k_{t+3}) satisfies (*). Straightforward calculations show that $\hat{c}$ changes only at dates $t+1$, t+2 and t+3. As we assume that a limit of a solution z^*_t exists for $t \to \infty$, at this limit $\kappa^* = \beta/(1-\beta)$, and for t large enough these changes in $\hat{c}$ are approximately equal to (dropping superscript *)

$$\begin{aligned}
d\hat{c}_{t+1} &= -dc_{t+1} - dk_{t+2} \\
d\hat{c}_{t+2} &= \frac{1}{\beta(1-\sigma)}[\hat{R}\beta(1-\sigma)dk_{t+2} - (1+\sigma\kappa)(-\sigma\kappa dc_{t+1}+dk_{t+2})] \\
d\hat{c}_{t+3} &= \frac{1}{\beta(1-\sigma)}\hat{R}\sigma\kappa(-\sigma\kappa dc_{t+1}+dk_{t+2}).
\end{aligned}$$

We now prove the following intermediate step. Let

$$Dv = (0, ..., 0, v_{\hat{c}t+1}, \beta v_{\hat{c}t+2}, \beta^2 v_{\hat{c}t+3}, 0...).$$

Auxiliary Claim: *Suppose that $\sigma > 1$ and $\kappa^* = \beta/(1-\beta)$ and $\hat{R}^* \geq \varepsilon/\beta$ for some $\varepsilon \in ((1+\beta)/2, 1]$. Then, there exist a date $t > 0$ and a change in $\triangle_{t+1}$ such that $Dv \cdot d\hat{c} \neq 0$.*

Proof: Drop superscript *. For $t>0$ large enough and a change in $\triangle_{t+1}$, the derivatives of $\hat{c}$ with respect to c_{t+1} and k_{t+2} are

$$\begin{bmatrix} -1 & -1 \\ a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \equiv \begin{bmatrix} -1 & -1 \\ \frac{1}{\beta(1-\sigma)}(1+\sigma\kappa)\sigma\kappa & \frac{1}{\beta(1-\sigma)}[\hat{R}\beta(1-\sigma)-(1+\sigma\kappa)] \\ -\frac{1}{\beta(1-\sigma)}\hat{R}(\sigma\kappa)^2 & \frac{1}{\beta(1-\sigma)}\hat{R}\sigma\kappa \end{bmatrix}.$$

The vector $(v_{\hat{c}t+1},\beta v_{\hat{c}t+2},\beta^2 v_{\hat{c}t+3})$ is orthogonal to the two columns in the matrix of derivatives if and only if

$$\begin{bmatrix} -1 & a_1 \\ -1 & b_1 \end{bmatrix}\begin{bmatrix} v_{\hat{c}t+1} \\ \beta v_{\hat{c}t+2} \end{bmatrix} = -\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\beta^2 v_{\hat{c}t+3}.$$

Since $\sigma>1$ implies $a_1<0$ and $b_1>0$, the determinant of the l.h.s. matrix is not zero, so the matrix is invertible and

$$\begin{bmatrix} v_{\hat{c}t+1} \\ \beta v_{\hat{c}t+2} \end{bmatrix} = \begin{bmatrix} \frac{a_1b_2-a_2b_1}{a_1-b_1} \\ -\frac{a_2-b_2}{a_1-b_1} \end{bmatrix}\beta^2 v_{\hat{c}t+3}.$$

Since $\sigma>1$ implies $a_2>0$ and $b_2<0$, the ratios $v_{\hat{c}t+1}/\beta v_{\hat{c}t+2}$ and $\beta v_{\hat{c}t+2}/\beta^2 v_{\hat{c}t+3}$ are well defined,

$$\begin{cases} \frac{v_{\hat{c}t+1}}{\beta v_{\hat{c}t+2}} = -\frac{a_1b_2-a_2b_1}{a_2-b_2} = \frac{\sigma\kappa}{1+\sigma\kappa} \\ \frac{v_{\hat{c}t+2}}{\beta v_{\hat{c}t+3}} = -\frac{a_2-b_2}{a_1-b_1} = \frac{\hat{R}(1+\sigma\kappa)\sigma\kappa}{(1+\sigma\kappa)^2-\hat{R}\beta(1-\sigma)} \end{cases}.$$

If the vector $(v_{\hat{c}t+1},\beta v_{\hat{c}t+2},\beta^2 v_{\hat{c}t+3})$ is orthogonal to the columns in the matrix *at every date t*, then the two ratios have to be identical,

$$\frac{v_{\hat{c}t+1}}{\beta v_{\hat{c}t+2}} = \frac{\sigma\kappa}{1+\sigma\kappa} = \frac{\hat{R}(1+\sigma\kappa)\sigma\kappa}{(1+\sigma\kappa)^2-\hat{R}\beta(1-\sigma)} = \frac{v_{\hat{c}t+2}}{\beta v_{\hat{c}t+3}}$$

or equivalently

$$\hat{R} = \frac{(1+\sigma\kappa)^2}{(1+\sigma\kappa)^2+\beta(1-\sigma)}.$$

Observe that $(1+\sigma\kappa)^2+\beta(1-\sigma)>0$: $(1+\sigma\kappa)^2>\beta(\sigma-1)$ if and only if $1+\beta<4$. Now $\hat{R}\geq\varepsilon/\beta$ is equivalent to $(\varepsilon-\beta)(1+\sigma\kappa)^2+\varepsilon\beta(1-\sigma)\leq 0$. Clearly, for $\sigma=1$ the inequality is violated. The derivative with respect to σ is $2(\varepsilon-\beta)\kappa(1+\sigma\kappa)-\varepsilon\beta$. As $\varepsilon\leq 1$, it is $\kappa\geq\varepsilon\beta/(1-\beta)$, and the derivative is positive for all $\sigma>1$. Therefore, the inequality is violated for all $\sigma>1$. Consequently, if $1\geq\varepsilon>(1+\beta)/2$ and $\hat{R}\beta\geq\varepsilon/\beta$ for some $\varepsilon>(1+\beta)/2$ and all $\sigma>1$, then there is $t>0$ such that $(v_{\hat{c}t+1},\beta v_{\hat{c}t+2},\beta^2 v_{\hat{c}t+3})$ is not orthogonal to the columns in the matrix.$\square$

It is now an immediate consequence of the Auxiliary Claim that if $\tau^{k*} > 0$, i.e., $\hat{R}^*\beta > 1$, then there is a date $t > 0$ and a change in $\triangle_{t+1}$ such that $Dv \cdot d\hat{c} \neq 0$. Thus, at $\theta = 0$ there exists $dz \in \mathfrak{Z}$ such that $DU \cdot dz = Dv \cdot d\hat{c} > 0$, and $dz \in \Delta U \cap TF^*_{\backslash 2,t}$. By optimality of z^*, it cannot be that $DF^*_{2,t} \cdot dz \geq 0$. When θ is close enough to zero, continuity concludes the proof. ■

Proof of Proposition 9: Again, throughout we write σ for σ_u. Let dz_2 be the (symmetric) change which occurs with probability $1-\mu$. Since $\frac{1}{2}dz_2 \cdot D^2U^* \cdot dz_2 > (2\mu - 1)DU^* \cdot dz_2$ and $DU^* \cdot dz_2 > 0$ while $dz_2 \cdot D^2U^* \cdot dz_2 < 0$, then $\mu < 1/2$. Now we are going to show that, under the stated conditions, $dz = dz_2$ can be chosen so that $d\tau_{t+1} > 0$ and $d\tau_{t+3} < 0$, proving the claim.

Let $dc_{t+1} < 0$. From $\beta \frac{u_{ct+1}}{u_{ct}} R_{t+1} = 1$ since c_t and k_{t+1} do not change, u_{ct+1} increases and R_{t+1} must decrease, so that $d\tau_{t+1} > 0$. From $DF_{2,t} \cdot dz < 0$, it must be $dk_{t+2} = (\sigma - 1 + \sigma\kappa)dc_{t+1} - \varepsilon$ for some $\varepsilon > 0$, and then $dk_{t+2} < 0$.

Next, from $\beta \frac{u_{ct+3}}{u_{ct+2}} R_{t+3} = 1$ and since c_{t+3} does not change, there will be a decrease $d\tau_{t+3} < 0$ if and only if the total differential $d\gamma$ of

$$\gamma(c_{t+1}, k_{t+2}) \equiv \frac{\hat{R}(k_{t+3}(c_{t+1}, k_{t+2}))}{u_c(c_{t+2}(c_{t+1}, k_{t+2}))}$$

is negative. Computations show that, at a steady state ($\kappa = \beta/(1-\beta)$),

$$D\gamma \cdot (dc_{t+1}, dk_{t+2}) = \frac{\sigma}{u_c} \frac{1}{\beta(\sigma - 1)} (\hat{f}''\kappa + \frac{\hat{f}'}{c})(2\sigma\kappa + \sigma - 1)dc_{t+1} - \tilde{\varepsilon},$$

and this is negative if and only if

$$\hat{f}''(k_g)k_g + \hat{f}'(k_g) > 0 \tag{*}$$

where k_g is the steady state where $\hat{R}\beta > 1$. If $f(k) = k^\alpha$, then, $\hat{f}' = \alpha\frac{f(k)}{k} + 1 - \delta$ and $\hat{f}'' = \alpha(\alpha - 1)\frac{f(k)}{k^2}$, thus

$$-\hat{f}''k = \alpha(1-\alpha)\frac{f(k)}{k} < \alpha\frac{f(k)}{k} \leq \hat{f}',$$

and condition (*) holds, i.e., $d\tau_{t+3} < 0$. ■

8 Appendix C: Optimal taxspots

Proof of Proposition 2: Throughout, we use c_t for consumption. Let a solution $\mu^* \in \Delta(C)$ to problem E-RP be given. Let $\omega^t = (g^t, c^{t-1})$ be a history of expenditures and consumptions up to date $t > 0$, i.e., $c^{t-1} = (c_0, c_1, ..., c_{t-1})$, where $c_\tau \in [0, 1-g_\tau]$, all $\tau < t$, and $\omega^0 = g_0$. Let Ω be the space of infinite such histories, and $\mathscr{F}_t$ be the σ-algebra generated by histories ω^t.

Given μ^*, we construct a payoff-equivalent process $\mu^*_t, t \geq 0$ of distributions over consumption plans c_t adapted to the filtration $(\mathscr{F}_t)_{t\in T}$ (that is, conditional on realizations ω^t). For any $t \geq 0$, let μ^t_π be the resulting probability measure over histories ω^t induced by π and $\mu^*_\tau, \tau \leq t$. Hence, in problem E-RP the planner equivalently has chosen process $\mu^*_t, t \geq 0$: at each t, given g^{t-1} exogenous shock g_t is realized according to transition π and, for $t > 0$, c^{t-1} has realized according to measures $\mu_\tau^*, \tau < t$. Then, the planner tosses coin $\mu_t^*(\omega^t)$ to assign consumption c_t, and the resulting leisure $x_t = 1 - g_t - c_t$. The representative agent's expected utility is

$$\sum_t \beta^t \int U_t(c_t) d\mu_\pi^t(g_0).$$

Lottery $\mu_t^*(\omega^t)$ on c_t after history $\omega^t = (g^t, c^{t-1})$ is a probability measure over $[0, 1-g_t] \subset \mathbb{R}_+$, a closed subset of $\mathbb{R}$. Thus, by Kuratowski Theorem (see Parthasarathy, 1967, Ch. 3) there exists a Borel measurable, invertible function $f : \mathscr{B}[0, 1-g_t] \to \mathscr{B}(\mathbb{R})$ with measurable inverse such that $\theta_t(\omega^t) = f_* \mu_t^*(\omega^t)$ is the corresponding probability measure on $\mathbb{R}$, i.e., $\theta_t(\omega^t)(B) = \mu_t^*(\omega^t)(f^{-1}(B))$ for every Borel $B \in \mathscr{B}(\mathbb{R})$.

Let $S = [0,1]$ and Leb be the Lebesgue measure on S. Using a standard argument (see, e.g., Williams, 1991, Ch. 3), it can be shown that there exists a random variable $c_t^-(\omega^t) : S \to \mathbb{R}$ such that

$$\theta_t(\omega^t) = f_* \mu_t^*(\omega^t) = c_t^-(\omega^t)_* Leb. \tag{7}$$

where $c_t^-(\omega^t)_* Leb$ is the (Lebesgue-Stieltjes) probability measure associated with the r.v. $c_t^-(\omega^t)$ when the underlying space has a uniform distribution.

Let $c_t(\omega^t) : S \to [0, 1-g_t]$ be defined via $c_t(\omega^t) = f^{-1} \circ c_t^-(\omega^t)$. Now (7) implies that, for every $B \in \mathscr{B}([0, 1-g_t])$,

$$\begin{aligned} c_t(\omega^t)_* Leb(B) &= Leb((c_t^-(\omega^t))^{-1} \circ f)(B)) \\ &= \theta_t(\omega^t)(f(B)) = f_* \mu_t^*(\omega^t)(f(B)) = \mu_t^*(\omega^t)(B), \end{aligned}$$

i.e., consumption function $c_t(\omega^t)$ is a function measurable relative to taxspot space $(S, \mathscr{B}(S), Leb)$, with values $c_t(\omega^t, s_t)$ and with values distribution equal to the lottery $\mu_t^*(\omega^t)$, where $c_0(\omega^0) = c_0(g_0)$ is a function with values $c_0(\omega^0, s_0)$ measurable with respect to $(S, \mathscr{B}(S), Leb)$ and with distribution given by μ_0^*.

Histories $\omega^t = (g^t, c^{t-1})$ with μ_π^t-positive probability are then generated by the space of histories (g^t, s^{t-1}), and we can write $\hat{c}_t, t \geq 0$ as an equivalent process on $\hat{\Omega}$, the set of all histories $\hat{\omega}^t = (g^t, s^t)$: for $t = 0$, $\hat{c}_0(\hat{\omega}_0) = c_0(\omega^0, s_0)$, and for any $t > 0$ and any $\omega^t = (g^t, c^{t-1})$ with $c^{t-1} = c^{t-1}(g^{t-1}, s^{t-1})$ for some $s^{t-1} = (s_0, ..., s_{t-1})$ it is $\hat{c}_t(\hat{\omega}^t) = c_t(g^t, c^{t-1}(g^{t-1}, s^{t-1}), s_t)$ for $\hat{\omega}^t = (g^t, s^t)$. By construction,

$$\sum_t \beta^t \int U_t(c_t) d\mu_\pi^t(g^t, c^{t-1}; g_0) = \sum_t \beta^t \int U_t(\hat{c}_t) d\nu_\pi^t(g^t, s^t; g_0)$$

for some process of probabilities ν_π^t derived from π and $\nu_{t'}^* = Leb$, $t' \leq t$ (see f.note 12). We now let (bounded) processes $(x_t, b_{t+1}, p_t, \tau_t)$ be defined as adapted to the filtration generated by histories $\hat{\omega}^t = (g^t, s^t)$ and derived via $\hat{c}_t(\hat{\omega}^t)$ using market clearing (for x_t) and the FOC (for p_t and τ_t), and for each $t \geq 0$ and $\hat{\omega}^t$, defining

$$b_t(\hat{\omega}^t) = \sum_{s \geq t} \mathbb{E}_{t, \nu_\pi^s} \frac{p_s}{p_t} [(\hat{c}_s - \hat{b}_{0,s}) - (1 - \tau_s)(1 - x_s)].$$

Note that, for each $s \geq t$, the conditional expectation operator is now dependent on probabilities $\nu^*_{t'}$, $t' \leq s$ and on π, and sometimes we denote it simply as $\mathbb{E}_{t,\nu}$. Then, b_{t+1} is a ν_π^{t+1} -integrable function, and it is verified from $F(\hat{c}) = 0$ –with a change of variables similar to the one applied for U also for F– that $p_t b_t + p_t(1-\tau_t)(1-x_t) = p_t \hat{c}_t + \mathbb{E}_{t,\nu_\pi^{t+1}} p_{t+1} b_{t+1}$, while $\lim_{t\to\infty} \mathbb{E}_{0,\nu_\pi^{t+1}} p_{t+1} b_{t+1} = 0$. Since FOC and transversality imply optimality, we have shown that the Ramsey lottery μ^* induces a competitive equilibrium with taxspots and taxes τ, where ν^* is the process of uniform distributions $\nu^*_t, t \geq 0$ over S.

Finally, take a lottery μ^* solving problem E-RP. Let

$$E = \{(u,r) \in \mathbb{R}^2 : u = U(c) \text{ and } r = F(c), \text{ some } c \in \bar{C}\}$$

where $\bar{C} \subset \overset{o}{C}$ is (product) compact, contains the support of μ^*, and both U and F are continuous on $\bar{C}$. Thus, E itself is compact. Let $(u^*, 0)$ be the point $u^* = \int_{\bar{C}} U(c) d\mu^*$ and $0 = \int_{\bar{C}} F(c) d\mu^*$. Then, as $\bar{C}$ is metrizable, by Thm 15.10 in Aliprantis and Border (2005), $(u^*, 0) = \lim_n (u_n, r_n)$ where $u_n = \int_{\bar{C}} U(c) d\mu_n$ and $r_n = \int_{\bar{C}} F(c) d\mu_n$, $\mu_n \to \mu^*$ in the weak* topology, and μ_n is a finite support probability measure for each n. Thus (u_n, r_n) is in the convex hull CoE of E, a subset of $\mathbb{R}^2$. By Carathéodory's Convexity Theorem, any point in CoE can be expressed as the convex combination of at most 3 points in E. Further, as E is closed, so is CoE, i.e., $(u^*, 0) \in CoE$, meaning that μ^* also has finite support with at most 3 points in E, and corresponding elements of $\bar{C}$, denoted c_j, $j = 1,2,3$.

Thus, at the optimal μ^*, this probability measure also solves, for given triplets $c_j \in \bar{C}$, $j \leq 3$,

$$\begin{gathered} \max_{\mu(c_j), j\leq 3} \textstyle\sum_{j\leq 3} U(c_j)\mu(c_j) \quad \text{s. to} \\ \textstyle\sum_{j\leq 3} F(c_j)\mu(c_j) = 0 \\ 1 - \textstyle\sum_{j\leq 3} \mu(c_j) = 0 \\ \mu \geq 0 \end{gathered}$$

This is a finite dimensional linear programming problem. Its solution is a vertex with at most 2 nonzero points, i.e., there are at most two $j, j' \leq 3$ such that $\mu(c_j)\mu(c_{j'}) > 0$.

If for some $t \geq 0$ and g^t it is $c^t_j \neq c^t_{j'}$, $j' \neq j$, then

$$\mu^*(c_j) \leq \mu^{t+t',*}(c_j^{t+t'}; \omega_0) \leq \mu^*_t(c_{j,t}|\omega^t)$$

implies $\mu^*_{t+t'}(c_{j,t+t'}|g^{t+t'}, c_j^{t+t'-1}) = 1$ for all $g^{t+t'}$, all $t' > 0$. This implies that there is at most one date $t \geq 0$ such that $\mu^*_t(\omega^t) \neq \delta_c$ (the Dirac measure on c) for some $c \in (0, 1-g_t]$. The conclusion follows.■

Proof of Lemma 5: The first step is to revert to the optimal lottery μ^* and derive FOC. Lottery μ^* equivalently solves

$$\max_{\mu\geq 0} \int \hat{U}(k) d\mu \text{ s. to } \int \tilde{F}(k) d\mu \geq 0 \text{ and } 1 - \mu(K) = 0$$

where $\tilde{F}(k) \equiv \hat{F}(k) - \hat{R}_0 k_0 u_c(\hat{f}_0(k_0) - k_1)$ and μ is a signed Borel measure on the subspace $K \subset L(\mathbb{R})$ of capital processes adapted to $\{\mathscr{G}_t\}_{t\geq 0}$, while $\mu \geq 0$ stands for the subset of nonnegative such measures. Equipped with the total variation norm, the space is Banach. By A3.a, we have assumed that there exists $\mu^0 \geq 0$ with $1 = \mu^0(K)$, and such that $\int \tilde{F}(k) d\mu^0 > 0$. As the Dirac δ_k are nonnegative for every $k \in L(\mathbb{R})$, for any solution μ^* it is $(1-\varepsilon)\mu^* \geq 0$ and $(1+\varepsilon)\mu^* \geq 0$ for small enough $\varepsilon > 0$. Observe that here the equality constraint is an affine function of μ. Then, by a separation theorem (see, e.g., Luenberger, Thm 1, p. 217, 1969), there exist scalars $\lambda \geq 0$ and U^* such that

$$\hat{U}(k) + \lambda \tilde{F}(k) \leq U^*, \text{ all } k \in K, \ = U^*, \mu^*\text{-a.e.}$$

In particular, as μ^* puts mass one on the interior of K by Lemma 4, there exist a scalar h and $k \in L(\mathbb{R})$ such that $k^* \pm hk \in K$. Apply the previous inequality to these two points where the direction of change is the zero process except for the $k_{t+1}(g^t)$ component. Using the differentiability of u and of $\hat{a}_t, \hat{a}_{t+1}$, and the interiority of the point , the effect on the objective function is only at $k_t^* = k_t(g^t)$, as

$$\begin{aligned} U_{t,t+1}(k_{t+1}^* + hk_{t+1}(g^t), \mu^*) &= u(\hat{f}_t(k_t^*) - (k_{t+1}^* + hk_{t+1}(g^t))) \\ &\quad + \beta \mathbb{E}_{t,\mu*} u(\hat{f}_{t+1}(k_{t+1}^* + hk_{t+1}(g^t)) - k_{t+2}), \end{aligned}$$

and its effect on the constraint only on

$$\begin{aligned} a_{t,t+1}(k_{t+1}^* + hk_{t+1}(g^t), \mu^*) &= \hat{a}_t(\hat{f}_t(k_t^*) - (k_{t+1}^* + hk_{t+1}(g^t))) \\ &\quad + \beta \mathbb{E}_{t,\mu*} \hat{a}_{t+1}(\hat{f}_{t+1}(k_{t+1}^* + hk_{t+1}(g^t)) - k_{t+2}), \end{aligned}$$

where $\mathbb{E}_{t,\mu*}$ is the expectation conditional on g^t, k^{*t+1} and using μ^* to average the realizations of k_{t+2}. Then, taking $h \to 0$, we obtain

$$\beta \mathbb{E}_{t,\mu^*} u_{ct+1} \hat{R}_{t+1} (1 + \lambda \frac{\hat{a}_{ct+1}}{u_{ct+1}}) = u_{ct}(1 + \lambda \frac{\hat{a}_{ct}}{u_{ct}}).$$

Rearranging terms, assuming that $1 + \lambda \frac{\hat{a}_{ct}}{u_{ct}} > 0$, as under (b), and using the definition of M_t,

$$\beta \mathbb{E}_{t,\mu^*} u_{ct+1} \hat{R}_{t+1} \frac{1 + \lambda M_{t+1}}{1 + \lambda M_t} = u_{ct}.$$

On the other hand, as a taxspot equilibrium, consumption must satisfy

$$\beta \mathbb{E}_{t,\nu^*} u_{ct+1} \hat{R}_{t+1} (1 - \tau_{t+1}) = u_{ct}.$$

Recalling that $\mathbb{E}_{t,\mu^*} = \mathbb{E}_{t,\nu^*}$, equating the two expressions and again rearranging,

$$\mathbb{E}_t^{\Psi(\nu^*)} \tau_{t+1} = \frac{\mathbb{E}_{t,\nu^*} u_{ct+1} \hat{R}_{t+1} \tau_{t+1}}{\mathbb{E}_{t,\nu^*} u_{ct+1} \hat{R}_{t+1}} = \frac{\lambda}{1 + \lambda M_t} \frac{\mathbb{E}_{t,\nu^*} u_{ct+1} \hat{R}_{t+1} (M_t - M_{t+1})}{\mathbb{E}_{t,\nu^*} u_{ct+1} \hat{R}_{t+1}} \geq 0$$

if and only if, when $\lambda > 0$ as under (a),

$$M_t \geq \frac{\mathbb{E}_{t,\nu^*} u_{ct+1} \hat{R}_{t+1} M_{t+1}}{\mathbb{E}_{t,\nu^*} u_{ct+1} \hat{R}_{t+1}}$$

as wanted. ■

Proof of Proposition 7: Existence of an interior solution under A3 is obtained following the logic of the proofs of Lemma 2 and 6. We now want to show that for every $t > 0$ there exists $t' > t$ such that $\mu^*_{t'} \neq \delta_{z_{t'}}$ (the Dirac on $z_{t'}$), all $z_{t'} \in Z_t$. If not, suppose μ^* is such that $\mu^*_{t'} = \delta_{z'_t}$ some $z'_{t'}$, all $t' \geq t$. Then, μ^* solves the problem

$$\begin{array}{cc} \max_{\hat{c},x,c,k} U(\hat{c},x,c,\theta) \equiv \sum_{t' \geq t} \beta^{t'} U_{t'}(\hat{c},x,c,\theta) & \text{s. to} \\ F_{t'}(\hat{c},x,c,k) \geq 0 & \text{all } t' \geq t, \\ \lim_{t' \to \infty} \beta^{t'} u_{ct'} k_{t'+1} = 0, & \end{array}$$

with $k_t > 0$ given. This is an M-RP-WK problem. By assumption, it is then $\tau^x_{t'} > 0$, and as (R) holds, $\det A_{t'} > 0$, $\mu^*_{t'}$-a.e., for all $t' \geq t$.

For all $t' \geq t$, $z^*_{t'}$ is in $[0,\bar{k}] \times [0,1] \times [0,\bar{k}] \times [0,\bar{k}]$, a compact set $\bar{Z}$. Thus, for every $\varepsilon > 0$ consider the finite subcover $\{N^i\}_{i=1}^n$ of $\bar{Z}$ with $\cup_i N^i = Z$ and $z^i \in N^i$ such that $||z^*_{t'} - z^i|| < \varepsilon$, all $t' \geq t$. At z^i we perturb utility v so that condition D-WK holds at z^i. Then, for ε small enough and since the functions involved are continuous and inequalities are strict, condition D-WK will hold at $z^*_{t'}$, and since we have not changed first and second derivatives, $\det A_{t'} > 0$ and $\tau^{x*}_{t'} > 0$, all $t' \geq t$. We finally apply Proposition 6 to get a lottery that Pareto improves over $\mu^*_{t'}$, hence over μ^*, a contradiction. ■

Proof of Proposition 11: We follow the same reasoning as in the proof of Lemma 5. Using a separation argument (see, e.g., Luenberger, Thm 1, p. 217, 1969), first order conditions for interior states in the support of μ^* at any history $z^t = (\hat{c}^t, c^t, k^{t+1})$ lead to

$$\begin{array}{rcl} c_t & : & \tilde{\lambda}_t = \tilde{\lambda}_{t-1}[\dfrac{\sigma_t - 1}{\sigma_t \kappa_{t+1}} + 1] + \dfrac{1}{\sigma_t \kappa_{t+1}} \dfrac{(1-\theta) v_{\hat{c}t} - \theta u_{ct}}{u_{ct}}, \\ k_{t+1} & : & \beta \mathbb{E}_{t,\mu^*} m_{t+1} \hat{R}_{t+1} \hat{M}_{t+1} - \hat{M}_t = \tilde{\lambda}_t - \tilde{\lambda}_{t-1}, \end{array}$$

μ^*-a.e., where $\tilde{\lambda}_t, t > 0$ is the process of multipliers for the Euler constraints, and $m_{t+1} = u_{ct+1}/u_{ct}$. Additionally, the constraint $\beta \mathbb{E}_{t-1,\mu^*} u_{ct}(\hat{f}_t(k_t) - \hat{c}_t) = u_{ct-1} k_t$ must also hold μ^*-a.e., all $t > 0$. Using the Euler equation with explicit taxes, we can then write the second equation as

$$\beta \mathbb{E}_{t,\mu^*} m_{t+1} \hat{R}_{t+1} \hat{M}_{t+1} - \hat{M}_t \beta \mathbb{E}_{t,\mu^*} m_{t+1} \hat{R}_{t+1} + \hat{M}_t \beta \mathbb{E}_t m_{t+1} \hat{R}_{t+1} \tau_{t+1} = \tilde{\lambda}_t - \tilde{\lambda}_{t-1},$$

For all realizations z^t with $\tilde{\lambda}_t - \tilde{\lambda}_{t-1} \geq 0$, we obtain

$$\hat{M}_t \frac{\mathbb{E}_{t,\mu^*} m_{t+1} \hat{R}_{t+1} \tau_{t+1}}{\mathbb{E}_{t,\mu^*} m_{t+1} \hat{R}_{t+1}} \geq \hat{M}_t - \frac{\mathbb{E}_{t,\mu^*} m_{t+1} \hat{R}_{t+1} \hat{M}_{t+1}}{\mathbb{E}_{t,\mu^*} m_{t+1} \hat{R}_{t+1}}$$

or, using the definition of expected tax $\mathbb{E}_t^{\Psi(v^*)}\tau_{t+1}$ and of density ψ_{t+1},

$$\hat{M}_t \mathbb{E}_t^{\Psi(v^*)}\tau_{t+1} \geq \hat{M}_t - \mathbb{E}_{t,v^*}\psi_{t+1}\hat{M}_{t+1}.$$

Thus, we conclude that for all trajectories where $\tilde{\lambda}_t - \tilde{\lambda}_{t-1} \geq 0$, it is $\mathbb{E}_t^{\Psi(v^*)}\tau_{t+1} \geq 0$ if $\hat{M}_t \geq \mathbb{E}_t^{\Psi(v^*)}\hat{M}_{t+1}$.

Finally, suppose at the optimal taxspot μ^* history $z^t \in B$ a Borel set with μ^*-positive probability and it is also on the optimal path without lotteries, for some $t \geq 0$, that is: $z_t = (c_t^{nl}, c_t^{nl}, k_{t+1}^{nl})$ and $\hat{\phi}_t^{nl} - \hat{\phi}_{t-1}^{nl} = \hat{\phi}_t^l - \hat{\phi}_{t-1}^l$, where indexes l, nl denote the 'lottery' and 'no lottery' problem at z_t. From the first order condition for capital in either problem $i = l, nl$,

$$\beta \mathbb{E}_{\mu_t^*} \frac{v_{\hat{c}t+1}}{u_{ct}} \hat{f}'_{t+1} - \frac{v_{\hat{c}t}}{u_{ct}} \geq \beta \frac{v_{\hat{c}t+1}^{nl}}{u_{ct}} \hat{f}'_{t+1} - \frac{v_{\hat{c}t}}{u_{ct}}.$$

Thus, from concavity of $v_{\hat{c}}$,

$$v_{\hat{c}}(\mathbb{E}_{\mu_t^*}\hat{c}_{t+1}) - v_{\hat{c}t+1} > \mathbb{E}_{\mu_t^*} v_{\hat{c}t+1} - v_{\hat{c}t+1}^{nl} \geq 0$$

and from concavity of v,

$$\mathbb{E}_{\mu_t^*}\hat{c}_{t+1} < \hat{c}_{t+1}^{nl}$$

and finally from market clearing

$$(1 - \tau_{t+1}^{nl})\hat{R}_{t+1} = \frac{c_{t+1}^{nl} + k_{t+2}^{nl}}{k_{t+1}} < \mathbb{E}_{\mu_t^*} \frac{c_{t+1} + k_{t+2}}{k_{t+1}} = (1 - \mathbb{E}_{\mu_t^*}\tau_{t+1})\hat{R}_{t+1},$$

i.e., $\tau_{t+1}^{nl} > \mathbb{E}_{\mu_t^*}\tau_{t+1}$, as wanted.■

9 Appendix D: A perturbation argument (for condition D)

How likely is condition D to arise? One can regard condition D as quite pervasive in the space of preferences, or utility functions, if vicinity is judged in the Whitney C^3-topology sense: two utility functions are close if their derivatives up to the third order are close on all compact sets. Then, for given economy we can perturb the initial utility function to tweak the constraint set around a solution $\hat{c}^*$, without changing it anywhere else, while staying in the vicinity of the original economy. We can formally show this in finite economy approximations of the given infinite horizon economy, as follows.

Lemma 10 *Let $T < +\infty$. Suppose that at an RP optimum $\hat{c}^*$ it is $U_{\hat{c}t}^* > 0$ for some date-event g^t, $t \geq 0$. Then there is an arbitrarily close economy such that $\hat{c}^*$ is still a solution to the RP problem and condition D is satisfied.*

Lemma 10 addresses the infinite dimensional case in providing dominance of a 2-point lottery in at least finitely many periods and state realizations g_t, as the perturbation can be done independently at any $\hat{c}_t^*$ that is locally isolated in $[0,1]$. Constant elasticity of substitution utilities bundle risk aversion and prudence under the same parameter, and any change in this parameter entails a change in both the second and third derivatives. Thus, the density argument must put us outside this class of functions.

The first step to prove Lemma 10 consists in finding a perturbation of a third derivative which does not alter first and second derivatives, or the other third derivatives. We illustrate the argument for $v_{\hat{c}\hat{c}\hat{c}}$ although it will be clear that the same argument applies to any third derivative of v.

For vicinity, we use density in the Whitney C^3-topology sense: we say that two functions v, $\hat{v}$ $\in \mathscr{C}^3(X,\mathbb{R})$ are *arbitrarily close* if for every $\varepsilon > 0$, it is

$$\|v-\hat{v}\|_{W,K} = \sup\{||D^j v(x) - D^j \hat{v}(x)|| : x \in K,\ j = 0,1,2,3\} < \varepsilon$$

all compact subsets K of X. We let $D^3_{i_3 i_2 i_1}$ denote the third partial with respect to variables i_j, $j = 1,2,3$, where $i_j \in \{\hat{c}, x\}$ is the coordinate variable at the j-th order of derivation.

Lemma 11 *Let $T < +\infty$. For any per-period utility u and any solution point $\hat{c}_t^*(g^t)$ of problem RP, there is a utility function $\hat{v}$ arbitrarily close to v and such that $D^j\hat{v}(\hat{c}^*,x^*) = D^j v(\hat{c}^*,x^*)$ for $j = 0,1,2$, and $D^3_{i_3i_2i_1}\hat{v}(\hat{c}^*,x^*) = D^3_{i_3i_2i_1}v(\hat{c}^*,x^*)$ for all $\{i_3,i_2,i_1\} \neq \{\hat{c},\hat{c},\hat{c}\}$, while*

$$D^3_{\hat{c}\hat{c}\hat{c}}\hat{v}(\hat{c}_t^*(g^t), 1-g_t-\hat{c}_t^*(g^t)) = v^*_{\hat{c}\hat{c}\hat{c}t} + \alpha$$

for some arbitrary scalar α.

Proof: For $\chi = (\acute{c},\alpha,\varepsilon)$ let the function $f_\chi : \mathbb{R}_+ \to \mathbb{R}$ be defined as

$$f_\chi(\hat{c}) = \begin{cases} \alpha\varepsilon(\hat{c}-\acute{c}+\varepsilon)^2 & \text{if } \hat{c}-\acute{c} < -\varepsilon \\ -\dfrac{1}{6}\alpha(\hat{c}-\acute{c}-\varepsilon)^3 & \text{if } -\varepsilon \le \hat{c}-\acute{c} \le \varepsilon \\ 0 & \text{if } \hat{c}-\acute{c} > \varepsilon. \end{cases}$$

It is verified that f_χ is twice continuously differentiable, and for fixed $(\acute{c},\alpha)$ the function f_χ converges to the zero function as ε converges to zero, in the Whitney C^3 topology.

Consider $\chi = (\hat{c}^*,\alpha,\varepsilon)$ and $\chi' = (\hat{c}^*+3\varepsilon/4, \alpha', \varepsilon/4)$. It is

$$f'_\chi(\hat{c}^*) = -\alpha\varepsilon^2/2,\ f''_\chi(\hat{c}^*) = \alpha\varepsilon,\ f'''_\chi(\hat{c}^*) = -\alpha,$$

while, since $\hat{c}^* - \acute{c} = -3\varepsilon/4 < -\varepsilon/4$,

$$f'_{\chi'}(\hat{c}^*) = -2\alpha'\varepsilon^2/8,\ f''_{\chi'}(\hat{c}^*) = 2\alpha'\varepsilon \text{ and } f'''_{\chi'}(\hat{c}^*) = 0.$$

Hence, for $\alpha' = 2\alpha$ it is $f'_{\chi'}(\hat{c}^*) = f'_{\chi}(\hat{c}^*)$ and $f''_{\chi'}(\hat{c}^*) = f''_{\chi}(\hat{c}^*)$. Now define a (smooth) bump function $b(\hat{c})$ which is zero outside the interval $(\hat{c}^* - \varepsilon, \hat{c}^* + \varepsilon)$ and $b(\hat{c}^*) = 1$. Define

$$V_t(\hat{c}) = \hat{v}(\hat{c}, 1 - g_t - \hat{c}) = U_t(\hat{c}) + b(\hat{c})[f_{\chi'}(\hat{c}) - f_{\chi}(\hat{c})].$$

The first- and second-order derivatives of U_t and V_t at $\hat{c}^*$, and the third-order derivatives $D^3_{i_3 i_2 i_1}$ for $\{i_3, i_2, i_1\} \neq \{\hat{c}, \hat{c}, \hat{c}\}$, are identical, while the third-order derivative $V_{\hat{c}\hat{c}\hat{c}}(\hat{c}^*, x^*) = v_{\hat{c}\hat{c}\hat{c}}(\hat{c}^*, x^*) + \alpha$, as wanted. ■

The above argument not only shows that the perturbed function is close to the initial one, but it also requires changing the function only in an arbitrarily small neighborhood around the initial point, $\hat{c}^*_t(g^t)$.

We are now going to apply the perturbation in Lemma 11 to any economy where utility violates condition D to obtain a nearby economy with the required property D and, in addition, with the property that the initial (locally unique) solution to the M-RP problem stays a solution after the perturbation. This can be done while staying within an arbitrary neighborhood of the initial economy, thus proving Lemma 10.

To this end, let a solution $(\hat{c}^*, x^*)$ to the M-RP problem and $g^t, t \geq 0$ be given. By Lemma 1.iii it is $F(\hat{c}^*, \hat{b}_0, g) = 0$. Let $\hat{c}_{-g^t}$ be process $\hat{c}$ without the element $\hat{c}_t(g^t)$. Suppose that $U^*_{\hat{c}t} > 0$ and $a^*_{\hat{c}t} \neq 0$ at $g^t, t > 0$. We also assume at no loss of generality that $\hat{c}^*_t(g^t)$ is locally isolated. In particular, $\hat{c}^*_\tau(g^\tau) \neq \hat{c}^*_t(g^t)$, all $(g^\tau, \tau) \neq (g^t, t)$. The latter conditions hold for a large (generic in endowments) set of economies when T is finite, via repeated applications of the Parametric Transversality Theorem. By the Implicit Function Theorem there exists h such that $\hat{c}_t(g^t) = h(\hat{c}_{-g^t})$ for any $\hat{c}_{-g^t}$ within a neighborhood of $\hat{c}^*_{-g^t}$, such that $F(\hat{c}_t(g^t), \hat{c}_{-g^t}, \hat{b}_0, g) = 0$, and

$$D_\tau h(\hat{c}^*_{-g^t}) = -\frac{\hat{\beta}^\tau a^*_{\hat{c}\tau}}{\hat{\beta}^t a^*_{\hat{c}t}} \text{ all } (g^\tau, \tau) \neq (g^t, t)$$

while

$$D^2_{\tau'\tau} h(\hat{c}^*_{-g^t}) = \begin{cases} D_\tau h[\frac{a^*_{\hat{c}\hat{c}\tau}}{a^*_{\hat{c}\tau}} - \frac{a^*_{\hat{c}\hat{c}t} D_{\tau'} h}{a^*_{\hat{c}t}}] & \text{if } \tau' = \tau \\ -D_\tau h \frac{a^*_{\hat{c}\hat{c}t} D_{\tau'} h}{a^*_{\hat{c}t}} & \text{otherwise.} \end{cases}.$$

After substitution of h in v for $\hat{c}_t(g^t)$ the M-RP problem becomes

$$\max \xi(\hat{c}_{-g^t}) \equiv \mathbb{E}_0 \sum_{(g^{t'}, t') \neq (g^t, t)} \beta^{t'} U_{t'}(\hat{c}_{t'}) + \hat{\beta}^t U_t(h(\hat{c}_{-g^t})).$$

Let $D^2\xi$ be the Hessian of the function ξ computed at the solution $\hat{c}^*$, and $|_k D^2 \xi_k|$ be the determinant of the k-principal submatrix of $D^2\xi$, where k ranges from 0 to g^T, T (and histories are totally ordered). For the initial point $\hat{c}^*$ to still be optimal after the (small) perturbation, the following first and second order conditions must be satisfied:

$$\hat{\beta}^\tau U_{\hat{c}\tau}(\hat{c}^*_\tau(g^\tau)) + \hat{\beta}^t U_{\hat{c}t}(\hat{c}^*_t(g^t)) D_\tau h = 0, \text{ all } (g^\tau, \tau) \neq (g^t, t) \tag{8}$$

$$(-)^k |_k D^2 \xi_k| > 0 \text{ for every } k \geq 1. \tag{9}$$

We start with a change of the third derivative of v around the point $\hat{c}_t^*(g^t), x_t^*(g^t)$ using $x_t^*(g^t) = 1 - g_t - \hat{c}_t^*(g^t)$. We apply Lemma 11 to change the appropriate derivative according to Corollary 1, perturbing $a^*_{\hat{c}\hat{c}t}$. The FOC do not change. If the second order conditions (SOC) are strict, we stop. Otherwise, we apply Lemma 11 again to change $a^*_{\hat{c}\hat{c}\tau}$, $\tau \neq t$, for τ and history g^τ corresponding to the last diagonal entry in the k-th principal, making $D^2_{\tau\tau}\xi < 0$ and large enough, so that (SOC) holds at the perturbed economy. Observe that

$$D^2_{\tau\tau}\xi = \hat{\beta}^\tau U_{\hat{c}\hat{c}\tau}(\hat{c}_\tau^*(g^\tau)) + \hat{\beta}^t [U_{\hat{c}\hat{c}t}(\hat{c}_t^*(g^t))(D_\tau h) + U_{\hat{c}t}(\hat{c}_t^*(g^t))D^2_{\tau\tau}h]$$

and $D^2_{\tau\tau}h$ is a function of $a^*_{\hat{c}\hat{c}\tau}$ (but to perturb $D^2_{\tau\tau}\xi$, it must be that $a^*_{\hat{c}\tau} \neq 0$. Again, as $T < +\infty$, this will be generically true).

By the first perturbation, we have guaranteed condition D at g^t. By the second perturbations, we have assured that $\hat{c}^*$ is still the only solution in a small enough neighborhood of $\hat{c}^*$. The conclusion follows. ■

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