Determinate Indeterminacy: Keynesian Search in Stochastic Overlapping Generations Models.

Parand Akbari,[1] **Genaro Basulto**[2], **Eungsik Kim**[3], **Stephen Spear**[4].

[1] *Tepper School of Business, Carnegie Mellon University, pakbari@andrew.cmu.edu.* [2] *Tepper School of Business, Carnegie Mellon University, gbasulto@andrew.cmu.edu.* [3] *Department of Economics, University of Kansas, eungsikk@ku.edu.* [4] *Tepper School of Business, Carnegie Mellon University, ss1f@andrew.cmu.edu.*

Abstract: In this paper, we show that in a stochastic overlapping generations (SOLG) model with Keynesian search, agents' beliefs must align with the endogenously generated wealth distribution. This requirement stems from the pecuniary wealth externality introduced by portfolio rebalancing decisions. Using a three-period SOLG model, we examine issues such as age-based disparities in hours worked and the impact of employment uncertainty across business cycles. Our model incorporates dual labor market structures and calibrates the hours worked by young agents to drive stochastic fluctuations. We also explore the role of matching efficiency shocks, providing an alternative to traditional TFP shocks. We show preliminary results for the model with three overlapping generations and propose a Neural Network approach to find the equilibrium in our full-fledged model.

1. Introduction

General equilibrium macroeconomic models in which search replaces classical assumptions about the operations of labor markets go back at least to the seminal work on search theory by Diamond, Mortenson and Pissarides in the late 1990's. One of the perceived drawbacks of these early models (and their subsequent refinement as New Keynesian macro models) was the assumption that wages were determined by bilateral bargaining between firm and potential employee. This assumption has been criticized on the grounds that it leads to far smoother employment and output time-series in models that incorporate it than is observed in actual data. It also implicitly preserves the general equilibrium linkage between worker supply and firm demand for labor, albeit now channeled through the search mechanism, particularly in models where workers' bargaining power is assumed to depend on the state of the labor market (i.e., on current or lagged aggregate shocks).

The basic assumption that employment matches are determined by search reflects the fact that labor markets are inherently incomplete, generally due to information asymmetries about the overall value of a match. Empirical evidence for labor market incompleteness goes back at least to Jovanovic (1979) [1] and Jovanovic and Mincer (1981) [2]. Both of these papers show that once one controls for regression bias due to spurious correlations between separation rates and job tenure, there is a negative structural relationship between these variables. Jovanovic (1979) [1] interprets this as evidence that employment matches, at least at entry levels or for unskilled workers, are experience goods, as opposed to pure search goods. Under this interpretation, employers and workers can only learn the value of an employment match by making it and seeing if it works. Under the pure search goods interpretation, the value of potential matches can be determined once a search meeting has occurred, with good matches accepted and bad ones rejected. This is similar to how matching models are used in search-based models of money. Under this interpretation, however, wages do serve to induce greater search effort by workers when firms need more labor, and hence the fundamental mechanism of market clearing is preserved in these models. In experience good models, wages serve a different purpose. As Jovanovic notes, in these models "employers can contract with workers on an individual basis. The employer is then able to reward a worker with whom he matches well by paying the worker relatively more." (Ibid p. 974) This function of wages is independent of any market-clearing role. There is additional evidence for the labor as experience good hypothesis in Kotlikoff (1979) [3]. In this paper, Kotlikoff examines data on the New Orleans slave market between 1804 and 1862. One striking observation in this data is that the sale of individual slaves generally came with a guarantee as to the slave's ability to work. As Kotlikoff notes, "In the case of slaves who were not fully guaranteed, the exclusions from the full guarantee are often stated." (Ibid,p. 497) Kotlikoff also notes that this data is generally considered reliable because it was required by law in order to establish each owner's right to the particular slave being purchased.

Based on this kind of empirical evidence, together with a trove of evidence in the finance literature on the causal role that financial markets play in predicting business cycles, Farmer (2008) [4] introduced an alternative search mechanism which explicitly broke the link between wages and market clearing, which he dubbed (Old) Keynesian Search theory. In this search mechanism, firms produce to meet demand, paying a wage determined by their own labor demand schedule, but hiring only enough workers to meet their expected demand in each period. This effectively removes the labor market equilibration from Farmer's otherwise conventional RBC-based model, and renders the steady state of the model indeterminate. Farmer closes the model by introducing a belief function by which agents forecast future wealth, which then determines demand. The indeterminacy in the model means that any belief function equilibrium will constitute a rational expectations equlibrium (REE). The arbitrariness of the belief function then allows Farmer (and others working in similar frameworks) to generate equilibrium time-series that match the data much more closely than convention RBC or NK models.

In this paper, we show that in a general stochastic overlapping generations (SOLG) model with Keynesian search, beliefs cannot be arbitrary. The reason for this is staightforward. General SOLG models require carrying along the wealth distribution as a state variable (in what is known as a recursive equilibrium) because of the pecuniary wealth externality introduced by agents' portfolio rebalancing decisions[1]. Since this is a fundamental difference between the SOLG and RBC-based DSGE models, it is worth unpacking further.

In the SOLG framework, portfolio rebalancing in the face of shock realizations makes the competitive equilibrium history dependent. Woodford (1986) showed how to solve these models using the infinite history of shocks as the state variable. Later work by Spear and Srivastava (1986), Duffie et. al. (1994), and Rios-Rull (1996) showed how to use lagged endogenous variables as sufficient statistics for the shock history. Rios-Rull's assumption that the wealth distribution would be the natural statistic was subsequently shown to be true generically by Citanna and Siconolfi (2010). It is, of course, true that standard DSGE models include lagged endogenous state variables – the capital stock is the most obvious – but these are not serving as sufficient statistics for shock histories in these models. Indeed, in the RBC setting, the capital stock is determined by the law of motion for capital together with the optimal choice of saving by households. It is also true that in the standard DSGE setting, portfolio rebalancing decisions generate income effects, but these effects are internalized by the infinte lived households in the model via adjustments to bequests in a way that keeps the intertemporal allocations ex ante optimal. This is the feature that is missing in the SOLG setting.

We will show that while the SOLG setting admits the same kind of steady state indeterminacy as in Farmer's RBC-based model, agents' expectations (or beliefs) about the equilibrium stochastic

[1]See Spear and Young (2023) for an examination of the historical development of these results

process must conform with the endogenously generated distribution of asset holdings in the model. This requires that the REE be a non-trivial fixed point in the mapping from expectations to temporary equilibria of the model. Finally, in order to avoid the confusion that can arise when trying to distinguish Keynesian models from New Keynesian models or old Keynesian models, we will refer to models in which some labor markets are incomplete as Incomplete Search-base Labor Market (ISLM) models.

We use the model to examine a number of issues. The first concern the disparity in hours worked by age. Rios-Rull (1996) [5] used a multi-period SOLG model to examine how life-cycle differences affected a number of macroeconomic variables, among them the hours worked by agents of different ages. He found that the model (which includes both aggregate and idiosyncratic shocks) matched the observed hours worked of older agents, but could not replicate the hours of younger workers. This discrepancy suggests that while conventional labor market models (whether classical or New Keynesian) do an adequate job of explaining employment of older workers, a new approach is needed to understand what is going on at the entry level. This observation then suggests that lifecycle considerations may be important for the emergence of dual labor markets consisting of a primary sector employing skilled, experienced workers and a secondary sector consisting of entry-level workers with "low skill levels, low earnings, easy entry, job impermanence, and low returns to education or experience" (Beer and Barringer [1970]). We model the dual labor market structure by working with a three-period lived SOLG model in which middle-aged agents act as managers who, among other activities, participate in the recruiting of new young agents via the Keynesian search process from Farmer's work [2]. This lifecycle interpretation is well-grounded in empirical work showing that young workers optimally sample different jobs/occupations, which generates a signicant degree of employment churn, before settling into long duration spells of employment with specific firms, or within specific industries. (See, e.g., Miller [1984] [7] for early work on this, or Sullivan [2010] [8] and references therein for a more recent survey of this work.) The model developed here then allows us to use hours worked by the young as a calibrated parameter that drives the stochastic fluctuations of the model. We then show how to use standard results from pure exchange SOLG models to determine the REE of the dual labor markets model, which, following Farmer, we then interpret as the belief function of firms that gives rise to the observed variation in hours worked by the young in equilibrium.

A second, more general set of issues that we can address in the model concern the relationship between employment uncertainty and employment across the business cycle. For this analysis, we assume that the aggregate shocks affecting production act not on TFP, but rather on the efficiency of the matching process. While this is a somewhat unusual approach, we find it more plausible than the alternative of accepting the fiction of large aggregate TFP shocks or assuming that belief-based shocks to aggregate demand account for all buisness cycle movements. There is a substantial literature on the empirical importance of uncertainty in the labor markets that suggests putting the shocks on the matching process as an alternative to standard TFP shocks – see, for example Ravn and Sterk (2017) [9] or Barnichon and Figuera (2010) [10]. As Barnichon and Figuera note, the existence of shocks to the matching function provides a mechanism to explain the apparent shifts in the Beveridge curve over time, and their estimates of these shocks provide a benchmark for calibrating our model. Given this calibration, we can then determine the stochastic process of equilibrium in the model, conditional on the state of the business cycle and ask whether the simulations match up with the data. In a richer version of the model where we calibrate production using the results from regime-switching econometric models of the business cycle,

[2]Firms' decisions on how much seasoned labor to allocate to search generates search intensity effects similar to those originally analyzed by Howitt and McAfee (1987) [6].

we can examine the extent to which time homogeneous shocks to matching are consistent with the non-linearity of the regime-switching process. Conversely, given data about the movement of the Beveridge curve over the business cycle, we can calibrate time-varying shocks to matching and then examine whether the resulting stochastic process of employment is consistent with the data. Finally, we note that it is entirely possible to interpret the shocks to matching as extrinsic uncertainty, i.e. sunspots. Examples of this include the apparent discrimination against the long-termed unemployed during the recovery from the recession induced by the 2008 financial crisis (see Kroft, Notowidigdo, and Katz [2014] [11]). A second example is the so-called "Big Quit" following the covid-19 pandemic lockdowns, in which large numbers of laid off workers either exited the work force, or spent above average amounts of time searching for new jobs. The most recent data on work force participation strongly suggests that this was a transient phenomenon. We view modeling these actions as random shocks as a way of cutting through the need to formally model the psychological or policy-driven motives behind these examples.

A third set of issues we can address involve the product and financial market side of the model. Once we specify how wages and profits are determined on the production side, the REE of the associated exchange economy will then generate expectations (possibly over sunspots or sentiment variables) such that demand is compatible with production. (In Farmer's interpretation of the model, these expectations would be the cause of the production variations over the business cycle.) The resulting REE will generate time-series data in simulations for the equilibrium stochastic process on asset prices as well as consumption, which lets us examine how well the asset markets in the model match up with empirical data on asset markets over the course of the business cycle.

We interpret the results we obtain as indicating that while Farmer's contention that beliefs are important in determining macroeconomic equilibrium is valid, in realistic settings – SOLG models or NKDSGE models with heterogeneity and incomplete markets – where the wealth distribution evolves endogenously, beliefs cannot be arbitrary. Rather, they are constrained by the rational expectations equilibrium assumption used to close the mapping from expectations to temporary equilibria. The fact that even in these more realistic settings Keynesian search leads to indeterminate steady state equilibria highlights the importance of Farmer's observation about the incompleteness of labor markets. Our take on this incompleteness, however, is based on the lifecycle considerations that have been shown to generate the excess volatility in hours worked by young people. In this view, established workers with occupation-specific human capital and established track records don't generate the moral hazard and adverse selection problems associated with entry level labor markets, and hence these agents transact over complete markets for their services.

The fact that the REE must be determined as a non-trivial fixed point of the temporary equilibrium mapping also suggests that there is an important role for learning in environments where the steady state equilibrium is indeterminate. Because of the hysteresis induced by the steady state indeterminacy, plausible learning mechanisms may lead to variations in effective demand when unanticipated shocks – wars, pandemics, new technological or scientific discovers – impinge on the operation of the economy and disrupt an established long-run rational expectations equilibrium.

2. Model.

2.1. Three-period Framework

To illustrate the basic workings of the model, we first examine a three-period version of it. The model is a stochastic version of the Diamond model, but, following Farmer (2011, 2013) [4, 12], we assume that the labor market for new entrants (i.e. young agents) is incomplete. In the absence of an organized market for entry level inputs, firms in the model must engage in costly search to determine the suitability, numbers hired, and wages paid to young workers. We model this via the ISLM search process outlined in Farmer (2011, 2013) [4, 12]. As outlined in the introduction, young workers who pass the search screen face a take-it-or-leave it decision on whether to accept employment at the wage offered by the firm. This wage can be determined in a number of different ways. Farmer (2011, 2013) [4, 12] assumes that firms pay competitive wages once the screened pool of appicants has been determined. Kim et. al. (2022) use a Shapley-Shubik market game with oligopolistic firms to determine the post-screening wage offer. We will follow Farmer in this paper and assume that wages are determined competitively. To simplify the exposition, we also assume that workers supply their labor inelastically (with respect to wages). Other than this assumption on wage determination, the search process in the model is standard. The costs of search in the model take the form of diverting seasoned (i.e. middle-aged) workers from production to search.

The market for seasoned labor (i.e. middle aged agents) is complete and functions normally. The empirical justification for this is based on the observed differences between the volatility of hours worked by middle-aged agents relative to young agents, together with Miller (1984)'s [7] findings on the relationship between age and job stability. As noted above, firms must allocate seasoned workers to either production acitivites or search activities, which requires that firms forecast the demand they expect to face, and then allocate search resources to hire entry-level workers who engage in production with seasoned workers allocated to production.

Let N_y and N_m denote the number of young and middle-aged workers employed in production, with K the amount of capital employed. The representative firm's production function is the given by

$$Y = F\left(N_y, N_m, K\right) \tag{1}$$

where F is C^2 and concave. For specificity, we work with a Cobb-Douglas formulation of production for our examples and simulations, so that

$$Y = AN_y^a N_m^b K^c \tag{2}$$

Following standard practice, we will work in per capita terms, though it will simplify our exposition to be able to work with the two types of labor as being relative to the total amount of each type employed. So, we write

$$Y = A\left(\frac{L_y}{L_y}N_y\right)^a \left(\frac{L_m}{L_m}N_m\right)^b \left(\frac{L_y+L_m}{L_y+L_m}K\right)^c = AL_y^a L_m^b \left(L_y+L_m\right)^c n_y^a n_m^b k^c \tag{3}$$

where L_i is the population of type $= y, m$,and

$$n_y = \frac{N_y}{L_y} \tag{4}$$

$$n_m = \frac{N_m}{L_m} \tag{5}$$

$$k = \frac{K}{L_y + L_m} \tag{6}$$

Dividing through by $L_y + L_m$ and defining

$$y = \frac{Y}{L_y + L_m} \tag{7}$$

equation (4) yields

$$y = Bn_y^a n_m^b k^c \tag{8}$$

where $B = A\frac{L_y^a L_m^b}{(L_y+L_m)^{1-c}}$. For the general production function, we write $y = f(n_y, n_n, k)$ to denote output in per capita terms. As long as population doesn't change and the relative populations of the two types of labor remains the same, B will be constant. For our analysis of competitive markets, we will assume that the production technology exhibits constant returns to scale, so that $a + b + c = 1$. As noted above, we follow Farmer in assuming that firms are myopic and act to maximize profit in the current period. More complicated versions of the model would relax this assumption. For an interesting model of more general forward-looking behavior in a model of optimal contracting with moral hazard, see Li and Wang (2022) [13].

We assume that there are a continuum of households of each age (with the measure of each normalized to 1). Consumption is the numeraire good, and agents in the model can save by either purchasing capital assets issued by the firm, or by trading investment bonds with zero net supply. Firms can convert each unit of assets sold into a unit of capital. We denote bond quantities sold at time t by b_t and assume they sell at time t at a price $\phi_t < 1$ and are paid off the following period at par. Firms assets, k_t, are in terms of the final good, i.e. they are sold at a price of 1, the firms pay r_t and capital depreciates at a rate δ. As we noted above, we assume that agents' wages are determined competitively, and will show the calculation of this below. Agents are endowed with 1 unit of possible labor input, and don't value this directly. For young agents just entering the labor market, they must first spend time searching for work, and, if hired, they supply their single unit of labor time. To keep the model relatively tractable and not have to worry about a government sector paying unemployment insurance, we will simplify the model by assuming that every household ends up supplying some fraction $0 \le n_y \le 1$ of their available labor depending on the employment level determined by the search process. For middle-aged agents, their labor supply (hours) can be denoted similarly as n_m, though in equilibrium, because the market for middle-aged labor is complete, this will be equal to 1 (i.e. full employment).

A typical young agent then faces budget constraints (omitting time subscripts for simplicity) of the form

$$c_y = w_y n_y - \phi b_y - k_y \tag{9}$$

$$c_m = w_m n_m + b_y - \phi b_m + k_y(1 - \delta + r) - k_m \tag{10}$$

$$c_r = b_m + k_m(1 - \delta + r) \tag{11}$$

Old agents don't work and must finance their retirement consumption from the returns on their asset holding.

Asset and bond market clearing conditions in this economy are

$$b_y + b_m = 0 \tag{12}$$

$$k_y + k_m = k_t \tag{13}$$

2.2. Search

The timing of production and consumption activities in the model is as follows. First, firms hire seasoned labor in the middle-aged market at the competitive market-clearing wage. Given the market completeness assumption, this will generate full employment of seasoned labor. Firms then forecast how much they believe they will need to produce, and allocate some measuren_{m1} to direct production activites, with the remainder $n_{m2} = 1 - n_{m1}$ allocated to searching for entry level workers.

The Keynesian search mechanism is taken from Farmer's work. Formally, we assume that the labor input of the middle-aged agents is divided between management activities n_{m1} (which contributed directly to the production of output), and recruiting activities n_{m2}which don't contribute directly to production but are necessary for firms to recruit young workers. As noted above, we require $n_{m1} + n_{m2} = 1$.

The search technology for recruiting young workers combines the number of young workers searching for jobs – in this case since we assume no disutility incurred from either searching or working this number will be the full population of young agents. The assumption that young agents all search is consistent with Farmer's assumption that workers are all hired anew in every period. It is also consistent with our notion of entry-level labor in the three-period OLG setting, though this would need to be adjusted in a mutli-period setting to be more realistic. – with the number of management workers allocated to search, to produce job matches according to

$$n_y = M\left(\Gamma n_{m2}, 1\right) \tag{14}$$

The matching function M has the usual properties, and, following Farmer, Γ denotes the recruiting productivity of middle-aged agents assigned to recruiting.

The representative firm's profit static maximization problem (taking output as the numeraire) is

$$\max_{n_{m1t}, n_{m2t}, k_t} f\left(q_t n_{m2t}, n_{m1t}, k_t\right) - w_{mt}\left[n_{m1t} + n_{m2t}\right] - r_t k_t \tag{15}$$

where q_t is a measure of the the productivity of recruiting that the individual firm takes as given, but which will be determined in equilibrium from the search process. Specifically, given the matching function and the recruiting productivity parameter Γ, allocating n_{m2t} of middle-aged labor to search (the firm's search intensity) yields $q_t n_{m2t}$ in young labor input for the firm, and the firm takes q_t as given. As in Farmer, we can interpret this parameter as reflecting search costs due to congestion in the labor market for young workers. As previously noted, we assume that the labor market for middle-aged workers functions normally. Given the three-period time-frame of the model, we assume that all capital is used up in production Does this mean depreciation is 100%?. Under the assumption that markets are competitive, if the production function is Cobb-Douglas,

$$y = B\left(q n_{m2}\right)^a n_{m1}^b k^c \tag{16}$$

then maximizing with respect to the allocation to recruiting (under the assumption that middle-aged agents are paid the same wage), the firm's first-order conditions are (ignoring time subscripts)

$$Bqf_1\left(\bullet\right) - qw_y - w_m = 0 \tag{17}$$

$$Bf_2(\bullet) - w_m = 0 \tag{18}$$

$$Bf_3(\bullet) - r_t = 0 \tag{19}$$

or

$$\frac{a}{n_{m2}} y = q w_y + w_m \tag{20}$$

$$\frac{b}{n_{m1}} y = w_m \tag{21}$$

$$\frac{c}{k} y = r_t \tag{22}$$

wherea, b, c are the Cobb-Douglas exponents on the labor input of young and middle-aged agents respectively. In the Farmer framework, firms meet demand and labor markets are competitive, so y and the wage variables in these equations are taken as given. This gives us three equations to determine the allocation of middle-aged workers between recruiting and management, plus capital. Focusing on the labor markets, for the Cobb-Douglas production function, the first two FOCs give

$$\frac{ay}{n_{m2}} - \frac{by}{1 - n_{m2}} - q w_y = 0$$

If we let $x = \frac{1}{n_{m2}}$ this reduces to a quadratic of the form

$$ayx^2 - \left[(b + a)\, y + q w_y\right] x + q w_y = 0$$

or

$$x^2 - \left[1 + \frac{b}{a} + \frac{q w_y}{ay}\right] x + \frac{q w_y}{ay} = 0$$

Note that wages, total employment of the middle-aged, and total output are fixed. In the search equilibrium, we determine q via $qn_{2m} = M\left(\Gamma n_{2m}, 1\right)$or $q = M\left(\Gamma, \frac{1}{n_{2m}}\right)$ via the degree one homogeneity assumption that is standard for matching functions. Since the second-order conditions take the form

$$D(FOC_{labor}) = \begin{bmatrix} q^2 f_{11} & q f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

the SOC matrix is negative definite, since the restriction of f to the labor markets is strictly concave and q is non-negative. So the positive solutions to the quadratic above will be local maxima for the profit maximization problem. In a standard model with all market complete, we would have the firm take the wages and outputs as given to determine employment demand contingent on overall demand for goods. The general equilibrium in the model would then determine market-clearing wages and output. With the incomplete entry-level labor market assumption, however, employment of young agents will be determined once firms know the demand for output, with wages then determined once employment levels are known. The determination of overall demand in the model will then depend, as in Farmer's work, on household expectations of future income and wealth. Indeed, in Farmer's DSGE framework, we would choose an expectations function for asset values, which, together with the wages generated on the production side, would completely determine agents wealth, and hence via standard utility maximization, their demands for output and investment bonds. Given the expectations, firms' employment and production decisions would then be consistent with expectations, so that the resulting equilibrium was a rational expectations equilibrium.

As we noted in the introduction, this will not be the case in general in the stochastic OLG framework. Because the SOLG model requires including lagged endogenous variables in the state description – we will work with the recursive equilibrium concept in which the state variables are the lagged wealth distribution – expectations of future economic conditions require that agents forecast the wealth distribution along with asset values and incomes. The temporary equilibrium based on these forecasts will endogenously generate a distribution of wealth which may or may not coincide with the forecasted distribution. If it does, we are at a rational expectations equilibrium. If it does not, forecasts will need to adjust until the economy is at a fixed point for the operator mapping forecasts into temporary equilibria. So, unlike the conventional DSGE framework, the SOLG equlibrium disciplines firm activity through the REE consistency requirement, which is imposed through the equilibration process on the product and financial markets.

In general, working with Farmer's algorithm of first specifying expectations and then determining household consumption and firm production activities will be intractable in the SOLG setting. But, the model will still support Farmer's contention that expectations cause movements in real economic variables in the rational expectations equilibrium by reversing the causality and first determining firm activities, given the demand for output and capital. If we fix capital for the moment (and assume that households save by purchasing equity shares in the firms which entitle them to shares in firms' profits), then determining firm activities boils down specifying an allocation forn_{m2},which then determines wages and employment. This determines household incomes, so that solving the model completely now involves finding the competitive equilibrium for a simple pure exchange economy.

In the general stochastic setting, we can drive the model by assuming that $n_y = \tilde{n}_y$ is given and stochastic – the simplest way to do this is to assume that n_y fluctuates at business cycle frequencies. This then determines at each time t, $n_{2m} = \frac{1}{q}\tilde{n}_y$ determined as above, and $n_{1m} = 1 - n_{2m}$, and output determined from the production function. The wages are then given from the marginal productivity conditions, or, equivalently, we can then find the REE for this model by looking at the related pure exchange model in which young agent have endowments equal to $w_y n_y$ while middle-aged agents have endowments w_m. In our actual simulations, we assume that the productivity parameter q is random, with $n_{m1}$and n_{m2}fixed (at calibrated values), so that variations in recruiting productivity drive variations in income. Channeling this variation through production and wages generates the income process for the SOLG product market, which in turn determines the rational expectations equilibrium expectations that will lead firms to optimally allocate recruiting resources in the way we have posited. We will also show below that when capital is variable, the Keynesian search process leads to a reduced form model in which household saving is done via the purchase or sale of firms' investment bonds. This model looks like a standard SOLG model with capital.

2.3. Calibration in the Three-period Model

We do a rudimentary calibration for the three period model.

1. We assume that production exhibits constant returns to scale in the two types of labor and capital. Based on this, we use the standard factor shares approach to calibrate the exponents in the Cobb-Douglas production function. For low-skilled workers, estimates of their shares are around 22% (see the Economic Policy Institute [2024] online share calculator), so we take

$a = 0.22$. With overall labor share at 66%, this implies that $b = 0.44$ with the capital share at the standard 33% value.

2. Given the search procedure for determining employment of the young/low-skilled workers, we need to calculate how much firm recuriting intensity must fluctuate to generate the standard business cycle fluctuations of 3%. The simplest way to model this is to assume that firm recruiting productivity is subject to shocks, i.e. to fluctuations in the q parameter. We note, however, that in actual firms, HR hiring productivity is likely endogenous, depending on the firm's expectations of its labor needs and output demand. Absent a detailed theory of how these expectations are formed, we follow the widely-applied approach of simply assuming these are stochastic. For this calibration, then, we start with

$$\ln y = a \ln q + a \ln n_{m2} + b \ln n_{m2} + c \ln k \tag{23}$$

Then

$$\frac{dy}{y} = a\frac{dq}{q} + c\frac{dk}{k} \tag{24}$$

With $\frac{cy}{k} = r$ from the first-order condition for capital, we get

$$dk = \frac{c}{r}dy - \frac{cy}{r^2}dr$$

so that

$$\frac{dk}{k} = \frac{\frac{c}{r}dy}{\frac{cy}{r}} - \frac{\frac{cy}{r^2}dr}{\frac{cy}{r}} = \frac{dy}{y} - \frac{dr}{r}$$

Putting this together with Equation 31, we have

$$\frac{dq}{q} = \frac{1}{a}\left[(1-c)\frac{dy}{y} + c\frac{dr}{r}\right] \tag{25}$$

Since r is determined endogenously in the model, we can handle calibration by picking a "ballpark" value (based on variability of real long-run interest rates) and then use the simulation based on this calibration to determine a new number for the bond price variance and iterate on this.

3. To calibrate q,we use HR/employee ratios from https://www.sesamehr.com/blog/hr-to-employee-ratio/(figures are per 100 employees)

- Fewer than 100 employees: 2.70
- 100 to 249 employees: 1.26
- 250 to 499 employees: 1.07
- 500 to 999 employees: 0.82
- 1,000 to 2,499 employees: 0.79
- 2,500 to 7,499 employees: 0.53
- 7,500 or more employees: 0.42

Since we are treating the firm as representative of the full economy, we should use 0.042-0.07 as the HR/employee ratio, so we take average $n_{m2} = 0.05$. To parametrize Γ for the matching function $M = \sqrt{\Gamma n_{2m}}$, we take $n_y = 0.5$ from U.S. employment data for low-skill labor (see `https://www.oecd.org/employment/ministerial/employment-in-figures.htm`), this yields an estimate of $\Gamma = 5$, so that $q = 10$.

4. From this, for output fluctations totaling 3%, and taking the volatility of asset prices to be 1% (based on recent data from DataTrek, https://datatrekresearch.com/long-term-look-at-us-equity-volatility/?v=7516fd43adaa), using fluctuations in search intensity to calibrate, we need

$$\frac{dq}{q} = \frac{1}{0.22}(\pm 0.5)\left[(0.66)(0.03) + (0.33)(0.01)\right] = 0.0525$$

This calibration will then yield employment levels for unseasoned workers with average $n_y^+ = 0.52625$ and $n_y^- = 0.47375$. Using the Chauvet-Guo estimates as of conditional standard deviations between booms and recessions of 2% and 5% respectively, we then get conditional employment levels of

$$n_y^- \in \{0.44875, 0.49875\}$$

$$n_y^+ \in \{0.51125, 0.54125\}$$

5. If capital were fixed (as in Farmer's model) we could use these calibrations directly to determine the endowments of middle-aged and young agents, with profits distributed as dividends and simulate the resulting pure exchange economy. With capital variable, we need firms to determine an optimal capital stock given the shock to q. For this, we start with a reduced form production function

$$g(q,k) = (0.05q)^a (0.95)^b (\phi k)^c$$

From the first-order conditions for the labor markets

$$\pi = g - \left[a - b\frac{n_{m2}}{n_{m1}}\right] g - \frac{b}{n_{m1}} g - k$$

Factoring out g and simplifying yields$\pi = [1 - a - b]\, g - k$.With g as specified above, we can then write

$$\pi = \Omega q^a (\phi k)^c - k$$

where $\Omega = (0.05)^a (0.95)^b (1 - a - b)$. Taking a first-order condition with respect to capital here and simplifying yields

$$\hat{k} = \frac{(c\phi\Omega q^a)^{1-c}}{\phi}$$

Simulations for this version of the model then require simultaneous determination of wages (given the employment levels based on realizations of the q shock), asset prices and then capital and output. Simulations also indicate how to calibrate q in order to generate the correct variation in aggregate output, since

$$y = (0.05q)^a (0.95)^b \left((c\phi\Omega q^a)^{1-c}\right)^c$$

Taking logs with respect to y and q,we get

$$\ln y = a\left(1 + c - c^2\right)\ln q$$

so that

$$\frac{d\ln y}{d\ln q} = a\left(1 + c - c^2\right) = 0.1$$

Hence, it takes a 10% change in q to generate a 1% change in y, so we would calibrate $\Delta q = \pm 15\%$ to generate a total variation in y of 3% over the business cycle.

6. For the model with capital fixed at $k = 1$, using these parameterizations, we get wage bills for each type of labor of

$$w_y n_y = \left[a - \frac{b n_{m2}}{n_{m1}}\right] y$$

and

$$w_m\left(n_{m1} + n_{m2}\right) = w_m = \frac{by}{n_{m1}}$$

from the first-order conditions derived above. Hence, for our calibrations, we would assign endowments to young and old and profits as

$State$	n_y	y	w_y	w_m	π
1	0.456	0.823	0.162	0.381	0.280
2	0.504	0.841	0.165	0.390	0.286
3	0.510	0.843	0.166	0.391	0.287
4	0.530	0.850	0.167	0.394	0.290

where states 1and 2 are recession states, and the 3 and 4 are boom states and $e_i, i = y, m$ are the implied endowments generated by each agent's labor income.
The transition probabilities for states is given as

	$S'1$	$S'2$	$S'3$	$S'4$
S	0.1764705882	0.1764705882	0.3235294118	0.3235294118

so that the economy is in a state of recession around a third of the time.

2.4. Simulation Results in the Three-Period Model

Following Henriksen (2012) [14] and Mott's doctoral thesis, we utilize Chebyshev polynomials and Neural Networks as functional approximations for the equilibrium decisions of agents in the three-period model with fixed capital, K=1 (see the Appendix for computation details). We present preliminary numerical results of the equilibrium for this model. Although these results are initial, they provide a first characterization of the equilibrium behavior and offer insights into what might be expected in the equilibrium of the full model. We estimate the equilibria with a time-preference parameter $\beta = (0.97)^{20} = 0.5$. We assume that utility functions are CRRs, of the form

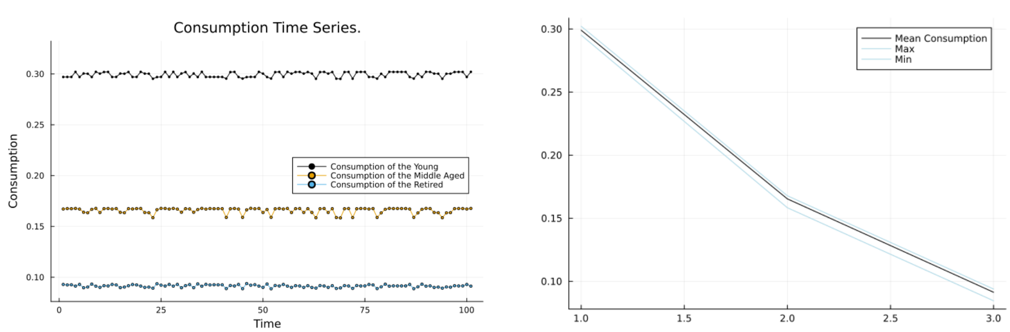
$$u(c_i) = \frac{c_i^{1-a}}{1-a} \quad \text{for} \quad i = y, m, r \tag{26}$$

and set the risk aversion coefficient $a = 2$.

The mean and standard deviation of the relevant variables in our model's simulated equilibrium, using fourth-degree Chebyshev Polynomials over 100 time periods, are reported in Table 1. The simulation achieves an error magnitude of 10^{-22} in the Euler equation for young agents (Euler 1), indicating optimal behavior, and 10^{-1} in the equilibrium equation for middle-aged agents. The average consumption of the young is approximately 0.12, supported by borrowing an average of 0.09 units of assets, with a consumption price of around 0.64 units. The middle-aged agents save for retirement by lending assets to the young generation and consume, on average, 0.14 units.

Table 1. Averages and Standard Deviations (in parenthesis) using Chebyshev Polynomials.

	Chev. Polynomials
Consumption of the young	0.2990
	(0.0026)
Consumption of the middle aged	0.1653
	(0.0032)
Consumption of the retired	0.0913
	(0.0014)
Asset prices	1.46298
	(0.0326)
Asset holdings of the young	-0.0912
	(0.0010)
Asset holdings of the middle aged	0.0912
	(0.0010)
Mean sq. error in Euler 1 (Chev. Nodes)	(0.0004)
	(0.0001)
Mean sq. error in Euler 2 (Chev. Nodes)	0.0003
	(0.0002)
Mean sq. error in Euler 1 (Simulations)	0.0009
	(0.0027)
Mean sq. error in Euler 2 (Simulations)	0.0089
	(0.0022)
Time Periods	1000



(a) Consumption profiles for the three generation equilibrium using fourth degree Chevyshev polynomials.

(b) Average, minimum, and maximum consumption by age group.

Fig. 1. Simulations of the Equilibirum profile using Chevyshev polynomials.

Figure 1a shows the consumption path of the three generations of agents over the 100 periods of the simulated economy using Chebyshev Polynomials. The Figure, together with Table 2, shows that the agents consume the most when they are middle-aged: in boom states of the economy, they consume around 0.155, while during recessions, they consume, on average, 0.12, a decrease of 13% from their consumption during booms. The young agents consume, on average, 0.13 during booms, but their consumption drops to 0.12 during recessions, a decrease of 8% compared to the periods when the economy performs well. Finally, the old agents consume the least in this economy but also experience the least volatility of consumption. They go from 0.09 during booms to 0.08 on average during recessions. Overall, the numerical results capture the observed volatilities of consumption, but they also show old agents consuming the least, a result that contrasts with former models in the literature. This result could be driven by the fact that the young agents do not face any borrowing constraints, so they can increase their consumption, presenting a hump-shaped consumption profile over the generations. Overall, the results from the Chevyshev polynomials in Table 2 show that the economy is more volatile during recessions and more stable during the boom periods. During recessions, the young agents borrow around 4% more asset than during the better states of the economy, and the asset prices rise from 0.58 to 0.7, an increase of 20%.

Table 2. Consumption, asset holdings and prices across all states. Based on 100 simulated equilibrium periods using Chevyshev Polynomials.

State	C_y	C_m	C_r	A_y	A_m	Asset Price
1	0.2953	0.1586	0.0890	-0.0894	0.0894	1.4901
	(9.37e-5)	(0.0002)	(0.0002)	(5.29e-5)	(5.29e-5)	(0.0001)
2	0.3003	0.1636	0.0900	-0.0902	0.0902	1.4991
	(0.0001)	(0.0003)	(0.0005)	(0.0001)	(0.0001)	(2.01e-5)
3	0.2970	0.1670	0.0929	-0.0924	0.0924	1.4177
	(5.15e-5)	(0.0004)	(0.0005)	(0.0001)	(0.0002)	(0.0019)
4	0.3019	0.1675	0.0914	-0.0914	0.0914	1.4760
	(8.74e-5)	(0.0001)	(0.0002)	(3.23e-5)	(3.23e-5)	(0.0004)

We believe that employing Neural Networks to simulate data is essential for accurately obtaining the equilibrium of the full model. Our primary objective was to compute the equilibrium recursively using a deep neural network. This approach involves approximating several critical functions, including a price function and two policies governing savings for young and middle-aged agents.

In our model, the state of the economy is represented by (Z_t) and the asset holdings of young and middle-aged from the previous period, denoted as $b_{t-1} = \{b^1_{t-1}, b^2_{t-1}\}$. The input to the neural network is thus $x_t = [Z_t, b^1_{t-1}, b^2_{t-1}]$. The output of the neural network includes the price function ϕ_t and the policy functions for the bond holdings in the current period, b^1_t and b^2_t.

To ensure the accuracy of the approximation, the policy functions produced by the network must be consistent with the Euler equations for young and middle-aged agents and satisfy the bond market clearing condition (See Appendix). Our neural network loss function is designed to minimize the sum of squared errors of the Euler equations, coupled with a penalty function to handle cases of negative consumption. Following the methodology outlined by Azinovic (2023) [15], we have integrated the market clearing conditions directly into the architecture of the neural network. This design choice ensures that these conditions are always met, up to numerical precision, thereby enhancing the reliability of our equilibrium computations.

Our preliminary results using Neural Networks closely align with those obtained using Chebyshev Polynomials for the three-generation model, as illustrated in Figure 2. This consistency reinforces the reliability of our findings and suggests that Neural Networks are capable of effectively handling the complexities of the full model.

Figure 2a shows the loss, represented by the sum of squared errors of the young and middle-aged Euler Equations during training epochs, which has reached 10^{-5}. The consumption profiles for the young (age group 0), middle-aged (age group 1), and retired (age group 2) are depicted in Figure 2b. As shown, the young consume the most, followed by the middle-aged, and then the retired. Figure 2c presents the bond holdings of the young and middle-aged. In this figure, the x-axis of the blue points represents b^0_{t-1}, which is zero, while the y-axis represents b^1_t. The orange and green points correspond to the bond holdings of the middle-aged and retired, respectively. Finally, Figure 2d depicts the price counts in the simulations. As our next steps, we will incorporate capital as a state variable into the Neural Network model and introduce a borrowing constraint to enhance the model's accuracy.

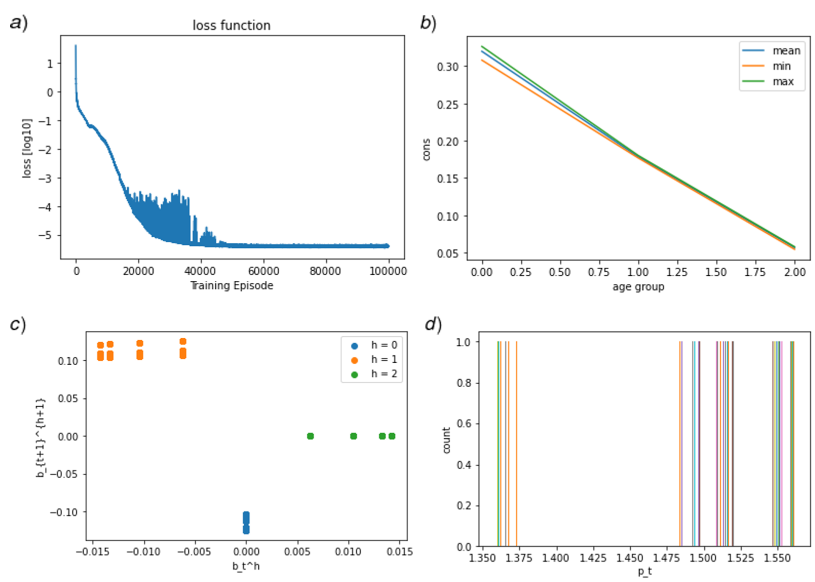


Fig. 2. Consumption Profiles for the three generation equilibrium using Neural Networtk.

3. The General Model

In this section, we extend the results to more realistic models by allowing an arbitrary lifetime, heterogeneity within cohorts, and a general shock process. In this general model, time is still discrete. Agents live L-period lives from the youngest age 1 to the oldest age L, with $M \geq 1$ different types of agents born each period who differ in terms of preferences. We maintain the basic structure of the production side of the economy as in the three-period model, but allow for a more general shock structure for the representative firm's search process. So, we assume there is a continuum of each type of agent, and thus they take prices as given. There are $S > 1$ states of exogenous shocks, $s \in \{z_1, \ldots, z_S\}$. The shock is generated by an IID process with the probability of state s occurring equal to π^s where $0 < \pi^s < 1$ for $\forall s$ and $\sum_{s \in \{z_1,\ldots,z_S\}} \pi^s = 1$. We denote the history of shocks affecting the economy at time t by $S^t = \{s_1, s_2, ..., s_t\}$.

Type-j agents born in time t and history S^t are considered entry-level workers for the first T_E periods of life, and are seasoned workers for the remaining $L - T_E$ periods. All entry level workers earn the (stochastic) entry level wage and, in keeping with the assumption that the market for these workers is incomplete, face the possibility of unemployment. All seasoned workers are fully employed and earn the same seasoned level wage. For this competitive setting, we maintain the assumption that the wage is determined by the marginal product in each period of the two types of labor. The employment levels and wages in each period thus determine the income available to each agent.

The consumption stream of a type-j agent born in time t and history S^t is denoted by $c_j(S^t) = \left\{\left(c_{i,j}\left(S^t, S_{t+1}^{t+i-1}\right)\right)_{S_{t+1}^{t+i-1}}\right\}_{i=1}^{L}$ where $c_{i,j}\left(S^t, S_{t+1}^{t+i-1}\right)$ is the consumption of a type-j agent in age i given a path of shocks for $(i-1)$ periods after the history S^t, $S_{t+1}^{t+i-1} = (s_{t+1}, \ldots, s_{t+i-1})$ for $i > 1$. When $i = 1$, we define $c_{1,j}\left(S^t, S_{t+1}^t\right) = c_{1,j}(S^t)$. Similarly, the lifetime portfolio of a type-j household is denoted by $a_j(S^t) = \left\{\left(a_j\left(S^t, S_{t+1}^{t+i-1}\right)\right)_{S_{t+1}^{t+i-1}}\right\}_{i=1}^{L-1}$ where $a_{i,j}\left(S^t, S_{t+1}^{t+i-1}\right)$ is the asset holding of a type-j agent in age i in node S^{t+i-1}. As above, we define $a_{1,j}\left(S^t, S_{t+1}^t\right) = a_{1,j}(S^t)$. Note that households do not save in period L, their last period of life. Given our assumptions on the production side of the economy, asset holdings will consist of either zero net supply bonds or capital.

The lifetime expected utility for a type-j individual is given by an additively time-separable von Neumann-Morgenstern utility function $V_j : \mathbb{R}_+^{(S^L-1)/(S-1)} \to \mathbb{R}$:

$$V_j\left(c_j\left(S^t\right)\right) = \mathbb{E}_t\left[\sum_{i=1}^{L} \beta_j^{i-1} u_j\left(c_{i,j}\left(S^t, S_{t+1}^{t+i-1}\right)\right)\right]$$

where $\beta_j \in (0, 1]$ is the discount factor of type j and the felicity functions of type j, u_j, satisfy the regularity conditions $u_j'(c) > 0$, $u_j''(c) < 0$, and $u_j'(0) = +\infty$.

Type-j agents maximize their lifetime expected utility subject to the following sequence of budget constraints in current-value prices:

$$\begin{aligned}
c_{1,j}\left(S^t\right) &= w_j\left(S^t\right) n_j\left(S^t\right) - \phi\left(S^t\right) b_{1j}\left(S^t\right) - k_{1j}\left(S^t\right) \\
c_{i,j}\left(S^t, S_{t+1}^{t+i-1}\right) &= w_j\left(S^t, S_{t+1}^{t+i-1}\right) n_j\left(S^t, S_{t+1}^{t+i-1}\right) + b_{(i-1)j}\left(S^t, S_{t+1}^{t+i-1}\right) + R\left(S^t, S_{t+1}^{t+i-1}\right) k_{(i-1)j}\left(S^t, S_{t+1}^{t+i-2}\right) \\
&\quad -\phi\left(S^t, S_{t+1}^{t+i-1}\right) b_{ij}\left(S^t, S_{t+1}^{t+i-1}\right) - k_{ij}\left(S^t, S_{t+1}^{t+i-1}\right) \text{ for } \forall S_{t+1}^{t+i-1} \text{ and } i \in \{2, \ldots, L\}
\end{aligned}$$

where $R\left(S^t, S_{t+1}^{t+i-1}\right) = R\left(S^{t+i-1}\right)$ is the gross return to capital (including depreciation, if any) in terms of the single good in node S^{t+i-1},and $\phi\left(S^t, S_{t+1}^{t+i-1}\right) = \phi\left(S^{t+i-1}\right)$ is the bond price. Note that we all for agents to have retirement income sources beyond the returns on their assets. This income will be a calibration input to the simulations.

Solving the optimization problem for type-j agents yields asset demand functions:

$$a_{i,j}\left(S^t, S_{t+1}^{t+i-1}\right) = a_{i,j}\left(\boldsymbol{P}_t^{t+L-1}\left(S^t\right); S_{t+1}^{t+i-1}\right) \tag{27}$$

for $\forall S_{t+1}^{t+i-1}$ and $i \in \{1, \ldots, L-1\}$ where $\boldsymbol{P}_t^{t+L-1}(S^t) = \left\{\phi(S^t), R(S^t) \ldots, \left(\phi\left(S^t, S_{t+1}^{t+L-1}\right), R\left(S^t, S_{t+1}^{t+L-1}\right)\right)_{S_{t+1}^{t+L-1}}\right\}$.
The equity demand functions are indexed by the history of shocks realized after the first period of life, S_{t+1}^{t+i-1} for $i > 1$. When $i = 1$, we define $a_{1,j}(S^t) = a_{1,j}\left(\boldsymbol{P}_t^{t+L-1}(S^t)\right)$.

For the extended model, the asset market-clearing condition in time t and node S^t requires that:

$$\sum_{i=1}^{L-1}\sum_{j=1}^{M} a_{ij}\left(\boldsymbol{P}_{t+1-i}^{t+L-i}\left(S^{t+1-i}\right); S_{t+2-i}^t\right) = \boldsymbol{\kappa} = \begin{bmatrix} 0 \\ k_t \end{bmatrix}$$

for $s_t \in \{z_1, \ldots, z_S\}$. Here, the zero element of $\boldsymbol{\kappa}$is the clearing condition for the bond market, while k_t is the demand for capital from the representative firm determined from its optimization.[3] Given out maintained assumptions about the production side of the economy, the representative firm's optimization problem remains the same as it was in the three-period setting.

With these notations, we define two equilibrium concepts as in the previous models. The competitive equilibrium in the generalized model is a sequence of firm outputs (and the wages and employment levels this determines), asset holdings and consumption for all types and asset prices in all nodes starting in time 0, $\left\{\left\{a_j(S^t), c_j(S^t)\right\}_j, \boldsymbol{p}(S^t) = [\phi(S^t), R(S^t)]\right\}$ for $\forall S^t$ and $t \geq 0$, satisfying:

- Individuals maximize their expected utility under budget constraints given the sequence of asset prices;
- Firms maximize profits given production outputs which meet demand; and
- The asset markets clear and the aggregate resource constraint holds.

We can also show the existence of a competitive equilibrium for this generalized model using a standard truncation method.

For the Markov equilibrium that we will define below, we take all but one of the asset holding quantities as the lagged endogenous state variables (via the asset market clearing condition). Specifically, we exclude the asset holding of the type-M agent in the second oldest cohort with age $L-1$ and let the set of the lagged endogenous state variables in time t be denoted as:

$$\xi_{t-1} = \left\{\left\{a_{i,j,t-1}\right\}_{j=1,\ldots,M}\right\}_{i=1,\ldots,L-1} \setminus \left\{a_{(L-1),M,t-1}\right\}. \tag{28}$$

With abuse of notation, we also use ξ_{t-1} to denote a $((L-1)M-1)$ vector for the lagged endogenous state variables in time t. The Markov equilibrium is defined by time-homogeneous policy functions for firm outputs, for asset holdings and consumptions for all types and asset prices, $\left\{\left\{a_{i,j}(\chi), c_{i,j}(\chi)\right\}_{i,j}, \boldsymbol{p}(\chi)\right\}$ which solve the household problem and clear both the asset and consumption markets, assuming as before that firms maximize profits given their desired production outputs. Here, $\chi = [\xi_{-1}, s] \in \hat{\Sigma} \subset \mathbb{R}^{(L-1)M}$ represents the state variables – the lagged asset holdings distribution and the realization of the current search shock, and $\hat{\Sigma}$ is the state space of both endogenous and exogenous state variables. We can write the equilibrium conditions for the Markov equilibrium as follows:

$$\boldsymbol{E}\left(S^t\right) = \xi\left(\xi_{t-1}, s_t\right) \tag{29}$$

$$\boldsymbol{\kappa} = \iota^T \xi\left(\xi_{t-1}, s_t\right) + a_{(L-1),M}\left(\boldsymbol{P}_{t-L+2}^{t+1}\left(S^{t-L+2}\right); S_{t-L+3}^t\right)$$

[3] By Walras' law, the market clearing condition for the consumption good will also hold.

where the first equation represents the optimality condition of the household problem and the second one is the asset market clearing condition. Here,

$$A\left(S^t\right) = \begin{bmatrix} \left\{a\left(\boldsymbol{P}_{t-L+2}^{t+1}\left(S^{t-L+2}\right); S_{t-L+3}^{t}\right)\right\}_{j=1,\ldots,M-1} \\ \left\{a_{(L-2),j}\left(\boldsymbol{P}_{t-L+3}^{t+2}\left(S^{t-L+3}\right); S_{t-L+4}^{t}\right)\right\}_{j=1,\ldots,M} \\ \vdots \\ \left\{a\left(\boldsymbol{P}_{t}^{t+L-1}\left(S^t\right)\right)\right\}_{j=1,\ldots,M} \end{bmatrix} \tag{30}$$

is the $((L-1)\,M-1)$ vector of asset demand functions for all but the type-M and age-$(L-1)$ agents in node S^t,

$$\xi\left(\xi_{t-1}, s_t\right) = \begin{bmatrix} \left\{a\left(\xi_{t-1}, s_t\right)\right\}_{j=1,\ldots,M-1} \\ \left\{a_{(L-2),j}\left(\xi_{t-1}, s_t\right)\right\}_{j=1,\ldots,M} \\ \vdots \\ \left\{a_{1,j}\left(\xi_{t-1}, s_t\right)\right\}_{j=1,\ldots,M} \end{bmatrix} \tag{31}$$

is the asset demand vector replaced with the policy functions in a ME and ι is an $((L-1)\,M-1)$ sum vector adding up the asset holdings distribution except for the demand by the type-M and age-$(L-1)$ agent.

For this generalized model, we can directly invoke the results in citanna2010recursive and citanna2012recursive to show the existence of a ME since this model exhibits sufficient heterogeneity within each cohort to satisfy their condition for the generic existence of a Markov equilibrium.

4. Conclusion

Our analysis demonstrates that in a stochastic overlapping generations model with Keynesian search, agents' beliefs must align with the endogenous wealth distribution, constrained by the rational expectations equilibrium. Preliminary results show the significance of beliefs in macroeconomic equilibrium and the role of lifecycle considerations in labor market dynamics. The inclusion of matching efficiency shocks offers a plausible alternative to traditional TFP shocks, further highlighting the complexities of labor markets and the need for realistic modeling approaches.

Additionally, we employed Neural Networks to approximate the equilibrium decisions of agents in the model. This approach provided results consistent with those obtained using Chebyshev polynomials, reinforcing the reliability of our findings. The use of Neural Networks demonstrates their potential to handle the complexities of the full-fledged model, suggesting they are a viable tool for future research in capturing the dynamic behavior of economic agents.

References

1. B. Jovanovic, "Job matching and the theory of turnover," J. Political Econ. **87**, 972–990 (1979).
2. B. Jovanovic and J. Mincer, "Labor mobility and wages," Stud. labor markets pp. 21–63 (1981).
3. L. J. Kotlikoff, "The structure of slave prices in new orleans, 1804 to 1862," Econ. Inq. **17**, 496–518 (1979).
4. R. E. A. Farmer, *Confidence, Crashes and Animal Spirits* (SSRN, 2008).
5. J.-V. Ríos-Rull, "Life-cycle economies and aggregate fluctuations," The Rev. Econ. Stud. **63**, 465–489 (1996).
6. P. Howitt and R. P. McAfee, "Costly search and recruiting," Int. Econ. Rev. pp. 89–107 (1987).
7. R. A. Miller, "Job matching and occupational choice," J. Political economy **92**, 1086–1120 (1984).
8. P. Sullivan, "Empirical evidence on occupation and industry specific human capital," Labour Econ. **17**, 567–580 (2010).

9. M. O. Ravn and V. Sterk, “Job uncertainty and deep recessions,” J. Monet. Econ. **90**, 125–141 (2017).
10. R. Barnichon and A. Figura, “What drives movements in the unemployment rate? a decomposition of the beveridge curve,” SSRN (2010).
11. K. Kroft, F. Lange, M. J. Notowidigdo, and L. F. Katz, “Long-term unemployment and the great recession: the role of composition, duration dependence, and nonparticipation,” J. Labor Econ. **34**, S7–S54 (2016).
12. R. E. Farmer, “Animal Spirits, Financial Crises and Persistent Unemployment,” The Econ. J. **123**, 317–340 (2013).
13. Y. Li and C. Wang, “Endogenous labor market cycles,” Int. Econ. Rev. **63**, 849–881 (2022).
14. E. Henriksen and S. Spear, “Endogenous market incompleteness without market frictions: Dynamic suboptimality of competitive equilibrium in multiperiod overlapping generations economies,” J. Econ. Theory **147**, 426–449 (2012).
15. A. Marlon and Z. Jan, “Economics-inspired neural networks with stabilizing homotopies,” SSRN (2023).

5. Appendix

5.1. Euler Equations

For the numerical work, we assume that utility funcitons are CRR, of the form

$$u(c_i) = \frac{c_i^{1-a}}{1-a} \quad \text{for} \quad i = y, m, r \tag{32}$$

The time-preference parameter $\beta = (0.97)^{20} = 0.5$, and the risk aversion coefficient $a = 2$.

$$\begin{aligned}
&\max_{c_y^s, c_m^{s,s'}, c_r^{s',s"}, a_y^s, a_m^{s'}} u(c_y^s) + \beta E[u(c_m^{s,s'})] + \beta^2 E[u(c_r^{s',s"})] \\
&\text{s.t.} \\
&c_y^s = w_y^s n_y^s - \phi(z_{t-1}, s) a_y^s \\
&c_m^{s,s'} = w_m^{s'} n_m^{s'} + a_y^s - \phi'(z_t, s') a_m^{s'} \\
&c_r^{s',s"} = a_m^{s'}
\end{aligned} \tag{33}$$

$$c_y^s + c_m^{s',s} + c_r^{s",s} = w_y + w_m$$
$$w_y^s n_y^s - \phi(z_{t-1}, s) a_y^s + w_m^{s'} n_m^{s'} + a_y^s - \phi(z_t, s') a_m^{s'} + a_m^{s'} = w_y^s n_y^s + w_y^s n_y^s$$

Using the budget constraint to turn this into an unconstrained optimization:

$$\max_{a_y^s, a_m^{s'}} u(w_y^s n_y^s - \phi(z_{t-1}, s) a_y^s) + \sum_{s'} \beta \pi^{s'} u(w_m^{s'} n_m^{s'} + a_y^s - \phi'(z_t, s') a_m^{s'}) + \sum_{s'} \sum_{s"} \beta^2 \pi^{s'} \pi^{s"} u(a_m^{s'}) \tag{34}$$

Taking FOC with respect to young bond holding we get:

$$\begin{aligned}
&[a_y^s] : -\phi(z_{t-1}, s)(c_y^s)^{-a} + \sum_{s'} \beta \pi^{s'} (c_m^{s,s'})^{-a} = 0 \\
&[a_y^s] : -\phi(z_{t-1}, s)(w_y^s n_y^s - \phi(z_{t-1}, s) a_y^s)^{-a} + \sum_{s'} \beta \pi^{s'} (w_m^{s'} n_m^{s'} + a_y^s - \phi'(z_t, s') a_m^{s'})^{-a} = 0
\end{aligned} \tag{35}$$

Taking FOC with respect to middle-aged bond holdings we get:

$$\begin{aligned}
&[a_m^{s'}] : -\phi'(z_t, s')(c_m^{s,s'})^{-a} + \sum_{s"} \beta \pi^{s"} (c_r^{s',s"})^{-a} = 0 \\
&[a_m^{s'}] : -\phi'(z_t, s')(w_m^{s'} n_m^{s'} + a_y^s - \phi'(z_t, s') a_m^{s'})^{-a} + \sum_{s"} \beta \pi^{s"} (a_m^{s'})^{-a} = 0
\end{aligned} \tag{36}$$

Market clearing requires that:

$$a_y^s + a_m^s = 0 \tag{37}$$

6. Notes of Model with Capital.

Simulations for this version of the model would then require simultaneous determination of wages (given the employment levels based on realizations of the q shock), asset prices and then capital and output.

The equation for capital is:

$$\hat{k} = \frac{(c\phi\Omega q^a)^{1-c}}{\phi}$$

Where $\Omega = (0.05)^a(0.95)^b(1-a-b)$, (Note: a, b are known parameters).

What we don't know is q (probably state variable? $q = 10$ and $\Delta q = +-15\%$). Also ϕ is the asset price (output of the NN). Once we have this we get k and output as:

$$y = (0.05q)^a\,(0.95)^b\left((c\phi\Omega q^a)^{1-c}\right)^c$$

$qn_{2m} = M(\Gamma n_{2m}, 1)$or $q = M\left(\Gamma, \frac{1}{n_{2m}}\right), n_{2m} = (1/q)n_y, n_{2m} = 1-n_{1m}$. $n_y = 0.5, n_{m2} = 0.05$, $\Gamma = 5$.

Pseudo-algorithm to solve the model with capital: States are q_t, a_m^t outputs are price $(\phi(q_t, a_m^t))$ and asset holdings $(a_m^{t+1}(q_t, a_m^t))$.

1. Fix $n_{m2} = 0.05$ and $\Gamma = 5$. Then $n_y = q * n_{2m}$ with q given as state.

2. Take NN output $\phi(q, a_m)$ and q and calculate

$$\hat{k} = \frac{(c\phi\Omega q^a)^{1-c}}{\phi}$$

   Then

$$y = (0.05q)^a\,(0.95)^b\left((c\phi\Omega q^a)^{1-c}\right)^c$$

3. Then wages solve

$$\frac{a}{n_{m2}}y = qw_y + w_m \tag{38}$$

$$\frac{b}{n_{m1}}y = w_m \tag{39}$$

4. Solve the consumption part of the model using the wages and fraction of employment given in states 1-4 to calculate the endowments of the agents.

5. Here, the model seems to indicate that the budget constraints and the market clearing constraints should be

$$\begin{aligned} c_y &= w_y n_y - \phi a_y \\ c_m &= w_m n_m + a_y - \phi' a_m \\ c_r &= a_m \\ k &= a_y + a_m \end{aligned}$$

Idea for the distribution of q is to make it so that we match the old values of ny, i.e $q = n_y/n_{2m} = n_y/0.05$. Eg. If $n_y = 0.51 \Rightarrow q = 10.2$.

$$q^- = \{9.12, 10.08\} \tag{40}$$

$$q^+ = \{10.2, 10.6\} \tag{41}$$

Same distribution as before.

6.1. Model with Capital

Household solves the following problem:

$$\max_{c_y, c_m, c_r, b_y^s, b_m^{s'}} u(c_y^s) + \beta E[u(c_m^{s,s'})] + \beta^2 E[u(c_r^{s',s"})]$$

$$\begin{aligned} &\text{s.t.} \\ &c_y^s = w_y^s n_y^s - \phi^s b_y^s - k_y^s \\ &c_m^{s,s'} = w_m^{s'} n_m^{s'} + b_y^s - \phi^{s'} b_m^{s'} + k_y^s(1 - \delta + r^{s'}) - k_m^{s'} \\ &c_r^{s',s"} = b_m^{s'} + k_m^{s'}(1 - \delta + r^{s"}) \end{aligned} \tag{42}$$

And firms solve the following problem:

$$\underset{n_{m1t}, n_{m2t}, k_t}{max} \; B(qn_{m2})^a n_{m1}^b k_t^c - w_{yt} q_t n_{m2t} - w_{mt} \left[n_{m1t} + n_{m2t}\right] - r_t k_t \tag{43}$$

HH Euler equations:

$$\begin{aligned} &[b_y^s] : -\phi^s (c_y^s)^{-a} + \beta \sum_{s'} \pi^{s'} (c_m^{s,s'})^{-a} = 0 \\ &[b_m^{s'}] : -\phi^{s'} (c_m^{s,s'})^{-a} + \beta \sum_{s"} \pi^{s"} (c_r^{s',s"})^{-a} = 0 \\ &[k_y^s] : -(c_y^s)^{-a} + \beta \sum_{s'} \pi^{s'} (1 - \delta + r^{s'}) (c_m^{s,s'})^{-a} = 0 \\ &[k_m^{s'}] : -(c_m^{s,s'})^{-a} + \beta \sum_{s"} \pi^{s"} (1 - \delta + r^{s"}) (c_r^{s',s"})^{-a} = 0 \end{aligned} \tag{44}$$

Firm problem FOCs:

$$\begin{aligned} &[n_{m2}] : \frac{aF}{n_{m2}} - w_y q_t - w_m = 0 \rightarrow aBq^a (n_{m2})^{a-1} n_{m1}^b k^c - w_{yt} q_t - w_{mt} = 0 \\ &[n_{m1}] : \frac{bF}{n_{m1}} - w_{mt} = 0 \rightarrow bB(qn_{m2})^a n_{m1}^{b-1} k^c - w_{mt} = 0 \\ &[k] : \frac{cF}{k} = r_t \rightarrow cB(qn_{m2})^a n_{m1}^b k^{c-1} = r_t \end{aligned} \tag{45}$$

So, for each t we have:

$$\begin{aligned} c_t^y &= n_t^y w_t^y + b_{t-1}^0 - \phi_t b_t^y + k_{t-1}^0 (1 - \delta + r_t) - k_t^y \\ c_t^m &= n_t^m w_t^m + b_{t-1}^y - \phi_t b_t^m + k_{t-1}^y (1 - \delta + r_t) - k_t^m \\ c_t^r &= n_t^r w_t^r + b_{t-1}^m - \phi_t b_t^r + k_{t-1}^m (1 - \delta + r_t) - k_t^r \end{aligned} \tag{46}$$

where b_{t-1}^0, k_{t-1}^0, b_t^r, k_t^r, n_t^r are zero but have been written for consistency.

Market clearing conditions:

$$\begin{aligned} b_t^y + b_t^m &= 0 \\ K_t &= k_t^y + k_t^m \end{aligned} \tag{47}$$

If we rewrite the budget constraint in a more general form as below:

$$\begin{aligned} c_t^h &= n_t^h w_t^h + b_{t-1}^{h-1} - \phi_t b_t^h + k_{t-1}^{h-1} (1 - \delta + r_t) - k_t^h \\ c_t^1 &= n_t^1 w_t^1 + b_{t-1}^0 - \phi_t b_t^1 + k_{t-1}^0 (1 - \delta + r_t) - k_t^1 \\ c_t^2 &= n_t^2 w_t^2 + b_{t-1}^1 - \phi_t b_t^2 + k_{t-1}^1 (1 - \delta + r_t) - k_t^2 \\ c_t^3 &= n_t^3 w_t^3 + b_{t-1}^2 - \phi_t b_t^3 + k_{t-1}^2 (1 - \delta + r_t) - k_t^3 \end{aligned} \tag{48}$$

Where $h = 1$ refers to the young generation, $h = 2$ refers to the middle-aged generation, and $h = 3$ refers to the old generation. The inputs to the neural network are as follows:

$$[z_t, b^0_{t-1}, b^1_{t-1}, b^2_{t-1}, k^0_{t-1}, k^1_{t-1}, k^2_{t-1}]$$

and Output would be:

$$[\phi^b_t, b^1_t, b^2_t, b^3_t, k^1_t, k^2_t, k^3_t]$$

Therefore, for each period t, using the aggregated capital values and the calibrated parameters for q and labor supply for each state, we can calculate the output y as shown in Eq. 16 , along with the young and middle-aged wages and the interest rate as specified in Eq. 45. Subsequently, we will be able to determine the consumption for each generation.