Efficient Investment, Search, and Sorting in Matching Markets

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September, 2024

Abstract

We study markets where heterogeneous agents first make investment decisions and then engage in costly search to form productive matches. The trading process is random search and bargaining with explicit search costs. Despite potential hold-up and matching problems, we prove a second welfare theorem: the constrained efficient allocation is an equilibrium. The agents' private incentives to invest and to accept/reject potential partners as they search are perfectly aligned with the social benefit. Furthermore, we establish a new sorting result for two-sided markets, equilibrium existence, conditions for uniqueness, and novel economic implications.

1 Introduction

This paper studies markets whose participants first make investment decisions and then engage in costly search to form productive matches. These two features are central in various settings. For example, in the marriage market, individuals make premarital investments in their education and career before looking for a partner. In the labor market, workers acquire human capital before searching for jobs, while firms adopt technologies before hiring workers. Likewise, in the real estate market, developers often build before finding prospective buyers; in venture capital markets, entrepreneurs invest time and money developing start-ups prior to seeking funding; and in product markets, buyers and sellers make ex-ante investments before meeting.

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In these examples, the productivity of a match depends on the prior investments, and the outcome each agent receives in the market depends upon who matches with whom and how long it take to find them. We aim to address fundamental questions for such settings: When is the market efficient? In an equilibrium, how do agents invest, who matches with whom and what is the search duration?

The main challenge to studying these markets is that efficiency depends upon the alignment of agents' private incentives with the social benefit regarding both the investment and search decisions. Since agents invest before meeting their partners, a potential hold-up problem may reduce the incentive to invest. In addition, when the prior investments vary within the population, finding a suitable partner might take some time, and there is a potential matching problem: an agent may search too much and reject efficient matches or search too little and accept inefficient matches or both.

Furthermore, the investment and search decisions should be studied together in equilibrium as they are mutually dependent. For example, the incentive to acquire a certain college degree depends on which types of firms may potentially hire the worker and how long it will take to find a job. Likewise, a firm's incentive to adopt a new technology depends on the skills of workers it may potentially hire and how long it will take to fill vacancies.

Previous work has extensively studied markets with either ex-ante investment or search-and-matching separately. Building on the foundational Diamond-Mortensen-Pissarides model, the prevailing view in the literature is that efficiency fails in markets with search frictions. First, there is under-investment: the private return on investment is less than its social value (see, e.g., Acemoglu 1996). Second, models with heterogeneous agents typically have some mismatches: the agents don't internalize the externalities they impose when they accept and reject partners (see, e.g., Shimer and Smith 2001).

In order to address the questions above, in this paper we develop a new and tractable model of investment, search, and matching. Contrary to the common view in the literature, we prove a new second welfare theorem: the constrained efficient allocation is obtainable in the market. In addition, we prove a new sorting result for two-sided markets (such as the labor market), establish new existence and uniqueness results, and demonstrate important economic applications. The model and results show that our framework can serve as a workhorse for studying these markets.

Description of the model

We develop a search-and-matching model with transfers between two populations of agents, called buyers and sellers, but one can equally consider workers and firms, men and women, or any two groups who invest and then match. What is important is that output is produced by pairs of agents, one from each side of the market. Agents invest in skills before entering the market and they are heterogeneous in their investment costs. Their match output depends upon the skills that they have acquired, but there is some sand in the wheels of the market: to form productive matches, the agents must engage in costly search.

We consider the standard random search and bargaining process with explicit search costs and without discounting, as in Atakan (2006). In every period, each agent in the market incurs the same search cost and randomly meets an agent from the other side. When two agents meet, they can either agree to match or continue searching. If both agree, then they exit the market and divide their output according to Nash bargaining. If at least one rejects, then they remain in the market and draw new partners in the next period. A new cohort of agents is born in every period, acquires skills, and then enters the market. We analyze a *steady-state equilibrium* where, for every skill, the inflow of agents to the market equals the outflow.

The term 'skill' refers to investments that enhance productivity. For instance, in the labor market, a worker's skill is their education level, while a firm's skill is their technology. In a product market, a seller's investment reduces their production cost, a buyer's investment increases their value, and the match output is the buyer's value minus the seller's cost. In the marriage market, we assume that men and women are ex-ante identical – they can acquire the same skills and have the same cost distribution.

The market is competitive in that every skill has a value and agents optimize given these values. An important and novel feature of our model is that the values serve *double duty*: creating incentives to invest and to accept/reject matches. First, each agent compares their marginal cost of acquiring a skill to its marginal value in the market. Second, two agents will accept (or reject) each other whenever the match output is greater (resp. smaller) than the sum of their values. As in standard search and matching models, these values are endogenously determined in an equilibrium and must be self-consistent.

Description of the main results

Despite potential inefficiencies, we prove that every constrained efficient allocation is an equilibrium outcome (Theorem 1).[1] The proof constructs market values that satisfy the standard equilibrium conditions while perfectly aligning the agents' incentives with the planner. Strikingly, these values *simultaneously* solve the investment and matching problems. This theorem also establishes the existence of equilibrium.

A key tension underlying this welfare result is that the decisions to invest and to accept or reject potential partners impose externalities on other agents. Regarding investment, if the planner increases the inflow of skill i buyers, then in the new steady state their relative size in the market increases which has a direct effect on skill i buyers and their partners. Just as important, there is an *indirect effect* on other agents' productivity and search costs because the relative size of their matching partners changes as well. The planner takes such steady state search externalities into account. Regarding matching, when the planner decides that two skills should reject, the planner forgoes their match output and incurs a higher search cost to form more productive partnerships, but must also consider the subsequent chain reaction affecting the steady-state skill composition. In contrast, in equilibrium, each agent invests and accepts or rejects partners simply by their private incentives, as determined by the value of each skill in the market. Remarkably, there are equilibrium values that incorporate both the direct effects and the indirect search externalities.

Our second main point is that the equilibria have a clear and simple structure. Theorem 2 establishes that if the production function is supermodular (or submodular), then there is positive (resp. negative) assortative matching. Importantly, this sorting result applies to two-population models, such as the labor market, whereas previous sorting results apply only to one-population models (e.g., Shimer and Smith 2000 and Atakan 2006). Furthermore, if the production function is additively separable, then the equilibrium is unique and achieves the first-best allocation. Theorem 3 shows that our main results regarding efficient investment, efficient matching, and sorting are robust to modifying the bargaining weights and search cost parameters.

Finally, the model is applicable to a wide variety of economic situations and has important implications. First, in the labor market, we establish sufficient conditions for

[1] The constrained efficient allocation maximizes utilitarian welfare subject to the steady state constraints. Since utility is transferable, the utilitarian and Pareto criteria coincide.

sorting – when will high-tech firms match with high-skill workers – and the framework allows us to analyze the mutual dependency between sorting and investments. Second, in the marriage market, we show that an occupational gender gap can arise and can even be efficient. That is, the two populations are ex-ante identical and the market outcome is discriminatory: men and women who acquire the same skill receive different payoffs in the market. Finally, in product markets, the production function is commonly presumed additively separable, and we establish a unique equilibrium. Economies with non-separable production functions can have multiple equilibria and the agents may fail to coordinate on the efficient one. Policy interventions may be helpful for alleviating such coordination problems.

Related Literature

Our paper is the first to provide a general and tractable model incorporating three components: (i) random search and bargaining, (ii) matching between heterogeneous agents, and (iii) pre-entry investments. These three components have not been studied together and have novel implications when studied jointly.[2] The table below summarizes the central papers in the strands of the literature most closely related to our work, random search or frictionless matching models with transferable utility.

Group	Papers	Search	Matching	Investment	Results
1	Cole et al. (2001) Noldeke and Samuelson (2015)	No	Yes	Yes	Efficiency
2	Shimer and Smith (2000) Atakan (2006)	Yes	Yes	No	Sorting (single population)
	Shimer and Smith (2001)	Yes	Yes	No	Inefficiency
3	Acemoglu (1996) Masters (1998) Acemoglu and Shimer (1999)	Yes	No	Yes	Inefficiency
4	Hosios (1990)	Yes	No	No	Efficiency (for a specific bargaining weight)
5	Gale (1987) Mortensen and Wright (2002) Lauermann (2013)	Yes	No	No	Convergence to First Best
6	This paper	Yes	Yes	Yes	Constrained Efficiency + Sorting + Robustness

Table 1: Literature Comparison

[2]For example, the efficiency of an occupational gender gap requires all three components.

The papers in group 1 extend the classical assignment model of Shapley and Shubik (1971) to settings with ex-ante investments. These models have perfect *frictionless matching* and typically find that the first-best allocation is a competitive equilibrium outcome, but there may exist additional inefficient equilibria (see also Mailath et al. 2013; Dizdar 2018; Chade and Lindenlaub 2022).[3] We contribute to this literature by adding search frictions and establishing that the *constrained efficient* allocation is an equilibrium outcome. One novel implication of our model is that in the symmetric marriage, the occupational gender gap can be efficient. In contrast, in frictionless models, the efficient outcome is always symmetric.

The papers in group 2 study the random search and bargaining model with heterogeneous agents but without investment (see also Burdett and Coles 1999). As in Atakan (2006), we consider a model with explicit search costs. We add pre-entry investment and prove novel efficiency, sorting, and existence results. All three are substantial contributions. In particular, our sorting result is for two-population models, wheres the previous sorting results of Shimer and Smith (2000) and Atakan (2006) are for one-population models. This is a major difference because the labor and product markets are two-sided, and the proof is novel and non-trivial (the proofs of previous results relied heavily on the one-population assumption).[4] In addition, establishing existence in search models is a tricky problem (see, e.g., Manea 2017 and Lauermann et al. 2020) and standard techniques don't apply to our model with an endogenous inflow.

Our efficiency result stands in contrast to previous results in the search literature. First, in the standard random search and bargaining model, Shimer and Smith (2001) show that agents mismatch: low-types reject too frequently while high-types accept too often. Second, in the same standard search model, but with *homogeneous* agents, the papers in group 3 show that the hold-up problem leads to under-investment. However, in our model, the equilibrium values simultaneously solve both the hold-up and the matching problems. The key difference is that our model has explicit search costs whereas those models have implicit search costs due to time discounting. Our results suggest that the hold-up and matching problems are not due to search frictions *per se*, but rather to discounting (see Section 6.3).

[3] In Chade and Lindenlaub (2022), utility is not perfectly transferable and therefore the first-best is generally unattainable but there does exist a Pareto efficient equilibrium.

[4] The one-population model is a special case of the two-population model.

Hosios (1990) considers a search model with *homogeneous* agents except that the two sides of the market may meet partners at different rates. The meeting rates create incentives to enter and hence affect the balance ratio in the market (the relative size of each population). The main result is that the "right" bargaining weights can achieve the efficient balance ratio in the market. Our paper is about a different problem: we study the investment and matching decisions in a model with *heterogeneous* agents. Indeed, the bargaining weights mechanically affect the balance ratio, but it is a secondary issue for our problems: our main results regarding efficient investment, efficient matching, and sorting do not depend on the bargaining weights (see Section 6.1).

The papers in group 5 study whether the random search and bargaining model converges, as the discount factor $\delta \to 1$, to the frictionless Walrasian outcome. Our paper shows that the constrained efficient allocation is achieved in a market with investment, search, and matching (and explicit search costs). There is an important literature on search with non-transferable utility and directed search, but these models differ extensively from ours. For instance, Burdett and Coles (2001) consider a marriage market with premarital investments, but they assume a very specific form of *non-transferable utility* and *homogeneous* investment costs. They show that an equilibrium exists and that it is inefficient.[5] In the literature on directed search, sellers post prices to attract buyers, and the equilibrium can achieve an efficient allocation (see, e.g., Acemoglu and Shimer 1999; Shi 2001; Jerez 2017) and sorting (see, e.g., Shimer 2005; Eeckhout and Kircher 2010; Cai et al. 2021). However, the matching process and the price-determination mechanism are substantially different than in the random search and bargaining model.

[5]In the non-trivial case of high investment costs, agents overinvest to appeal to better partners, and they search too much in that agents are too selective.

2 The Model

The economy consists of two populations, buyers and sellers, who first invest in acquiring skills and then search for a partner with whom they may form a productive match. Formally, each buyer acquires one skill from a finite set $I \subset \mathbb{N}$, and each seller acquires one skill from a finite set $J \subset \mathbb{N}$. The agents are heterogeneous in their investment cost: each buyer has a type $\beta \sim F^b$ and incurs the cost $C^b(i, \beta)$ from acquiring skill $i \in I$. Likewise, each seller has a type $\sigma \sim F^s$ and incurs the cost $C^s(j, \sigma)$ from acquiring skill $j \in J$. Output is produced by buyer-seller pairs according to their skills and is summarized by the matrix $G = [g_{ij}]$, where the entry $g_{ij} \geq 0$ denotes the output of a pair with skills i, j. Agents have transferable utility and incur a fixed per-period search cost $c > 0$.

Definition. An *economy* is a tuple $\langle F^b, F^s, I, J, C^b, C^s, G, c\rangle$ consisting of prior distributions, skill sets, investment cost functions, the output function, and a search cost. The economy is *symmetric* if $F^b = F^s, I = J$, $C^b = C^s$, and $g_{ij} = g_{ji}, \forall i, j$.

The type distributions F^b and F^s are continuous and strictly increasing over their connected supports: $\mathcal{B} = supp(F^b) \subseteq \mathbb{R}$ and $\mathcal{S} = supp(F^s) \subseteq \mathbb{R}$. The match output g_{ij} is strictly increasing in skills. The cost functions are non-negative, strictly increasing in both arguments, bounded and continuous. Furthermore, they satisfy increasing differences: the difference $C^b(i', \beta) - C^b(i, \beta)$ is strictly increasing in β whenever $i' > i$ and the difference $C^s(j', \sigma) - C^s(j, \sigma)$ is strictly increasing in σ whenever $j' > j$. That is, a higher skill enhances match output, but is more costly to acquire, and higher types have higher costs and higher marginal costs.

Timing. The model takes place in discrete time periods over an infinite horizon. In every period, a measure one population of buyers and a measure one population of sellers are born. Each newborn agent chooses a skill and then enters the matching market. Each agent in the market incurs the search cost c and randomly meets a partner. When two agents meet, they can either accept the match or continue searching in the hope of finding a better partner. If both agents accept the match, then they exit the market and divide their output according to Nash bargaining. If at least one rejects, then they both remain in the market. In the next period, a new cohort is born and the process repeats itself. Note that only the new-born agents invest, agents in the market cannot change their prior investments, and mutually accepted matches are once and for all. We refer to the agents in the market as the stock population, the agents entering the market as the inflow population, and the agents exiting the market as the outflow population.

Steady State. The economy is in a *steady state* if in the stock population the measure of agents with each skill is constant over time. Therefore, for each skill, the inflow of agents equals the outflow. In a steady state, we denote the measures of skill i buyers and skill j sellers in the stock population by b_i and s_j. The total measures of buyers and sellers in the market are $B = \sum_{i \in I} b_i$ and $S = \sum_{j \in J} s_j$, and the *proportions* of skill i buyers and skill j sellers are $x_i = b_i/B$ and $y_j = s_j/S$ (notice that $B \geq 1$ and $S \geq 1$). The notation (x_i) and (y_j) denotes the profile of buyer and seller proportions. We let $z = \langle (x_i), (y_j), B, S \rangle$ be the *state variable* where the set of all state variables is $\mathcal{Z} = \Delta(I) \times \Delta(J) \times [1, \infty)^2$.

Meetings. An agent can meet at most one partner in each period and pairs meet at random. The total number of meetings per period is $\mu(B, S) = \min(B, S)$. Therefore, if the market is balanced, $B = S$, then every agent randomly draws a partner in each period; but if it is unbalanced, $B \neq S$, then agents on the long side of the market would be rationed. We will show that an unbalanced market is not an equilibrium outcome (see Lemma 1), and therefore, to simplify the notation, we assume without loss of generality that the market is balanced, and denote the market size by $N = B = S$ and the state by $z = \langle (x_i), (y_j), N \rangle$. In Section 6.2, we extend the analysis to consider more general meeting functions.

Strategies. An agent's strategy specifies their choice of skill and which agents they accept. We assume Markov strategies. The *investment strategy* of buyer β is $\mathcal{I}^\beta : \mathcal{Z} \to I$ and that of seller σ is $\mathcal{I}^\sigma : \mathcal{Z} \to J$. The *acceptance strategy* of a buyer with skill i is $A_i^b : \mathcal{Z} \times J \to [0, 1]$, which specifies the probability she accepts a seller with skill j upon meeting. For a seller with skill j, it is $A_j^s : \mathcal{Z} \times I \to [0, 1]$. Note that the acceptance strategies do not depend on the agents' identities because the match output depends only on skills. To simplify, we will suppress the state variable in the strategies. It will be convenient to summarize the acceptance strategies by a matching matrix $M = [m_{ij}]$, where the element $m_{ij} = A_i^b(j) \cdot A_j^s(i)$ is the probability that buyer i and seller j both agree to match, conditional on meeting.

Remark 1. The search cost c captures various costs incurred explicitly from search. These include the opportunity cost of time (think of the man-hours firms spend screening and interviewing candidates; while candidates forgo some income, say from driving an Uber, as they go through ads, apply, and prepare to interview); flow payments and fees (subscriptions to online search platforms, hiring talent recruiters, or advertisement fees); cognitive effort costs (browsing and comparing products online for hours, or the negative mental health impact of unemployment); or even singles paying per date. In contrast, in a model with time discounting, agents incur an implicit search cost as their payoffs are delayed. Which costs are more salient depends upon the economic situation being modeled, but there are certainly situations where additive costs are predominant.[6]

Remark 2. We assumed that all agents enter the market and actively search. In Section 4.1, we allow agents to not enter the market and receive an exogenous outside option. The analysis and key results are the same (except the overall inflow can be less than one). This extension also covers the case where the agents in the market can choose to not search, incurring zero search costs and meeting no one, which is equivalent to an outside option of zero.

2.1 Equilibrium

Every skill has a value in the market and agents optimize given the values and the steady state. We denote the values of a skill i buyer by v_i, and of a skill j seller by w_j. The profiles of buyer and seller values are (v_i) and (w_j), respectively. As is standard in the search and matching literature, we define an equilibrium using the matching matrix and values, rather than the strategies.

Definition. A *steady state equilibrium* $\langle z, M, (v_i), (w_j) \rangle$ consists of a state variable, a matching matrix, and market values satisfying conditions (1), (3), and (4) below.

The first condition is that acceptance decisions are individually optimal. When two agents with skills i and j meet, the *surplus* is $s_{ij} = g_{ij} - v_i - w_j$, and the acceptance decisions satisfies the *Efficient Matching* condition:

[6] For example, when search transpires over a short period of time and does not affect the consumption date (think of the time spent searching online for a product that will be delivered tomorrow or college students applying for jobs which they will take after graduation).

$$m_{ij} = \begin{cases} 1 & \text{if } s_{ij} > 0 \\ 0 & \text{if } s_{ij} < 0 \end{cases} \tag{1}$$

The condition is intuitive because an agent will accept a match precisely when her payoff from doing so is greater than her continuation value. When the surplus is negative, i.e. $v_i + w_j > g_{ij}$, the match is always rejected because both agents cannot receive at least their value, while when the surplus is positive, the agents will reach a mutually beneficial agreement. If the surplus is exactly zero, then m_{ij} is unrestricted, i.e. $0 \leq m_{ij} \leq 1$.

When two agents accept each other, each receives their own value and half of the match surplus. This division rule is the Nash bargaining solution and also is a subgame perfect equilibrium of a strategic bargaining game (see, e.g., Atakan 2006). The second condition is that the values are self-consistent, and therefore satisfy the following recursive equation:

$$\begin{aligned} v_i &= \sum_{j \in J} y_j \left[m_{ij} \left(v_i + \tfrac{s_{ij}}{2} \right) + (1 - m_{ij}) v_i \right] - c, \forall i \\ w_j &= \sum_{i \in I} x_i \left[m_{ij} \left(w_j + \tfrac{s_{ij}}{2} \right) + (1 - m_{ij}) w_j \right] - c, \forall j \end{aligned} \tag{2}$$

That is, in every period, buyer i pays the search cost c and meets a seller.[7] The probability of meeting seller j is y_j. If a match is accepted, the buyer receives her continuation value and half of the surplus, whereas if the match is rejected, she attains her continuation value v_i. Simplifying, we obtain the *Constant Surplus* equations:

$$\begin{aligned} \sum_{j \in J} y_j m_{ij} s_{ij} &= 2c, \forall i \\ \sum_{i \in I} x_i m_{ij} s_{ij} &= 2c, \forall j \end{aligned} \tag{3}$$

The investment decisions are individually optimal: $\mathcal{I}^\beta \in \arg\max_{i \in I} v_i - C(i, \beta), \forall \beta$ and $\mathcal{I}^\sigma \in \arg\max_{j \in J} w_j - C(j, \sigma), \forall \sigma$. Since the cost function satisfies strictly increasing differences, the set of cost types who choose each skill is an interval (and hence measurable). Furthermore, at most one type can be indifferent between any

[7] These equation presume that the market is balanced, $B = S$, so no one is rationed. If the market were unbalanced, then the long side would be rationed and the value equations would also have a meeting probability (but this does not occur in equilibrium, see Lemma 1).

two skills,[8] and thus the values (v_i) and (w_j) uniquely determine the inflows (up to measure zero). Formally, we denote by $F^b(A) = \int_A dF^b$ the measure of set A according to F^b. The measure of buyers who choose skill i is $F^b\left(\{\beta : \mathcal{I}^\beta = i\}\right) = F^b\left(\{\beta : i \in \arg\max_{i'\in I} v_{i'} - C^b(i',\beta)\}\right)$, and analogously for sellers.

The final set of conditions is that the economy is in a steady state. We refer to Equations (4) as the *Inflow=Outflow* equations:

$$\begin{aligned} \overbrace{F^b\left(\left\{\beta : i \in \arg\max_{i'\in I} v_{i'} - C^b(i',\beta)\right\}\right)}^{inflow} &= \overbrace{Nx_i \sum_{j\in J} y_j m_{ij}}^{outflow}, \forall i \in I \\ F^s\left(\left\{\sigma : j \in \arg\max_{j'\in j} w_{j'} - C^s(j',\sigma)\right\}\right) &= \ Ny_j \sum_{i\in I} x_i m_{ij}, \forall j \in J \end{aligned} \tag{4}$$

The inflow is the measure of buyers who choose skill i. The outflow is the measure of skill i buyers in the market, Nx_i, times the probability of exiting (each buyer meets a skill j with probability, y_j, and they accept each other with probability, m_{ij}). The seller Inflow=Outflow equations are analogous.

2.2 Equilibrium Properties

The next two lemmas will be useful. The first states that unbalanced states do not occur in equilibria.

Lemma 1. *(No Rationing) In any equilibrium, $B = S$.*

Proof. WLOG, suppose that $B \geq S$. Then, a buyer meets a seller with probability $\rho = S/B$, and a seller meets a buyer with probability 1. Therefore, the values satisfy:

$$\begin{aligned} \forall i : v_i =& \rho \sum_{j\in J} y_j \left[m_{ij}\left(v_i + \frac{s_{ij}}{2}\right) + (1-m_{ij})v_i\right] + (1-\rho)v_i - c \Rightarrow \sum_{j\in J} y_j m_{ij} s_{ij} = \frac{2c}{\rho} \\ \forall j : w_j =& \sum_{i\in I} x_i \left[m_{ij}\left(w_j + \frac{s_{ij}}{2}\right) + (1-m_{ij})w_j\right] - c \Rightarrow \sum_{i\in I} x_i m_{ij} s_{ij} = 2c \end{aligned}$$

Therefore, since $\sum_{i\in I} x_i = \sum_{j\in J} y_j = 1$:

$$\frac{2c}{\rho} = \sum_{i\in I} x_i \sum_{j\in J} y_j m_{ij} s_{ij} = \sum_{j\in J} y_j \left(\sum_{i\in I} x_i m_{ij} s_{ij}\right) = 2c \Rightarrow B = S \qquad \square$$

[8] If buyer $\hat\beta$ is indifferent between acquiring skills i and i', where $i' > i$, then all buyers $\beta < \hat\beta$ strictly prefer skill i' to skill i and all buyers $\beta > \hat\beta$ strictly prefer skill i to skill i'.

The next lemma states that, in equilibrium, the agents' values are increasing and the marginal values are bounded by the expected marginal productivity.

Lemma 2. *In any equilibrium,*

$$\frac{\sum_{j\in J} y_j m_{i'j}(g_{i'j}-g_{ij})}{\sum_{j\in J} y_j m_{i'j}} \geq v_{i'} - v_i \geq \frac{\sum_{j\in J} y_j m_{ij}(g_{i'j}-g_{ij})}{\sum_{j\in J} y_j m_{ij}} > 0,\ \forall i' > i$$

$$\frac{\sum_{i\in I} x_i m_{ij'}(g_{ij'}-g_{ij})}{\sum_{i\in I} x_i m_{ij'}} \geq w_{j'} - w_j \geq \frac{\sum_{i\in I} x_i m_{ij}(g_{ij'}-g_{ij})}{\sum_{i\in I} x_i m_{ij}} > 0,\ \forall j' > j$$

In particular, if $m_{ij} = 1$, $\forall i, j$, then the marginal value equals the expected marginal productivity: $v_{i'} - v_i = \sum_{j\in J} y_j(g_{i'j} - g_{ij})$ and $w_{j'} - w_j = \sum_{i\in I} x_i(g_{ij'} - g_{ij})$.

Proof. The Constant Surplus and Efficient Matching conditions imply that:

$$\sum_{j\in J} y_j m_{ij} s_{ij} = 2c = \sum_{j\in J} y_j m_{i'j} s_{i'j} \geq \sum_{j\in J} y_j m_{ij} s_{i'j}$$

Subtracting the RHS from the LHS, and normalizing:

$$v_{i'} - v_i \geq \frac{\sum_j y_j m_{ij}(g_{i'j}-g_{ij})}{\sum_j y_j m_{ij}} > 0$$

The upper bound is derived analogously by switching i and i'. □

Remark 3. Lemma 2 implies the bound: $\max_j g_{i'j} - g_{ij} \geq v_{i'} - v_i \geq \min_j g_{i'j} - g_{ij}$.

Remark 4. The Constant Surplus equations have two further implications: First, they determine the values for unchosen (measure 0) skills as what agents would receive in equilibrium if they were to invest in such a skill. Therefore, such values cannot be set arbitrarily (for instance, to minus infinity). Second, every agent has at least one partner with whom the surplus is positive. Furthermore, that partner is not of measure 0, which implies that there are no pathological equilibria where an agent searches forever.

Remark 5. If $\langle z, M, (v_i), (w_j)\rangle$ is an equilibrium, then so is $\langle z, M, (v_i+t), (w_j-t)\rangle$ for any transfer $t \in \mathbb{R}$. Therefore, there is at least one degree of freedom in the equilibrium values. We now show that there is in fact exactly one degree of freedom. This is because the marginal values, i.e. Δv_i, are uniquely pinned down by the investment decisions and a Constant Surplus equation imposes an additional condition on the value functions.

3 Illustrative Examples

We now illustrate the model with two examples. We consider a symmetric economy with two skills, $I = J = \{0, 1\}$. Each agent can either invest and become skilled, $i = j = 1$, or not invest and remain unskilled, $i = j = 0$. The cost of becoming skilled is the agent's type, which is uniformly distributed $\beta, \sigma \sim U[a, d]$. To simplify notation, denote by $x = x_1$ and $y = y_1$ the proportion of skilled buyers and skilled sellers. We consider the following supermodular and submodular production matrices:

$$G^{sup} = \begin{bmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad G^{sub} = \begin{bmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$$

In both matrices, skilled-skilled pairs produce $g_{11} = 4$ and unskilled-unskilled pairs produce $g_{00} = 1$. The first production matrix is *supermodular* because skilled-unskilled pairs produce $g_{10} = g_{01} = 2$, so an agent's marginal productivity is greater when matched with a skilled agent than when matched with an unskilled agent, $g_{11} - g_{01} = 2 > 1 = g_{10} - g_{00}$. Conversely, the second production matrix is *submodular* because $g_{10} = g_{01} = 3$, and so $g_{11} - g_{01} = 1 < 2 = g_{10} - g_{00}$.

In each case, we will demonstrate the constrained efficient allocation and the equilibria. The constrained efficient allocation solves the planner's problem: choose the investment thresholds, matching rule, and state to maximize per-period welfare:[9]

$$W(x, y, N, [m_{ij}], \beta_1, \sigma_1) = \overbrace{\sum_{i=0}^{1} \sum_{j=0}^{1} N x_i y_j m_{ij} g_{ij}}^{\text{Productivity}} \quad - \quad \overbrace{2Nc}^{\text{Search Cost}} \quad - \quad \overbrace{\int_a^{\beta_1} \beta dF(\beta) - \int_a^{\sigma_1} \sigma dF(\sigma)}^{\text{Investment Cost}}$$

subject to the steady state constraints:

$$N x \sum_{j=0}^{1} y_j m_{1j} = F(\beta_1), \qquad N(1 - x) \sum_{j=0}^{1} y_j m_{0j} = 1 - F(\beta_1),$$

$$N y \sum_{i=0}^{1} x_i m_{i1} = F(\sigma_1), \qquad N(1 - y) \sum_{i=0}^{1} x_i m_{j0} = 1 - F(\sigma_1)$$

[9]First term: $N x_i y_j m_{ij}$ is the measure of accepted matches between skills i, j and g_{ij} is their output.

Supermodular Production

We take $G = G^{sup}$ and fix the distribution parameters $a = 0.8$ and $d = 2.8$. The planner's optimal policy is spanned by two allocations:

1) **All Skills Match:** Agents accept any partner $m_{ij} = 1, \forall i, j$, total welfare is

$$\mathcal{W}^{All} = N\left[4xy + 2y(1-x) + 2x(1-y) + (1-x)(1-y)\right] - 2Nc - \int_a^{\beta_1} \beta dF(\beta) - \int_a^{\sigma_1} \sigma dF(\sigma)$$

The steady state equations reduce to $N = 1$ and $x = F(\beta_1)$ and $y = F(\sigma_1)$. The optimal skill distribution and investment thresholds for this case are: $x = y = 0.2$ and $\beta_1 = \sigma_1 = 1.2$.

2) **Positive Assortative Matching (PAM):** Same skills accept $m_{11} = m_{00} = 1$ and opposite skills reject $m_{01} = m_{10} = 0$, total welfare is

$$\mathcal{W}^{PAM} = N\left[4xy + (1-x)(1-y)\right] - 2Nc - \int_a^{\beta_1} \beta dF(\beta) - \int_a^{\sigma_1} \sigma dF(\sigma)$$

The steady state equations are:

$$\begin{aligned} Nxy &= F(\beta_1), \quad N(1-x)(1-y) = 1 - F(\beta_1) \\ Nyx &= F(\sigma_1), \quad N(1-y)(1-x) = 1 - F(\sigma_1) \end{aligned}$$

which imply $\beta_1 = \sigma_1$ and $N = \frac{1}{xy+(1-x)(1-y)}$, and so the planner's optimization problem is three dimensional (β_1, x, y). The optimal skill distribution and investment thresholds for PAM are symmetric and depend on c, we denote them by $x = y \equiv x^{*PAM}(c)$ and $\beta_1 = \sigma_1 \equiv \beta^{*PAM}(c)$.

Figure 1 depicts the welfare of these two allocations (using the optimal investment thresholds and skill distribution for each allocation) as a function of the search cost c.

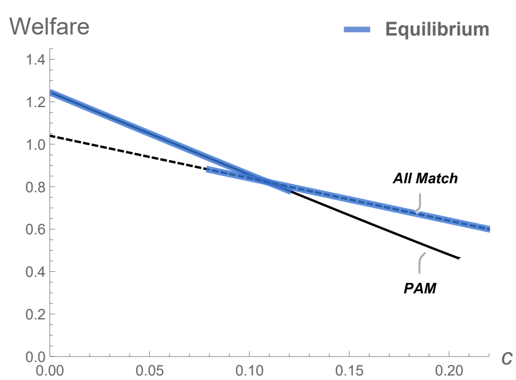


Figure 1: Welfare and Equilibrium

The upper envelope of these curves is the constrained efficient allocation. The trade-off is between higher productivity (PAM) and lower search cost (All Match). The shaded regions are where each allocation is an equilibrium. This figure visually demonstrate a Second Welfare Theorem: the upper envelope is always an equilibrium.[10] The following claim derives the equilibrium regions.

Claim. The All Match allocation is an equilibrium if and only if $c \geq 0.08$**.** The PAM allocation is an equilibrium if and only if $c \leq c_1 \approx 0.12$**.**

Proof. First, in an equilibrium where all skills match, Lemma 2 implies that the marginal values must equal the marginal productivities:

$$\Delta v = y(g_{11} - g_{01}) + (1 - y)(g_{10} - g_{00}) = 1 + y$$
$$\Delta w = x(g_{11} - g_{10}) + (1 - x)(g_{01} - g_{00}) = 1 + x$$

and the steady state equations are $F(\Delta v) = x$ and $F(\Delta w) = y$. These equations have a unique solution $\Delta v = \Delta w = 1.2$ and $x = y = 0.2$, and the candidate state $x = 0.2$ induces values $(\bar{v}), (\bar{w})$ that must solve: i) the corresponding Constant Surplus equations; and ii) the Efficient Matching conditions so that all pairs indeed want to match (i.e., $\bar{v}_i + \bar{w}_j \leq g_{ij}, \forall i, j$).

"$\Leftarrow$" If $c < 0.08$, the values that solve the Constant Surplus equations are too high for an All Skills Match equilibrium because agents with opposite skills reject each other (i.e., $\bar{v}_i + \bar{w}_j > g_{ij}$ for $i \neq j$).

"$\Rightarrow$" If $c \geq 0.08$, the induced values are sufficiently low to satisfy the efficient matching conditions. The state $x = y = 0.2$ and marginal values $\Delta\bar{v} = \Delta\bar{w} = 1.2$ constitute the unique equilibrium where all pairs match. Remarkably, this equilibrium coincides with the planner's optimal All Match policy.

Second, in an equilibrium with PAM, it must be that $\Delta v = \Delta w$ (because the outflow of skilled buyers = the outflow of skilled sellers); and $x = y$ (because of the Constant Surplus equations $xs_{11} = 2c = ys_{11}$); together implying[11]

[10] We depict a certain economy parameterized by $F = U[0.8, 2.8]$ and G^{sup}, but a similar picture would arise for most two-skill supermodular symmetric economies. Recall that in the All Match allocation (1), the planner's optimal policy does not depend on c, and so this curve is linear. In contrast, in the PAM allocation (2), the planner's optimal stocks and flows change with c, and thus the PAM curve is convex (though it is hard to see in this graph).

$$\overbrace{F(\Delta v)}^{\text{Inflow}} = \overbrace{\frac{x^2}{x^2 + (1-x)^2}}^{\text{Outflow}} \text{ and } \Delta v = \frac{g_{11}}{2} - \frac{c}{x} - \left(\frac{g_{00}}{2} - \frac{c}{1-x} \right)$$

The states x that solves these two equations are the only candidates for an equilibrium with PAM and every candidate x induces values that must solve: i) the corresponding Constant Surplus equations; and ii) the Efficient Matching conditions so that same skills accept (i.e., $v_i + w_j \leq g_{ij}$ for $i = j$) and opposite skills reject (i.e., $v_i + w_j \geq g_{ij}$ for $i \neq j$).

"$\Leftarrow$" If $c > c_1$, then the values that solve the Constant Surplus equation are too low for a PAM equilibrium because agents with opposite skills accept each other.

"$\Rightarrow$" If $c \leq c_1$, then there is a *unique* candidate state $\hat{x}(c)$ whose induced values $(\hat{v}), (\hat{w})$ satisfy these two conditions. The skill distribution $y = x = \hat{x}(c)$, market size $N = \frac{1}{x^2+(1-x)^2}$, and values $(\hat{v}), (\hat{w})$ constitute the unique equilibrium with PAM. Remarkably, this equilibrium coincides with the planner's optimal PAM policy, $\hat{x}(c) = x^{*PAM}(c)$ and $\Delta\hat{v} = \Delta\hat{w} = \beta^{*PAM}(c)$. □

Remark 6. In Figure 1, there can be multiple equilibria, but the overlap region is small. The planner could implement various other policies, varying either the investment thresholds or the matching rule, but those would generate lower welfare, and they cannot be supported by an equilibrium (generically).[12]

Submodular Production

We now take $G = G^{sub}$ and fix the average cost type to be $(a + d)/2 = 1.5$. The constrained efficient allocation is spanned by the following three simple allocations:

1) **Negative Assortative Matching (NAM):** Only the agents with below-average costs invest and only opposite skills match, $m_{10} = m_{01} = 1$ and $m_{00} = m_{11} = 0$. The steady state is $x = y = 1/2$ and $N = 2$. Per-period welfare:

$$\mathcal{W}^{NAM} = g_{10} - 2Nc - \int_a^{\mu} \beta f(\beta) d\beta - \int_a^{\mu} \sigma f(\sigma) d\sigma = 3 - 4c - \frac{a+\mu}{2}$$

[11]LHS: Divide the two steady state equations $Nx^2 = F(\Delta v)$ and $N(1-x)^2 = 1 - F(\Delta v)$. RHS: Subtract the two Constant Surplus equations $xs_{11} = 2c$ and $(1-x)s_{00} = 2c$ and use $\Delta v = \Delta w$.

[12]In this example, for any c, there are at most three equilibria: the two depicted above and possibly a mixed one where skilled-unskilled pairs match with a strictly positive probability (which has lower welfare).

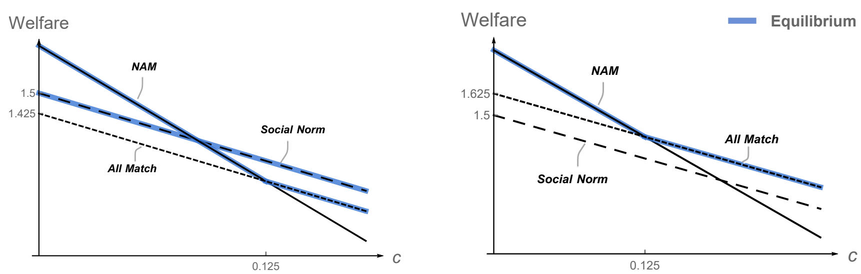
2) **All Skills Match:** The same investment thresholds as the NAM allocation, but now all pairs match, $m_{ij} = 1, \forall i, j$. The stock population is $N = 1$, and $x = y = 1/2$. Per-period welfare:

$$\mathcal{W}^{All} = \tfrac{1}{4}\left(g_{11} + g_{10} + g_{01} + g_{00}\right) - 2Nc - \int_a^\mu \beta f(\beta) d\beta - \int_a^\mu \sigma f(\sigma) d\sigma = 2.75 - 2c - \tfrac{a+\mu}{2}$$

3) **Social Norm (one-sided investment):** Every buyer invests and becomes skilled and every seller does not invest and remains unskilled. Agents accept any partner. Since the market clears in every period, the stock population is $N = 1$. Per-period welfare:

$$\mathcal{W}^{SN} = g_{10} - 2Nc - \int_a^d \beta f(\beta) d\beta = 3 - 2c - \mu$$

Figure 2 illustrates the welfare of these allocations as a function of c. The equilibrium regions are shaded blue. Panel (a) depicts the case where the distribution F has small support, $l = d - a < 1$ and Panel (b) depicts a large support $l > 1$.



(a) $l < 1$ (b) $l > 1$

Figure 2: Equilibrium Regions

Notice each allocation is supported by an equilibrium whenever it is efficient[13]

$$\begin{aligned}
\mathcal{W}^{NAM} &= 3 \;\; -4c - \left(\tfrac{a+\mu}{2}\right) \\
\mathcal{W}^{All} &= 2.75 - 2c - \left(\tfrac{a+\mu}{2}\right) \\
\mathcal{W}^{SN} &= 3 \;\; -2c - \;\; \mu
\end{aligned}$$

- The NAM allocation is an equilibrium if and only if the search cost $c \leq 1/8$. This allocation maximizes productivity but has a higher search cost.

[13]The specific parameters illustrated are $l = 0.5$ and $l = 1.5$. The arguments for the NAM and the All Match equilibria regions are the same as in the supermodular case (see above). To see why the Social Norm equilibrium depends on the support $l = d - a$: By Lemma 2, the marginal values are bounded by the marginal productivities: $1 \leq \Delta v \leq 2$. Therefore, if $l > 1$, then $a < 1$ (because the average cost-type is 1.5) and all types less than 1 will invest in every equilibrium. If $l \leq 1$, the values $v_1 = 2.5 - c$, $v_0 = 0.5 - c$, $w_1 = 1.5 - c$, and $w_0 = 0.5 - c$ satisfy the equilibrium conditions. Notice that since the average cost-type is 1.5 and $l < 1$, it follows that $1 < a < d < 2$, and therefore all buyers want to invest because $\Delta v = 2 \geq d \geq \beta$ and no seller wants to invest because $\Delta w = 1 \leq a \leq \sigma$.

- The All Match allocation is an equilibrium if and only if the search cost $c \geq 1/8$. This allocation benefits from lower search costs but has lower productivity because agents mismatch (both unskilled-unskilled and skilled-skilled matches occur).
- The Social Norm allocation is an equilibrium if and only if the support $l \leq 1$. This allocation maximizes productivity and minimizes the search cost but has a higher total investment cost: high-cost buyers invest while low-cost sellers do not. This misallocation of talent is more severe for a wider cost distribution.

These three allocations demonstrate the trade-off between productivity, investment cost, and search cost. Each allocation optimizes two components at the expense of the third. The takeaways from the examples are:

1) Efficiency: The constrained efficient allocation, depicted by the upper envelope of the lines, is an equilibrium.

2) Assortative Matching: Under the supermodular production function G^{sup}, agents with the same skills matched; and under the submodular function G^{sub}, agents with opposite skills matched.

3) Economic Implications: When the production function is submodular, the efficient outcome can be discriminatory. Discrimination induces the two groups to invest differently and thereby minimizes search costs and enhances productivity, but at the expense of higher investment costs. Finally, there can be multiple equilibria but the equilibria set is small and tractable.

4 The Second Welfare Theorem

To simplify, skills are labelled as $I = \{0, 1, \ldots, |I| - 1\}$ and $J = \{0, 1, \ldots, |J| - 1\}$. The constrained efficient allocation is the solution to the problem of a social planner who chooses the investment and acceptance strategies and sets the stock in the matching market, in order to maximize per-period total welfare, subject to the condition that the economy is in a steady state.

Without loss of generality: i) the planner chooses a balanced state, $B = S = N$ (rationing would not benefit the planner); ii) the matching strategies are represented by a matching matrix; and iii) the planner's optimal investment strategies can be defined by thresholds $\beta_0 \geq \beta_1 \geq \ldots \geq \beta_I$ and $\sigma_0 \geq \sigma_1 \geq \ldots \geq \sigma_J$, so that all buyers of type $\beta \in (\beta_{i+1}, \beta_i)$ choose skill i and all sellers of type $\sigma \in (\sigma_{j+1}, \sigma_j)$ choose skill j (types and skills are inversely related because costs increase with type).

Thus, the planner chooses $\langle z, M, (\beta_i), (\sigma_j)\rangle$ to maximize:

$$\mathcal{W}\left(\langle z, M, (\beta_i), (\sigma_j)\rangle\right) = \sum_{i\in I}\sum_{j\in J} N x_i y_j m_{ij} g_{ij} - 2Nc - \sum_{i\in I}\int_{\beta_{i+1}}^{\beta_i} C^b(i,\beta) f^b(\beta) d\beta \qquad (5)$$
$$- \sum_{j\in J}\int_{\sigma_{j+1}}^{\sigma_j} C^s(j,\sigma) f^s(\sigma) d\sigma$$

subject to
$$flow_i^b := \left(F^b(\beta_i) - F^b(\beta_{i+1})\right) - N x_i \sum_{j\in J} y_j m_{ij} = 0, \forall i \qquad (6)$$
$$flow_j^s := \left(F^s(\sigma_j) - F^s(\sigma_{j+1})\right) - N y_j \sum_{i\in I} x_i m_{ij} = 0, \forall j \qquad (7)$$
$$x_i \geq 0, \forall i \qquad (8)$$
$$y_j \geq 0, \forall j \qquad (9)$$
$$X := 1 - \sum_{i\in I} x_i = 0 \qquad (10)$$
$$Y := 1 - \sum_{j\in J} y_j = 0 \qquad (11)$$
$$1 \geq m_{ij} \geq 0, \forall i, j \qquad (12)$$
$$F^b(\beta_{|I|}) = F^s(\sigma_{|J|}) = 0 \qquad (13)$$
$$F^b(\beta_0) = F^s(\sigma_0) = 1 \qquad (14)$$

The first term in the objective function is per-period total output (the measure of formed matches between buyer i and seller j is $N x_i y_j m_{ij}$ and their match output is g_{ij}), the second term is the per-period total search cost, and the last two terms are the per-period total investment costs. The first constraint is that inflow equals outflow. The other conditions stipulate that x_i, y_j are proportions, m_{ij} are probabilities, and that the planner must assign a skill to every agent. Note that the maximization problem does not explicitly require that $\beta_0 \geq \beta_1 \geq \ldots \geq \beta_I$ and $\sigma_0 \geq \sigma_1 \geq \ldots \geq \sigma_J$, nor $N > 0$, because these conditions are implied by the other constraints (see proof).

Theorem 1. *(Second Welfare Theorem) For every economy* $\langle F^b, F^s, I, J, C^b, C^s, G, c\rangle$*:*

i) There exists an optimal policy $\langle z, M, (\beta_i), (\sigma_j)\rangle$*.*

ii) Every optimal policy $\langle z, M, (\beta_i), (\sigma_j)\rangle$ *can be decentralized. That is, there are values* (v_i^*)*,* (w_j^*)*, and a matching matrix* M^* *such that* $\langle z, M^*, (v_i^*), (w_j^*)\rangle$ *is an equilibrium, where* $m_{ij}^* = m_{ij}$ *for all* i, j *such that* x_i*,* $y_j > 0$*.*

The proof shows that shadow values of the flow constraints can serve as the equilibrium values that decentralize the optimal allocation. That is, these shadow values satisfy the Constant Surplus equations and IC conditions: If the planner's optimal policy specifies that buyer β chooses skill i^* then $i^* \in \arg\max_{i \in I} v_i - C^b(i, \beta)$, and if seller σ chooses skill j^*, then $j^* \in \arg\max_{j \in J} w_j - C^s(j, \sigma)$. Furthermore, if the planner's policy specifies that i^* and j^* should accept (or reject) each other, then $v_{i^*} + w_{j^*} \geq g_{i^*j^*}$ (resp. $v_{i^*} + w_{j^*} \leq g_{i^*j^*}$).

Proof. First, we show that the constraints of the problem imply that $N > 0$, and $\beta_i \geq \beta_{i+1}$ for all i, and $\sigma_j \geq \sigma_{j+1}$ for all j. To see this, observe that $F^b(\beta_{|I|}) = 0$ and $F^b(\beta_0) = 1$, and so there exists a skill i such that $F(\beta_i) > F(\beta_{i+1})$. By constraint $flow_i^b$, it must be that $N x_i \sum_{j \in J} y_j m_{ij} > 0$. Since x_i, y_j, m_{ij} are all non-negative, it follows that $N > 0$. Thus, the outflow of every skill is non-negative, and from the flow conditions, it must be that $\beta_i \geq \beta_{i+1}$ for all i, and likewise $\sigma_j \geq \sigma_{j+1}$ for all j.

(i) Existence: To demonstrate existence, since the objective is continuous, all we need to show is that the policy space is compact. First, there is a uniform upper bound $\overline{N}$ so that in any optimum, $N \leq \overline{N}$ (recall that $N \geq 0$). For the upper bound, notice that the Inflow=Outflow constraints imply $\sum_{i \in I} \sum_{j \in J} N x_i y_j m_{ij} = 1$, and therefore the first term in the welfare expression is a convex combination of g_{ij} and therefore is uniformly bounded by $\max g_{ij}$. Thus, $\lim_{N \to \infty} \mathcal{W} = -\infty$ and so the optimal policy cannot involve arbitrarily large N. The planner can choose quantiles $F(\beta_i)$ instead of thresholds β_i, and since the objective is also continuous in the quantiles and the quantile space is bounded, a maximum indeed exists.

(ii) Decentralizing optimal allocations: The dual problem is

$$
\begin{aligned}
\mathcal{L}\left(\langle z, M, (\beta_i), (\sigma_j)\rangle\right) = & \sum_{i\in I}\sum_{j\in J} N x_i y_j m_{ij} g_{ij} - 2Nc \\
& - \sum_{i\in I}\int_{B_i} C^b(i,\beta) f^b(\beta) d\beta - \sum_{j\in J}\int_{S_j} C^s(j,\sigma) f^s(\sigma) d\sigma \\
& + \sum_{i\in I} v_i \cdot flow_i^b + \sum_{j\in J} w_j \cdot flow_j^s + \sum_i \phi_i x_i + \sum_j \psi_j y_j + \gamma X + \lambda Y \\
& + \sum_{i\in I}\sum_{j\in J} \left(\eta_{ij} m_{ij} + \hat{\eta}_{ij}(1 - m_{ij})\right)
\end{aligned}
$$

We will first show that a constraint qualification holds and then construct an equilibrium using the shadow values from the KKT conditions.

1) The Constraint Qualifications: Since the problem is not convex, we use the constant rank regularity condition, which requires that for each subset of the gradients of the active inequality constraints and the equality constraints, the rank in the vicinity of the optimal point is constant (Janin 1984). The formal proof is given in Lemma 3 in the Appendix.

2) Deriving values from the KKT conditions: Due to the constraint qualification above, the first order conditions (FOC) of the dual problem $\mathcal{L}$ are necessary at any optimum:

FOC(N): $$\sum_{i\in I}\sum_{j\in J} x_i y_j m_{ij} g_{ij} - 2c - \sum_{i\in I} v_i \left(x_i \sum_{j\in J} y_j m_{ij}\right) - \sum_{j\in J} w_j \left(y_j \sum_{i\in I} x_i m_{ij}\right) = 0$$

$$\iff \sum_i \sum_j x_i y_j m_{ij}(g_{ij} - v_i - w_j) = 2c$$

FOC(x_i): $$N\sum_j y_j m_{ij} g_{ij} - v_i N \sum_j y_j m_{ij} - N \sum_j w_j m_{ij} y_j - \gamma + \phi_i = 0$$

$$\iff N\sum_j y_j m_{ij}\left(g_{ij} - v_i - w_j\right) = \gamma - \phi_i$$

FOC(y_j): $$N\sum_i x_i m_{ij} g_{ij} - N\sum_i v_i x_i m_{ij} - w_j N \sum_i m_{ij} x_i - \lambda + \psi_j = 0$$

$$\iff N\sum_i x_i m_{ij}\left(g_{ij} - v_i - w_j\right) = \lambda - \psi_j$$

Complementary slackness: $\phi_i x_i = 0$ and $y_j \psi_j = 0$ and $\phi_i, \psi_j \geq 0$.

FOC(m_{ij}): $$Nx_iy_jg_{ij} - v_iNx_iy_j - w_jNx_iy_j + \eta_{ij} - \hat{\eta}_{ij} = 0$$
$$\iff Nx_iy_j(g_{ij} - v_i - w_j) = -\eta_{ij} + \hat{\eta}_{ij}$$

Complementary slackness: $\eta_{ij}m_{ij} = 0$ and $\hat{\eta}_{ij}(1 - m_{ij}) = 0$ and $\eta_{ij}, \hat{\eta}_{ij} \geq 0$.

FOC(β_i): $$f^b(\beta_i)(v_i - v_{i-1}) = f^b(\beta_i)\left(C(i, \beta_i) - C(i-1, \beta_i)\right), \text{ for } i \in \{1, \dots, I-1\}$$

FOC(σ_j): $$f^s(\sigma_j)(w_j - w_{j-1}) = f^s(\sigma_j)\left(C(j, \sigma_j) - C(j-1, \sigma_j)\right), \text{ for } j \in \{1, \dots, J-1\}$$

We now show that the shadow values v_i, w_j, together with the matching matrix M and state z, constitute an equilibrium.

Decentralizing the constrained optimal allocation when z is interior (ii): To verify the Constant Surplus equations, notice that:

$$N \cdot 2c = N\sum_I\sum_J x_iy_jm_{ij}(g_{ij} - v_i - w_j) = \sum_I x_iN\sum_J y_jm_{ij}(g_{ij} - v_i - w_j)$$
$$= \sum_I x_i(\gamma + \phi_i) = \sum_I \gamma x_i + \phi_ix_i = \sum_I \gamma x_i = \gamma$$

The first line uses FOC(N), while the second line uses FOC(x_i), complementary slackness ($\phi_ix_i = 0$), and the condition $\sum_I x_i = 1$. Therefore $\gamma = 2cN$. Since z is interior, $\phi_i = 0$, and the FOC(x_i) is $\sum_J y_jm_{ij}\left(g_{ij} - v_i - w_j\right) = 2c$, which is the Constant Surplus equation for skill i. An analogous argument holds for the sellers.
To verify the Efficient Matching conditions, notice that if $g_{ij} - v_i - w_j > 0$, the FOC for m_{ij} requires that $\hat{\eta}_{ij} > 0$ and therefore $m_{ij} = 1$. Similarly, if $g_{ij} - v_i - w_j < 0$, the FOC for m_{ij} requires that $\eta_{ij} > 0$ and therefore $m_{ij} = 0$.

To verify that the investments are incentive compatible, we show that for any type $\beta \in [\beta_{i+1}, \beta_i]$, their most preferred skill is i. To see this, for any lower skill, $i' \leq i$, the FOC for the threshold $\beta_{i'}$ is $f(\beta_{i'})(v_{i'} - v_{i'-1}) = f(\beta_{i'})(C(i', \beta_{i'}) - C(i'-1, \beta_{i'}))$ and recall that $\beta_{i'} \geq \beta$. Since $f > 0$ everywhere, this can be simplified to $v_{i'} - C(i', \beta_{i'}) = v_{i'-1} - C(i'-1, \beta_{i'})$. Since type $\beta_{i'}$ is indifferent between the skills i' and $i'-1$, by single-crossing, type β weakly prefers skill i' to skill $i'-1$. Thus, type β weakly prefers i to any lower skill i'. An analogous argument applies for higher skills.

The case of a non-interior z can be found in the Appendix. □

The following are immediate consequences:

Corollary 1. *An equilibrium exists.*

Proposition 1. *The welfare function $\mathcal{W}$ is continuous, strictly decreasing, and convex in c. Moreover, the population size N is weakly decreasing in c.*

The proof is in the Appendix. It relies on the observation that $\partial\mathcal{W}/\partial c = -2N$, which follows immediately from the envelope theorem, implying that a shock to c has greater impact on welfare when c is small than when c is large.

Remark 7. In the Appendix, we construct the matching and values for unchosen skills. If the planner's solution does not use a certain skill, then a simple approach would be to set the values of that skill to $-\infty$ so that no agent would choose it. However, we cannot take this approach as we require that the equilibrium conditions (the Constant Surplus equations and Efficient Matching conditions) apply for all skills, including unchosen ones. That is, the value of an unchosen skill is what an agent would receive in equilibrium if he were to invest in such a skill. Theorem 1 proves that any optimum can be decentralized except that the planner can match unchosen skills in any fashion because they have no impact on welfare, whereas an equilibrium requires that unchosen types still satisfy the Efficient Matching conditions.

4.1 Outside Options and Endogenous Entry

We now turn to the case where agents have outside options. Suppose that every new-born agent can either invest and enter the market or opt out and receive the outside payoff equal to u^b for buyers and u^s for sellers. In equilibrium, buyer β enters the market if and only if $\max_i v_i - C(i,\beta) \geq u^b$, and seller σ enters if and only if $\max w_j - C(j,\sigma) \geq u^s$. We focus on the interesting case where there are gains to trade, and so for at least two types, β and σ, $\max_{i\in I, j\in J} g_{ij} - 2c - C(i,\beta) - C(j,\sigma) > u^b + u^s$. The only difference from the baseline model is that the planner now also chooses the entry thresholds β_0 and σ_0 to maximize:

$$\mathcal{W} = N\sum_{i\in I}\sum_{j\in J} x_i y_j m_{ij} g_{ij} - 2Nc - \sum_{i\in I}\int_{\beta_{i+1}}^{\beta_i} C(i,\beta) f^b(\beta)d\beta - \sum_{j\in J}\int_{\sigma_{j+1}}^{\sigma_j} C(j,\sigma) f^s(\sigma)d\sigma$$
$$+ \int_{\beta_0}^{\infty} u^b f^b(\beta)d\beta + \int_{\sigma_0}^{\infty} u^s f^s(\sigma)d\sigma$$

and the boundary conditions $F^b(\beta_0) = 1$ and $F^s(\sigma_0) = 1$ are removed.

Corollary 2. *In a model with outside options, the constrained efficient outcome is an equilibrium.*

The proof shows that the shadow values still constitute an equilibrium (see Appendix). As before, v_0 is the shadow value of the skill 0 flow constraint. However, there is an additional first-order condition since β_0 is now endogenous: $v_0 - C(0, \beta_0) = u^b$ which is precisely the equilibrium entry condition for buyers. An analogous argument holds for sellers.

Remark 8. In the baseline model, there is exactly one degree of freedom in the equilibrium values (see Remark 5). In the model with outside options, there is an additional entry condition and thus the values are unique (assuming that at least one agent take the outside option).

5 Equilibrium Sorting and Uniqueness

In this section, we show that the equilibria have a clear and simple structure: Section 5.1 shows that if the production function is super/submodular, then every equilibrium exhibits assortative matching. Section 5.2 shows that when production is additively separable (product market), the equilibrium is unique. While in some cases the model may admit multiple equilibria, these results and the previous examples illustrate that the efficient allocation is not caught in a widely cast net.

5.1 Assortative Matching

Denote the matching set of skill-i buyers by $M_i = \{j : m_{ij} > 0\} \subseteq J$, this is the set of seller skills with whom buyer i matches. Similarly, for sellers, $M_j = \{i : m_{ij} > 0\} \subseteq I$. The maxima and minima of these sets are denoted $\overline{m}_i = \max M_i$, $\underline{m}_i = \min M_i$, $\overline{m}_j = \max M_j$ and $\underline{m}_j = \min M_j$. We say that a buyer's matching set M_i is convex if $\underline{m}_i < j < \overline{m}_i$ implies that $m_{ij} = 1$ (this is stronger than stating that the matching sets are intervals because it requires that only boundary types can match probabilistically). Convexity is defined analogously for sellers. A matching matrix M exhibits *positive assortative matching* (PAM) if the matching sets are convex and the maxima/minima are weakly increasing. Likewise, M exhibits *negative assortative matching* (NAM) if the matching sets are convex and the maxima/minima are weakly decreasing. Finally, we say that *All Skills Match* if $m_{ij} = 1$ for all i, j.[14]

m_{ij}	j_1	j_2	j_3	j_4	j_5
i_1	[blue]	[green]			
i_2	[blue]	[blue]	[blue]	[blue]	
i_3		[green]	[blue]	[blue]	
i_4			[green]	[blue]	[blue]
i_5					[blue]

Table 2: A PAM matrix: $m_{ij} = 1$ (blue), $0 < m_{ij} < 1$ (green), and $m_{ij} = 0$ (blank)

In Table 2, we depict a matching matrix that satisfies PAM. To maintain PAM, this matrix cannot be modified so that buyer 1 matches with seller 3 (pure or mixed) because that would violate the convexity condition for buyer 1. Likewise, it cannot be that buyer 2 matches with seller 5 because that would violate monotonicity.

The production function G is *supermodular (or submodular)* if the marginal productivity of every skill i, $g_{(i+1)j} - g_{ij}$, is strictly increasing (resp. decreasing) in j, and the marginal productivity of every skill j, $g_{i(j+1)} - g_{ij}$, is strictly increasing (resp. decreasing) in i; G is *separable* if the marginal productivity of every skill is constant.

Theorem 2. *In equilibrium, there is PAM whenever G is supermodular, NAM whenever G is submodular, and All Skills Match whenever G is separable.*

Outline of the proof: to establish monotonicity, we use the observation that the surplus function s_{ij} inherits super/submodularity from G. We prove convexity from algebraic manipulations of the Constant Surplus equations.

Proof. Throughout, we will use the following fact: if G is supermodular, then so are the surpluses $[s_{ij}]$.

[14] Note that we are using the standard definition of PAM and NAM in random search models, which is based on matching sets. In particular, if there is PAM or NAM according to our definition, then the matching sets M_i, M_j satisfy the lattice condition in Shimer and Smith (2000).

Increasing Upper Bounds: Fix two buyer skills $i_2 > i_1$. Suppose that $\overline{m}_{i_2} < \overline{m}_{i_1}$. Define $k = \overline{m}_{i_1}$. By Efficient Matching, it must be that $s_{i_1 k} \geq 0 \geq s_{i_2 k}$. By supermodularity of s, it holds that $s_{i_1 j} > s_{i_2 j}$ for every $j < k$ and thus $s_{i_1 j} > s_{i_2 j}$ for every $j \in M_{i_2}$. This violates the Constant Surplus equations because

$$2c = \sum_{j \in J} y_j m_{i_2 j} s_{i_2 j} = \sum_{j \in M_{i_2}} y_j s_{i_2 j} < \sum_{j \in M_{i_2}} y_j s_{i_1 j} \leq \sum_{j \in M_{i_1}} y_j s_{i_1 j} = \sum_{j \in J} y_j m_{i_1 j} s_{i_1 j} = 2c$$

Demonstrating increasing lower bounds is similar. Demonstrating decreasing bounds for the submodular G case is analogous.

Convexity: Suppose not. That is, there is a buyer i and sellers $j_1 < j < j_2$ such that $m_{ij} < 1$, and $m_{ij_1}, m_{ij_2} > 0$. Then, it must be the case that seller j has a strictly positive surplus with a lower buyer and that buyer is present with non-zero measure. Otherwise

$$2c = \sum_{i' > i} x_i s^+_{i'j} < \sum_{i' > i} x_i s^+_{i'j_2} \leq 2c$$

with the inequality being due to the fact that $s_{i'j_2} \geq s_{ij} + s_{i'j_2} > s_{ij_2} + s_{i'j} \geq s_{ij_2}$ for every $i' > i$ due to the supermodularity of s. Therefore, there is some $i' < i$ such that $x_{i'} > 0$ and $s_{i'j} > 0$.

An analogous argument demonstrates that there is:

1. A higher buyer $i' > i$ such that $x_{i'} > 0$ and $s_{i'j} > 0$.
2. A lower seller $j' < j$ such that $y_{j'} > 0$ and $s_{ij'} > 0$.
3. A higher seller $j' > j$ such that $y_{j'} > 0$ and $s_{ij'} > 0$.

Let $\underline{j} = arg \max_{j' \leq j} s_{ij'}$ and likewise $\overline{j} = arg \max_{j' \geq j} s_{ij'}$. Similarly, let $\underline{i} = arg \max_{i' \leq i} s_{i'j}$ and likewise $\overline{i} = arg \max_{i' \geq i} s_{i'j}$. See below for an illustration of the matching matrix.

	$\dots$	$\underline{j}$	$\dots$	j	$\dots$	$\overline{j}$	$\dots$
$\dots$				0			
$\underline{i}$				1			
$\dots$							
i	0	1		$m_{ij} < 1$		1	0
$\dots$							
$\overline{i}$				1			
$\dots$				0			

Define $y = y_j$, $\underline{y} = \sum_{j'<j} y_{j'}$ and $\overline{y} = \sum_{j'>j} y_{j'}$. Similarly, $x = x_i$, $\underline{x} = \sum_{i'<i} x_{i'}$, and $\overline{x} = \sum_{i'>i} x_{i'}$. Notice that $\overline{x}, \underline{x}, \overline{y}, \underline{y} > 0$ as shown above.

By the supermodularity of s, for any $i' > i$, it is the case that $s_{i'\overline{j}} + s_{ij} > s_{i'j} + s_{i\overline{j}}$ and since $s_{ij} \leq 0$, it follows that $s_{i'\overline{j}} > s_{i'j} + s_{i\overline{j}}$. Thus,

$$2c \geq \sum_{i' \geq i} x_{i'} s_{i'\overline{j}} > \sum_{i' \geq i} x_{i'}(s_{i'j} + s_{i\overline{j}}) = \left(\sum_{i' \geq i} x_{i'} s_{i'j} \right) + (x + \overline{x}) s_{i\overline{j}} \tag{15}$$

The strict inequality use the fact that $x_{i'} > 0$ for some $i' > i$. Next, notice that $s_{\underline{i}j} \geq s_{i'j}$ for all $i' < i$. Therefore,

$$\underline{x} s_{\underline{i}j} = \sum_{i'<i} x_{i'} s_{\underline{i}j} \geq \sum_{i'<i} x_{i'} s_{i'j} \tag{16}$$

Adding equations (15) and (16) gives

$$2c + \underline{x} s_{\underline{i}j} > \sum_{i'} x_{i'} s_{i'j} + (x + \overline{x}) s_{i\overline{j}}$$

And therefore,

$$\underline{x} s_{\underline{i}j} > (x + \overline{x}) s_{i\overline{j}} \tag{17}$$

Similarly, it can be observed that:

$$s_{\overline{i}j'} > s_{ij'} + s_{\overline{i}j} \text{ for all } j > j'$$
$$s_{\underline{i}j'} > s_{ij'} + s_{\underline{i}j} \text{ for all } j' > j$$
$$s_{i'\underline{j}} > s_{i'j} + s_{i\underline{j}} \text{ for all } j' < j$$

Repeating the same arguments:

$$\overline{y} s_{i\underline{j}} > (\underline{y} + y) s_{\overline{i}j} \tag{18}$$
$$\overline{y} s_{i\overline{j}} > (y + \underline{y}) s_{\underline{i}j} \tag{19}$$
$$\overline{x} s_{\overline{i}j} > (\underline{x} + x) s_{i\underline{j}} \tag{20}$$

As shown earlier, all of the surpluses, $s_{\underline{i}j}, s_{\overline{i}j}, s_{i\underline{j}}, s_{i\overline{j}}$ are positive. Taking the product of Inequalities (17)–(20) and dividing by the surpluses yields:

$$\underline{x}\overline{x}\underline{y}\overline{y} > (\underline{x} + x)(\overline{x} + x)(\underline{y} + y)(\overline{y} + y)$$

which is a contradiction due to the strict inequality.

Separability Implies All Skills Match: By Lemma 2, it is the case that for any two sellers, $w_{j'} - w_j = g_{j'} - g_j$. Therefore, the surplus function is constant $s_{ij'} = g_i + g_{j'} - v_i - w_{j'} = g_i + g_j - v_i - w_j$ and by the Constant Surplus equations, it must be that $s_{ij} = 2c$ for all i, j. So, every pair of agents accept their match. □

Remark 9. Previous work established sufficient conditions for positive/negative assortative matching for a single-population model (Shimer and Smith 2000, Atakan 2006). However, these results do not apply to settings with two different populations, such as labor or product markets.[15] To our knowledge, our paper is the first to prove these results in a two-population model. While it may be intuitive to expect that sorting holds in a two-population model with super/submodular production, the extension is not trivial as the proofs in those papers rely heavily on symmetry of the matching sets. In contrast, we prove convexity from the Constant Surplus equations.[16] Of course, the applied literature has studied sorting in two-population models by using a weaker definition of assortative matching (based on the frequency at which different matches are observed).

Remark 10. The assortative matching result is useful for numerical analysis. For example, in the 5×5 case depicted in Table 2, there are $2^{25} \approx 33.6$ million pure matching matrices, but only 2,762 of them satisfy PAM. In the 5×7 case, there are $2^{35} \approx 34$ trillion pure matching matrices, of which only 21,659 satisfy PAM.[17]

5.2 Uniqueness: Separable Production

We now demonstrate that when the production function is separable, i.e. $g_{ij} = g_i + g_j$, there is a unique equilibrium. To relate to previous work, we phrase this subsection in the language of a product market, as in Rubinstein and Wolinsky (1985) and Gale (1987). Each seller can produce exactly one unit of a homogeneous good and each buyer desires a single unit. A buyer that invests in skill i receives the payoff α_i from consuming the good and a seller that invests in skill j can produce the good at a cost κ_j. The consumption value α_i is increasing in i and the cost κ_j is decreasing in j. When a buyer and seller meet, their output is $g_{ij} = \alpha_i - \kappa_j$. This production function is separable because marginal productivity $g_{i'j} - g_{ij}$ is independent of j. There is endogenous entry, with outside options u^b for buyers and u^s for sellers. To focus

[15] Notice that even if two populations are ex-ante symmetric, their investments may be asymmetric and hence the equilibrium will not be symmetric (see Example 2).

[16] In the model with time discounting, to show that the matching sets are convex, Shimer and Smith (2000) place further restriction on the production function which imply that the surplus function s_{ij} is quasi-concave whereas our proof works without further restrictions. In fact, there are examples where G is supermodular and s_{ij} is not convex, and yet there is PAM.

[17] At 1000 calculations per second, this is the difference between a program taking a millennium and 21 seconds.

on the interesting case, we ignore the trivial equilibrium where no agent enters, and we assume that there are gains to trade, and so for at least two types, β and σ, $\max_{i\in I, j\in J} g_{ij} - 2c - C(i,\beta) - C(j,\sigma) > u^b + u^s$ and that not all agents enter, so there are at least two types for which the opposite inequality holds.

Proposition 2. *Any economy with a separable production function (with or without outside options) has a* ***unique*** *equilibrium and its allocation achieves the first best.*

Theorem 2 demonstrates that with a separable production function, in any equilibrium, All Skills Match. The rest of the proof follows: since All Skills Match, the marginal values equal marginal productivities (Lemma 2), and by separability, $\Delta v_i = g_i$ and $\Delta w_j = g_j$. Thus, the flows and stocks are uniquely pinned down. Moreover, the surpluses s_{ij} are constant, and so **a law of one price** prevails (all trades occur at one price) and endogenous entry uniquely pins down the price that equates supply and demand. Finally, the agents' private incentives to invest are exactly aligned with the planner, so the equilibrium achieves the first-best. The proof is in the Appendix.

6 Robustness

In this section, we extend the baseline model in several directions: Section 6.1 considers asymmetric search costs and bargaining weights, Section 6.2 considers other CRS meeting functions, and Section 6.3 adds time discounting. We will show that our main results regarding efficient investment, efficient matching, sorting, and existence are robust to modifying the bargaining weights, search costs, and meeting function (satisfying CRS). However, efficiency fails when agents discount time and we provide examples to illustrate why.

6.1 Asymmetric Search Costs and Bargaining Weights

We extend the baseline model by allowing asymmetric search costs and bargaining weights. In every period, each buyer incurs the search cost $c^b > 0$ and each seller incurs the search cost $c^s > 0$. When a buyer with skill i and a seller with skill j accept each other, the buyer receives $v_i + \alpha s_{ij}$ and the seller receives $w_j + (1-\alpha)s_{ij}$.

In the baseline model, $c^b = c^s = c$ and $\alpha = 1 - \alpha = 1/2$, and Lemma 1 established that in any equilibrium, the number of buyers B equals the number of sellers S. When the bargaining weights or search costs are asymmetric, the equilibrium state can be unbalanced, $B \neq S$. Recall that in an unbalanced market, the long side of the market is rationed, e.g., if $B < S$, then in each period, every buyer meets a seller with probability 1 and every seller meets a buyer with probability B/S (and vice-versa if $B > S$).

Theorem 3. *For every economy $\langle F^b, F^s, I, J, C^b, C^s, G, c^b, c^s, \alpha \rangle$, let $r \equiv \frac{\alpha}{1-\alpha}\frac{c^s}{c^b}$:*

1. *Every equilibrium has the same balance ratio $\frac{B}{S} = r$.*
2. *Given the balance ratio r, the constrained efficient investments, matching, and steady state are an equilibrium outcome. That is, let $\langle z, M, (\beta_i), (\sigma_j) \rangle$ maximize total welfare under the previous constraints (6)-(14) and the additional constraint $\frac{B}{S} = r$. There are values (v_i^*), (w_j^*), and a matching matrix M^* such that $\langle z, M^*, (v_i^*), (w_j^*) \rangle$ is an equilibrium, where $m_{ij}^* = m_{ij}$ for all i, j such that x_i, $y_j > 0$.*

Proof. Define $\mu = min(B, S)$. In equilibrium, the values satisfy:

$$v_i = (\mu/B)\left(\sum_{j\in J} y_j\left[m_{ij}\left(v_i + \alpha s_{ij}\right) + (1-m_{ij})v_i\right]\right) + \left(1-\mu/B\right)v_i - c^b, \forall i$$

$$w_j = (\mu/S)\left(\sum_{i\in I} x_i\left[m_{ij}\left(w_j + (1-\alpha)s_{ij}\right) + (1-m_{ij})w_j\right]\right) + (1-\mu/S)w_j - c^s, \forall j$$

Rewriting, we obtain the modified Constant Surplus equations:

$$\sum_{j\in J} y_j m_{ij} s_{ij} = \frac{c^b}{\alpha\left(\mu/B\right)}, \forall i \tag{21}$$

$$\sum_{i\in I} x_i m_{ij} s_{ij} = \frac{c^s}{\left(1-\alpha\right)\left(\mu/S\right)}, \forall j$$

$$\Rightarrow \frac{c^b}{\alpha\left(\mu/B\right)} = \sum_{i\in I} x_i \sum_{j\in J} y_j m_{ij} s_{ij} = \sum_{j\in J} y_j \sum_{i\in I} x_i m_{ij} s_{ij} = \frac{c^s}{\left(1-\alpha\right)\left(\mu/S\right)}$$

$$\Rightarrow \frac{B}{S} = \frac{\alpha}{1-\alpha}\cdot\frac{c^s}{c^b} \tag{22}$$

The rest of the proof follows a similar argument as the proof of Theorem 1 and is given in the Appendix. □

Therefore, the search costs and bargaining weights uniquely pin down the balance ratio $r = \frac{c^s}{c^b}\frac{\alpha}{1-\alpha}$. The market is balanced $B = S$ iff the bargaining weight equals the search cost ratio $\alpha = \frac{c^b}{c^b+c^s}$. Under any other bargaining weight, the market is imbalanced which is inefficient because one side of the market is rationed. However, imbalance is the only inefficiency: given the balance ratio, there are values that decentralize the efficient investment decisions, search decisions, and steady state as an equilibrium.

Existence and sorting hold as is.

Corollary 3. *An equilibrium exists.*

Theorem 4. *For any economy $\langle F^b, F^s, I, J, C^b, C^s, G, c^b, c^s, \alpha\rangle$, in equilibrium, there is PAM whenever G is supermodular, NAM whenever G is submodular, and All Skills Match whenever G is separable.*

The proof of Theorem 4 is essentially the same as the proof of Theorem 2 using the modified constant surplus equations. Finally, the next Proposition states that any economy with asymmetric search costs and bargaining weights has an equivalent economy with symmetric bargaining weights and search costs.

Proposition 3. *Given the economy $\mathcal{E}^{asym}$ with asymmetric search costs c^b and c^s and bargaining weight α, let $\mathcal{E}^{sym}$ denote the same economy only with a symmetric bargaining weight and symmetric search costs $c = \max\{\frac{c^b}{2\alpha}, \frac{c^s}{2(1-\alpha)}\}$. These two economies have the same equilibrium allocations and welfare.*

Proof. Notice that the constant surplus equations of both economies are the same. □

The readers might find the results in this section surprising. For instance, starting with symmetric costs and bargaining weights, one may expect that tilting the bargaining weight to favor one side would cause multiple inefficiencies: the incentives to invest may be too strong for one side and too weak for the other side; and mismatches may also occur since the equilibrium values change. Theorem 3 demonstrates that the only inefficiency is imbalance. Specifically, in the new equilibrium, the balance ratio will adjust $B/S = \alpha c^s/(1-\alpha)c^b$, and the steady state skill distributions and values also change. However, given the new balance ratio, the new equilibrium values perfectly align the agents' private incentives to invest and to search with the social benefit.

Since an imbalanced market is inefficient, a natural question is whether government subsidies can improve efficiency and if so how? First, search cost subsidies can be overall beneficial, in contrast to the balanced market case where the cost of such subsidies always outweighs the social benefits. Regarding how, subsidizing the search cost of the short side of the market can improve the welfare of all agents, while subsidizing the long side improves neither. In particular, counter-intuitively, if buyers are less numerous than sellers, then not only would buyers prefer to receive any search cost subsidy, but sellers would also prefer that any search cost subsidies be directed at the buyers. Finally, any investment cost subsidies reduce overall welfare.

6.2 Meeting Function

Finally, we consider a general meeting function where $\mu(B,S)$ is the total number of meetings in a period. In every period, each agent can meet at most one other agent, and so $\mu(B,S) \le \min\{B,S\}$. Meetings are still random and the probability that a buyer meets a seller is $\mu(B,S)/B$, while the probability that a seller meets a buyer is $\mu(B,S)/S$. As is standard, we take μ to be homogeneous of degree 1.

Corollary 4. *For every economy* $\langle F^b, F^s, I, J, C^b, C^s, G, c^b, c^s, \alpha, \mu\rangle$, *let* $r \equiv \frac{\alpha}{1-\alpha}\frac{c^s}{c^b}$

1. *Every equilibrium has the same balance ratio* $\frac{B}{S} = r$.
2. *Given the balance ratio* r, *the constrained efficient investments, matching, and steady state are an equilibrium outcome. That is, let* $\langle z, M, (\beta_i), (\sigma_j)\rangle$ *maximize total welfare under the previous constraints (6)-(14) and the additional constraint* $\frac{B}{S} = r$. *There are values* (v_i^*), (w_j^*), *and a matching matrix* M^* *such that* $\langle z, M^*, (v_i^*), (w_j^*)\rangle$ *is an equilibrium, where* $m_{ij}^* = m_{ij}$ *for all* i, j *such that* x_i, $y_j > 0$.

The proof closely follows that of Theorem 3 (see Appendix). This result clarifies the relationship between our work and Hosios' condition. As mentioned, Hosios (1990) considers a model with homogeneous agents and shows that the right bargaining weight achieves the efficient balance ratio in the market. An analogous condition applies in our model: the equilibrium balance ratio is the efficient one whenever the bargaining weight of each side equals their share of the overall search costs (at the constrained efficient state B, S):[18]

$$\alpha = \frac{Bc^b}{Bc^b + Sc^s} = \frac{\partial\mu(B,S)/\partial B}{\mu(B,S)/B} \tag{23}$$

Therefore, our result and Hosios' result are different and complementary. Hosios' result

[18] Assuming that μ is differentiable, the second equality holds at the optimum and is the familiar Hosios' condition.

is about achieving the efficient balance ratio in the market which requires condition (23). Our result is about decentralizing the efficient investment and matching decisions and it does not depend on the bargaining weights or search costs.

6.3 Discounting

We extend the baseline model to an economy $\mathcal{E}_{\delta,c} = \langle F^b, F^s, I, J, C^b, C^S, G, c, \delta \rangle$ where agents incur an additive search cost c and discount time at the rate $\delta \in [0,1]$. To reduce notation, we return to the case of symmetric search costs and bargaining weights and the standard meeting function. Since the agents' continuation values are now discounted, the match surplus becomes $s_{ij} = g_{ij} - \delta v_i - \delta w_j$. The efficient matching conditions are the same:

$$s_{ij} > 0 \rightarrow m_{ij} = 1; \text{ and } s_{ij} < 0 \rightarrow m_{ij} = 0$$

but the surplus equations are now:

$$v_i = \sum_{j \in J} y_j \left[m_{ij} \left(\delta v_i + \frac{s_{ij}}{2} \right) + (1 - m_{ij}) \, \delta v_i \right] - c$$

$$\Rightarrow \sum_{j \in J} y_j m_{ij} s_{ij} = 2 \left[c + (1-\delta) v_i \right]$$

Notice that agents incur both an explicit search cost of c and an implicit search cost of $(1-\delta)v_i$ because their payoffs are delayed. The implicit search costs are increasing in values and this inefficiently distorts the equilibrium investment and matching decisions. First, acquiring a higher skill entails a higher implicit search cost, which reduces the incentive to invest. Second, high-skill agents have high implicit search costs and may accept too often, while low-skill agents have low implicit search costs and may reject too often (see Shimer and Smith 2001).

To illustrate these inefficiencies, consider a symmetric economy with two skills $I = J = \{0, 1\}$ and a separable production function, $g_{ij} = g_i + g_j$.

Inefficient Matching: Since the production function is separable, for any discount factor and search cost, the efficient matching rule is All Skills Match. However, for a high productivity ratio g_1/g_0, All Skills Match is not supported by an equilibrium. To see why, plugging $m_{ij} = 1$, $\forall i, j$ into the constant surplus equations and differencing, we get the marginal values $\Delta w = \Delta v = \frac{g_1 - g_0}{2 - \delta}$. The equilibrium is therefore symmetric,

$x = F(\Delta v) = F(\Delta w) = y$, and WLOG $v_0 = w_0$, $v_1 = w_1$.[19] Solving these equations, we get: $v_0 = -c + g_0 + \frac{1-\delta}{2-\delta}(g_1 - g_0)x$ and $v_1 = -c + g_1 - \frac{1-\delta}{2-\delta}(g_1 - g_0)(1-x)$. Setting $c \approx 0$, unskilled agents will reject each other whenever:

$$\frac{g_1}{g_0} > 1 + \frac{1}{x}\frac{2-\delta}{\delta} \iff \delta v_0 > g_0 \iff 0 > s_{00}$$

Therefore, when the productivity ration g_1/g_0 is high, all skills match is not an equilibrium. Instead, unskilled agent match only with skilled agents while skilled agents accept each other and unskilled agents. The matching matrix is: $M = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. Intuitively, this mismatch happens because unskilled agents have low implicit search costs and therefore "hunt" for skilled agents from whom they can extract more surplus (because the skilled agents have high implicit search costs).

Non-assortative Matching: The hunting equilibrium is not only inefficient, but it also violates assortative matching: the matching sets are not monotonic because skilled buyers match with unskilled sellers but unskilled buyers do not. Notice that this example is not a knife edge case. For instance, taking $x \geq 0.9$, $\delta \geq 0.9$, and $g_0 = 1$ and $c \approx 0$, then both efficiency and sorting fail whenever $g_1 > 2.36$.

Under-investment: The efficient investment rule is that the agents with marginal costs below the marginal productivity $g_1 - g_0$ should invest. However, in this example, even if All Skills Match is an equilibrium, the agents will under-invest because the marginal value is less than the marginal productivity, $\Delta v = \Delta w = \frac{g_1 - g_0}{2-\delta} < g_1 - g_0$.

However, all is not lost. As $\delta \to 1$, the equilibrium converges to the efficient outcome. More generally, in $\mathcal{E}_{\delta,c}$, for any $c > 0$ and under appropriate conditions, as $\delta \to 1$, the equilibrium converges to our model. Since the proof is involved, we omit this result.

7 Discussion

This paper developed and analyzed a model where heterogeneous agents acquire skills and then engage in costly search to form productive matches in the market. Despite potential hold-up and matching problems, the main result is that the constrained

[19] Notice that given any symmetric All Skills match equilibrium with values $(\hat{v}_0, \hat{v}_1, \hat{w}_0, \hat{w}_1)$, there is an equivalent equilibrium where $v_0 = w_0 = (\hat{v}_0 + \hat{w}_0)/2$ and $v_1 = w_1 = (\hat{v}_1 + \hat{w}_1)/2$.

efficient allocation is an equilibrium of the decentralized market. In addition, regarding the equilibrium structure: we established assortative matching for super/submodular production functions and uniqueness for separable production functions. Importantly, the sorting result applies to two-population models, such as the labor market. Our main results regarding efficient investment, efficient matching, and sorting are robust: they do not depend on the search costs nor the bargaining weights. We mention below several implications and takeaways from our model and results.

Search Externalities: A key tension underlying the efficiency result is that the decisions to acquire skills and to accept or reject potential partners impose externalities on other agents. For instance, if less buyers acquire skill i and more buyers acquire skill $i+1$, then the pool of agents in the market changes: the number of buyers with skill i decreases, the number of buyers with skill $i+1$ increases, and the number of buyers and sellers with other skills may also change because the relative size of their matching partners may increase or decrease. Alternatively, if buyers with skill i accept more partners, then their number in the search pool decreases, and subsequently the number of agents with other skills may increase or decrease since the relative size of their matching partners may change. The planner's solution takes such steady-state search externalities into account. In contrast, in equilibrium, each agent invests and accepts or rejects partners simply by their private incentives, as determined by the value of each skill in the market. In order to achieve the efficient outcome, the equilibrium values must incentivize the agents to internalize these externalities, which is perhaps the most surprising and powerful aspect of the welfare theorem.

Explicit and Implicit Search Costs: The search literature departs from the frictionless matching benchmark in assuming that search is costly and time consuming. *Explicit search costs* reflect a wide range of costs people incur per unit of time as they search (see Remark 1 for examples). On the other hand, when agents discount time, they incur *implicit search costs* as their payoffs are delayed. These implicit search costs depend upon each agent's continuation value, but explicit search costs do not. This difference has significant consequences for both efficiency and sorting. In models with discounting, agents under-invest because acquiring a better skill also entails acquiring a higher implicit search cost. In addition, agents may mismatch and sorting may fail even when production is additively separable. For instance, in the "hunting" example in Section 6.3, low-skill agents have low implicit costs and search too much.[20] An important takeaway from our paper is that inefficiencies are not caused by search frictions *per se* but by discounting. Both explicit and implicit search costs are important in many applications and which is more salient depends upon the economic situation being modeled.[21]

The comparison between explicit and implicit time costs is important in the bargaining literature. Rubinstein's (1982) seminal paper studied a bilateral bargaining game with (i) discounting and (ii) explicit time costs. He found that the model with discounting has a unique SPE that depends smoothly and intuitively on the discount factors, whereas the model with explicit costs has a stark SPE: the player with the smaller cost receives almost the entire pie, and if both players have the same cost, then almost any split of the pie is an SPE. The bargaining literature naturally gravitated towards the discounting model. However, search and bargaining models are another story: our paper demonstrated that the explicit search cost model is simple, tractable, and it delivers sharp results.

Labor, Product, and Marriage Markets: For the labor market, a fundamental question is about sorting – when will high-tech firms match with high-skill workers? Theorem 2 establishes a simple sufficient condition for sorting in two-population models: if the production function is supermodular (or submodular), then the matching set of each skill is a "discrete interval" and higher skills match with higher (resp. lower) intervals. The matching sets still depend on the search cost but they have a simple structure, each set is defined by an upper and lower bound (see Remark 10).

For product markets, match output is typically taken to be the gains from trade, $g_{ij} = v_i - c_j$, which is additively separable. In our framework, buyers may invest to increase their valuations while sellers may invest to reduce their costs. Proposition 2 establishes that there is a unique equilibrium which achieves the first-best allocation.

[20] The same "hunting equilibrium" also occurs with strictly supermodular production functions, for instance, $g_{ij} = (i + j)^n$ when $n > 1$. In models with discounting, to guarantee sorting, supermodular g is not enough, but the log of its first and second derivatives must also be supermodular (see Shimer and Smith 2000). For instance, $g_{ij} = ij$ satisfies these conditions and $g_{ij} = (i + j)^n$ does not.

[21] When search transpires over a short time window and does not affect the consumption date, explicit search costs are important. For instance, think of the time spent today searching online for a product that will be delivered tomorrow or college students applying for jobs which they will take after graduation.

For a symmetric marriage market, Section 3 demonstrated that an occupational gender gap can arise and can even be efficient. This highlighted a basic tradeoff between investment, search, and productivity: asymmetric investments may facilitate search and enhance productivity at the expense of higher investment costs (due to a misallocation of talent). This market outcome is discriminatory as the returns on investment depends on gender. The occupational gender gap can also arise when it is inefficient and in some cases can be corrected by a policy intervention.

Policy Intervention: In our model, inefficiencies may result from two possible sources. First, there can be a coordination failure and a policy intervention may move the economy away from an inefficient equilibrium. For example, an investment subsidy can rectify the previously mentioned occupational gender gap. Second, in the case of asymmetric bargaining weights or search costs, the market can be imbalanced, $B \neq S$. In this case, a small search cost subsidy for the short side of the market is generally net beneficial (for both sides of the market), but a search subsidy targeted at the long side never is.

Applications and Simulations: The welfare and sorting results are useful for computational analyses. In particular, solving the planner's problem solely requires finding an allocation whereas solving for an equilibrium also requires finding values subject to additional constraints. Notice that for an n-skill economy, the endogenous variables $N, (x_i), (y_j), (\beta_i), (\sigma_j)$ are of order n, but the matching matrix $[m_{ij}]$ is of order n^2. The assortative matching result reduces the number of matching variables from n^2 to $2n$, which brings the dimensionality of the whole problem from $O(n^2)$ to $O(n)$. A further advantage of the welfare theorem is that seeing the economy through the planner's lens may provide intuition that is not evident from the equilibrium conditions. Calibration of the model to fit empirical data lies beyond the scope of the current paper, but the theoretical results found here offer promise.

References

Acemoglu, D. (1996). A microfoundation for social increasing returns in human capital accumulation. *The Quarterly Journal of Economics 111*(3), 779–804.

Acemoglu, D. and R. Shimer (1999). Holdups and efficiency with search frictions. *International Economic Review 40*(4), 827–849.

Atakan, A. E. (2006). Assortative matching with explicit search costs. *Econometrica 74*(3), 667–680.

Burdett, K. and M. G. Coles (1999). Long-term partnership formation: Marriage and employment. *The Economic Journal 109*(456), 307–334.

Burdett, K. and M. G. Coles (2001). Transplants and implants: The economics of self-improvement. *International Economic Review 42*(3), 597–616.

Cai, X., P. Gautier, and R. P. Wolthoff (2021). Search, screening and sorting.

Chade, H. and I. Lindenlaub (2022). Risky matching. *The Review of Economic Studies 89*(2), 626–665.

Cole, H. L., G. J. Mailath, and A. Postlewaite (2001). Efficient non-contractible investments in large economies. *Journal of Economic Theory 101*(2), 333 – 373.

Diamond, P. A. (1982). Wage determination and efficiency in search equilibrium. *The Review of Economic Studies 49*(2), 217–227.

Dizdar, D. (2018, August). Two-sided investment and matching with multidimensional cost types and attributes. *American Economic Journal: Microeconomics 10*(3), 86–123.

Eeckhout, J. and P. Kircher (2010). Sorting and decentralized price competition. *Econometrica 78*(2), 539–574.

Gale, D. (1987). Limit theorems for markets with sequential bargaining. *Journal of Economic Theory 43*(1), 20 – 54.

Hosios, A. J. (1990). On the efficiency of matching and related models of search and unemployment. *The Review of Economic Studies 57*(2), 279–298.

Janin, R. (1984). Directional derivative of the marginal function in nonlinear programming. In *Sensitivity, Stability and Parametric Analysis*, pp. 110–126. Springer.

Jerez, B. (2017). Competitive search equilibrium with multidimensional heterogeneity and two-sided ex-ante investments. *Journal of Economic Theory 172*, 202 – 219.

Lauermann, S. (2013). Dynamic matching and bargaining games: A general approach. *American Economic Review 103*(2), 663–89.

Lauermann, S., G. Nöldeke, and T. Tröger (2020). The balance condition in search-and-matching models. *Econometrica 88*(2), 595–618.

Mailath, G. J., A. Postlewaite, and L. Samuelson (2013). Pricing and investments in matching markets. *Theoretical Economics 8*(2), 535–590.

Manea, M. (2017). Steady states in matching and bargaining. *Journal of Economic Theory 167*, 206 – 228.

Masters, A. M. (1998). Efficiency of investment in human and physical capital in a model of bilateral search and bargaining. *International Economic Review 39*(2), 477–494.

Mortensen, D. T. (1982a). The matching process as a noncooperative bargaining game. In *The Economics of Information and Uncertainty*, pp. 233–258. University of Chicago Press.

Mortensen, D. T. (1982b). Property rights and efficiency in mating, racing, and related games. *The American Economic Review 72*(5), 968–979.

Mortensen, D. T. and R. Wright (2002). Competitive pricing and efficiency in search equilibrium. *International Economic Review 43*(1), 1–20.

Noldeke, G. and L. Samuelson (2015). Investment and competitive matching. *Econometrica 83*(3), 835–896.

Pissarides, C. A. (1985). Short-run equilibrium dynamics of unemployment, vacancies, and real wages. *The American Economic Review 75*(4), 676–690.

Rubinstein, A. and A. Wolinsky (1985). Equilibrium in a market with sequential bargaining. *Econometrica: Journal of the Econometric Society*, 1133–1150.

Shapley, L. S. and M. Shubik (1971). The assignment game i: The core. *International Journal of game theory 1*(1), 111–130.

Shi, S. (2001). Frictional assignment i efficiency. *Journal of Economic Theory 98*(2), 232 – 260.

Shimer, R. (2005). The assignment of workers to jobs in an economy with coordination frictions. *Journal of Political Economy 113*(5), 996–1025.

Shimer, R. and L. Smith (2000). Assortative matching and search. *Econometrica 68*(2), 343–369.

Shimer, R. and L. Smith (2001). Matching, search, and heterogeneity. *The BE Journal of Macroeconomics 1*(1).

8 Appendix

Remaining Proofs for Theorem 1:

We first prove the non-interior case and then the constant rank constraint qualification.

Proof. z **is non-interior:**

Given any optimal policy $\langle z, M, (\beta_i), (\sigma_j)\rangle$, the FOCs imply that there are shadow values (v_i), (w_j) such that (see proof of Theorem 1 in text):

$$\sum_j y_j m_{ij} (g_{ij} - v_i - w_j) \geq 2c \text{ with equality when } x_i > 0$$
$$\sum_i x_i m_{ij} (g_{ij} - v_i - w_j) \geq 2c \text{ with equality when } y_j > 0$$
$$N x_i y_j (g_{ij} - v_i - w_j) = -\eta_{ij} + \hat{\eta}_{ij}$$

where $\eta_{ij} m_{ij} = 0$ and $\hat{\eta}_{ij}(1 - m_{ij}) = 0$ and $\eta_{ij}, \hat{\eta}_{ij} \geq 0$.

The above equations demonstrate the Constant Surplus equations for all i where $x_i > 0$. But, the Constant Surplus equation may not hold for skills i where $x_i = 0$. Therefore, for any skill i where $x_i = 0$, we define v_i^* to be the unique value which solves $\sum_j y_j \max\{g_{ij} - v_i^* - w_j, 0\} = 2c$. For any skill i where $x_i > 0$, we define $v_i^* = v_i$. Likewise, for sellers j where $y_j = 0$, define w_j^* to be the unique value which solves $\sum_j y_j \max\left(g_{ij} - v_i - w_j^*, 0\right) = 2c$ $y_j > 0$. For sellers j where $y_j > 0$, define $w_j^* = w_j$. Define a matching matrix by $m_{ij}^* = \mathbf{1}_{g_{ij} - v_i^* - w_j^* > 0}$ whenever $x_i = 0$ or $y_j = 0$ and setting $m_{ij}^* = m_{ij}$ otherwise.

It now remains to be seen that $\langle z, M^*, (v_i^*), (w_j^*)\rangle$ satisfies the equilibrium constraints.

The Constant Surplus Equations hold: For any skill i where $x_i > 0$, from the above, we have that $\sum_j y_j m_{ij}^* \left(g_{ij} - v_i^* - w_j^*\right) = \sum_j y_j m_{ij} (g_{ij} - v_i - w_j) = 2c$ because $v_i^* = v_i$ and whenever $y_j > 0$, then $m_{ij} = m_{ij}^*$ and $w_j = w_j^*$. For any skill i where $x_i = 0$,

$$\begin{aligned}\sum_j y_j m_{ij}^* \left(g_{ij} - v_i^* - w_j^*\right) &= \sum_j y_j \max\left(g_{ij} - v_i^* - w_j^*, 0\right) \\ &= \sum_j y_j \max\left(g_{ij} - v_i^* - w_j, 0\right) = 2c\end{aligned}$$

because $w_j^* = w_j$ whenever $y_j > 0$. The same argument demonstrates the Constant Surplus equations for the sellers.

Efficient Matching holds: For any two skills i, j where $x_i = 0$ or $y_j = 0$, the efficient matching condition holds by definition. For any two skills i, j where $x_i > 0$ and $y_j > 0$, then $v_i^* = v_i$, $w_j^* = w_j$, and $m_{ij}^* = m_{ij}$ and the Efficient Matching condition is a direct consequence of FOC(m_{ij}).

Optimal Investments: Regarding optimal investments, just as in the proof in the main section, here the values (v_i) satisfy incentive compatibility for investments. However, it is not readily evident that the values (v_i^*) satisfy incentive compatibility because the values for unrealized skills are modified, and may be increased. We now show that for all unrealized skills $v_i \geq v_i^*$.

Since $m_{ij}x_iy_j = m_{ij}^*x_iy_j$ for any two skills i, j, the policy $\langle z, M^*, (\beta_i), (\sigma_j)\rangle$ is admissible and optimal. By the constraint qualifications, there are values $(\hat{v}_i)$, $(\hat{w}_j)$ which satisfy the FOCs for $\langle z, M^*, (\beta_i), (\sigma_j)\rangle$. From FOC($\beta_i$), we have that the marginal values are equal for all i, $\hat{v}_i - \hat{v}_{i-1} = C(i, \beta_i) - C(i-1, \beta_i) = v_i - v_{i-1}$. Likewise, for all sellers j, $\hat{w}_j - \hat{w}_{j-1} = w_j - w_{j-1}$. Thus, there is a constant t such that $\hat{v}_i + \hat{w}_j = v_i + w_j + t$ for all i, j. For any skill i such that $x_i > 0$,

$$\begin{aligned} 2c &= \sum_j y_j m_{ij}^*(g_{ij} - \hat{v}_i - \hat{w}_j) = \sum_j y_j m_{ij}^*(g_{ij} - v_i - w_j - t) \\ &= \sum_j y_j m_{ij}(g_{ij} - v_i - w_j - t) = 2c - t \sum_{ij} y_j m_{ij} \end{aligned}$$

Therefore, $t = 0$ and so $\hat{v}_i + \hat{w}_j = v_i + w_j$ for all i, j.

For any unchosen skill i,

$$\sum_j y_j m_{ij}^* \left(g_{ij} - v_i^* - w_j\right) = 2c \geq \sum_j y_j m_{ij}^* \left(g_{ij} - \hat{v}_i - \hat{w}_j\right) = \sum_j y_j m_{ij}^* \left(g_{ij} - v_i - w_j\right)$$

Therefore, we can conclude that $v_i \geq v_i^*$. This demonstrates incentive compatibility. For every skill i, $v_i \geq v_i^*$ with equality if $x_i > 0$. As (v_i) satisfied incentive compatibility and (v_i^*) differs by only lowering the value of unrealized skills, the values (v_i^*) also satisfy incentive compatibility. This establishes that for the values $(v_i^*), (w_j^*)$, no agent wishes to choose any unchosen skill and completes the proof. □

Constraint Qualification

Lemma 3. *The planner's optimization problem satisfies the Constant Rank Constraint Qualification.*

Proof. We show that for each subset of the gradients of the active inequality constraints and the equality constraints, the rank in a vicinity of the optimal point is constant (Janin (1984)).

There is an immediate linear dependency among the gradients:

$$\sum_{i\in I} \alpha \nabla flow_i^b - \sum_{j\in J} \alpha \nabla flow_j^s = 0$$

which follows from

$$\sum_{i\in I} flow_i^b - \sum_{j\in J} flow_j^s = 0$$

We will show that this is the only linear dependency, which suffices for the constant rank constraint qualification. Suppose that $\sum_n \alpha_n \nabla_n = 0$ where the summation is over all the active gradients. To simplify notation, we label the skills as $I = \{0, \dots, k\}$ and $J = \{0, \dots, l\}$. Notice first that (β_i) and (σ_j) appear only in the flow constraints:

∇	β_1	β_2	β_3	$\dots$	β_k	N	$\sigma_j, x_i,$ y_j, m_{ij}
$\nabla flow_0^b$	$-f^b(\beta_1)$	0	0	0	0	$-x_0 \sum_{j\in J} y_j m_{0j}$	$\dots$
$\nabla flow_1^b$	$f^b(\beta_1)$	$-f^b(\beta_2)$	0	0	0	$-x_1 \sum_{j\in J} y_j m_{1j}$	$\dots$
$\nabla flow_2^b$	0	$f^b(\beta_2)$	$-f^b(\beta_3)$	0	0	$-x_2 \sum_{j\in J} y_j m_{2j}$	$\dots$
$\dots$	0	0	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\nabla flow_{k-1}^b$	0	0	0	$f^b(\beta_{k-1})$	$-f^b(\beta_k)$	$-x_{k-1} \sum_{j\in J} y_j m_{k-1,j}$	$\dots$
$\nabla flow_k^b$	0	0	0	0	$f^b(\beta_k)$	$-x_k \sum_{j\in J} y_j m_{k,j}$	$\dots$

Since β_i only shows up in up in $flow_i^b$, $flow_{i-1}^b$ it must be that

$$0 = \sum_n \alpha_n \frac{\partial f_n}{\partial \beta_{i'}} = \sum_{i\in I} \alpha_i \frac{\partial flow_i^b}{\partial \beta_{i'}} = f(\beta_{i'})\alpha_{i'} - f(\beta_{i'})\alpha_{i'+1} \text{ for all } i'$$

Thus, there is an α such that $\alpha_i = \alpha$ for all the coefficients of the constraints $\nabla flow_i^b$. Similarly, there is a χ so that $\alpha_j = \chi$ for all the coefficients of the constraints $\nabla flow_j^s$.

Furthermore, N only shows up in the flow constraints, so it must be that

$$-\alpha \sum_i x_i \sum_j y_j m_{ij} - \chi \sum_j y_j \sum_i x_i m_{ij} = 0$$

which implies $\chi = -\alpha$ (notice that $\sum_i x_i \sum_j y_j m_{ij} = 1/N$). Therefore, there is exactly one linear dependency

$$\sum \alpha_i \nabla flow_i^b + \sum_j \alpha_j \nabla flow_j^s = \alpha \left(\sum_i \nabla flow_i^b - \sum_j \nabla flow_j^s \right) = 0$$

Second, the coefficients on $\nabla(x_i \geq 0)$ and ∇X are all zeros. The reason is that x_i appears in the flow constraints and the constraints $x_i \geq 0$ and $X = 0$. By the previous step, in any linear dependence, the flow constraints cancel each other out, so only the constraints $x_i \geq 0$ and $X = 0$ are relevant. Therefore, if $\sum_i \xi_i \nabla(x_i \geq 0) + \xi \nabla X = 0$, then $0 = \xi_i \frac{\partial x_i}{\partial x_i} + \xi \frac{\partial X}{\partial x_i} = \xi_i - \xi$, and so $\xi_i = \xi$ for all i. If $\xi \neq 0$, then it must be that every inequality on x is active, so $x_i = 0$ for every i, contradicting $0 = X = 1 - \sum_i x_i$, which holds in any admissible tuple. The same argument applies to the y_j. So $\xi_i = \xi = \xi_j = 0$ for all i, j.

Third, the coefficients on the m_{ij} constraints are zeros. The reason is that the variable m_{ij} appears only in the flow equations and the inequality constraints on m_{ij}. The flow constraints cancel each other out. For the m_{ij} constraints, $\nabla(1 \geq m_{ij} \geq 0) = (0, \ldots 0, \pm 1, 0 \ldots)$ and at most one of the m_{ij} constraints can be active where the only non-zero element is in the m_{ij} coordinate and therefore these gradients coefficients must be 0. $\square$

Proof of Proposition 1:

Proof. Consider the economies $\Gamma_c = \langle F^b, F^s, I, J, C^b, C^s, G, c\rangle$ indexed by their search cost c and denote its constrained efficient welfare as $\mathcal{W}_c$. Denote an optimal allocation as x_c with associated population N_c (there may be multiple optimal allocations). Notice that by an imitation argument, $\mathcal{W}_c \geq \mathcal{W}_{c'} + 2N(c')(c' - c)$ because the planner could implement $x_{c'}$ when faced with the economy x_c. This implies that welfare is decreasing in c, as expected. Reversing c and c' gives $2N(c)(c'-c)+\mathcal{W}_{c'} \geq \mathcal{W}_c$. Taking $c' > c$. this implies that $|\mathcal{W}_c - \mathcal{W}_{c'}| \leq 2N(c)(c'-c)$. That is, when $N(c)$ is unique, it is the case that $\frac{\partial \mathcal{W}_c}{\partial c} = -2N(c)$ and otherwise the left-derivative is $\sup -2N(c)$ and the right-derivative is $\inf -2N(c)$. To see convexity of $\mathcal{W}_c$, it suffices to demonstrate that N is increasing in c. Take $c' > c$. Since $\mathcal{W}_c \geq \mathcal{W}_{c'}+2N(c')(c'-c)$, and similarly $\mathcal{W}_{c'} \geq \mathcal{W}_c+2N(c)(c-c')$. Adding these two equations together gives $0 > 2(N(c') - N(c))(c' - c)$ and therefore $N(c) \geq N(c')$. □

Proof of Proposition 2:

Proof. The first-best allocation is unique and satisfies:

First-Best Matching: All pairs match. Since the marginal productivity of an agent is not affected by the skills of her partner, all pairs match to minimize the search cost.

First-Best Investment: Buyer β and seller σ acquire the skills: $i^*(\beta) = \arg\max_i \alpha_i - C^b(i, \beta)$ and $j^*(\sigma) = \arg\max_j -\kappa_j - C^s(j, \sigma)$. Denote by $C^{b*}(\beta) = C^b(i^*(\beta), \beta)$ the investment cost buyer β pays to acquire the efficient skill, and likewise $C^{s*}(\sigma) = C^s(j^*(\sigma), \sigma)$.

The social welfare of a match between buyer β and seller σ is $\omega(\beta, \sigma) = \alpha_{i^*(\beta)} - C^{b*}(\beta) - \kappa_{j^*(\sigma)} - C^{s*}(\sigma) - 2c$. The assumption before the proof implies that there are types, $\beta', \sigma', \hat{\beta}, \hat{\sigma}$ such that $\omega(\beta', \sigma') > u^b + u^s > \omega(\hat{\beta}, \hat{\sigma})$. So, in the first-best, some agents enter and others don't.[22]

First-Best Entry: Buyer β and seller σ enter iff $\beta \leq \beta_0$ and $\sigma \leq \sigma_0$. The entry thresholds are pinned down by[23] $F^b(\beta_0) = F^s(\sigma_0)$ and $\omega(\beta_0, \sigma_0) = u^b + u^s$.

[22] The case where everyone enters is trivial.

[23] Since buyers and sellers exit in equal numbers, in a steady state they must also enter in equal numbers.

Since g is separable, Lemma 2 implies that in equilibrium, the marginal value equal the marginal productivity: $\Delta v_i = \alpha_{i+1} - \alpha_i$, for every i, and $\Delta w_j = -(\kappa_{j+1} - \kappa_j)$, for every j. Therefore, the match surplus $s_{ij} = \alpha_i - \kappa_j - v_i - w_j$ is constant. As a result:

Equilibrium Matching: Theorem 2 demonstrates that in every equilibrium, all skills match.

Equilibrium Investment: The individually optimal investments satisfy

$$\arg\max_i \left\{v_i - C^b(i,\beta)\right\} = \arg\max_i \left\{\alpha_i - C^b(i,\beta)\right\}, \text{ for evey } \beta$$
$$\arg\max_j \left\{w_j - C^s(j,\sigma)\right\} = \arg\max_j \left\{-\kappa_j - C^s(j,\sigma)\right\}, \text{ for every } \sigma$$

The maximizers are equal because $\alpha_i - v_i$ and $-\kappa_j - w_j$ are constant.

Equilibrium Entry: First, we show that there is entry. If not, then $v_{i^*(\beta)} - C^{b*}(\beta) \leq u^b$ and $w_{j^*(\sigma)} - C^{s*}(\sigma) \leq u^s$, for all β, σ, and so $v_{i^*(\beta)} - C^{b*}(\beta) + w_{j^*(\sigma)} - C^{s*}(\sigma) \leq u^b + u^s$. Substituting in the Constant Surplus equations, it follows that, $\alpha_{i^*(\beta)} - C^{b*}(\beta) - \kappa_{j^*(\sigma)} - C^{s*}(\sigma) - 2c \leq u^b + u^s$, which violates the assumption that there are types, β', σ' such that $\omega(\beta', \sigma') > u^b + u^s$. By a similar argument, it cannot be that all agents enter. Second, since some agents enter and others do not, denote by $\underline{\beta}, \underline{\sigma}$ the threshold types for whom the entry constraints hold with equality, notice that

$$\begin{aligned} u^b + u^s &= v_{i^*(\underline{\beta})} - C^{b*}\left(\underline{\beta}\right) + w_{j^*(\underline{\sigma})} - C^{s*}\left(\underline{\sigma}\right) \\ &= \alpha_{i^*(\underline{\beta})} - C^{b*}\left(\underline{\beta}\right) - \kappa_{j^*(\underline{\sigma})} - C^{s*}\left(\underline{\sigma}\right) - 2c = \omega(\underline{\beta}, \underline{\sigma}) \end{aligned}$$

The second equality follows from the Constant Surplus equation, $v_i + w_j = \alpha_i - \kappa_j - 2c$. In a steady state, the same measure of buyers and sellers enter, $F^b(\underline{\beta}) = F^s(\underline{\sigma})$. These two equations are the same as the equations that characterized the first-best entry decisions, and therefore it must be that $\underline{\beta} = \beta_0$ and $\underline{\sigma} = \sigma_0$. □

9 Online Appendix

This Section proves Theorem 3 and Corollaries 2 and 4. We extend the baseline model to the generalized economy $\mathcal{E} = \langle F^b, F^s, I, J, C^b, C^s, G, c^b, c^s, \alpha, \mu, u^b, u^s \rangle$ by adding the following additional features:

- Asymmetric search costs c^b and c^s and bargaining weight α (as in Section 6.1).
- A meeting function $\mu(B, S)$ specifying the total number of meetings in each period and satisfying constant returns to scale (as in Section 6.2).
- Agents have outside options u^b and u^s and so entry is endogenous (as in Section 4.1). To avoid trivial outcomes, we maintain the assumption that there are gains to trade for at least two types, β and σ, so that

$$\max_{i \in I, j \in J} \mu(1,1) g_{ij} - c^b - c^s - C(i, \beta) - C(j, \sigma) > u^b + u^s$$

We will now prove a more general version of the the previous results.

Corollary. *For every generalized economy, let* $r \equiv \frac{\alpha}{1-\alpha} \frac{c^s}{c^b}$*:*

1. *Every equilibrium has the same balance ratio* $\frac{B}{S} = r$*.*
2. *Given the balance ratio* r*, the constrained efficient investments, matching, and steady state are an equilibrium outcome. That is, let* $\langle z, M, (\beta_i), (\sigma_j) \rangle$ *maximize total welfare under the previous constraints (6)-(14) and the additional constraint* $\frac{B}{S} = r$*. There are values* (v_i^*)*,* (w_j^*)*, and a matching matrix* M^* *such that* $\langle z, M^*, (v_i^*), (w_j^*) \rangle$ *is an equilibrium, where* $m_{ij}^* = m_{ij}$ *for all* i, j *such that* x_i*,* $y_j > 0$*.*

Proof. 1) Let $\mu = \mu(B, S)$. As we previously showed in Section 6.1, in equilibrium, the values satisfy:

$$v_i = (\mu/B) \left(\sum_{j \in J} y_j \left[m_{ij} \left(v_i + \alpha s_{ij} \right) + (1 - m_{ij}) v_i \right] \right) + (1 - \mu/B)\, v_i - c^b, \forall i$$

$$w_j = (\mu/S) \left(\sum_{i \in I} x_i \left[m_{ij} \left(w_j + (1 - \alpha) s_{ij} \right) + (1 - m_{ij}) w_j \right] \right) + (1 - \mu/S) w_j - c^s, \forall j$$

Rewriting, we obtain the modified Constant Surplus equations:

$$\sum_{j\in J} y_j m_{ij} s_{ij} = \frac{c^b}{\alpha\left(\mu/B\right)}, \forall i \tag{24}$$
$$\sum_{i\in I} x_i m_{ij} s_{ij} = \frac{c^s}{\left(1-\alpha\right)\left(\mu/S\right)}, \forall j$$

$$\Rightarrow \frac{c^b}{\alpha\left(\mu/B\right)} = \sum_{i\in I} x_i \sum_{j\in J} y_j m_{ij} s_{ij} = \sum_{j\in J} y_j \sum_{i\in I} x_i m_{ij} s_{ij} = \frac{c^s}{\left(1-\alpha\right)\left(\mu/S\right)}$$

$$\Rightarrow \frac{B}{S} = \frac{\alpha}{1-\alpha} \cdot \frac{c^s}{c^b} \tag{25}$$

2) Decentralizing the efficient allocation given r. To simplify, we focus on the case where the state is interior and the proof repeats that argument with the appropriate modifications. The same could be done for the boundary case as well. The original planner's problem 5 is modified because the agents have an outside option and there is a general meeting function, and so the measure of buyers B need not equal the measure of sellers S. The planner now chooses the state $z = (B, S, (x_i), (y_j))$ instead of $z = (N, (x_i), (y_j))$, the investment thresholds, and the matching rule to maximize

$$\mathcal{W} = \mu(B,S) \sum_{i\in I} \sum_{j\in J} x_i y_j m_{ij} g_{ij} - Bc^b - Sc^s - \sum_{i\in I} \int_{\beta_{i+1}}^{\beta_i} C^b(i,\beta) f^b(\beta) d\beta$$
$$- \sum_{j\in J} \int_{\sigma_{j+1}}^{\sigma_j} C^s(j,\sigma) f^s(\sigma) d\sigma + \int_{\beta_0}^{\infty} u^b f^b(\beta) d\beta + \int_{\sigma_0}^{\infty} u^s f^s(\sigma) d\sigma$$

subject to the steady state conditions,

$$
\begin{aligned}
flow_i &= \int_{\beta_{i+1}}^{\beta_i} f^b(\beta)d\beta - x_i\mu(B,S)\sum_{j\in J} y_j m_{ij} = 0, \forall i \\
flow_j &= \int_{\sigma_{j+1}}^{\sigma_j} f^s(\sigma)d\sigma - y_j\mu(B,S)\sum_{i\in I} x_i m_{ij} = 0, \forall j \\
&B, S \geq 0 \\
&x_i \geq 0, \forall i \\
&y_j \geq 0, \forall j \\
X &= 1 - \sum_{i\in I} x_i = 0 \\
Y &= 1 - \sum_{j\in J} y_j = 0 \\
&1 \geq m_{ij} \geq 0, \forall i, j \\
F^b(\beta_{|I|}) &= F^s(\sigma_{|J|}) = 0 \\
B - rS &= 0
\end{aligned}
$$

Notice that taking weighted sums of the flow conditions implies that $F^b(\beta_0) = F^s(\sigma_0)$. The planner's problem is modified in four ways: i) agents can take an outside option which is included in the objective function and the conditions $F(\beta_0) = 1$ and $F(\sigma_0) = 1$ are removed; ii) the measure of buyers B and sellers S may differ and since we assumed that the are gains to trade, the conditions $B, S \geq 0$ will not bind at the efficient solution; iii) we add the balance ratio constraint $\frac{B}{S} = r$; and iv) the Inflow=Outflow equations are modified because the outflow of buyers and sellers is

$$
\begin{aligned}
(Bx_i)\left(\frac{\mu(B,S)}{B}\right)\sum_{j\in J} y_j m_{ij} &= x_i\mu(B,S)\sum_{j\in J} y_j m_{ij},\ \forall i \\
(Sy_j)\left(\frac{\mu(B,S)}{S}\right)\sum_{i\in I} x_i m_{ij} &= y_j\mu(B,S)\sum_{i\in I} x_i m_{ij},\ \forall j
\end{aligned}
$$

The KKT regularity conditions continue to hold, by the same arguments as in Theorem 1 (because the linear dependencies of the gradients do not change).

Replacing $B = rS$ in the objective:

$$\mathcal{W} = \mu(rS, S) \sum_{i \in I} \sum_{j \in J} x_i y_j m_{ij} g_{ij} - rSc^b - Sc^s - \sum_{i \in I} \int_{\beta_{i+1}}^{\beta_i} C^b(i, \beta) f^b(\beta) d\beta$$
$$- \sum_{j \in J} \int_{\sigma_{j+1}}^{\sigma_j} C^s(j, \sigma) f^s(\sigma) d\sigma + \int_{\beta_0}^{\infty} u^b f^b(\beta) d\beta + \int_{\sigma_0}^{\infty} u^s f^s(\sigma) d\sigma$$
$$= S\mu(r, 1) \sum_{i \in I} \sum_{j \in J} x_i y_j m_{ij} g_{ij} - rSc^b - Sc^s - \sum_{i \in I} \int_{\beta_{i+1}}^{\beta_i} C^b(i, \beta) f^b(\beta) d\beta$$
$$- \sum_{j \in J} \int_{\sigma_{j+1}}^{\sigma_j} C^s(j, \sigma) f^s(\sigma) d\sigma + \int_{\beta_0}^{\infty} u^b f^b(\beta) d\beta + \int_{\sigma_0}^{\infty} u^s f^s(\sigma) d\sigma$$

As in the proof of Theorem 1, we define the Lagrangian and taking the FOC we get:

FOC(S):
$$\mu(r, 1) \left(\sum_{i \in I} \sum_{j \in J} x_i y_j m_{ij} \left(g_{ij} - v_i - w_j \right) \right) - rc^b - c^s = 0$$

(Recall that $S > 0$ and so the multiplies on this constraint is 0).

So,

$$\sum_{i \in I} \sum_{j \in J} x_i y_j m_{ij} s_{ij} = \frac{rc^b + c^s}{\mu(r, 1)}$$

FOC($\mathbf{x_i}$):
$$\mu(rS, S) \sum_{j \in J} y_j m_{ij} g_{ij} - v_i \mu(rS, S) \sum_{j \in J} y_j m_{ij} - \mu(rS, S)) \sum_{j \in J} w_j y_j m_{ij} - \gamma - \phi_i = 0$$

where $\phi_i x_i = 0$. Therefore,

$$S\mu(r, 1) \sum_j y_j m_{ij} s_{ij} = \gamma + \phi_i \tag{26}$$

Multiplying by x_i and summing,

$$\gamma = S\mu(r, 1) \sum_{i \in I} \sum_{j \in J} x_i m_{ij} y_j s_{ij}$$

Substituting in from $FOC(S)$:

$$\gamma = S\mu(r, 1) \frac{rc^b + c^s}{\mu(r, 1)} = S(rc^b + c^s) = Bc^b + Sc^s$$

Therefore, from equation (26) we get that

$$\sum_j y_j m_{ij} s_{ij} = \frac{Bc^b + Sc^s}{\mu(rS,S)} = \frac{Bc^b + Sc^s}{\mu(B,S)} \tag{27}$$

Likewise:

FOC($\mathbf{y_i}$): $$\mu(rS,S)\sum_{i\in I} x_i m_{ij} g_{ij} - w_j \mu(rS,S)\sum_{i\in I} x_i m_{ij} - \mu(rS,S)\sum_{i\in I} v_i x_i m_{ij} - \eta - \psi_j = 0$$

$$\sum_{i\in I} x_i m_{ij} s_{ij} = \frac{\lambda + \psi_j}{\mu(rS,S)}$$

Using the previous two equations and multiplying by y_j and then summing, we obtain:

$$\frac{Bc^b + Sc^s}{\mu(rS,S)} = \sum_{i\in I}\sum_{j\in J} x_i y_j m_{ij} s_{ij} = \frac{\lambda}{\mu(rS,S)}$$

Therefore, for interior allocations, $\psi_j = 0$ and:

$$\sum_i x_i m_{ij} s_{ij} = \frac{Bc^b + Sc^s}{\mu(rS,S)} = \frac{Bc^b + Sc^s}{\mu(B,S)} \tag{28}$$

To decentralize the optimal allocation, we show that the shadow values of the flow constraints $(v_i), (w_j)$ together with the matching matrix M and state z constitute an equilibrium. Notice that the balance ratio $\frac{B}{S} = r \equiv \frac{\alpha c^s}{(1-\alpha)c^b} \iff \alpha = \frac{Bc^b}{Bc^b + Sc^s}$

We first show that the equilibrium constant surplus equations 24 hold, that is,

$$\sum_{j\in J} y_j m_{ij} s_{ij} = \frac{Bc^b}{\alpha\mu(B,S)}, \forall i$$

$$\sum_{i\in I} x_i m_{ij} s_{ij} = \frac{Sc^s}{(1-\alpha)\,\mu(B,S)}, \forall j$$

Notice that these equations coincide with equations 27 and 28 whenever $\alpha = \frac{Bc^b}{Bc^b + Sc^s}$.

The FOC(β_0) condition is precisely the equilibrium entry condition for buyers, $v_0 - C(0,\beta_0) = u^b$, that is, the shadow value v_0 makes the threshold type β_0 indifferent. Likewise, the FOC(σ_0) condition is precisely the equilibrium entry condition for sellers.

The rest of the proof uses the same argument as in Theorem 1: the FOC[m_{ij}] and the complementary slackness conditions imply that the values and matching matrix satisfy the equilibrium matching conditions $s_{ij} > 0 \to m_{ij} = 1$ and $s_{ij} < 0 \to m_{ij} = 0$; and FOC[$\beta_i$] and FOC[$\sigma_j$] imply that the constrained efficient investments are incentive compatible. □