

# LEARNING ABOUT INFORMATIVENESS

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**ABSTRACT.** We study whether individuals can learn the informativeness of their information technology through social learning. As in the classic sequential social learning model, rational agents arrive in order and make decisions based on the past actions of others and their private signals. There is uncertainty regarding the informativeness of the common signal-generating process. We show that in this setting asymptotic learning about informativeness is not guaranteed and depends crucially on the relative tail distributions of private beliefs induced by uninformative and informative signals. We identify the phenomenon of perpetual disagreement as the cause of learning and characterize learning in the canonical Gaussian environment.

## 1. INTRODUCTION

Social learning plays a vital role in the dissemination and aggregation of information. The behavior of others reflects their private knowledge about an unknown state of the world, and so by observing others, individuals can acquire additional information, enabling them to make better-informed decisions. A key assumption in most existing social learning models is the presence of an informative source that provides a useful private signal to each individual. In this paper, we explore how the possibility that the source is uninformative interferes with learning, and study the conditions under which individuals can eventually distinguish an uninformative source from an informative one. This question is particularly relevant today due to the proliferation of novel information technologies, raising concerns about the accuracy and credibility of the information they provide.<sup>1</sup>

Formally, we introduce uncertainty regarding the informativeness of the source into the classic sequential social learning model (Banerjee, 1992; Bikhchandani, Hirshleifer, and Welch, 1992). As usual, a sequence of short-lived agents arrives in order, each acting once by choosing an action to match an unknown payoff-relevant state that can be either good

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*Date:* October 6, 2024.

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<sup>1</sup>For example, the recent surge in the popularity of ChatGPT, a generative AI language model, has led to its widespread usage by a wide range of individuals, including laypeople, artists, and college students. Despite the model’s disclaimer stating that “ChatGPT may produce inaccurate information about people, places, or facts,” its adoption continues to grow.

or bad. Before making their decisions, each agent observes the past actions of her predecessors and receives a private signal from a common source of information. However, unlike in the usual setting, there is uncertainty surrounding this common information source. In particular, we assume that this source can be either informative, generating private signals that are independent and identically distributed (i.i.d.) conditioned on the payoff-relevant state, or uninformative, producing private signals that are i.i.d. but independent of the payoff-relevant state. Both the payoff-relevant state and the informativeness of the source are realized independently at the outset and are assumed to be fixed throughout.

If an outside observer, who aims to evaluate the informativeness of the source, were to have access to the private signals received by the agents, he would gradually accumulate empirical evidence about the source and eventually learn its informativeness. However, when only the history of past actions is observable, his inference problem becomes more challenging—not only because there is less information available, but also because these past actions are correlated with each other. This correlation arises from social learning behavior, where agents’ decisions are influenced by the inferences they draw from observing others’ actions. We say that *asymptotic learning of informativeness* holds if the outside observer’s belief about the source’s informativeness converges to the truth, i.e., it converges almost surely to one when the source is informative and to zero when it is uninformative. The questions we aim to address are: Can learning about informativeness be achieved asymptotically, and if so, under what conditions? Furthermore, what are the behavioral implications of such learning?

Our question of learning about informativeness is new within the realm of social learning. In the absence of information uncertainty, a common inquiry in this literature is whether agents engaged in social learning behavior ultimately choose the *correct* action, i.e., the action that matches the payoff-relevant state. However, in the presence of information uncertainty, it is unclear whether agents will eventually choose the same action, let alone whether that action is correct. In fact, we show that even when agents eventually take the correct action, they may still remain uncertain about its correctness, as this now depends on whether the source’s informativeness is fully revealed. This result highlights the close connection between learning about informativeness and learning about the payoff-relevant state, as we will discuss in Section 5. Therefore, to establish societal confidence in decision-making, it is important to understand the conditions under which asymptotic learning of informativeness holds.

We consider *unbounded signals* (Smith and Sørensen, 2000) under which the agent’s private belief induced by an informative signal can be arbitrarily strong. We focus on this setting since otherwise, learning about the source’s informativeness can easily be precluded by agents’ lack of response to their private signals. As shown by Smith and

Sørensen (2000), with bounded signals, an information cascade—where agents stop responding to their private signals—would be triggered, blocking further information aggregation. In contrast, with unbounded signals, cascades do not form, thus allowing information to continue accumulating through agents’ actions.

Our main result (Theorem 1) shows that even with unbounded signals, achieving asymptotic learning of informativeness is far from guaranteed. The key factor in determining such learning lies in the tail distributions of agents’ private beliefs. Specifically, it depends on whether the belief distribution induced by uninformative signals has *fatter* or *thinner* tails compared to that induced by informative signals. More precisely, we show that asymptotic learning of informativeness holds when uninformative signals have fatter tails than informative signals, but fails when uninformative signals have thinner tails.

For example, consider an informative source that generates Gaussian signals with unit variance and a mean of +1 if the payoff-relevant state is good and a mean of −1 if the state is bad. Meanwhile, the uninformative source generates Gaussian signals with mean 0, independent of the state. If the uninformative source generates signals with a variance strictly greater than one, then the uninformative signals have fatter tails, and thus asymptotic learning of informativeness holds. In contrast, when the uninformative signals have a variance strictly less than one, they exhibit thinner tails, and so asymptotic learning of informativeness fails.

As another illustration of the main result, consider the case where the informative signals follow the same distributions as before, but the uninformative signals are chosen uniformly from the bounded interval  $[-\varepsilon, \varepsilon]$  for some small  $\varepsilon > 0$ . In this case, the distribution of private beliefs induced by these uninformative signals also has bounded support. Consequently, it can be viewed as having *extremely* thinner tails compared to the informative Gaussian signals. Hence, Theorem 1 implies that the informativeness of the source cannot be fully revealed. Nevertheless, under such an informative source, almost all agents individually learn its informativeness: Once they receive a signal outside the support  $[-\varepsilon, \varepsilon]$ , which is highly probable for small  $\varepsilon$ , they infer that it can only come from the normal distribution, indicating that the source must be informative. However, an outside observer who only observes the agents’ actions is unable to determine the informativeness of the source.

The mechanism behind the main result is as follows. First, in our model, despite information uncertainty, agents always act as if the signals are informative. Therefore, when the source is indeed informative and generates unbounded signals, agents will eventually take the correct action. Now, suppose the source is uninformative and generates signals with thinner tails. In this case, it is unlikely that agents will receive signals extreme enough to break the consensus on actions, so they usually mimic their predecessors. As a result, an outside observer who only observes agents’ actions cannot discern that the source is uninformative, as an action consensus will be reached under both uninformative

and informative sources. In contrast, suppose the source is uninformative but generates signals with fatter tails. In this scenario, extreme signals are more likely, allowing agents to break the consensus; in fact, both actions will be taken infinitely often, so agents will never settle on an action consensus. Hence, an outside observer who observes an infinite number of action switches learns that the source is uninformative.

For some private belief distributions, their relative tail thickness is neither thinner nor fatter. For these, we show that the same holds: Asymptotic learning of informativeness is achieved if and only if conditioned on the source being uninformative, agents never settle on an action consensus (Proposition 1). In the Gaussian setting where informative signals are symmetric and the relative tail thickness is incomparable, we complement our main result by showing that learning holds if the uninformative signals have mean zero (Theorem 2). As for agents’ behavior, when the source is informative, as mentioned before, they eventually choose the correct action, regardless of information uncertainty. Nevertheless, we show that in this case they are certain that the action is correct if and only if the source’s informativeness is fully revealed (Proposition 2). In contrast, when the source is uninformative, agents are clearly not guaranteed to settle on the correct action; in fact, their actions may or may not converge at all. Proposition 1 shows that an outside observer eventually learns the informativeness of the source if and only if the agents’ actions do not converge when the source is uninformative.

**Related Literature.** Our paper contributes to a rich literature on sequential social learning. Assuming that the common source of information is always informative, the primary focus of this literature has been on determining whether agents can eventually learn to choose the correct action. Various factors, such as the information structure (Banerjee, 1992; Bikhchandani, Hirshleifer, and Welch, 1992; Smith and Sørensen, 2000) and the observational networks (Çelen and Kariv, 2004; Acemoglu, Dahleh, Lobel, and Ozdaglar, 2011; Lobel and Sadler, 2015), have been extensively studied to analyze their impact on information aggregation, including its efficiency (Rosenberg and Vieille, 2019) and the speed of learning (e.g., Vives, 1993; Hann-Caruthers, Martynov, and Tamuz, 2018). However, the question of learning about the informativeness of the source—which is the focus of this paper—remains largely unexplored.<sup>2</sup>

A few papers explore the idea of agents having access to multiple sources of information in the context of social learning. For example, Liang and Mu (2020) consider a model in which agents endogenously choose from a set of correlated information sources, and the acquired information is then made public and learned by other agents. They focus on the externality in agents’ information acquisition decisions and show that information complementarity can result in either efficient information aggregation or “learning traps,” in which society gets stuck in choosing suboptimal information structures. In a different

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<sup>2</sup>For comprehensive surveys on recent developments in the social learning literature, see e.g., Golub and Sadler (2017); Bikhchandani, Hirshleifer, Tamuz, and Welch (2021).

setting, [Chen \(2022\)](#) examines a sequential social learning model in which ambiguity-averse agents have access to different sources of information. Consequently, information uncertainty arises in his model because agents are unsure about the signal precision of their predecessors. He shows that under sufficient ambiguity aversion, there can be information cascades even with unbounded signals. Our paper differs from these prior works as we focus on rational agents with access to a common source of information of unknown informativeness.

Another way of viewing our model is by considering a social learning model with four states: The source is either informative with the good or bad state, or uninformative with either the good or bad state. In such multi-state settings, recent work by [Arieli and Mueller-Frank \(2021\)](#) demonstrates that pairwise unbounded signals are necessary and sufficient for learning, when the decision problem that agents face includes a distinct action that is uniquely optimal for each state. This is not the case in our model, because the same action is optimal in different states, e.g., when the source is uninformative, and so even when agents observe a very strong signal indicating that the state is uninformative, they do not reveal it in their behavior.

More recently, [Kartik, Lee, Liu, and Rappoport \(2022\)](#) consider a setting with multiple states and actions on general sequential observational networks. They identify a sufficient condition for learning —“excludability” —that jointly depends on the information structure and agents’ preferences. Roughly speaking, this condition ensures that agents can always displace the wrong action, which is their driving force for learning. In our model, when the source is uninformative, agents cannot displace the wrong action as all signals are pure noise.<sup>3</sup> Conceptually, our approach differs from theirs as we are interested in identifying the uninformative state from the informative one, instead of identifying the payoff-relevant state.

Our paper is also related to the growing literature on social learning with misspecified models. [Bohren \(2016\)](#) investigates a model where agents fail to account for the correlation between actions, demonstrating that different degrees of misperception can lead to distinct learning outcomes. In a broader framework, [Bohren and Hauser \(2021\)](#) show that learning remains robust to minor misspecifications. In contrast, [Frick, Iijima, and Ishii \(2020\)](#) find that an incorrect understanding of other agents’ preferences or types can result in a severe breakdown of information aggregation, even with a small amount of misperception. Later, [Frick, Iijima, and Ishii \(2023\)](#) propose a unified approach to establish convergence results in misspecified learning environments where the standard martingale approach fails to hold. On a more positive note, [Arieli, Babichenko, Müller, Pourbabaee, and Tamuz \(2023\)](#) illustrate that by being mildly condescending—misperceiving others as having slightly lower-quality of information—agents may perform better in the sense that on average, only finitely many of them take incorrect actions.

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<sup>3</sup>This observation can also be seen from Theorem 2 in [Kartik, Lee, Liu, and Rappoport \(2022\)](#).

## 2. MODEL

**2.1. Setup.** There is an unknown binary state of the world  $\theta \in \{\mathbf{g}, \mathbf{b}\}$ , chosen at time 0 with equal probability. We refer to  $\mathbf{g}$  as the good state and  $\mathbf{b}$  as the bad state. A countably infinite set of agents indexed by time  $t \in \mathbb{N} = \{1, 2, \dots\}$  arrive in order, each acting once. The action of agent  $t$  is  $a_t \in A = \{\mathbf{g}, \mathbf{b}\}$ , with a payoff of one if her action matches the state  $\theta$  and zero otherwise. Before agent  $t$  chooses an action, she observes the history of actions made by her predecessors  $H_t = (a_1, \dots, a_{t-1})$  and receives a private signal  $s_t$ , taking values in a measurable space  $(S, \Sigma)$ .

A pure strategy of agent  $t$  is a measurable function  $\sigma_t : A^{t-1} \times S \rightarrow A$  that selects an action for each possible pair of observed history and private signal. A pure strategy profile  $\sigma = (\sigma_t)_{t \in \mathbb{N}}$  is a collection of pure strategies of all agents. A strategy profile is a Bayesian Nash equilibrium—referred to as equilibrium hereafter—if no agent can unilaterally deviate from this profile and obtain a strictly higher expected payoff conditioned on their information. Given that each agent acts only once, the existence of an equilibrium is guaranteed by a simple inductive argument. In equilibrium, each agent  $t$  chooses the action  $a_t$  that maximizes her expected payoff given the available information:

$$a_t \in \arg \max_{a \in A} \mathbb{E}[\mathbb{1}(\theta = a) | H_t, s_t]. \quad (1)$$

Below, we make a continuity assumption which implies that agents are never indifferent, and so there is a unique equilibrium.

**2.2. The Informativeness of the Source.** So far, the above setup is the canonical setting of the sequential social learning model (Banerjee, 1992; Bikhchandani, Hirshleifer, and Welch, 1992; Smith and Sørensen, 2000). Our model builds on this setting and introduces another dimension of uncertainty regarding the informativeness of a common source. Specifically, at time 0, independent of the payoff-relevant state  $\theta$ , nature chooses an additional binary state  $\omega \in \{0, 1\}$ , where  $\omega = 1$  with probability  $\gamma \in (0, 1)$ .<sup>4</sup> When  $\omega = 1$ , the source is informative and sends i.i.d. signals across agents conditional on  $\theta$ , with distribution  $\mu_\theta$ . When  $\omega = 0$ , the source is uninformative and still sends i.i.d. signals, but independently of  $\theta$ , with distribution  $\mu_0$ . The realization of  $\omega$  determines the signal-generating process for all agents. Throughout, we denote by  $\mathbb{P}_0[\cdot] := \mathbb{P}[\cdot | \omega = 0]$  and  $\mathbb{P}_1[\cdot] := \mathbb{P}[\cdot | \omega = 1]$  the conditional probability distributions given  $\omega = 0$  and  $\omega = 1$ , respectively. Similarly, we use the notation  $\mathbb{P}_{1,\mathbf{g}}[\cdot] := \mathbb{P}[\cdot | \omega = 1, \theta = \mathbf{g}]$  to denote the conditional probability distribution given  $\omega = 1$  and  $\theta = \mathbf{g}$ . We use an analogous notation for  $\omega = 1$  and  $\theta = \mathbf{b}$ .

We first observe that, despite the uncertainty regarding the informativeness of the source, in equilibrium, each agent chooses the action that is most likely to match the state, *conditional* on the source being informative.

<sup>4</sup>Our results do not depend on the independence assumption between  $\theta$  and  $\omega$ . They hold true as long as conditioned on  $\omega = 0$ , both realizations of  $\theta$  are equally likely.

**Lemma 1.** *The equilibrium action for each agent  $t$  is*

$$a_t \in \arg \max_{a \in A} \mathbb{P}_1[\theta = a | H_t, s_t]. \quad (2)$$

That is, agents always act as if signals are informative, irrespective of the underlying signal-generating process. Intuitively, treating signals as informative—even when they are pure noise—does not adversely affect agents’ payoffs, since in the absence of any useful information, each agent with a uniform prior is indifferent between the available actions. Essentially, this result ensures that agents always respond to their private signals. Later, in Section 7, we show that when the prior is not uniform, agents eventually stop responding to signals, and consequently, the informativeness of the source can never be fully revealed.

**2.3. Information Structure.** Assume the distributions  $\mu_{\mathbf{g}}$ ,  $\mu_{\mathbf{b}}$ , and  $\mu_0$  are distinct<sup>5</sup> and mutually absolutely continuous, so no signal fully reveals either state  $\theta$  or  $\omega$ . As a consequence, conditioned on  $\omega = 1$ , the log-likelihood ratio of any signal

$$\ell_t = \log \frac{d\mu_{\mathbf{g}}}{d\mu_{\mathbf{b}}}(s_t),$$

is well-defined, and we call it the *agent’s private log-likelihood ratio*. Since regardless of the realization of  $\omega$ , agents always act as if the signals are informative (Lemma 1), it is sufficient to consider  $\ell_t$  to capture their behavior.<sup>6</sup> We denote by  $F_{\mathbf{g}}$  and  $F_{\mathbf{b}}$  the cumulative distribution functions (CDFs) of  $\ell_t$  conditioned on the event that  $\omega = 1$  and  $\theta = \mathbf{g}$  and the event that  $\omega = 1$  and  $\theta = \mathbf{b}$ , respectively. We denote by  $F_0$  the CDF of  $\ell_t$  conditioned on  $\omega = 0$ . All conditional CDFs  $F_{\mathbf{g}}$ ,  $F_{\mathbf{b}}$ , and  $F_0$  are mutually absolutely continuous, as  $\mu_{\mathbf{g}}$ ,  $\mu_{\mathbf{b}}$  and  $\mu_0$  are. Let  $f_{\mathbf{g}}$ ,  $f_{\mathbf{b}}$  and  $f_0$  denote the corresponding density functions of  $F_{\mathbf{g}}$ ,  $F_{\mathbf{b}}$  and  $F_0$  whenever they are differentiable.

We focus on *unbounded signals* in the sense that the agent’s private log-likelihood ratio can take on arbitrarily large or small values, i.e., for any  $M \in \mathbb{R}$ , there exists a positive probability that  $\ell_t > M$  and a positive probability that  $\ell_t < -M$ . We informally refer to a signal  $s_t$  as *extreme* when the corresponding  $\ell_t$  it induces has a large absolute value. A common example of unbounded private signals is the case of Gaussian signals, where  $s_t$  follows a normal distribution  $\mathcal{N}(m_{(\omega, \theta)}, \sigma^2)$  with variance  $\sigma^2$  and mean  $m_{(\omega, \theta)}$  that depends on the pair of states  $(\omega, \theta)$ . An extreme Gaussian signal is a signal that is, for example,  $5 - \sigma$  away from the mean  $m_{(\omega, \theta)}$ .

We make two assumptions for expository simplicity. First, we assume that the pair  $(F_{\mathbf{g}}, F_{\mathbf{b}})$  of informative conditional CDFs is symmetric around zero, i.e.,  $F_{\mathbf{g}}(x) + F_{\mathbf{b}}(-x) =$

<sup>5</sup>Formally, two distributions are distinct if there exists a set to which they assign a different measure. From an economic perspective, ensuring that  $\mu_0$  is distinct from both  $\mu_{\mathbf{g}}$  and  $\mu_{\mathbf{b}}$  prevents a situation in which there is a ‘fake’ information technology that mimics the informative source at one of the realizations of  $\theta$  but is actually uninformative. Similarly,  $\mu_{\mathbf{g}}$  being distinct from  $\mu_{\mathbf{b}}$  means that in the informative state, the signals are actually informative.

<sup>6</sup>Formally, the sequence of actions  $a_1, \dots, a_t$  is determined by  $\ell_1, \dots, \ell_t$ .

1. Given that the prior on  $\theta$  is uniform, this assumption is equivalent to requiring our model to be invariant with respect to a relabeling of the payoff-relevant state. Second, we assume that all conditional CDFs— $F_{\mathfrak{g}}$ ,  $F_{\mathfrak{b}}$ , and  $F_0$ —are continuous, so agents are never indifferent between actions.

In addition, we assume that  $F_{\mathfrak{b}}$  has a differentiable left tail, i.e., it is differentiable for all  $x$  negative enough, and its probability density function  $f_{\mathfrak{b}}$  satisfies the condition that  $f_{\mathfrak{b}}(-x) < 1$  for all  $x$  large enough. By symmetry, this implies that  $F_{\mathfrak{g}}$  also has a differentiable right tail and its density function  $f_{\mathfrak{g}}$  satisfies the condition that  $f_{\mathfrak{g}}(x) < 1$  for all  $x$  large enough. This is a mild technical assumption that holds for every non-atomic distribution commonly used in the literature, including the Gaussian distribution. It holds, for instance, whenever the density tends to zero at infinity.

**2.4. Asymptotic Learning of Informativeness.** We denote by  $q_t := \mathbb{P}[\omega = 1 | H_t]$  the belief that an outside observer assigns to the source being informative after observing the history of agents' actions from time 1 to  $t - 1$ . As this observer collects more information over time, his belief  $q_t$  converges almost surely since it is a bounded martingale. To ensure that he eventually learns the truth regarding the informativeness of the source, we introduce the following notion of learning.

**Definition 1.** *Asymptotic learning of informativeness* holds if for all  $\omega \in \{0, 1\}$ ,

$$\lim_{t \rightarrow \infty} q_t = \omega \quad \mathbb{P}_{\omega}\text{-almost surely.}$$

That is, conditioned on an informative source, the outside observer's belief that the source is informative converges to one almost surely; meanwhile, conditioned on an uninformative source, his belief that the source is informative converges to zero almost surely. As we explain below in Section 4.1, when all signals are publicly observable, asymptotic learning of informativeness is always attainable, regardless of the underlying information structure.

### 3. RELATIVE TAIL THICKNESS

To study the conditions for achieving asymptotic learning of informativeness, it is crucial to understand the concept of relative tail thickness, which compares the tail distributions of agents' private log-likelihood ratios induced by different signals. This comparison is important because it captures the relative likelihood of generating extreme signals from different sources.

Formally, for any pair of CDFs  $(F_0, F_{\theta})$  where  $\theta \in \{\mathfrak{g}, \mathfrak{b}\}$  and some  $x \in \mathbb{R}_+$ , we denote their corresponding ratios by

$$L_{\theta}(x) := \frac{F_0(-x)}{F_{\theta}(-x)} \quad \text{and} \quad R_{\theta}(x) := \frac{1 - F_0(x)}{1 - F_{\theta}(x)}.$$

For large  $x$ ,  $L_{\theta}(x)$  and  $R_{\theta}(x)$  represent the left and right tail ratios of  $F_0$  over  $F_{\theta}$ , respectively. The following definitions of fatter and thinner tails describe situations in which



extreme signals are either more or less likely to occur under an uninformative source compared to an informative one.<sup>7</sup>

**Definition 2.** (i) The uninformative signals have *fatter tails* than the informative signals if there exists  $\varepsilon > 0$  such that

$$L_{\mathfrak{b}}(x) \geq \varepsilon \quad \text{for all } x \text{ large enough,}$$

and  $R_{\mathfrak{g}}(x) \geq \varepsilon \quad \text{for all } x \text{ large enough.}$

(ii) The uninformative signals have *thinner tails* than the informative signals if there exists  $\varepsilon > 0$  such that

$$\text{either } L_{\mathfrak{g}}(x) \leq 1/\varepsilon \quad \text{for all } x \text{ large enough,}$$

or  $R_{\mathfrak{b}}(x) \leq 1/\varepsilon \quad \text{for all } x \text{ large enough.}$

That is, for the uninformative signals to have fatter tails, both their corresponding left *and* right tail ratios must eventually be bounded from below. Conversely, for the uninformative signals to have thinner tails, at least one of the tail ratios—either left *or* right—must eventually be bounded from above. The “and” condition is important for defining fatter tails, as it ensures that extreme signals of both types (positive and negative) occur frequently enough to prevent agents from reaching an action consensus. In contrast, the “or” condition for thinner tails requires only the infrequency of at least one type of extreme signal, thus allowing agents to reach a consensus with positive probability. As discussed in Section 4.2, determining whether settling on an action consensus is possible under an uninformative source is key to understanding whether asymptotic learning of informativeness holds.

Note that the first condition  $L_{\mathfrak{b}}(x) \geq \varepsilon$  implies that  $L_{\mathfrak{g}}(x) \geq \varepsilon$ . This follows from the well-known fact that  $F_{\mathfrak{g}}$  exhibits first-order stochastic dominance over  $F_{\mathfrak{b}}$ , i.e.,  $F_{\mathfrak{g}}(x) \leq F_{\mathfrak{b}}(x)$  for all  $x \in \mathbb{R}$  (see, e.g., [Smith and Sørensen, 2000](#); [Chamley, 2004](#); [Rosenberg and Vieille, 2019](#)). Similar statements apply to the remaining three conditions. Furthermore, note that the uninformative signals cannot have both fatter and thinner tails simultaneously, as  $F_{\mathfrak{g}}$  and  $F_{\mathfrak{b}}$  represent distributions of the agent’s private log-likelihood ratio.<sup>8</sup> However, there are distributions under which the uninformative signals have neither fatter nor thinner tails. For these cases, we characterize the conditions for asymptotic learning of informativeness in the canonical Gaussian environment (see Section 4.3).

Intuitively, compared to informative signals, uninformative signals with fatter tails are more likely to exhibit extreme values. Thus, by Bayes’ Theorem, observing an extreme

<sup>7</sup>In statistics, the notion of relative tail thickness has also been explored. Our definition of thinner tails is closest to that of [Rojo \(1992\)](#), where a CDF  $F$  is considered not more heavily tailed than  $G$  if  $\limsup_{x \rightarrow \infty} (1 - F(x))/(1 - G(x)) < \infty$ . Other notions of relative tail thickness, represented in terms of density quantile functions, can be found in [Parzen \(1979\)](#) and [Lehmann \(1988\)](#). See [Rojo \(1992\)](#) for a discussion of the relationship between these existing notions.

<sup>8</sup>In particular,  $F_{\mathfrak{g}}$  and  $F_{\mathfrak{b}}$  satisfy the following inequality:  $e^x F_{\mathfrak{g}}(-x) \leq F_{\mathfrak{b}}(-x)$ , for all  $x \in \mathbb{R}$ .

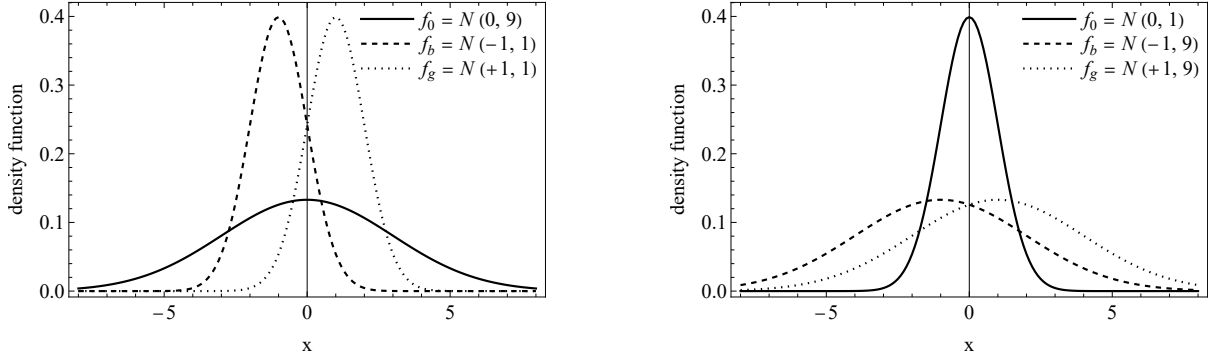


FIGURE 3.1. Probability density functions of informative Gaussian signals and their corresponding uninformative Gaussian signals with fatter tails (on the left) and with thinner tails (on the right).

signal suggests that the source is uninformative. In contrast, uninformative signals with thinner tails tend to exhibit moderate values, so observing an extreme signal in this case suggests that the source is informative. Next, we provide three examples of uninformative signals with either fatter or thinner tails.

**Example 1** (Gaussian Signals). Consider the case where  $F_g$  is normal with mean  $+1$  and unit standard deviation and  $F_b$  is also normal with mean  $-1$  and unit standard deviation.

Suppose that  $F_0$  has zero mean. If it has a standard deviation of  $3$ , then the uninformative signals have fatter tails. In this scenario, observing an extreme signal, such as anything around  $11$ , indicates that it is much more likely that the source is uninformative—where the signal is less than  $4 - \sigma$  away—than for an informative  $10 - \sigma$  signal to be generated under  $F_g$ . Conversely, if the standard deviation of  $F_0$  is  $1/3$ , the uninformative signals have thinner tails, and thus an extreme signal suggests that the source is informative. A graphical example of uninformative Gaussian signals with fatter and thinner tails is shown in Figure 3.1.<sup>9</sup>

**Example 2** (First-Order Stochastically Dominated (FOSD) Signals). Suppose that  $F_0$  first-order stochastically dominates  $F_g$ , i.e.,  $F_0(x) \leq F_g(x)$ , for all  $x \in \mathbb{R}$ . In other words, the uninformative signals are more likely to exhibit high values than the informative signals associated with the good state. It follows immediately from  $L_g(x) \leq 1$  for all  $x \in \mathbb{R}$  that the uninformative signals have thinner tails than the informative signals.

Now, suppose that an extreme positive signal is observed. It is highly unlikely that the source is informative and associated with the bad state. Instead, this suggests that the source is more likely to be either uninformative or informative but associated with the good state. However, one cannot exclude either possibility based on such an observation, and hence it does not provide conclusive evidence about the source's informativeness.

<sup>9</sup>See Lemma 8 in the appendix for a more general comparison of the relative tail thickness for Gaussian signals with different variances.

**Example 3** (Mixture Signals). For any pair of distributions  $(F_{\mathbf{g}}, F_{\mathbf{b}})$  and any  $\alpha \in (0, 1)$ , let  $F_0 = \alpha F_{\mathbf{g}} + (1 - \alpha)F_{\mathbf{b}}$ . Observe that the uninformative signals represented by  $F_0$  have fatter tails than the informative signals.<sup>10</sup> In particular, when  $\alpha = 1/2$ , the corresponding mixture distribution  $F_0 = (F_{\mathbf{g}} + F_{\mathbf{b}})/2$  coincides with the unconditional distribution of  $\ell_t$  generated by an informative source. Thus, we can think of this uninformative source as being a priori indistinguishable from the informative one.

Alternatively, the mixture distribution  $F_0$  can be viewed as generating conditionally i.i.d. signals, but instead of conditioning on the state  $\theta$ , they are generated conditioned on a different state  $\eta$ . Specifically, suppose that in each period, the state  $\eta$  is randomly drawn from the same set  $\{\mathbf{g}, \mathbf{b}\}$ , independent of  $\theta$ . Say, with probability  $\alpha$ , the event  $\eta = \mathbf{g}$  occurs and a signal is drawn from the distribution  $F_{\mathbf{g}}$ . With the complementary probability, the event  $\eta = \mathbf{b}$  occurs, and a signal is drawn from the distribution  $F_{\mathbf{b}}$ .<sup>11</sup>

## 4. MAIN RESULTS

**4.1. A Benchmark.** As a benchmark, we briefly discuss the case where all signals are observed by the outside observer.<sup>12</sup> Depending on the realizations of  $\theta$  and  $\omega$ , these signals are distributed according to either  $\mu_{\mathbf{g}}$ ,  $\mu_{\mathbf{b}}$  or  $\mu_0$ . Since these measures are distinct, as the sample size grows, this observer eventually learns which distribution is being sampled. Formally, at time  $t$ , the empirical distribution of the signals  $\hat{\mu}_t$  assigns to a measurable set  $B \subseteq S$  the probability

$$\hat{\mu}_t(B) := \frac{1}{t} \sum_{\tau=1}^t \mathbb{1}(s_{\tau} \in B).$$

Conditional on both states  $\omega$  and  $\theta$ , this is the empirical mean of i.i.d. Bernoulli random variables. Hence, by the strong law of large numbers, for every measurable set  $B \subseteq S$ ,

$$\lim_{t \rightarrow \infty} \hat{\mu}_t(B) = \mu_{(\omega, \theta)}(B) \quad \text{almost surely,}$$

where  $\mu_{(1, \mathbf{g})} = \mu_{\mathbf{g}}$ ,  $\mu_{(1, \mathbf{b})} = \mu_{\mathbf{b}}$  and  $\mu_{(\cdot, 0)} = \mu_0$ .

Thus, regardless of the underlying signal-generating process, any uncertainty concerning the informativeness of the source is eventually resolved if all signals are publicly observable. Next, we turn to our main setting where the signals remain private and only the actions are observable. Clearly, in this setting, less information is available—observed actions contain less information than private signals. In addition, one needs to take into account the positive correlation between these observed actions.

<sup>10</sup>To see this, fix any constant  $\alpha \in (0, 1)$  and let  $F_0 = \alpha F_{\mathbf{g}} + (1 - \alpha)F_{\mathbf{b}}$ . Since CDFs always take nonnegative values, for any  $x \in \mathbb{R}$ ,  $F_0(x) \geq (1 - \alpha)F_{\mathbf{b}}(x)$ . Similarly,  $1 - F_0(x) = \alpha(1 - F_{\mathbf{g}}(x)) + (1 - \alpha)(1 - F_{\mathbf{b}}(x)) \geq \alpha(1 - F_{\mathbf{g}}(x))$ . Let  $\varepsilon = \min\{\alpha, 1 - \alpha\}$ , and thus by definition  $F_0$  has fatter tails.

<sup>11</sup>When  $\alpha = 1/2$ , we can think of these uninformative signals as being generated based on a sequence of fair and independent coin tosses.

<sup>12</sup>Equivalently, one can let the outside observer observe all agents' actions in addition to their signals. Since actions contain no additional payoff relevant information, it suffices to only consider the signals.

**4.2. Public Actions.** We now present our main result (Theorem 1). In contrast to the public signal benchmark, our main result shows that achieving asymptotic learning of informativeness is no longer guaranteed. In fact, the key determinant of learning about informativeness is the relative tail thickness between the uninformative and informative signals, as introduced in Definition 2.

**Theorem 1.** *When the uninformative signals have fatter (thinner) tails than the informative signals, asymptotic learning of informativeness holds (fails).*

Theorem 1 demonstrates that an outside observer, who learns from observing agents' actions, eventually distinguishes between informative and uninformative sources if the latter source generates signals with greater dispersion than the former. In contrast, when the uninformative signals are relatively concentrated compared to informative ones, such differentiation becomes unattainable for the observer.

For example, consider informative signals that follow a normal distribution with unit variance and mean  $+1$  and  $-1$ , respectively. When the uninformative signals follow a normal distribution with a variance of 2, they have fatter tails, and thus Theorem 1 implies that asymptotic learning of informativeness holds. In contrast, when the uninformative signals follow a normal distribution with a variance of  $1/2$ , they have thinner tails, and Theorem 1 implies that such learning fails. More generally, Theorem 1 can be immediately applied to cases where the variance of the uninformative Gaussian signals,  $\tau^2$ , is strictly higher or lower than that of the informative Gaussian signals,  $\sigma^2$ . This is because the relative tail thickness of normal distributions is determined solely by their variances. That is, in this setting, asymptotic learning of informativeness holds if  $\tau > \sigma$  and fails if  $\tau < \sigma$ .<sup>13</sup>

The idea behind our proof of Theorem 1 is as follows. First consider the case where the source is informative. In this case, the likelihood of generating extreme signals that overturn a long streak of correct action consensus decreases rapidly. Consequently, agents will eventually choose the correct action since they always treat signals as informative. Now, suppose that the source is uninformative, and instead of reaching a consensus, agents continue to disagree indefinitely, leading to both actions being taken infinitely often. If this were the case, an outside observer would eventually be able to distinguish between informative and uninformative sources, as they induce distinct behavioral patterns among agents. Whether these disagreements persist or not depends on whether the tails of the uninformative signals are thick enough to generate these overturning extreme signals.

In summary, when the tails of uninformative signals are sufficiently thick, overturning extreme signals occur frequently enough so that disagreements persist. Conversely, when the tails are relatively thin, these signals are less likely to occur, and disagreements

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<sup>13</sup>For more details, see Lemma 8 in the appendix.

eventually cease. Hence, we conclude that the relative tail thickness between uninformative and informative signals plays an important role in determining the achievement of asymptotic learning of informativeness.

*Perpetual Disagreement.* As mentioned above, if agents' actions do not converge under an uninformative source, then intuitively, the outside observer can infer that the source is uninformative. To formally establish the key mechanism underlying our main result, we define the event in which agents' actions never converge as *perpetual disagreement*. Let  $S := \sum_{t=1}^{\infty} \mathbb{1}(a_t \neq a_{t+1})$  denote the total number of action switches, so that the event  $\{S = \infty\}$  is the perpetual disagreement event. It turns out that perpetual disagreement under an uninformative source is not only a sufficient condition for asymptotic learning of informativeness but also a necessary condition, as shown in the following proposition.

**Proposition 1.** *Asymptotic learning of informativeness holds if and only if conditioned on  $\omega = 0$ , the perpetual disagreement event occurs almost surely.*

The proof of Proposition 1 uses the idea of an outside observer whose goal is to guess the informativeness of the source. Note that achieving asymptotic learning of informativeness means that eventually, this outside observer can always make the correct guess. Since he observes all agents' actions, he expects to see the agents form an action consensus—i.e., their actions converge—when the source is informative. Therefore, if he observes that actions never converge, he infers that the source must be uninformative and makes the correct guess. Now, suppose to the contrary that agents' actions could also converge when the source is uninformative. This would imply that action convergence is possible under both informative and uninformative sources. As a result, the observer would no longer be certain of the source's informativeness, contradicting the hypothesis that he always makes the correct guess.

**4.3. Gaussian Private Signals.** While Theorem 1 provides valuable insight into the role of relative tail thickness in learning about informativeness, there are situations where uninformative signals have neither fatter nor thinner tails compared to informative signals, rendering Theorem 1 inapplicable. For example, consider a scenario where  $F_0$ ,  $F_g$ , and  $F_b$  are normal distributions with the same variance and mean 0, 1, and  $-1$ , respectively. As  $x$  approaches infinity, both  $F_0(-x)$  and  $F_b(-x)$  approach zero, but the former goes to zero much faster than the latter, leading  $L_b(x)$  to converge to zero. As a result,  $F_0$  does not have fatter tails. Similarly, both  $L_g(x)$  and  $R_b(x)$  tend to infinity as  $x$  approaches infinity, so  $F_0$  does not have thinner tails either.

To complement the findings of Theorem 1, we focus on the Gaussian environment where all signals are normal and share the same variance  $\sigma^2$ . The informative signals are symmetric with mean  $+1$  and  $-1$ , respectively, while the uninformative signals have

mean  $m_0 \in (-1, 1)$ .<sup>14</sup> In this setting, a simple calculation shows that the agent’s private log-likelihood ratio is directly proportional to the private signal:  $\ell_t = 2s_t/\sigma^2$ . As a consequence, all distributions  $F_g$ ,  $F_b$ , and  $F_0$  are also Gaussian. In the following result, we show that asymptotic learning of informativeness is achieved if and only if the uninformative signals are symmetric around zero.

**Theorem 2.** *Suppose all private signals are Gaussian with the same variance, where informative signals have means  $-1$  and  $+1$ , and uninformative signals have mean  $m_0 \in (-1, 1)$ . Then, asymptotic learning of informativeness holds if and only if  $m_0 = 0$ .*

Theorem 2 shows that in a Gaussian setting with symmetric informative signals, asymptotic learning of informativeness occurs only when the uninformative signals are also symmetric around zero. The intuition is similar to that used in establishing Proposition 1: Given that all signals have the same variance, any deviation of the mean of  $F_0$  from zero shifts  $F_0$  closer to either  $F_g$  or  $F_b$ , causing the uninformative signals to resemble the corresponding informative signals. As a result, under such an uninformative source, agents will also reach an action consensus, which in turn prevents an outside observer from fully distinguishing between informative and uninformative sources.

**4.4. Numerical Simulation.** To further illustrate our main result, we use Monte Carlo simulations to numerically examine the belief process of an outside observer and the corresponding action switches among agents. We fix the pair of informative Gaussian signal distributions to be  $(\mu_g, \mu_b)$  where  $\mu_g = \mathcal{N}(+1, 2)$  and  $\mu_b = \mathcal{N}(-1, 2)$ . These processes are simulated under two scenarios: one with fatter-tailed uninformative Gaussian signals with distribution  $\mu_0 = \mathcal{N}(0, 3)$ , and the other with thinner-tailed uninformative Gaussian signals with distribution  $\mu_0 = \mathcal{N}(0, 1)$ .<sup>15</sup> For each uninformative source, we conduct these simulations 1,000 times, redrawing the payoff-relevant state each time, and calculate the averages for each period. This yields approximations for the expected belief and the expected total number of action switches in the presence of uninformative signals. Figure 4.1 displays the results of these simulations.

What immediately stands out is that under fatter-tailed uninformative signals, the belief of the outside observer that the source is informative decreases much faster compared to thinner-tailed uninformative signals. By period 60, this belief is approximately 0.3 under fatter-tailed uninformative signals, which is less than two-thirds of the belief observed under thinner-tailed uninformative signals. These findings align with the predictions of Theorem 1, suggesting that in the former case the observer will eventually learn that the source is uninformative. In contrast, with thinner-tailed uninformative signals, the

<sup>14</sup>In the case where all Gaussian signals share the same variance and the absolute value of  $m_0$  is strictly greater than one, the uninformative signals clearly have thinner tails. For example, when  $m_0 = 2$ , it reduces to Example 2. Thus, our main result implies that asymptotic learning of informativeness fails.

<sup>15</sup>Note that in this case, the agent’s private log-likelihood ratio  $\ell_t = s_t$ , so  $F_0, F_g$  and  $F_b$  have the same distribution as  $\mu_0, \mu_g$  and  $\mu_b$ , respectively.

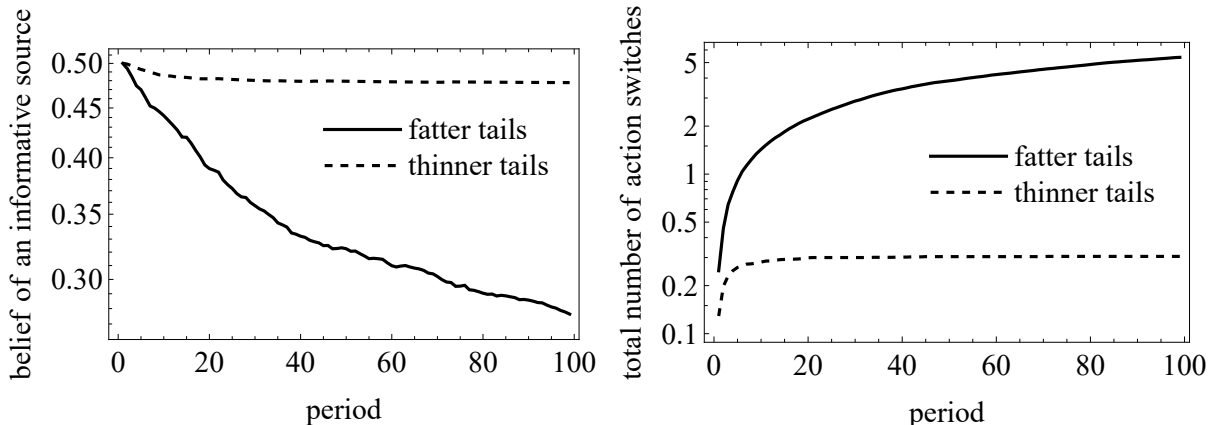


FIGURE 4.1. The belief that the source is informative (on the left) and the total number of action switches (on the right) under uninformative signals with fatter and thinner tails.

decline in the belief about the source’s informativeness plateaus around 0.48 after period 20, suggesting that the observer will not be able to learn that the source is uninformative.

As shown in Proposition 1, the key mechanism driving learning about informativeness is the persistence of disagreements under an uninformative source. Recall that the intuition is that the uninformative source with fatter tails has a higher probability of generating extreme signals, which in turn, prevents agents from converging to a consensus. This phenomenon is evident in the right plot of Figure 4.1, where the average total number of action switches under fatter-tailed uninformative signals increases over time. In contrast, under the thinner-tailed uninformative signals, the total number of switches plateaus in a short amount of time, suggesting that perpetual disagreement does not occur in this case.

## 5. CONNECTIONS TO OTHER NOTIONS OF LEARNING

In this section, we discuss the relationship between asymptotic learning of informativeness and other notions of learning that have appeared in the literature. One common notion, which concerns the convergence of agents’ actions, is known as herding. Another notion, which refers to the convergence of agents’ beliefs, is called complete learning. Both of these notions focus on the payoff-relevant state. As is well-known, when the source is always informative and generates unbounded signals, agents will eventually choose the correct action, and their beliefs will converge to the truth. By contrast, we will demonstrate that in the presence of information uncertainty, agents may eventually choose the correct action without their beliefs converging to the truth.

Formally, we say that *correct herding* holds if  $\lim_{t \rightarrow \infty} a_t = \theta$  almost surely. That is, the event that all but finitely many agents take the correct action occurs with probability one. Meanwhile, let  $p_t = \mathbb{P}[\theta = \mathbf{g} | H_t]$  denote the *social belief* of agent  $t$ . This is the belief assigned by agent  $t$  to the good state, given only the history of past actions. We say

that *complete learning* holds if agents' social beliefs converge to the truth, i.e.,  $p_t$  almost surely converges to one when  $\theta = \mathbf{g}$  and to zero when  $\theta = \mathbf{b}$ .

Note that in our model, since agents always behave as if the signals are informative, when the source is indeed informative and generates unbounded signals, correct herding also holds:

$$\lim_{t \rightarrow \infty} a_t = \theta \quad \mathbb{P}_1\text{-almost surely.}$$

Hence, when the source is informative, agents will eventually choose the correct action, regardless of whether asymptotic learning of informativeness is achieved. In contrast, when the source is uninformative, agents are clearly not guaranteed to choose the correct action; in fact, their actions may or may not converge at all. Indeed, as suggested by Proposition 1, in this case, their actions do not converge if asymptotic learning of informativeness holds, or they may converge to the wrong action if such learning fails.

Nevertheless, achieving correct herding under an informative source does not imply the attainment of complete learning. In this case, although agents eventually take the correct action, they remain uncertain about its correctness unless asymptotic learning of informativeness holds. Conversely, if the informativeness of the source remains uncertain, so do agents' beliefs about the correctness of their action consensus. This relationship is summarized in Proposition 2. On the other hand, when the source is uninformative, Proposition 3 shows that achieving asymptotic learning of informativeness is equivalent to the social belief converging to the uniform prior. Together, these two propositions demonstrate the connections between learning about informativeness and learning about the payoff-relevant state.

**Proposition 2.** *Asymptotic learning of informativeness holds if and only if conditioned on  $\omega = 1$ , complete learning holds.*

The proof of Proposition 2 uses the idea of an outside observer who observes the same public information as agent  $t$ , but does not observe her private signal. So, at time  $t$ , the observer's belief is equal to the social belief. As long as the observer remains uncertain about the source's informativeness, he cannot fully trust the public information, which consists of agents' actions. Conversely, once the source's informativeness is confirmed, this public information becomes highly accurate, thus enabling the observer's belief to converge to the truth.

**Proposition 3.** *Asymptotic learning of informativeness holds if and only if conditioned on  $\omega = 0$ ,  $\lim_{t \rightarrow \infty} p_t = 1/2$  almost surely.*

The proof of Proposition 3 follows a similar approach to Proposition 2 and applies the result of Proposition 1. One direction is straightforward: if the observer learns that the source is uninformative, then the social belief must converge to the prior as the action history contains no information about the payoff-relevant state. For the other direction, suppose by contraposition that the observer does not learn that the source is



uninformative. Then, Proposition 1 implies that agents may eventually reach a consensus on some action. Since this is also possible under an informative source, the observer cannot fully dismiss the possibility that agents' actions are informative. As a result, his belief about the payoff-relevant state is not guaranteed to remain at the prior.

## 6. ANALYSIS

In this section we first examine how agents update their beliefs. We present two standard yet useful properties of their belief updating process, namely, the *overturning principle* and *stationarity*. Then, based on Proposition 1, we characterize asymptotic learning of informativeness in terms of *immediate agreement*—the event in which all agents immediately reach a consensus on some action—which simplifies the problem. Finally, we provide a proof sketch of our main result (Theorem 1) at the end of this section.

**6.1. Agents' Beliefs Dynamics.** Since agents always act as if signals are informative (Lemma 1), we focus on how agents update their beliefs conditioned on an informative source. We denote by  $\pi_t$  the *public belief* that agent  $t$  assigns to the good state after observing the history of actions  $H_t$ , i.e.,  $\pi_t := \mathbb{P}_1[\theta = \mathbf{g}|H_t]$ . The corresponding log-likelihood ratio of  $\pi_t$  is given by:

$$r_t := \log \frac{\pi_t}{1 - \pi_t} = \log \frac{\mathbb{P}_1[\theta = \mathbf{g}|H_t]}{\mathbb{P}_1[\theta = \mathbf{b}|H_t]}.$$

Furthermore, let  $L_t$  denote the log-likelihood ratio of the posterior belief that agent  $t$  assigns to the good state over the bad state after observing both the history of actions and her private signal:

$$L_t := \log \frac{\mathbb{P}_1[\theta = \mathbf{g}|H_t, s_t]}{\mathbb{P}_1[\theta = \mathbf{b}|H_t, s_t]}.$$

Recall that  $\ell_t$  is the log-likelihood ratio of agent  $t$ 's private belief induced by a signal  $s_t$  conditioned on  $\omega = 1$ . By Bayes' rule, we can write

$$L_t = r_t + \ell_t.$$

It follows from (2) that, in equilibrium, agent  $t$  chooses action  $\mathbf{g}$  if  $\ell_t \geq -r_t$  and action  $\mathbf{b}$  if  $\ell_t < -r_t$ . Hence, conditioned on the state  $\theta$  and the event that  $\omega = 1$ , the probability that agent  $t$  chooses action  $\mathbf{g}$  is  $1 - F_\theta(-r_t)$  and the probability that she chooses action  $\mathbf{b}$  is  $F_\theta(-r_t)$ . As a consequence, the agents' public log-likelihood ratios ( $r_t$ ) evolve as follows:

$$r_{t+1} = r_t + D_{\mathbf{g}}(r_t) \quad \text{if } a_t = \mathbf{g}, \tag{3}$$

$$r_{t+1} = r_t + D_{\mathbf{b}}(r_t) \quad \text{if } a_t = \mathbf{b}, \tag{4}$$

where

$$D_{\mathfrak{g}}(r) := \log \frac{1 - F_{\mathfrak{g}}(-r)}{1 - F_{\mathfrak{b}}(-r)} \quad \text{and} \quad D_{\mathfrak{b}}(r) := \log \frac{F_{\mathfrak{g}}(-r)}{F_{\mathfrak{b}}(-r)}.$$

Note that  $D_{\mathfrak{g}}$  always takes positive values and  $D_{\mathfrak{b}}$  always takes negative values as  $F_{\mathfrak{g}}$  first-order stochastically dominates  $F_{\mathfrak{b}}$ .

*Overturning Principle and Stationarity.* One important property held by the agent's public belief is known as the *overturning principle* (Sørensen, 1996; Smith and Sørensen, 2000), which asserts that a single action switch is sufficient to change the verdict of  $\pi_t$ .

**Lemma 2** (Overturning Principle). *For each agent  $t$ , if  $a_t = \mathfrak{g}$ , then  $\pi_{t+1} \geq 1/2$ . Similarly, if  $a_t = \mathfrak{b}$ , then  $\pi_{t+1} \leq 1/2$ .*

Another important property held by  $\pi_t$  is *stationarity*—the value of  $\pi_t$  captures all past information about the payoff-relevant state, independent of time. This holds in our model because, regardless of the informativeness of the source, agents always update their public log-likelihood ratios according to either (3) or (4). We further write  $\mathbb{P}_{\tilde{\omega}, \tilde{\theta}, \pi}$  to denote the conditional probability distribution given the pair of state realizations  $(\tilde{\omega}, \tilde{\theta})$  while highlighting the different values of the prior  $\pi$ .

**Lemma 3** (Stationarity). *For any fixed sequence  $(b_\tau)_{\tau=1}^k$  of  $k$  actions in  $\{\mathfrak{g}, \mathfrak{b}\}$ , any prior  $\pi \in (0, 1)$  and any pair  $(\tilde{\omega}, \tilde{\theta}) \in \{0, 1\} \times \{\mathfrak{g}, \mathfrak{b}\}$*

$$\mathbb{P}_{\tilde{\omega}, \tilde{\theta}}[a_{t+1} = b_1, \dots, a_{t+k} = b_k | \pi_t = \pi] = \mathbb{P}_{\tilde{\omega}, \tilde{\theta}, \pi}[a_1 = b_1, \dots, a_k = b_k].$$

This lemma states that regardless of the source's informativeness, if agent  $t$ 's public belief is equal to  $\pi$ , then the probability of observing a sequence  $(b_1, \dots, b_k)$  of actions of length  $k$  is the same as observing this sequence starting from time 1, given that the agents' prior on the payoff-relevant state is  $\pi$ .

**6.2. Immediate Agreement.** In our model, since the agent's public belief  $\pi_t$  evolves as in the standard model, it remains a martingale when the source is informative. However, when the source is uninformative,  $\pi_t$  ceases to be a martingale under the measure  $\mathbb{P}_0$ . Therefore, we need to employ a different analytical approach to understand what ensures asymptotic learning of informativeness.<sup>16</sup> To do so, we focus on the event  $\{a_1 = a_2 = \dots = a\}$  in which all agents immediately reach a consensus on some action  $a \in \{\mathfrak{g}, \mathfrak{b}\}$ . We denote such an event by  $\{\bar{a} = a\}$  and refer to it as *immediate agreement on action  $a$* . Note that conditioned on this event, the process of public log-likelihood ratios  $(r_t)$  is deterministic and evolves according to either (3) or (4).

The following lemma shows that conditioned on an informative source, immediate agreement on the wrong action is impossible, whereas immediate agreement on the correct

<sup>16</sup>A similar approach can be found in Arieli, Babichenko, Müller, Pourbabaee, and Tamuz (2023), where the agent's public belief also ceases to be a martingale under the correct measure because overconfident agents have misspecified beliefs.

action is possible, at least for some prior. For brevity, we state this result only for the case where  $\theta = \mathbf{g}$ . By symmetry, analogous statements hold for  $\theta = \mathbf{b}$ .

**Lemma 4.** *Conditioned on  $\omega = 1$  and  $\theta = \mathbf{g}$ , the following two conditions hold:*

- (i) *Immediate agreement on action  $\mathbf{b}$  is impossible.*
- (ii) *Immediate agreement on action  $\mathbf{g}$  is possible for some prior  $\pi \in (0, 1)$ .*

The first part of Lemma 4 holds since, as mentioned before, conditioned on an informative source, all but finitely many agents take the correct action. This immediately implies that agents cannot reach a consensus on the wrong action from the outset. Likewise, the second part of Lemma 4 holds because if agents eventually reach a consensus on the correct action, by stationarity, they can also do so immediately at least for some prior.<sup>17</sup>

Next, we focus on the immediate agreement event conditioned on an uninformative source. Recall that in Proposition 1, we establish the relationship between learning about informativeness and perpetual disagreement. Building on this, we now characterize asymptotic learning of informativeness in terms of immediate agreement, which is a crucial step in proving our main result.

**Proposition 4.** *Asymptotic learning of informativeness holds if and only if conditioned on  $\omega = 0$ , immediate agreement on any action is impossible.*

This proposition states that achieving asymptotic learning of informativeness is equivalent to the absence of immediate agreement starting from a uniform prior given an uninformative source. The proof of Proposition 4 utilizes the idea that the agent’s belief updating process is eventually monotonic—a technical property that we establish in Lemma 6 in the appendix. This property ensures that if immediate agreement is impossible for some prior, e.g., the uniform prior, it becomes impossible for *all* prior. By applying stationarity, this implies that agents cannot reach a consensus on any actions for any prior. Consequently, perpetual disagreement occurs with probability one, and it follows from Proposition 1 that this is equivalent to achieving asymptotic learning of informativeness.

Therefore, we have reduced the problem of learning about informativeness to determining the possibility of immediate agreement conditioned on an uninformative source, which is much easier to analyze. Specifically, conditioned on the event  $\{\bar{a} = \mathbf{g}\}$ , let  $r_t^{\mathbf{g}}$  denote the deterministic process of  $r_t$  based on (3). Recall that agent  $t$  chooses  $a_t = \mathbf{g}$  if  $\ell_t \geq -r_t$  and  $a_t = \mathbf{b}$  otherwise. Consequently, the probability of  $\{\bar{a} = \mathbf{g}\}$  is equal to the probability that  $\ell_t \geq -r_t^{\mathbf{g}}$  for all  $t \geq 1$ . Moreover, conditioned on an uninformative source, since private signals are i.i.d., the corresponding private log-likelihood ratios are

<sup>17</sup>In fact, part (ii) of Lemma 4 holds not only for some prior but also for any prior. This can be seen by applying a similar argument used in the proof of Lemma 7 in the appendix. We omit the stronger statement here, as it is not required to prove our main result.

also i.i.d. Thus, conditioned on  $\omega = 0$ , the probability of immediate agreement on action  $\mathbf{g}$  is

$$\mathbb{P}_0[\bar{a} = \mathbf{g}] = \prod_{t=1}^{\infty} (1 - F_0(-r_t^{\mathbf{g}})).$$

To determine whether the above probability is positive or zero, by a standard approximation argument, it is equivalent to examining whether the sum of the probabilities of the following events is finite or infinite:

$$\mathbb{P}_0[\bar{a} = \mathbf{g}] > 0 (= 0) \Leftrightarrow \sum_{t=1}^{\infty} F_0(-r_t^{\mathbf{g}}) < \infty (= \infty). \quad (5)$$

By symmetry of the model, we also have

$$\mathbb{P}_0[\bar{a} = \mathbf{b}] > 0 (= 0) \Leftrightarrow \sum_{t=1}^{\infty} (1 - F_0(r_t^{\mathbf{g}})) < \infty (= \infty). \quad (6)$$

In summary, asymptotic learning of informativeness holds if and only if conditioned on the source being uninformative, the probability of generating extreme signals decreases slowly enough so that both sums in (5) and (6) are infinite. As we discuss below, for uninformative signals with fatter tails, these sums diverge, and for signals with thinner tails, at least one of these sums converges.

**6.3. Proof Sketch of Theorem 1.** We conclude this section by providing a sketch of the proof of Theorem 1. On the one hand, by part (i) of Lemma 4, the probability of generating extreme signals that match the payoff-relevant state decreases relatively slowly under informative signals. Hence, if the source is uninformative and generates signals with fatter tails, this probability declines even more slowly. Consequently, both the sums in (5) and (6) are infinite, which means that no immediate agreement is possible. Thus, by Proposition 4, asymptotic learning of informativeness holds. On the other hand, by part (ii) of Lemma 4, the probability of generating extreme signals that mismatch the state decreases relatively fast under informative signals. Hence, if the source is uninformative and generates signals with thinner tails, this probability decreases even more rapidly, at least for some type of extreme signals. Consequently, either the sum in (5) or the sum in (6) (or both) is finite, which means that immediate agreement on some action is possible. Hence, by Proposition 4, asymptotic learning of informativeness fails.

## 7. NON-UNIFORM PRIOR

So far, we have focused on the situation in which agents possess a uniform prior over the payoff-relevant state, leading them to act as if all signals are informative (Lemma 1). As a result, agents always respond to their private signals, and this responsiveness turns out to be essential for an outside observer aiming to infer the source's informativeness. Intuitively, if agents have a non-uniform prior, those who learn that the source is uninformative should stop responding to their private signals—since they recognize

that these signals contain no information—and instead choose the action favored by their prior; however, as seen before, action consensus under an uninformative source hinders the outside observer’s ability to learn the source’s informativeness, and thus, asymptotic learning of informativeness should not hold. We formalize this intuition in the following proposition.

**Proposition 5.** *Asymptotic learning of informativeness fails whenever the prior on the payoff-relevant state  $\theta$  is non-uniform.*

Thus, when agents have a non-uniform prior—even if it is weak and close to indifference—no signal structure can guarantee asymptotic learning of informativeness. Again, the proof of this proposition uses ideas similar to those in Proposition 1. We refer to the action favored by the prior as the default action. Suppose to the contrary that asymptotic learning of informativeness holds. In that case, when the source is uninformative, agents will eventually learn this and choose the default action. However, when the source is informative, agents will also take the default action with positive probability, since it is the correct action in some states. But this contradicts the hypothesis that the outside observer’s belief about the source’s informativeness almost surely converges to the truth, and thus we conclude that asymptotic learning of informativeness fails.

## 8. CONCLUSION

In this paper, we study the sequential social learning problem in the presence of a potentially uninformative source. We show that achieving asymptotic learning of informativeness—where an outside observer eventually discerns whether the source is informative or uninformative—is not guaranteed and depends on the relationship between the conditional distributions of the private signals. In particular, it hinges on the relative tail distribution of signals: When uninformative signals have fatter tails than their informative counterparts, the observer eventually learns the source’s informativeness; conversely, when they have thinner tails, this learning fails. We also characterize the conditions for asymptotic learning of informativeness in the canonical case of Gaussian private signals, where the relative tail thickness is incomparable.

More generally, our analysis suggests that irregular behavior, such as an action switch (or a disagreement) following a prolonged sequence of identical actions, is the driving force behind learning about the source’s informativeness. Indeed, contrary to the public-signal benchmark where asymptotic learning of informativeness is always achieved, if agents’ signals remain private, an outside observer can only learn that the source is uninformative by observing agents switching their actions. We show that conditioned on an uninformative source, when action switches accumulate indefinitely (or disagreements

persist), the observer eventually learns the informativeness of the source, and vice versa. We view this characterization as the key mechanism behind our main result.<sup>18</sup>

We make the symmetry assumption on the informative distributions  $F_{\mathfrak{g}}$  and  $F_{\mathfrak{b}}$  for expositional purposes. Since our notion of relative tail thickness does not require this symmetry, our main result (Theorem 1) can be easily extended to the non-symmetric case. A more substantial assumption is that agents, when conditioned on an uninformative source, are indifferent between both actions. As shown in Section 7, relaxing this assumption leads to the failure of asymptotic learning of informativeness. Nevertheless, it seems reasonable to conjecture that, when the prior is very close to uniform, an outside observer would almost learn the source’s informativeness. Indeed, for a sufficiently weak non-uniform prior, the likelihood of observing the same action history should closely resemble that under a uniform prior. It would be interesting to understand the conditions under which learning about informativeness almost holds—not almost surely converging to the truth but close enough to the truth—suggesting an alternative notion of asymptotic learning.

A limitation of our results is that they apply only asymptotically. Our numerical simulations suggest that in the case of uninformative Gaussian signals with fatter tails, the asymptotics can already kick in relatively early in the process. Thus, it would also be interesting to understand the speed at which an outside observer learns about the informativeness of the source. Another promising direction for future research is to explore varying degrees of informativeness, beyond the extreme cases considered in this paper. For example, instead of assuming the source is either informative or completely uninformative, one could consider weakly or strongly informative sources and ask whether learning about the source’s informativeness can still be achieved. We conjecture that our current notion of asymptotic learning would fail in this case, as the “perpetual disagreement” argument would not hold, given that a weakly informative source with unbounded signals would still lead to action convergence. However, it seems plausible that if the frequency of action switches differs significantly across different informative sources, the observer could almost learn to distinguish between them.<sup>19</sup> We leave these interesting questions for future research.

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<sup>18</sup>In the context of scientific paradigms, this mechanism is reminiscent of [Kuhn \(1962\)](#)’s idea that the accumulation of anomalies may trigger scientific revolutions and paradigm shifts. See [Ba \(2022\)](#) for a study on the rationale behind the persistence of a misspecified model, e.g., a wrong scientific paradigm.

<sup>19</sup>For example, one way to generate different frequencies of action switches is to allow multiple draws of the payoff-relevant state. We conjecture that this would provide more information to the outside observer, potentially speeding up learning. This is because, compared to an uninformative source, agents’ actions are more likely to change under an informative source, as they are more likely to receive opposing signals when the payoff-relevant state changes.

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## APPENDIX A. PROOFS

**Proof of Lemma 1.** Recall that in equilibrium, each agent  $t$  chooses the action that is most likely to match the state  $\theta$  conditioned on the available information  $(H_t, s_t)$ . By Bayes’ rule, the relative likelihood between the good state and the bad state for agent  $t$  is

$$\frac{\mathbb{P}[\theta = \mathbf{g}|H_t, s_t]}{\mathbb{P}[\theta = \mathbf{b}|H_t, s_t]} = \frac{\sum_{\tilde{\omega} \in \{0,1\}} \mathbb{P}_{\tilde{\omega}}[\theta = \mathbf{g}|H_t, s_t] \cdot \mathbb{P}[\omega = \tilde{\omega}|H_t, s_t]}{\sum_{\tilde{\omega} \in \{0,1\}} \mathbb{P}_{\tilde{\omega}}[\theta = \mathbf{b}|H_t, s_t] \cdot \mathbb{P}[\omega = \tilde{\omega}|H_t, s_t]}.$$

Note that  $\mathbb{P}_0[\theta = \mathbf{g}|H_t, s_t] = \mathbb{P}_0[\theta = \mathbf{g}]$  and  $\mathbb{P}_0[\theta = \mathbf{b}|H_t, s_t] = \mathbb{P}_0[\theta = \mathbf{b}]$  as conditioned on  $\omega = 0$ , neither the history  $H_t$  nor the signal  $s_t$  contains any information about the payoff-relevant state  $\theta$ . Since the states  $\omega$  and  $\theta$  are independent of each other and the prior on  $\theta$  is uniform,  $\mathbb{P}_0[\theta = \mathbf{g}|H_t, s_t] = \mathbb{P}_0[\theta = \mathbf{b}|H_t, s_t] = 1/2$ . Thus, it follows from the above equation that

$$\frac{\mathbb{P}[\theta = \mathbf{g}|H_t, s_t]}{\mathbb{P}[\theta = \mathbf{b}|H_t, s_t]} \geq 1 \Leftrightarrow \frac{\mathbb{P}_1[\theta = \mathbf{g}|H_t, s_t]}{\mathbb{P}_1[\theta = \mathbf{b}|H_t, s_t]} \geq 1.$$

That is, in equilibrium, each agent chooses the most likely action conditioned on the available information and the source being informative.  $\square$



**Proof of Lemma 2.** For any  $t \geq 1$ , one has that

$$\pi_{t+1} = \mathbb{E}_1[\mathbb{1}(\theta = \mathbf{g})|H_{t+1}] = \mathbb{E}_1[\mathbb{E}_1[\mathbb{1}(\theta = \mathbf{g})|H_t, s_t]|H_{t+1}],$$

where the second equality follows from the law of the iterated expectation. Thus, if  $a_t = \mathbf{g}$ , by Lemma 1,  $\mathbb{P}_1[\theta = \mathbf{g}|H_t, s_t] \geq \mathbb{P}_1[\theta = \mathbf{b}|H_t, s_t]$ . It follows from the above equation that  $\pi_{t+1} \geq 1 - \pi_{t+1}$ , which implies that  $\pi_{t+1} \geq 1/2$ . The case where  $a_t = \mathbf{b}$  implies that  $\pi_{t+1} \leq 1/2$  follows from a symmetric argument.  $\square$

The following simple claim will be useful in proving Proposition 1. It employs an idea similar to the *no introspection principle* in Sørensen (1996). Recall that  $q_t = \mathbb{P}[\omega = 1|H_t]$  is the belief that an outside observer assigns to the source being informative based on the history of actions from time 1 to  $t - 1$  and  $\gamma = \mathbb{P}[\omega = 1] \in (0, 1)$  is the prior that the source is informative.

**Claim 1.** For any  $a \in (0, 1/2)$  and any  $b \in (1/2, 1)$ ,

$$\begin{aligned} \mathbb{P}_0[q_t = \tilde{q}] &\leq \frac{1-a}{a} \frac{\gamma}{1-\gamma} \cdot \mathbb{P}_1[q_t = \tilde{q}], \text{ for all } \tilde{q} \in [a, 1/2]; \\ \mathbb{P}_0[q_t = \tilde{q}] &\geq \frac{1-b}{b} \frac{\gamma}{1-\gamma} \cdot \mathbb{P}_1[q_t = \tilde{q}], \text{ for all } \tilde{q} \in [1/2, b]. \end{aligned}$$

**Proof.** Fix a prior  $\gamma \in (0, 1)$ . For any  $\tilde{q} \in (0, 1)$ , let  $\tilde{H}_t$  be a history of actions such that the associated belief  $q_t$  is equal to  $\tilde{q}$ . By the law of total expectation,

$$\mathbb{P}[\omega = 1|q_t = \tilde{q}] = \mathbb{E}[\mathbb{E}[\mathbb{1}(\omega = 1)|\tilde{H}_t]|q_t = \tilde{q}] = \mathbb{E}[\tilde{q}|q_t = \tilde{q}] = \tilde{q}.$$

It follows from Bayes' rule that

$$\frac{\mathbb{P}_0[q_t = \tilde{q}]}{\mathbb{P}_1[q_t = \tilde{q}]} = \frac{\mathbb{P}[\omega = 0|q_t = \tilde{q}]}{\mathbb{P}[\omega = 1|q_t = \tilde{q}]} \cdot \frac{\mathbb{P}[\omega = 1]}{\mathbb{P}[\omega = 0]} = \frac{1-\tilde{q}}{\tilde{q}} \frac{\gamma}{1-\gamma}.$$

Since for any  $a \in (0, 1/2)$  and any  $\tilde{q} \in [a, 1/2]$ ,  $1 \leq \frac{1-\tilde{q}}{\tilde{q}} \leq \frac{1-a}{a}$ , it thus follows from the above equation that

$$\mathbb{P}_0[q_t = \tilde{q}] = \frac{1-\tilde{q}}{\tilde{q}} \frac{\gamma}{1-\gamma} \cdot \mathbb{P}_1[q_t = \tilde{q}] \leq \frac{1-a}{a} \frac{\gamma}{1-\gamma} \cdot \mathbb{P}_1[q_t = \tilde{q}].$$

The second inequality follows from an identical argument.  $\square$

In the following proofs for Proposition 1, 2 and 3, we use extensively the idea of an outside observer, say observer  $x$ , who observes everyone's actions. The information available to him at time  $t$  is thus  $H_t = (a_1, \dots, a_{t-1})$  and at time infinity is  $H_\infty = \cup_t H_t$ . He gets a utility of one if his guess about  $\omega$  is correct and zero otherwise. Furthermore, we denote by  $q_\infty := \mathbb{P}[\omega = 1|H_\infty]$  the belief that he has at time infinity about the event  $\omega = 1$ . Similarly, we denote by  $p_\infty := \mathbb{P}[\theta = \mathbf{g}|H_\infty]$  the belief that he has at time infinity about the event  $\theta = \mathbf{g}$ .

**Proof of Proposition 1.** We first show the if direction. Recall that  $S = \sum_{t=1}^{\infty} \mathbb{1}(a_t \neq a_{t+1})$  and suppose that  $\mathbb{P}_0[S = \infty] = 1$ . Denote by  $a_t^x$  the guess that the outside observer

$x$  would make to maximize his probability of guessing  $\omega$  correctly at time  $t$ . Fix a large positive integer  $k \in \mathbb{N}$ . Let  $A_t(k)$  denote the event that there have been at least  $k$  action switches before time  $t$  and denote its complementary event by  $A_t^c(k)$ .

Consider the following strategy  $\tilde{a}_\infty^x(k)$  for  $x$  at time infinity:  $\tilde{a}_\infty^x(k) = 0$  if  $A_\infty(k)$  occurs and  $\tilde{a}_\infty^x(k) = 1$  otherwise. That is, the observer would guess 0 if there are at least  $k$  action switches at time infinity and guess 1 otherwise. The expected payoff of  $x$  under this strategy is

$$\mathbb{P}[\tilde{a}_\infty^x(k) = \omega] = \mathbb{P}_0[A_\infty(k)] \cdot \mathbb{P}[\omega = 0] + \mathbb{P}_1[A_\infty^c(k)] \cdot \mathbb{P}[\omega = 1]. \quad (\text{A.1})$$

Since conditioned on  $\omega = 1$ , agents eventually reach a consensus almost surely, it follows that for all  $k$  large enough,

$$\mathbb{P}_1[A_\infty^c(k)] = 1. \quad (\text{A.2})$$

By assumption,  $\mathbb{P}_0[S = \infty] = 1$ , which implies that  $\mathbb{P}_0[A_\infty(k)] = 1$  for all  $k$ . Thus, it follows from (A.1) and (A.2) that for all  $k$  large enough,  $\mathbb{P}[\tilde{a}_\infty^x(k) = \omega] = 1$ . In other words, for all  $k$  large enough, the strategy  $\tilde{a}_\infty^x(k)$  achieves the maximal payoff for  $x$ .

Meanwhile, note that the optimal strategy for  $x$  at time infinity is to make a guess that he believes is most likely, given the information  $H_\infty$ :  $a_\infty^x = 1$  if  $q_\infty \geq 1/2$  and  $a_\infty^x = 0$  otherwise. Since  $\mathbb{P}[\tilde{a}_\infty^x(k) = \omega] = 1$  for all  $k$  large enough, the optimal strategy  $a_\infty^x$  must also achieve the maximal payoff of one:

$$1 = \mathbb{P}[a_\infty^x = \omega] = \mathbb{P}_1[q_\infty \geq 1/2] \cdot \mathbb{P}[\omega = 1] + \mathbb{P}_0[q_\infty < 1/2] \cdot \mathbb{P}[\omega = 0]. \quad (\text{A.3})$$

It follows from (A.3) that  $\mathbb{P}_0[q_\infty < 1/2] = \mathbb{P}_1[q_\infty \geq 1/2] = 1$ . It remains to show that  $\mathbb{P}_0[q_\infty = 0] = 1$  and  $\mathbb{P}_1[q_\infty = 1] = 1$ . To this end, first notice that  $\mathbb{P}_0[q_\infty < 1/2] = 1$  implies that  $\mathbb{P}_0[q_\infty \geq 1/2] = 0$ . Thus, by Claim 1, it further implies that for any  $b \in (1/2, 1)$  and all  $\tilde{q} \in [1/2, b]$ ,  $\mathbb{P}_1[q_\infty = \tilde{q}] = 0$ . Consequently, it follows from  $\mathbb{P}_1[q_\infty \geq 1/2] = 1$  that  $\mathbb{P}_1[q_\infty = 1] = 1$ . The case that  $\mathbb{P}_0[q_\infty = 0] = 1$  follows from an identical argument. Together, we conclude that asymptotic learning of informativeness holds.

Next, we show the only-if direction. Suppose by contraposition that  $\mathbb{P}_0[S < \infty] > 0$ . Again, since conditioned on  $\omega = 1$ , agents eventually reach a consensus on the correct action, this implies that  $\mathbb{P}_1[S < \infty] = 1$ . Thus, there exists a history of actions at time infinity  $\tilde{H}_\infty$  that is possible under both probability measures  $\mathbb{P}_0$  and  $\mathbb{P}_1$ :  $\mathbb{P}_0[\tilde{H}_\infty] > 0$  and  $\mathbb{P}_1[\tilde{H}_\infty] > 0$ . It follows from Bayes' rule that  $\mathbb{P}[\omega = 1 | \tilde{H}_\infty] < 1$  and  $\mathbb{P}[\omega = 0 | \tilde{H}_\infty] < 1$ .

Assume without loss of generality that under this history  $\tilde{H}_\infty$ , the corresponding belief  $\tilde{q}_\infty \geq 1/2$ . Therefore, given  $\tilde{H}_\infty$ , the observer  $x$  would guess  $a_\infty^x = 1$ . As a consequence, the probability of  $x$  guessing correctly about  $\omega$  is strictly less than one:

$$\begin{aligned} \mathbb{P}[a_\infty^x = \omega] &= \mathbb{P}[a_\infty^x = \omega, \tilde{H}_\infty] + \mathbb{P}[a_\infty^x = \omega, \tilde{H}_\infty^c] \\ &= \mathbb{P}[\omega = 1 | \tilde{H}_\infty] \cdot \mathbb{P}[\tilde{H}_\infty] + \mathbb{P}[a_\infty^x = \omega, \tilde{H}_\infty^c] \\ &< \mathbb{P}[\tilde{H}_\infty] + \mathbb{P}[\tilde{H}_\infty^c] = 1. \end{aligned}$$

This implies that asymptotic learning of informativeness fails since otherwise, by definition  $\mathbb{P}[a_\infty^x = \omega] = 1$ , which is in contradiction with the above strict inequality.  $\square$

The following equation will be useful in proving Proposition 2, 3, and later in Proposition 5. Recall that  $q_\infty$  and  $p_\infty$  are the beliefs of the observer  $x$  assigned to the events  $\omega = 1$  and  $\theta = \mathbf{g}$  at time infinity, respectively. By the law of total probability,

$$p_\infty = \mathbb{P}_1[\theta = \mathbf{g}|H_\infty] \cdot q_\infty + \mathbb{P}_0[\theta = \mathbf{g}|H_\infty] \cdot (1 - q_\infty) \quad (\text{A.4})$$

$$= \pi_\infty \cdot q_\infty + \frac{1}{2} \cdot (1 - q_\infty). \quad (\text{A.5})$$

where  $\pi_\infty = \mathbb{P}_1[\theta = \mathbf{g}|H_\infty]$ , and the second equality follows from the fact that, conditioned on  $\omega = 0$ , no action contains any information about  $\theta$  and the prior on  $\theta$  is uniform.

**Proof of Proposition 2.** Since agents always act as if signals are informative (Lemma 1), the agent's public belief  $\pi_t = \mathbb{P}_1[\theta = \mathbf{g}|H_t]$  remains a martingale under the measure  $\mathbb{P}_1$ . Using a standard martingale convergence argument with unbounded signals (Smith and Sørensen, 2000), we have that (i) conditioned on  $\omega = 1$  and  $\theta = \mathbf{g}$ ,  $\pi_\infty = 1$  almost surely and (ii) conditioned on  $\omega = 1$  and  $\theta = \mathbf{b}$ ,  $\pi_\infty = 0$  almost surely.

Suppose that conditioned on  $\omega = 1$ , complete learning holds. That is, (i) conditioned on  $\omega = 1$  and  $\theta = \mathbf{g}$ ,  $p_\infty = 1$  almost surely and (ii) conditioned on  $\omega = 1$  and  $\theta = \mathbf{b}$ ,  $p_\infty = 0$  almost surely. It follows from (A.5) that conditioned on  $\omega = 1$ ,  $q_\infty = 1$  almost surely so that asymptotic learning of informativeness holds. Conversely, suppose that asymptotic learning of informativeness holds. By definition, conditioned on  $\omega = 1$ ,  $q_\infty = 1$  almost surely. It then follows from (A.5) that conditioned on  $\omega = 1$ ,  $p_\infty = \pi_\infty$  almost surely, and thus complete learning holds.  $\square$

**Proof of Proposition 3.** The only-if direction is straightforward: Suppose asymptotic learning of informativeness holds. By definition, conditioned on  $\omega = 0$ ,  $q_\infty = 0$  almost surely, and thus it follows from (A.5) that  $p_\infty = 1/2$  almost surely.

Now, we prove the if direction. Suppose by contrapositive that asymptotic learning of informativeness fails. By Proposition 1, conditioned on  $\omega = 0$ , there exists a history  $\tilde{H}_\infty$  that is possible at time infinity in which agents eventually reach a consensus on some action. Since such history is also possible conditioned on  $\omega = 1$ , the corresponding belief of an outside observer  $\tilde{q}_\infty \in (0, 1)$ .<sup>20</sup> Furthermore, note that depending on the action to which the history  $\tilde{H}_\infty$  converges, the corresponding public belief of the agent  $\tilde{\pi}_\infty$  takes values in  $\{0, 1\}$ . Hence, by (A.5), the event  $\{p_\infty \neq 1/2\}$  occurs with positive probability.  $\square$

**Proof of Lemma 4.** The proof idea is similar to the proof of Lemma 10 in Arieli, Babichenko, Müller, Pourbabaee, and Tamuz (2023). Recall that we use  $\mathbb{P}_{1,\mathbf{g}}$  to denote the conditional probability distribution given  $\omega = 1$  and  $\theta = \mathbf{g}$  and we use  $\mathbb{P}_{1,\mathbf{g},\pi}$  to

<sup>20</sup>Note that conditioned on  $\omega = 0$ ,  $q_\infty$  has support  $\subseteq [0, 1)$  as  $\mathbb{P}[\omega = 0|q_\infty = 1] = 0$ .

denote the same conditional probability distribution while emphasizing the prior value. Since conditioned on  $\omega = 1$ , correct herding holds, this means that  $\mathbb{P}_{1,\mathbf{g}}[\lim_t a_t = \mathbf{g}] = 1$ . As a consequence, part (i) follows directly from the fact that the events  $\{\bar{a} = \mathbf{b}\}$  and  $\{\lim_t a_t = \mathbf{g}\}$  are disjoint, and thus  $\mathbb{P}_{1,\mathbf{g}}[\bar{a} = \mathbf{b}] = 0$ .

For part (ii), let  $\tau < \infty$  denote the last random time at which the agent chooses the wrong action  $\mathbf{b}$ . It is well-defined as correct herding holds. Hence,  $1 = \mathbb{P}_{1,\mathbf{g}}[\lim_t a_t = \mathbf{g}] = \sum_{k=1}^{\infty} \mathbb{P}_{1,\mathbf{g}}[\tau = k]$ . By the overturning principle (Lemma 2),  $a_\tau = \mathbf{b}$  implies that  $\pi_{\tau+1} \leq 1/2$ . As a consequence,

$$\begin{aligned} 1 &= \sum_{k=1}^{\infty} \mathbb{P}_{1,\mathbf{g}}[\tau = k] = \sum_{k=1}^{\infty} \mathbb{P}_{1,\mathbf{g}}[a_{k+1} = a_{k+2} = \dots = \mathbf{g}, \pi_{k+1} \leq 1/2] \\ &= \sum_{k=1}^{\infty} \mathbb{E}_{1,\mathbf{g}}[\mathbb{P}_{1,\mathbf{g}}[a_{k+1} = a_{k+2} = \dots = \mathbf{g}, \pi_{k+1} \leq 1/2 \mid \pi_{k+1}]] \\ &= \sum_{k=1}^{\infty} \mathbb{E}_{1,\mathbf{g}}[\mathbb{1}(\pi_{k+1} \leq 1/2) \cdot \mathbb{P}_{1,\mathbf{g}}[a_{k+1} = a_{k+2} = \dots = \mathbf{g} \mid \pi_{k+1}]] \\ &= \sum_{k=1}^{\infty} \mathbb{E}_{1,\mathbf{g}}[\mathbb{1}(\pi_{k+1} \leq 1/2) \cdot \mathbb{P}_{1,\mathbf{g},\pi_{k+1}}[\bar{a} = \mathbf{g}]], \end{aligned}$$

where the second equality follows from the law of total expectation, and the last equality follows from the stationarity property (Lemma 3). Suppose that for all prior  $\pi \in (0, 1)$ ,  $\mathbb{P}_{1,\mathbf{g},\pi}[\bar{a} = \mathbf{g}] = 0$ . This implies that the above equation equals zero, a contradiction.  $\square$

**A.1. Proof of Theorem 1.** In this section, we prove Proposition 4 and Theorem 1. To prove Proposition 4, we will first prove the following proposition (Proposition 6). Together with Proposition 1, they jointly imply Proposition 4. The proof of Theorem 1 is presented at the end of this section. We write  $\mathbb{P}_{0,\pi}$  to denote the conditional probability distribution given  $\omega = 0$  while highlighting the value of the prior  $\pi$  on  $\theta$ .<sup>21</sup>

**Proposition 6.** *The following are equivalent.*

- (i) For any action  $a \in \{\mathbf{g}, \mathbf{b}\}$ ,  $\mathbb{P}_{0,\pi}[\bar{a} = a] = 0$  for all prior  $\pi \in (0, 1)$ .
- (ii)  $\mathbb{P}_0[S = \infty] = 1$ .
- (iii) For any action  $a \in \{\mathbf{g}, \mathbf{b}\}$ ,  $\mathbb{P}_{0,\pi}[\bar{a} = a] = 0$  for some prior  $\pi \in (0, 1)$ .

To prove this proposition, we first establish some preliminary results on the process of the agents' public log-likelihood ratios conditioned on the event of immediate agreement. These results lead to Lemma 7, a crucial part in establishing the equivalence between no immediate agreement and perpetual disagreement conditioned on an uninformative source. We present the proof of Proposition 6 towards the end of this section.

*Preliminaries.* Recall that conditioned on  $\{\bar{a} = \mathbf{g}\}$ , the process of the agent's public log-likelihood ratio  $r_t$  evolves deterministically according to (3), which we denote by  $r_t^{\mathbf{g}}$ . Let

<sup>21</sup>We continue to omit the prior  $\pi$  when it is uniform.

the corresponding updating function be

$$\phi(x) := x + D_{\mathfrak{g}}(x).$$

That is,  $r_{t+1}^{\mathfrak{g}} = \phi(r_t^{\mathfrak{g}})$  for all  $t \geq 1$ . Since the entire sequence  $(r_t^{\mathfrak{g}})$  is determined once its initial value  $r_1^{\mathfrak{g}}$  is specified, we denote the value of  $r_t^{\mathfrak{g}}$  with an initial value  $r_1^{\mathfrak{g}} = r$  by  $r_t^{\mathfrak{g}}(r)$ . We can thus write  $r_t^{\mathfrak{g}}(r) = \phi^{t-1}(r)$  for all  $t \geq 1$ , where  $\phi^t$  is its  $t$ -th composition and  $\phi^0(r) = r$ .

We remind the readers of two standard properties of the sequence  $(r_t^{\mathfrak{g}})$ , as summarized in the following lemma. The first part of this lemma states that  $(r_t^{\mathfrak{g}})$  tends to infinity as  $t$  tends to infinity, and the second part shows that it takes only some bounded time for the sequence  $(r_t^{\mathfrak{g}})$  to reach any positive value.

**Lemma 5** (The Long-Run and Short-Run Behaviors of  $r_t^{\mathfrak{g}}$ ).

(i)  $\lim_{t \rightarrow \infty} r_t^{\mathfrak{g}} = \infty$ .

(ii) For any  $\bar{r} \geq 0$ , there exists  $t_0$  such that  $r_{t_0}^{\mathfrak{g}}(r) \geq \bar{r}$  for all  $r \geq 0$ .

**Proof.** See Lemma 6 and Lemma 12 in [Rosenberg and Vieille \(2019\)](#). □

Note that although the sequence  $(r_t^{\mathfrak{g}})$  eventually approaches infinity, it may not do so monotonically without additional assumptions on the distributions  $F_{\mathfrak{g}}$  and  $F_{\mathfrak{b}}$ .<sup>22</sup> The next lemma shows that, under some mild technical assumptions on the left tail of  $F_{\mathfrak{b}}$ , the function  $\phi(x)$  eventually increases monotonically.

**Lemma 6** (Eventual Monotonicity). *Suppose that  $F_{\mathfrak{b}}$  has a differentiable left tail and its probability density function  $f_{\mathfrak{b}}$  satisfies the condition that, for all  $x$  large enough,  $f_{\mathfrak{b}}(-x) < 1$ . Then,  $\phi(x) := x + D_{\mathfrak{g}}(x)$  increases monotonically for all  $x$  large enough.*

**Proof.** By assumption, we can find a constant  $\rho < 1$  such that for all  $x$  large enough,  $f_{\mathfrak{b}}(-x) \leq \rho$ . By definition,  $D_{\mathfrak{g}}(x) = \log \frac{1 - F_{\mathfrak{g}}(-x)}{1 - F_{\mathfrak{b}}(-x)}$ . Taking the derivative of  $D_{\mathfrak{g}}$ ,

$$D'_{\mathfrak{g}}(x) = \frac{f_{\mathfrak{g}}(-x)}{1 - F_{\mathfrak{g}}(-x)} - \frac{f_{\mathfrak{b}}(-x)}{1 - F_{\mathfrak{b}}(-x)}.$$

Observe that the log-likelihood ratio of the agent's private log-likelihood ratio  $\ell_t$  is the log-likelihood ratio itself (see, e.g., [Chamley \(2004\)](#)):

$$\log \frac{dF_{\mathfrak{g}}}{dF_{\mathfrak{b}}}(x) = x.$$

It follows that

$$-D'_{\mathfrak{g}}(x) = f_{\mathfrak{b}}(-x) \left( \frac{1}{1 - F_{\mathfrak{b}}(-x)} - \frac{e^{-x}}{1 - F_{\mathfrak{g}}(-x)} \right) \leq \frac{f_{\mathfrak{b}}(-x)}{1 - F_{\mathfrak{b}}(-x)}.$$

<sup>22</sup>In the case of binary states and actions, [Herrera and Hörner \(2012\)](#) show that the property of increasing hazard ratio is equivalent to the monotonicity of this updating function. See [Smith, Sørensen, and Tian \(2021\)](#) for a general treatment.

Fix some  $\varepsilon > 0$  small enough so that  $(1 + \varepsilon)\rho \leq 1$ . It follows from the above inequality that there exists some  $x$  large enough such that  $-D'_g(x) \leq (1 + \varepsilon)f_b(-x)$ . Furthermore, for all  $x' \geq x$ ,

$$\begin{aligned} D_g(x) &= D_g(x') - \int_x^{x'} D'_g(y) dy \\ &\leq D_g(x') + (1 + \varepsilon) \int_x^{x'} f_b(-x) dx \\ &= D_g(x') - (1 + \varepsilon)(F_b(-x') - F_b(-x)). \end{aligned}$$

Rearranging the above equation,

$$\begin{aligned} D_g(x) - D_g(x') &\leq (1 + \varepsilon)(F_b(-x) - F_b(-x')) \\ &\leq (1 + \varepsilon)\rho(x' - x), \end{aligned}$$

where the second last inequality follows from the fact that  $f_b(-x) \leq \rho < 1$ . Since  $(1 + \varepsilon)\rho \leq 1$ , the above inequality implies that there exists some  $x$  large enough such that  $D_g(x) + x \leq D_g(x') + x'$  for all  $x' \geq x$ . That is,  $\phi(x)$  eventually increases monotonically.  $\square$

Given these lemmas, we are ready to prove the following result. It shows that conditioned on an uninformative source, the possibility of immediate agreement is independent of the prior belief.

**Lemma 7.** *For any action  $a \in \{\mathbf{g}, \mathbf{b}\}$ , the following statements are equivalent:*

- (i)  $\mathbb{P}_{0,\pi}[\bar{a} = a] > 0$ , for some prior  $\pi \in (0, 1)$ ;
- (ii)  $\mathbb{P}_{0,\pi}[\bar{a} = a] > 0$ , for all prior  $\pi \in (0, 1)$ .

**Proof.** The second implication, namely, (ii)  $\Rightarrow$  (i) is immediate. We will show the first implication, (i)  $\Rightarrow$  (ii). Fix some prior  $\tilde{\pi} \in (0, 1)$  such that  $\mathbb{P}_{0,\tilde{\pi}}[\bar{a} = \mathbf{g}] > 0$  and let  $\tilde{r} = \log \frac{\tilde{\pi}}{1-\tilde{\pi}}$ . Since  $r_t^g(\tilde{r})$  is a deterministic process, the event  $\{\bar{a} = \mathbf{g}\}$  initiated at the prior  $\pi_1 = \tilde{\pi}$  is equivalent to the event  $\{\ell_t \geq -r_t^g(\tilde{r}), \forall t \geq 1\}$ . Conditioned on  $\omega = 0$ , since signals are i.i.d., so are the agents' private log-likelihood ratios. Thus, we have

$$\mathbb{P}_{0,\tilde{\pi}}[\bar{a} = \mathbf{g}] = \prod_{t=1}^{\infty} (1 - F_0(-r_t^g(\tilde{r}))). \quad (\text{A.6})$$

As a consequence,  $\mathbb{P}_{0,\tilde{\pi}}[\bar{a} = \mathbf{g}] > 0$  if and only if there exists  $M < \infty$  such that

$$-\sum_{t=1}^{\infty} \log(1 - F_0(-r_t^g(\tilde{r}))) < M.$$

For two sequences  $(a_t)$  and  $(b_t)$ , we write  $a_t \approx b_t$  if  $\lim_{t \rightarrow \infty} (a_t/b_t) = 1$ . Since  $r_t^g(\tilde{r}) \rightarrow \infty$  (this follows from part (i) of Lemma 5),  $\log(1 - F_0(-r_t^g(\tilde{r}))) \approx -F_0(-r_t^g(\tilde{r}))$ . Thus, the

above sum is finite if and only if

$$\sum_{t=1}^{\infty} F_0(-r_t^{\mathfrak{g}}(\tilde{r})) < M. \quad (\text{A.7})$$

By the overturning principle (Lemma 2), it suffices to show that (A.7) implies that

$$\sum_{t=1}^{\infty} F_0(-r_t^{\mathfrak{g}}(r)) < M, \quad \text{for any } r \geq 0.$$

By the eventual monotonicity of  $\phi$  (Lemma 6) and the fact that  $r_t^{\mathfrak{g}}(\tilde{r}) \rightarrow \infty$ , we can find a large enough  $\bar{t}$  such that  $r_{\bar{t}}^{\mathfrak{g}}(\tilde{r}) := \bar{r} \geq 0$  and  $\phi(r) \geq \phi(\bar{r})$  for all  $r \geq \bar{r}$ . By part (ii) of Lemma 5, there exists  $t_0 \in \mathbb{N}$  such that  $r_{t_0}^{\mathfrak{g}}(r) \geq \bar{r}$  for all  $r \geq 0$ . Since above  $\bar{r}$ ,  $\phi$  is monotonically increasing, one has  $\phi(r_{t_0}^{\mathfrak{g}}(r)) \geq \phi(\bar{r})$  for any  $r \geq 0$ . Consequently, for all  $\tau \geq 1$ ,  $r_{\tau+t_0}^{\mathfrak{g}}(r) = \phi^{\tau}(r_{t_0}^{\mathfrak{g}}(r)) \geq \phi^{\tau}(\bar{r}) = r_{\tau+1}^{\mathfrak{g}}(\bar{r})$ . Since  $r_{\tau+1}^{\mathfrak{g}}(\bar{r}) = r_{\tau+1}^{\mathfrak{g}}(r_{\bar{t}}^{\mathfrak{g}}(\tilde{r})) = r_{\tau+\bar{t}}^{\mathfrak{g}}(\tilde{r})$ , it follows that

$$F_0(-r_{\tau+t_0}^{\mathfrak{g}}(r)) \leq F_0(-r_{\tau+\bar{t}}^{\mathfrak{g}}(\tilde{r})).$$

Thus, it follows from (A.7) that for any  $r \geq 0$ ,  $\sum_{t=1}^{\infty} F_0(-r_t^{\mathfrak{g}}(r)) < \infty$ , as required. The case for action  $\mathfrak{b}$  follows from a symmetric argument.  $\square$

Now, we are ready to prove Proposition 6.

**Proof of Proposition 6.** We show that (i)  $\Rightarrow$  (ii), (ii)  $\Rightarrow$  (iii), and (iii)  $\Rightarrow$  (i). To show the first implication, we prove the contrapositive statement. Suppose that  $\mathbb{P}_0[S < \infty] > 0$ . This implies that there exists a sequence of action realizations  $(b_1, b_2, \dots, b_{k-1}, b_k = \dots = a)$  for some action  $a \in \{\mathfrak{b}, \mathfrak{g}\}$  such that

$$\mathbb{P}_0[a_t = b_t, \forall t \geq 1] > 0.$$

By stationarity, there exists some  $\pi' \in (0, 1)$  such that

$$\mathbb{P}_{0, \pi'}[\bar{a} = a] > 0,$$

which contradicts (i).

To show the second implication, suppose towards a contradiction that there exists some action  $a \in \{\mathfrak{g}, \mathfrak{b}\}$  such that  $\mathbb{P}_{0, \pi}[\bar{a} = a] > 0$  for all prior  $\pi \in (0, 1)$ . In particular, it holds for the uniform prior. Since the event  $\{\bar{a} = a\}$  is contained in the event  $\{S < \infty\}$ ,

$$0 < \mathbb{P}_0[\bar{a} = a] \leq \mathbb{P}_0[S < \infty].$$

This implies that  $\mathbb{P}_0[S = \infty] < 1$ , a contradiction to (ii).

Finally, we show the last implication by contraposition. Suppose that there exists some  $a \in \{\mathfrak{g}, \mathfrak{b}\}$  such that  $\mathbb{P}_{0, \pi}[\bar{a} = a] > 0$  for some prior  $\pi \in (0, 1)$ . By Lemma 7, it also holds for all prior  $\pi \in (0, 1)$ , which is a contradiction to (iii). This concludes the proof of Proposition 6.  $\square$

**Proof of Proposition 4.** By the equivalence between (ii) and (iii) in Proposition 6 and Proposition 1, we have shown Proposition 4.  $\square$

Given Proposition 4, we are now ready to prove our main result.

**Proof of Theorem 1.** By part (i) of Lemma 4,  $\mathbb{P}_{1,\mathfrak{b}}[\bar{a} = \mathfrak{g}] = 0$  and  $\mathbb{P}_{1,\mathfrak{g}}[\bar{a} = \mathfrak{b}] = 0$ . Following a similar argument that led to (A.6), one has

$$0 = \mathbb{P}_{1,\mathfrak{b}}[\bar{a} = \mathfrak{g}] = \prod_{t=1}^{\infty} (1 - F_{\mathfrak{b}}(-r_t^{\mathfrak{g}})).$$

Taking the logarithm on both sides, the above equation is equivalent to  $-\sum_{t=1}^{\infty} \log(1 - F_{\mathfrak{b}}(-r_t^{\mathfrak{g}})) = \infty$ . Since  $r_t^{\mathfrak{g}} \rightarrow \infty$ ,  $\log(1 - F_{\mathfrak{b}}(-r_t^{\mathfrak{g}})) \approx -F_{\mathfrak{b}}(-r_t^{\mathfrak{g}})$  and the previous sum is infinite if and only if

$$\sum_{t=1}^{\infty} F_{\mathfrak{b}}(-r_t^{\mathfrak{g}}) = \infty. \quad (\text{A.8})$$

Similarly, we have that  $\mathbb{P}_{1,\mathfrak{g}}[\bar{a} = \mathfrak{b}] = 0$  if and only if  $\sum_{t=1}^{\infty} (1 - F_{\mathfrak{g}}(-r_t^{\mathfrak{b}})) = \infty$ , where  $r_t^{\mathfrak{b}}$  denotes the deterministic process of  $r_t$  conditioned on the event  $\{\bar{a} = \mathfrak{b}\}$ . By symmetry,  $r_t^{\mathfrak{b}} = -r_t^{\mathfrak{g}}$  for all  $t \geq 1$ . Hence,  $\mathbb{P}_{1,\mathfrak{g}}[\bar{a} = \mathfrak{b}] = 0$  if and only if

$$\sum_{t=1}^{\infty} (1 - F_{\mathfrak{g}}(r_t^{\mathfrak{g}})) = \infty, \quad (\text{A.9})$$

Suppose that the uninformative signals have fatter tails than the informative signals. By definition, there exists  $\varepsilon > 0$  such that for all  $x$  large enough,  $F_0(-x) \geq \varepsilon \cdot F_{\mathfrak{b}}(-x)$  and  $1 - F_0(x) \geq \varepsilon \cdot (1 - F_{\mathfrak{g}}(x))$ . It then follows from (A.8) and (A.9) that

$$\sum_{t=1}^{\infty} F_0(-r_t^{\mathfrak{g}}) = \infty \quad \text{and} \quad \sum_{t=1}^{\infty} (1 - F_0(r_t^{\mathfrak{g}})) = \infty.$$

Using the same logic we used to deduce (A.8) and (A.9), having these two divergent sums is equivalent to  $\mathbb{P}_0[\bar{a} = \mathfrak{g}] = 0$  and  $\mathbb{P}_0[\bar{a} = \mathfrak{b}] = 0$ . Thus, by Proposition 4, asymptotic learning of informativeness holds.

By part (ii) of Lemma 4, there exist  $\pi, \pi' \in (0, 1)$  such that  $\mathbb{P}_{1,\mathfrak{g},\pi}[\bar{a} = \mathfrak{g}] > 0$  and  $\mathbb{P}_{1,\mathfrak{b},\pi'}[\bar{a} = \mathfrak{b}] > 0$ . Let  $r = \log \frac{\pi}{1-\pi}$  and  $r' = \log \frac{\pi'}{1-\pi'}$ . Following a similar argument that led to (A.7), these are equivalent to

$$\sum_{t=1}^{\infty} F_{\mathfrak{g}}(-r_t^{\mathfrak{g}}(r)) < \infty \quad \text{and} \quad \sum_{t=1}^{\infty} (1 - F_{\mathfrak{b}}(r_t^{\mathfrak{g}}(r'))) < \infty.$$

Now, suppose that the uninformative signals have thinner tails than the informative signals. By definition, there exists  $\varepsilon > 0$  such that either (i)  $F_0(-x) \leq (1/\varepsilon) \cdot F_{\mathfrak{g}}(-x)$  for all  $x$  large enough, or (ii)  $1 - F_0(x) \leq (1/\varepsilon) \cdot (1 - F_{\mathfrak{b}}(x))$  for all  $x$  large enough. It then follows from the above inequalities that either (i)  $\sum_{t=1}^{\infty} F_0(-r_t^{\mathfrak{g}}(r)) < \infty$  or (ii)  $\sum_{t=1}^{\infty} (1 - F_0(r_t^{\mathfrak{g}}(r'))) < \infty$ . Equivalently, we have either (i)  $\mathbb{P}_{0,\pi}[\bar{a} = \mathfrak{g}] > 0$  for some  $\pi \in (0, 1)$  or (ii)  $\mathbb{P}_{0,\pi'}[\bar{a} = \mathfrak{b}] > 0$  for some  $\pi' \in (0, 1)$ . By Lemma 7, these also hold for all prior



$\pi, \pi' \in (0, 1)$ , including the uniform prior. So we have either (i)  $\mathbb{P}_0[\bar{a} = \mathfrak{g}] > 0$  or (ii)  $\mathbb{P}_0[\bar{a} = \mathfrak{b}] > 0$ . Thus, by Proposition 4, asymptotic learning of informativeness fails.  $\square$

**A.2. Gaussian Private Signals.** In this section, we consider the canonical environment in which all signal distributions  $\mu_{\mathfrak{g}}, \mu_{\mathfrak{b}}$  and  $\mu_0$  are normal. In particular, the distributions  $\mu_{\mathfrak{g}}$  and  $\mu_{\mathfrak{b}}$  share the same variance  $\sigma^2$  and have mean  $+1$  and  $-1$ , respectively. Meanwhile,  $\mu_0$  has mean  $m_0 \in (-1, 1)$  and variance  $\tau^2$ . Note that in this case, the agent's private log-likelihood ratio induced by a signal  $s_t$  is

$$\ell_t = \log \frac{f_{\mathfrak{g}}(s_t)}{f_{\mathfrak{b}}(s_t)} = \frac{2}{\sigma^2} s_t. \quad (\text{A.10})$$

Since  $\ell_t$  is proportional to  $s_t$ , the distributions  $F_{\theta}$  and  $F_0$  are also normal, with a variance of  $4/\sigma^2$  and  $4\tau^2/\sigma^4$ , respectively.

We first consider the case where  $\tau \neq \sigma$ . In this case, the relative tail thickness between the uninformative and informative signals is solely determined by their variances, as shown by the following lemma.

**Lemma 8.** *Suppose  $F$  and  $G$  are two Gaussian cumulative distribution functions with means  $\mu_1$  and  $\mu_2$  and variances  $\sigma_1^2$  and  $\sigma_2^2$ , respectively. If  $\sigma_1 > \sigma_2$ , then  $F$  has fatter tails than  $G$ . Meanwhile,  $G$  has thinner tails than  $F$ .*

**Proof.** Let  $f$  and  $g$  denote the probability density functions of  $F$  and  $G$ . Their ratio evaluated at  $x \in \mathbb{R}$  is

$$\frac{f(x)}{g(x)} = \frac{\sigma_2}{\sigma_1} \exp \left( \left( \frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2} \right) \frac{x^2}{2} + \left( \frac{\mu_1}{\sigma_1^2} - \frac{\mu_2}{\sigma_2^2} \right) x + \frac{1}{2} \left( \frac{\mu_2^2}{\sigma_2^2} - \frac{\mu_1^2}{\sigma_1^2} \right) \right).$$

Suppose  $\sigma_1 > \sigma_2$ . It follows from the above equation that  $\lim_{x \rightarrow \infty} \frac{f(-x)}{g(-x)} = \infty$  and  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \infty$ . This clearly implies that  $F(-x) \geq G(-x)$  and  $1 - F(x) \geq 1 - G(x)$  for all  $x$  large enough. By Definition 2,  $F$  has fatter tails than  $G$ , and conversely,  $G$  has thinner tails than  $F$ .  $\square$

We henceforth focus on the case where  $\tau = \sigma$ . When all private signals are Gaussian, [Hann-Caruthers, Martynov, and Tamuz \(2018\)](#) show that one can approximate the sequence  $r_t^{\mathfrak{g}}$  by  $(2\sqrt{2}/\sigma) \cdot \sqrt{\log t}$  for all  $t$  large enough (see their Theorem 4):

$$\lim_{t \rightarrow \infty} \frac{r_t^{\mathfrak{g}}}{(2\sqrt{2}/\sigma) \cdot \sqrt{\log t}} = 1. \quad (\text{A.11})$$

Given this approximation and Proposition 4, we are ready to prove Theorem 2.

**Proof of Theorem 2.** In this proof we use the Landau notation, so that  $O(g(t))$  stands for some function  $f : \mathbb{N} \rightarrow \mathbb{R}$  such that there exists a positive  $M \in \mathbb{R}$  and  $t_0 \in \mathbb{N}$  such that  $|f(t)| \leq M \cdot g(t)$  for all  $t \geq t_0$ .

Note that by (A.10), we can write

$$F_0(-r_t^{\mathfrak{g}}) = \mathbb{P}_0[\ell_t \leq -r_t^{\mathfrak{g}}] = \mathbb{P}_0[s_t \leq -(\sigma^2/2) \cdot r_t^{\mathfrak{g}}].$$

By (A.11), we have that for all  $t$  large enough,

$$F_0(-r_t^g) = \mathbb{P}_0[s_t \leq -\sigma\sqrt{2\log t}] =: \mu_0(-\sigma\sqrt{2\log t}),$$

where  $\mu_0$  is the CDF of  $s_t$  conditioned on  $\omega = 0$ . Since  $\mu_0$  is the normal distribution with mean  $m_0 \in (-1, 1)$  and variance  $\sigma^2$ , observe that  $\mu_0(x) = \frac{1}{2} \operatorname{erfc}(-\frac{x-m_0}{\sigma\sqrt{2}})$ , where  $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$  is the complementary error function.

Applying a standard asymptotic expansion of the complementary error function, i.e.,  $\operatorname{erfc}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} + O(e^{-x^2}/x^3)$ , we obtain that for all  $t$  large enough,

$$\mu_0(-\sigma\sqrt{2\log t}) = \frac{e^{-\left(\frac{m_0}{\sigma}\sqrt{2\log t} + \frac{m_0^2}{2\sigma^2}\right)}}{t(\sqrt{\pi}\log t + \delta \cdot m_0)} + O\left(\frac{e^{-m_0\sqrt{2\log t}}}{t(\sigma\sqrt{2\log t} + m_0)^3}\right), \quad (\text{A.12})$$

where  $\delta > 0$  is a constant.

Case (i): suppose  $m_0 = 0$ . Then (A.12) becomes  $\frac{1}{t\sqrt{\pi}\log t} + O\left(\frac{1}{t(\log t)^{3/2}}\right)$ . Since the series  $\frac{1}{t\log t}$  is divergent and  $\frac{1}{t\log t} \leq \frac{1}{t\sqrt{\log t}}$  for all  $t \geq 2$ , the sum of the first term also diverges. Hence, the sum of (A.12) diverges, which implies  $\sum_{t=1}^\infty F_0(-r_t^g) = \infty$ . By the same approximation argument used in the proof of Theorem 1, this is equivalent to  $\mathbb{P}_0[\bar{a} = \mathbf{g}] = 0$ . Using an analogous argument, we obtain  $\mathbb{P}_0[\bar{a} = \mathbf{b}] = 0$ . Together, by Proposition 4, we conclude that asymptotic learning of informativeness holds.

Case (ii): suppose  $m_0 \neq 0$  and  $m_0 \in (-1, 1)$ . Let  $c = \frac{m_0\sqrt{2}}{\sigma}$ . By the change of variable  $x = \sqrt{\log t}$ ,

$$\int_2^\infty \frac{e^{-c\sqrt{\log t}}}{t\sqrt{\log t}} dt = 2 \int_{\sqrt{\log 2}}^\infty e^{-cx} dx.$$

If  $m_0 > 0$ , then  $c = \frac{m_0\sqrt{2}}{\sigma} > 0$ . By the integral test, the sum in (A.12) converges, and thus  $\sum_{t=1}^\infty F_0(-r_t^g) < \infty$ . Again, by the same logic that we use to deduce (A.7), this is equivalent to  $\mathbb{P}_0[\bar{a} = \mathbf{g}] > 0$ . If  $m_0 < 0$ , it follows from a symmetric argument that  $\sum_{t=1}^\infty (1 - F_0(r_t^g)) < \infty$ , which is equivalent to  $\mathbb{P}_0[\bar{a} = \mathbf{b}] > 0$ . In either case, it follows from Proposition 4 that asymptotic learning of informativeness fails.  $\square$

**A.3. Non-Uniform Prior.** In this section, we prove Proposition 5. We relax the uniform prior assumption on  $\theta$  while maintaining the independence assumption between  $\theta$  and  $\omega$ . Let  $a^d$  denote the default action that is favored by the non-uniform prior.

Recall that  $p_t = \mathbb{P}[\theta = \mathbf{g}|H_t]$  is the social belief that  $\theta = \mathbf{g}$  at time  $t$  and  $q_t = \mathbb{P}[\omega = 1|H_t]$  is the outside observer's belief that  $\omega = 1$  at time  $t$ . Given that the prior on  $\theta$  is not uniform, each agent  $t$  chooses an action according to (1). It immediately follows that  $p_t$  also satisfies the overturning principle (see Lemma 2). The following lemma is useful in proving Proposition 5.

**Lemma 9.** *Given a non-uniform prior on  $\theta$ , the following statements hold:*

- (i) *Conditioned on  $\lim_{t \rightarrow \infty} q_t = 0$  almost surely, actions almost surely converge to  $a^d$ .*
- (ii) *Conditioned on  $\lim_{t \rightarrow \infty} q_t = 1$  almost surely, actions almost surely converge to  $\theta$ .*

**Proof.** Fix a prior  $\pi = \mathbb{P}[\theta = \mathbf{g}] > 1/2$ , and so  $a^d = \mathbf{g}$ . For part (i), suppose towards a contradiction that actions do not converge to  $\mathbf{g}$  almost surely. There are two possible cases: either  $\lim_{t \rightarrow \infty} a_t = \mathbf{b}$ , or the actions never converge, i.e.,  $S = \infty$ . By the overturning principle, these scenarios imply that either  $\lim_{t \rightarrow \infty} p_t = p_\infty \leq 1/2$ , or  $p_\infty = 1/2$ . Together they imply that the event  $\{p_\infty \leq 1/2\}$  occurs with positive probability. Since  $\lim_{t \rightarrow \infty} q_t = q_\infty = 0$  almost surely, it follows from (A.4) that

$$p_\infty = \mathbb{P}_0[\theta = \mathbf{g} | H_\infty] = \pi > 1/2 \quad \text{almost surely,}$$

a contradiction to the event  $\{p_\infty \leq 1/2\}$  occurring with positive probability. Hence, we conclude that  $\lim_{t \rightarrow \infty} a_t = \mathbf{g}$  almost surely.

For part (ii), consider the case where  $\theta = \mathbf{g}$ . Assume by contradiction, that either  $\lim_{t \rightarrow \infty} a_t = \mathbf{b}$  or  $S = \infty$  occurs with positive probability. Again, by the overturning principle, these imply that the event  $\{p_\infty \leq 1/2\}$  occurs with positive probability. Since  $\lim_{t \rightarrow \infty} q_t = 1$  almost surely, it follows from (A.4) that  $\pi_\infty = p_\infty \leq 1/2$  with positive probability. By a standard martingale convergence argument with unbounded signals (Smith and Sørensen, 2000), we know that conditioned on  $\theta = \mathbf{g}$ ,  $\pi_\infty = 1$  almost surely, which contradicts to the event  $\{\pi_\infty \leq 1/2\}$  occurring with positive probability. An analogous argument applies to the case where  $\theta = \mathbf{b}$ .  $\square$

**Proof of Proposition 5.** Suppose by contradiction that asymptotic learning of informativeness holds. By definition and Lemma 9, we have that conditioned on  $\omega = 0$ ,  $\lim_t a_t = a^d$  almost surely; similarly, conditioned on  $\omega = 1$ ,  $\lim_t a_t = \theta$  almost surely. However, since  $\mathbb{P}[\theta = a^d | \omega = 1] > 0$ , there exists a history of actions at time infinity  $H_\infty^d$  in which  $\lim_t a_t = a^d$  and such history is possible under both probability measures  $\mathbb{P}_0$  and  $\mathbb{P}_1$ . Following a similar argument to that used in the proof of Proposition 1 (for the only-if direction), this implies that asymptotic learning of informativeness fails, a contradiction.  $\square$