The Strategy of Single Transferable Vote *

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Abstract

In a single transferable vote (STV) election each voter's ballot is a rank ordering of the candidates. Each stage eliminates the remaining candidate with the fewest ballots listing her as the favorite, among those candidates that have not been eliminated, until one candidate remains. We study the quantitative manipulability of STV in comparison with plurality and other systems, and we study the relative importance of manipulation at the stages of three and four remaining candidates. We find that STV is less manipulable than the other systems. For STV the dominant mode of manipulation is at the round of three remaining candidates, with the manipulator pushing a weak candidate into the round of two, then benefitting when they lose.

Key Words: Manipulability; Single Transferable Vote.

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1 Introduction

The Single Transferable Vote (STV) is an electoral system where each voter submits a ballot ranking the candidates in order of preference. Initially, each candidate receives the ballots listing them as the top choice. The candidate with the fewest ballots is then eliminated, and her ballots are reallocated to the second choice on each ballot. This process continues: at each step, the candidate with the fewest ballots is eliminated, and her ballots are reallocated to the next preferred candidate still in the race. This continues until the number of remaining candidates matches the number of available seats. For a single seat, this system is known as instant runoff voting. There are many variations of STV worldwide, including limiting the number of candidates a voter can rank and allowing voters to rank only a subset of candidates, ensuring their ballot is never reallocated to certain candidates under any circumstances.

The invention of the Single Transferable Vote (STV) is generally attributed to Thomas Hare, though it appears to have been previously considered, albeit negatively, by Condorcet. STV has been employed in various countries for many years and has seen a rise in popularity recently. It is used in elections for the Australian House of Representatives and in presidential elections in Ireland and Malta. Notably, STV was used in the 2021 Democratic primary for the New York City mayoral election. The advantages and disadvantages of STV, both as a practical electoral system and as an embodiment of democratic values, can be analyzed from multiple perspectives.

In many electoral systems (with approval voting being an exception), the winning candidate is determined based on a profile of strict preference orderings submitted by the voters. A voter can *manipulate* the outcome by submitting an ordering that does not reflect their true preferences, thereby changing the winning candidate to one they prefer over the candidate who would have won if she had submitted their true preference. For example, in a plurality election, the winner is the candidate ranked first by the most voters. A voter might manipulate the outcome by listing as their top choice the candidate who is her preferred option among the leading contenders, rather than their actual favorite.

The famous Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975) implies that there are some preference profiles for which STV is manipulable. This theorem effectively assumes that each voter knows the profile of other voters' preferences. However, in elections with more than two candidates, opinion polling provides information that is, at best, much more uncertain and vague. One might wonder if this uncertainty could make manipulation impossible, but McLennan (2011) has shown that any voting system can still create circumstances where a voter can achieve a better (expected) outcome by manipulating, even under various types of uncertainty. For example, one might assume that opinion polls accurately reflect the preferences of the overall population, but the voters who actually show up at the polls are a random sample from this population. Since all electoral systems other than dictatorship are manipulable, comparisons of the extent of manipulability and the complexity of manipulations must be quantitative.

In this paper, we study the manipulability of STV quantitatively, comparing it with plurality and several other electoral systems. The system we refer to as "runoff" involves sending the two candidates with the most first-place rankings to a second election, which is won by the candidate preferred by most voters over the other finalist. In an antiplurality election, the winner is the candidate with the fewest ballots ranking her last. We also consider two variants of antiplurality. In "antiplurality runoff," the finalists are the two candidates ranked last on the fewest ballots. The Coombs rule is an iterative elimination process similar to STV, but at each stage, the candidate with the most last-place rankings among the remaining candidates is eliminated. These systems all produce the same result when there are two candidates. When there are three candidates, STV and runoff are identical, and antiplurality runoff and the Coombs rule yield the same outcome. We will demonstrate that, according to our measures, for elections with three or four candidates, STV is the least manipulable, while plurality and antiplurality are significantly more manipulable than the other systems.

The possibility of manipulation undermines the fairness and transparency of an electoral system. Ideally, voters should be able to vote their true preferences without any strategic considerations, but for any nondictatorial system, such advice would be misleading. The best approach is to provide voters with strategic advice that is both simple and accurate. In our view, an electoral system cannot be considered truly democratic if it grants significant advantages to those with cleverness and skill in manipulating the vote.

Bartholdi and Orlin (1991) have demonstrated that determining the possibility of successful manipulation in an election is NP-complete. They note that this complexity can be seen as a advantage, as it makes manipulation more challenging, or as a disadvantage, as it might confuse less sophisticated voters. However, this complexity is largely irrelevant in large elections, where the influence of a single vote on eliminating a candidate in any given round is minimal. We define the elimination of one candidate when there are k remaining candidates as the *round of* k. In large elections, the likelihood that one vote will alter the outcome of any round k is extremely low, and thus the chance that manipulation could affect multiple rounds is negligible. When we limit the problem inputs to those that are manipulable in only one round, the complexity of determining possible manipulation diminishes significantly.

Unlike plurality voting, in STV elections, it is impossible to beneficially alter the outcome at the final round by misreporting preferences. This suggests, and our quantitative analysis supports, that STV is less susceptible to manipulation compared to plurality. In a plurality election, the common strategic advice is straightforward: voting for any candidate other than the top two contenders is considered a waste of your vote. However, what strategic advice can be offered to voters in an STV election?

Example Consider a scenario with three candidates: a, b, and c such that:

(a) 40% of voters have either of the preference orderings $c \succ a \succ b$ or $c \succ b \succ a$;

(b) 30% of voters have the preference ordering $a \succ b \succ c$;

(c) 15% of voters have the preference ordering $b \succ a \succ c$;

(d) 15% of voters have the preference ordering $b \succ c \succ a$.

Either a or b could be the candidate eliminated at the round of 3, and if a is eliminated first, then b prevails in the round of 2, but if b is eliminated first, then c defeats a in the round of 2. Thus a voter can manipulate if, by reporting a false preference, she can affect who survives the round of 3, and this changes the final winner in a direction favorable to her true preference. Specifically, the following manipulations are possible:

(a) with true preference $a \succ b \succ c$, submit either $b \succ a \succ c$ or $b \succ c \succ a$;

(b) with true preference either $c \succ a \succ b$ or $c \succ b \succ a$, submit either $a \succ b \succ c$ or $a \succ c \succ b$.

The general idea is that instead of voting for the candidate that is truly preferred, the manipulator pushes a weakly weak candidate into the round of 2, then benefits when they lose. Note that type (a) manipulations change the vote difference between a and b by two votes ?, and in this sense are twice as powerful as type (b) manipulations.

When manipulation can affect the process at only one round, say the round of k, a manipulation has the following character: there are two paths of elimination that agree at all rounds prior to k and have different sets of remaining candidates at all subsequent rounds (otherwise the two paths reunite and go to the same outcome) and different winners. Our main theoretical result is that any such pair of paths is possible, in the relevant sense.

As a practical matter, skillful manipulation at the round $k > 3$, perhaps using public opinion polling data, is very difficult at best. Changing from one path to the other requires the manipulator to foresee the candidates that will remain along both paths. Manipulations that attempt to attain a better final winner, in expectation, by averaging over multiple scenarios, are perhaps more plausible in a quantitative sense, but the amount and accuracy of information that is required is still enormous. Furthermore, there are various ways that attempts to manipulate might backfire. Indeed, various authors have pointed out that coalitional manipulations of STV can be delicate, insofar as they require some members of a group of like minded voters to manipulate while others do not (citations?).

One of our main quantitative findings reinforces this. We show that in a four candidate election, scenarios with manipulation at the round of 4 are, in aggregate, less than half as likely as scenarios with manipulation at the round of 3. We suspect that this is also the case when there are more than 4 candidates, and that the pattern continues, with manipulation at the round of 5 being less important, in aggregate, than manipulation at the round of 4, and so forth. However, at present we are unable to verify these conjectures either theoretically of computationally. Nevertheless, it seems reasonable to advise voters to forget about manipulations at any round other than the round of 3, because such manipulations are both difficult to foresee and, in aggregate, quite unlikely.

Related Literature Numerous criteria have been proposed to evaluate voting systems' desirability, with STV being analyzed from both theoretical and practical perspectives. Our emphasis is on studies examining the quantitative manipulability of STV compared to other systems, specifically considering individual manipulation rather than coalition manipulation.

Chamberlin (1985) examines the manipulability of four voting rules-Borda count, Coombs, STV, and plurality-using Monte Carlo methods to generate voter preferences with three candidates. Their findings show that STV is the least manipulable, while Borda count is the most manipulable. Nitzan (1985) defines the manipulability measure of a deterministic voting scheme as the ratio of manipulable preference profiles to possible preference profiles. This measure, also adopted by Kelly (1993), is known as the *Kelly-Nitzan index*. While the Kelly-Nitzan index is intuitively understandable for a finite set of voters, its computation is challenging. Many studies, including Walsh (2010)– whose empirical results demonstrate that NP-complete manipulation problems for STV can be solved quickly for many problem instances– employ Monte Carlo methods to generate random profiles and assess STV's manipulability for finite voter sets. Similarly, Aleskerov et al. (2018) uses random sampling to evaluate the manipulability of ten collective decision rules. However, Monte Carlo methods require a small number of voters to ensure a high probability of generating a manipulable profile.

Huang and Chua (2000) emphasized the analytical foundation of certain probability computations in social choice theory, providing a thorough characterization of the vulnerability properties of the four scoring rules analyzed in Lepelley and Mbih (1994) concerning manipulation by coalitions in a 3-alternative, n-agent society. However, Lepelley et al. (2008) noted that Ehrhart (1962) had already laid out the theoretical basis for these computations. They further explored various applications of this method, including the computation of coalition manipulability. The efficiency of this approach has significantly improved with advancements in algorithms and software for computing polytope volumes (Bruns and Ichim, 2010). Notably, recent progress has enabled the computation of such volumes in elections with four candidates (Bruns et al., 2019) and, more recently, in elections with five candidates (Bruns and Ichim, 2021).

To our knowledge, this is the first paper to define and compute measures of individual manipulability using polytope volumes. These volumes offer several advantages: they are relevant for elections with many voters, have simple and precise definitions, and can be exactly computed when there are four five or fewer candidates, providing a standard or canonical measure. In contrast, studies using Monte Carlo generation of random profiles differ in various dimensions and offer random, hence approximate, estimates of the quantities of interest.

Organization of the paper The remainder is organized as follows. Section 2 presents the model and defines the measures of manipulability for each voting rule. Section 4 provides the analytical result. Section 5 presents the computational results of manipulability measures for each voting rule. Section 6 concludes.

2 The Model

Let A be a finite set containing $n \geqslant 3$ candidates, denoted by typical elements a, b, c, etc. Define $\mathcal{O}$ as the set of all strict preference orderings of A, where typical elements are $\succ, \succ'$, etc. Assume $\varnothing \neq S \subset A$ and $\succ \in \mathcal{O}$. Define $\varphi_S(\succ)$ as the favorite element of S for some $\succ \in \mathcal{O}$, and $\omega_S(\succ)$ as the worst element of S for some $\succ \in \mathcal{O}$. Specifically, $\varphi_S(\succ)$ is the element of S such that $\varphi_S(\succ) \succ a$ for all $a \in S \backslash \{\varphi_S(\succ)\}$, and $\omega_S(\succ)$ is the element of S such that $a \succ \omega_S(\succ)$ for all $a \in S \backslash \{\omega_S(\succ)\}$. Thus,

$$\mathcal{F}_{a,S} = \{\succ \in \mathcal{O} : \varphi_S(\succ) = a\} \quad \text{and} \quad \mathcal{W}_{S,a} = \{\succ \in \mathcal{O} : \omega_S(\succ) = a\}$$

represent the preferences in $\mathcal{O}$ where a is the favorite and worst element of S, respectively.

Let $\mathcal{N}$ be a finite set of voters with N elements. The set of possible *profiles* is $\mathcal{O}^{\mathcal{N}}$, where a typical element is $\succ = (\succ_i)_{i\in\mathcal{N}}$. Whether $\succ$ denotes a preference or a profile will always be clear from context. Given a profile $\succ$, $\succ_{-i}$ denotes the $(N-1)$-tuple of preferences of agents other than i, and for $\succ_i' \in \mathcal{O}$, $(\succ_i', \succ_{-i})$ denotes the profile with the indicated components.

A *voting rule* is a function $f : \mathcal{O}^{\mathcal{N}} \to \Delta(A)$, where $\Delta(A)$ is the set of probability measures on A. Such a rule is *deterministic* if $f(\mathcal{O}^{\mathcal{N}}) \subset A$, where we are identifying A with the set of vertices of the simplex $\Delta(A)$. The voting rules we study are deterministic in spirit, but random tie breaking is necessary in order for them to be symmetric with respect to interchange of voters, even though in the end the details of tie breaking are unimportant. In order to be precise we need to specify that, in the following definitions, all random choices assign equal probability to all possibilities (e.g., the set of pairs that could go to a runoff) and all random events are statistically independent. The voting rules we study are:

- *Plurality* f_P is the voting rule in which $f_P(\succ)$ is chosen randomly from the set of alternatives a that are maximal for $|\{\, i \in \mathcal{N} : \varphi_A(\succ_i) = a \,\}|$.

- *Runoff* f_R is the voting rule in which a pair of alternatives $\{a_1, a_2\}$ is chosen randomly from the set of alternatives a that are maximal for $|\{\, i \in \mathcal{N} : \varphi_A(\succ_i) = a \,\}|$, and the winner is chosen randomly from the set of $a \in \{a_1, a_2\}$ that are maximal for $|\{\, i \in \mathcal{N} : \varphi_{\{a_1,a_2\}}(\succ_i) = a \,\}|$.

- *Single transferable vote* f_S is defined recursively: we set $A_1(\succ) = A$, for $j = 1, \ldots, n-1$ we set $A_{j+1}(\succ) = A_j(\succ) \setminus \{e_j(\succ)\}$, where $e_j(\succ)$ is chosen randomly from the set of alternatives $a \in A_j(\succ)$ that are minimal for $|\{\, i \in \mathcal{N} : \varphi_{A_j(\succ)}(\succ_i) = a \,\}|$, and $f_S(\succ)$ is the unique element of $A_n(\succ)$.

- *Antiplurality* f_A is the voting rule in which $f_A(\succ)$ is chosen randomly from the set of alternatives a that are minimal for $|\{\, i \in \mathcal{N} : \omega_A(\succ_i) = a \,\}|$.

- *Coombs rule* f_C is defined recursively: we set $A_1(\succ) = A$, for $j = 1, \ldots, n-1$ we set $A_{j+1}(\succ) = A_j(\succ) \setminus \{e_j(\succ)\}$, where $e_j(\succ)$ is chosen randomly from the set of alternatives $a \in A_j(\succ)$ that are maximal for $|\{\, i \in \mathcal{N} : \omega_{A_j(\succ)}(\succ_i) = a \,\}|$, and $f_C(\succ)$ is the unique element of $A_n(\succ)$.

Voter i can *manipulate* a deterministic voting rule f at a profile $\succ$ if there is a $\succ_i' \in \mathcal{O}$ such that $f(\succ_i', \succ_{-i}) \succ_i f(\succ)$. We say that f is *manipulable* at $\succ$ if some i can manipulate f at $\succ$. The formal statement of the Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite,

1975) is that if $n \geqslant 3$, f is deterministic, $f(\mathcal{O}^{\mathcal{N}}) = A$, and f is not manipulable at any profile, then f is dictatorial: there is some $i \in \mathcal{N}$ such that $f(\succ) = \varphi_A(\succ_i)$ for all profiles $\succ$.

Gibbard (1977) expanded the Gibbard-Satterthwaite theorem to include nondeterministic voting rules. Let $\mu, \mu' \in \Delta(A)$ and $\succ \in \mathcal{O}$. We say that μ is *potentially superior to* μ' for $\succ$ if $\sum_{a \in A} u(a)\mu(a) > \sum_{a \in A} u(a)\mu'(a)$ for some utility function $u \colon A \to \mathbb{R}$ with $u(a) > u(b)$ for all $a, b \in A$ such that $a \succ b$. Voter i can *potentially manipulate* a voting rule f at a profile $\succ$ if there exists $\succ_i' \in \mathcal{O}$ such that $f(\succ_i', \succ_{-i})$ is potentially superior to $f(\succ)$ for $\succ_i$. Gibbard's extension asserts that if f is not potentially manipulable at any profile, then it can be expressed as a probabilistic combination of schemes that are unilateral (depending on a single agent's preference) and duple (choosing between two alternatives). Consequently, this result implies that f_P, f_R, f_S, f_A, and f_C are potentially manipulable at certain profiles.

For nondeterministic voting rules, there exists a stronger form of manipulation. We say that μ *first-order stochastically dominates* μ' for $\succ$ if

$$\mu(\{\, b \in A : b \succ a \,\}) \geqslant \mu'(\{\, b \in A : b \succ a \,\})$$

for all $a \in A$, with strict inequality for at least one a. Slightly abusing notation, we denote this by $\mu \succ \mu'$. Voter i can *manipulate* a voting rule f at a profile $\succ$ if there exists $\succ_i' \in \mathcal{O}$ such that $f(\succ_i', \succ_{-i}) \succ_i f(\succ)$. Moreover, f is *manipulable* at $\succ$ if there exists at least one voter i who can manipulate f at $\succ$.

The definitions below use manipulability rather than potential manipulability, but for us the distinction is not important. Below we will show that for large N, the preponderance of misrepresentations that alter the outcome pass either from one pure outcome to another or between a pure outcome and an 50-50 lottery of that outcome and some other outcome. Such a misrepresentation is a manipulation if and only if it is a potential manipulation.

Let $\mathcal{P} = \{\, p \in \mathbb{R}_+^{\mathcal{O}} : \sum_{\succ \in \mathcal{O}} p_\succ = 1 \,\}$ be the simplex over $\mathcal{O}$. We sometimes treat elements of $\mathcal{P}$, notationally, as measures: for $p \in \mathcal{P}$ and $T \subset \mathcal{O}$, $p(T) = \sum_{\succ \in T} p_\succ$. In particular, for $\varnothing \neq S \subset A$ and $a \in S$ let $p_{a,S} = p(\mathcal{F}_{a,S})$ and $p_{S,a} = p(\mathcal{W}_{a,S})$. Let $\mathcal{P}^N$ be the set of elements of $\mathcal{P}$ whose components are all multiples of $1/N$. Let $p^{\mathcal{N}} \colon \mathcal{O}^{\mathcal{N}} \to \mathcal{P}^N$ be the function defined by letting $p^{\mathcal{N}}(\succ)$ be the element of $\mathcal{P}^N$ with components

$$p^{\mathcal{N}}_{\succ}(\succ) = \frac{1}{N} |\{\, i \in \mathcal{N} : \succ_i = \succ \,\}|.$$

Thus, $p^{\mathcal{N}}(\succ)$ represents the proportion of each preference ordering in the profile. A voting rule f is *anonymous* if there is a function $F \colon \mathcal{P}^N \to \Delta(A)$ such that $f = F \circ p^{\mathcal{N}}$. In other words, a voting rule is anonymous because the outcome depends only on the distribution of the votes, not on which particular voter casts which vote. The function F associated with each rule takes the distribution of preferences $p \in \mathcal{P}^N$ and maps it to an outcome in $\Delta(A)$, ensuring that permutations of voters do not affect the result. It is easy to see that f_P, f_R, f_S, f_A and f_C are anonymous; let F_P, F_R, F_S, F_A, and F_C be the corresponding functions from $\mathcal{P}^N$ to $\Delta(A)$. We say that an anonymous voting rule is *manipulable* at $p \in \mathcal{P}^N$ if it is manipulable at any $\succ \in \mathcal{O}^{\mathcal{N}}$ such that $p^{\mathcal{N}}(\succ) = p$.

Let $\beta = (\frac{1}{n!}, \ldots, \frac{1}{n!})$ be the *barycenter* of $\mathcal{P}$. Let $\mathcal{H} = \{ p \in \mathbb{R}^{\mathcal{O}} : \sum_{\succ \in \mathcal{O}} p_{\succ} = 1 \}$ be the hyperplane in $\mathbb{R}^{\mathcal{O}}$ that contains $\mathcal{P}$. A *polyhedral cone emanating from* β is the set C of solutions in $\mathcal{H}$ of a finite system of weak linear inequalities that are satisfied exactly by β. The *dimension* of C is the dimension of its affine hull. A *face* of C is the intersection of C with one of its bounding hyperplanes. A *conical decomposition* of $\mathcal{H}$ emanating from β is finite collection $\mathcal{C}$ of polyhedral cones emanating from β such that:

(a) for each $C \in \mathcal{C}$, each face of C is an element of $\mathcal{C}$;

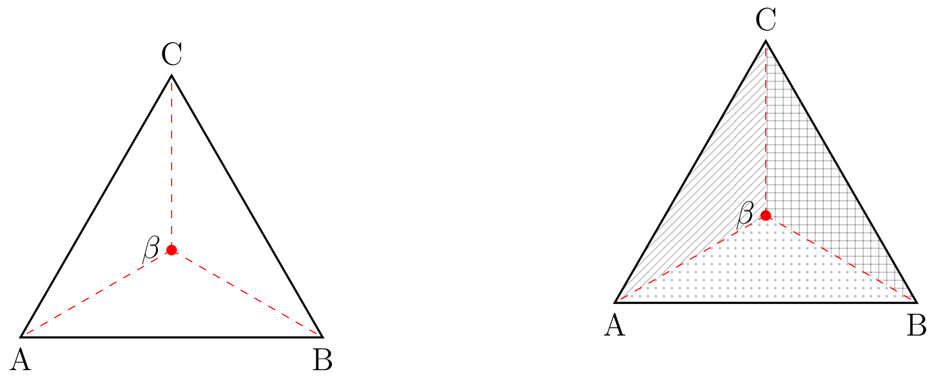
(b) for all $C_1, C_2 \in \mathcal{C}$, $C_1 \cap C_2 \in \mathcal{C}$;

(c) $\bigcup_{C \in \mathcal{C}} C = \mathcal{H}$.

An anonymous voting rule f with associated function F is *majoritarian* if there is a conical decomposition $\mathcal{C}$ of $\mathcal{H}$ such that for each $(n! - 1)$-dimensional $C \in \mathcal{C}$ there is an $a_C \in A$ such that $F(p) = a_C$ for all $p \in \mathcal{P}^N$ in the interior of C. Evidently f_P, f_R, f_S, f_A, and f_C are majoritarian; let $\mathcal{C}_P$, $\mathcal{C}_R$, $\mathcal{C}_S$, $\mathcal{C}_A$, and $\mathcal{C}_C$ be the associated conical decompositions of $\mathcal{H}$.

Our measures of relative manipulability are based on the concept of *Impartial Anonymous Culture* (IAC), where all elements of $\mathcal{P}^N$ are considered equally likely. Conversely, in the *Impartial Culture* (IC) framework introduced by Garman and Kamien (1968), all profiles in $\mathcal{O}^{\mathcal{N}}$ are assumed to be equally likely. Within the IC framework, Nitzan (1985) defines the manipulability of a deterministic voting scheme as the ratio of the number of manipulable preference profiles to the total number of possible preference profiles. This measure, also adopted by Kelly (1993), is known as the *Kelly-Nitzan index*. While the Kelly-Nitzan index provides an intuitive measure of manipulability for a finite set of voters, computing it is notably challenging since determining the possibility of a successful manipulation is an NP-complete problem.

We compute the Kelly-Nitzan indices for IAC without relying on random sampling. Instead, we exploit the property that these indices can be effectively estimated for large N by considering the volumes of polytopes corresponding to the voting rules in question. For small n, we utilize



(a) Conical decomposition.

(b) Majoritarian rule.

Figure 1: Panel (a) represents the conical decomposition of the preference simplex. The vertices of the triangle correspond to the pure preference orderings, where each vertex represents a situation where one alternative is the most preferred by all voters. The barycenter is the center of the triangle and represents a uniform distribution of preferences. The dashed lines from the barycenter to each vertex divide the triangle into conical regions. Each region represents a set of preference distributions where the preference orderings are consistent within that conical section. Panel (b) shows the application of a majoritarian rule to the same preference simplex. The triangle is divided into regions filled with different dashed patterns. These patterns represent the areas where each alternativen is the majority choice according to the majoritarian rule.

existing software to compute these indices numerically. As we will illustrate, as N tends to infinity, the measures of relative manipulability for various anonymous majoritarian voting rules under IAC, and the relative significance of different types of manipulations within a single voting rule, converge to comparisons of the volumes of intersections of the polytope $\mathcal{P}$ with $(n!-2)$-dimensional cones in H emanating from β.

By the central limit theorem, if $\succ$ follows a uniform distribution in $\mathcal{O}^{\mathcal{N}}$, then the distribution of $(p^{\mathcal{N}}(\succ)-\beta)/\|p^{\mathcal{N}}(\succ)-\beta\|$ converges to the uniform distribution on the unit sphere in H centered at β. Therefore, for IC, the assessment of relative manipulability of anonymous majoritarian voting rules simplifies to comparing the volumes of intersections of this sphere with $(n!-2)$-dimensional cones in H emanating from β. Thus, it is reasonable to expect that both approaches will yield qualitatively similar results, especially for small n. Our decision to use the software for computing volumes of polytopes primarily stems from its availability and practical feasibility.

3 Manipulation Paths

We now describe manipulations for STV formally, as pairs of paths of eliminations that agree up to a certain round, then diverge, arriving at different outcomes. Roughly, we take the probability of such a manipulation to be the limit as $N \to \infty$ of N times the probability of a profile for which this manipulation is possible. We show that, in this sense, all manipulation paths have positive probability.

An *elimination path* is an ordered list $j = j_1, \ldots, j_n$ of all candidates. A profile $p \in P$ is *strictly consistent* with j at round $h = 2, \ldots, n$ if

$$p_{j_h,\{j_1,\ldots,j_h\}} < p_{j_g,\{j_1,\ldots,j_h\}} \quad \text{for } g = 1, \ldots, h-1,$$

which means that in an STV election with distribution p, candidate j_n is eliminated first, then j_{n-1}, and so forth, until j_1 prevails.

A profile $p \in P$ is *consistent* with j at round $h = 2, \ldots, n$ if

$$p_{j_h,\{j_1,\ldots,j_h\}} \leqslant p_{j_g,\{j_1,\ldots,j_h\}} \quad \text{for } g = 1, \ldots, h-1.$$

A *manipulation* at the round of h is a pair (j, k) of elimination paths such that $j_\ell = k_\ell$ for all $\ell > h$ and $\{j_1, \ldots, j_\ell\} \neq \{k_1, \ldots, k_\ell\}$ for all $\ell < h$. there is no definition of manipulation here (Note that this implies that $j_h \neq k_h$.) We are imagining a situation in which there is a tied election at the stage of h, but not at any other stage, so if $\{j_1, \ldots, j_\ell\} = \{k_1, \ldots, k_\ell\}$ for some $\ell < h$, then the subsequent eliminations and the eventual winner will also be the same for j and k. We say that p is *consistent* with (j, k) if it is consistent with both j and k at every round, and it is *strictly consistent* with (j, k) if it is consistent with j and k at the round of h and strictly consistent with j and k at all other rounds.

Let $P_{(j,k)}$ and $P^\circ_{(j,k)}$ be the sets of p that are consistent and strictly consistent with (j, k) respectively. As a subset of the $(n!-2)$-dimensional polytope $\{\, p \in P : p_{j_h,\{j_1,\ldots,j_h\}} = p_{k_h,\{k_1,\ldots,k_h\}} \,\}$, $P_{(j,k)}$ is defined by a finite conjunction of weak linearinequalities, so it is a polytope. The next result asserts that $P^\circ_{(j,k)} \neq \varnothing$, so $P_{(j,k)}$ has positive $(n! - 2)$-dimensional volume, which we denote by $\lambda(P_{(j,k)})$ or $\lambda_{(j,k)}$.

Theorem 1. *For any $h = 1, \ldots, k$ and any manipulation (j, k) at the round of h there is a $p \in P$ that is strictly consistent with (j, k).*

Proof. For the sake of simplicity we consider only the case $h = n$; it will be obvious that an even more cumbersome construction is possible for general h. For $\ell = 0, \ldots, n-1$ let $\succ_{j,\ell}$ and $\succ_{k,\ell}$ be the elements of $\mathcal{O}$ such that

$$j_{n-\ell} \succ_{j,\ell} j_{n-\ell-1} \succ_{j,\ell} \cdots \succ_{j,\ell} j_1 \succ_{j,\ell} j_n \succ_{j,\ell} j_{n-1} \succ_{j,\ell} \cdots \succ_{j,\ell} j_{n-\ell+1}$$

and

$$k_{n-\ell} \succ_{k,\ell} k_{n-\ell-1} \succ_{k,\ell} \cdots \succ_{k,\ell} k_1 \succ_{k,\ell} k_n \succ_{k,\ell} k_{n-1} \succ_{k,\ell} \cdots \succ_{k,\ell} k_{n-\ell+1}.$$

For a sufficiently small $\varepsilon > 0$ let p be the profile given by

$$p_\succ = \begin{cases} \frac{1}{n!}(1 + \frac{2(1-\varepsilon^n)}{1-\varepsilon}) - \varepsilon^{\ell+1}, & \succ \in \{\succ_{j,\ell}, \succ_{k,\ell}\} \text{ for some } \ell, \\ \frac{1}{n!}(1 + \frac{2(1-\varepsilon^n)}{1-\varepsilon}), & \text{otherwise.} \end{cases}$$

By symmetry, for a particular $\ell = n, \ldots, 2$ and $m = \ell - 1, \ldots, 1$ it suffices to show that $p_{j_\ell,\{j_\ell,\ldots,1\}} < p_{j_m,\{j_\ell,\ldots,1\}}$. The number of $\succ \in \mathcal{O}$ such that $\varphi_{\{j_\ell,\ldots,j_1\}}(\succ) = j_\ell$ is equal to the number of $\succ \in \mathcal{O}$ such that $\varphi_{\{j_\ell,\ldots,j_1\}}(\succ) = j_m$. We have $\varphi_{\{j_\ell,\ldots,j_1\}}(\succ_{j,p}) = j_\ell$ for all $p = 0, \ldots, \ell - 1$, $\varphi_{\{j_\ell,\ldots,j_1\}}(\succ_{j,m-1}) = j_m$, and $\varphi_{\{j_\ell,\ldots,j_1\}}(\succ_{j,p}) \notin \{j_\ell, j_m\}$ for all other p. Therefore (provided ε is sufficiently small) we certainly have $p_{j_\ell,\{j_\ell,\ldots,1\}} < p_{j_m,\{j_\ell,\ldots,1\}}$ unless $\varphi_{\{j_\ell,\ldots,j_1\}}(\succ_{k,p}) = j_\ell$ for all $p = 0, \ldots, \ell - 1$. If this is the case, then $k_p \notin \{j_{\ell-1}, \ldots, j_1\}$ for all $p = n, \ldots, \ell + 1$, and $k_p \neq j_\ell$ for all such p, so that $\{k_n, \ldots, k_{\ell+1}\} = \{j_n, \ldots, j_{\ell+1}\}$, after which $\varphi_{\{j_\ell,\ldots,j_1\}}(\succ_{k,\ell-1}) = j_\ell$ implies that $k_\ell = j_\ell$. But $\{k_n, \ldots, k_{\ell+1}\} = \{j_n, \ldots, j_{\ell+1}\}$ is impossible if $\ell < n$ because (j, k) is a manipulation, and $k_n = j_n$ is also impossible because (j, k) is a manipulation at the round of n. The proof is complete. □

There is a parallel analysis for ASTV. A profile $p \in P$ is *strictly anticonsistent* with an elimination path j at $h = 2, \ldots, n$ if

$$p_{\{j_1,\ldots,j_h\},j_h} > p_{\{j_1,\ldots,j_h\},j_g} \quad (g = 1, \ldots, h - 1).$$

As before, if p is strictly anticonsistent with j at all h, then in an ASTV election with distribution p, j_n is eliminated first, then j_{n-1}, and eventually j_1 prevails. A profile $p \in P$ is *anticonsistent* with j at $h = 2, \ldots, n$ if

$$p_{\{j_1,\ldots,j_h\},j_h} \geqslant p_{\{j_1,\ldots,j_h\},j_g} \quad (g = 1, \ldots, h - 1).$$

If (j, k) is a manipulation at the round of h, we say that p is *anticonsistent* with (j, k) if it is antianticonsistent with both j and k at every round, and it is *strictly anticonsistent* with (j, k) if it is anticonsistent with j and k at the round of h and strictly anticonsistent with j and k at all other rounds. Let $Q_{(j,k)}$ and $Q^\circ_{(j,k)}$ be the sets of p that are anticonsistent and strictly anticonsistent with (j, k) respectively. As before, $Q_{(j,k)}$ is a polytope, and $Q^\circ_{(j,k)} \neq \varnothing$, so $Q_{(j,k)}$ has positive $(n! - 2)$-dimensional volume. The proof of the following is similar to what we saw above, so we omit it.

Theorem 2. *For any $h = 1, \ldots, k$ and any manipulation (j, k) at the round of h there is a $p \in P$ that is strictly anticonsistent with (j, k).*

One of the key insights in understanding the manipulability of STV lies in how the probability of successful manipulation diminishes as the number of voters increases. Theorem 3 formalizes this idea, showing that the likelihood of manipulation is closely tied to the volume of the region in the preference space that allows for manipulation and that this probability decays exponentially with the number of voters.

Theorem 3. *For any election with $n \geqslant 3$ candidates and a sufficiently large number of voters N, the probability that a randomly chosen voter can successfully manipulate the STV outcome at a given round k is at most proportional to the volume ratio*

$$\frac{\lambda(P_{j,k})}{\lambda(P)},$$

where $P_{j,k}$ is the set of preference distributions allowing a manipulation at round k, and P is the full preference simplex. Moreover, for large N, this probability decreases exponentially in N.

Proof. Let P be the preference simplex, representing all possible distributions of voter preferences over n candidates. Let $P_{j,k}$ denote the subset of P where manipulation at round k is possible, meaning that a voter can alter the elimination path in STV by misreporting their preference.

Since we assume a uniform distribution over P, the probability of drawing a profile from $P_{j,k}$ is given by:

$$\mathbb{P}(p \in P_{j,k}) = \frac{\lambda(P_{j,k})}{\lambda(P)}. \tag{1}$$

A successful manipulation requires a voter to change the elimination order at round k. If X_j denotes the number of first-choice votes received by candidate j, then for large N, X_j follows an approximately normal distribution:

$$X_j \sim \mathcal{N}(Np_j, Np_j(1 - p_j)). \tag{2}$$

A manipulation at round k requires shifting the elimination threshold, which occurs when the vote count difference between two candidates is within a margin of order $O(1)$. Standard

Gaussian tail bounds imply that the probability of such a shift is at most:

$$\mathbb{P}_N(p) \approx e^{-cN}, \tag{3}$$

for some constant $c > 0$.

Thus, the overall probability of manipulation satisfies:

$$\mathbb{P}(\text{Successful Manipulation}) \leqslant \frac{\lambda(P_{j,k})}{\lambda(P)} \cdot e^{-cN}. \tag{4}$$

□

Understanding how manipulable profiles are distributed across different rounds offers valuable insight into strategic behavior in STV elections. Manipulation tends to be more likely in earlier rounds, when more candidates are in play and ties are easier to influence. Theorem 4 highlights this pattern, showing that the volume of profiles allowing manipulation is significantly larger in the round of 3 than in the round of 4. In fact, more than half of all manipulable scenarios arise during the transition from three to two candidates, making this stage the most critical point for strategic influence.

Theorem 4. *Let f_S be the single transferable vote (STV) rule with 4 candidates, and let $P_{j,k}$ denote the set of preference profiles where a voter can manipulate the outcome by changing the elimination path from j to k at round h. Then:*

$$\lambda(P_{j,3}) > \lambda(P_{j,4}),$$

where $\lambda(\cdot)$ denotes the Lebesgue measure on the preference simplex. Consequently:

$$\frac{\lambda(P_{j,3})}{\sum_{k=3}^{4} \lambda(P_{j,k})} > \frac{1}{2}.$$

Proof. Let P be the preference simplex, representing the set of all possible distributions of voter preferences. A manipulation at round h occurs if a voter can misreport their preferences to change the elimination order at that round, leading to a different final winner. We want to show that the measure of manipulable profiles at the round of 3 is strictly larger than at the round of 4.

For a manipulation to occur at round 3, the profile must lie on the hyperplane where two out of three remaining candidates are tied. Let $j = (j_1, j_2, j_3, j_4)$ and $k = (k_1, k_2, k_3, k_4)$ be two distinct elimination paths that differ at round 3. The set of preference distributions where a voter can change the path from j to k by breaking a tie is:

$$P_{j,k,3} = \{p \in P : p_{j_3} = p_{k_3},\ p_{j_i} \neq p_{k_i} \text{ for } i < 3\}.$$

Similarly, for round 4:

$$P_{j,k,4} = \{p \in P : p_{j_4} = p_{k_4},\ p_{j_i} \neq p_{k_i} \text{ for } i < 4\}.$$

The sets $P_{j,k,3}$ and $P_{j,k,4}$ are convex polytopes obtained as intersections of the simplex with tie-breaking hyperplanes. Since these hyperplanes split the simplex, the relative volume of the manipulable region decreases as the number of remaining candidates grows. Specifically, the region $P_{j,k,4}$ is a lower-dimensional slice of the simplex, as it requires a more restrictive tie condition involving four candidates rather than three.

Moreover, the number of distinct elimination paths that can diverge due to a tie is greater at round 3 than at round 4. In round 3, manipulation can occur whenever two out of three remaining candidates tie, while in round 4, manipulation requires a tie between the two lowest-ranked candidates among four. The combinatorial structure of the elimination tree implies that there are more opportunities to manipulate earlier in the process.

Since the measure of the manipulable region is proportional to the number of such tie-breaking opportunities and the dimension of the hyperplane intersections, it follows that:

$$\lambda(P_{j,3}) = \sum_{(j,k) \text{ paths}} \lambda(P_{j,k,3}) > \sum_{(j,k) \text{ paths}} \lambda(P_{j,k,4}) = \lambda(P_{j,4}).$$

Thus, the measure of manipulable profiles is larger at round 3 than at round 4, and the probability that a random profile is manipulable is correspondingly higher:

$$\frac{\lambda(P_{j,3})}{\lambda(P_{j,3}) + \lambda(P_{j,4})} > \frac{1}{2}.$$

□

4 Quantifying Manipulability

In this section we explain our quantifications of manipulability. Let L be a rational *lattice* in $\mathbb{R}^d$: L is the set of integral linear combinations of d linearly independent *generators* $g_1, \ldots, g_d \in \mathbb{Q}^d$. Suppose that P is a rational polytope in $\mathbb{R}^d$, i.e., the convex hull of finitely many points in

$\mathbb{Q}^d$, and that P is d-dimensional. Ehrhart (1962) showed that for positive integers t the number $L(P,t)$ of points in $tP \cap L$ is a rational *quasipolynomial* function of t. That is, there are rational valued periodic functions $c_0(t), \ldots, c_d(t)$ of t such that $L(P,t) = c_d(t)t^d + \cdots + c_1(t)t + c_0(t)$. In addition $c_d(t)$ is a constant function whose value is the *normalized volume* of P, which is the volume of P divided by the volume of the *fundamental region* $\{ c_1 g_1 + \cdots + c_d g_d : 0 \leqslant c_1, \ldots, c_d \leqslant 1 \}$.

Although it would be possible to go into great detail, the main consequences of this for us are not complicated. For the sake of concreteness we discuss STV, but the discussion pertains equally to all of the voting rules we are studying. Let (j,k) be a manipulation at the round of h. There are five "layers" of profiles in P^N that are in $P_{(j,k)}$ or differ from an element of $P_{(j,k)} \cap P^N$ by changing the preferences of one or two agents in a way that creates a vote count difference between j_h and k_h at the round of h of one or two. Ehrhart's result implies that the number of such profiles is well approximated by $N^{n!-2}$ times the normalized volume of P. Any profile that is manipulable by virtue of changing the path or elimination from j to k or vice versa is in this set, and with minor exceptions (e.g., the preferences that would benefit from manipulating have probability zero, or the manipulation would trigger undesired changes of the outcomes in subsequent rounds) all the profiles in this set are manipulable. Manipulations that change the outcome in multiple rounds are confined to a neighborhood of the union of the $(n!-3)$-dimensional elements of $\mathcal{C}_S$, so their number is bounded by a constant times $N^{n!-3}$.

We are primarily interested in relative manipulability, such as the ratio of the manipulability of STV to the manipulability of plurality. In this ratio the other terms described above cancel, leaving the ratio of the sum of the volumes of the various $P_{(j,k)}$ to the volume of the relevant polytope for plurality. Similarly, if (j',k') is a second manipulation, the relative importance of (j,k) in comparison with (j',k') is the ratio of the volumes of $P_{(j,k)}$ and $P_{(j',k')}$.

One possibility that does not seem to have much (or perhaps any) precedent in previous literature is to weight manipulable profiles by the number of agents who are able to manipulate. For example, if $p \in P^N$ is manipulable by manipulations that change the elimination path from j to k, then summing $p_\succ$ over all $\succ$ for which such a manipulation is desirable gives a weighted measure of the manipulability of p. Let

$$P_{(j,k),\succ} = \{ (p,t) \in P_{(j,k)} \times [0,1] : t \leqslant p_\succ \}.$$

An average weighted manipulability can be computed by summing the volumes of the $P_{(j,k),\succ}$ over all $\succ$ for which changing the elimination path from j to k is beneficial. In this sum it seems appropriate to count the volume of $P_{(j,k),\succ}$ twice if $\varphi_{\{j_1,\ldots,j_h\}}(\succ) = j_h$ because the manipulation of such a $\succ$ changes the relevant vote count difference by two.

5 The Computational Results

This section reports and discusses the volume computations for elections with three and four candidates. All computations were performed by Normaliz (Bruns and Ichim, 2010). Although it would not be appropriate to describe the underlying algorithms here, an important point is that the power of Normaliz has increased substantially in recent years due to its implementation of the *Lawrence algorithm* (Lawrence, 1991; Filliman, 1992) in exact arithmetic. In this algorithm the volume of a polytope is represented as a signed sum of volumes of simplices, where the sum is over the full dimensional simplices of a triangulation of the dual polytope. This algorithm works well when the polytope is described by the inequalities requiring that all variables are nonnegative and a small number of additional facet inequalities, as is typical in applications to social choice. El Ouafdi et al. (2020) use Normaliz and other softwares to compute other volumes related to elections with four candidates. As we mentioned previously, Bruns and Ichim (2021) have computed 119-dimensional volumes related to elections with five candidates and 8 facet inequalities in addition to the nonnegativity constraints, but the relevant polytopes for manipulability of STV with five candidates have 10 additional facet inequalities, which puts them just beyond the current range of feasibility.

Any sort of manipulation is a matter of a voter changing the outcome by breaking some tie. Fix two candidates, say a and b. We consider the subset of $\mathcal{P}$ consisting of those profiles p that assign equal total weight to the preferences that have a as the favorite and the preferences that have b as a favorite. This an $(n!-2)$-dimensional polytope. For each type of manipulation, the set of manipulable profiles is a subset of this polytope defined by additional linear inequalities. For example, in order for p to be manipulable for plurality, it must be the case for each other candidate c, p assigns less total weight to preferences for which c is the favorite than it assigns to preferences that have a as the favorite. In the tables below manipulability is quantified as the volume of the set of profiles at which the manipulation is possible as a fraction of the volume of the polytope at which the relevant two candidates are tied. This unit of measurement is an arbitrary convention, but we are primarily interested in the relative manipulability of different voting systems, and the relative importance of different manipulation scenarios, which are ratios that are unaffected by this choice of “numeraire.”

We first discuss the case of three candidates. Table 1 shows the fractions of tie volumes for the three different manipulation scenarios for STV in which a and b are tied for elimination in the round of 3.

j	k	Fraction of Tie Volume
c, b, a	a, c, b	0.078
b, c, a	c, a, b	0.078
b, c, a	a, c, b	0.043
	Total	0.199

Table 1: Manipulability of STV With 3 Candidates

Table 2 shows the fractions of tie volumes for the same three manipulation scenarios for Coombs. Evidently Coombs is about 1.5 times as manipulable.

j	k	Fraction of Tie Volume
c, b, a	a, c, b	0.114
b, c, a	c, a, b	0.114
b, c, a	a, c, b	0.057
	Total	0.285

Table 2: Manipulability of Coombs With 3 Candidates

The total tie volumes for plurality and antiplurality are 0.0623 and 0.0907 respectively, so STV is indeed significantly less manipulable as STV.

An interesting point is that for both STV and Coombs, in the third, less likely scenario, the two orderings of the candidates are more distant in the metric on permutations given by the minimum number of transpositions of adjacent elements required to pass from one to the other.

We now take up the case of four candidates. Table 3 shows the tie volumes for the various manipulations at the round of 3. Again we see that a manipulation is less likely if the two orderings of the candidates are more distant. Here the relevant total measure of manipulability for plurality is 0.260, the measure for runoff is 0.150, and the measure for Coombs is 0.114.

Table 4 shows the manipulability of STV, under various scenarios, at the round of 3. The main finding is that the total manipulability of this sort is less than half the manipulability at the round of 3.

6 Concluding Remarks

We have studied the manipulability of STV, in comparison with a wide variety of voting systems, using measures that reduce to volumes of polytopes. In contrast with studies that

j	k	Fraction of Tie Volume
d, a, b, c	b, d, a, c	0.024
a, d, b, c	d, b, a, c	0.024
b, d, a, c	a, d, b, c	0.012
c, b, a, d	a, c, b, d	0.024
b, c, a, d	c, a, b, d	0.024
b, c, a, d	a, c, b, d	0.012
	Total	0.120

Table 3: Manipulability of STV With 4 Candidates, Round of 3

j	k	Fraction of Tie Volume	j	k	Fraction of Tie Volume
d, c, b, a	a, d, c, b	0.0044	b, d, c, a	a, d, c, b	0.0013
d, c, b, a	c, a, d, b	0.0025	b, d, c, a	c, a, d, b	0.0006
d, c, b, a	a, c, d, b	0.0020	b, d, c, a	a, c, d, b	0.0005
c, d, b, a	d, a, c, b	0.0025	c, b, d, a	d, c, a, b	0.0025
c, d, b, a	a, d, c, b	0.0020	c, b, d, a	d, a, c, b	0.0007
c, d, b, a	a, c, d, b	0.0044	c, b, d, a	a, d, c, b	0.0006
d, b, c, a	c, d, a, b	0.0025	c, b, d, a	a, c, d, b	0.0016
d, b, c, a	a, d, c, b	0.0016	b, c, d, a	d, c, a, b	0.0020
d, b, c, a	c, a, d, b	0.0007	b, c, d, a	c, d, a, b	0.0044
d, b, c, a	a, c, d, b	0.0006	b, c, d, a	d, a, c, b	0.0006
b, d, c, a	d, c, a, b	0.0044	b, c, d, a	a, d, c, b	0.0005
b, d, c, a	c, d, a, b	0.0020	b, c, d, a	c, a, d, b	0.0016
b, d, c, a	d, a, c, b	0.0016	b, c, d, a	a, c, d, b	0.0013
		Total			0.0491

Table 4: Manipulability of STV With 4 Candidates, Round of 4

use random sampling, our measures provide an exactly defined standard that we are able compute precisely. We find that for elections with 4 candidates, manipulation at the round of 4 is less than half as likely as manipulation at the round of 3, which reinforces the practical considerations suggesting that manipulation at rounds other than 3 are not worth considering.

Obviously one would like to extend the quantitative analysis to elections with 5 or more candidates. The current rate of progress for algorithms that compute volumes of polytopes, and their implementation in actual software, gives rise to optimism.

There is considerable scope for extending the analysis to different models of voter preference. In particular, a reason for preferring STV to runoff that does not come through in our analysis is that if there are many similar centrist parties, one or both finalists in a runoff election may be extreme, and unrepresentative of the consensus views of the voters.

Many other issues could be mentioned.

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