Understanding Speculative Bubbles: Insights from Theory and Empirical Research*

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Abstract: A speculative bubble arises when the price of an asset exceeds every trader's valuation measured by her willingness to pay if obliged to hold the asset forever. A speculative bubble implies the presence of speculative trade - whoever holds the asset intends to sell it at a later date. This article develops a general theory of speculative trade and bubbles in dynamic asset markets with short-sale constraints and risk-neutral agents with heterogeneous beliefs. Moreover, we review empirical evidence from several well-known recent episodes of speculative bubbles in real-world markets.

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1. Introduction

Heterogeneity of beliefs among market participants in financial markets is a widespread phenomenon. It may arise due to asymmetric information, differences in the speed of learning, or varying interpretations of common information. Standard theories of asset pricing are based on the assumption that agents have common beliefs about assets' future returns. Miller (1977) was among the first to argue that the functioning of financial markets with heterogeneous beliefs can differ significantly from those with homogeneous beliefs. This is particularly true when there are binding restrictions on short selling. In such markets, asset prices are not determined by the valuation of the marginal investor, but rather by that of the most optimistic investor. This mechanism is most transparent when investors are risk-neutral. Agents with the most optimistic beliefs purchase the available supply of the asset, while those with more pessimistic beliefs wish to short sell but are constrained by short-sale restrictions and thus remain on the sidelines. Equilibrium asset prices reflect the valuations of the most optimistic investors, while the pessimists are literally out of the market.

Investors' beliefs change over time due to the arrival of new information and belief updating. The set of most optimistic agents in the market may change from one trading date to the next. If so, the former optimists will sell the assets to the newly more optimistic agents. This gives rise to speculative trade: agents buy the asset in order to sell it at a future date. Harrison and Kreps (1978) pointed out that speculative trade leads to asset prices exceeding valuations of the most optimistic agents. This resale premium, in excess of the valuation as if the asset were to be held forever, has more recently been referred to as a speculative bubble. Speculative trade and speculative bubbles are inherently linked and arise together in speculative markets.

Speculative bubbles are the most widely accepted explanation for some well-known episodes of price bubbles in recent history. The most prominent example is the dot-com bubble of 1999-2001. Ofek and Richardson (2003) presented compelling evidence that it exhibited all the features of a speculative bubble. Speculative bubbles are also the generally accepted explanation for the Chinese warrants bubble of 2005-2008 described in Xiong and Yu (2011). We review these two episodes in Section 2 and provide some other examples.

Heterogeneous beliefs in asset markets may arise through three distinct mechanisms. The first mechanism is dogmatic beliefs. This was adopted by Harrison and Kreps (1978), where agents have Markovian beliefs with different transition matrices. The beliefs are dogmatic meaning that they remain unchanged despite agents' observing outcomes of the underlying stochastic process. The second, more compelling mechanism arises in the Bayesian model of learning with heterogeneous prior distributions. This has been put forward by Morris (1996). The third arises from updating a common prior on public information arriving over time, when agents hold heterogeneous beliefs about the relationship between information signals and the asset's valuation. This framework was adopted in Hong, Scheinkman and Xiong (2006) and earlier in Scheinkman and Xiong (2003).

Speculative trade and bubbles arise in markets with heterogeneous beliefs only if beliefs change over time and the set of most optimistic agents changes along the way. It is important to distinguish between short-run and long-run optimism in asset markets. Short-run optimism concerns beliefs about next period asset payoff, comprising the price and the dividend. Changes in short-run optimism over time generate speculative trade and bubbles. It is, however, an endogenous feature of market equilibrium as it involves asset prices. The long-run optimism refers to an agent's valuation of the asset, that is, the expected discounted value of future dividends, and is an exogenous aspect of the belief. In an important result, we show in Theorem 1 that changes in long-run optimism, called *valuation switching*, are a sufficient condition for switching of short-run optimism and thereby for generating speculative bubble. It is, however, not a necessary condition, as illustrated by the example due to Harrison and Kreps (1978), in which short-run optimism arises due to differences in beliefs about next-period dividends.

For Markovian beliefs, the condition of valuation switching can be expressed in terms of belief transition matrices, see Section 3. In the Bayesian model of learning, valuation switching is closely related to the monotone likelihood ratio order of priors. In general, the absence of an MLR-dominant prior is a necessary condition for valuation switching, and is sufficient in some cases of interest, see Morris (1996) and Section 4. Under differential interpretation of information, valuation switching arises from differences in the strength of agents' reactions to signals. Such differences may result from over- or underconfidence in the precision of those signals.

Our discussion of speculative bubbles covers both the finite-time and the infinite-time asset markets. The finite-time horizon is included because most real-world examples of speculative bubbles, such as the Chinese warrants bubble, involved assets with finite maturity. In contrast, most of the theoretical literature on speculative trade and bubbles focuses on infinite-horizon settings. In a finite-horizon market, speculative bubble naturally disappears ("crashes") at the date immediately before maturity, whereas in an infinite-horizon market, bubbles may persist or crash at any time. For Markovian beliefs in infinite time, speculative bubbles persist over time. In the Bayesian model of learning, asymptotic properties of speculative bubbles are closely related to consistency of prior beliefs at the true parameter: bubbles may vanish over time if priors are well-specified, but they can persist indefinitely if priors are misspecified, see Werner (2024). Models of markets with differential interpretation of information typically assume a finite time horizon.

The paper is organized as follows. In Section 2, we describe the dot-com bubble and the Chinese warrants bubble, highlighting their distinctive features as speculative bubbles arising from heterogeneous beliefs. In Section 3 we present the model of dynamic asset markets with short-sale restrictions and risk-neutral agents with heterogeneous beliefs. We introduce a formal definition of a speculative bubble and prove that valuation switching is a sufficient condition for a speculative bubble to arise in equilibrium. In Sections 4-6, we study speculative trade and bubbles arising from three mechanisms generating heterogeneous beliefs and valuation switching: Markovian beliefs, Bayesian learning and differential interpretation of public information. We present several examples. Section 7 offers discussion and concluding remarks.

Literature on the Theory of Speculative Trade and Bubbles

The notion of speculative trade was introduced in Keynes (1936, Chapter 12), who warned that such activity skews the capital markets away from serving the real economy and makes them inherently unstable. Miller (1977) echoed these concerns, emphasizing the role of heterogeneous beliefs in generating speculative trade. Harrison and Kreps (1978) formalized the notion of speculative trade in an infinite time horizon model of asset markets with Markovian beliefs and observed that speculative trade is associated with asset prices higher than the expected discounted value of dividends of the most optimistic investor. They provided an example to illustrate this observation. Harrison and Kreps did not use the term "speculative bubble" for the difference between the asset price and the optimist's fundamental value; the term was introduced only more recently. Wu and Guo (2003) extended the model with Markovian beliefs beyond the example of Harrison and Kreps (1978), generalizing it to finite-state Markov dividend processes. Further, they demonstrated the possibility of speculative trade and bubbles in a rational belief equilibrium of Kurz (1994).

Morris (1996) introduced parametric Bayesian learning with heterogeneous priors and showed that such priors naturally lead to posterior beliefs changing over time in a way that gives rise to valuation switching and speculative bubbles. He argued that the model explains the overpricing observed in the initial trading period following an IPO, when short-sale restrictions are strongest. Slawski (2009) extended the model of Bayesian learning from the i.i.d distribution of dividends to two-state Markov processes and presented an example of persistent speculation under heterogeneous misspecified priors. Werner (2024) developed further generalizations of the model and pointed out the relevance of the statistical concept of consistency of the prior at the true parameter for the asymptotic properties of asset prices and valuations in speculative markets.

Speculative trade can arise in finite-horizon markets with short-sale constraints and heterogeneous beliefs resulting from differential interpretation of public information. Harris and Raviv (1993) developed a model assuming an exponential distribution of the information signals and focusing on the volume and turnover of speculative trade, without addressing price bubbles. Hong, Scheinkman and Xiong (2006) developed a three-period model with differential interpretation of public information and risk-averse agents to rationalize the run-up and the collapse of the dot-com bubble. The model features normal distributions of the signals and the dividends, with heterogeneous beliefs about the signal precision, and a limited asset float that increases over time. Palfrey and Wang (2012) present a laboratory test of speculative trade and bubbles arising from differential interpretations of public signals. Their theoretical model features binary signals and heterogeneous non-Bayesian belief updating, permitting both over- and underreaction to signals. The experimental results provide strong evidence of speculative bubbles.

Werner (2022) showed that speculative trade and bubbles may arise with ambiguous beliefs that are common to all traders. These beliefs are modeled as sets of probability measures. To illustrate the mechanism, Werner (2022) replicated the example of Harrison and Kreps (1978), replacing heterogeneous beliefs with common ambiguous beliefs. Speculative bubbles may also arise in rational expectations equilibrium under asymmetric information, see Allen, Morris and Postlewaite (1993) and Conlon (2004). These models feature short-sales constraints and risk-averse agents, and are concerned with rational expectations equilibrium.

An excellent survey of different theories of asset price bubbles in competitive markets is Barlevy (2015), who also provides an insightful discussion of some policy implications. In a different context, Biais and Bossaerts (1998) study speculative trading in a non-Walrasian market with strategic traders and hierarchies of beliefs, showing how disagreement about others' beliefs can give rise to a resale premium.

2. The Dot-com Bubble and the Chinese Warrants Bubble

The dot-com bubble of 1999-2001 is the most prominent example of a speculative bubble in recent history. Fueled by the widespread excitement about the internet, stock prices of internet-based companies rose by almost 1000 % over the period of two years between early 1998 and 2000. The bubble peaked on March 10, 2000, and then burst, with prices going back to the levels comparable to 1998. Numerous companies went bankrupt. Ofek and Richardson (2003) provided convincing evidence that the dot-com bubble exhibited all the characteristics of a speculative bubble. The volume of trade in internet stocks between January 1998 and February 2000 accounted for almost 40 % of total volume in US stocks, nearly double the share before 1997. The volume per stock was three times higher for internet stocks than for their non-internet counterparts. Short selling was constrained by lock-up restrictions on newly issued IPOs, which constituted a significant portion of internet stocks. The IPOs largely contributed to the extraordinary run-up of internet stocks, as they averaged a stunning 89% average first-day return. The short sale restrictions were binding, as indicated, for instance, by unusually frequent violations of the put-call parity relation in the options markets. Short interest, measured as fraction of outstanding shares that have been sold short, was twice as high for internet stocks as for their non-internet counterparts. Ofek and Richardson (2003) argue that investors' beliefs were highly diverse because of an unusually high participation of retail investors rather than financial institutions. For example, in March 2000, the median ownership of internet companies shares by institutions was 25% compared with 40% for non-internet companies.

Griffin et al. (2011) provide a detailed analysis of the trading patterns of institutional and retail investors in internet stocks during the dot-com period. They show that institutional investors were the largest buyers of internet stocks in the run-up from January 1998 to March 2000. Among them, hedge funds were the most aggressive traders. Institutions began pulling out of the market after the peak, while individual investors became the primary buyers. Differences in trading patterns between institutional and individual investors were also significant around news announcement. Cochrane (2003) views the dot-com bubble as an episode in which some internet stocks functioned as highly liquid assets, commanding a convenience yield (or liquidity premium) due to intense trading rather than resale premia.

Another example of a speculative bubble is the Chinese warrants bubble, described in detail in Xiong and Yu (2011). In 2005-2006, 18 Chinese companies issued put warrants for trading on the Shanghai and Shenzhen Stock Exchanges. Soon after issuance, there was a boom in the stock market and the warrants, which had maturity of 6-24 months, became essentially worthless. Yet, they continued to be traded at relatively high prices, with astonishing turnover and volume. The turnover rate on some warrants was more than 300 times higher than that of NYSE stocks. Xiong and Yu (2011) argue that there was a dispersion of investors' beliefs about the likelihood of the warrants having non-zero payoff at maturity. Short selling of them was strictly prohibited. Pearson, Yang and Zhang (2020) study trading patterns in these warrants using data from brokerage accounts. They document extremely high turnover and substantial differences in trading patterns between institutional and retail investors.

Mei, Scheinkman and Xiong (2006) study dual-class shares listed on the Shanghai and Shenzhen Stock Exchanges during 1993–2001, using brokerage account data to examine speculative trade. They show that heterogeneous beliefs, together with short-sale constraints, led prices to deviate from fundamental values and produced violations of the law of one price.

There have been many other episodes of rapid asset price increases that bear all the signs of a speculative bubble, but they are less transparent than the dot-com and the Chinese warrants bubbles because the price surges were not followed by major crashes. For example, the spectacular 600 % price increase of Tesla stock in 2020 was accompanied by a well-publicized diversity of beliefs about the company, high trading volume, and high short interest. More recently, claims have been made about an emerging "AI bubble," with comparisons frequently drawn to the dot-com episode and the alleged overpricing of firms associated with artificial intelligence. Investment in these stocks has been massive, suggesting that prices largely reflect the valuations of optimistic investors and that the market is dominated by optimists. At the same time, however, there is little evidence of widespread short selling or other forms of pessimistic speculation in these assets. The absence of a strong presence of pessimists limits the case for interpreting current price dynamics as a speculative bubble. That said, the jury is still out, and the ultimate assessment will depend on how the market evolves over time.

3. Speculative Trade and Bubbles

Time is discrete and begins at date 0. Time horizon T may be finite or infinite. The set of possible states at each date is a finite set S. The product set S^T represents all T-sequences of states. For a sequence (or path) of states with $s_t \in S$ at date t, we use s^t to denote the partial history $(s_0, \dots, s_t)$ through date t. Partial histories are date-t events. The set S^T together with the σ-field Σ of T-products of subsets of S is the measurable space describing the uncertainty.

There is a single risky asset available for trade at every date in every state. Date-t dividend of the risky asset is a random variable d_t on (S^T, Σ) assumed measurable with respect to $\mathcal{F}_t$, the σ-field of date-t events, and bounded over time. When the time horizon is finite, we often assume that the asset pays a single non-zero dividend at the maturity date, taken to be the terminal date T. The ex-dividend price of the asset at date t is a random variable p_t, measurable with respect to $\mathcal{F}_t$. There is also a one-period risk-free asset available for trade at every date, with a unit payoff at the next date. Date-t price of the risk-free asset is a

$\mathcal{F}_t$-measurable random variable q_t. If the time horizon is finite, we set $p_T = 0$ and $q_T = 0$, which enable us to treat the terminal date like any other trading date.

There are I agents. Each agent i is risk-neutral and discounts future consumption by discount factor $\beta < 1$, common to all agents. Agent i's beliefs are represented by a probability measure P^i on (S^T, Σ) such that $P_i(s^t) > 0$ for every s^t. Agent i's utility function of a consumption plan $c = \{c_t\}_{t=0}^T$, which is bounded, positive, and adapted to $\mathcal{F}_t$, is

$$\sum_{t=0}^{T} \beta^t E_i[c_t], \tag{1}$$

where E_i denotes the expectation under probability measure P^i. Endowments e_t^i are measurable w.r. to $\mathcal{F}_t$, positive, and bounded over time. Holdings of the risky asset and the risk-free asset are $h = \{h_t\}_{t=0}^T$ and $b = \{b_t\}_{t=0}^T$, respectively, with h_t and b_t adapted to $\mathcal{F}_t$. Initial holdings of the risky asset are $\hat{h}_0^i \geq 0$ and the supply is $\hat{h}_0 = \sum_i \hat{h}_0^i$, assumed strictly positive. The supply and initial holdings of the risk-free asset are zero.

The agent faces the following budget and portfolio constraints for the choice of an optimal consumption plan c and portfolio holdings (b, h),

$$\begin{aligned} c_0 + p_0 h_0 + q_0 b_0 &\leq e_0^i + p_0 \hat{h}_0^i, \\ c_t(s^t) + p_t(s^t) h_t(s^t) + q_t(s^t) b_t(s^t) &\leq e_t^i(s^t) + [p_t(s^t) + d_t(s^t)] h_{t-1}(s_-^t) + b_{t-1}(s_-^t), \\ h_t(s^t) \geq 0, \ b_t(s^t) &\geq -B \qquad \forall s^t, \quad \forall t \leq T, \end{aligned} \tag{2}$$

where s_-^t denotes the predecessor event of s^t at date $t-1$. Condition (2) represents the short-sales constraints, with the bound B on short selling of the risk-free asset assumed strictly positive.[1]

An equilibrium consists of prices p and consumption-portfolio allocation $\{c^i, h^i, b^i\}$ such that plans (c^i, h^i, b^i) are optimal and markets clear. Market clearing conditions are

$$\sum_i c_t^i = \bar{e}_t^i + \hat{h}_0 d_t, \qquad \forall\ t \leq T,$$

[1] In practice, short-sale constraints are often more complex. Short selling requires locating a broker willing to lend shares and posting substantial collateral. We abstract from these important features.

and

$$\sum_i h_t^i = \hat{h}_0, \text{ and } \sum_i b_t^i = 0, \qquad \forall\ t < T.$$

In an equilibrium with strictly positive consumption of all agents, equilibrium price of the risk-free asset is

$$q_t(s^t) = \beta$$

for every s^t and $t < T$, so that $1/\beta$ is the risk-free (gross) return. Because of the short-sales constraint, equilibrium prices of the risky asset satisfy the relationship

$$p_t(s^t) = \max_i \beta E_i[p_{t+1} + d_{t+1}|s^t], \tag{3}$$

for every s^t, $t < T$. The agent (or agents) whose one-period-ahead conditional belief is the maximizing one on the right-hand side of (3) holds the asset in s^t while the other agents whose conditional beliefs give lower expectation have zero holding. We call the agent whose beliefs is the maximizing one the *short-run optimist* at s^t.

Market belief at s^t is the maximizing one-period-ahead conditional probability in (3), i.e., the optimist's belief, and is denoted by $\hat{P}^{+1}(\cdot|s^t)$. Let $\hat{P}$ be the probability measure on S^T derived from one-period-ahead probabilities $\hat{P}^{+1}(\cdot|s^t)$.[2] It follows that $\hat{P}$ is a risk-neutral pricing measure (or state-price process) for p. If the time horizon is finite, then repeated application of equation (3) leads to

$$p_t(s^t) = \sum_{\tau=t+1}^{T} \beta^{\tau-t} E_{\hat{P}}[d_\tau|s^t], \tag{4}$$

for every s^t and $t < T$. That is, the asset price equals the sum of discounted expected dividends under the market belief. If the time horizon is infinite, then eq. (4) follows from the no-bubble theorem (see Theorem 3.3 in Santos and Woodford (1997), or Theorem 30.6.1 in LeRoy and Werner (2014)).[3], applicable in this context since the present valued of aggregate resources is finite and asset supply is strictly positive. Equation (4) shows that there is no rational price bubble in equilibrium.

[2] If $T = \infty$, the existence of probability measure $\hat{P}$ on S^∞ follows from the Kolmogorov Extension Theorem.

[3] These theorems are concerned either with borrowing or debt constraints on portfolio holdings. However, one can verify that they apply to short-sales constraints as long as the existence of state prices is assured as it is in our model.

The fundamental value of the asset under agent's i belief is the discounted sum of expected future dividends conditional on event s^t, that is,

$$V_t^i(s^t) = \sum_{\tau=t+1}^{T} \beta^{\tau-t} E_i[d_\tau | s^t],$$

for $t < T$. If the time horizon is finite, we set $V_T = 0$. Because of risk-neutral utilities, agents' fundamental values represent their willingness to pay for the asset if obliged to hold it forever[4]. It follows from (3) that

$$p_t(s^t) \geq V_t^i(s^t), \tag{5}$$

for every i, every s^t. The following lemma will be used in the analysis to follow.

Lemma 1: *If $p_t(s^t) > V_t^i(s^t)$ for agent i in some event s^t, then $p_\tau(s^\tau) > V_\tau^i(s^\tau)$ for every predecessor event s^τ of s^t, where $\tau < t$.*

Proof: We first prove that $p_{t-1}(s^{t-1}) > V_{t-1}^i(s^{t-1})$ for the immediate predecessor of s^t. From (3) we have

$$p_{t-1}(s^{t-1}) \geq \beta E_i[p_t + d_t | s^{t-1}] > \beta E_i[V_t^i + d_t | s^{t-1}] = V_{t-1}^i(s^{t-1}), \tag{6}$$

where we used (5) and, for the strict inequality, the assumption that $p_t(s^t) > V_t^i(s_t)$. The proof for non-immediate predecessor events is an iteration of the argument in (6). □.

We say that there is a *speculative bubble* in event s^t if

$$p_t(s^t) > \max_i V_t^i(s^t). \tag{7}$$

It follows from Lemma 1 that if there is a speculative bubble in event s^t at date t, then there is a speculative bubble at every date $\tau < t$, in each predecessor event. Thus, a speculative bubble must originate at date 0, or more generally at the time of the initial offering, but it may cease to exist (i.e., "burst") at a later date.

If (7) holds, then the short-run optimist who buys the asset at s^t pays the price exceeding her valuation of the asset, if she were to hold it forever. This means, of course, that she intends to sell the asset at a later date. Thus, a speculative

[4] If the time horizon is finite, then "forever" means up to the terminal date T.

bubble implies the presence of speculative trade. The exact pattern of speculative trade can be traced from equation (3). Trade occurs whenever there is a change in the composition of the group of short-run optimistic agents, that is, agents whose beliefs are the maximizing ones in (3). Those who no longer are the optimists sell the asset up to the limit of the short-sale constraint, while the new optimists buy it.

Agent i is (weakly) *valuation dominant from date-t on* in event s^t if

$$V_\tau^i(s^\tau) \geq V_\tau^j(s^\tau), \qquad \forall\, \tau \geq t,\ \forall\, j, \tag{8}$$

for every event s^τ which is a successor of s^t.[5] If there is no valuation dominant agent from date-t on in event s^t, then we say that agents' beliefs exhibit *valuation switching* in event s^t. If the time horizon T is finite, then no valuation switching can occur at date $T-1$, and there is no speculative bubble at that date either. Furthermore, if no single agent's expected value of the dividend exceeds those of all other agents in every state at date $T-1$, then valuation switching occurs at every date prior to $T-1$.

Theorem 1: *If agents' beliefs exhibit valuation switching in event s^t, then in equilibrium there is speculative bubble in s^t.*

PROOF: Suppose by contradiction that $p_t(s^t) = V_t^i(s^t)$ for some agent i. It follows from Lemma 1 that $p_\tau(s^\tau) = V_\tau^i(s^\tau)$ for every successor event s^τ. Since agent i is not valuation dominant, there exists agent j and a successor event s^τ such that $V_\tau^j(s^\tau) > V_\tau^i(s^\tau) = p_\tau(s^\tau)$. But this contradicts (5). $\square$.

The condition of valuation switching is not necessary for the existence of speculative bubbles. There may be a valuation dominant agent and yet speculative bubble and trade may emerge in equilibrium. This has been demonstrated by Harrison and Kreps (1978) in an example which we reproduce in Section 4.

4. Speculative Bubbles with Markovian Beliefs

Markovian beliefs of agent i are specified by a matrix of transition probabilities $Q_i : S \times S \to S$. We assume that $Q_i(s, s') > 0$ for every $s, s' \in S$ and every

[5] We often omit the phrase "from date-t on" and simply say valuation dominant.

i. Probability measure P_i on (S^T, Σ) obtains by taking path-dependent products of Q_i. It makes the state variable s_t a Markov process. If the dividend d_t is a time-invariant function of the current state, i.e., $d_t(s_t) = d(s_t)$ for some function $d : S \to R_+$, then d_t is a stationary Markov process under Q_i.

If the time horizon is infinite and dividends are a time-invariant function of the state, then the fundamental value of the asset is time-invariant as well, and it satisfies the recursive equation $V^i = \beta Q_i[V^i + d]$, where V^i denotes the S-vector of values $V^i(s)$. The solution to this equation is[6]

$$V^i = \beta(I - \beta Q_i)^{-1} Q_i d. \tag{9}$$

The condition of valuation switching in Theorem 1 holds in this case if and only if there is no single agent whose valuation V^i, given be (9), exceeds those of all other agents in every state. Wu and Guo (2003) provide a weaker sufficient condition for a speculative bubble with Markovian beliefs.

If the time horizon is finite and there is a single non-zero dividend paid at the terminal date, then the fundamental value is

$$V_t^i = \beta^{T-t} Q_i^{T-t} d_T, \tag{10}$$

for every $t < T$. A sufficient condition for valuation switching is that there is no single agent whose expected value of the dividend at date $T-1$, which is $\beta Q_i d_T$, exceeds those of all other agents in every state, see Section 3.

The following example of speculative trade and bubbles with Markovian beliefs is due to Harrison and Kreps (1978).

Example 1: Suppose that the time horizon is infinite and the dividend process d_t is a Markov chain taking values 0 or 1 for every $t \geq 1$. There are two agents, with beliefs described by transition matrices Q_1 and Q_2. We take

$$Q_1 = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} \quad \text{and} \quad Q_2 = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

where the rows correspond to the states 0 and 1 being transitioned from, and the columns to the states 0 and 1 being transitioned to. Note that agent 1 is more

[6] Wu and Guo (2003) show that the matrix $I - \beta Q_i$ is invertible for every $\beta < 1$.

optimistic than agent 2 about next-period high dividend when current state is 0 and vice versa when the current state is 1. Discount factor is $\beta = 0.9$.

Fundamental values of the asset depend only on the current dividend and can be solved from eq. (9). They are $V^1(0) = 4.66$, $V^1(1) = 4.34$, $V^2(0) = 4.09$, and $V^2(1) = 4.91$. Since $V^1(0) > V^2(0)$ and $V^2(1) > V^1(1)$, there is valuation switching at every date and state.

In equilibrium, the agent who is more optimistic about the next-period dividend is also the short-run optimist (about price plus dividend), and holds the asset. Equilibrium prices can be found from equation (3). We have

$$p(0) = \beta[\frac{1}{4}p(0) + \frac{3}{4}(p(1)+1)], \text{ and } p(1) = \beta[\frac{1}{4}p(0) + \frac{3}{4}(p(1)+1)]. \quad (11)$$

Because of the symmetry of agents' beliefs, equilibrium price is state independent and equal to $p(0) = p(1) = 6\frac{3}{4}$. One can easily verify that the right-hand sides of equations (11) are the respective maximal values among the two agents. We have

$$p(0) > \max_i V^i(0) \text{ and } p(1) > \max_i V^i(1).$$

There is a speculative bubble in accordance with Theorem 1.

We consider next the original example of Harrison and Kreps (1978). Let

$$Q_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \quad \text{and} \quad Q_2 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

The discount factor is $\beta = 0.75$. As before, agent 1 is the optimist about the next-period high dividend when the current dividend is 0, while agent 2 is the optimist when it is 1. Fundamental values are obtained from (9), and they are $V^1(0) = \frac{4}{3}, V^1(1) = \frac{11}{9}$, and $V^2(0) = \frac{16}{11}, V^2(1) = \frac{21}{11}$. It follows that agent 2 is valuation dominant (the long-run optimist) at every date and state. Yet, in equilibrium the agent who holds the asset is the one who is more optimistic about the next-period dividend. Equilibrium prices are found again from equation (3). They are $p(0) = \frac{24}{13}$ and $p(1) = \frac{27}{13}$. There is a speculative bubble. □

Example 1 shows that valuation switching is not necessary for the existence of speculative bubbles. Moreover, it demonstrates that the agent with the highest valuation need not be the one holding the asset in equilibrium, that is, the long-run optimist need not be the short-run optimist.

5. Speculative Bubbles with Bayesian Learning

We describe the model of Bayesian learning of i.i.d. distribution. An exposition of the general non-i.i.d. case can be found in Werner (2024).

There is a family of probability measures π_θ on the state space $(S, 2^S)$ parametrized by θ in the set of parameters Θ. The set Θ can be finite or infinite. There is a σ-field $\mathcal{G}$ of subsets of Θ, and the mapping $\theta \to \pi_\theta(A)$ is measurable for every $A \in 2^S$. We assume that $\pi_\theta(s) > 0$ for every θ and every $s \in S$. Let P_θ denote the product measure π_θ^T on (S^T, Σ) which makes state variable s_t independently and identically distributed with π_θ at every t. An agent, who does not know the true probability measure of the i.i.d. process s_t, has a prior belief μ on $(\Theta, \mathcal{G})$. The support of prior μ is the smallest closed subset of Θ of μ-measure 1, or equivalently a closed set $C \subset \Theta$ with $\mu(C) = 1$ and such that if $\theta \in C$, then $\mu(U) > 0$ for every neighborhood U of θ in Θ.

Prior μ induces a joint distribution Π_μ of parameters and sequences of states defined by

$$\Pi_\mu(A \times B) = \int_A P_\theta(B)\mu(d\theta),$$

for $A \in \mathcal{G}$ and $B \in \Sigma$. Conditional probability on $\mathcal{G} \times \Sigma$ upon observing date-t history of states s^t is $\Pi_\mu(\cdot|s^t)$. It induces the posterior belief on Θ denoted by $\mu_t(\cdot|s^t)$ and conditional probability of the future given the past on Σ denoted by $P_\mu(\cdot|s^t)$. For example, if μ is a Dirac point-mass measure at some θ, then $\mu_t = \mu$ for every t and $P_\mu(\cdot|s^t) = P_\theta(\cdot|s^t)$. This is a "dogmatic" belief.

Let agent's i prior belief be μ_i on $(\Theta, \mathcal{G})$. We use E_i to denote the expectation under probability measure P_{μ_i} and $E_i[\cdot|s^t]$ for conditional expectation under $P_{\mu_i}(\cdot|s^t)$. The fundamental value $V_t^i(s^t)$ is the sum of discounted expected future dividends under agent's i updated belief $P_{\mu_i}(\cdot|s^t)$. We assume either an infinite time horizon, in which the dividend is a time-invariant function of the state,[7] or a finite horizon, with a single non-zero dividend paid at the terminal date. In the case of an infinite horizon, the fundamental value of the asset is

$$V_t^i(s^t) = \frac{\beta}{1-\beta} E_i[d|s^t], \tag{12}$$

[7]This can be extended to dividends paid at a constant frequency greater than one period, which is empirically more relevant, see Werner (2024).

for every s^t. In the case of a finite horizon, it is

$$V_t^i(s^t) = \beta^{T-t} E_i[d_T|s^t] \tag{13}$$

for every s^t where $t < T$.

A sufficient condition for a speculative bubble under Bayesian learning is valuation switching, see Theorem 1. Valuation dominance - which defines the switching condition - is closely related to the monotone likelihood ratio order of priors (see Morris (1996)). Suppose that $\Theta \subset R$ and that each prior μ_i has strictly positive density function f_i on Θ. Thus, all priors have full support Θ. We say that prior μ_i dominates μ_j in the monotone likelihood ratio (MLR) order if

$$\frac{f_i(\theta')}{f_i(\theta)} \geq \frac{f_j(\theta')}{f_j(\theta)} \quad \text{whenever} \quad \theta' \geq \theta,$$

for every $\theta, \theta' \in \Theta$.

Let $V_0^\theta(s)$ denote the date-0 fundamental value of the asset in state s, which is, in cases (12) and (13), the discounted expected value of the dividend under π_θ; that is $\frac{\beta}{1-\beta}E_\theta[d]$ or $\beta^T E_\theta[d_T]$.

Proposition 1: *Suppose that $V_0^{\theta'}(s) \geq V_0^\theta(s)$ for every $\theta' \geq \theta$ and every $s \in S$. If agent's i prior μ_i MLR-dominates every other agents' prior, then agent i is valuation dominant in every $s^t \in S^T$.*

Proof: It is well-known that if μ_i MLR-dominates μ_j, then μ_i dominates μ_j in the sense of first-order stochastic dominance. Since $V_0^\theta(s)$ is increasing in θ, it follows that

$$V_0^i(s^0) = \int V_0^\theta(s^0) f_i(\theta) d\theta \geq \int V_0^\theta(s^0) f_j(\theta) d\theta = V_0^j(s^0) \tag{14}$$

for every j and every s^0. Further, if μ_i MLR-dominates μ_j, then the posterior $\mu_i(\cdot|s^t)$ MLR-dominates the posterior $\mu_j(\cdot|s^t)$ for every s^t, see Shaked and Shanthikumar (2007). As in (14), this implies $V_t^i(s^t) \geq V_t^j(s^t)$ for every j and every s^t. □

The following example of speculative bubbles under Bayesian learning with heterogeneous priors is due to Morris(1996).

Example 2: Suppose that the time horizon is infinite and that dividends are a time-invariant function of the state, taking values 0 or 1 at every date. Probability measures on the state-space $S = \{0, 1\}$ is parametrized by the probability θ of high dividend 1 where $\theta \in [0, 1] = \Theta$. Consider a prior μ on $[0, 1]$ with density function f. Expected value of the dividend $E_\mu[d|s^t]$ conditional on s^t is equal to the conditional probability of next-period high dividend. That probability depends only on the number of high dividends from date 0 through t, and is denoted by $\nu(t, k)$ for $k \leq t$. We have, by Bayesian updating,

$$\nu(t,k) = \frac{\int_0^1 \theta^{k+1}(1-\theta)^{t-k} f(\theta) d\theta}{\int_0^1 \theta^{k}(1-\theta)^{t-k} f(\theta) d\theta}.$$

An important class of priors on the interval $[0, 1]$ are beta priors with density functions of the form $f(\theta) \sim \theta^{\alpha-1}(1-\theta)^{\beta-1}$ for $\alpha > 0$ and $\beta > 0$. The posterior probability of high dividend under beta prior is

$$\nu(t,k) = \frac{k+\alpha}{t+\alpha+\beta}, \tag{15}$$

see Ghosh and Ramamoorthi (2003). The fundamental value of the asset conditional on (t, k) is $\frac{\beta}{1-\beta}\nu(t, k)$.

For two priors μ and μ' with beta distributions with (α, β) and (α', β') respectively, one can show that μ valuation-dominates μ' at date 0 if and only if $\alpha \geq \alpha'$ and $\beta \leq \beta'$. Otherwise, there is valuation switching at every date and state. It is well known that prior μ dominates μ' in the MLR-order if and only if $\alpha \geq \alpha'$ and $\beta \leq \beta'$. Thus, MLR-order dominance and valuation dominance are equivalent for beta priors on $[0, 1]$.

Consider two popular priors under ignorance: the uniform prior with density $f_u(\theta) \equiv 1$ and the Jeffreys prior with density $f_J(\theta) \sim \frac{1}{\sqrt{\theta(1-\theta)}}$, see Morris (1996). These are beta priors with $(\alpha_u, \beta_u) = (1, 1)$ and $(\alpha_J, \beta_J) = (\frac{1}{2}, \frac{1}{2})$ for which the posterior expected values of the dividend are $\nu_u(t, k) = \frac{k+1}{t+2}$ and $\nu_J(t, k) = \frac{k+\frac{1}{2}}{t+1}$ They give rise to permanent valuation switching. In a market where some agents have a uniform prior and others have Jeffreys prior, there is, by Theorem 1, a speculative bubble at every date and event. A closed-form expression for equilibrium prices is not available. □

Speculative bubbles can arise when agents' priors have different supports. Suppose that the time horizon is infinite. Let $C_i \subset \Theta$ be the support of agent's i prior belief μ_i and let

$$M_i = \max_{\theta \in C_i} E_\theta[d], \quad \text{and } m_i = \min_{\theta \in C_i} E_\theta[d].$$

be the maximal and the minimal expected value of the dividend over all priors in the support.

We have

Proposition 2: *Suppose that the time horizon is infinite and that dividends are a time-invariant function of the state. Further, suppose that the mapping $\theta \rightarrow \pi_\theta$ is 1-to-1 and continuous. If there is no single agent i such that $M_i \geq M_j$ and $m_i \geq m_j$ for all $j \neq i$, then there is valuation switching and a speculative bubble at every date and state in an equilibrium.*

The proof has been relegated to the Appendix. The hypothesis of Proposition 2 holds, in particular, if one agent's prior has full support Θ, and another agent's prior is a Dirac point mass at some parameter in the interior of Θ. For example, these agents could be a Bayesian learner with an ignorant prior and another agent who knows the true distribution. Example 3 illustrates the result.

Example 3: Consider again the setting of Example 2 with infinite time horizon and time-invariant dividend taking values 0 or 1. Suppose that agents have uniform priors on different intervals of parameters. More specific, there are two type of agents, i and j, with i having the uniform prior on $[0, 1]$ and j having uniform prior on an interval $[a, b]$ where $0 < a < b < 1$. Thus, type-j agents have more concentrated prior. Since the posterior belief at t conditional on k high dividends has strictly positive density on the interval $[a, b]$ and $\nu_j(t, k)$ is its mean, it follows that $a < \nu_j(t, k) < b$. Using (15), we see that for every (t, k), there exist (τ', k') with $\tau' \geq t$ and $k' \geq k$ such that $\nu_j(\tau', k') < a$. Similarly, there exists (τ'', k'') with $\tau'' \geq t$ and $k'' \geq k$ such that $b < \nu_i(\tau'', k'')$. This implies valuation switching and a speculative bubble in equilibrium at every date and state, in accordance with Proposition 2. If $a = b$, then agent's j prior is the point-mass Dirac measure on a. The argument for valuation switching continues to hold. $\square$

An important issue arising when investors's beliefs adhere to Bayesian learning is whether updated beliefs merge with the true distribution as time goes to infinity.[8] The key statistical concept underlying this analysis is consistency of the prior at the true parameter. It requires that the posterior distribution over the parameters converges weakly to the true parameter almost surely, under the true distribution of states. Werner (2024) shows that if every agent's prior belief is consistent at the true parameter and these priors are mutually absolutely continuous, then each agent's fundamental value and the equilibrium asset price converge to the true fundamental value. If there is a speculative bubble, it vanishes in the limit as time goes to infinity, even though speculative trade may persist. By the classical Theorem of Doob (see Ghosh and Ramamoorthi (2003)), every prior on a finite-dimensional parameter space is consistent at almost every parameter in its support. In most cases in our setting, the consistency holds at every prior in the support by a result of Freedman (1963). We refer the reader to Werner (2024) for more details of the results on asymptotic properties of speculative bubbles and trade under Bayesian learning, including a discussion of misspecified priors. We provide here an example that illustrates these results.

Example 4: Consider the setting of Example 2. The uniform prior and the Jeffreys prior are absolutely continuous with respect to each other. Further, they are consistent at the true parameter θ_0 for every $\theta_0 \in [0,1]$. This follows from Freedman (1963), but it can also argued directly as follows. By the strong Law of Large Numbers, the frequency k/t of high states converges to θ_0 with π_0–probability 1. Therefore the means of the posteriors of μ_u and μ_J - which are given by (15) - converge to θ_0. Since the variances of the posteriors converge to zero (see Ghosh and Ramamoorthi (2003)), the consistency of μ_u and μ_J at every $\theta_0 \in [0,1]$ follows.

Fundamental values of the asset are $\frac{\beta}{1-\beta}\nu_u(k,t)$ and $\frac{\beta}{1-\beta}\nu_J(k,t)$ and they converge to the true value $\frac{\beta}{1-\beta}\theta_0$, with π_0^∞-probability 1. The price of the asset converges to the true value as well. There is speculative trade at every date and state, but the speculative bubble vanishes in the limit. □

[8] The interpretation of this issue in the setting with a finite time horizon is that the number of trading dates T diverges to infinity, making the frequency of trade infinitely high.

6. Speculative Bubbles with Differential Interpretation of Common Signals

In this section we assume that the time horizon is finite and there is a single non-zero dividend paid at the terminal date T. At each date $t = 1, \ldots T-1$, agents observe the realization of a public signal y_t. Agents have heterogeneous beliefs about the joint distribution of the signals and the dividend. We assume, as is standard in the literature, that the signals are independent and have identical distribution conditional on the dividend. That is, conditional on d_T, the signals $(y_1, \ldots, y_{T-1})$ are i.i.d. Further, the marginal distributions of the dividend d_T are the same for all agents, so they share a common prior on the dividend but differ in their interpretation of the public signals.

The fundamental value of the asset conditional on observing the history of signals $y^t = (y_1, \ldots, y_t)$ is

$$V_t^i(y^t) = \beta^{T-t} E_i[d_T | y^t] \tag{16}$$

for $t < T$.

Suppose that the conditional distribution of the signal has a density function $f_i(y|d_T)$ for every i. If the the density function $f_i(y|d')$ MLR-dominates the density $f_i(y|d)$ whenever $d' > d$, for every values d and d' of d_T, then the posterior distribution of the dividend conditional on y' first-order stochastically dominates the distribution conditional on y whenever $y' \geq y$. Thus, higher values of the signal are more favorable (or more informative) to the dividend, see Milgrom (1981). Because of the i.i.d assumption on the signals, higher values of the date-t signal y_t are also more favorable to the dividend. Consequently, the fundamental value of the asset, given by (16), is an increasing function of the date-t signal for every t.

A sufficient condition for valuation switching is that there is no single agent whose expected value of the dividend at date $T-1$ exceeds those of all other agents in every state. Such differences in expected values may arise from different strengths of reaction to the signal. This is illustrated in the following example based on Harris and Raviv (1993).

Example 5: Suppose that the dividend d_T takes two values H or L such that $L < H$. Agents have common prior probabilities π_L and π_H of d_T, with $0 < \pi_H < 1$.

Information signals are conditionally independent and identically distributed given the dividend, with support on the entire real line. The conditional distribution of y given d_T is two-sided exponential, with the following densities parametrized by a_i and b_i:

$$f_i(y|H) = \begin{cases} k_i a_i^y & \text{if } \; y \geq 0, \\ k_i b_i^{-y} & \text{if } \; y < 0, \end{cases}$$

and

$$f_i(y|L) = \begin{cases} k_i b_i^y & \text{if } \; y \geq 0, \\ k_i a_i^{-y} & \text{if } \; y < 0, \end{cases}$$

where $0 < a_i < 1$ and $0 < b_i < 1$, and where $k_i = 1/[\frac{1}{\ln(1/a_i)} + \frac{1}{\ln(1/b_i)}]$ is a normalization factor. Parameters a_i and b_i can be different across agents.

The posterior probability of the dividend equal to H after observing the history of signals y^t for $t < T$, can be derived using the Bayes rule, and it is

$$\pi_i(H|y^t) = \frac{1}{1 + \frac{\pi_L}{\pi_H}(\frac{b_i}{a_i})^{m_t}}, \tag{17}$$

where $m_t = y_1 + \ldots y_t$. The posterior probability depends only on the cumulative value m_t of the past and current signals, and we write $\pi_i(H|m_t)$. We assume that $\frac{b_i}{a_i} < 1$ for every i. Under this condition, the density $f_i(y|H)$ MLR-dominates $f_i(y|L)$ implying, as shown in (17), that higher values of the signal are more favorable to the dividend.

The ratio $\frac{b_i}{a_i}$ can be interpreted as the strength of reaction to the signal. The higher the ratio, the greater the probability assigned to the high dividend when the cumulative signal is positive, and to the low dividend when the cumulative signal is negative.

Agent i's fundamental value of the asset at date t depends only on the cumulative signal and is given by

$$V_t^i(m_t) = \beta^{T-t}[L(1 - \pi_i(H|m_t) + H\pi_i(H|m_t)] \tag{18}$$

for every $t < T$. If agents differ in the strength of reaction to the signal, so that agent i over-react relative to agent j, i.e.,

$$\frac{b_i}{a_i} > \frac{b_j}{a_j}$$

for some i and j, then $\pi_i(H|m_t) > \pi_j(H|m_t)$ and $V_t^i(m_t) > V_t^j(m_t)$ for $m_t > 0$, and the inequalities are reversed for $m_t < 0$. This implies that there is valuation switching at every date $t < T-1$ for every history of signals y^t. It follows from Theorem 1 that there is a speculative bubble in equilibrium.

An explicit solution for equilibrium prices is not available, though they can be computed numerically. Harris and Raviv (1993) characterize asset trading under the assumption that agents act as price setters, setting prices equal to the highest fundamental value, so that there is no speculative bubble. □

If the dividend is normally distributed and the conditional distribution of the signal on the dividend is normal with zero-excess-mean, i.e., y is d_T plus a zero-mean normal noise, then the signal is favorable to the dividend. The strength of reaction to the signal depends on the belief about the variance of the noise and expresses over- or underconfidence in the precision of the signal. Heterogeneous beliefs about the signal precision generate valuation switching and speculative bubbles, see Hong, Scheinkman and Xiong (2006).

7. Discussion and Concluding Remarks

Risk Aversion Speculative trade can be considered with risk-averse agents, as in Hong, Scheinkman and Xiong (2006). An agent's willingness to pay for the asset, if obliged to hold it forever, can be defined using state-dependent intertemporal marginal rates of substitution. It depends not only on the agent's belief but also on her risk tolerance. It remains true that if the price of the asset exceeds every agent's thus defined fundamental value, then an agent who holds the asset intends to sell it up to the limit of the short-sale constraint at some date and event in the future. Yet in the presence of risk aversion, there will generally be other trades motivated by hedging, and it appears artificial to distinguish between hedging and speculative trade solely based on whether the short-sale limit is reached.

Rational versus Speculative Bubble As noted in Section 3, the speculative bubble in the setting we consider is not a rational bubble. The argument we provided depends, however, on the assumption that there is a single long-lived risky asset. If there are multiple assets with short-sales constraints on each, there may not exist a risk-neutral measure that defines the fundamental value of every asset in the sense of a rational pricing bubble. The literature on rational pricing bubbles typically considers markets with debt or borrowing constraints that ensure the existence of a risk-neutral measure.

Interestingly, the classical example of a rational bubble on a zero-dividend asset (or fiat money) is also a speculative bubble albeit with risk-averse agents. The fundamental value defined as the willingness to pay for the asset if obliged to hold it forever, is clearly zero. The equilibrium price is strictly positive and short-sale constraints are, in fact, alternately binding for the two agents in the market. Yet the reason for trade is consumption smoothing. Werner (2014) presents an example of a rational price bubble on an asset in zero supply that exhibits no features of a speculative bubble, as the portfolio constraints are never binding.

Concluding Remarks This paper presents a unified framework for speculative trade and bubbles in dynamic asset markets with heterogeneous beliefs and short-sale constraints. A speculative bubble arises when the market price exceeds every trader's long-run valuation, and valuation switching is a key factor behind such price dynamics. Among the mechanisms that generate belief heterogeneity—Markovian beliefs, Bayesian learning, and differential interpretation of public information—the same principle applies: shifts in long-run optimism induce changes in short-run optimism and speculation. The analysis spans both finite- and infinite-horizon markets, showing that valuation switching provides a common foundation for speculative bubbles.

Empirical evidence from the dot-com and Chinese warrants episodes supports this perspective. Both reveal how differences in belief formation and interpretation of information can produce rapid price surges. Taken together, the theoretical and empirical findings suggest that speculative bubbles can arise naturally when investors hold different beliefs and face trading constraints.

8. Appendix

PROOF OF PROPOSITION 2: Consider an arbitrary event s^t and let i be the agent with the highest valuation $V_t^i(s^t)$. Since $V_t^i(s^t) = \frac{\beta}{1-\beta} E_i[d|s^t]$, the expected value $E_i[d|s^t]$ is the highest as well. We shall prove that there exists a successor event s^τ for some $\tau > t$ such that $E_j[d|s^\tau] > E_i[d|s^\tau]$ for some agent j. By assumption, there exists an agent j such that either $M_j > M_i$ or $m_j > m_i$. Consider the former case first. Let $\theta \in C_j$ be such that $E_\theta[d] > M_i$. Note that $M_i \geq E_i[d|s^\tau]$ for every s^τ since the support of the posterior $\mu_i(\cdot|s^\tau)$ is C_i. By the Theorem of Doob (see Ghosh and Ramamoorthi (2003)), θ can be selected so that the prior μ_j is consistent at θ. This implies that the set $\{s^\infty : \lim_{t\to\infty} E_j[d|s^t] = E_\theta[d]\}$ has P_θ-probability 1. It follows that for τ large enough, the date-τ event $\{s^\tau : E_j[d|s^\tau] > M_i\}$ has strictly positive P_θ-probability. This implies that $E_j[d|s^\tau] > E_i[d|s^\tau]$ and consequently $V_\tau^j(s^\tau) > V_\tau^i(s^\tau)$. This concludes the proof of the first case. In the second case we have $m_j > m_i$. Then there exists $\theta \in C_i$ such that $m_j > E_\theta[d]$. The same argument as before shows that there is s^τ for some $\tau > t$ such that $V_\tau^j(s^\tau) > V_\tau^i(s^\tau)$. This concludes the proof. □

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