Monotonicity and Implementability: Beyond Convex Domains

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Abstract

We use insights from combinatorial Hodge theory and algebraic topology to analyze how incentive compatibility constraints shape the set of implementable deterministic allocation rules. For simply-connected domains, we show that 2-cyclic monotonicity is sufficient for implementability under an additional condition on an allocation rule. The additional condition ensures that local incentive compatibility implies global incentive compatibility. The additional condition is redundant for convex and single-peaked preferences domains.

1 Introduction

One of the main goals of mechanism design is to study the properties of optimal mechanisms maximizing a given objective, e.g. revenue or total welfare. The difficulty in deriving such mechanisms results from the designer having lack of information about agents' preferences. Hence, a well-designed mechanism should take into account agents' ability to hide their privately held information, often called incentive compatibility constraints. This paper analyzes how the incentive compatibility constraints shape the set of deterministic mechanisms available to the designer.

Myerson (1981) shows that in private value settings with one-dimensional types any *non-decreasing* allocation rule can be implemented, i.e. there exists a payment rule that is combined with the allocation rule produces a direct mechanism where truthtelling is in the best interests of agents. In multi-dimensional settings Rochet (1987) shows that a condition called *cycle monotonicity* is necessary and sufficient for an allocation rule to be implementable. To define this condition,

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let us associate with each agent a complete directed graph with types being vertices and a directed edge between two vertices having a length equal to the benefits from truthtelling compared to pretending to be the other vertex's type. The cycle monotoncity condition then says that the length of any finite cycle in this graph has a non-negative length. Though the cycle monotonicity is an elegant condition, it is usually tedious to verify. An important contribution by Saks and Yu (2005) (see also Bikhchandani et al. (2006), Ashlagi et al. (2010), Cuff et al. (2012), Archer and Kleinberg (2014)) is to show that for *convex domains* with finite set of outcomes it is sufficient to verify only the cycles of length two - *2-cycle monotonicity* condition.

In this paper, we extend the analysis of incentive-compatibility to *non-convex domains*. In particular, we consider *simply-connected domains*, which are path-connected domains where any two curves connecting any two points can be continuously transformed one into another. For simply-connected domains with finite set of outcomes, if any intersection of outcome sets (subsets of types leading to the same outcome) is simply-connected, an allocation rule satisfies 2-cycle monotonicity and *decomposition monotonicity* conditions, then the allocation rule is implementable.[1] The decomposition monotonicity condition can be interpreted as a condition ensuring that local incentive compatibility implies global incentive compatibility.

We show that decomposition monotonicity is implied by 2-cycle monotonicity if for any pair of outcome sets and for any type in one of them there exists a type in the other outcome set such that the interval connecting these two types belongs to the domain. Using this geometric interpretation, we show that any 2-cycle monotonicity is necessary and sufficient for implementability for any convex domain and for the domain of single peaked preferences, thus reestablishing the previous important results of Saks and Yu (2005) and Mishra, Pramanik and Roy (2014).

In addition to a novel analysis of incentive compatibility applicable beyond convex domains, we also introduce novel techniques to economics literature. An important step of our proofs uses a seminal Nerve theorem from algebraic topology (Bjorner, 1995) to translate topological properties of a system of sets to a graph associated with this system. We then analyze the properties of edge flows on this graph using the Helmholtz decomposition theorem from a recently developed combinatorial Hodge theory Jiang et al. (2011). These techniques are novel and can be of a special interests to people interested in mechanism design, networks, and possibly other areas of economics.

The paper proceeds as follows. Section 2 presents the model and states our main result. Section 3 shows that 2-cycle monotonicity implies decomposition monotonicity for convex domains. We also provide a geometric interpretation of decomposition monotonicity in this section. We then apply this interpretation to single-peaked preference domain in Section 4. Section 5 concludes and discusses related literature. The Appendix contains the proofs.

[1] A variant of the decomposition monotonicity condition was first proposed by Müller, Perea and Wolf (2007). It was also used in Berger, Müller and Naeemi (2009) and Mishra, Pramanik and Roy (2014).

2 Monotonicity and Implementability

In this section we first present a general framework, some essential definitions, and some important results of the previous literature. We then state our main result and explain the intuition behind its proof.

Since we study incentive compatibility we consider only the perspective of a single player.[2] Let A be a finite set of social outcomes and $T \subseteq R^{|A|}$ be the set of possible agent types, where $|A|$ denotes the cardinality of set A. We assume that if agent has type $\mathbf{t} \in T$ his utility for outcome $a \in A$ equals

$$U(a, \mathbf{t}, p) = t_a - p$$

where p is agent's payment.

A direct mechanism is characterized by two functions: an allocation rule, $f : T \to A$, mapping agent's reported type to the set of outcomes, and a payment rule, $p : T \to \mathcal{R}$, mapping the agent's reported type to the set of real numbers. We consider only deterministic allocation rules and do not allow randomizations over outcomes in A. For convenience, we also use the following notation: if allocation rule f chooses outcome a for type $\mathbf{t}$, we consider vector $\mathbf{f}(\mathbf{t}) \in \{0, 1\}^{|\mathbf{A}|}$ having ath component equal to 1 and all other components equal to 0. Using this notation, agent's utility from reporting type $\mathbf{t}'$ when his true type is $\mathbf{t}$ equals

$$U(\mathbf{t}'|\mathbf{t}) = \mathbf{f}(\mathbf{t}')\mathbf{t} - p(\mathbf{t}')$$

where $\mathbf{f}(\mathbf{t}')\mathbf{t}$ refers to the usual vector product. We call allocation rule f *implementable* if there exists a payment rule p such that mechanism (f, p) is incentive compatible, i.e.

$$\mathbf{f}(\mathbf{t})\mathbf{t} - p(\mathbf{t}) \geq \mathbf{f}(\mathbf{t}')\mathbf{t} - p(\mathbf{t}') \quad \forall \mathbf{t}, \mathbf{t}' \in T$$

Considering only deviations $\mathbf{t} \to \mathbf{t}'$ and $\mathbf{t}' \to \mathbf{t}$ we can eliminate payments to obtain a necessary condition for implementability, called *2-cyclic monotonicity*

$$(\mathbf{f}(\mathbf{t}) - \mathbf{f}(\mathbf{t}'))\,\mathbf{t} + (\mathbf{f}(\mathbf{t}') - \mathbf{f}(\mathbf{t}))\,\mathbf{t}' \geq \mathbf{0} \quad \forall \mathbf{t}, \mathbf{t}' \in T \tag{1}$$

The name of this condition comes from the following observation. One could consider a complete directed network with types being vertices and a directed edge from type $\mathbf{t}$ to type $\mathbf{t}'$ having length equal to the benefits from truthtelling compared to pretending to be type $\mathbf{t}'$. For this network inequality (1) corresponds to a condition that the length of each 2-cycle is non-negative. One could straightforwardly extend this condition to cycles of any length: if the length of any finite cycle is non-negative allocation f is called *cyclically monotone.* Rochet (1987) shows that cyclical monotonicity is not only necessary, but also sufficient condition for an allocation rule to be implementable.

[2] See Carroll (2012) on how the single agent analysis extends to settings with many agents.

Theorem 1 (Rochet (1987)) *A necessary and sufficient condition for f to be implementable is being cyclical monotone, i.e. for any M and for all cycles $\mathbf{t}_0, ..., \mathbf{t}_M = \mathbf{t}_0$ in T*

$$\sum_{k=0}^{M-1} (\mathbf{f}(\mathbf{t}_{k+1}) - \mathbf{f}(\mathbf{t}_k))\mathbf{t}_{k+1} \geq 0 \tag{2}$$

Though cyclical monotonicity fully characterizes implementable allocation rules this condition is tedious to verify. The original and important contribution of Saks and Yu (2005) is to establish that 2-cyclic monotonicity is sufficient for implementability for convex domains. Let us denote the closure of set S as $\overline{S}$.

Theorem 2 (Saks and Yu, 2005; Ashlagi et al., 2010) *If the closure of domain $\overline{T}$ is convex, a necessary and sufficient condition for f to be implementable is being 2-cyclic monotone.*

Conditions (1) and (2) can be conveniently reformulated using the notion of an outcome graph (see, e.g., Vohra (2011)). To define the outcome graph, let us denote outcome set T_a as the set of types for which outcome a is chosen. Without loss of generality we assume that each outcome $a \in A$ is chosen for at least one type. The outcome graph Γ_f of allocation rule f is then defined as a directed graph with set A being its vertices and the length of a directed edge from a to b equal to

$$l_{ab} = \inf_{\mathbf{t} \in T_a} (t_a - t_b)$$

The cyclical monotonicity of function f translates to the outcome graph. Graph Γ_f satisfies cycle-monotonicity property if for any natural M the length of M-cycle $\{a_0, a_1, ..., a_M\}$ with $a_i \in A$ for $i = 0, ..., M$ and $a_0 \equiv a_M$ satisfies

$$\sum_{k=0}^{M-1} l_{a_k a_{k+1}} \geq 0$$

which is equivalent to the existence of a function $s : A \to R$ such that

$$s_a - s_b \leq l_{ab}$$

for all a and b in A (see Chapter 3 in Vohra (2011)).

Moreover, it is easy to verify that allocation rule f is M-cyclic monotone if and only if the induced outcome graph Γ_f is M-cyclic monotone. We mainly use the outcome graph to analyze implementability of a given allocation rule.

To formulate our main theorem we need some definitions. We call two outcomes to be *adjacent* if the closure of corresponding outcome sets have non-empty intersection, and *distant* otherwise.

Definition 1 *If for any two* distant *outcomes $a, b \in A$ and* some *path $\{a \equiv a_0, ..., a_k \equiv b\}$ such that $\overline{T}_{a_j} \cap \overline{T}_{a_{j+1}} \neq \emptyset$, $j = 0, ..., k-1$ we have $l_{ab} \geq \sum_{j=0}^{k-1} l_{a_j a_{j+1}}$ we say that allocation rule f satisfies the* ***decomposition monotonicity*** *property.*

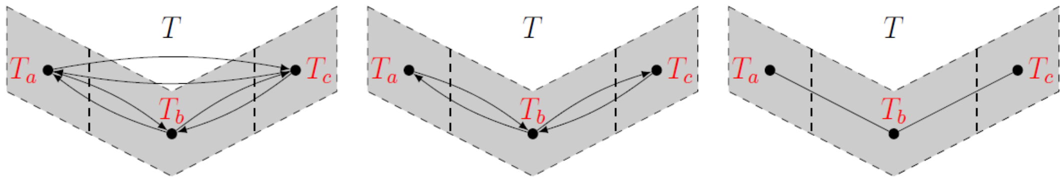

Figure 1: The outcome and neighborhood graphs.

The decomposition monotonicity property requires local incentive compatibility to imply global incentive compatibility. Indeed, we can then restate the decomposition property as follows: the gains from deviating for distant outcomes a and b $(-l_{ab})$ has to be not larger than the total gains from deviating along a path of adjacent outcomes connecting outcomes a and b. This property is interpreted using network terms in the Appendix.

We also define the notions of path-connected and simply connected sets. A set S is *path-connected* if any two points $x \in S$ and $y \in S$ can be connected with a curve lying inside set S. A set is *simply-connected* if it is path-connected and any two continuous curves both connecting points x and y can be continuously transformed one into the other. We are now ready to formulate our main theorem.

Theorem 3 *Assume that the closure of domain $\overline{T}$ is simply-connected, and that any non-empty intersection $\cap_a \overline{T}_a$ is simply-connected. Then a deterministic allocation rule f is implementable if and only if*

1) f is decomposition monotone, and

2) f is 2-cyclic monotone.

To establish the statement of the theorem (see a complete proof in Appendix) we show that any allocation rule satisfying conditions 1), 2) also satisfies cyclic monotonicity condition and, hence, implementable (see Theorem 1). To check cyclic monotonicity one has to verify that all possible cycles of outcome graph Γ_f (see the left panel of Figure 1) has a non-negative length. The main idea of the proof is to consider first only cycles induced by a neighborhood graph Γ_f^N having the same vertices A and edges connecting only adjacent outcomes, i.e. edges $ab, ba \in \Gamma_f^N$ if and only if $\overline{T}_a \cap \overline{T}_b \neq \emptyset$ (see the middle panel of Figure 1). A crucial property of neighborhood graph Γ_f^N is that if allocation rule f is 2-cyclic monotone then any 2-cycle of the neighborhood graph has exactly zero length, i.e. $l_{ab} + l_{ba} = 0$ for any $ab \in \Gamma_f^N$. This allows us to regard graph Γ_f^N as an undirected graph with an edge flow l, where edge flow l_{ab} between vertices a and b equals minus edge flow $(-l_{ba})$ between vertices b and a (see the right panel of Figure 1).

To analyze properties of cycles we also complete the neighborhood graph Γ_f^N with triples abc such that the closures of corresponding sets have non-empty intersection $\overline{T}_a \cap \overline{T}_b \cap \overline{T}_c \neq \emptyset$. This

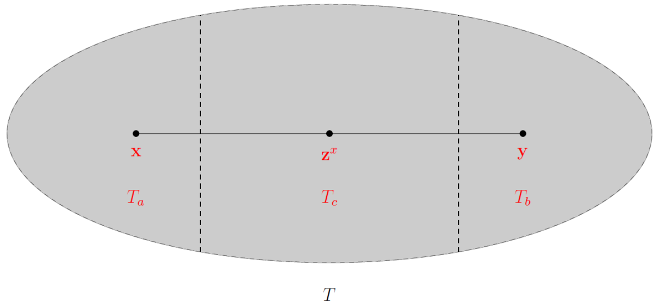

Figure 2: 2-cycle monotonicity imlies decomposition monotonicity for convex domains.

leads us to an important construction in algebraic topology - *the nerve*: the nerve N of a system of sets $\{\overline{T}_a\}_{a\in A}$ consists of duples ab such that $\overline{T}_a \cap \overline{T}_b \neq \emptyset$ and triples abc such that $\overline{T}_a \cap \overline{T}_b \cap \overline{T}_c \neq \emptyset$.[3]

In the proof of Theorem 3 we establish two important properties of the nerve N and edge flow l on neighborhood graph Γ_f^N. First, all 3-cycle defined on triples belonging to nerve N has 0 length. Second, the Nerve theorem from algebraic topology ensures that, given any non-empty intersection $\cap_a \overline{T}_a$ being simply-connected (the first condition of Theorem 3), nerve N inherits topological properties of the closure of domain $\overline{T}$. Given 3-cycles having 0 length and the topological properties of nerve N we then use the Helmholtz decomposition theorem from recently developed Hodge theory to establish that edge flow l is a gradient flow, i.e. the edge flow can be represented as $l_{ab} = s_a - s_b$ for some function $s : A \to R$. Hence, all cycles of neighborhood graph Γ_f^N has 0 length. As the final step, we show that the decomposition monotonicity property ensures that all cycles of outcome graph Γ_f have non-negative length.

3 Decomposition Monotonicity

We now analyze the decomposition monotonicity property. We first prove that 2-cycle monotonicity implies decomposition monotonicity for convex domains. The intuition behind this proof will then give us a more general characterization of the property.

Let us consider some domain with a convex closure $\overline{T}$ and assume there are only three possible outcomes a,b, and c.[4] Take two outcomes a and b whose outcomes sets $\overline{T}_a$ and $\overline{T}_b$ have empty intersection (if all outcome sets intersect the decomposition monotonicity property is trivially satisfied). For each type $\mathbf{x} \in T_a$ consider some type $\mathbf{y} \in T_b$. Since set $\overline{T}$ is convex interval $[\mathbf{x}, \mathbf{y}]$ lies within set $\overline{T}$ and $[\mathbf{x}, \mathbf{y}] \cap \overline{T}_c \neq \emptyset$ (see Figure 2). In addition, we denote vector $\mathbf{f}_a \in R^{|A|}$ having ath component equal 1 and all other components equal 0. We then have

$$\mathbf{x}(\mathbf{f}_a - \mathbf{f}_b) = \mathbf{x}(\mathbf{f}_a - \mathbf{f}_c) + \mathbf{x}(\mathbf{f}_c - \mathbf{f}_b)$$

[3] The nerve of system of sets $\{T_a\}_{a\in A}$ generally consists of all possible subsets $\sigma \subset A$ such that $\cap_{a\in\sigma} T_a$. We can restrict ouselves only to duples and triples in our analysis.

[4] Note that any convex set is also simply-connected.

Notice that for any $\mathbf{z}^x \in [\mathbf{x}, \mathbf{y}] \cap \overline{T}_c$ there exists $l^x \geq 0$ such that $(\mathbf{x} - \mathbf{z}^x) = l^x(\mathbf{z}^x - \mathbf{y})$. The 2-cycle monotonicity then implies

$$(\mathbf{x} - \mathbf{z}^x)(\mathbf{f}_c - \mathbf{f}_b) = l^x(\mathbf{z}^x - \mathbf{y})(\mathbf{f}_c - \mathbf{f}_b) \geq 0.$$

This allows us to obtain

$$\mathbf{x}(\mathbf{f}_a - \mathbf{f}_b) \geq \mathbf{x}(\mathbf{f}_a - \mathbf{f}_c) + \mathbf{z}^x(\mathbf{f}_c - \mathbf{f}_b).$$

We finally have

$$\begin{aligned} l_{ab} &= \inf_{\mathbf{x} \in T_a} \mathbf{x}(\mathbf{f}_a - \mathbf{f}_b) \\ &\geq \inf_{\mathbf{x} \in T_a} \mathbf{x}(\mathbf{f}_a - \mathbf{f}_c) + \inf_{\mathbf{x} \in T_a} \mathbf{z}^x(\mathbf{f}_c - \mathbf{f}_b) \\ &\geq \inf_{\mathbf{x} \in T_a} \mathbf{x}(\mathbf{f}_a - \mathbf{f}_c) + \inf_{\mathbf{z} \in \overline{T}_c} \mathbf{z}(\mathbf{f}_c - \mathbf{f}_b) \\ &= l_{ac} + l_{cb}. \end{aligned}$$

The first inequality follows from the fact that infimum of a sum is not smaller than the sum of infimums; the second inequality follows from the fact that infimum over set T_c is smaller than infimum over its subset. The final equality follows from the definition of edge's length and $\inf_{\mathbf{z} \in \overline{T}_c} \mathbf{z}(\mathbf{f}_c - \mathbf{f}_b) = \inf_{\mathbf{z} \in T_c} \mathbf{z}(\mathbf{f}_c - \mathbf{f}_b)$. Hence, 2-cycle monotonicity implies decomposition property when there are three outcomes. The proof for any finite number of outcomes follows the same steps and can be found in Appendix.

Theorem 4 (Convex domains) *2-cycle monotonicity implies decomposition monotonicity for domains with convex closures.*

We finally note that the closure of the outcome set for alternative a can be defined as $\overline{T}_a = \{\mathbf{t} \in R^{|A|} : t_a - t_b \geq l_{ab},\ \forall b \neq a\} \cap \overline{T}$. Since an intersection of convex sets is a convex set we have that all outcome sets $\overline{T}_a$ as well as any non-empty intersection $\cap_a \overline{T}_a$ are convex. Since any convex set is simply-connected the first condition of Theorem 3 is automatically satisfied for convex domains. This allows us to reestablish the important result of Saks and Yu (2005) stated in Theorem 2.

A attentive reader notices that we do not really need the closure of domain $\overline{T}$ be convex for the proof of Theorem 4 be valid. The above proof works without any changes when allocation f satisfies the following condition.

Definition 2 *Allocation rule f satisfies Property 1 if for any two outcomes $a, b \in A$ we have that for any type $\mathbf{x} \in T_a$ there exists a type $\mathbf{y} \in T_b$ such that interval $[\mathbf{x}, \mathbf{y}] \in \overline{T}$.*

For future reference, we state this implication as a separate theorem.

Theorem 5 *If an allocation rule satisfies 2-cycle monotonicity and Property 1 it also satisfies decomposition monotonicity.*

We use Property 1 to show that 2-cycle monotonicity implies decomposition monotonicity for single-peaked preference domains in the following subsection.

4 Single-Peaked Preferences

Let us define the domain of single-peaked preferences. We consider some ordering over a finite set of outcomes A. The domain of types T is single-peaked if for each $\mathbf{t} \in T$ there exists an outcome $p \in A$ such that for any q, $q' \in A$ such that $q < q' \leq p$ or $q > q' \geq p$ we have $t_{q'} > t_q$. Alternative p is called the peak of type $\mathbf{t}$. Note that domain T consist only of strict types. We now show that the single-peaked preference domain satisfies Property 1.

Let us consider outcome $a \in A$ and some type $\mathbf{t} \in T_a$ with the peak at $p \in A$. We also consider some outcome $b \neq a$. Though we cannot claim that each type in set T_b has its peak at b, 2-cycle monotonicity ensures that there always exists such type. Let us take some type $\bar{\mathbf{t}} \in T_b$. Then, 2-cycle monotonicity ensures that if type $\bar{\mathbf{t}} + \boldsymbol{\alpha}$ is single-peaked for some $\boldsymbol{\alpha}$ such that $\alpha_b > \alpha_c$ for any $c \in A \backslash \{b\}$, then type $\bar{\mathbf{t}} + \boldsymbol{\alpha}$ also belongs to T_b (see Lemma A1). Hence, we can always choose $\bar{\mathbf{t}} \in T_b$ with the peak at b.

Let us consider the case when $b = p$. Then both t_q and $\bar{t}_q$ are increasing for $q \leq p$ and decreasing for $q \geq p$. Hence, for all $\beta \in [0,1]$ all types $(1-\beta)t_q + \beta \bar{t}_q$ are increasing for $q \leq p$ and decreasing for $q \geq p$ (as long as strict). Hence, all types in $[\mathbf{t}, \bar{\mathbf{t}}]$ are single-peaked and $[\mathbf{t}, \bar{\mathbf{t}}] \in \overline{T}$.

Now consider the case $b > p$ (a similar argument applies in case $b < p$). Both t_q and $\bar{t}_q$ are increasing for $q \leq p$ and decreasing for $q \geq b$, so is their convex combination. For $p \leq q \leq b$ type t_q is decreasing in q and $\bar{t}_q$ is increasing in q. We now construct a new type $\mathbf{t}' \in T_b$. We pick t'_b arbitrary and choose t'_{b-1} such that $t'_b - t'_{b-1} > \bar{t}_b - \bar{t}_{b-1}$. If $p < b-1$ we then choose t'_q for $q = b-2, ..., p$ satisfying to inequalities

$$\frac{t'_{q+2} - t'_{q+1}}{t_{q+1} - t_{q+2}} < \frac{t'_{q+1} - t'_q}{t_q - t_{q+1}} \tag{3}$$

$$\bar{t}_{q+1} - \bar{t}_q < t'_{q+1} - t'_q \tag{4}$$

This can be done by choosing t'_q low enough at each step. As we show below, these inequalities ensure that $[t, t'] \subset T$ and $t' \in T_b$. We finally choose t'_q such that $\bar{t}_{q+1} - \bar{t}_q < t'_{q+1} - t'_q$ for $q < p$ and $\bar{t}_q - \bar{t}_{q+1} < t'_q - t'_{q+1}$for $q > b$, which also ensures that $\mathbf{t}'$ is single peaked.

We now show that inequalities (3) and (4) ensure that for $\beta \in [0,1]$ $(1-\beta)\mathbf{t} + \beta\mathbf{t}'$ is single-peaked whenever it is strict. Note that both t_q and t'_q are increasing for $q \leq p$ and decreasing for $q \geq b$, so is their convex combination. Let us assume that there exists $q \in \{p, ..., b-2\}$ such that

$$(1-\beta)t_q + \beta t'_q > (1-\beta)t_{q+1} + \beta t'_{q+1} \tag{5}$$

and

$$(1-\beta)t_{q+1} + \beta t'_{q+1} < (1-\beta)t_{q+2} + \beta t'_{q+2} \tag{6}$$

We can rearrange inequalities (5) and (6) to obtain

$$\frac{1-\beta}{\beta} > \frac{t'_{q+1} - t'_q}{t_q - t_{q+1}}$$

and

$$\frac{1-\beta}{\beta} < \frac{t'_{q+2} - t'_{q+1}}{t_{q+1} - t_{q+2}}$$

where we used $t_q > t_{q+1} > t_{q+2}$. We also know that $\beta > 0$ because inequality (6) is violated for $\beta = 0$. The above two inequalities contradict (3). Hence, for any $\mathbf{t} \in T_a$ we found $\mathbf{t}'$ such that $[\mathbf{t}, \mathbf{t}'] \in T$. Note also that inequalities (4) guarantee that for any outcome $c \neq b$ we have $t'_b - t'_c > \bar{t}_b - \bar{t}_c$. Hence, if outcome c would be chosen for type $\mathbf{t}'$ that would violate 2-cycle monotonicity. Hence, $\mathbf{t}' \in \mathbf{T}_b$.

Overall, the single-peaked preference domain satisfies Property 1 as well as decomposition monotonicity. Finally, Lemma **??** in Appendix shows that if an allocation rule satisfies 2-cycle monotonicity it also satisfies the first condition of Theorem 3, i.e. any non-empty intersection $\cap_a \overline{T}_a$ is simply-connected, which leads to the following result.

Theorem 6 *For the domain of single-peaked preferences, a necessary and sufficient condition for an allocation rule to be implementable is being 2-cyclic monotone.*

5 Conclusion

Using insights from combinatorial Hodge theory (Jiang et al. (2011)) and algebraic topology (Bjorner (1995)) we provide a novel analysis of implementable allocation rules. Our main result (Theorem 3) shows that if both a type domain and any finite non-empty intersection outcome sets are simply-connected, any allocation rule satisfying decomposition monotonicity and 2-cycle monotonicity is implementable. This result provides conditions on both a type domain and an allocation rule, which is in contrast to existing literature analyzing only conditions on type domains where every 2-cycle monotone rule is implementable. Such Saks and Yu (2005) show that any 2-cycle monotone allocation rule is implementable for convex domains.[5] Mishra, Pramanik and Roy (2014) prove the same result for the domain of single-peaked preferences and Mu'alem and Schapira (2008) and Koppe, Queyranne and Ryan (2015) for some discrete domains. In contrast

[5]See also Bikhchandani et al. (2006) and Archer and Kleinberg (2014) for relevant characterizations. Ashlagi et al. (2010) also prove that if every 2-cyclical monotone allocation rule (including random allocations) is implementable on some domain then this domain has a convex closure.

to these papers, we provide a criterion to check the implementability of an allocation rule even if it does not belong to a domain where every 2-cycle monotone rule is implementable.

The application of our approach is limited to settings with a finite set of outcomes. For settings with infinite set of outcomes, it is hard to define a neighborhood graph. In addition, we can apply the Nerve theorem (Theorem 7) only to a finite collection outcomes sets. We refer readers interested in domains with an infinite set of outcomes to Berger, Müller and Naeemi (2009), Archer and Kleinberg (2014), Carbajal and Müller (2015), and Carbajal and Müller (2016).

The decomposition monotonicity is first mentioned by Müller, Perea and Wolf (2007) who highlights its importance in characterizing Bayes-Nash incentive compatibility (see also Vohra, 2011). Our characterization of decomposition monotonicity (Theorems 4 and 5) borrows many insights from Carroll (2012) who analyzes domains where local incentive compatibility implies global incentive compatibility. This connection one more time highlights that decomposition monotonicity is a condition demanding local incentive compatibility to imply global incentive compatibility.[6]

Overall, we provide a systematic approach to analyze incentive compatibility in multidimensional domains that go beyond convexity assumption. We believe this approach will broaden the knowledge of domains where 2-cycle monotonicity implies implementability. We leave this exciting perspective for future research.

[6] Carroll (2012) analyzes domains without transfers when local incentive compatibility implies global incentive compatibility. Mishra, Pramanik and Roy (2015) analyzes the role of local incentive compatibility constraints in environments with transfers highlighting the importance of taxation principle.

6 Appendix

More on decomposition monotonicity. The decomposition monotonicity property can be interpreted in terms of network theory as follows. Consider the restriction of the network to adjacent outcomes endowed with a length l^R which is simply the restriction of l. For any pair of outcomes a and b, define the reduced length $\tilde{l}^R_{ab}$, which is the length of the shortest path from a to b on the restricted network. Then the decomposition monotonicity property is equivalent to $l \geq \tilde{l}^R$. Note that we also have $\tilde{l}^R \geq \tilde{l}$, where $\tilde{l}$ is the reduced length in the full network, and that cyclical monotonicity is equivalent to $\tilde{l}_{aa} \geq 0$ for any a.

Proposition 1 *The decomposition monotonicity property is equivalent to*

$$\tilde{l} = \tilde{l}^R$$

where $\tilde{l}$ is the reduced length in the full network, and $\tilde{l}^R$ is the reduced length in the network restricted to adjacent nodes.

Proof. Assume the decomposition monotonicity property holds; then by the previous discussion, $l \geq \tilde{l}^R$. Thus, $\tilde{l} \geq \tilde{l}^R$. But we have $\tilde{l}^R \geq \tilde{l}$, thus $\tilde{l} = \tilde{l}^R$. Conversely, if $\tilde{l} = \tilde{l}^R$, then $l \geq \tilde{l} = \tilde{l}^R$, thus the decomposition monotonicity property holds. ■

Proof of Theorem 3. We show that for a domain with a simply-connected closure $\overline{T}$ if allocation rule f satisfies conditions 1) and 2) then all M-cycles of the outcome graph Γ_f have non-negative length. Theorem 1 then establishes that f is implementable.

The main idea of the proof is to first analyze cycles on a neighborhood graph $\Gamma_f^N = \{A, E\}$ with the set of vertices equal to the set of outcomes A, and the set of edges E connecting only neighborhood outcomes, i.e. $ab, ba \in E$ if and only if the intersection of the closure of the outcome sets is non-empty $\overline{T}_a \cap \overline{T}_b \neq \varnothing$. A convenient property of the neighborhood graph Γ_f^N is that 2-cyclic monotonicity implies that every 2-cycle has exactly 0 length. To show this, let us consider sets $\overline{T}_a$ and $\overline{T}_b$ having non-empty intersection. For any $\mathbf{t} \in \overline{T}_a \cap \overline{T}_b$ the definition of l weight implies

$$\begin{aligned} t_a - t_b &\geq l_{ab} \\ t_b - t_a &\geq l_{ba} \end{aligned}$$

Hence, $l_{ab} + l_{ba} \leq 0$. Combining this inequality with 2-cyclic monotonicity we conclude that $l_{ba} + l_{ab} = 0$. The zero length of 2-cycles allows us to regard graph Γ_f^N as non-directed graph with an edge flow l where the flow between vertices a and b equals weight l_{ab}, and the flow between vertices b and a equals l_{ba} or $-l_{ab}$. (We use a non-standard definition of the edge flow: the amount of flow into a vertex does not need to equal the amount of flow out of it.)

To analyze the length of 3-cycles we also complete the neighborhood graph Γ_f^N with triples $H = \{abc \mid \overline{T}_a \cap \overline{T}_b \cap \overline{T}_c \neq \varnothing\}$. We refer to system $N = (A, E, H)$ as the **nerve** of a family of sets $(\overline{T}_a)_{a \in A}$. Condition 1) of the theorem stating that any non-empty intersection of sets $\cap_a \overline{T}_a$ is simply-connected allows us to apply the Nerve theorem from algebraic topology (see Wu (1962) or Bjorner (1995)) to establish that the nerve inherits the topological properties of the closure of the domain $\overline{T}$.[7]

[7] We avoid a formal definition of a nerve being simply-connected because it leads us to a long chain of supportive definitions. We refer an interested reader to Jonsson (2008) for a detailed treatment.

Theorem 7 (Nerve theorem) *Let the closure of the domain $\overline{T}$ be simply-connected. If every non-empty intersection $\cap_a \overline{T}_a$ is simply-connected, then nerve N corresponding to system $(\overline{T}_a)_{a \in A}$ is simply-connected.*

The nerve theorem allows us to conclude that nerve $N = (A, E, H)$ is simply-connected. In addition, 2-cycle monotonicity implies that any 3-cycle defined for any triple of the nerve, i.e. $abc \in H$, has 0 length. To show this let us take some $abc \in H$ and some $\mathbf{t} \in \overline{T}_a \cap \overline{T}_b \cap \overline{T}_c$. Since $\mathbf{t} \in \overline{T}_a \cap \overline{T}_b$ we have $l_{ab} = -l_{ba} = t_a - t_b$. Similar expressions hold for l_{bc} and l_{ca}. Hence, for any $abc \in H$ we have $l_{ab} + l_{bc} + l_{ca} = 0$.

Using the fact that all 3-cycles on nerve have 0 length we now apply the seminal result from Hodge theory - Helmholtz decomposition theorem (see Theorem 4 in Jiang et al. (2011)).

Theorem 8 (Helmholtz decomposition) *If (A, E, H) is simply-connected and each 3-cycle $abc \in H$ has 0 length the edge flow l is a gradient flow, i.e. $l_{ab} = s_a - s_b$ for some function $s : A \to R$.*

If edge flow l is a gradient flow, for any M-cycle of the neighborhood graph Γ_f^N we have

$$\sum_{j=0}^{M-1} l_{a_j a_{j+1}} = \sum_{j=0}^{M-1} (s_{a_j} - s_{a_{j+1}}) = 0,$$

which establishes that any M-cycle of the neighborhood graph Γ_f^N has zero length.

Let us now consider some M-cycle $\{b_0, b_1, ..., b_M = b_0\}$ of outcome graph Γ_f. Since the closure of domain $\overline{T}$ is path-connected, for each edge $\{b_j, b_{j+1}\}$ there exists a path $\{b_j = a_j^0, ..., a_j^{K_j} \equiv b_{j+1}\}$ connecting b_j and b_{j+1} such that $\overline{T}_{a_j^k} \cap \overline{T}_{a_j^{k+1}} \neq \varnothing$ for each k. Note that $a_j^k a_j^{k+1} \in \Gamma_f^N$. Given the decomposition monotonicity property and that any cycle on graph Γ_f^N has zero length we have

$$\sum_{j=0}^{M-1} l_{b_j b_{j+1}} \geq \sum_{j=0}^{M-1} \sum_{k^j=0}^{K^j} l_{a_j^{k^j} a_j^{k^j+1}} = 0$$

This completes the proof of the theorem. □

Proof of Theorem 4. Let us consider settings with a finite number of outcomes. Consider some outcomes $a, b \in A$ with sets $\overline{T}_a$ and $\overline{T}_b$ having empty intersection. Take some $\mathbf{x} \in T_a$ and $\mathbf{y} \in T_b$. Since the closure of domain $\overline{T}$ is convex we have that $[\mathbf{x}, \mathbf{y}] \in \overline{T}$. Hence, for any $\mathbf{x} \in T_a$ there exists a sequence $s^x = \{a \equiv a_0^x, ..., a_k^x \equiv b\}$ such that $\overline{T}_{a_j^x} \cap \overline{T}_{a_{j+1}^x} \neq \emptyset$, $j = 0, ..., k-1$ and $[\mathbf{x}, \mathbf{y}] \cap \overline{T}_{a_j^x} \neq 0$. We then have

$$\mathbf{x}(\mathbf{f}_a - \mathbf{f}_b) = \sum_{j=0}^{k-1} \mathbf{x}(\mathbf{f}_{a_j^x} - \mathbf{f}_{a_{j+1}^x})$$

Notice that for any $\mathbf{z}_j^x \in [\mathbf{x}, \mathbf{y}] \cap \overline{T}_{a_j^x}$ there exists some $l_j^x > 0$ such that vectors $\mathbf{x} - \mathbf{z}_j^x = l_j^x(\mathbf{z}_j^x - \mathbf{z}_{j+1}^x)$. The 2-cycle monotonicity then implies

$$(\mathbf{x} - \mathbf{z}_j^x)(\mathbf{f}_{a_j^x} - \mathbf{f}_{a_{j+1}^x}) = l_j^x(\mathbf{z}_j^x - \mathbf{z}_{j+1}^x)(\mathbf{f}_{a_j^x} - \mathbf{f}_{a_{j+1}^x}) \geq 0$$

This allows us to obtain

$$\mathbf{x}(\mathbf{f}_b - \mathbf{f}_a) \geq \sum_{j=0}^{k-1} \mathbf{z}_j^x (\mathbf{f}_{a_j^x} - \mathbf{f}_{a_{j+1}^x})$$

Let us denote the finite set of sequences $S = \{s^x : \mathbf{x} \in T_a\}$ where s^x is a sequence defined above. We then have

$$\begin{aligned} l_{ab} &= \inf_{\mathbf{x} \in T_b} \mathbf{x}(\mathbf{f}_b - \mathbf{f}_a) \\ &\geq \inf_{\mathbf{x} \in T_b} \sum_{j=0}^{k-1} \mathbf{z}_j^x (\mathbf{f}_{a_j^x} - \mathbf{f}_{a_{j+1}^x}) \\ &\geq \inf_{\mathbf{x} \in T_b} \sum_{j=0}^{k-1} \inf_{z_j \in \overline{T}_{a_j^x}} \mathbf{z}_j (\mathbf{f}_{a_j^x} - \mathbf{f}_{a_{j+1}^x}) \\ &\geq \min_{\{a_0, \ldots, a_k\} \in S} \sum_{j=0}^{k-1} \inf_{\mathbf{z}_j \in \overline{T}_{a_j}} \mathbf{z}_j (\mathbf{f}_{a_j} - \mathbf{f}_{a_{j+1}}) \\ &= \min_{\{a_0, \ldots, a_k\} \in S} \sum_{j=0}^{k-1} l_{a_j a_{j+1}} \end{aligned}$$

where we used $l_{a_j a_{j+1}} = \inf_{\mathbf{z}_j \in T_{a_j}} \mathbf{z}_j (\mathbf{f}_{a_j} - \mathbf{f}_{a_{j+1}}) = \inf_{\mathbf{z}_j \in \overline{T}_{a_j}} \mathbf{z}_j (\mathbf{f}_{a_j} - \mathbf{f}_{a_{j+1}})$. Since, the set of sequences S is finite, the minimum is achieved for some sequence. This completes the proof of the theorem.□

Lemma A1 . *Assume allocation rule satisfies 2-cycle monotonicity and take some* $\mathbf{t} \in T_b$ *and* $\mathbf{t}' = \mathbf{t} + \boldsymbol{\alpha} \in T$, *where* $\boldsymbol{\alpha} : \alpha_b > \alpha_a$ *for any* $a \in A \backslash \{b\}$. *We then must have that* $\mathbf{t}' \in T_b$.

Proof. Let us assume that $\mathbf{t}' \in T_c$ for some $c \neq b$. Then 2-cycle monotonicity implies that $t'_c - t'_b \geq t_c - t_b$, which contradicts $\alpha_b > \alpha_c$. □

□

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