# Large Elections with Costly Information and Conflict of Interest

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**Abstract**

This paper studies large elections with costly information acquisition in a general model that allows for conflict of interest between voters and for supermajority rules. Generically, there exists a cursed limit equilibrium that corresponds to voting according to the prior. Only when the cost of information is "high", a Downsian paradox of voting prevails and, generically, in all limit equilibria, the same outcome is elected in each state. When the cost of information is "low", non-constant limit equilibria are either utilitarian or information is 'too cheap' to protect the minority interests and the limit outcome is utilitarian with probability zero. When the cost of information is "intermediate", the limit equilibria with maximal information acquisition and the entailed welfare can be described by a generalization of the product logarithm function as a function of the primitives of the model. This allows for rich comparative statics: more conflict of interest between voters increases information acquisition ('competition effect'), but reduces utilitarian welfare; voter groups with a stronger ideology have more voting power; more consensus among voters can reduce the voters' welfare. Looking beyond utilitarianism, the paper characterizes the social welfare functions that are implementable by varying information cost; thereby voting with costly information is related to specific axioms of social choice.

Political elections face at least two significant challenges: First, typically voting is costly for voters and even more so informed voting. As a consequence, elections involve a free-rider problem. Second, voters have a conflicting interest such that the election has to screen voters for efficient decision-making. This paper studies a setup in which both problems are present:[1]voters preferences depend on unknown information ('the state'), the preferences of voters are conflicting when the state is known, and information about the state can be acquired at a cost. The model generalizes Martinelli [2006], most importantly by allowing for supermajority rules and a conflict of interest of voters. More precisely, voters are separated into two groups with opposing preferences when the state is known, a 'majority group' and a 'minority group'.

We show that 'information is power': the more information a group of voters acquired, the more likely the election outcome is the one preferred by the group (Lemma 4). We relate the amount of information acquired by the groups to the primitives of the model and define a measure of 'voting power' in terms of the primitives.

We characterize all limit equilibria of the model. When the cost of information is 'high', a variant of the Downsian paradox of voting prevails: voters acquire so few information such that difference in vote shares of any given alternative is small across all states. We show that as a consequence, voters

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learn almost nothing about the state from conditioning on the election being tied. Generically, for any equilibrium sequence, the same outcome is elected with probability 1 in each state (Corollary 2) are equlibria are not welfare-efficient. The first main result of this paper is a weak converse: there can exist equilibrium sequences that solve both the free-rider problem as well as the screening problem partially when cost is"intermediate", and fully when cost is"low" (Theorem 3, Corollary 1 and Theorem 4). However, we also show that when the cost of information is"low", voting power is non-linear in intensities. As a consequence, the following scenario occurs: the utilitarian outcome is the one preferred by the minority group $g$; however, the amount of information acquired by the majority group exceeds the amount of information acquired by the minority group. Therefore, we show that in any non-Downsian equilibrium, the voting outcome is the one preferred by the majority group with probability close to 1. Information is 'too cheap' to protect minority interests (Theorem 4).

As the second main result, we uncover a correspondence between limit equilibria when the cost of information are intermediate and a class of transcendental equations. Three hundred years ago, Lambert and Euler studied the equation

$$ze^z = \nu. \tag{1}$$

The inverse of the function on the left hand side of (1) is called the product logarithm or the Lambert-$W$ function and denoted $W_0$[2]. To my knowledge, this paper is the first to discover the Lambert function in an application in economics. The Lambert $W$-function appears in many physical and mathematical problems, see for example the survey of Corless et al. [1996][3] and possesses many known useful properties.[4] When the cost of information is "intermediate", there exist equilibria with substantial information acquisition and their unique limit can be described by generalisations of the Lambert $W$-function (Theorem 2). This explicit description allows for rich comparative statics.
To do so, we show that we can consider the best response function as a function in the margins of victory in each state. Intuitively, this is, because the margins of victory determine the probability that a single voter decides the election in each state, and these probabilities pin down the best response. The possibility of deciding the election incentivizes voters to acquire information. Since, uninformed citizen vote in the same way in each state, differences in the margin of victory across states are a function of the information acquired. The fixed point equation of the best response equates the difference in the margin of victory with a function of the information acquired by the voters. We rewrite the limit fixed point equation into an equation that generalizes the Lambert equation (1) in two main steps: Let $2n + 1$ be the size of the electorate. First, for any state $\omega$ and any margin of victory $|q_\omega - \frac{1}{2}|$, the probability that a single vote decides the election is the probability that the binomial distribution $\mathcal{B}(2n, q_\omega)$ takes the value $n$. We use the local version of the central limit theorem to express the limit of this probability in terms of the density of the standard normal distribution. The solution to the voters' maximization problem shows how the probabilities that a single vote is decisive in each state, translate into information acquisition. This way, we arrive at an explicit description for the aggregate information acquired by voters as a function of the margins of victory, and hence at a description of the left-hand side of the fixed point equation as a function of the margins of victory. Second, we show that the expected margin of victory has to be zero in each

[1]Similarly, Krishna and Morgan [2011] and Krishna and Morgan [2015] study a setup in which both problems are present. Two main differences are that in their model voting and not information is costly, and preferences do not depend on beliefs about the state. So, inherently, there is no information aggregation problem. We discuss the relation to models with voting cost in Section 7.

[2]More precisely, $W_0$ is the principal branch of the inverse relation of the function on the left-hand side of (1). However, we only consider real numbers $z \geq 0$ such that the left-hand side is strictly increasing and the inverse is a well-defined function.

[3]The Lambert $W$-function e.g. appears as a solution to a type of one-dimensional Schroedinger equation, in connection to the distribution of prime numbers or in models of enzyme kinetics.

[4]E.g. the product logarithm is approximated by $\log(\nu) - \log(\log(\nu)) \leq W_0(\nu) \leq \log(\nu)$. We have derived properties of the generalized product logarithm functions that appear in the voting setting, but have not included them in this paper: e.g. the $m$-th derivatives can be described recursively by a system of partial differential equations.

state in the limit of any equilibrium sequence with substantial information acquisition. Intuitively, if the expected margin of victory is positive in all states, the local central limit theorem entails that the probability that the realized margin of victory is zero converges to zero exponentially fast. Similarly, the probability that a single vote is decisive, hence the incentives of voters to acquire information are exponentially small. we show that this 'zero-margin-of-victory' condition pins down the posterior belief about the state conditional on the election being tied, and therefore the relation of the margin of victory in one state to the other. Finally, this allows writing the fixed point equation in one variable, the margin of victory in a given state. Surprisingly, the resulting equation is a generalization of the Lambert equation (1) and coincides with the Lambert equation when the prior is symmetric.

The second set of results describes the welfare properties of the election as a function of the primitives of the model. First, we see that depending on the conflict of interest of voters, election outcomes can be arbitrarily efficient or arbitrarily inefficient. A priori one might think that the presence of a conflict of interest might increase welfare since the competition among voter groups might fuel information acquisition of voters. While in fact, information acquisition is higher when there is a conflict of interest between voters due to a *competition effect*, interestingly a higher conflict of interest never improves welfare. This is because the information acquired by groups with opposed interests does 'cancels' out each other in the sense that one group might vote more often for the welfare-maximizing outcome in each state when better informed whereas the other group votes more often against it when better informed. We show that this *screening effect* dominates the effect of increased information acquisition (*competition effect*) such that a higher degree of the conflict of interest unambiguously reduces social welfare. Second, the voting power of a group is inversely related to the fraction of voters of the group for which the preferences over the alternatives are state-independent: Naturally, these voters have no incentive whatsoever to acquire any information. This translates into voting power since 'information is power'. Third, we can understand the degree of consensus of the electorate as the probability that a random pair of voters has the same ordinal preferences when the state is known. More formally, the degree of the consensus is captured by the relative size of the majority and the minority group. We show that a higher degree of consensus can reduce social welfare: generically, there exists an open interval of the degree of consensus on which the efficiency of all equilibria with substantial information acquisition is decreasing when the consensus increases.

Finally, based on the characterization of limit equilibria, we characterize the social welfare functions that can be asymptotically implemented by choice of the information cost. We show that these are the (state-dependent) Bergson social welfare functions with parameter $0 \leq \rho < 1$ (Theorem 5). Roughly speaking, these are the social welfare functions that weights intensities of voters by a parameter $\rho$ (for a formal definition see Section 4). Importantly, the Bergson social welfare rules possess an axiomatization that was provided by Roberts [1980] and Moulin [1991]. Therefore, the importance of Theorem 5 is in building a bridge between the typically distant worlds of the axiomatic social choice theory and the game-theoretic analysis of elections.

The rest of the paper is organized as follows: Section 1 introduces the model. Section 2 shows that all equilibria take a cut-off form. Then, in Section 2.1 we use the local central limit theorem to derive a formula for the probability that a single vote is decisive when the electorate grows large. Section 2.2 explains in which sense 'information is power' in the model of this paper and derives formulas for the amount of information acquired by each voter group. Section 2.3 shows that any equilibrium sequence that does not converge to voting according to the prior must satisfy that the limit of the margin-of-victory is zero in expectation. Section 3 provides the first set of results: Section 3.2 proves the existence of equilibria with substantial information acquisition when cost are not "high". Section 3.1 shows that, when cost is intermediate, these equilibria are characterized by a class of generalized Lambert equations. Section 3.3 discusses the welfare properties of these equilibria with substantial information acquisition. Section 4 characterizes the social welfare functions that can be asymptotically implemented by choice of the information cost. Section 5 discusses the welfare properties as a function of the primitives of the model. Section 6 discusses all other limit equilibria of the model: In particular, Section 6.1 shows that generically a sequence of cursed equilibria exists for which information acquisition is exponentially low and that such low information acquisition is then self-confirming. Section 6.2 shows that, when cost is high, all limit equilibria correspond to voting according to the prior. Section 7 discusses the related literature, in particular Martinelli [2006], Oliveros [2013], Krishna and Morgan [2011] and Bhattacharya [2013].

# 1 Model

There are $2n+1$ voters, two possible election outcomes $A$ and $B$, and two states of the world $\omega \in \{\alpha, \beta\} = \Omega$. Voters hold a common prior. The prior probability of $\alpha$ is $p_0 \in (0,1)$, and the probability of $\beta$ is $1-p_0$. Voters have heterogeneous preferences. The preferences are private information. A preference type is a pair $t = (y, \lambda) \in [0,1] \times \{g, G\}$. A voter either belongs to a majority group $G$ or to a minority group $g$. The utility of a voter of type $t$ from the outcome $x \in \{A, B\}$ in $\omega \in \{\alpha, \beta\}$ is denoted by $u(t, x, \omega)$. For voters $t = (y, G)$ of the majority group,

$$u(t,z,\omega) = \begin{cases} 1-y & \text{if} \quad (z,\omega) = (A,\alpha), \\ y & \text{if} \quad (z,\omega) = (B,\beta), \\ 0 & \text{else,} \end{cases} \tag{2}$$

For voters $t = (y, g)$ of the minority group,

$$u(t,z,\omega) = \begin{cases} k \cdot (1-y) & \text{if} \quad (z,\omega) = (B,\alpha), \\ k \cdot y & \text{if} \quad (z,\omega) = (A,\beta), \\ 0 & \text{else.} \end{cases} \tag{3}$$

for some $k > 0$. Note that voters of different groups have opposed interest when the state is known. Preference types are independently and identically distributed across voters. The payoff type $y$ of a voter is independently drawn from the group type $z$, according to a commonly known distribution $F$ that has a strictly positive, continuous density $f$. We denote by $\Pr(\lambda) \in [0,1]$ the probability that a random voter is of group type $\lambda \in \{g, G\}$, where $\Pr(G) > \frac{1}{2}$.

*Ideology and Taste.* The preference specification includes a case with a taste component $t$ and an ideology component $e$: Let $i = \frac{1}{2}$ and $e \in [-i, i]$. Let $y = i + e$, then $1 - y = i - e$. The ideology component is common within a group and captures which outcome a group prefers in each state. The personal taste component might capture state-independent beliefs about differences in the competence of two candidates $A$ and $B$.[5]

**Threshold of Doubt.** The payoff parameter describes a threshold of doubt. For a given belief $p$ about $\alpha$, a voter of group type $G$ prefers $A$ if and only if

$$\begin{aligned} p(1-y) &\geq (1-p)y \\ \Leftrightarrow \frac{p}{1-p} &\geq \frac{y}{1-y} \\ \Leftrightarrow p &\geq y. \end{aligned} \tag{4}$$

[5] Note that the restriction $e \in [-i, i]$ is without loss, since otherwise, the personal taste component is such that the voter always prefers the same candidate in both states $\omega$. The model presented can be easily extended to include such *partisan voters*.

Similarly, a voter of group type $G$ prefers $A$ if and only if

$$p \leq y. \tag{5}$$

For a given belief $p$ about $\alpha$, the probability that a random citizen prefers $A$ is

$$\psi(p) := \Pr(G)F(p) + \Pr(g)(1 - F(p)). \tag{6}$$

Note that the assumption $\Pr(G) > \Pr(g)$ together with the assumption that $G$ has a strictly positive density $f$ implies that the derivative $\frac{\delta}{\delta p}\psi(p) = (\Pr(G) - \Pr(g))f(p)$ is strictly positive. To make the analysis interesting, we assume that $\Pr(G) > \tau > \Pr(g)$ such that there exists a belief $p_\tau \in (0,1)$ for which $\psi(p_\tau) = \tau$. It follows from the strict monotonicity of $\psi$ that such a belief is unique.

**Timing.** Firstly, the state of the world $z$ and the private preference types realise. Then, each voter can acquire information of quality $x$ at cost $c(x) = \frac{\kappa}{d}x^d$ for some given $d > 1$[6] and $\kappa > 0$. Then, each voter receives a binary signal $s \in \{a, b\}$ with $\Pr(a|\alpha) = \Pr(b|\beta) = \frac{1}{2} + x$. Signals are private information of the voters. Then, voters simultaneously decide if to vote for $A$ or $B$. The election outcome is chosen by $\tau$-majority rule for $\tau \in (0,1)$ with $\tau n \in \mathbb{N}$ and $(1-\tau)n \in \mathbb{N}$.[7]

**Strategies.** A symmetric strategy is a function from $[0,1] \times \{g, G\}$ to $[0, \frac{1}{2}] \times [0,1]^2$. The first component of a strategy is denoted $x(t)$ and desribes the quality of information $x$ acquired by a voter of type $t$. The second component is denoted $\sigma(a,t)$ and describes the probability to vote $A$ after receiving $a$. The third component is denoted $\sigma(b,t)$ and describes the probability to vote for $A$ after receiving $b$. For any $\omega \in \Omega$, we slightly abuse notation and use $\sigma$ as a generic symbol for strategies and write

$$\begin{aligned} q_\alpha(\sigma) \quad &:= \quad \Pr(G) \int_{t \in [y,G] \in [0,1] \times \{G\}} (\frac{1}{2} + x(t))\sigma(a,t) + (\frac{1}{2} - x(t))\sigma(b,t)) dF(y) \\ &+ \quad \Pr(g) \int_{t \in [y,g] \in [0,1] \times \{G\}} (\frac{1}{2} + x(t))\sigma(a,t) + (\frac{1}{2} - x(t))\sigma(b,t)) dF(y) \end{aligned} \tag{7}$$

for the probability that a random citizen votes $A$ in $\alpha$. Similarly, we write

$$\begin{aligned} q_\beta(\sigma) \quad &:= \quad \Pr(G) \int_{t \in [y,G] \in [0,1] \times \{G\}} (\frac{1}{2} - x(t))\sigma(a,t) + (\frac{1}{2} + x(t))\sigma(b,t)) dF(y) \\ &+ \quad \Pr(g) \int_{t \in [y,g] \in [0,1] \times \{G\}} (\frac{1}{2} - x(t))\sigma(a,t) + (\frac{1}{2} + x(t))\sigma(b,t)) dF(y) \end{aligned} \tag{8}$$

for the probability that a random citizen votes $A$ in $\beta$.

# 2 Equilibrium Characterization

We analyse the symmetric Bayes-Nash-equilibria of the Bayesian game of voters in nondegenerate strategies and call them (voting) equilibria. A strategy is non-degenerate if the probability that a random citizen votes $A$ is neither 0 nor 1. For a given strategy $\sigma$, we use piv to denote the event

[6] As discussed in the literature, if the marginal cost of information at 0 are non-zero, that is if $c'(0) > 0$, then for sufficiently large $n$ no information acquisition is possible. Intuitively, this is true, because marginal benefits converge to 0 as the probability that a single vote is decisive converges to 0 (see e.g. Martinelli [2006]).

[7] Note that the restriction on $\tau$ is innocuous since we consider the case of large elections, that is when $n \to \infty$.

in which, from the viewpoint of a given voter, $n$ of the other $2n$ voters vote for $A$ and $n$ for $B$. It follows from the independence of types and signals across voters that

$$\Pr(\text{piv}|\omega;\sigma) = \binom{2n}{\tau n} q_\omega^{\tau n}(1-q_\omega)^{(1-\tau)n}, \tag{9}$$

where we also used that $\tau n \in \mathbb{N}$ and $(1-\tau)n \in \mathbb{N}$.

**Theorem 1.** *(Cut-off Characterisation)*
*For any equilibrium $\sigma$, there exist $\underline{y}(\lambda) < \Pr(\alpha|piv;\sigma_n) < \overline{y}(\lambda)$ for any $\lambda \in \{G,g\}$ such that:*

1. *Any voter of type $t=(y,G)$ votes $A$ if $y<\underline{y}(G)$, votes $B$ if $y>\overline{y}(G)$, and acquires information $x^*(t)>0$ with*

$$\Pr(\text{piv}|\sigma)[\Pr(\alpha|\text{piv})(1-y)+\Pr(\beta|\text{piv})y] = c'(x^*(t)),$$

   *and votes $A$ after receiving $a$, and $B$ after receiving $b$ if $[y \in \underline{y}(G), \overline{y}(G)]$.*

2. *Any voter of type $t=(y,g)$ votes $B$ if $y<\underline{y}(g)$, votes $A$ if $y>\overline{y}(g)$, and acquires information $x^*(t)>0$ with*

$$\Pr(\text{piv}|\sigma)[\Pr(\alpha|\text{piv})k(1-y)+\Pr(\beta|\text{piv})ky] = c'(x^*(t)).$$

   *and votes $B$ after receiving $a$, and $A$ after receiving $b$ if $[y \in \underline{y}(g), \overline{y}(g)]$.*

**A Balance Condition.** Note that Proposition 1 implies that in any equilibrium a random voter plays a pure strategy with probability 1. It also follows from Proposition 1 that in any equilibrium, a voter of $t=(y,\lambda)$ only changes his vote with the signal if $\underline{y}(\lambda) \le y \le \overline{y}(\lambda)$. We call

$$I(\lambda) := \int_{y\in[0,1]} x(y,\lambda)dF(y)$$

the *total information* acquired by a group $\lambda$. Then,

$$\Pr(\sigma(s,t)=1|\alpha) - \Pr(\sigma(s,t)=1|\beta) \quad = \quad 2(\Pr(G)I(G) - \Pr(g)I(g)). \tag{10}$$

Here we used that the difference between the probability that a random voter of group $\lambda$ receives an $a$-signal in $\alpha$ and the probability that a random voter of type $\lambda$ receives an $a$-signal in $\beta$ is given by $2I(\lambda)$. The equation (10) shows that the difference of the voting probabilities exactly balances out with (twice) the net information $\Pr(G)I(G) - \Pr(g)I(g)$ acquired by the voters.
**Proof.**

*Optimal Voting Behaviour.* Recall that for a given strategy $\sigma$, we use piv to denote the event in which, from the viewpoint of a given voter, $n$ of the other $2n$ voters vote for $A$ and $n$ for $B$. In this event, if she votes $A$, the outcome is $A$, if she votes $B$, the outcome is $B$. In any other event, the outcome is independent of her vote. Thus, a strategy is optimal if and only if it is optimal conditional on piv. Given $\sigma$, a voter of type $t=(y,\lambda)$ who acquired information of quality $x$ and received $s$ weakly prefers to vote $A$ if and only if

$$\Pr(\alpha|s,x,\text{piv};\sigma) \begin{cases} \ge y & \text{if} \quad \lambda = G, \\ \le y & \text{if} \quad \lambda = g. \end{cases} \tag{11}$$

The upper inequality follows from the inequality(4), and the lower inequality follows from the inequality (5). Note that the probability of being pivotal is strictly positive under any non-degenerate strategy $\sigma$, hence in any equilibrium. Note that in equilibrium, any type $t=(y,G)$ that acquires information, that is for which $x(t)>0$, votes $A$ after $a$ and $B$ after $b$. It follows from the inequalities

(11) that $t$ otherwise weakly prefers to vote for the same alternative after any signal. But then, $t$ is strictly better off simply voting for this alternative without acquiring information. Similarly, in any equilibrium $\sigma$, any type $t = (y, g)$ that acquires information votes $B$ after $a$ and $A$ after $b$. It also follows from the inequalities (11) and the the continuity of $G$, that in any equilibrium the voting behaviour of a random voter that does not acquire information is pure with probability 1.

*Optimal Information Acquisition.* For a type $t = (y, G)$ let $x^*(t)$ be the maximizer of

$$\Pr(\text{piv}|\sigma)[\Pr(\alpha|\text{piv})(1-y) + \Pr(\beta|\text{piv})y](\frac{1}{2} + x) - c(x) \tag{12}$$

across all $x \in [0, \frac{1}{2}]$ (we will show instantaneously that a maximizer of 12 is unique.). That is, $x^*(t)$ is the optimal choice of information quality conditional on information acquisition being optimal.[8] When $t$ chooses quality $x$, with probability

$$\Pr(\text{piv}|\sigma)\Pr(\alpha|\text{piv})(\frac{1}{2} + x),$$

the vote of $t$ is decisive, $\alpha$ holds and $t$ received signal $a$. After $a$ it is optimal for $t$ to vote $A$ and this yields utility of $1 - y$. With probability

$$\Pr(\text{piv}|\sigma)\Pr(\beta|\text{piv})(\frac{1}{2} + x),$$

the vote of $t$ is decisive, $\beta$ holds and $t$ received signal $b$. After $b$ it is optimal for $t$ to vote $B$ and this yields utility of $y$. When receiving $b$ in $\alpha$, $t$ votes $B$ and receives utility 0. Similarly, when receiving $a$ in $\beta$, $t$ votes $A$ and receives utility 0. We equate marginal benefit and marginal cost of quality $x$,

$$\Pr(\text{piv}|\sigma)[\Pr(\alpha|\text{piv})(1-y) + \Pr(\beta|\text{piv})y] = c'(x^*(t)). \tag{13}$$

Since $c$ is strictly increasing, the maximizer of (13) is unique. It follows from (9) that $\Pr(\text{piv}|\omega;\sigma) > 0$ for any non-degenerate strategy and any $\omega \in \{\alpha, \beta\}$. Then, it follows from $c'(0) = 0$ that $x^*(t) > 0$. The analogous derivation shows that for voter of type $t = (y, g)$,

$$\Pr(\text{piv}|\sigma)[\Pr(\alpha|\text{piv})k(1-y) + \Pr(\beta|\text{piv})ky] = c'(x^*(t)). \tag{14}$$

*Information Acquisition Cutoffs.* A voter of type $t = (y, G)$ is indifferent between not acquiring information and voting $A$ after both signals and acquiring the optimal positive amount of information $x^*(t)$ and voting $A$ after $a$ and $B$ after $b$ if

$$\begin{aligned}
& \Pr(\text{piv}|\sigma)\Pr(\alpha|\text{piv})(1-y) \\
= \quad & \Pr(\text{piv}|\sigma)[\Pr(\alpha|\text{piv})(1-y) + \Pr(\beta|\text{piv})y](\frac{1}{2} + x^*(t)) - c(x^*(t)) \\
\Leftrightarrow \quad & \frac{\Pr(\alpha|\text{piv})(1-y)}{\Pr(\alpha|\text{piv})(1-y) + \Pr(\beta|\text{piv})y} = \frac{1}{2} + x^*(t) - \frac{c(x^*(t)}{c'(x^*(t))} \\
\Leftrightarrow \quad & \frac{\Pr(\alpha|\text{piv})(1-y)}{\Pr(\alpha|\text{piv})(1-y) + \Pr(\beta|\text{piv})y} = \frac{1}{2} + \frac{d-1}{d}x^*(t).
\end{aligned} \tag{15}$$

where we used the first order condition (13) for the equivalence on the third line. The equivalence on the last line follows since $c(x) = x^d$. Similarly, a voter of type $t = (y, G)$ is indifferent between

[8] Recall that in equilibrium $\Pr(piv|\alpha) > 0$. Hence, it follows from $c(x) = 0$ that $x^*$ is never zero. On the other hand the equilibrium choice of information quality $x(t)$ is zero whenever voting uninformedly strictly dominates any strategy with positive information acquisition.

not acquiring information and voting $B$ after both signals and acquiring he optimal positive amount of information $x^*(t)$ and voting $A$ after $a$ and $B$ after $b$ if

$$\begin{aligned}
& \Pr(\text{piv}|\sigma)\Pr(\beta|\text{piv})y \\
= \quad & \Pr(\text{piv}|\sigma)[\Pr(\alpha|\text{piv})(1-y)+\Pr(\beta|\text{piv})y](\frac{1}{2}+x^*(t))-c(x^*(t)) \\
\Leftrightarrow \quad & \frac{(1-\Pr(\alpha|\text{piv}))y}{\Pr(\alpha|\text{piv})(1-y)+\Pr(\beta|\text{piv})y} = \frac{1}{2}+x^*(t)-\frac{c(x^*(t)}{c'(x^*(t))} \\
\Leftrightarrow \quad & \frac{(1-\Pr(\alpha|\text{piv}))y}{\Pr(\alpha|\text{piv})(1-y)+\Pr(\beta|\text{piv})y} = \frac{1}{2}+\frac{d-1}{d}x^*(t).
\end{aligned} \tag{16}$$

Note that we can interpret the left hand side $h(y) := \frac{\Pr(\alpha|\text{piv})(1-y)}{\Pr(\alpha|\text{piv})(1-y)+\Pr(\beta|\text{piv})y}$ of the indifference condition (15) as the posterior probability of $\alpha$ conditional on being pivotal and a signal $s$ with $\Pr(s|\alpha) = 1-y$, and $\Pr(s|\beta) = y$. Intuitively, the posterior $h(y)$ is strictly increasing in $1-y$. Note that $h(\Pr(\alpha|\text{piv})) = \frac{1}{2}$. Since the probability of being pivotal is strictly positive in equilibrium, it follows from the first-order condition (15) that the optimal information quality $x^*(t)$ is strictly larger than zero in any equilibrium. So $y = \Pr(\alpha|\text{piv})$ does not solve (15). It follows from an application of the implicit function theorem that, in any equilibrium and for any $n$ sufficiently large, the indifference equation (15) has a unique solution

$$\underline{y}(G) < \Pr(\alpha|\text{piv}). \tag{17}$$

Details are provided in the Appendix. Analogously, in any equilibrium and for any sufficiently large $n$, the indifference equation (16) has a unique solution

$$\overline{y}(G) > \Pr(\alpha|\text{piv}). \tag{18}$$

Analogously, for any equilbrium $\sigma$ there exist cutoffs $\underline{y}(g)$ and $\overline{y}(g)$ such that under $\sigma$ any type $t = (y, g)$ acquires information $x(t) > 0$ if and only if $y \in [\underline{y}(g), \overline{y}(g)]$.
■

Note that Proposition 1 implies that for any equilibrium and any $\lambda \in \{\alpha, \beta\}$, we have $x(\Pr(\alpha|\text{piv}, \lambda)) > 0$. Moreover, it follows from the first-order conditions (13) and (14) and from $c'(x) = \kappa x^{d-1}$ that for any $y \in [0, 1]$,

$$\begin{aligned}
x(y,g)^{d-1} &= kx(y,G)^{d-1}, \\
\Leftrightarrow x(y,g) &= k^{\frac{1}{d-1}}x(y,G).
\end{aligned} \tag{19}$$

**Best Response is a Function of the Voting Probabilities.** The characterisation of the best response through the inequalities (11), the first-order conditions (13) and (14) and the indifference equations (15) and (16) shows that the probabilities of being pivotal in $\alpha$ and $\beta$ are a sufficient statistic for the best response. Since, for any strategy $\sigma$, the probability of being pivotal in any state $\omega$ is a function of the probability that a random citizen votes $A$ in $\omega$, we can write the best response as a function of these voting probabilities $q_\omega$.

## 2.1 Local Central Limit Theorem

**Pivot Probabilities.** The following lemmata describe the pivot probabilities in each state when the electorate grows large, that is the probabilities of the event piv conditional on $\omega \in \{\alpha, \beta\}$. It is useful to introduce the notation

$$\delta_\omega := \lim_{n\to\infty} (q_\omega - \tau) n^{\frac{1}{2}} \in \mathbb{R} \cup \{\infty, -\infty\}. \tag{20}$$

**Lemma 1.** *Consider any strategy sequence $\sigma_n$. Then,*

$$\lim_{n\to\infty} \Pr(\text{piv}|\omega;\sigma_n) n^{\frac{1}{2}} = \begin{cases} \phi_{0,\tau(1-\tau)}(\delta_\omega) & \text{if} \quad \delta \in \mathbb{R}, \\ 0 & \text{if} \quad \delta \in \{\infty, -\infty\} \end{cases}$$

*where $\phi_{0,\tau(1-\tau)}$ denotes the density of the normal distribution with mean 0 and variance $\tau(1-\tau)$.*

**Proof.** Note that $\Pr(\text{piv}|\omega;\sigma_n) = \Pr(\text{B}(n, q_\omega(\sigma_n) = \tau n)$ where $\text{B}(n,p)$ denotes the binomial distribution with parameters $n$ and $p$. The result follows from the local central limit theorem for triangular arrays of integer-valued random variables; see for example Theorem 2 in Davis and McDonald [1995]. ■

In the Online Supplement, we provide a self-contained proof of Lemma 1 that relies on Stirling's formula. The proof in the Online Supplement also yields

**Lemma 2.** *Consider any strategy sequence $\sigma_n$. Then, for any $\omega \in \{\alpha, \beta\}$*

$$\lim_{n\to\infty} \Pr(\text{piv}|\omega;\sigma_n) n^{\frac{1}{2}} = \lim_{n\to\infty} (2\pi)^{-\frac{1}{2}} (\tau(1-\tau))^{-\frac{1}{2}} \Big[ (\frac{q_\omega(\sigma_n)}{\tau})^\tau (\frac{1-q_\omega(\sigma_n)}{1-\tau})^{(1-\tau)} \Big]^n.$$

**Proof.** In the Online Supplement. ■

## 2.2 Information and Voting Power

**Swing Voters and Vanishing Information Acquisition.** Note that the function $q^\tau(1-q)^{1-\tau}$ has a unique maximum at $q = \tau$. Consequently, $(\frac{q_n}{\tau})^\tau(\frac{1-q_n}{1-\tau})^{(1-\tau)} \leq 1$. It follows from this observation and Lemma 2 that for any strategy sequence $\sigma_n$, the probability of being pivotal converges to 0 for $n \to \infty$. Hence, it follows from the assumption that $c(x) = x^d$ and the first-order conditions (13) and (14) that

$$x^*(t) \text{ is converging to 0 uniformly in } t. \tag{21}$$

Note that it is only rational for a voter to acquire information $x^*(t) > 0$ if, given $x^*(t)$, one of the signals $s \in \{a, b\}$ reverts the preference over $A$ and $B$: recall the threshold of doubt interpretation of the payoff types $y$, see the inequalities (4) and (5). Then, more precisely, this means that $\Pr(\alpha|\text{piv}, b; x^*(y,\lambda)) \leq y \leq \Pr(\alpha|\text{piv}, a; x^*(y,\lambda))$. Therefore, it follows from the observation that $x^*(t)$ is converging to 0 uniformly that the mass of voters that acquire information, that is the *swing voters*, converges to 0.

Recall the definition of $h(y)$ after the equation (16) which we can rewrite as $h(y) - \frac{1}{2} = \frac{d-1}{d} x^*(\underline{y}(\lambda), \lambda))$. Recall that $h(\Pr(\alpha|\text{piv})) = \frac{1}{2}$. A Taylor approximation of the left hand side at the root $\tilde{y} := \Pr(\alpha|\text{piv})$ gives

$$\begin{aligned} \underline{y}(\lambda) - \tilde{y}) &\approx \frac{1}{h'(\tilde{y})} \frac{d-1}{d} x^*(\underline{y}(\lambda), \lambda) \\ &\approx \frac{1}{h'(\tilde{y})} \frac{d-1}{d} x^*(\tilde{y}) \end{aligned} \tag{22}$$

where the approximation on the second line follows since $x(y)$ is continuous in $y$ and since the mass of swing voters converges to 0 which implies that $\lim_{n\to\infty} \underline{y}(\lambda) = \lim_{n\to\infty} \tilde{y}$. Further Taylor approximations of $F(\overline{y}(\lambda))$ and $F(\underline{y}(\lambda))$ at $\tilde{y}$ yield[9]

$$\begin{aligned} F(\overline{y}(\lambda)) - F(\underline{y}(\lambda)) &\approx \overline{y}(\lambda) - \underline{y}(\lambda) f(\tilde{y}) \\ &\approx \frac{f(\tilde{y})}{h'(\tilde{y})} \frac{d-1}{d} x^*(\tilde{y}), \end{aligned} \tag{23}$$

where the approximation on the last line follows from the approximation (22). We conclude that the mass of swing voters is proportional to the information acquired by the type $\tilde{y} = \Pr(\alpha|\text{piv})$ which is indifferent before receiving a signal. Intuitively, then, the total amount of information $I_\lambda = \int_{\underline{y}(\lambda)\le y\le\overline{y}(\lambda)} x(y,\lambda)dF(y)$ acquired by a group $\lambda$ is proportional to the square of the information acquired by the indifferent voter since any type that acquire information is arbitrarily close to $\tilde{y}$ for $n$ arbitrarily large.

**Lemma 3.** *Let $\tilde{y} = \Pr(\alpha|\text{piv})$. Then, for any sequence of strategies $\sigma_n$ and any $\lambda \in \{G, g\}$, the best response satsifies*

$$I_\lambda = \frac{2(d-1)}{d} \frac{f(\tilde{y})}{h'(\tilde{y})} \cdot x(\tilde{y},\lambda)^2 + O(\Pr(\text{piv}|\sigma_n)^{\frac{2}{d-1}})o(1). \tag{24}$$

**Proof.** In the Appendix. ■

Let $\tilde{y} = \Pr(\alpha|\text{piv})$. We have

$$\begin{aligned} \lim_{n\to\infty} q_\alpha(\text{BR}(\sigma_n)) &= \lim_{n\to\infty} \Pr(G)F(\tilde{y}) + \Pr(g)(1-F(\tilde{y})) \\ &= \lim_{n\to\infty} \psi(\tilde{y}) \end{aligned} \tag{25}$$

where the equality on the first line follows from Theorem 1 and since the mass of swing voters converges to zero , see the equation (22) and (21). The equality on the last line holds by the definition of the function $\psi$. Analogously,

$$\lim_{n\to\infty} q_\beta(\text{BR}(\sigma_n)) = \lim_{n\to\infty} \psi(\tilde{y}). \tag{26}$$

**Order of the Voting Probabilities.** Recall the equation (10) which showed that the difference of the voting probabilities is the same as twice the net information $\Pr(G)I_G - \Pr(g)I_g$ acquired by the voters in equilibrium. We will now show that the net amount of information can be expressed by the information acquired by the indifferent voter $\tilde{y} = \Pr(\alpha|\text{piv})$ of the majority group $G$ only: First, recall that the first-order conditions (13) and (14) imply for any pay-off type $y \in [0,1]$, the voter of the minority group $g$ acquired $k^{\frac{1}{d-1}}$ times as much information as the voter of the majority group $G$, i.e. $x(y,g) = k^{\frac{1}{d-1}} y(y,G)$; see the equation (19). Combining the equation (10), the formula in Lemma 3 for the total amount of information $I_\lambda$ acquired by each group $\lambda$ and the equation (19) gives[10]

[9] In the proof of Lemma 3 we show that the error term of the approximation (23) is of order $O(\Pr(\text{piv}|\sigma_n)^{\frac{2}{d-1}})o(1)$.

[10] More precisely,

$$\begin{aligned} q_\alpha - q_\beta &= 2(\Pr(G)I(G) - \Pr(g)I(g)) \\ &= \frac{4(d-1)}{d} \frac{f(\tilde{y})}{h'(\tilde{y})} (\Pr(G)x(\tilde{y},G)^2 - \Pr(g)x(\tilde{y},g)^2) + O(\Pr(\text{piv}|\sigma_n)^{\frac{2}{d}})o(1) \\ &= \frac{4(d-1)}{d} \frac{f(\tilde{y})}{h'(\tilde{y})} (\Pr(G) - k\Pr(g))x(\tilde{y},G)^2 + O(\Pr(\text{piv}|\sigma_n)^{\frac{2}{d}})o(1), \end{aligned} \tag{27}$$

where the equality on the first line follows from the equation (10), the equality second line follows from Lemma 3. The equality on the last line follows from the equation (19).

**Lemma 4.** *Let* $\Pr(G) - k^{\frac{2}{d-1}}\Pr(g) > (<)0$*, and* $d > 1$*. For any strategy sequence* $\sigma_n$*,*

$$\begin{aligned} & q_\alpha(\mathrm{BR}(\sigma_n)) - q_\beta(\mathrm{BR}(\sigma_n)) \\ = \quad & \frac{4(d-1)}{d}\frac{f(\tilde{y})}{h'(\tilde{y})}(\Pr(G) - k^{\frac{2}{d-1}}\Pr(g))x(\tilde{y}, G)^2 + O(\Pr(\mathrm{piv}|\sigma_n)^{\frac{2}{d}})o(1), \end{aligned} \tag{28}$$

*where* $BR(\sigma_n)$ *is the best reponse to* $\sigma_n$*.*

It follows from the first-order conditions (13) and (14) that $x(\tilde{y}, G)$ is asymptotically equivalent to $\Pr(\mathrm{piv}|\sigma_n)$, that is $x(\tilde{y}, G) \in \Theta(\Pr(\mathrm{piv}|\sigma_n))$.[11] Hence, if $\Pr(G) - k^{\frac{2}{d-1}}\Pr(g) > 0$, and $d > 1$, it follows from the equality (27) that there exists $\bar{n} \in \mathbb{N}$ such that $q_\alpha - q_\beta > 0$ for all $n \geq \bar{n}$. Analogously, $\Pr(G) - k^{\frac{2}{d-1}}\Pr(g) < 0$, and $d > 1$, it follows it follows from the equality (27) that there exists $\bar{n} \in \mathbb{N}$ such that $q_\alpha - q_\beta < 0$ for all $n \geq \bar{n}$.

**Voting Power.** Consider any equilibrium sequence for which $\mathrm{sgn}(q_\alpha - \tau) \neq \mathrm{sgn}(q_\beta - \tau)$. Suppose that group $G$ acquires more information in expectation, that is $\Pr(G)I_G - \Pr(g)I_g > 0$. It follows from the equation (10) that $q_\beta \leq \tau \leq q_\alpha$; consequently in any state the outcome is more likely to be the one prefered by group $G$, that is $A$ in $\alpha$ and $B$ in $\beta$. Now Lemma 4 shows that the order of the voting probabilities in the two states and the order the amount of information acquired by each group respectively is the same as the order of $\Pr(G)$ and $\Pr(g)k^{\frac{2}{d-1}}$ when the electorate is large. Motivated by this observation, we call $\Pr(G)$ and $\Pr(g)k^{\frac{2}{d-1}}$ the *voting power* of the majority and the minority group respectively.

## 2.3 Close Elections

This section shows that in any equilibrium sequence that involves substantial information acquisition the election is necessarily close to being tied when the electorate is large. Intuitively, a voter acquires information to make a better voting decision. But he knows that this only matters when his vote actually decides the election. If the election has a positive margin of victory in expectation, this event has a probability that is exponentially small (see e.g. Lemma 2).

**Lemma 5.** *Let* $\psi(p_0) \neq \frac{1}{2}$*. Any equilibrium sequence for which* $I_\lambda$ *does not converge to zero exponentially fast for some* $\lambda \in \{G, g\}$*, satisfies* $\lim_{n\to\infty} q_\omega = \tau$ *for all* $\omega \in \{\alpha, \beta\}$*.*

**Proof.** Consider any equilibrium sequence with $\lim_{n\to\infty} q_\omega \neq \tau$ for some $\omega \in \{\alpha, \beta\}$. Since the amount of information acquired converges to zero (see (21), $\lim_{n\to\infty} q_\omega \neq \tau$ for any $\omega \in \{\alpha, \beta\}$ (see also (25) and (26)). Let $\epsilon > 0$ such that $|\psi(p_0) - \tau| > 2\epsilon$. Then, there exists $n(\epsilon) \in \mathbb{N}$ such that $|q_\omega - \tau| > \epsilon$ for all $n \geq n(\epsilon)$. Then, it follows from Lemma 2 that the pivot probabilities converge to zero exponentially fast. It follows from the first-order conditions (13) and (14) that the amount $x(t)$ of information acquired by each type $t$ converges to zero exponentially fast. This finishes the proof of the Lemma. ■

Suppose that there exists an equilibrium sequence $\sigma_n$ with $\lim_{n\to\infty} q_\omega(\sigma_n) = \tau$ for all $\omega \in \{\alpha, \beta\}$. It follows from the equation (25) that

$$\lim_{n\to\infty} \psi(\Pr(\mathrm{piv}|\alpha; \sigma_n)) = \tau. \tag{29}$$

We draw several implications from the condition (29) that will be useful for the proof of the first main result: it follows from the strict monotonicity of $\psi$ that

[11] We use the Knuth-Landau notation $(x_n)_{n\in\mathbb{N}} \in \Theta((y_n)_{n\in\mathbb{N}})$ to express that a sequence $(x_n)_{n\in\mathbb{N}}$ is asymptotically equivalent to another sequence $(y_n)_{n\in\mathbb{N}}$, which means that $\lim_{n\to\infty} \frac{x_n}{y_n} \in \mathbb{R}$.

$$\lim_{n\to\infty} \Pr(\text{piv}|\alpha) = p_\tau, \tag{30}$$

where we defined $p_\tau$ as the unique $p \in [0,1]$ for which $\psi(p) = \tau$. Then,

$$\begin{aligned} \lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha)}{\Pr(\text{piv}|\beta)} \frac{p_0}{1-p_0} &= \frac{p_\tau}{1-p_\tau} \\ \Leftrightarrow \frac{\phi(\delta_\alpha)}{\phi(\delta_\beta)} \frac{p_0}{1-p_0} &= \frac{p_\tau}{1-p_\tau} \\ \Leftrightarrow \phi(\delta_\alpha) &= \phi(\delta_\beta) \frac{1-p_0}{p_0} \frac{p_\tau}{1-p_\tau}, \end{aligned} \tag{31}$$

where the equivalence on the second line follows from Lemma 1. We can rewrite the equation (31) as

$$\begin{aligned} e^{-\frac{\delta_\alpha^2}{2\tau(1-\tau)}} &= e^{-\frac{\delta_\beta^2}{2\tau(1-\tau)}} \frac{1-p_0}{p_0} \frac{p_\tau}{1-p_\tau} \\ \Leftrightarrow \delta_\alpha^2 - \delta_\beta^2 &= [\ln(\frac{p_0}{1-p_0}) - \ln(\frac{p_\tau}{1-p_\tau})] 2\tau(1-\tau). \end{aligned} \tag{32}$$

# 3 Equilibria with Much Information

## 3.1 Generalized Lambert Equations

The balance condition (10) says that the difference in the voting probabilities $q_\alpha - q_\beta$ equals (twice) the net information $\Pr(G)I_G - \Pr(g)I_g$ acquired by the voters in equilibrium. If we impose the weak efficiency requirement that $q_\alpha \geq \tau \geq q_\beta$, the margins of victory $|q_\omega - \tau|$ are small when the difference in the voting probabilities is small and vice versa. Then, the relation between the difference in the voting probabilities and the incentives to acquire information is antiproportional. This captures the free-rider aspect of the election: when many voters acquire information and as a consequence, the election outcome is more close to the outcome when states are known and less likely to be tied, incentives to acquire information are small. Intuitively, this complementarity implies uniqueness of the equilibrium. Theorem 2 shows that when the cost of information are intermediate ($d = 3$), the unique limit equilibrium that satisfies the weak efficiency requirement is identified by a unique scaled margin of victory $\lim_{n\to\infty}(q_\omega - \tau)n^{\frac{1}{2}} = \delta_\omega$ for some $\omega \in \{\alpha, \beta\}$ if such an equilibrium exists. Moreover, we show that $\delta_\omega = W^\eta(\nu)$ where $W^\eta(\nu)$ is a function that generalises the product logarithm function and $\nu$ and $\eta$ are functions that capture the primitives of the model. To show this, we rewrite the balance condition (10) into an equation in the variable $\delta_\omega$ only that generalises Lambert's equation.

**Theorem 2.** *Let $d = 3$ and $\Pr(G) - k\Pr(g) > 0$.*[12] *Suppose that there exists an equilibrium sequence $\sigma_n$ and $\bar{n} \in \mathbb{N}$ such that for any $n \geq \bar{n}$ and any state $\omega$, the utilitarian outcome is more likely to be elected in $\omega$ than not, given $\sigma_n$. Let $\nu = \frac{8(d-1)}{d} \frac{f(p_\tau)}{h'(p_\tau)} (\Pr(G) - k\Pr(g)) \frac{1}{\kappa} (\pi)^{-\frac{1}{2}}$ and $\eta = \left[\ln(\frac{p_\tau}{1-p_\tau}) - \ln(\frac{p_0}{1-p_0})\right] 2\tau(1-\tau)$.*

[12]When $\Pr(G) - k\Pr(g) > 0$, the analogous result holds where the only modification is that we multiply the left hand side of the following equations 33 and 34 by $(-1)$.

1. *If $p_0 \leq p_\tau$, the limit of the equilibrium sequence satisfies the generalised Lambert equation*

$$(w + (w^2 + \eta)^{\frac{1}{2}}) e^{\frac{1}{2\tau(1-\tau)} w^2} = \nu p_0 (1 - p_\tau), \tag{33}$$

*with $w = \delta_\alpha$.*

2. *If $p_0 \geq p_\tau$, the limit of the equilibrium sequence satisfies the generalised Lambert equation*

$$(w + (w^2 - \eta)^{\frac{1}{2}}) e^{\frac{1}{2\tau(1-\tau)} w^2} = \nu p_\tau (1 - p_0), \tag{34}$$

*with $w = \delta_\beta$.*

Note that the functions on the left hand side of the generalised Lambert equations (33) and (34) are strictly increasing in $w$. For $p_0 \leq p_\tau$, the parameter $\eta$ is positive and we denote by $W^\eta$ the inverse of the function on the left hand side of (33). For $p_0 \geq p_\tau$, the parameter $\eta$ is negative and we denote by $W^\eta$ the inverse of the function on the left hand side of (34). We call $W^\eta(\nu)$ a generalised Lambert function.

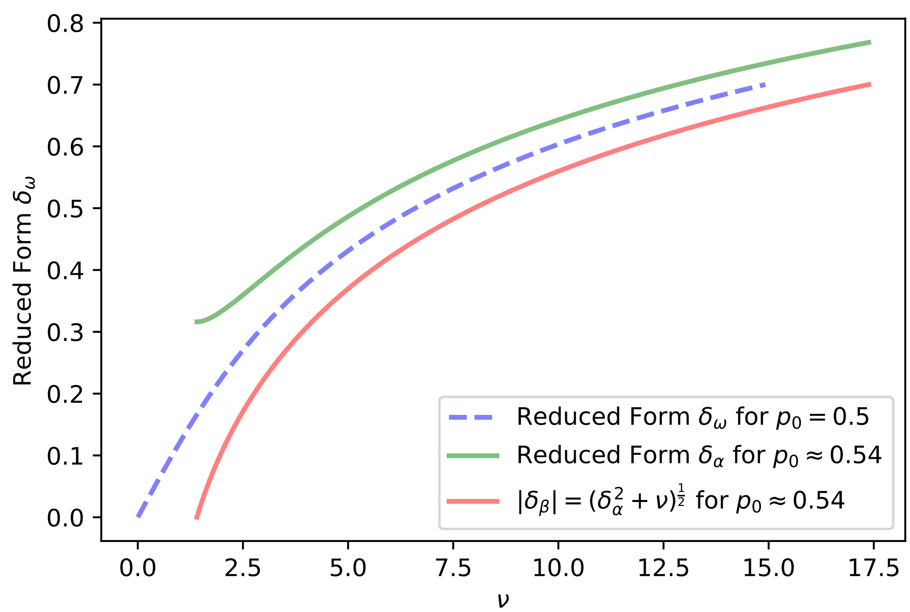


Figure 1: Let $d = 3$ and $\tau = p_\tau = \frac{1}{2}$. Let $\Pr(G) - k\Pr(g) > 0$. Figure 1 illustrates the reduced form description $\delta_\beta = \mathrm{W}^\eta((\nu p_\tau(1 - p_0))$ and $|\delta_\beta| = (\delta_\alpha^2 + \nu)^{\frac{1}{2}}$ of the Lambert-type limit equilibrium as a function of $\nu$ and $\eta$: first, when the prior is unbiased relative to the voting rule (dotted line); second, when the prior is biased relative to the voting rule (straight lines).

**Proof of Theorem 2..** Suppose that there exists an equilibrium sequence as in the Theorem. Recall the balance condition (10) which says that, in equilibrium, the difference of the voting probabilities in $\alpha$ and $\beta$ equals twice the 'net' information acquired by the voters,

$$q_\alpha - q_\beta = 2(\Pr(G) I_G - \Pr(g) I_g). \tag{35}$$

Recall that we can understand the best response as a function of the voting probabilities in each state, that is of $q_\alpha$ and $q_\beta$. The equation (35) has to be satisfied by each fixed point of the best response. Multiplication by $n^{\frac{1}{2}}$ and taking limits $n \to \infty$ gives

$$|\delta_\alpha| + |\delta_\beta| = \lim_{n\to\infty} 2(\Pr(G)I_G \Pr(g)I_g)n^{\frac{1}{2}}, \tag{36}$$

where we used the assumption of the theorem that $\operatorname{sgn}(q_\alpha - \tau) \neq \operatorname{sgn}(q_\alpha - \tau)$ for any $n$ large enough. We will now use three observations from the sections 2.1, 2.2 and 2.3 to rewrite the 'limit fixed point equation' (36): First, the total amount of information $I_\lambda$ acquired by a group $\lambda$ is proportional to the square of the information acquired by the indifferent voter type $\tilde{y} = \Pr(\alpha|\text{piv})$; see Lemma 3. Second, it follows from the weak requirement that the utilitarian outcome is more likely to be elected in each state, i.e. $q_\alpha \geq \tau \geq q_\beta$ and from equation (25) that the election is neccessarily close, that is $\lim_{n\to\infty} q_\omega = p_\tau$ where $p_\tau$ is the unique belief $p \in [0,1]$ for which $\psi(p_\tau) = \tau$. Intuitively, only a close election creates enough incentives for information acquisition such that the election outcome can satisfy the weak requirement. This pins down the posterior conditional on the election being tied,

$$\lim_{n\to\infty} \Pr(\text{piv}|\alpha) = p_\tau,$$

see equation (30). On the one hand, this pins down the indifferent voter type $\tilde{y} = \Pr(\alpha|\text{piv})$. So, we can express how much information $\tilde{y}$ acquires by use of the first-order condition (13) for $p_\tau$, for example for group $G$,

$$x(\tilde{y}, G)^2 \quad = \quad p_0 \Pr(\text{piv}|\alpha)(1 - p_\tau) + (1 - p_0)\Pr(\text{piv}|\beta)p_\tau \frac{1}{\kappa}, \tag{37}$$

where we used the standing assumption of Theorem 2 that $d = 3$ or equivalently that $c(x) = \frac{\kappa}{d}x^d$. On the other hand, this pins down the relation between the margin of victory in $\alpha$ and the margin of victory in $\beta$, and therefore of the scaled margins of victory $\delta_\alpha$ and $\delta_\beta$, see the equations (31) and (32). Third, we use the local limit theorem to express the probability of the election being tied in $\omega$ as a function of $\delta_\omega$; see Lemma 9. All taken together, we can express the information acquired by the voter $\tilde{y} \approx p_\tau$ that is indifferent before receiving a signal, and hence the net information acquired, as a function of $\delta_\alpha$ and $\delta_\beta$. Furthermore, as just observed, we can express $\delta_\beta$ as a function of $\delta_\alpha$. So,

$$\begin{aligned} & \lim_{n\to\infty} 2(\Pr(G)I_G + \Pr(g)I_g)n^{\frac{1}{2}} \\ = \; & \lim_{n\to\infty} [\frac{4(d-1)}{d} \frac{f(\tilde{y})}{h'(\tilde{y})}(\Pr(G) - k^{\frac{2}{d-1}}\Pr(g))x(\tilde{y}, G)^2 n^{\frac{1}{2}} \\ = \; & [\frac{4(d-1)}{d} \frac{f(\tilde{y})}{h'(\tilde{y})}(\Pr(G) - k^{\frac{2}{d-1}}\Pr(g))2\phi(\delta_\alpha)(p_0(1 - p_\tau))\frac{1}{\kappa} \end{aligned} \tag{38}$$

where the equation on the second line follows from Lemma 3[13] and the equation on the second line follows from the first-order condition (13) (or (52)) and the local limit theorem Lemma 1 .[14] Suppose

[13]Note that $[O(\Pr(\text{piv}|\sigma_n)^{\frac{2}{d-1}})o(1)]n^{\frac{1}{2}} = 0$, since the probability of being pivotal is strictly bounded above by $2n^{-\frac{1}{2}}\phi(0)$ for $n$ sufficiently large (see e.g. Lemma 2).

[14]More precisely,

$$\begin{aligned} \lim_{n\to\infty} x(\tilde{y}, G)^{d-1} n^{\frac{1}{2}} &= (n^{\frac{1}{2}} p_0 \Pr(\text{piv}|\alpha)(1 - p_\tau) + (1 - p_0)\Pr(\text{piv}|\beta)p_\tau)\frac{1}{\kappa} \\ &= (p_0\phi(\delta_\alpha)(1 - p_\tau) + (1 - p_0)\phi(\delta_\beta)p_\tau)\frac{1}{\kappa} \\ &= \Big[\phi(\delta_\alpha)(p_0(1 - p_\tau) + (1 - p_0)p_\tau \frac{p_0}{1 - p_0}\frac{1 - p_\tau}{p_\tau})\Big]\frac{1}{\kappa} \\ &= 2\phi(\delta_\alpha)(p_0(1 - p_\tau))\frac{1}{\kappa}, \end{aligned} \tag{39}$$

that the outcome $A$ is favored under the prior relative to the voting rule, that is $p_0 \leq p_\tau$; hence, $\eta = \left[\ln(\frac{p_\tau}{1-p_\tau}) - \ln(\frac{p_0}{1-p_0})\right] 2\tau(1-\tau) \geq 0$. If we use the equation (32) and express $\delta_\beta$ as a function of $\delta_\alpha$ on the left hand side of the 'limit fixed point equation' (36), we obtain

$$
\begin{aligned}
& (\delta_\alpha(1+(\delta_\alpha+\eta)^{\frac{1}{2}}) && (40)\\
= \quad & [\frac{4(d-1)}{d}\frac{f(\tilde{y})}{h'(\tilde{y})}(\Pr(G) - k^{\frac{2}{d-1}}\Pr(g))2\phi(\delta_\alpha)(p_0(1-p_\tau))\frac{1}{\kappa} \\
\Leftrightarrow \quad & (\delta_\alpha(1+(\delta_\alpha+\eta)^{\frac{1}{2}})e^{\delta_\alpha^2\frac{1}{2\tau(1-\tau)}} \\
= \quad & [\frac{8(d-1)}{d}\frac{f(\tilde{y})}{h'(\tilde{y})}(\Pr(G) - k^{\frac{2}{d-1}}\Pr(g))(p_0(1-p_\tau))\frac{1}{\kappa}(\pi)^{-\frac{1}{2}} \\
\Leftrightarrow \quad & (\delta_\alpha(1+(\delta_\alpha+\eta)^{\frac{1}{2}})e^{\delta_\alpha^2\frac{1}{2\tau(1-\tau)}} = \nu(p_0(1-p_\tau)), && (41)
\end{aligned}
$$

where the equality on the first line follows from the equation (54). The first equivalence follows from the formula for the density of the normal distribution. The equality on the last line follows from the observation that $\lim_{n\to\infty}\Pr(\alpha|\text{piv}) = \lim_{n\to\infty}\tilde{y} = p_\tau$ and since we defined $\nu = \frac{8(d-1)}{d}\frac{f(p_\tau)}{h'(p_\tau)}(\Pr(G) - k\Pr(g))\frac{1}{\kappa}(\pi)^{-\frac{1}{2}}$. Analogously, one shows that for $p_0 \geq p_\tau$ it must hold that

$$
(\delta_\beta(1+(\delta_\beta-\eta)^{\frac{1}{2}})e^{\delta_\beta^2\frac{1}{2\tau(1-\tau)}} = \nu(p_\tau(1-p_0)). \tag{42}
$$

■

**The classical Lambert Equation arises when the prior is uniform.** Let $\eta = 0$. Then the generalised Lambert equation (33) can be written as

$$
\begin{aligned}
2we^{\frac{1}{2\tau(1-\tau)}w^2} &= \nu p_\tau(1-p_0)\\
\Leftrightarrow \frac{1}{\tau(1-\tau)}w^2 e^{\frac{1}{\tau(1-\tau)}w^2} &= \frac{1}{4\tau(1-\tau)}(\nu p_\tau(1-p_0))^2\\
\Leftrightarrow ze^z &= \frac{1}{4\tau(1-\tau)}(\nu p_\tau(1-p_0))^2
\end{aligned}
$$

for $z = \frac{1}{\tau(1-\tau)}w^2$. Note that this is the classical Lambert equation as studied by Euler and Lambert (see Lambert [1758] and Euler [1783]).

**Remark 1.** *The converse of Theorem 3 also holds generically. Suppose, that there exists an equilibrium sequence $(\sigma_n)$ that satisfies the generalised Lambert equation, that is (33) or (33) respectively. Suppose that $\nu p_0(1-p_\tau) \neq 0$ or $\nu p_\tau(1-p_0) \neq 0$. This implies that $|\delta_\alpha| > 0$ and $|\delta_\beta| > 0$. Suppose that there exists a subsequence for which $\operatorname{sgn}(q_\alpha - \tau) = \operatorname{sgn}(q_\alpha - \tau)$. Then, for this subsequence, $\delta_\alpha < 0$ and $\delta_\beta < 0$ or $\delta_\alpha > 0$ and $\delta_\beta > 0$. In any case,*

$$
|\delta_\alpha| + |\delta_\beta| > \delta_\alpha - \delta_\beta = \lim_{n\to\infty}(q_\alpha - q_\beta)n^{\frac{1}{2}}. \tag{43}
$$

*Now, the generalised Lambert equation is satisfied if and only if the sum of the margin of victories equals twice the net information acquired by the voters,*

$$
|\delta_\alpha| + |\delta_\beta = 2(\Pr(G)I_G + \Pr(g)I_g), \tag{44}
$$

*whereas in equilibrium, the difference in the margin of victories has to equal twice the net information acquired by the voters,*

$$
\delta_\alpha - \delta_\beta = 2(\Pr(G)I_G + \Pr(g)I_g), \tag{45}
$$

*Clearly, (43), (44) and (45) cannot hold simultaneously, hence we arrive at a contradiction.*

where the equality on the first line follows from the first-order condition (13) and from $c(x) = \frac{\kappa}{d}x^d$. The equality on the second line follows from Lemma 1. The equality on the third line follows from the equation (31).

## 3.2 Existence for Intermediate and Low Cost

Suppose that the cost of information are intermediate ($d = 3$) and $p_0 \le p_\tau$. Suppose that the generalised Lambert equations (33) and (34) would be a sufficient equilibrium condition for the existence of a limit equilibrium. Whenever $p_0(1 - p_\tau)\nu > \eta^{\frac{1}{2}}$, it follows from the intermediate value theorem that there exists $w = \delta_\alpha$ such that the Lambert equation is satisfied, and hence, a limit equilibrium with the scaled margin of victory $\delta_\alpha$. As argued in the remark 1, under the limit equilibrium strategy, it must hold $\mathrm{sgn}(\delta_\alpha) \neq \mathrm{sgn}(\delta_\beta)$. Thus, in any state, the utilitarian outcome is more likely to be elected than not. The following Theorem 3 shows that this heuristic intuition for existence of a limit equilibrium with $\mathrm{sgn}(\delta_\alpha) \neq \mathrm{sgn}(\delta_\beta)$ is indeed correct.

**Theorem 3.** *1. Let $d = 3$. There exists an equilibrium sequence for which in any state $\omega \in \{\alpha, \beta\}$, the utilitarian outcome is more likely to be elected if either $p_0 \le p_\tau$ and $p_\tau(1 - p_0)\nu > (-\eta)^{\frac{1}{2}}$, or $p_0 \ge p_\tau$ and $p_0(1 - p_\tau)\nu > (-\eta)^{\frac{1}{2}}$.*

*2. Let $d > 3$. There exists an equilibrium sequence with $\mathrm{sgn}(q_\alpha - \tau) \neq \mathrm{sgn}(q_\alpha - \tau)$.*

Note that in the case when the cost of information are intermediate ($d = 3$), Theorem 2 already shows that the conditions $p_0 \le p_\tau$ and $p_\tau(1-p_0)\nu \ge (-\eta)^{\frac{1}{2}}$, or $p_0 \ge p_\tau$ and $p_0(1-p_\tau)\nu \ge (-\eta)^{\frac{1}{2}}$ are necessary for the existence of equilibrium sequences for which in any state $\omega \in \{\alpha, \beta\}$, the utilitarian outcome is more likely to be elected. This follows, since otherwise the respective generalised Lambert equation does not have a solution $\delta_\omega \ge 0$. Now, Theorem 3 shows that the strict versions of these conditions are also sufficient for existence, completing the characterisation of such equilibrium sequences.

**Proof.** W.l.o.g., we restrict to the situation when $\Pr(G) - k^{\frac{2}{d-1}} \Pr(g)$ such that the order of the voting probabilities under the best reponse is given by

$$q_\alpha \ge q_\beta \tag{46}$$

when the electorate is large; see Lemma 3. The remaining cases are analogous.
The main trick of the proof is to modify the best response function such that the modified version satisfies the condition of Theorem 2 that in each state the utilitarian outcome is more likely to be elected, i.e. it satisfies $q_\alpha \ge \tau \ge q_\beta$. For this, we consider the best response as a function in the voting probabilities $q_\alpha$ and $q_\beta$ and denote it $\mathrm{BR}(q_\alpha, q_\beta)$. For any voting probabilities $(q_\alpha, q_\beta)$, we let the modified best response be the function that maps $(q_\alpha, q_\beta)$ to the pair

$$\tilde{q_\alpha} = \max(\tau, q_\alpha(\mathrm{BR}(q_\alpha, q_\beta)), \tag{47}$$
$$\tilde{q_\beta} = \min(\tau, q_\beta(\mathrm{BR}(q_\alpha, q_\beta)) \tag{48}$$

The modified best reponse is continuous in $q_\alpha$ and $q_\beta$. It follows from the Brouwer fixed point theorem that there exists a sequence of fixed points $(\tilde{q_\alpha}^*, \tilde{q_\beta}^*)$ of the modified best reponse.

**Lemma 6.** *Any fixed point of the modified best reponse is interior when $n$ is large enough.*

Given the Lemma, the strategies $\mathrm{BR}(\tilde{q_\alpha}^*, \tilde{q_\beta}^*)$ are equilibria of the voting game with $q_\alpha(\mathrm{BR}(\tilde{q_\alpha}^*, \tilde{q_\beta}^*)) \ge \tau \ge q_\beta(\mathrm{BR}(\tilde{q_\alpha}^*, \tilde{q_\beta}^*))$. This will finish the proof of Theorem 3. To prove the Lemma, first, we show

**Claim 1.** *Suppose that there exists a sequence of non-interior fixed point $(\tilde{q_\alpha}_n, \tilde{q_\beta}_n)$ of the modified best response. Then, $\lim_{n\to\infty}(\tilde{q_\alpha}_n - \tau)n^{\frac{1}{2}} \in \mathbb{R}$. In particular, $\lim_{n\to\infty} \tilde{q_\omega}_n = \tau$ for any $\omega \in \{\alpha, \beta\}$.*

**Proof.** W.l.o.g. suppose that $(\tilde{q_\beta})_n = \tau$ for any $n$ large enough. From the definition of the modified best response, $\tilde{q_\beta}_n = \tau$ is eqivalent to $q_\beta(\mathrm{BR}((\tilde{q_\alpha}_n, \tilde{q_\beta}_n)) \ge \tau$. It follows from (46) that

$q_\alpha(\mathrm{BR}((\tilde{q_\alpha}_n, \tilde{q_\beta}_n)) > \tau$, hence from the definition of the modified best response, $q_\beta(\mathrm{BR}((\tilde{q_\alpha}_n, \tilde{q_\beta}_n)) = \tilde{q_\alpha}_n$. Suppose that $\delta_\alpha = \lim_{n\to\infty}(\tilde{q_\alpha}_n - \tau)n^{\frac{1}{2}} = \infty$. Then,

$$\begin{aligned}\lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha)}{\Pr(\text{piv}|\beta)} &= \lim_{n\to\infty} \frac{\phi(\delta_\alpha)}{\phi(0)} \\ &= 0. \end{aligned} \tag{49}$$

Consequently, $\lim_{n\to\infty} \Pr(\alpha|\text{piv}) = 0$. Then, it follows from the equation (25) that $q_\beta(\mathrm{BR}(\sigma_n)) = \lim_{n\to\infty} \psi(\Pr(\alpha|\text{piv})) = \psi(0) < \tau$. However, this contradicts with the assumption that $\lim_{n\to\infty} \tilde{q_\beta} = \tau$ or equivalently that $\lim_{n\to\infty} q_\beta(\mathrm{BR}(\sigma_n)) \geq \tau$. Consequently, $\delta_\alpha \in \mathbb{R}$.

■ **Proof of Lemma 6.**

Here, we provide the proof for the case when the cost are intermediate ($d = 3$). We will explain how the insights easily generalise to the situation when cost are low ($d > 3$) along the way. Details of the proof for $d > 3$ can be found in the Appendix.

**Case 1.** Suppose that there exists a sequence of fixed points $((\tilde{q_\alpha})_n, (\tilde{q_\beta})_n)$ of the modified best reponse and $\tilde{n} \in \mathbb{N}$ such that $q_\beta(\mathrm{BR}(((\tilde{q_\alpha})_n, (\tilde{q_\beta})_n))) \geq \tau$ for any $n \geq \tilde{n}$.

First, we will derive a lower bound for $\delta_\alpha$ by rewriting the balance condition (10) using the same three observations that we used to derive the generalised Lambert equations in Theorem 2.[15] First, the total amount of information $I_\lambda$ acquired by a group $\lambda$ is proportional to the square of the information acquired by the indifferent voter type $\tilde{y} = \Pr(\alpha|\text{piv})$. As a consequence,

$$\begin{aligned} &\lim_{n\to\infty} (q_\alpha(\mathrm{BR}(\sigma_n)) - q_\beta(\mathrm{BR}(\sigma_n)))n^{\frac{1}{2}} \\ = \;& [\frac{4(d-1)}{d}\frac{f(\tilde{y})}{h'(\tilde{y})}(\Pr(G) - k\Pr(g))x(\tilde{y}, G)^2 + O(\Pr(\text{piv}|\sigma_n)^{\frac{2}{d}})o(1)]n^{\frac{1}{2}} \end{aligned} \tag{50}$$

where the equality on the second line follows from (10) and then restates the equation (28) from Lemma 4. Second, Claim 1 together with 25 and the strict monotonicity of $\psi$ implies that

$$\lim_{n\to\infty} \Pr(\alpha|\text{piv}; ((\tilde{q_\alpha})_n, (\tilde{q_\beta})_n) = p_\tau. \tag{51}$$

The equation (51) on the one hand pins down limit of the indifferent voter type $\tilde{y} = \Pr(\alpha|\text{piv})$. So, we can express how much information $\tilde{y}$ acquires by use of the first-order condition (13) for $p_\tau$, for example for group $G$,

$$x(\tilde{y}, G)^2 = p_0\Pr(\text{piv}|\alpha)(1 - p_\tau) + (1 - p_0)\Pr(\text{piv}|\beta)p_\tau\frac{1}{\kappa}, \tag{52}$$

where we used that a standing assumption that $d = 3$ or equivalently that $c(x) = \kappa x^d$. On the other hand, the equation (51) pins down the relation between the margin of victory in $\alpha$ and the margin of victory in $\beta$, and therefore of $\tilde{\delta}_\alpha$ and $\tilde{\delta}_\beta$, see the equations (31) and (32). Third, we use the local limit theorem to express the probability of the election being tied in $\omega$ as a function of $\tilde{\delta}_\omega$; see Lemma 9. Taken together, we can express the information acquired by the indifferent voter $\tilde{y}$, and hence the net information acquired, as a function of $\tilde{\delta}_\alpha$ and $\tilde{\delta}_\beta$. Furthermore, we can express $\tilde{\delta}_\alpha$ as a function of $\tilde{\delta}_\beta$. So,

$$\begin{aligned} &\lim_{n\to\infty}[\frac{4(d-1)}{d}\frac{f(\tilde{y})}{h'(\tilde{y})}(\Pr(G) - k^{\frac{2}{d-1}}\Pr(g))x(\tilde{y}, G)^2 \\ = \;& [\frac{4(d-1)}{d}\frac{f(\tilde{y})}{h'(\tilde{y})}(\Pr(G) - k^{\frac{2}{d-1}}\Pr(g))2\phi(\tilde{\delta}_\beta)(p_\tau(1 - p_0))\frac{1}{\kappa} \end{aligned} \tag{53}$$

[15]The reader may want to jump to the derivation (54) directly.

where the equation on the first line follows from Lemma 3[16] and the equation on the second line follows from the equations (52) and (31) and the local limit theorem Lemma 9 .[17] Now,

$$\begin{aligned}
\delta_\alpha &= \lim_{n\to\infty} |q_\alpha(\mathrm{BR}(\sigma_n)) - \tau| n^{\frac{1}{2}} \\
&\geq \lim_{n\to\infty} (q_\alpha(\mathrm{BR}(\sigma_n)) - q_\beta(\mathrm{BR}(\sigma_n))) n^{\frac{1}{2}} \\
&\geq [\frac{4(d-1)}{d}\frac{f(\tilde{y})}{h'(\tilde{y})}(\Pr(G) - k\Pr(g))2(p_\tau(1-p_0)] n^{\frac{1}{2}} \\
&= \nu(p_\tau(1-p_0) \\
&> \eta^{\frac{1}{2}} \\
&= \delta_\alpha, \qquad (54)
\end{aligned}$$

The equality on the third line follows from the equations (50) and (53) and the assumption that $\tilde{\delta_\beta} = 0$. The equality on the fourth line follows from the definition of $\nu$. The inequality on the fifth line is the assumption that $(p_\tau(1-p_0)\nu > \eta^{\frac{1}{2}}$. Now, the inequality (54) is a contradiction since clearly $\delta_\alpha = \delta_\alpha$.

Loosely speaking, (54) shows that the margins of victory and (twice) the net information acquired under the best response are not 'balanced' for any potential non-interior fixed point. The incentives to acquire information are so large when the electorate is split in state $\beta$, this is when $(\tilde{q}_\beta)_n = \tau$, that the difference in the margin of the victory under the best response is larger than under the candidate fixed point $((\tilde{q}_\alpha)_n, (\tilde{q}_\beta)_n)$. Intuitively, when the cost is lower ($d > 3$), voters acquire even more information, and the margin of victory under the best response is even larger. Details of the proof when $d > 3$ are given in the Appendix.

**Case 2.** Suppose that there exists a sequence of fixed points $(\tilde{q_\alpha}_n, \tilde{q_\beta}_n)$ of the modified best reponse and $\tilde{n} \in \mathbb{N}$ such that $q_\alpha(\mathrm{BR}((\tilde{q_\alpha}_n, \tilde{q_\beta}_n))) \leq \tau$ for any $n \geq \tilde{n}$.
For the ease of exposition, consider the case when $p_0 \neq p_\tau$, and suppose w.l.o.g. that $p_0 > p_\tau$. The case $p_\tau = p_0$ is proven in the Appendix. Now, $\tilde{\delta_\alpha} = 0$ implies that the margin of victory in $\alpha$ is zero, hence the probability of the election being tied is weakly larger in $\alpha$ than in $\beta$. This implies that the posterior conditional on being pivotal weakly exceeds the prior, i.e. $\Pr(\alpha|\mathrm{piv}) \geq p_0$. It follows from the assumption that $p_0 > p_\tau$ that $\Pr(\alpha|\mathrm{piv}) > p_\tau$. But then it follows from the equation (25) that under the best response the margin of victory in $\alpha$ is strictly positive. But then it follows from the definition that the modified best response to the fixed point does not coincide with the fixed point. This yields a contradiction. ■

[16] Note that when $d = 3$, we have $[O(\Pr(\mathrm{piv}|\sigma_n)^{\frac{2}{d-1}})o(1)]n^{\frac{1}{2}} = 0$, since the probability of being pivotal is strictly bounded above by $2n^{-\frac{1}{2}}\phi(0)$ for $n$ sufficiently large (see e.g. Lemma 2).

[17] More precisely,

$$\begin{aligned}
\lim_{n\to\infty} x(\tilde{y}, G)^{d-1} n^{\frac{1}{2}} &= n^{\frac{1}{2}} p_0 \Pr(\mathrm{piv}|\alpha)(1-p_\tau) + (1-p_0)\Pr(\mathrm{piv}|\beta)p_\tau \frac{1}{\kappa} \\
&= (p_0\phi(\tilde{\delta}_\alpha)(1-p_\tau) + (1-p_0)\phi(\tilde{\delta}_\beta)p_\tau)\frac{1}{\kappa} \\
&= \left[\phi(\tilde{\delta}_\beta)(p_0(1-p_\tau)\frac{1-p_0}{p_0}\frac{p_\tau}{1-p_\tau} + (1-p_0)p_\tau)\right]\frac{1}{\kappa} \\
&= 2\phi(\tilde{\delta}_\beta)(1-p_0)p_\tau\frac{1}{\kappa},
\end{aligned}$$

where the equality on the first line restates the equation (52). The equality on the second line follows from Lemma 1. The equality on the third line follows from the equation (31).

## 3.3 Utilitarian Welfare

The outcome $z \in \{A, B\}$ is a *utilitarian outcome* in state $\omega$ if $z$ maximizes ex-ante expected utility of voters across outcomes. It follows from the specification of the utilities of each voter type in (2) and (3) that $\mathrm{E}(u(t, A, \alpha)) = \Pr(G)\mathrm{E}_F(y)$, that $\mathrm{E}(u(t, B, \alpha)) = \Pr(g)k\mathrm{E}_F(y)$, that $\mathrm{E}(u(t, A, \beta)) = \Pr(G)(1 - \mathrm{E}_F(y))$ and that $\mathrm{E}(u(t, B, \beta)) = \Pr(g)k(1 - \mathrm{E}_F(y))$. Hence,

$$
\begin{aligned}
& \mathrm{E}(u(t, A, \omega)) > \mathrm{E}(u(t, B, \omega)) \\
\Leftrightarrow \quad & \begin{cases} \Pr(G) - \Pr(g)k > 0 & \text{if} \quad \omega = \alpha, \\ \Pr(G) - \Pr(g)k < 0 & \text{if} \quad \omega = \beta. \end{cases}
\end{aligned}
$$

So the utilitarian outcome is $A$ in $\alpha$ and $B$ in $\beta$ if $\Pr(R) - \Pr(L)k > 0$. The utilitarian outcome is $B$ in $\alpha$ and $A$ in $\beta$ if $\Pr(R) - \Pr(L)k < 0$. For any sequence of strategies, let $\delta'_\omega = \lim_{n\to\infty} q_\omega - \tau$. N

**Lemma 7.** *For any sequence of strategies $\sigma_n$, we have $\lim_{n\to\infty} \Pr(A \text{ is elected}|\omega; \sigma_n) = \Phi((\frac{2}{\tau(1-\tau)})^{\frac{1}{2}} \delta_\omega)$ where $\Phi$ is the cumulative distribution function of the standard normal distribution and $W^\eta$ is the generalised Lambert function (see Theorem 2).*

**Proof.** In the Appendix. ■

### Low Cost ($d > 3$).

**Theorem 4.** *Let $d > 3$. Consider any equilibrium sequence $\sigma_n$ for which there exists $\bar{n} \in \mathbb{N}$ such that $\mathrm{sgn}(q_\alpha - \tau) \neq \mathrm{sgn}(q_\beta - \tau)$ for all $n \geq \bar{n}$. If $\mathrm{sgn}(\Pr(G) - k^{\frac{2}{d-1}}\Pr(g)) = \mathrm{sgn}(\Pr(G) - k\Pr(g)$,*

$$
\begin{aligned}
\lim_{n\to\infty} \Pr(A \text{ is elected}|\alpha; \sigma_n) &= 1, \\
\lim_{n\to\infty} \Pr(B \text{ is elected}|\beta; \sigma_n) &= 1.
\end{aligned}
$$

*If $\mathrm{sgn}(\Pr(G) - k^{\frac{2}{d-1}}\Pr(g)) \neq \mathrm{sgn}(\Pr(G) - k\Pr(g)$,*

$$
\begin{aligned}
\lim_{n\to\infty} \Pr(A \text{ is elected}|\alpha; \sigma_n) &= 0, \\
\lim_{n\to\infty} \Pr(B \text{ is elected}|\beta; \sigma_n) &= 0.
\end{aligned}
$$

**Proof.** Consider any equilibrium sequence $\sigma_n$ for which there exists $\bar{n} \in \mathbb{N}$ such that $\mathrm{sgn}(q_\alpha - \tau) \neq \mathrm{sgn}(q_\beta - \tau)$ for all $n \geq \bar{n}$.

**Claim 1.** For any $\omega \in \{\alpha, \beta\}$, it holds $\lim_{n\to\infty} \delta_\omega \in \{\infty, -\infty\}$.

Suppose w.l.o.g. that $\delta_\beta \in \mathbb{R}$. Suppose also that $\delta_\alpha \in \{\infty, -\infty\}$. At first, we will lead the second assumption to a contradiction. Under both assumptions,

$$
\begin{aligned}
\lim_{n\to\infty} \frac{\Pr(\alpha|\mathrm{piv}; \sigma_n)}{\Pr(\beta|\mathrm{piv}; \sigma_n)} &= \frac{p_0}{1 - p_0} \frac{\Pr(\mathrm{piv}|\alpha; \sigma_n) n^{\frac{1}{2}}}{\Pr(\mathrm{piv}|\beta; \sigma_n) n^{\frac{1}{2}}} \\
&= \frac{\phi(\delta_\alpha)}{\phi(\delta_\beta)} \\
&= 0,
\end{aligned}
$$

where the equality on the second line follows from Lemma 1 and the equality on the third line follows from the assumption that $\delta_\beta \in \mathbb{R}$ and that $\delta_\alpha \in \{\infty, -\infty\}$. But, then it follows from the equation (26) that $\lim_{n\to\infty} q_\beta = \psi(0) > \tau$, hence $\delta_\beta = \infty$. This contradicts with the initial assumption that

$\delta_\beta \in \mathbb{R}$. Hence, $\delta_\beta \in \mathbb{R}$ implies that $\delta_\alpha \in \mathbb{R}$. Then, the probability that the election is tied is of the order of $n^{-\frac{1}{2}}$, that is

$$\begin{aligned} \lim_{n\to\infty} \Pr(\text{piv}|\sigma_n)n^{\frac{1}{2}} &= \lim_{n\to\infty} [p_0 \Pr(\text{piv}|\alpha;\sigma_n) + (1-p_0)\Pr(\text{piv}|\beta;\sigma_n)]n^{\frac{1}{2}} \\ &= p_0\phi(\delta_\alpha) + (1-p_0)\phi(\delta_\beta) \in \mathbb{R}, \end{aligned} \tag{55}$$

where the equality on the second line follows from Lemma 1. Note that $\delta_\alpha \in \mathbb{R}$ implies that $\lim_{n\to\infty} q_\alpha = \tau$. Since $\psi$ is strictly increasing, $\lim_{n\to\infty} q_\alpha = \tau$ together with the equation (25) implies that $\lim_{n\to\infty} \Pr(\alpha|\text{piv};\sigma) = p_\tau \in (0,1)$. Now, the marginal cost $c'(x(\tilde{y},G))$ of the type $\tilde{y} = \Pr(\alpha|\text{piv};\sigma_n)$ are of the order of $n^{-\frac{1}{2}}$ since

$$\begin{aligned} \lim_{n\to\infty} c'(x(\tilde{y},G))^{d-1}n^{\frac{1}{2}} &= \lim_{n\to\infty} \Pr(\text{piv}|\sigma_n)(\tilde{y}(1-\tilde{y}) + (1-\tilde{y})\tilde{y})n^{\frac{1}{2}} \\ &= 2p_\tau(1-p_\tau)(p_0\phi(\delta_\alpha) + (1-p_0)\phi(\delta_\beta)) \in \mathbb{R}, \end{aligned}$$

where the equality on the first line follows from the first-order condition (13). The equality on the second line follows from the equation (55) and from the earlier implication $\lim_{n\to\infty} \Pr(\alpha|\text{piv};\sigma) = p_\tau \in (0,1)$. Since $c'(x) = x^{d-1}$, we see that for any $d > 3$, the square of the information acquired by the type $(\tilde{y}, G)$, that is $x(\tilde{y},G)^2$, is of an order larger than $n^{-\frac{1}{2}}$, that is

$$\lim_{n\to\infty} x(\tilde{y},G)^2 n^{\frac{1}{2}} = \infty. \tag{56}$$

It follows from Lemma 3 that the difference in the expected vote share for $A$ in $\alpha$ and the expected vote share for $A$ in $\beta$ is of an order larger than $n^{-\frac{1}{2}}$, that is

$$\lim_{n\to\infty} (q_\alpha - q_\beta)n^{\frac{1}{2}} = \infty. \tag{57}$$

The assumption that $\text{sgn}(q_\alpha - \tau) \neq \text{sgn}(q_\beta - \tau)$ implies that $\lim_{n\to\infty}(q_\alpha - q_\beta)n^{\frac{1}{2}} = |\delta_\alpha| + |\delta_\beta|$. However, this contradicts with the earlier observation that $\delta_\alpha \in \mathbb{R}$ or with the initial assumption that $\delta_\beta \in \mathbb{R}$. Hence, $\delta_\alpha \in \{\infty, -\infty\}$. This finishes the proof of the claim.

Suppose that $\text{sgn}(\Pr(G) - k^{\frac{2}{d-1}}\Pr(g)) = \text{sgn}(\Pr(G) - k\Pr(g)$. W.l.o.g. suppose that $\Pr(G) - k\Pr(g) > 0$. It follows from Lemma 4 that $q_\alpha > q_\beta$ for any $n$ sufficiently large. Hence, $\delta_\beta \geq \delta_\beta$. It follows from the assumption that $\text{sgn}(q_\alpha - \tau) \neq \text{sgn}(q_\beta - \tau)$ and Claim 1 that $|\delta_\alpha| = \infty$ and $|\delta_\beta| = -\infty$. Then, it follows from Lemma 7 that the probability that $A$ gets elected in $\alpha$ converges to $\Phi((\frac{2}{\tau(1-\tau)})^{\frac{1}{2}}\delta_\alpha) = 1$. Similarly, the probability that $A$ gets elected in $\beta$ converges to $\Phi((\frac{2}{\tau(1-\tau)})^{\frac{1}{2}}\delta_\beta) = 0$. All other cases are proved analogously. ■

## Intermediate Cost ($d = 3$).

For intermediate cost of information ($d = 3$), we illustrate how the efficiency of the election depends on the primitives of the model (captured by $\nu = \frac{8(d-1)}{d}\frac{f(p_\tau)}{h'(p_\tau)}(\Pr(G) - k\Pr(g))\frac{1}{\kappa}$ and $\eta = \Big[\ln(\frac{p_\tau}{1-p_\tau}) - \ln(\frac{p_0}{1-p_0})\Big]2\tau(1-\tau)$ ):

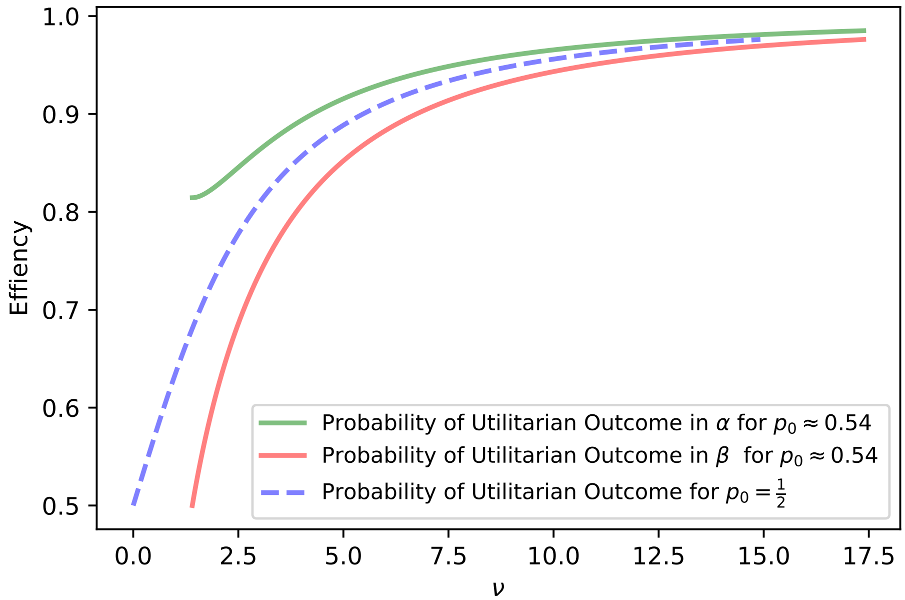


Figure 2: Let $d = 3$ and $\tau = p_\tau = \frac{1}{2}$. Let $\Pr(G) - k\Pr(g) > 0$. Figure 2 illustrates the utilitarian efficiency of the unique limit equilibrium for which, in each state, the utilitarian outcome is more likely to be elected as a function of $\nu$ and $\eta$: first, when the prior is unbiased relative to the voting rule (blue dotted line); second, when the prior is biased relative to the voting rule (red and green line).

The following corollary provides the explicit formula for the distribution of the election outcome in each state in the Lambert-type limit equilibria of Theorem 2, hence, in particular, the explicit formula underlying the graphs in Figure 2.

**Corollary 1.** *Let $d = 3$. Let $\Pr(G) - k\Pr(g) > 0$.*[18] *For any equilibrium sequence such that there exists $\bar{n} \in \mathbb{N}$ and $\operatorname{sgn}(q_\alpha - \tau) \neq \operatorname{sgn}(q_\beta - \tau)$ for any $n \geq \bar{n}$: the probability that $A$ gets elected in $\alpha$*[19] *converges to*

$$\Phi\left(\left(\frac{2}{\tau(1-\tau)}\right)^{\frac{1}{2}} W^\eta(\nu p_0(1-p_\tau))\right) \qquad \text{if } \eta > 0,$$

$$\Phi\left(\left(\frac{2}{\tau(1-\tau)}\right)^{\frac{1}{2}} (W^\eta(\nu p_\tau(1-p_0))^2 - \eta)^{\frac{1}{2}}\right) \qquad \text{if } \eta < 0.$$

*The probability that $A$ gets elected in $\beta$ converges to*

$$\Phi\left(\left(\frac{2}{\tau(1-\tau)}\right)^{\frac{1}{2}} (W^\eta(\nu p_0(1-p_\tau))^2 + \eta)^{\frac{1}{2}}\right) \qquad \text{if } \eta > 0,$$

$$-\Phi\left(\left(\frac{2}{\tau(1-\tau)}\right)^{\frac{1}{2}} W^\eta(\nu p_\tau(1-p_0))\right) \qquad \text{if } \eta < 0.$$

*where $\Phi$ is the cumulative distribution function of the standard normal distribution.*

[18] For $\Pr(G) - k^{\frac{2}{d-1}}\Pr(g) < 0$, the formulas (58) below describe the probability that $B$ gets elected in $\alpha$.
[19] Formulas for the probability that $A$ gets elected in $\beta$ are derived analogously.

**Proof.** Note that $\eta > 0$ is equivalent to $p_0 < p_\tau$. Consider the case when $\eta > 0$. Consider any equilibrium sequence such that there exists $\bar{n} \in \mathbb{N}$ and $\text{sgn}(q_\alpha - \tau) \neq \text{sgn}(q_\beta - \tau)$ for any $n \geq \bar{n}$. It follows from Theorem 2 that the limit of the equilibrium sequence satisfies the Lambert equation (33) if $\eta > 0$. Hence, the limit of the sequence is fully described by

$$\delta_\alpha = W^\eta(\nu p_0(1 - p_\tau)).$$

It follows from the assumption that $\Pr(G) - k^{\frac{2}{d-1}} \Pr(g) > 0$ and from Lemma 4 that $q_\alpha > q_\beta$ for any $n$ sufficiently large. The assumption that $\text{sgn}(q_\alpha - \tau) \neq \text{sgn}(q_\beta - \tau)$ for any $n \geq \bar{n}$ implies that $q_\alpha > \tau$ for any $n$ sufficiently large. It follows from Lemma 7 that the probability that $A$ gets elected in $\alpha$ converges to

$$\Phi((\frac{2}{\tau(1-\tau)})^{\frac{1}{2}} \delta_\alpha) \quad = \quad \Phi((\frac{2}{\tau(1-\tau)})^{\frac{1}{2}} W^\eta(\nu p_0(1 - p_\tau))). \tag{58}$$

The formulas for all other cases are derived analogously. ■

# 4 Implementation of Social Choice Rules

This section characterises which social choice rules are implementable by choice of the cost function $c$ for information. For each $\rho \in \mathbb{R}_+$, define the (state-dependent) Bergson welfare function $B_\rho : ([0,1] \times \{G, g\})^{2n+1} \times \{\alpha, \beta\} \to \mathbb{R}$ (Burk [1936]) by

$$B_\rho((y_i, \lambda_i), \omega) = \sum_{i=1,\ldots,2n+1} \begin{cases} \sum_{i:\lambda_i = G}(1 - y_i)^\rho + \sum_{i:\lambda_i = g} -(k(1 - y_i))^\rho & \text{if} \quad \omega = \alpha, \\ \sum_{i:\lambda_i = G} -(y_i)^\rho + \sum_{i:\lambda_i = g}(k y_i)^\rho & \text{if} \quad \omega = \beta \end{cases}$$

Fix some $\omega \in \{\alpha, \beta\}$. For any $\rho$, the restriction of the Bergson welfare function to $B_\rho(-, \omega)$ for some $\omega \in \{\alpha, \beta\}$, induces a mapping from preference profiles of the voters to (social) preferences over election outcomes. Roberts [1980], Theorem 6 axiomatizes the mappings from strictly positive utility profiles to (social) preferences over outcomes that are implementable by the Bergson welfare functions: the implementable mappings are the only ones that satisfy continuity, anonymity, neutrality, monotonicity, separability and scale invariance.[20] A social choice rule is a function $f : ([0,1] \times \{G, g\})^{2n+1} \times \{\alpha, \beta\} \to 2^{\{A,B\}}$. For each $\rho \in \mathbb{R}_+$, define the (state-dependent) Bergson social choice rule $f_\rho$ by

$$f_\rho((y_i, \lambda_i), \omega) = \begin{cases} \{A\} & \text{if} \quad B_\rho((y_i, \lambda_i), \omega) > 0, \\ \{A, B\} & \text{if} \quad B_\rho((y_i, \lambda_i), \omega) = 0, \\ \{B\} & \text{if} \quad B_\rho((y_i, \lambda_i), \omega) < 0. \end{cases}$$

A cost function $c$ robustly implements a social choice rule $f$ if given $c$, for any $p_0 \in (0, 1)$, there exists an equilibrium sequence $\sigma_n$ such that

$$\lim_{n \to \infty} \Pr_{(t_i)_{i=1,\ldots,2n+1}}(\text{some } x \in f_\rho((t_i), \omega) \text{ is elected} | \omega; \sigma_n) = 1.$$

**Lemma 8.** *Let $\rho = \frac{2}{d-1}$. The cost function $c(x) = x^d$ with $d > 3$ robustly implement the Bergson social choice rule with $\rho = \frac{2}{d-1}$.*

**Proof.** Note that the value of the Bergson welfare function equals the sum of the utility differences of the type profile in a state $\omega$, that is

$$\mathrm{B}_\rho((y_i, \lambda_i), \omega) = \sum_{i=1,\ldots,2n+1} (u(t_i, A, \omega) - u(t_i, A, \omega))^\rho. \tag{59}$$

[20] The exact definitions of these axioms are given in the cited papers.

Hence, $B_\rho((y_i,\lambda_i),\omega) > 0$ if and only if $\frac{1}{2n+1}\sum_{i=1,\ldots,2n+1}(u(t_i,A,\omega)-u(t_i,A,\omega))^\rho > 0$. It follows from the weak law of large numbers that $\lim_{n\to\infty} B_\rho((y_i,\lambda_i),\omega) > 0$ almost surely when $E_t((u(t,A,\omega)-u(t,B,\omega))^\rho) > 0$ and that $\lim_{n\to\infty} B_\rho((y_i,\lambda_i),\omega) < 0$ almost surely when $E_t((u(t,A,\omega)-u(t,B,\omega))^\rho) < 0$. Now,

$$\begin{aligned} E_t((u(t,A,\alpha)-u(t,B,\alpha))^\rho) &= \Pr(G)E((1-y)^\rho) - \Pr(g)E(k^\rho(1-y)^\rho) \\ &= \Pr(G) - k^\rho \Pr(g)E((1-y)^\rho), \\ E_t((u(t,A,\beta)-u(t,B,\beta))^\rho) &= -\Pr(G)E(y^\rho) + \Pr(g)E(k^\rho y^\rho) \\ &= -(\Pr(G) - k^\rho\Pr(g))E(y^\rho). \end{aligned}$$

**Case 1.** Let $\Pr(G) - k^\rho \Pr(g) > 0$.
First, it follows from the above observations that $\lim_{n\to\infty} B_\rho((y_i,\lambda_i),\omega) > 0$ almost surely in $\alpha$ and $\lim_{n\to\infty} B_\rho((y_i,\lambda_i),\omega) < 0$ almost surely in $\beta$. Hence the Bergson social choice correspondence selects $\{A\}$ almost surely in $\alpha$ and to $\{B\}$ almost surely in $\beta$ as $n\to\infty$. Recall that $\rho = \frac{2}{d-1}$. It follows from Theorem 2 and Theorem 3 and the assumption of this case that there exists an equilibrium sequence for which $A$ is elected in $\alpha$ with probability converging to 1 and for which $B$ is elected in $\beta$ with probability converging to 1. This shows that $c(x) = x^d$ implements $f_\rho$ asymptotically. ■

Clearly, besides the Bergson social choice rules with $0 < \rho < 1$, also the constant social choice rule that implements $A$ in both states with probability 1 and the constant social choice rule that implements $B$ in both states with probability 1 is robustly implementable, namely through the trivial equilibria in which all citizens vote for the same alternative. The Condorcet Jury Theorem that holds in the setup of this model when cost is zero (see Bhattacharya [2013]) implies that the Bergson social choice rule with parameter $\rho = 0$ (which corresponds to 'full information equivalence') is robustly implemented by the cost function $c(x) = 0$. The following theorem shows that however besides the constant and the Bergson social choice rules no social choice rules are robustly implementable.

**Theorem 5.** *Let $\psi(p_0) \neq \tau$.*

1. *The only social choice rules that are robustly implementable by some cost function are the constant social choice rules and the Bergson social choice rules with parameter $0 \leq \rho < 1$.*[21]

2. *For the cost function $c(x) = x^3$, there exists an equilibrium sequence for which the election outcome coincides with the Bergson social choice rule with $\rho = 1$ (utilitarian welfare) with probability strictly larger than $\frac{1}{2}$ as $n\to\infty$. When $\eta\to\infty$, this probability converges to 1.*

**Proof.**

1. We use results from the following Section 6 that finishes the characterisation of all equilibria for any cost function $c(x) = x^d$: Recall the assumption that $\psi(p_0) \neq \tau$ from the statement of this theorem.
   Let $d < 3$. Corollary 2 states that all equilibrium sequences implement a constant outcome across states.
   For $d = 3$, there exist at most three distinct limit equilibria: a limit equilibrium that corresponds to voting according to the prior (see Theorem 6), a Lambert type limit equilibrium (see Theorem 2) and a limit equilibrium for which a fixed alternative $x \in \{A, B\}$ is elected with probability $\frac{1}{2} \leq p_\omega < 1$ in each state (see Remark 2). Importantly, we conclude that for any $d \leq 3$, there is no non-constant distribution of election outcomes such that for any prior $p_0 \in (0,1)$ there exists a limit equilibrium that implements the distribution. Hence, there does not exist a non-constant social choice rule that is robustly implemented by a cost function

[21] Interestingly, a similar result has been independently shown by Eguia and Xefteris [2018] in a different context. The limit equilibria under vote-buying correspond to the Bergson social choice rules (without restriction on $\rho$).

$c(x) = x^d$ with $d \leq 3$.
For $d > 3$, it follows from Theorem 8 and it proof that any limit equilibrium either implements the constant social choice rule that selects the outcome that is favored by the prior or the equilibrium sequence satisfies $\text{sgn}(q_\alpha - \tau) \neq \text{sgn}(q_\beta - \tau)$. It follows from Lemma 8 that, for $d > 3$, any equilibrium sequence with $\text{sgn}(q_\alpha - \tau) \neq \text{sgn}(q_\beta - \tau)$ robustly implements the Bergson social choice rule with $\rho = \frac{2}{d-1}$. Note that the function that maps $d > 3$ to $\frac{2}{d-1}$ is surjective on $(0, 1)$. It follows from the Condorcet Jury Theorem (Bhattacharya [2013]) that the Bergson social choice rule with $\rho = 0$ (full information outcome) is robustly implemented by the cost function $c(x) = 0$.

2. The claim directly follows from Theorem 3 and from Corollary 1 which also, for each state, specifies the limit of the probabilities that the election outcome in the equilibrium sequence is utilitarian as $n \to \infty$ as a function of the primitives.

■

# 5 Determinants of Utilitarian Welfare

In this section, we study the social welfare implications of varying the preference distribution in two ways: First, we capture the degree of the conflict of interest or the competitiveness of the election by the difference in the aggregate intensities of both groups and show that social welfare unambiguously decreases with the conflict of interest. However, since the election is more competitive when the conflict of interest is higher, we show that also information acquisition increases, effectively mitigating the welfare loss.
*Conflict of Interest.* Let $d = 3$. Consider the Lambert-type limit equilibrium $\sigma_C$ as a function of the *conflict of interest* $C = (\Pr(G) - k\Pr(g))^{-1} > 0$. Fix some $C'$ and let agents play $\sigma_{C'}$. Suppose that the conflict of interest increases from $C'$ to $C > C'$ (everything else being fixed). Then, the election is in expectation more close to being tied as $n \to \infty$: formally,

$$\begin{aligned}
0 \quad &< \quad \Big[\Pr(\sigma(s,t) = 1|\alpha; \sigma_{C'}, C) - \Pr(\sigma(s,t) = 1|\alpha; \sigma_{C'}, C)\Big] n^{\frac{1}{2}} \\
&= \quad \lim_{n\to\infty} 2(\Pr(G) I_G(\sigma_{C'}) - \Pr(\sigma_{C'}(g) I_g(\sigma'_C)) n^{\frac{1}{2}} \\
&= \quad \lim_{n\to\infty} \frac{4(d-1)}{d} \frac{f(\tilde{y})}{h'(\tilde{y})} C^{-1} x(\tilde{y}, G)^2 n^{\frac{1}{2}} \\
&< \quad \lim_{n\to\infty} \frac{4(d-1)}{d} \frac{f(\tilde{y})}{h'(\tilde{y})} (C')^{-1} x(\tilde{y}, G)^2 n^{\frac{1}{2}} \\
&= \quad \lim_{n\to\infty} 2(\Pr(G) I_G(\sigma_C) - \Pr(\sigma_{C'}(g) I_g(\sigma_C)) n^{\frac{1}{2}} \\
&= \quad \lim_{n\to\infty} \Big[\Pr(\sigma(s,t) = 1|\alpha; \sigma_{C'}, C') - \Pr(\sigma(s,t) = 1|\alpha; \sigma_{C'}, C)\Big] n^{\frac{1}{2}} \in \mathbb{R}.
\end{aligned}$$

where the equality on the second line restates the equation (10). The equality on the third and fifth line follow from Lemma 4. The inequality on the fourth line follows from $C > C'$.We see that the election is more close to being tied, because the difference in the amount of information acquired by the majority group, that is $\Pr(G)I_G$, and the amount of information acquired by the minority group, that is $\Pr(g)I_g$ is smaller. Hence, the screening problem worsens when the conflict of interest increases ('screening effect'). However, as the election is also in expectation more close to being tied, as an indirect effect the incentives to acquire information increase ('competition effect') or, put differently, the free-rider problem relaxes. However, intuitively, the indirect effect on information acquisition is second-order and dominated by the direct effect of a higher conflict of interest. This is, in fact, true and Figure 3 shows how the utilitarian efficiency of the Lambert-type limit equilibrium is strictly decreasing as a function of the conflict of interest $C$. Formally, this follows since the reduced form $\delta_\omega$ of the Lambert-type equilibrium is strictly decreasing in $C$ (compare with Figure 1) and since the utilitarian welfare of the equilibrium is strictly increasing in $\delta_\omega$ as Lemma 7 shows.

Moreover, the election outcome converges to the random election outcome in each state as $C \to \infty$.

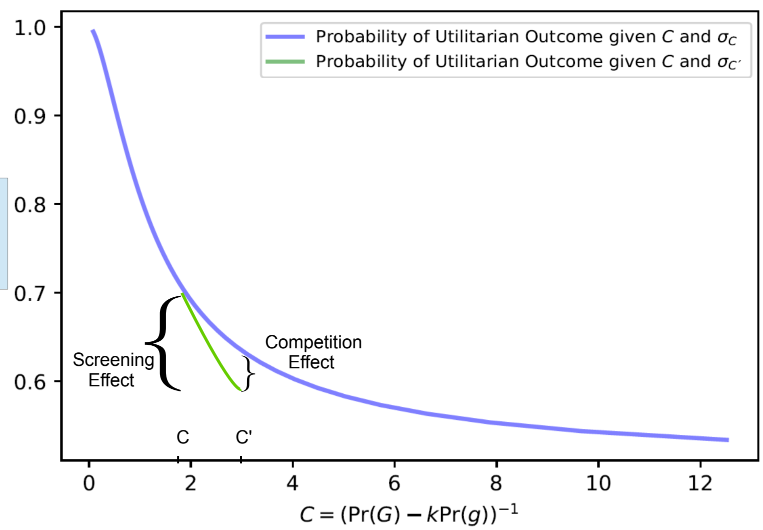


Figure 3: Let $d = 3$ and $\tau = p_\tau = p_0 = \frac{1}{2}$. Let $\Pr(G) - k\Pr(g) > 0$. Figure 3 shows the utilitarian efficiency of the Lambert-type limit equilibrium of Theorem 2 as a function of $C = (\Pr(G) - k\Pr(g))^{-1} > 0$. Figure 3 shows also how the change in utilitarian efficiency as the conflict of interest increases from $C$ to $C' > C$ decomposes into a direct effect (the 'screening effect') and an indirect effect (the 'competition effect') from more information acquisition.

Let $d > 3$. Theorem 4 shows how the conflict of interest interacts with the cost of information. Whenever the utilitarian outcome is the one preferred by the minority group $g$, that is when $\Pr(G) - k\Pr(g) < 0$, the following can happen: When costs are low, that is when $d > 3$, the amount of information acquired by the majority group, that is $\Pr(G)I_G$, might exceed the amount of information acquired by the minority group, that is $\Pr(g)I_g$. This is the case when $\Pr(G) - k^{\frac{2}{d-1}}\Pr(g) > 0$; see Lemma 3). But then, the order of the voting probabilities is given by $q_\alpha > q_\beta$ for any $n$ large enough; see Lemma 4, so, in any equilibrium with $\mathrm{sgn}(q_\alpha - \tau) \neq \mathrm{sgn}(q_\beta - \tau)$ in each state the outcome prefered by the majority group is more likely. In fact, Theorem 4 shows that, in these cases, the probability that the utilitarian outcome is elected converges to zero. Figure 4 illustrates the intervals of the parameter $\Pr(G)$ for which the election outcome is inefficient when $k = 2$ as a function of the level of cost $d$. When $d$ grows to infinity, that is information becomes arbitrarily cheap, the election outcome converges to the full-information outcome. Hence, for $d \to \infty$ the minority prefered outcome is not elected.

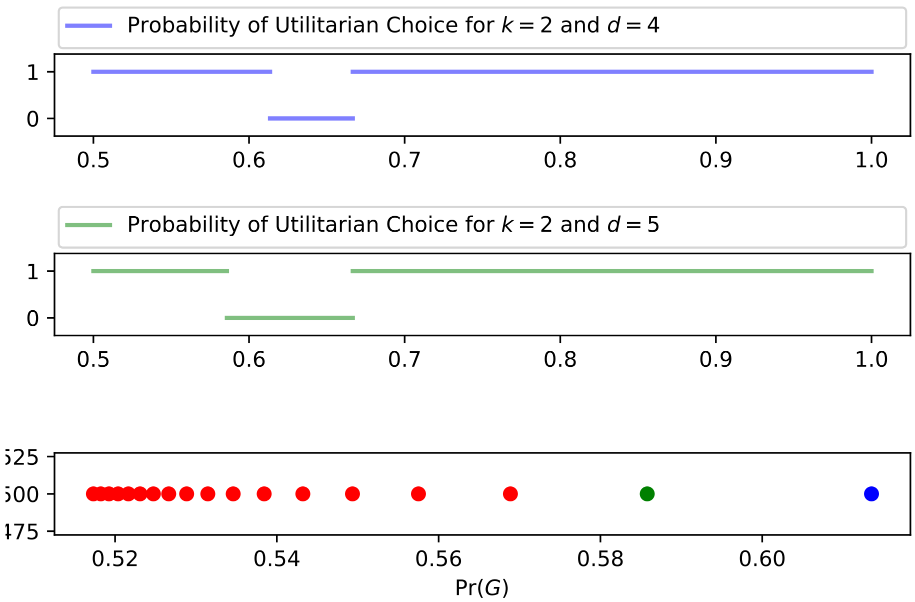


Figure 4: Let $d > 3$, $k = 2$ and $\Pr(G) > \Pr(g)$. So, the utilitarian outcome in each state is the one prefered by minority group if $\Pr(G) - 2\Pr(g) < 0 \Leftrightarrow \Pr(G) < \frac{2}{3}$. The figures illustrate the utilitarian efficiency of the election in the unique limit equilibrium for which the election outcome is not constant across states as a function of $d$ and $\Pr(G)$. The limit equilibrium is inefficient on the open interval $(\frac{2^{\frac{2}{d-1}}}{1+2^{\frac{2}{d-1}}}, \frac{2}{3})$. The third figure depicts the lower bound $\frac{2^{\frac{2}{d-1}}}{1+2^{\frac{2}{d-1}}}$ as a function of $d$; wee see that the equilibrium outcome converges to the full information outcome for $d \to \infty$.

*The Strength of Ideology and Voting Power.* Recall the ideology and taste interpretation of the payoff types: $y = i + e$ for some normalised ideology component $i = \frac{1}{2}$, and a taste component $e \in [-i, i]$. Whenever the taste component of a voter matters weakly more than ideology, that is when $|e| \geq |i|$ or equivalently when $y \in \{0, 1\}$, the voter prefers either $A$ independently of his belief about the state or $B$ independently of his belief about the state. Suppose that the distribution of types differs across groups, that is $y \sim F_\lambda$ for $\lambda \in \{G, g\}$. Denote $m(\lambda)$ the mass of types for which taste dominates ideology, that is of the types for which $e \in \{-\frac{1}{2}, \frac{1}{2}\}$. Suppose that the conditional distributions $F_\lambda(-, y \in (0,1))$ are identical across $\lambda \in \{g, G\}$. Denote by $f$ the densities of the conditional distributions $(F_\lambda)_{y \in (0,1)}$. It is easy to see that for this extension of the model, the approximation (28) of Lemma 4 generalises to

$$\begin{aligned} & q_\alpha(\mathrm{BR}(\sigma_n)) - q_\beta(\mathrm{BR}(\sigma_n)) \\ \approx \; & \frac{4(d-1)}{d} \frac{1}{h'(\tilde{y})}((1 - m(G))f(\tilde{y})\Pr(G) - k^{\frac{2}{d-1}}(1 - m(g))f(\tilde{y})\Pr(g))x(\tilde{y}, G)^2. \end{aligned} \tag{60}$$

Hence, for any strategy sequence $\sigma_n$, we have

$$\begin{aligned} q_\alpha(\mathrm{BR}(\sigma_n)) & \overset{(<)}{>} q_\beta(\mathrm{BR}(\sigma_n)) \\ \Leftrightarrow (1 - m(G))\Pr(G) - k^{\frac{2}{d-1}}(1 - m(g))\Pr(g) & \overset{(<)}{>} 0. \end{aligned} \tag{61}$$

We can understand $(1-m(G))\Pr(G)$ as the *voting power* of group $G$ and $(1-m(g))k\Pr(g)$ as the voting power of group $g$. The approximation (60) implies that the group with the higher voting power determines which outcome is more likely to be elected in each state. The following observations generalise to the extended model: For $d > 3$, it follows from Proposition 4 that the more powerful group can ensure election of the prefered outcome with probability 1 in each state. For $d = 3$, the outcome probabilities are continuous in the difference of the voting power (see Corollary 1). A group with more voters for which taste dominates ideology has a lower voting power, since $(1-m(\lambda))$ decreases. In this sense, a stronger ideology of a group is directly related to voting power and a perfect substitute for intensities or group size.

*More Consensus can be Bad.* To be inserted.

# 6 Discussion and Remarks

## 6.1 Cursed Downsian Equilibria

**Lemma 9.** *For any sequence of strategies $\sigma_n$ with $\lim_{n\to\infty} q_\omega \in (0,1)$ for any $\omega \in \{\alpha,\beta\}$: If $\lim_{n\to\infty} |q_\alpha(\sigma_n)-\tau| - |q_\beta(\sigma_n)-\tau|n = 0$ then $\lim_{n\to\infty} \Pr(\alpha|\text{piv},\sigma_n) = p_0$.*

**Proof.** In the Appendix. ■

Lemma 9 formalizes the intuition that voters cannot learn anything about the state from conditioning on the election being tied, for $n \to \infty$, if the probability that a random citizen votes $A$ in $\alpha$ is sufficiently close to the probability that a random voter random citizen votes $A$ in $\beta$.

*Low Information Acquisition implies no Learning about the State.* Intuitively, if the total amount of information $I(\lambda) = \int_{y\in[0,1]} x(y, \lambda dF(y)$ acquired by any group is very small, then the probability that a random citizen votes $A$ in $\alpha$ and the probability that a random citizen votes $A$ in $\beta$ are very close. In fact, if $\lim_{n\to\infty} I_\lambda(\sigma_n)\cdot n = 0$ for all $\lambda \in \{g, G\}$, it follows from the equality (10) that

$$\lim_{n\to\infty} (|q_\alpha(\sigma_n)-\tau| - |q_\beta(\sigma_n)-\tau|)n = 0 \tag{62}$$

which is the sufficient condition for no learning about the state from Lemma 9.
Roughly speaking, the equation (62) shows that sufficiently low information acquisition implies no learning about the state in equilibrium. We claim that generically the converse is also true.

*Generically, Voting According to the Prior implies Low Information Acquisition.* Let $|\psi(p_0)-\tau| > 2\epsilon$ for some $\epsilon > 0$: This means that a random voter prefers $A$ with probability unequal to $\tau$ under the prior (this is what we mean with 'generic'). Suppose that voters follow a strategy with $|q_\omega - \psi(p_0)| < \epsilon$ for any $\omega \in \{\alpha,\beta\}$. It follows from the assumption $|\psi(p_0)-\tau| > 2\epsilon$ that $q_\omega \neq \tau$, and hence $\frac{\psi(p_0)}{\tau}^{\tau} \frac{1-p_0}{1-\tau}^{1-\tau} < 1$. It follows from Lemma 2 that the probability of being pivotal converges to zero exponentially fast. It follows from the first-order conditions (13) and (14) that any type $t$ either acquires no information or an information of a quality $x^*(t)$ such that $c'(x^*(t)) = x^*(t)^{d-1} < 2\Pr(\text{piv}|\sigma)$. Consequently, the total information $I_\lambda$ acquired by any group $\lambda$, converges to 0 exponentially fast.

*Generically, Low Information Acquisition is Self-confirming.* We claim that voting approximately according to the prior is self-confirming. For this, consider $\epsilon > 0$ with $|\psi(p_0)-\tau| > 2\epsilon$. Recall that we can understand the best response as a function of the voting probalities $q_\alpha$ and $q_\beta$ (see the discussion at the end of Section 2). Consider any strategy with $|q_\omega - \tau| > \epsilon$ for any $\omega \in \{\alpha,\beta\}$. For example, any strategy with $|q_\omega - \psi(p_0)| < \epsilon$ satisfies this assumption. We just argued in the previous paragraph that, under the best response, $I_\lambda$ converges to zero exponentially fast for any $\lambda \in \{G, g\}$. It follows from the formula (10) that the difference $q_\alpha - q_\beta$ converges to zero exponentially fast. It follows from Lemma 9 that voters cannot learn anything from conditioning on the election being tied, for $n \to \infty$, that is $\lim_{n\to\infty} \Pr(\alpha|\text{piv}, \sigma_n) = p_0$. Recall the observation (21) that the quality of information acquired by any type $t$ converges to 0 in equilibrium. It follows from (25) and (26) that $\lim_{n\to\infty} q_\omega = \psi(\Pr(\alpha|piv)) = \psi(p_0)$. Hence, there exists $n(\epsilon)$ such that for any $n \geq n(\epsilon)$, the best response to $(q_\omega)_{\omega\in\{\alpha,\beta\}}$ is a strategy with voting probabilities $\epsilon$-close to $\psi(p_0)$. It follows from a fixed point argument, that there exists a sequence of equilibria that converge to voting according to the prior.

We call an equilibrium sequence $\sigma_n$ *cursed* if, for any $\lambda \in \{G, g\}$, the total information $I(\lambda)$ converges to zero exponentially fast under $\sigma_n$, and conclude

**Theorem 6.** *(Cursed Equilibrium Sequences)*
*If $\phi(p_0) \neq \frac{1}{2}$, there exists a cursed equilibrium sequence $\sigma_n$ and $\lim_{n\to\infty} q_\omega = \psi(p_0)$ for any $\omega \in \{\alpha, \beta\}$.*

Note that it follows from the law of large numbers that any equilibrium sequence $\sigma_n$ with $\lim_{n\to\infty} q_\omega = \psi(p_0)$ for any $\omega \in \{\alpha, \beta\}$, satisfies

$$\Pr(A \text{ is elected}|\alpha) = \Pr(A \text{ is elected}|\beta) = \begin{cases} 1 & \text{if} \quad \psi(p_0) > \tau, \\ 0 & \text{if} \quad \psi(p_0) < \tau. \end{cases} \tag{63}$$

Note that it follows from the equation (10) that for any cursed equilibrium sequence, the difference in the voting probabilities $q_\alpha$ and $q_\beta$ is converging to zero exponentially fast. Therefore, it follows from Lemma 9 that $\lim_{n\to\infty} \Pr(\alpha|\text{piv}, \sigma_n) = p_0$. It follows from the equations (25) and (26) that any cursed equilibrium sequence satisfies

$$\lim_{n\to\infty} q_\omega = \psi(p_0). \tag{64}$$

## 6.2 High Cost

**Theorem 7.** *(Equilibria when Costs are High)*
*Let $d < 3$ and $\sigma_n$ be an equilibrium sequence. Then,*

$$\lim_{n\to\infty} \Pr(A \textit{ is elected}|\alpha; \sigma_n) = \lim_{n\to\infty} \Pr(A \textit{ is elected}|\beta; \sigma_n) \in \{0, \frac{1}{2}, 1\}.$$

**Proof.**

**Claim 2.** *Let $d < 3$ and $\sigma_n$ an equilibrium sequence. Then, $\delta_\alpha((\sigma_n)) = \delta_\beta((\sigma_n))$.*

It follows from the first-order condition (13) that $c'(x(\tilde{y}, G)) = x(\tilde{y}, G)^{d-1} \in \Theta(\Pr(\text{piv}|\sigma_n))$. The probability that a voter is pivotal is stricly bounded above as a consequence of Lemma 2, $\Pr(\text{piv}|\sigma_n) \in O(n^{-\frac{1}{2}})$. As a consequence, $x(\tilde{y}, G)^2 \in O(n^{-\frac{1}{d-1}})$. From Lemma 7 it follows that the difference of the expected vote share for $A$ in $\alpha$ and the expected vote share for $A$ in $\beta$ are converging to zero at a rate larger than $n^{-\frac{1}{d-1}}$, that is $|q_\alpha - q_\beta| \in O(n^{-\frac{1}{d-1}})$. For any $d > 3$, this implies $|q_\alpha - q_\beta| \in O(n^{-\frac{1}{2}})$. So, in particular

$$\delta_\alpha - \delta_\beta = \lim_{n\to\infty} \Big[(q_\alpha - \tau) - (q_\beta - \tau)\Big] n^{\frac{1}{2}} = (q_\alpha - q_\beta) n^{\frac{1}{2}} = 0.$$

This finishes the proof of Claim 1. Now, Theorem (7) follows from Claim 1 as follows: suppose that $\delta_\alpha = \delta_\beta \in \{-\infty, \infty\}$. Then, it follows from Lemma 7 that $\lim_{n\to\infty} \Pr(\text{A is elected}|\alpha; \sigma_n) =$

$\lim_{n\to\infty} \Pr(\text{A is elected}|\beta;\sigma_n) \in \{0,1\}$. Suppose that $\delta_\alpha = \delta_\beta \in \mathbb{R}$. It follows from Lemma 1 that $\lim_{n\to\infty} \Pr(\alpha|\text{piv};\sigma_n) = p_0$. It follows from (25) and (26) that $\lim_{n\to\infty} q_\omega = \psi(p_0)$. Hence, $\psi(p_0) = \tau$, since otherwise $\delta_\omega \notin \mathbb{R}$ which contradicts the initial assumption. It follows from the strict monotonicity of $\psi$ that $p_0 = p_\tau$. Suppose that that $\Pr(G) - k^{\frac{2}{d-1}} \Pr(g) > 0$. It follows from Lemma (4) that $q_\alpha > q_\beta$ for any $n$ sufficiently large. Now, suppose that $\delta_\omega < 0$. Hence, for any $n$ sufficiently large, $q_\beta < q_\alpha < \tau$. But, then $\Pr(\alpha|\text{piv};\sigma_n) > p_0 = p_\tau$. So, it follows from the strict monotonicity of $\psi$ that $\psi(\Pr(\alpha|\text{piv};\sigma_n) > \tau$. It follows from (25) that $\lim_{n\to\infty} q_\alpha = \lim_{n\to\infty} \psi(\Pr(\alpha|\text{piv};\sigma_n) \geq \tau$. But this contradicts with the initial assumption that $\delta_\omega < 0$. Analogously, one leads the assumption that $\delta_\omega > 0$ to a contradiction. Analogously, one arrives at contradictions for the case when $\Pr(G) - k^{\frac{2}{d-1}} \Pr(g) < 0$. We conclude that $\delta_\alpha = \delta_\beta \in \mathbb{R}$ implies $\delta_\alpha = \delta_\beta = 0$. It follows from Lemma 7 that this implies that $\lim_{n\to\infty} \Pr(\text{A is elected}|\omega) = \frac{1}{2}$ for any $\omega \in \{\alpha,\beta\}$. This finishes the proof of Theorem 7. ■

**Corollary 2.** *Let $d < 3$. If $\psi(p_0) \neq \tau$, then for any equilibrium sequence, $\lim_{n\to\infty} \Pr(A \text{ is elected}|\omega) = 1$ for all $\omega \in \{\alpha,\beta\}$ or $\lim_{n\to\infty} \Pr(A \text{ is elected}|\omega) = 0$ for all $\omega \in \{\alpha,\beta\}$.*

**Proof.** W.l.o.g. let $\psi(p_0) > \tau$, that is $p_0 \geq p_\tau$. The other case is proven in the same way. Recall Claim 1 of Theorem 7. It follows from Lemma 7 that it suffices to show that $\delta_\alpha = \delta_\beta \notin\in \mathbb{R}$. Suppose the opposite. It follows from Lemma 1 that

$$\begin{aligned} \lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha)}{\Pr(\text{piv}|\beta)} &= \frac{\phi(\delta_\alpha}{\phi(\delta_\beta)} \\ &= 1. \end{aligned} \tag{65}$$

Hence, $\lim_{n\to\infty} \Pr(\alpha|\text{piv}) = p_0$. It follows from (25) and (26) that $\lim_{n\to\infty} q_\omega = \psi(p_0)$ for any $\omega \in \{\alpha,\beta\}$. But this contradicts with the initial assumption that $\delta_\omega \in \mathbb{R}$, given that $\psi(p_0) > \tau$. ■

## 6.3 Other Equilibria

Let me finish the characterization of the limit equilibrium outcomes.

**Theorem 8.** *Let $d > 3$. For any equilibrium sequence $\sigma_n$, denote $\lim_{n\to\infty} \Pr(A \text{ elected}|\omega) = z_\omega$. Then, $(z_\alpha, z_\beta) \in \{(1,0),(0,1),(0,0),(1,1)\}$.*

**Proof.** Consider any equilibrium sequence $\sigma_n$. If $\lim_{n\to\infty} \Pr(\alpha|\text{piv};\sigma_n) \neq p_\tau$, then it follows from (25) and (26) that $\lim_{n\to\infty} q_\omega \neq \tau$ for any $\omega \in \{\alpha,\beta\}$. Then, it follows from the weak law of large numbers that $((z_\alpha, z_\beta) \in \{(0,0),(1,1)\}$. If $\lim_{n\to\infty} \Pr(\alpha|\text{piv};\sigma_n) = p_\tau$, this implies

$$\begin{aligned} \frac{1-p_0}{p_0}\frac{p_\tau}{1-p_\tau} &= \lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha;\sigma_n)n^{-\frac{1}{2}}}{\Pr(\text{piv}|\alpha;\sigma_n)n^{-\frac{1}{2}}} \\ &= \frac{\phi(\delta_\alpha)}{\phi(\delta_\beta)}, \end{aligned} \tag{66}$$

whenever $\delta_\beta \in \mathbb{R}$. Together with the converse argument, this implies that $\delta_\alpha \in \{\infty, -\infty\}$ is equivalent to $\delta_\beta \in \{\infty, -\infty\}$. Now, it follows by the same line of argument as used to proof Claim 1 in the proof of Theorem 4 that

$$\lim_{n\to\infty} (q_\alpha - q_\beta) n^{\frac{1}{2}} = \infty \tag{67}$$

(see (57)). Intuitively, cost are low enough ($d > 3$) such that the net amount of total information acquired and therefore the difference in vote shares (recall 10) becomes infinitely larger than $n^{-\frac{1}{2}}$. If $\text{sgn}(q_\alpha - \tau) \neq \text{sgn}(q_\beta - \tau)$, it follows from (67) and Lemma 7 that $(z_\alpha, z_\beta) = (1,0)$. If $\text{sgn}(q_\alpha - \tau) = \text{sgn}(q_\beta - \tau)$, it follows from (67) and Lemma 7 that $(z_\alpha, z_\beta) \in \{(0,0),(1,1)\}$. ■

**Remark 2.** *Let $d = 3$ and $\Pr(G) - k \Pr(g) > 0$.*[22]

[22]The analogous remark applies when $\Pr(G) - k \Pr(g) < 0$.

1. *Theorem 2 characterizes all equilibrium sequences for which the utilitarian outcome is more likely to be elected in each state for any $n$ large.*

2. *It follows from Lemma 4 that $\Pr(G) - k\Pr(g) > 0$ implies that $q_\alpha > q_\beta$ for any $n$ sufficiently large. Consequently, it follows from Lemma 7 that any equilibrium sequence, for which the utilitarian outcome is less likely to be elected in some state, satisfies $\text{sgn}(q_\alpha - \tau) > 0 \Leftrightarrow \text{sgn}(q_\beta - \tau) > 0$ for any $n$ sufficiently large and $\text{sgn}(q_\alpha - \tau) < 0 \Leftrightarrow \text{sgn}(q_\beta - \tau) < 0$ for any $n$ sufficiently large. Therefore, the limit $\lim_{n\to\infty}(q_\alpha - q_\beta)n^{\frac{1}{2}}$ of the scaled net information is given by $(|\delta_\alpha| - |\delta_\beta|)$ (whenever $\delta_\alpha$ and $\delta_\beta$ are finite real numbers). Everything else being the same, it follows from the proof of Theorem 2 that:*

   1. *If $p_0 \geq p_\tau$, the equilibrium sequence must satisfy the equation*

   $$w - (w^2 + \eta)^{\frac{1}{2}} = \nu(p_\tau(1 - p_0)) \tag{68}$$

   *for $w = \delta_\alpha$.*

   2. *If $p_0 \leq p_\tau$, the equilibrium sequence must satisfy the equation*

   $$w - (w^2 + \eta)^{\frac{1}{2}} = \nu(p_0(1 - p_\tau)) \tag{69}$$

   *for $w = \delta_\beta$.*

   *It is easy to show that $\delta_\alpha \in \mathbb{R} \Leftrightarrow \delta_\beta \in \mathbb{R}$ (see e.g. the proof of Claim 1 after Theorem 3). If $\delta_\alpha = \delta_\beta = \infty$, it follows from Lemma 7 that $\lim_{n\to\infty}\Pr(A \textit{ is elected}|\omega) = 1$ for all $\omega \in \{\alpha, \beta\}$. If $\delta_\alpha = \delta_\beta = -\infty$, it follows from Lemma 7 that $\lim_{n\to\infty}\Pr(A \textit{ is elected}|\omega) = 0$ for all $\omega \in \{\alpha, \beta\}$.*

# 7 Literature

To be inserted.

# 8 Conclusion

In this paper, we study large elections with costly information acquisition in a canonical and general model that allows for conflict of interest between voters and supermajority rules. Unlike in earlier work, both a screening and a free-riding problem are present and it is not known from the previous literature if, in such a setting, a Downsian paradox of voting is present. The paper provides several sets of results. First, we characterize all limit equilibria and show, in particular, when non-Downsian limit equilibria exist. By doing so, we uncover a previously unknown relationship between the limit equilibria of voting games with information cost and generalizations of the product logarithm or Lambert $W$-function $W_0$. The product logarithm function and its generalizations possess useful technical properties, e.g. they follow specific differential equations. In this sense, I consider the uncovering of the relation to voting games as a technical advance that creates room for future work. Second, the paper provides several comparative statics results. Most importantly, we study the effect of varying the degree of conflict of interest of voters. Thereby, we uncover a *competition effect*, which shows that information acquisition increases with the conflict of interest. However, the competition effect is dominated by a *screening effect*, that is, the screening problem worsens and hence, overall a higher conflict of interest implies lower social welfare in equilibrium.
Third, based on the characterization of the limit equilibria, we characterize the social welfare functions that are asymptotically implementable (Theorem 5). This characterization relates the equilibria of elections with costly information to a specific class of social welfare rules, the Bergson social

welfare rules with parameters $0 \leq \rho < 1$. Importantly, the Bergson social welfare rules possess an axiomatization that was provided by Roberts [1980]. Therefore, the importance of Theorem 5 is in building a bridge between the typically distant worlds of the axiomatic social choice theory and the game-theoretic analysis of elections.

# 9 Appendix

## Appendix A: Equilibrium Characterisation

**Lemma 10.** *We have that for all $y \in [0,1]$: $\frac{\delta h(y)}{\delta y} < 0$ and $\frac{\delta h(\Pr(\alpha|piv))}{\delta y} \leq -1$.*

**Proof.** For the ease of notation, set $p := \Pr(\alpha|piv))$ for this proof.

$$\frac{\delta h(y)}{\delta y} = \frac{-p}{(1-y)p + y(1-p)} - (1-2p) \cdot \frac{(1-y)p}{((1-y)p + y(1-p))^2}, \tag{70}$$

$$\frac{\delta \tilde{h}(y)}{\delta y} = \frac{1-p}{(1-y)p + y(1-p)} - (1-2p) \cdot \frac{y(1-p)}{((1-y)p + y(1-p))^2}. \tag{71}$$

The formulas show that $\frac{\delta h(y)}{\delta y} < 0$ for any $p \leq \frac{1}{2}$. They also show that $\frac{\delta \tilde{h}(y)}{\delta y} > 0$ for any $p \leq \frac{1}{2}$, which implies that $\frac{\delta h(y)}{\delta y} = -\frac{\delta \tilde{h}(y)}{\delta y} < 0$ for any $p \geq \frac{1}{2}$. We conclude that $\frac{\delta h(y)}{\delta y} < 0$ for any $p \in [0,1]$. Plugging in $y = p$ into (70) yields

$$\begin{aligned} \frac{\delta h(\Pr(\alpha|\text{piv}))}{\delta y} &= \frac{-p}{2(1-p)p} - (1-2p)\frac{(1-p)p}{(2(1-p)p)^2} \\ &= -\frac{2p-(1-2p)}{4(1-p)p} \\ &= -\frac{1}{4p(1-p)} \\ &\leq -1, \end{aligned}$$

where we used that the maximum of the function $x(1-x)$ on the interval $[0,1]$ is $\frac{1}{4}$. ■

For any type $t = (y, G)$,

$$\begin{aligned} \frac{\delta x^*(t)}{\delta y} &= \frac{\Pr(\text{piv}|\sigma)(1-2\Pr(\alpha|\text{piv}))}{d(d-1)x^*(t)^{d-2}} \\ &= \frac{dx^*(t)^{d-1}\frac{(1-2\Pr(\alpha|piv))}{\Pr(\alpha|\text{piv})(1-y)+\Pr(\beta|\text{piv})y}}{d(d-1)x^*(t)^{d-2}} \\ &= \frac{1}{d-1}x^*(t)\frac{(1-2\Pr(\alpha|\text{piv}))}{\Pr(\alpha|\text{piv})(1-y)+\Pr(\beta|\text{piv})y}. \end{aligned} \tag{72}$$

where the equality on the first line follows from the implicit function. The equality on the second line follows from the first-order condition (13) and the assumption that $c(x) = x^d$. Since, $x^*(t)$ is uniformly converging to 0 in any equilibrium sequence $\sigma_n$ for $n \to \infty$, it follows from the equality (72) that $\frac{\delta x^*(t)}{\delta y}$ is uniformly converging to zero. Then, it follows from Lemma 10 that the difference of the left hand side and the right hand side of the indifference equation (15) is strictly decreasing in $y$ around $y = \Pr(\alpha|\text{piv})$ in equilibrium, for any $n$ sufficiently large. Recall that $h(\Pr(\alpha|\text{piv}) = \frac{1}{2}$ and that the optimal information quality $x^*(t)$ is strictly larger than zero in any equilibrium. Hence, the difference of the left hand side and the right hand side of the indifference equation (15) is negative for $y = \Pr(\alpha|\text{piv})$. It follows that (15) has a unique solution $\underline{y}(G) < \Pr(\alpha|\text{piv})$ for any $n$ sufficiently large. Analogously, one shows that the indifference equation (16) has a unique solution $\underline{y}(G) > \Pr(\alpha|\text{piv})$ for any $n$ sufficiently large.

**Lemma 3.** *Let $\tilde{y} = \Pr(\alpha|\text{piv})$. Then, for any sequence of strategies $\sigma_n$ and any $\lambda \in \{G, g\}$, the best response satsifies*

$$I_\lambda = \frac{2(d-1)}{d}\frac{f(\tilde{y})}{h'(\tilde{y})} \cdot x(\tilde{y}, \lambda)^2 + O(\Pr(\text{piv}|\sigma_n)^{\frac{2}{d-1}})o(1). \tag{24}$$

**Proof.** Recall that we defined $h(y) = \frac{\Pr(\alpha|\text{piv})(1-y)}{\Pr(\alpha|\text{piv})(1-y)+\Pr(\beta|\text{piv})y}$, and the equation (16) which we rewrite as

$$h(y) - \frac{1}{2} \quad = \quad \frac{d-1}{d} x^*(y,\lambda). \tag{73}$$

A Taylor approximation of the left hand side at the root $\tilde{y} := \Pr(\alpha|\text{piv})$ gives

$$h'(\tilde{y})(\underline{y}(\lambda) - \tilde{y}) + O((\underline{y}(\lambda) - \tilde{y}))^2 = \frac{d-1}{d} x^*(\underline{y}(\lambda), \lambda). \tag{74}$$

Analogously, we obtain

$$h'(\tilde{y})(\tilde{y} - \overline{y}(\lambda)) + O((\overline{y}(\lambda) - \tilde{y})^2 = \frac{d-1}{d} x^*(\overline{y}(\lambda), \lambda). \tag{75}$$

Hence,

$$\begin{aligned} \underline{y}(\lambda) - \overline{y}(\lambda) \quad &= \quad \frac{d-1}{d}(x^*(\underline{y}(\lambda),\lambda) + x^*(\overline{y}(\lambda),\lambda))\frac{1}{h(\tilde{y})} + O((\overline{y}(\lambda) - \tilde{y})^2 + O((\underline{y}(\lambda) - \tilde{y})^2 \\ &= \quad \frac{2(d-1)}{d} x(\tilde{y},\lambda)(1+o(1))\frac{1}{h'(\tilde{y})} + O((\overline{y}(\lambda) - \tilde{y})^2 + O((\underline{y}(\lambda) - \tilde{y})^2. \end{aligned} \tag{76}$$

where the first line follows from the equations (73) and (75). The equality on the second line follows, since $x(y)$ is continuous and since $\lim_{n\to\infty} \underline{y}(\lambda) = \lim_{n\to\infty} \overline{y}(\lambda) = \tilde{y}$. Taylor approximations of $F(\underline{y}(\lambda)$ and $F(\overline{y}(\lambda))$ at $\tilde{y}$ yield

$$F(\overline{y}(\lambda)) - F(\underline{y}(\lambda) \quad = \quad (\overline{y}(\lambda) - \underline{y}(\lambda))f(\tilde{y}) + O(\overline{y}(\lambda) - \underline{y}(\lambda))^2) \tag{77}$$

Note that it follows from the observation that the interval of swing voters vanishes for $n \to \infty$ (see Section 2.2) that $\lim_{n\to\infty} \frac{(\underline{y}(R)-\tilde{y})^2}{\underline{y}(R)-\tilde{y})} = \underline{y}(R) - \tilde{y} = 0$. Therefore the approximations (75) and (74) imply that

$$h'(\tilde{y})(\underline{y}(\lambda) - \tilde{y})(1 + o(1)) = \frac{d-1}{d} x(\underline{y}(\lambda)). \tag{78}$$

This however in turn implies that $(\underline{y}(R) - \tilde{y}) \in O(x(\underline{y}(\lambda)))$. But, it follows from the first-order conditions (13) and (14) that $(x(\underline{y}(\lambda),\lambda) \in O(\Pr(piv|\sigma_n)^{\frac{1}{d}})$. We conclude that $(\underline{y}(R) - \tilde{y})^2 \in O(\Pr(piv|\sigma_n)^{\frac{2}{d}})$. Therefore, we can rewrite the approximation (77) further,

$$\begin{aligned} F(\overline{y}(\lambda)) - F(\underline{y}(\lambda) \quad &= \quad (\overline{y}(\lambda) - \underline{y}(\lambda))f(\tilde{y}) + O(\Pr(\text{piv}|\sigma)^{\frac{2}{d}}) \\ &= \quad \frac{2(d-1)}{d}\frac{f(\tilde{y})}{h'(\tilde{y})} x^*(\tilde{y},\lambda)(1+o(1)) + O(\Pr(\text{piv}|\sigma)^{\frac{2}{d}}). \end{aligned} \tag{79}$$

where the equality on the last line follows from the equality (76) and the observation that $(\underline{y}(R) - \tilde{y})^2 \in O(\Pr(piv|\sigma_n)^{\frac{2}{d}})$.

We have

$$
\begin{aligned}
I_\lambda &= \int_{\underline{y}(\lambda)\le y\le \overline{y}(\lambda)} x(y,\lambda)dF(y), \\
&= (F(\underline{y}(\lambda) - F(\overline{y}(\lambda)))x(\tilde{y},\lambda)(1+o(1)), \qquad (80)
\end{aligned}
$$

where we used that $\underline{y}(\lambda) \le \tilde{y} \le \overline{y}(\lambda)$ (see Proposition 1) and that for all strategy sequences $\sigma_n$ we have $\lim_{n\to\infty} |\underline{y}(\lambda) - \overline{y}(\lambda)| = 0$ (see the discussion in Section 2.2). We obtain that

$$
\begin{aligned}
I_\lambda &= \Big[\frac{2(d-1)}{d}\frac{f(\tilde{y})}{h'(\tilde{y})}x^*(\tilde{y},\lambda)(1+o(1)) + O(\Pr(\text{piv}|\sigma)^{\frac{2}{d}})\Big]x(\tilde{y},\lambda)(1+o(1)) \\
&= (\frac{2(d-1)}{d}\frac{f(\tilde{y})}{h'(\tilde{y})})x^*(\tilde{y},\lambda)^2(1+o(1)) + O(\Pr(\text{piv}|\sigma)^{\frac{2}{d}})o(1) \\
&= (\frac{2(d-1)}{d}\frac{f(\tilde{y})}{h'(\tilde{y})})x^*(\tilde{y},\lambda)^2 + O(\Pr(\text{piv}|\sigma)^{\frac{2}{d}})o(1) \qquad (81)
\end{aligned}
$$

where the equality on the last line follows since $(x(\underline{y}(\lambda),\lambda) \in O(\Pr(piv|\sigma_n)^{\frac{1}{d}})$. This equation (81) had to be shown. ■

# Appendix B: Equilibria with Much Information

**Lemma 7.** *For any sequence of strategies $\sigma_n$, we have $\lim_{n\to\infty} \Pr(A \text{ is elected}|\omega;\sigma_n) = \Phi((\frac{2}{\tau(1-\tau)})^{\frac{1}{2}}\delta_\omega)$ where $\Phi$ is the cumulative distribution function of the standard normal distribution and $W^\eta$ is the generalised Lambert function (see Theorem 2).*

**Proof.** Let $q_n := \Pr(\sigma_n(s,t) = 1|\omega)$. By using the normal approximation[23]

$$\mathcal{B}(n+1,q_n) \simeq \mathcal{N}((n+1)q_n,(n+1)q_n(1-q_n)),$$

we see that the probability that $A$ wins the election in $\omega$ converges to

$$\Phi(\frac{(n+1)q-(n+1)\cdot q_n}{((n+1)q_n(1-q_n))^{\frac{1}{2}}}).$$

Taking limits $n\to\infty$, gives us

$$
\begin{aligned}
\lim_{n\to\infty}\Phi(\frac{(n+1)q-(n+1)\cdot x_n}{(n+1)q_n(1-q_n))^{\frac{1}{2}}}) &= \lim_{n\to\infty}\Phi(\frac{(n+1)\frac{1}{2}-(n+1)(\frac{1}{2}+(q_n-\frac{1}{2}))}{(n+1)^{\frac{1}{2}}(q_n(1-q_n))^{\frac{1}{2}}}) \\
&= \lim_{n\to\infty}\Phi((n+1)^{\frac{1}{2}}(q_n-\frac{1}{2})(q_n(1-q_n))^{-\frac{1}{2}}) \\
&= \lim_{n\to\infty}\Phi(2^{\frac{1}{2}}\delta_\omega(q_n(1-q_n))^{-\frac{1}{2}}) \\
&= \Phi((\frac{2}{\tau(1-\tau)})^{\frac{1}{2}}\delta_\omega),
\end{aligned}
$$

where the last equality holds, because either $\delta_\omega \in \{\infty,-\infty\}$, or $\lim_{n\to\infty} q_n = \frac{1}{2}$. ■

[23] For this normal approximation we cannot rely on the standard central limit theorem, because $q_n$ varies with $n$. However, the central limit theorem for triangular sequences holds for triangular sequences of Bernoulli distributions $\mathcal{B}(r,q)$ with $q$ bounded away from 0 and 1, by an application of the Berry-Esseen-Theorem. However, we will see instantaneously in Section 2.3 that for any equilibrium sequence, we have $\lim_{n\to\infty} q_\omega \in \{\psi(p_0),\tau\}$ (see Lemma 5 and its proof).

**Omitted Parts of the Proof of Theorem 3.**

Now, we will prove Case 1 of Lemma 6 for the situation when the cost of information are low ($d > 3$).
**Case 1.** Suppose that there exists a sequence of fixed points $(\tilde{q_{\alpha}}_n, \tilde{q_{\beta}}_n)$ of the modified best reponse and $\tilde{n} \in \mathbb{N}$ such that $q_\beta(\text{BR}((\tilde{q_{\alpha}}_n, \tilde{q_{\beta}}_n))) \geq \tau$ for any $n \geq \tilde{n}$.
It follows from the first-order condition (13) and the observation that $\lim_{n\to\infty} \tilde{y} = p_\tau$ that

$$\lim_{n\to\infty} \frac{x(\tilde{y}, G)^{d-1}}{2\text{Pr}(\text{piv})(\tilde{y}(1-\tilde{y})} = 1. \tag{82}$$

Since, for any strategy sequence, $x(t)$ converges uniformly to zero (see (21)), we have

$$\begin{aligned} \lim_{n\to\infty} \frac{x(\tilde{y}, G)^2}{\text{Pr}(\text{piv})} &= \lim_{n\to\infty} \frac{x(\tilde{y}, G)^2}{x(\tilde{y}, G)^{d-1}} \frac{x(\tilde{y}, G)^{d-1}}{\text{Pr}(\text{piv})} \\ &= \infty, \end{aligned} \tag{83}$$

where the equality on the second line follows from (82) and since $d > 3$. Also, it follows from the assumption that $\tilde{q}_\beta = \tau$ and from Lemma 2 together that the probability of being pivotal is a nonnegative multiple of $n^{-\frac{1}{2}}$ for any $n$. Hence, it follows from the equation (83) that $\lim_{n\to\infty} x(\tilde{y}, G)^2 n^{\frac{1}{2}} = \infty$. Then, it follows from the inequality (50) that $\delta_\alpha = \infty$. However, this yields a contradiction with Claim 1 which showed that $\delta_\alpha \in \mathbb{R}$.

Now, we will prove Case 2 of Lemma 6 for the situation when the cost of information are low or intermediate ($d \geq 3$).
**Case 2.** Suppose that there exists $\tilde{n} \in \mathbb{N}$ such that $q_\alpha = \tau$ for any $n \geq \tilde{n}$. Relative to the sketch of proof in the main text, the following proof is more involved since it does not use the assumption that the prior favors one of the alternatives, that is the assumption that $\psi(p_0) \neq \tau$. However, the contradiction is obtained in essentially the same way, by showing that the posterior conditional on being pivotal would have to exceed the prior, and hence the voting probability under the best reponse would be strictly unequal to $\tau$ when the electorate is large.
It follows from the construction of the modified best reponse and from the equation (46) that

$$\tilde{q_\beta} < \tau \tag{84}$$

for any $n \geq \bar{n}$. Then,

$$\begin{aligned} \delta_\beta(\text{BR}(\sigma_n)) &= \lim_{n\to\infty} |q_\beta(\text{BR}(\sigma_n)) - \tau| n^{\frac{1}{2}} \\ &\geq \lim_{n\to\infty} (q_\alpha(\text{BR}(\sigma_n)) - q_\beta(\text{BR}(\sigma_n)) n^{\frac{1}{2}} \\ &= \Big[\frac{4(d-1)}{d} \frac{f(\tilde{y})}{h'(\tilde{y})} (\text{Pr}(G) - k^{\frac{2}{d-1}} \text{Pr}(g)) x(\tilde{y}, G)^2 + O(\text{Pr}(\text{piv}|\sigma_n)^{\frac{2}{d-1}}) o(1)\Big] n^{\frac{1}{2}}, \end{aligned} \tag{85}$$

where the inequality on the second line follows from the assumption that $\tilde{q_\alpha} = \tau$ which is equivalent to $q_\alpha(\text{BR}(\sigma_n)) \leq \tau$. The equality on the third line follows from the formula for the difference in voting probabilities in Lemma 3. It follows from the first-order condition (13) that $x(\tilde{y}, G) \in \Theta(\text{Pr}(\text{piv}))^{\frac{1}{d-1}}$. Consequently, $\lim_{n\to\infty} \frac{x(\tilde{y}, G)^2}{O(\text{Pr}(\text{piv}|\sigma_n)^{\frac{2}{d-1}}) o(1)} = \infty$. It follows from the inequality (85) that for any $d \geq 3$,

$$\delta_\beta(\text{BR}(\sigma_n)) \geq \Big[\frac{4(d-1)}{d} \frac{f(\tilde{y})}{h'(\tilde{y})} (\text{Pr}(G) - k^{\frac{2}{d-1}} \text{Pr}(g)) x(\tilde{y}, G)^2\Big] n^{\frac{1}{2}}. \tag{86}$$

Now, it follows from the assumption that $\tilde{q_\alpha} = \tau$ and from Lemma 2 that the probability of being pivotal is a nonnegative multiple of $n^{-\frac{1}{2}}$. Then, $x(\tilde{y}, G) \in \Theta(\text{Pr}(\text{piv})^{\frac{1}{d-1}}$ and the inequality (86) imply that

$$\begin{aligned}\delta_\beta(\mathrm{BR}(\sigma_n)) &\geq \lim_{n\to\infty}(mn^{-\frac{1}{2}})n^{\frac{1}{2}} \\ &= m, \end{aligned}\tag{87}$$

for some $m > 0$ and any $d \geq 3$. Then,

$$\begin{aligned}\lim_{n\to\infty}\frac{\Pr(\text{piv}|\beta;(\tilde{q}_\beta,\tilde{q}_\alpha))}{\Pr(\text{piv}|\alpha;(\tilde{q}_\beta,\tilde{q}_\alpha))} &= \lim_{n\to\infty}\frac{\phi(\delta_\beta(\mathrm{BR}(\sigma_n))}{\phi(0)} \\ &< 1, \end{aligned}\tag{88}$$

where the equality on the first line follows from Lemma 1 and the assumption that $q_\alpha = \tau$ for any $n \geq \tilde{n}$. The inequality on the second line follows from (84) and since the density $\phi$ of the standard normal distribution has a unique and strict maximum at zero. Consequently, $\lim_{n\to\infty}\Pr(\text{piv}|\alpha) > p_0$. It follows from the strict monotonicity of $\psi$ that $\psi(\Pr(\text{piv}|\alpha)) > \psi(p_0) \geq \tau$. Then, it follows from the equation (25) that $\lim_{n\to\infty} q_\alpha(\mathrm{BR}(\sigma_n)) > \tau$. However, this contradicts with the assumption that $\lim_{n\to\infty}\tilde{q_\alpha} = \tau$ or equivalently that $\lim_{n\to\infty} q_\alpha(\mathrm{BR}(\sigma_n)) \leq \tau$.
This finishes the proof of the claim that there exists $\bar{n}$ such that, for any $n \geq \bar{n}$, any fixed point is interior.

# Appendix C: Discussion and Remarks

**Lemma 9.** *For any sequence of strategies $\sigma_n$ with $\lim_{n\to\infty} q_\omega \in (0,1)$ for any $\omega \in \{\alpha,\beta\}$: If $\lim_{n\to\infty}|q_\alpha(\sigma_n)-\tau| - |q_\beta(\sigma_n)-\tau|n = 0$ then $\lim_{n\to\infty}\Pr(\alpha|\text{piv},\sigma_n) = p_0$.*

**Proof.** We have

$$\lim_{n\to\infty}\frac{\Pr(\alpha|\text{piv},\sigma_n)}{\Pr(\beta|\text{piv},\sigma_n)} = \frac{p_0}{1-p_0}\cdot\frac{\Pr(\text{piv}|\alpha,\sigma_n)}{\Pr(\text{piv}|\beta\sigma_n)},\tag{89}$$

where we used Bayes formula. Let $q_n := \Pr(\sigma(s,t)=1|\alpha;\sigma_n)$ and $v_n := \Pr(\sigma(s,t)=1|\beta;\sigma_n)$. It follows from Lemma 2 that

$$\lim_{n\to\infty}\frac{\Pr(\text{piv}|\alpha,\sigma_n)}{\Pr(\text{piv}|\beta,\sigma_n)} = \lim_{n\to\infty}(\frac{q_n^\tau(1-q_n)^{1-\tau}}{v_n^\tau(1-v_n)^{1-\tau}})^n.\tag{90}$$

Consider $\epsilon > 0$ such that $\lim_{n\to\infty} q_n \in [\epsilon, 1-\epsilon]$ and $\lim_{n\to\infty} v_n \in [\epsilon, 1-\epsilon]$. Then, for any $n$ sufficiently large,

$$v_n^\tau(1-v_n)^{1-\tau} \geq \epsilon^2.\tag{91}$$

Let $L \in \mathbb{R}$ denote the maximum of the absolute value of the derivative of the function $x^\tau(1-x)^{1-\tau}$ on $[\epsilon, 1-\epsilon]$. Then,

$$\begin{aligned} &\lim_{n\to\infty}\frac{q_n^\tau(1-q_n)^{1-\tau}}{v_n^\tau(1-v_n)^{1-\tau}} \\ =\ &\lim_{n\to\infty}(1+\frac{q_n^\tau(1-q_n)^{1-\tau}-v_n^\tau(1-v_n)^{1-\tau}}{v_n^\tau(1-v_n)^{1-\tau}})^n \\ \leq\ &\lim_{n\to\infty}(1+\frac{L}{\epsilon^2}|q_n-v_n|)^n \\ \leq\ &\lim_{n\to\infty}(1+\frac{m}{n})^n \quad \text{for any} \quad m>0, \\ =\ &e^m, \end{aligned}\tag{92}$$

where the inequality on the third line follows from the definition of $L$ and the inequality (91). The inequalities on the fourth line follow from the standing assumption that $\lim_{n\to\infty}|q_n - v_n|n = 0$. Similarly,

$$\begin{aligned} & \lim_{n\to\infty} \frac{q_n^\tau(1-q_n)^{1-\tau}}{v_n^\tau(1-v_n)^{1-\tau}} \\ = & \lim_{n\to\infty}(1+\frac{q_n^\tau(1-q_n)^{1-\tau}-v_n^\tau(1-v_n)^{1-\tau}}{v_n^\tau(1-v_n)^{1-\tau}})^n \\ \geq & \lim_{n\to\infty}(1-\frac{L}{\epsilon^2}|q_n-v_n|)^n \\ \geq & \lim_{n\to\infty}(1-\frac{m}{n})^n \quad \text{for any} \quad m>0, \\ = & e^{-m}, \end{aligned} \tag{93}$$

It follows from the inequalities (92) and (93) that

$$\lim_{n\to\infty} \frac{q_n^\tau(1-q_n)^{1-\tau}}{q_n^\tau(1-q_n)^{1-\tau}} = 1. \tag{94}$$

It follows from the equalities (89), (90) and (94) that

$$\lim_{n\to\infty} \Pr(\alpha|\text{piv}, \sigma_n) = p_0. \tag{95}$$

■

# 10 Online Supplement

The online supplement provides a self-contained proof of the local central limit theorem for triangular arrays of Bernoulli distributions that relies Stirling's formula.

**Lemma 2.** *Consider any strategy sequence $\sigma_n$. Then, for any $\omega \in \{\alpha, \beta\}$*

$$\lim_{n\to\infty} \Pr(\text{piv}|\omega;\sigma_n)n^{\frac{1}{2}} = \lim_{n\to\infty}(2\pi)^{-\frac{1}{2}}(\tau(1-\tau))^{-\frac{1}{2}}\Big[(\frac{q_\omega(\sigma_n)}{\tau})^\tau(\frac{1-q_\omega(\sigma_n)}{1-\tau})^{(1-\tau)}\Big]^n.$$

**Proof.** We have

$$\begin{aligned} \binom{n}{\tau n} &= \frac{n!}{(\tau n)!((1-\tau)n)!} \\ &\simeq (2\pi)^{-\frac{1}{2}} \cdot \frac{n^{n+\frac{1}{2}}}{(\tau n)^{\tau n+\frac{1}{2}}}((1-\tau)n)^{(1-\tau)n+\frac{1}{2}} \cdot \frac{e^{-n}}{e^{-\tau n}e^{-(1-\tau)n}} \\ &= (2\pi)^{-\frac{1}{2}}n^{-\frac{1}{2}}(\tau n)^{-\tau n-\frac{1}{2}}(1-\tau)^{-(1-\tau)-\frac{1}{2}} \end{aligned}$$

where we used Stirling approximation

$$n! \simeq (2\pi)^{\frac{1}{2}}n^{n+\frac{1}{2}}e^{-n}$$

on the second line. Consequently,

$$\begin{aligned} \Pr(\text{piv}|\omega;\sigma_n) &= \Pr(\text{B}(n,q_n)=\tau n) \\ &= \binom{n}{\tau n}q_n^{\tau n}(1-q_n)^{(1-\tau)n} \\ &\simeq n^{-\frac{1}{2}}(2\pi)^{-\frac{1}{2}}(\tau(1-\tau))^{-\frac{1}{2}}(\frac{q_n}{\tau})^{\tau n}(\frac{1-q_n}{1-\tau})^{(1-\tau)n}, \end{aligned} \tag{96}$$

where we again used a Stirling approximation on the third line. This finishes the proof. ■

**Lemma 1.** *Consider any strategy sequence $\sigma_n$. Then,*

$$\lim_{n\to\infty} \Pr(\text{piv}|\omega;\sigma_n)n^{\frac{1}{2}} = \begin{cases} \phi_{0,\tau(1-\tau)}(\delta_\omega) & \text{if} \quad \delta\in\mathbb{R}, \\ 0 & \text{if} \quad \delta\in\{\infty,-\infty\} \end{cases}$$

*where $\phi_{0,\tau(1-\tau)}$ denotes the density of the normal distribution with mean* 0 *and variance* $\tau(1-\tau)$.

**Proof.** Recall the formula (96), where we used a Stirling approximation to describe $\Pr(\mathrm{B}(n,q_n)=\tau n)\cdot n^{\frac{1}{2}}$. We express the multiplicands on the right hand side of the approximation (96) one by one:

$$\begin{aligned}
(\frac{q_n}{\tau})^\tau &= (1+n^{-\frac{1}{2}}\frac{\delta_n}{\tau})^\tau \\
&= 1+\tau n^{-\frac{1}{2}}\frac{\delta}{\tau}+\frac{\tau(\tau-1)}{2!}\cdot n^{-1}(\frac{\delta_n}{\tau})^2+O((\frac{q_n-\tau}{\tau})^3) \\
&= 1+n^{-\frac{1}{2}}\delta_n-n^{-1}\frac{1-\tau}{2\tau}\delta_n^2+O((\frac{q_n-\tau}{\tau})^3),
\end{aligned}$$

where we use an exact Taylor approximation on the second line. For this note that for any $0\le z\le n^{-\frac{1}{2}}\frac{\delta_n}{\tau}=\frac{q_n-\tau}{\tau}$, we have $\frac{1}{3!}\tau(\tau-1)(\tau-2)(1+z)^{\tau-3}z^3\in O((\frac{q_n-\tau}{\tau})^3)$. Similarly,

$$\begin{aligned}
(\frac{1-q_n}{1-\tau})^{(1-\tau)} &= (1-n^{-\frac{1}{2}}\frac{\delta_n}{1-\tau})^{(1-\tau)} \\
&= 1-(1-\tau)n^{-\frac{1}{2}}\frac{\delta_n}{(1-\tau)}+\frac{(1-\tau)(-\tau)}{2!}\cdot n^{-1}(\frac{\delta_n}{1-\tau})^2+O(\frac{q_n-\tau}{\tau})^3 \\
&= 1-n^{-\frac{1}{2}}\delta_n-n^{-1}\frac{\tau}{2(1-\tau)}\delta_n^2+O((\frac{q_n-\tau}{\tau})^3)
\end{aligned}$$

where we again use an exact Taylor approximation on the second line. Consequently,

$$\begin{aligned}
&(\frac{q_n}{\tau})^\tau(\frac{1-q_n}{1-\tau})^{(1-\tau)} \\
=\ & (1+n^{-\frac{1}{2}}\delta_n-n^{-1}\frac{1-\tau}{2\tau}\delta_n^2)\cdot(1-n^{-\frac{1}{2}}\delta_n-n^{-1}\frac{\tau}{2(1-\tau)}\delta_n^2)+O((\frac{q_n-\tau}{\tau})^3) \\
=\ & 1-n^{-1}\delta_n^2(1+\frac{\tau^2}{2\tau(1-\tau)}+\frac{(1-\tau)^2}{2\tau(1-\tau)})+O((\frac{q_n-\tau}{\tau})^3) \\
=\ & 1-n^{-1}\delta_n^2\frac{1}{2\tau(1-\tau)}+O((\frac{q_n-\tau}{\tau})^3). \qquad (97)
\end{aligned}$$

where we use the binomial formula for the equality on the last line.
**Case 1.** If $\lim_{n\to\infty}\delta_n=\delta\in\mathbb{R}$, then

$$\begin{aligned}
e^{-\frac{\delta^2}{2\tau(1-\tau)}} &= \lim_{n\to\infty}(1-n^{-1}\delta^2\frac{1}{2\tau(1-\tau)})^n \\
&= \lim_{n\to\infty}(1-n^{-1}\delta_n^2\frac{1}{2\tau(1-\tau)}+O(n^{-\frac{3}{2}}(\frac{\delta_n}{\tau})^3))^n \\
&= \lim_{n\to\infty}(1-n^{-1}\delta_n^2\frac{1}{2\tau(1-\tau)}+O((\frac{q_n-\tau}{\tau})^3))^n \\
&= \lim_{n\to\infty}[(\frac{q_n}{\tau})^\tau(\frac{1-q_n}{1-\tau})^{(1-\tau)}]^n. \qquad (98)
\end{aligned}$$

where we used the limit characterisation of the $e$-function for the equality on the first and the second line: For all $x\in\mathbb{R}$, $\lim_{n\to\infty}(1+\frac{x}{n})^n=e^x$. The equality on the third line follows from the definition

$\delta_n = n^{\frac{1}{2}}(q_n - \tau)$. The equality on the fourth line follows from (97). We insert the equality (98) into (96) and obtain

$$\begin{aligned}\Pr(\mathrm{B}(n, q_n) = \tau n) &\simeq n^{-\frac{1}{2}}(2\pi)^{-\frac{1}{2}}(\tau(1-\tau))^{-\frac{1}{2}}e^{-\frac{\delta^2}{2\tau(1-\tau)}} \\ &= n^{-\frac{1}{2}}\phi_{0,\tau(1-\tau)}(\delta) \end{aligned} \quad (99)$$

**Case 2.** If $\lim_{n\to\infty}\delta_n \in \{\infty, -\infty\}$ and $\lim_{n\to\infty} q_n = \tau$, then for any $\delta \in \mathbb{R}$, we have

$$\begin{aligned} e^{-\frac{\delta^2}{2\tau(1-\tau)}} &= \lim_{n\to\infty}(1 - n^{-1}\delta^2\frac{1}{2\tau(1-\tau)})^n \\ &\geq \lim_{n\to\infty}(1 - n^{-1}\delta_n^2\frac{1}{2\tau(1-\tau)})^n \\ &= \lim_{n\to\infty}(1 - n^{-1}\delta_n^2\frac{1}{2\tau(1-\tau)} + O(n^{-\frac{3}{2}}(\frac{\delta_n}{\tau})^3))^n \\ &= \lim_{n\to\infty}[(\frac{q_n}{\tau})^\tau(\frac{1-q_n}{1-\tau})^{(1-\tau)}]^n, \end{aligned}$$

where the equality on the third line follows from $\lim_{n\to\infty}\frac{n^{-\frac{3}{2}}\delta_n^3}{n^{-1}\delta_n^2} = \lim_{n\to\infty}\frac{(q_n-\tau)^3}{(q_n-\tau)^2} = \lim_{n\to\infty} q_n - \tau = 0$. We insert the inequality (100) into (96) and obtain that

$$\begin{aligned}\lim_{n\to\infty}\Pr(\mathrm{B}(n, q_n) = \tau n)n^{\frac{1}{2}} &\leq (2\pi)^{-\frac{1}{2}}(\tau(1-\tau))^{-\frac{1}{2}}e^{-\frac{\delta^2}{2\tau(1-\tau)}} \\ &= n^{-\frac{1}{2}}\phi_{0,\tau(1-\tau)}(\delta) \end{aligned} \quad (100)$$

for any $\delta \in \mathbb{R}$.

**Case 3.** Suppose that $\lim_{n\to\infty} q_n \neq \tau$. Note that the function $x^\tau(1-x)^{1-\tau}$ has a unique maximum at $x = \tau$. Hence, $\lim_{n\to\infty}(\frac{q_n}{\tau})^\tau(\frac{1-q_n}{1-\tau})^{(1-\tau)} < 1$, and therefore

$$\lim_{n\to\infty}[(\frac{q_n}{\tau})^\tau(\frac{1-q_n}{1-\tau})^{(1-\tau)}]^n = 0. \quad (101)$$

We insert the equality (101) into (96) and obtain that

$$\lim_{n\to\infty}\Pr(\mathrm{B}(n, q_n) = \tau n)n^{\frac{1}{2}} = 0.$$

■

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