Endogenous credit constraints and equilibrium indeterminacy

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March 1, 2019

Abstract

We point out that the equilibrium definition applied by Miao and Wang [9] in their model of stock price bubbles involves an implicit assumption about the formulation of an endogenous credit constraint. By dropping this assumption, one can construct infinitely many additional equilibria, all of which exhibit stock price bubbles. The underlying reason for this result is informational non-uniqueness, a phenomenon known from the literature on dynamic games.

Keywords: endogenous credit constraint; equilibrium indeterminacy; informational non-uniquenes; stock price bubble

JEL Classification: D25, E22, E44, G10

1 Introduction

During the last two decades, macroeconomists have increasingly analyzed how financial market imperfections affect the business cycle and the functionality of monetary policy. In many of these studies, the market imperfections take the form of an endogenous credit constraint; see, e.g., Albuquerque and Hopenhayn [1], Alvarez and Jermann [2], Carli and Modesto [4], Gertler and Karadi [5], Jermann and Quadrini [6], Kiyotaki and Moore [7], Kocherlakota [8], and Miao and Wang [9]. Such a constraint limits the size of a loan by an endogenous function of the borrower's own choice variables. The aforementioned papers differ from each other, however, in the way how this constraint is formulated and how equilibria are defined. Jermann and Quadrini [6], for example, use a recursive equilibrium definition, according to which the decisions of all agents as well as the credit constraint are described by functions of the individual and aggregate states of the economy. Similarly, Albuquerque and Hopenhayn [1] consider fully state contingent contracts. Carli and Modesto [4], Kiyotaki and Moore [7], and Miao and Wang [9], on the other hand, define equilibria as sets of time-dependent functions, which satisfy the feasibility, optimality, and market clearing conditions in all periods (perfect foresight equilibria). It is known that recursively defined equilibria are Markov-perfect by construction, whereas perfect foresight equilibria typically are not. The purpose of the present paper is to demonstrate that perfect foresight equilibria of economies containing an endogenous credit constraint may also suffer from a strong form of equilibrium indeterminacy.

To achieve our goal we describe an economy very similar to the one studied by Miao and Wang [9]. At each point in time, the maximal investment which a firm can carry out is limited by the stock market value of the firm after a certain fraction of its capital stock has been deducted. Miao and Wang [9] provide a microfoundation for this kind of constraint, which is based on optimal contracts with limited commitment between agents interacting on an imperfect credit market. Since they use a non-recursive equilibrium formulation, the credit constraint is only required to hold along the equilibrium path. This opens up the door for informational non-uniqueness, a phenomenon first detected by Başar [3] in dynamic games. It originates from the fact that, in a deterministic model, an agent's action can be described in infinitely many different ways as a function of both time and individual states. The information set is so big that it allows for redundancies. But this is not the end of the story. Because different representations of the actions of one agent lead to different incentives for other agents, the resulting equilibrium paths differ from each other as well. In the case of endogenous credit constraints, the lenders can – through the choice of the representation of the constraint – affect the behavior of the borrowers. This is what we illustrate in the present paper.

The rest of the paper is organized as follows. In section 2 we formulate the model, which is a simplified version of Miao and Wang [9]. After describing the decision problems of households and firms and stating all market clearing conditions, we provide a detailed discussion of the endogenous credit constraint. In section 3 we show that there exists a continuum of mutually different equilibria. We start by establishing the existence of a continuum of stationary equilibria and then study the local dynamics around them. Finally, we discuss the reasons for equilibrium indeterminacy and point out that all of the equilibria fail to be Markov perfect. Section 4 concludes the paper. All proofs are relegated to the appendix.

2 Model formulation

In this section we describe the model which will be analyzed in the rest of the paper. It is similar to the model used by Miao and Wang [9] and we shall describe the essential differences between the two models in subsection 2.4.

Time is a continuous variable on the domain $\mathbf{T} = \mathbb{R}_+$. The economy is populated by a unit interval of households and a unit interval of firms. Firms use the input factors capital and labor to produce a single output good. The latter can be used for consumption and for investment and it serves as numeraire. Households are endowed with labor and they own the firms. Firms own their capital and rent labor services from the households. There exist two traded assets in the economy: bonds, which are available in zero net supply, and firm equity. Only the households have access to these two asset markets.

2.1 Households

There exists a unit interval of identical and infinitely-lived households. The representative household is endowed with a constant flow of one unit of labor per period. Initially, at time $t = 0$, it owns equally many shares of all firms in the economy and holds no bonds. The household has the instantaneous utility function $U : \mathbb{R}_+ \mapsto \mathbb{R}$ and the time-preference rate $\rho > 0$.[1] Let us denote by $c(t)$, $s(t)$, and $b(t)$ the rate of consumption, the share holdings, and the bond holdings of the representative household at time $t \in \mathbf{T}$. Furthermore, we denote by $r(t)$, $w(t)$, $\pi(t)$, and $v(t)$ the real interest rate, the wage rate, the dividend flow, and the share price at time $t \in \mathbf{T}$. The household takes the functions $r : \mathbf{T} \mapsto \mathbb{R}$, $w : \mathbf{T} \mapsto \mathbb{R}$, $\pi : \mathbf{T} \mapsto \mathbb{R}$, and $v : \mathbf{T} \mapsto \mathbb{R}$ as given and chooses the functions $c : \mathbf{T} \mapsto \mathbb{R}$, $s : \mathbf{T} \mapsto \mathbb{R}$, and $b : \mathbf{T} \mapsto \mathbb{R}$ so as to maximize its lifetime utility

$$\int_0^{+\infty} e^{-\rho t} U(c(t))\,\mathrm{d}t \tag{1}$$

subject to the flow budget constraint

$$\dot{b}(t) + v(t)\dot{s}(t) + c(t) = r(t)b(t) + \pi(t)s(t) + w(t) \tag{2}$$

and the initial conditions

$$s(0) = 1 \ \text{ and } \ b(0) = 0. \tag{3}$$

2.2 Firms

There exists a unit interval of identical and infinitely-lived firms which produce output from capital and labor. The firms own their capital stocks and they rent the labor services from the households. We denote the capital stock and the labor demand of the representative firm at time $t \in \mathbf{T}$ by $k(t)$ and $\ell(t)$, respectively. Output of the representative firm at time $t \in \mathbf{T}$ is given by $F(k(t), \ell(t))$ and the rate of investment is denoted by $i(t)$. The production function $F : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ satisfies all standard assumptions (including, in particular, linear homogeneity) and will later be chosen to be of Cobb-Douglas form. At time $t \in \mathbf{T}$ the firm sells the amount $F(k(t), \ell(t)) - i(t)$ on the output market, which generates the flow of profits

$$\pi(t) = F(k(t), \ell(t)) - i(t) - w(t)\ell(t). \tag{4}$$

The capital stock of the firm evolves according to

$$\dot{k}(t) = i(t) - \delta k(t), \tag{5}$$

where $\delta > 0$ is the rate of capital depreciation. The initial capital stock is given by

$$k(0) = \bar{k}, \tag{6}$$

[1] Later on we will assume the utility function to be linear.

where $\bar{k} > 0$ is an exogenous parameter. It is assumed that investment is non-negative and that it is bounded from above by a constraint of the form

$$i(t) \leq I(k(t), t). \tag{7}$$

This constraint will be discussed in great detail in subsection 2.4.

For notational convenience we introduce the state space $\mathbf{K} = \mathbb{R}_+$. The representative firm takes the functions $r : \mathbf{T} \mapsto \mathbb{R}$, $w : \mathbf{T} \mapsto \mathbb{R}$, and $I : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}$ as given and chooses functions $k : \mathbf{T} \mapsto \mathbb{R}_+$, $\ell : \mathbf{T} \mapsto \mathbb{R}_+$, and $i : \mathbf{T} \mapsto \mathbb{R}_+$ so as to maximize its shareholder value

$$\int_0^{+\infty} e^{-\int_0^t r(\tau)\,\mathrm{d}\tau} \pi(t)\,\mathrm{d}t \tag{8}$$

subject to (4)-(7). The optimal value function of this optimization problem will be denoted by $V : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}$, that is,

$$\begin{aligned} & V(\kappa, \tau) \\ = \;& \max \left\{ \int_\tau^{+\infty} e^{-\int_\tau^t r(t')\,\mathrm{d}t'} \pi(t)\,\mathrm{d}t \;\middle|\; \text{subject to (4)-(5), (7), and } k(\tau) = \kappa \right\}. \end{aligned} \tag{9}$$

2.3 Market clearing

The labor market clears at time $t \in \mathbf{T}$ if

$$\ell(t) = 1, \tag{10}$$

the asset markets clear if

$$b(t) = 0, \tag{11}$$

$$s(t) = 1, \tag{12}$$

$$v(t) = V(k(t), t), \tag{13}$$

and the output market clears if

$$F(k(t), \ell(t)) = c(t) + i(t).$$

Due to Walras' law, one of the market clearing conditions is redundant and we will therefore disregard the output market clearing condition in the analysis.

Finally, we have to make sure that households keep their firms running. This will be the case if the market value of the firms, which are held by the representative household, is at least as large as the market value of the capital installed in those firms. Since the value of capital outside the firms is equal to 1, this means that

$$V(k(t), t) \geq k(t) \tag{14}$$

must hold for all $t \in \mathbf{T}$.

2.4 The endogenous credit constraint

A major contribution of the paper by Miao and Wang [9] is the microfoundation of the constraint

$$i(t) \leq V(\xi k(t), t), \tag{15}$$

which enters their model instead of (7).[2] They derive (15) from an optimal contract with limited commitment between firms that want to invest and others which cannot invest. In order to formalize this idea, the authors assume that idiosyncratic investment opportunities arrive randomly at the firms and that the fraction $1 - \xi \in [0, 1)$ of a defaulting firm's capital gets lost if the lenders seize the assets of the firm. Without going into the details of their construction, let us simply point out that the firms' optimization problem in Miao and Wang [9] takes the form

$$\begin{aligned} & V(\kappa, \tau) \\ = \quad & \max \left\{ \int_{\tau}^{+\infty} e^{-\int_{\tau}^{t} r(t')\,\mathrm{d}t'} \pi(t)\,\mathrm{d}t \,\middle|\, \text{subject to (4)-(5), (15), and } k(\tau) = \kappa \right\}. \end{aligned} \tag{16}$$

Note that the optimal value function V of this optimization problem appears in one of the constraints that define the problem, namely in (15). It is this self-referential feature that generates multiple equilibria, some of which can be interpreted as stock price bubbles. Since the story in Miao and Wang [9] depends crucially an imperfect credit market and since the optimal value function V is endogenously determined, inequality (15) is referred to as an endogenous credit constraint.

Another observation about Miao and Wang [9] is that the right-hand side of (15) is a function of the firm's capital stock $k(t)$ and time t.[3] We adopt the very same property for the constraint (7), which is used in the present paper. In order to capture the

[2] See equations (16)-(17) in Miao and Wang [9].

[3] The paper by Miao and Wang [9] is a bit imprecise in this respect. On page 2599 the authors write "Let the ex ante market value of firm j prior to the realization of an investment opportunity shock be $V_t(K_t^j)$, where we suppress aggregate state variables in the argument." This sentence could be a hint that the authors intended to write the right-hand side of (15) as a function of the individual

essence of Miao and Wang's [9] story about credit market imperfections, we link the credit limit $I : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}$ to the shareholder value $V : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}$ by the requirement that

$$I(k(t), t) = V(\xi k(t), t) \tag{17}$$

must hold for all $t \in \mathbf{T}$. It is obvious that (7) and (17) together imply (15). One may interpret the function $I : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}$ as the contract with which lenders specify how big an investment they are willing to finance, i.e., as the strategy chosen by the lenders. One possible contract that is obviously consistent with (17) is defined by the requirement that

$$I(\kappa, t) = V(\xi\kappa, t) \tag{18}$$

holds for all $(\kappa, t) \in \mathbf{K} \times \mathbf{T}$. This is the contract that Miao and Wang [9] use in their paper. But, as will be shown in the following section, there exist other contracts that also ensure the validity of (17) and, hence, of (15). In order to do that, we now formally define equilibria.

Definition 1 Let $\bar{k} \in \mathbf{K} \setminus \{0\}$ be a given initial capital stock. An equilibrium from $\bar{k}$ is a 10-tuple of real-valued functions $(c, s, b, k, \ell, i, r, w, \pi, v)$ with domain $\mathbf{T}$ together with two functions $V : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}$ and $I : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}$ such that the following conditions are satisfied.
(a) Given (r, w, π, v) it holds that (c, s, b) maximizes (1) subject to (2)-(3).
(b) Given (r, w) and I it holds that (k, ℓ, i) maximizes (8) subject to (4)-(7) and the function V satisfies (9).
(c) The contract I satisfies condition (17).
(d) The market clearing conditions (10)-(14) hold.

To simplify the presentation we will drop the reference to the initial capital stock $\bar{k}$ and henceforth simply speak of equilibria.

3 Model analysis

In this section we construct a continuum of equilibria according to definition 1. This infinite set of equilibria contains those studied by Miao and Wang [9] as special cases.

capital stock of the firm and the aggregate state of the economy, and that the dependence on the aggregate state is simply disguised by the inclusion of the time variable. As a matter of fact, however, the entire analysis in Miao and Wang [9] is carried out with time t being one of the arguments of the optimal value function.

Our construction is carried out under the assumptions of risk neutral households and a Cobb-Douglas technology.[4] More specifically, we assume that $U(\gamma) = \gamma$ and $F(\kappa, \lambda) = \kappa^\alpha \lambda^{1-\alpha}$ hold for all $(\gamma, \kappa, \lambda) \in \mathbb{R}^3_+$, where $\alpha \in (0, 1)$ is a constant.

3.1 A parametric family of contracts

Following Miao and Wang [9] we look for equilibria in which the value function $V : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}$ takes the linear form

$$V(\kappa, t) = Q(t)\kappa + q(t) \tag{19}$$

with $Q : \mathbf{T} \mapsto \mathbb{R}$ and $q : \mathbf{T} \mapsto \mathbb{R}$ satisfying

$$Q(t) \geq 1 \text{ and } q(t) \geq 0 \tag{20}$$

for all $t \in \mathbf{T}$. The variable $Q(t)$ denotes the value of capital inside the firm, i.e., $Q(t)$ is Tobin's marginal Q. The variable $q(t)$ describes those components of the firm's value that do not originate from its capital stock. For this reason, an equilibrium with $q(t) \neq 0$ is said to contain a bubble. If $q(t) = 0$ holds, then it follows that $Q(t)$ coincides with Tobin's average Q

As for the possible contracts I, we restrict ourselves to the linearly parameterized family $\{I_\mu : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}_+ \,|\, \mu \in \mathbb{R}\}$ defined by

$$I_\mu(\kappa, t) = (1 - \mu)V(\xi k_e(t), t) + \mu V(\xi\kappa, t) = \xi Q(t)[(1 - \mu)k_e(t) + \mu\kappa] + q(t). \tag{21}$$

In order to avoid misunderstanding, let us emphasize that the borrowing firm's individual capital stock enters the credit limit I_μ as its first argument; see the investment constraint (7). The variable in definition (21) that corresponds to the individual capital stock is therefore κ. The variable $k_e(t)$ on the right-hand side of (21) refers to the *equilibrium value* of the borrowing firm's capital stock at time t as it is perfectly foreseen by the creditors.[5] In other words, the function $k_e : \mathbf{T} \mapsto \mathbb{R}$ is the equilibrium trajectory, i.e., a *fixed function of time*. By definition, it holds in equilibrium for all $t \in \mathbf{T}$ that $k(t)$ coincides with $k_e(t)$ and, hence,

$$I_\mu(\kappa, t) = \xi Q(t)[(1 - \mu)k(t) + \mu\kappa] + q(t) \tag{22}$$

[4]The main text of Miao and Wang [9] also restricts the analysis to the case of risk neutrality. The case of risk averse households is treated in appendix D of Miao and Wang [9] and in Sorger [11]. Both of these papers assume also a Cobb-Douglas technology.

[5]Since there exists a unit interval of identical firms, $k_e(t)$ in (21) could also be interpreted as the equilibrium value of the aggregate capital stock in the economy at time t.

is satisfied for all $(\kappa, t) \in \mathbf{K} \times \mathbf{T}$.

Since the representative firm's own capital stock enters the right-hand side of the credit constraint (7), capital accumulation does not only influence the firm's production possibilities but also its investment possibilities. According to (21) and (22), the marginal relaxation of constraint (7) by one additional unit of capital in period t is given by $\mu\xi Q(t)$, which is the partial derivative of $I_\mu(\kappa, t)$ with respect to the firm's capital stock κ. Note that the marginal relaxation is an increasing function of μ and that it is negative whenever μ is negative. In other words, if the parameter μ is positive, the firm can enhance its future investment possibilities via capital accumulation. In the case where $\mu = 0$ holds, the firm cannot affect its investment possibilities at all, as the function $I_0(\kappa, t)$ is a pure time-function which is independent of the firm's capital stock κ. In accordance with the dynamic games literature one could refer to $\mu = 0$ as the open-loop case. Finally, if $\mu < 0$ holds, accumulating more capital makes the credit constraint for the borrowers even tighter, i.e., capital accumulation at time t reduces the maximal investment volume after time t.

It is obvious that the specification in (21) or (22) ensures the validity of condition (c) in definition 1 and that the contract (18) considered by Miao-Wang [9] arises as the special case of (21) and (22) corresponding to $\mu = 1$.[6] The following theorem presents a characterization of equilibria by means of a three-dimensional boundary value problem.

Theorem 1 *Let a triple of real-valued functions (k, Q, q) on the time domain $\mathbf{T}$ be given such that the conditions $k(t) \geq 0$ and (20) hold for all $t \in \mathbf{T}$. Furthermore, let μ be an arbitrary real number and define the function $V : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}$ by (19) and the function $I : \mathbf{K} \times \mathbf{T} \mapsto \mathbb{R}$ by $I = I_\mu$ and (22).*
(a) *If $(c, s, b, k, \ell, i, r, w, \pi, v, V, I)$ is an equilibrium, then it follows that the functions k, Q, and q satisfy the dynamical system*

$$\dot{k}(t) \leq \xi Q(t)k(t) + q(t) - \delta k(t), \tag{23}$$

$$[Q(t) - 1][\xi Q(t)k(t) + q(t) - \delta k(t) - \dot{k}(t)] = 0, \tag{24}$$

$$\dot{Q}(t) = (\delta + \rho)Q(t) - \mu\xi Q(t)[Q(t) - 1] - \alpha k(t)^{\alpha-1}, \tag{25}$$

$$\dot{q}(t) = \rho q(t) - [Q(t) - 1][\xi(1 - \mu)k(t)Q(t) + q(t)] \tag{26}$$

[6] The specification proposed in (21) is just one of many possibilities which ensure the validity of condition (c) in definition 1. Indeed, for every function $G : \mathbb{R}^2 \mapsto \mathbb{R}_+$ satisfying $G(x, x) = x$ one could define $I(\kappa, t) = G(V(\xi k_e(t), t), V(\xi\kappa, t))$ or, alternatively, $I(\kappa, t) = V(\xi G(k_e(t), \kappa), t)$. Equation (21) arises as the special case where $G(x, y) = (1 - \mu)x + \mu y$.

for all $t \in \mathbf{T}$ *as well as the boundary conditions*

$$k(0) = \bar{k}, \tag{27}$$

$$\lim_{t \to +\infty} e^{-\rho t}[Q(t)k(t) + q(t)] = 0. \tag{28}$$

(b) *Conversely, if the functions* k*,* Q*, and* q *satisfy (23)-(28), then there exist real-valued functions* $(c, s, b, \ell, i, r, w, \pi, v)$ *defined on the domain* $\mathbf{T}$ *such that the 12-tuple* $(c, s, b, k, \ell, i, r, w, \pi, v, V, I)$ *is an equilibrium.*

3.2 Stationary equilibria

With the help of theorem 1 we can now study for which values of the parameter μ an equilibrium with credit limit I_μ exists. In a first step, we disregard the initial condition (27) and look for constant solutions of (23)-(26).[7] As usual, we shall refer to such solutions as stationary equilibria. To formulate our next main result, we need the following auxiliary lemma.

Lemma 1 *Assume that* $\xi \leq \delta$ *is satisfied and define*

$$\bar{\mu} = \frac{\delta(1+2\rho) - \rho\xi - 2\sqrt{\delta\rho(1+\rho)(\delta-\xi)}}{\xi},$$

$$Q_-(\mu) = \begin{cases} \dfrac{\delta + (\mu+\rho)\xi - \sqrt{[\delta + (\mu+\rho)\xi]^2 - 4\delta\mu\xi(1+\rho)}}{2\mu\xi} & \text{if } \mu \neq 0 \\ \dfrac{\delta(1+\rho)}{\delta+\rho\xi} & \text{if } \mu = 0, \end{cases}$$

$$Q_+(\mu) = \frac{\delta + (\mu+\rho)\xi + \sqrt{[\delta + (\mu+\rho)\xi]^2 - 4\delta\mu\xi(1+\rho)}}{2\mu\xi} \quad \text{for } \mu \neq 0.$$

(a) *The inequality* $\bar{\mu} \geq 1$ *holds, and it holds with equality if and only if* $\xi = \delta/(1+\rho)$*.*
(b) *For all* $\mu \leq \bar{\mu}$ *it holds that* $Q_-(\mu)$ *is a real number and the function* $Q_- : (-\infty, \bar{\mu}] \mapsto \mathbb{R}$ *is continuous and strictly increasing. Moreover, it holds that*

$$\lim_{\mu \to -\infty} Q_-(\mu) = 1,$$

$$Q_-(1) = \begin{cases} \dfrac{\delta}{\xi} & \text{if } \xi \geq \dfrac{\delta}{1+\rho}, \\ 1+\rho & \text{if } \xi < \dfrac{\delta}{1+\rho}. \end{cases}$$

[7] Note that any constant solution of (23)-(26) trivially satisfies the transversality condition (28).

(c) *For all $\mu \in (-\infty, 0) \cup (0, \bar{\mu}]$ it holds that $Q_+(\mu)$ is a real number and the function $Q_+ : (0, \bar{\mu}] \mapsto \mathbb{R}$ is continuous and strictly decreasing. Moreover, it holds that*

$$\lim_{\mu \searrow 0} Q_+(\mu) = +\infty,$$

$$Q_+(1) = \begin{cases} \dfrac{\delta}{\xi} & \text{if } \xi < \dfrac{\delta}{1+\rho}, \\ 1+\rho & \text{if } \xi \geq \dfrac{\delta}{1+\rho}. \end{cases}$$

(d) *It holds that $Q_-(\bar{\mu}) = Q_+(\bar{\mu})$.*

The graphs of the functions Q_- and Q_+ are shown in figures 1 and 2, respectively. Figure 1 applies to the case $\xi > \delta/(1+\rho)$ whereas figure 2 illustrates the situation when $\xi < \delta/(1+\rho)$ holds. The bold curves in these two figures represent the continua of stationary equilibria that will be discussed in the following theorem.

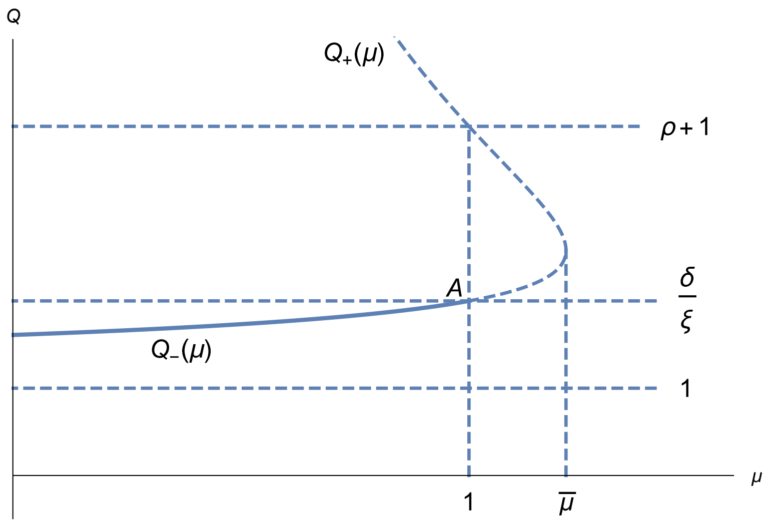

Figure 1: Illustration of lemma 1 in the case where $\delta/(1+\rho) < \xi < \delta/\rho$.

Theorem 2 **(a)** *If $\xi \geq \delta$ is satisfied then there exists a stationary equilibrium defined by*

$$k(t) = k_* = \left(\frac{\alpha}{\delta + \rho} \right)^{1/(1-\alpha)}, \ Q(t) = 1\,, \ q(t) = 0.$$

This equilibrium is independent of the parameter μ and it is the only stationary equilibrium satisfying $Q(t) = 1$.

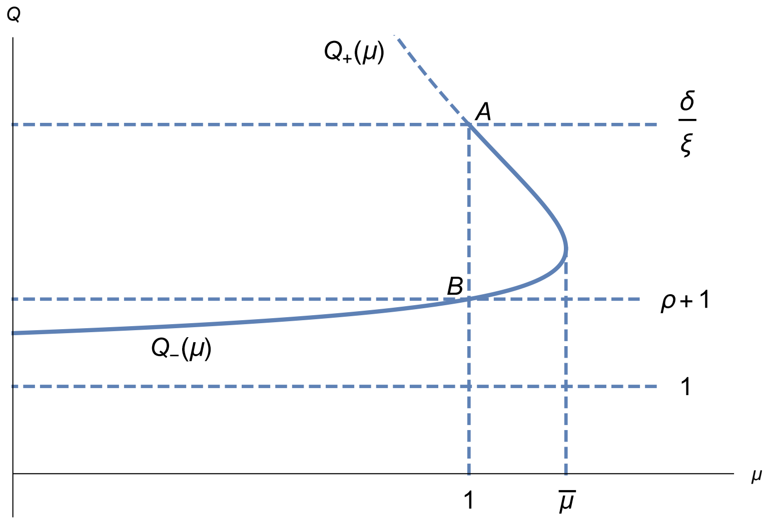

Figure 2: Illustration of lemma 1 in the case where $\xi < \delta/(1+\rho)$.

(b) *If $\xi < \delta$ holds, then there exists a stationary equilibrium for every $\mu \in (-\infty, 1]$. This stationary equilibrium is given by*

$$k(t) = k_-(\mu)\,,\ Q(t) = Q_-(\mu)\,,\ q(t) = q_-(\mu),$$

where

$$\begin{aligned} k_-(\mu) &= \left[\frac{\alpha}{\delta(1+\rho)+\rho(1-\xi)Q_-(\mu)}\right]^{1/(1-\alpha)}, \\ q_-(\mu) &= k_-(\mu)[\delta - \xi Q_-(\mu)]. \end{aligned}$$

(c) *If $\xi < \delta/(1+\rho)$ holds, then there exist in addition to the stationary equilibria from part (b) two stationary equilibria for every $\mu \in (1, \bar{\mu}]$. These stationary equilibria are given by*

$$k(t) = k_-(\mu)\,,\ Q(t) = Q_-(\mu)\,,\ q(t) = q_-(\mu),$$

and

$$k(t) = k_+(\mu)\,,\ Q(t) = Q_+(\mu)\,,\ q(t) = q_+(\mu),$$

respectively, where $k_-(\mu)$ and $q_-(\mu)$ have been defined in part (b) above and where

$$\begin{aligned} k_+(\mu) &= \left[\frac{\alpha}{\delta(1+\rho)+\rho(1-\xi)Q_+(\mu)}\right]^{1/(1-\alpha)}, \\ q_+(\mu) &= k_+(\mu)[\delta - \xi Q_+(\mu)]. \end{aligned}$$

For $\mu \in (1, \bar{\mu})$ it holds that the two stationary equilibria $(k_-(\mu), Q_-(\mu), q_-(\mu))$ and $(k_+(\mu), Q_+(\mu), q_+(\mu))$ differ from each other whereas for $\mu = \bar{\mu}$ they coincide.

In the stationary equilibrium from part (a) of the theorem, the credit constraint is not binding and Tobin's Q is equal to 1. Hence, capital inside a firm is as valuable as capital outside of firms. The more interesting cases are those in which the credit constraint is binding. These equilibria are described in parts (b) and (c) of the theorem. For $\mu = 1$ we obtain (not surprisingly) the same stationary equilibria that have already been identified by Miao and Wang [9]. Whenever $\xi < \delta$ holds, the stationary equilibrium from Miao and Wang [9] satisfies $Q(t) = \delta/\xi$, which corresponds to point A in figures 1 and 2. If the credit constraint is so tight that $\xi < \delta/(1+\rho)$ is satisfied, then Miao and Wang [9] have found another stationary equilibrium, which satisfies $Q(t) = 1 + \rho$ and therefore corresponds to point B in figure 2. Theorem 2 demonstrates that there exist infinitely many other stationary equilibria as well which, however, violate assumption (18), which has been implicitly imposed by Miao and Wang [9]. More specifically, within the parametric family of contracts defined by (21) there exist stationary equilibria for all $\mu \in (-\infty, 1]$ if $\delta/(1+\rho) \leq \xi < \delta$ holds, and for all $\mu \in (-\infty, \bar{\mu}]$ in the case where $\xi < \delta/(1+\rho)$ is satisfied. In figures 1 and 2 the Q-values corresponding to these stationary equilibria are indicated by the bold curves.

Next we add a few observations on the possible stationary equilibrium values of the key variables, whereby we restrict the discussion to those equilibria with $Q(t) > 1$. We see from lemma 1 that Tobin's marginal Q can take any value between 1 and δ/ξ. Correspondingly, the stationary equilibrium capital stocks can take any value between

$$\left[\frac{\alpha}{\delta(1+\rho) + \rho(1-\xi)}\right]^{1/(1-\alpha)} \quad \text{and} \quad \left[\frac{\alpha\xi}{\delta(\rho+\xi)}\right]^{1/(1-\alpha)}.$$

As in Miao and Wang [9] it holds that higher values of $Q(t)$ go along with lower values of the capital stock. It is moreover straightforward to see that consumption in the stationary equilibrium is also a decreasing function of $Q(t)$. Hence, we obtain the interesting result that both consumption and the capital stock in a stationary equilibrium can be increased by making the parameter μ in the credit limit I_μ smaller. More specifically, very negative values of μ generate consumption and capital values close to their respective suprema. This is surprising, because a very negative value of μ means that the credit constraint provides strong disincentives to capital accumulation.

Finally, we consider the stationary equilibrium values of $q(t)$. Since $q(t) = k(t)[\delta - \xi Q(t)]$ holds for all stationary equilibria with $Q(t) > 1$ and since both $k(t)$ and $\delta - \xi Q(t)$ are positive and decreasing functions of $Q(t)$, it follows that $q(t)$ is also a decreasing function of $Q(t)$. The minimal value of $q(t)$ is zero and is attained when $\mu = 1$ and $Q(t) = Q_-(1)$ or $Q(t) = Q_+(1)$ depending on whether $\xi \geq \delta/(1+\rho)$ or $\xi < \delta/(1+\rho)$

is satisfied. In all other cases it holds that $q(t) > 0$. Adopting the interpretation of Miao and Wang [9], we therefore see that all but one of the infinitely many stationary equilibria described in parts (b) and (c) of theorem 2 feature stock price bubbles. From the stated monotonicity properties it follows also that higher values of the bubble component $q(t)$ go along with higher values of capital and consumption. This shows that the stock price bubbles are productive ones.

3.3 Local dynamics

In this subsection we analyze the dynamic stability of the stationary equilibria listed in theorem 2. Since the dynamical system (23)-(26) has one predetermined variable ($k(t)$) and two jump variables ($Q(t)$ and $q(t)$), a stationary equilibrium is locally asymptotically stable and determinate if the Jacobian matrix has exactly one stable eigenvalue and it is locally asymptotically stable and indeterminate if it has more than one stable eigenvalue. Miao and Wang [9] and Sorger [11] have proved that $(k_-(1), Q_-(1), q_-(1))$ is locally asymptotically stable and determinate (one stable eigenvalue) whereas $(k_+(1), Q_+(1), q_+(1))$ is locally asymptotically stable and indeterminate (two stable eigenvalues). These facts give rise to the following conjecture:

Conjecture 1 **(a)** Whenever it exists, the stationary equilibrium $(k_-(\mu), Q_-(\mu), q_-(\mu))$ is locally asymptotically stable and determinate.
(b) Whenever it exists, the stationary equilibrium $(k_+(\mu), Q_+(\mu), q_+(\mu))$ is locally asymptotically stable and indeterminate.

Unfortunately, we have not been able to find a complete proof of this conjecture. However, the conjecture can be verified analytically in certain special cases and there is numerical confirmation for many other cases. If $Q(t) > 1$ holds, which is the case in all the stationary equilibria mentioned in parts (b)-(c) of theorem 2, then the Jacobian matrix of system (23)-(26) is given by

$$M(\mu) = \begin{pmatrix} \xi Q - \delta & \xi k & 1 \\ \alpha(1-\alpha)k^{\alpha-2} & \delta + \rho + \mu\xi(1-2Q) & 0 \\ -(1-\mu)\xi Q(Q-1) & (1-\mu)\xi k(1-2Q) - q & 1 + \rho - Q \end{pmatrix}, \tag{29}$$

where we have omitted the time variable from $k(t)$, $Q(t)$, and $q(t)$ for simplicity of presentation. Using the facts that

$$k^{\alpha-2} = \frac{\delta(1+\rho) + \rho(1-\xi)Q}{\alpha k}$$

and

$$q = k(\delta - \xi Q)$$

hold in all stationary equilibria from theorem 2(b-c) one can rewrite the Jacobian matrix as

$$M(\mu) = \begin{pmatrix} \xi Q - \delta & \xi k & 1 \\ \frac{(1-\alpha)[\delta(1+\rho)+\rho(1-\xi)Q]}{k} & \delta + \rho + \mu\xi(1-2Q) & 0 \\ -(1-\mu)\xi Q(Q-1) & k[(1-\mu)\xi(1-2Q) - \delta + \xi Q] & 1 + \rho - Q \end{pmatrix}.$$

Now consider for example the open-loop case $\mu = 0$ and $Q = Q_-(0) = \delta(1+\rho)/(\delta + \rho\xi)$. Substituting the value of Q into the expression for $M(0)$ and computing the characteristic polynomial $\mathcal{P}(z)$ we obtain

$$\mathcal{P}(z) = D_0 + D_1 z + D_2 z^2 - z^3,$$

where

$$\begin{aligned} D_0 &= -(1-\alpha)\delta(1+\rho)(\delta+\rho) < 0, \\ D_1 &= \frac{(\delta+\rho)[\delta^2 - \rho\xi(1+\rho) - \delta\xi(\alpha + \alpha\rho - \rho)]}{\delta + \rho\xi}, \\ D_2 &= \frac{\rho\xi(1+2\rho) + \delta[\rho + \xi(1+\rho)]}{\delta + \rho\xi} > 0. \end{aligned}$$

Using the results from Strelitz [13] and Weisstein [15] it has been argued in Sorger [11] that $D_0 < 0$ and $D_2 > 0$ together are sufficient for the matrix $M(0)$ to have exactly one stable eigenvalue. This confirms conjecture (a) in the case $\mu = 0$.

Next consider the case where $\mu = \bar{\mu}$ and $Q = Q_-(\bar{\mu}) = Q_+(\bar{\mu}) = [\delta + (\bar{\mu} + \rho)\xi]/(2\bar{\mu}\xi)$. Substituting the Q-value into the expression for $M(\bar{\mu})$ we find that the determinant of $M(\bar{\mu})$ is zero, which supports both parts of conjecture 1, as it shows that $\bar{\mu}$ is a bifurcation point.

As has already been mentioned, the case $\mu = 1$ has been dealt with in Miao and Wang [9] and Sorger [11].

Finally, we have numerically checked conjecture 1 for many parameter values and found it always to be confirmed.

We close this subsection by pointing out the similarity of the results from conjecture 1 to the situation in Tirole [14]. In the latter paper it holds that there exists a bubble-less, asymptotically stable, and determinate stationary equilibrium if the saving rate is lower than the Golden Rule value. If the saving rate increases and passes the Golden Rule value, this stationary equilibrium becomes indeterminate and a bubbly stationary equilibrium emerges. The latter inherits determinacy whereas the bubbleless one becomes indeterminate. The distinction between the saving rate in Tirole's [14] model being below or above the Golden Rule value corresponds in the present model to the distinction between credit market imperfections being mild ($\xi > \delta/(1+\rho)$) and being severe ($\xi < \delta/(1+\rho)$), respectively. An important difference betwenn the model in Tirole [14] and the present version of the Miao and Wang [9] economy is that in the former model there exist only one or two stationary equilibria whereas in the latter one there exists a continuum of stationary equilibria.

3.4 Informational non-uniqueness and Markov perfection

Let us now discuss the reason for the existence of infinitely equilibria. The key observations are (i) that the credit limit (i.e., the contract) in (7) is specified as a function of the individual capital stock of the firm at time t, $k(t)$, and of time t itself and (ii) that Miao and Wang's [9] optimality condition for contracts, (15), has to hold only along the equilibrium path (i.e., for all $t \in \mathbf{T}$). The lenders can therefore choose a contract I defined on the two-dimensional domain $\mathbf{K} \times \mathbf{T}$ under the only restriction that condition (15) holds on a one-dimensional subset of that domain. This allows many degrees of freedom for the specification of optimal contracts. By separating the crucial inequality (15) into the two components (7) and (17), we were able to illustrate that contracts other than the one defined by (18) can be used to ensure the validity of (15).

The endogeneity of the credit constraint, i.e., the feedback from the optimal value function to the credit limit, plays a crucial role for equilibrium indeterminacy. Since different contracts provide different incentives for the borrowing firms, they also induce different behavior of the firms which, in turn, leads to different optimal value functions. The bottom line of the analysis carried out in the previous subsections is that the aforementioned feedback loop generates a fixed point problem which has infinitely many solutions. In the literature on dynamic games this phenomenon is known as "informational non-uniqueness"; see Başar [3] or Sorger [10, chapter 7]. In this respect it is worth repeating that we have used a very specific functional form in (21) but that the same construction can also be performed using many other functional forms; see footnote 6.

How can informational non-uniqueness be avoided? The most obvious approach would be to impose Markov perfection on the equilibria. This would entail (i) that the credit limit I is defined as a function on the complete state space of the model consisting of all pairs of individual and aggregate states and (ii) that the equilibrium is defined recursively rather than as a perfect foresight equilibrium. In the present model, the aggregate state of the economy is the aggregate capital stock so that the credit limit would have to be written as a function $I : \mathbf{K} \times \mathbf{K} \mapsto \mathbb{R}$, which does not contain the time variable as an argument. We use such an approach in Sorger [12] to analyze Markov perfect equilibria on the basis of a recursive equilibrium definition.

In contrast, all of the infinitely many equilibria studied in the present paper (including the equilibria corresponding to $\mu = 1$ which have been discussed by Miao and Wang [9]) fail to be Markov-perfect. Indeed, if a shock at some time τ changes the aggregate state of the economy (i.e., the aggregate capital stock), the constraint $i(t) \leq I(k(t), t)$ for $t > \tau$ does no longer serve its intended purpose, namely to limit the firm's investment by the market value of a hypothetical firm owning the fraction ξ of the borrower's capital. This is formally demonstrated in Sorger [12] and has an obvious reason: since the function I depends on time but *not* on the aggregate state of the economy, an aggregate shock is not reflected by a corresponding change of the credit limit $I(k(t), t)$.

Miao and Wang [9], on the other hand, do not use a recursive equilibrium formulation to ensure Markov perfection and to rule out informational non-uniqueness, but they impose condition (18), instead.

4 Concluding remarks

In this paper we have argued that the equilibrium definition in Miao and Wang [9] involves an implicit assumption about the formulation of the crucial endogenous credit constraint. By dropping this assumption, the equilibrium set is considerably enlarged. The underlying reason for such a strong form of equilibrium indeterminacy is the possibility of representing the credit limit in infinitely many different ways as a function of time and states (closed-loop representations). The phenomenon, which is known by the name of informational non-uniqueness, is familiar from the dynamic games literature; see Başar [3]. It would also arise in other models with endogenous credit constraints provided that the constraint is formulated in a closed-loop form. As has been argued in subsection 3.4 and demonstrated in Sorger [12], one can avoid informational non-uniqueness by applying a recursive equilibrium definition. This has the additional advantage of leading to Markov perfect equilibria.

Appendix

Proof of theorem 1

(a) Because of (10)-(13) and (19) it follows that $\ell(t) = 1$, $b(t) = 0$, $s(t) = 1$, and $v(t) = Q(t)k(t) + q(t)$ hold for all $t \in \mathbf{T}$. This, in turn, shows that the representative household's wealth at time t is equal to $b(t)+v(t)s(t) = Q(t)k(t)+q(t)$. Since the triple (c, s, b) solves the representative household's optimization problem, the Euler equation and the transversality condition must hold. Due to risk neutrality the Euler equation boils down to $r(t) = \rho$ for all $t \in \mathbf{T}$. Because of this result and the fact that the household's wealth at time t is equal to $Q(t)k(t) + q(t)$, the transversality condition is given by (28).

Let us no turn to the representative firm's optimization problem. Condition (27) is identical to (6). The Hamilton-Jacobi-Bellman (HJB) equation for the firm's problem is given by

$$\begin{aligned} & \rho V(\kappa,t) - V_2(\kappa,t) \\ = \; & \max\left\{\kappa^\alpha \lambda^{1-\alpha} - w(t)\lambda - \iota + V_1(\kappa,t)(\iota - \delta\kappa)\,|\, 0 \leq \lambda,\, 0 \leq \iota \leq I_\mu(\kappa,t)\right\}, \end{aligned}$$

where we have already used $r(t) = \rho$. This equation has to hold for all $(\kappa, t) \in \mathbf{K} \times \mathbf{T}$. Noting that $V_1(\kappa, t) = Q(t) \geq 1$ holds according to (19)-(20), the necessary and sufficient first-order conditions for the maximization on the right-hand side of the HJB equation are

$$\lambda = \left[\frac{1-\alpha}{w(t)}\right]^{1/\alpha} \kappa \tag{30}$$

and

$$I_\mu(\kappa,t) \geq \iota \text{ and } [Q(t) - 1][I_\mu(\kappa,t) - \iota] = 0. \tag{31}$$

Substituting these relations along with $V_1(\kappa, t) = Q(t)$ back into the HJB equation we obtain

$$\rho V(\kappa,t) - V_2(\kappa,t) = \alpha \left[\frac{1-\alpha}{w(t)}\right]^{(1-\alpha)/\alpha} \kappa + [Q(t)-1]I_\mu(\kappa,t) - \delta Q(t)\kappa. \tag{32}$$

For $\kappa = k(t)$ we must have $(\lambda, \iota) = (\ell(t), i(t))$. Thus, we obtain from (5), (10), (22), and (30)-(31) that (23)-(24) and

$$\ell(t) = \left[\frac{1-\alpha}{w(t)}\right]^{1/\alpha} k(t) = 1$$

must hold for all $t \in \mathbf{T}$. The latter condition can be solved as $w(t) = (1-\alpha)k(t)^{\alpha}$. Substituting this result together with (19) and (22) into (32) we obtain

$$\begin{aligned} & \rho[Q(t)\kappa + q(t)] - \dot{Q}(t)\kappa - \dot{q}(t) \\ = \; & \alpha k(t)^{\alpha-1}\kappa + [Q(t) - 1]\left\{\xi Q(t)[(1-\mu)k(t) + \mu\kappa] + q(t)\right\} - \delta Q(t)\kappa. \end{aligned}$$

Since this equation has to hold for all $(\kappa, t) \in \mathbf{K} \times \mathbf{T}$ it is straightforward to derive (25)-(26). This completes the proof of part (a).

(b) It has already been mentioned that definition (22) ensures that $I_{\mu}(k(t), t) = V(\xi k(t), t)$ holds for all $t \in \mathbf{T}$ so that condition (c) of definition 1 is satisfied. It remains to be shown that there exist functions $(c, s, b, \ell, i, r, w, \pi, v)$ defined on the domain $\mathbf{T}$ such that $(c, s, b, k, \ell, i, r, w, \pi, v, V, I)$ with $I = I_{\mu}$ satisfies conditions (a), (b), and (d) of definition 1. To this end we define the functions c, s, b, ℓ, i, r, w, π, and v by

$$\begin{aligned} & s(t) = 1\,,\; b(t) = 0 \\ & r(t) = \rho\,,\; w(t) = (1-\alpha)k(t)^{\alpha}\,, \\ & \ell(t) = 1\,,\; i(t) = \dot{k}(t) + \delta k(t)\,, \\ & v(t) = Q(t)k(t) + q(t)\,,\; \pi(t) = \rho v(t) - \dot{v}(t)\,, \\ & c(t) = \pi(t) + w(t). \end{aligned}$$

Obviously, (c, s, b) is a feasible solution for the representative household's optimization problem. The Euler equation of this problem holds due to $r(t) = \rho$, and the transversality condition holds due to $v(t) = Q(t)k(t) + q(t)$ and (28). Finally, because of $\pi(t) = \rho v(t) - \dot{v}(t)$ the two assets have the same return at all times $t \in \mathbf{T}$ so that the household is indifferent regarding its portfolio. This shows that condition (a) of definition 1 holds.

To verify condition (b) of definition 1 one needs to show that the HJB equation holds and that $(\lambda, \iota) = (\ell(t), i(t))$ maximizes its right-hand side. This follows by noting that the first-order optimality conditions for the representative firm's optimization problem presented in the proof of part (a) above are necessary and sufficient.

The market clearing conditions (10)-(13) hold because of (19) and the specification of the functions $(c, s, b, \ell, i, r, w, \pi)$, and condition (14) follows from (19) and (20). This proves that condition (d) of definition 1 is satisfied as well. The proof of the theorem is now complete.

Proof of lemma 1

(a) The inequality $\bar{\mu} \geq 1$ is equivalent to

$$\delta(1+2\rho) - \xi(1+\rho) \geq 2\sqrt{\delta\rho(1+\rho)(\delta-\xi)}.$$

Because of $\delta \geq \xi$, the left-hand side is non-negative. Taking squares on both sides is therefore an equivalence transformation which leads after simplifications to $[\delta - (1+\rho)\xi]^2 \geq 0$. This proves part (a).

(b) The discriminant in the definitions of $Q_-(\mu)$ and $Q_+(\mu)$ is non-negative whenever $\mu \leq \bar{\mu}$. This proves that $Q_-(\mu)$ is a real number for all $\mu \leq \bar{\mu}$. The continuity of the function Q_- on $(-\infty, \bar{\mu}]$ is obvious for all $\mu \neq 0$ and it follows from the rule of de l'Hopital for $\mu = 0$. Let us denote the square root appearing in the definition of $Q_-(\mu)$ by S. Then we can write the derivative of $Q_-(\mu)$ as

$$\frac{1}{2\mu^2\xi}\left\{S - \delta - (\mu+\rho)\xi + \mu\xi\left[1 + \frac{\delta(1+2\rho) - (\mu+\rho)\xi}{S}\right]\right\}.$$

This expression is positive if and only if

$$S[S - \delta - (\mu+\rho)\xi] + \mu\xi[S + \delta(1+2\rho) - (\mu+\rho)\xi]$$

is positive. Using the fact that $S^2 = [\delta + (\mu+\rho)\xi]^2 - 4\delta\mu\xi(1+\rho)$, this condition can be written as

$$S < \frac{(\delta+\rho\xi)^2 - \mu\xi[\delta(1+2\rho) - \rho\xi]}{\delta+\rho\xi}. \tag{33}$$

Note that the assumption $\delta \geq \xi$ implies that the term $\delta(1+2\rho) - \rho\xi$ is positive. Combining this with $\mu \leq \bar{\mu}$ it follows that the right-hand side of (33) is positive provided that $(\delta+\rho\xi)^2 - \bar{\mu}\xi[\delta(1+2\rho) - \rho\xi]$ is positive which, according to the definition of $\bar{\mu}$, is the case if and only if

$$[\delta(1+2\rho) - \rho\xi]^2 - (\delta+\rho\xi)^2 < 2[\delta(1+2\rho) - \rho\xi]\sqrt{\delta\rho(1+\rho)(\delta-\xi)}$$

holds. Straightforward algebraic manipulations show that this inequality is indeed true. Hence, we can take squares on both sides of (33) without changing the inequality. Doing that shows that (33) holds and it follows that Q_- is strictly increasing on $(-\infty, \bar{\mu}]$.

As μ approaches $-\infty$ the square root appearing in the definition of $Q_-(\mu)$ behaves asymptotically as $-\mu\xi$. Hence, $Q_-(\mu)$ behaves asymptotically as $(\mu\xi + \mu\xi)/(2\mu\xi) = 1$. The formula for $Q_-(1)$ can easily be verified by substitution of $\mu = 1$ into the definition of $Q_-(\mu)$.

(c) Analogously to the proof of part (b) we see that $Q_+(\mu)$ is real for all $\mu \leq \bar{\mu}$ except possibly for $\mu = 0$. It is also clear that the function Q_+ is continuous on the interval $(0, \bar{\mu}]$. The derivative of $Q_+(\mu)$ is

$$-\frac{1}{2\mu^2\xi}\left[S + \delta + (\mu+\rho)\xi - \mu\xi\left(1 - \frac{\delta(1+2\rho) - (\mu+\rho)\xi}{S}\right)\right].$$

Following the same steps as in part (b) we see that this expression is negative if and only if

$$S > -\frac{(\delta + \rho\xi)^2 - \mu\xi[\delta(1+2\rho) - \rho\xi]}{\delta + \rho\xi}.$$

We have shown in the proof of part (b) that the right-hand side of this inequality is negative. Since S is positive, the inequality holds and it follows that the function Q_+ is strictly decreasing on $(0, \bar{\mu}]$. It is obvious from its definition that $Q_+(\mu)$ approaches $+\infty$ as μ approaches 0 from above and that the value $Q_+(1)$ is as stated in the lemma.

(d) This statement follows immediately from the observation that $S = 0$ holds for $\mu = \bar{\mu}$.

Proof of theorem 2

(a) It is easy to verify that $(k(t), Q(t), q(t)) = (k_*, 1, 0)$ satisfies conditions (20) and (23)-(26) independently of the value of μ. Moreover, if $Q(t) = 1$ holds for all $t \in \mathbf{T}$, then it follows from (25) and (26) that $q(t) = 0$ and $k(t) = k_*$ must hold. This proves that there cannot be any other stationary equilibrium satisfying $Q(t) = 1$.

Now suppose that $Q(t) > 1$ is satisfied in a stationary equilibrium. Then it follows from (23)-(24) that (23) holds with equality. Because of stationarity, (23) implies that $q(t) = k(t)[\delta - \xi Q(t)]$. Substituting this result into (26) one obtains $k(t) = 0$ or

$$\mu\xi Q(t)^2 - [\delta + (\mu+\rho)\xi]Q(t) + \delta(1+\rho) = 0. \tag{34}$$

We can rule out $k(t) = 0$, because this is inconsistent with (25). The above quadratic equation for $Q(t)$ has the solutions $Q_-(\mu)$ and $Q_+(\mu)$. Combining the observations $q(t) = k(t)[\delta - \xi Q(t)]$, $k(t) > 0$, and $Q(t) > 1$ with the equilibrium condition $q(t) \geq 0$ we see that $\delta > \xi$ must be satisfied.

(b) Assume that $\xi < \delta$ holds. From lemma 1 it follows that $1 < Q_-(\mu) \leq \delta/\xi$ holds for all $\mu \leq 1$. Furthermore, we have

$$(\delta + \rho)Q_-(\mu) - \mu\xi Q_-(\mu)[Q_-(\mu) - 1] = \delta(1 + \rho) + \rho(1 - \xi)Q_-(\mu) > 0,$$

where we have used the fact that $Q_-(\mu)$ is a root of equation (34). Hence, by defining $k(t) = k_-(\mu)$ and $q(t) = q_-(\mu)$ for all $t \in \mathbf{T}$ it follows that conditions (23)-(26) of theorem 1 are satisfied.

(c) This case can be proved analogously to case (b) by noting that lemma 1 implies

$$1 < Q_-(\mu) \leq Q_+(\mu) < \frac{\delta}{\xi}$$

for all $\mu \in (1, \bar{\mu}]$, where the weak inequality holds strictly unless $\mu = \bar{\mu}$.

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