# Manipulation of social choice functions under incomplete information*

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**Abstract**

We introduce and study a new property for social choice functions, called PC-strategy-proofness, which is weaker than strategy-proofness. A social choice function is PC-strategy-proof if it cannot be manipulated by an individual whose information about the preferences of the other members of the society is limited to the knowledge, for every pair of alternatives, of the number of individuals preferring the first alternative to the second one. We prove that, when at least three alternatives are considered, there is no Pareto optimal, anonymous and PC-strategy-proof social choice function.

**Keywords**: social choice function; manipulability; strategy-proofness; pairwise comparison; anonymity; Pareto optimality.

**JEL Classification Numbers**: D71, D72

## 1 Introduction

Consider a society whose purpose is to select a unique alternative among the ones in a given set. Assume that such a selection must be based only on individuals' preferences, expressed via rankings of the alternatives. Any procedure associating an alternative with each preference profile, namely a complete list of individuals' preferences, is called social choice function (SCF). Gibbard-Satterthwaite Theorem (Gibbard, 1973, Satterthwaite, 1975) is definitely one of the most celebrated results about SCFs. It states that, when at least three alternatives are considered, any strategy-proof and surjective SCF is dictatorial. Thus, even though highly desirable, strategy-proofness is a strong requirement, as dictatorships are the only (surjective) SCFs fulfilling it.

Efforts were made by several authors to weaken that property in interesting directions. Guided by the intuition that if a rule does not allow large enough gains from manipulating then individuals may decide not to support the cost for gathering the needed information about the others' preferences, Campbell and Kelly (2009) focus on the size of the potential gains an individual can achieve from manipulating Condorcet and scoring SCFs. Reffgen (2011) considers those manipulations which allow individuals to get a very good result and proves that every non-dictatorial and surjective SCF can be manipulated by an individual who can obtain her best or second best alternative via the manipulation. Sato (2013), in line with Kelly (1988), studies the implications of limiting the options for misrepresenting. He introduces the concept of AM-proofness, a condition weaker than strategy-proofness which imposes that individuals are reluctant to tell big lies, and proves an impossibility

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result on the universal domain (see also Carroll, 2012). Muto and Sato (2017) introduce the concept of bounded response which states that a switch between two alternatives in the preference order of an individual does not change the outcome or leads to select an alternative which is consecutively ranked to the original outcome in the (not modified) preference order of the considered individual. The authors prove that bounded response is strictly weaker than strategy-proofness but, along with Pareto optimality, still implies dictatorship. Following the ideas of Sato (2013), they also introduce a property weaker than AM-proofness mentioned above and prove that it still implies dictatorship when joined with Pareto Optimality. Campbell et al. (2018) introduce a concept of stability which refers to the idea that a small change in one of the individual preferences should have a small impact in the outcome. Their concept of stability is strictly weaker than strategy-proofness (indeed, it is weaker than bounded response). They prove that the plurality, the Condorcet, the Maximin and the Borda social choice correspondences[1] are stable when restricted to the set of preference profiles where they are resolute but none of these restrictions can be extended to the whole domain preserving stability. They also prove that if a SCF satisfies stability, Pareto optimality, monotonicity and tops-only, then it is dictatorial.

The mentioned results basically explore the effects of restricting the possible choice of false preferences for the individuals. This line of research seems to be unable to easily escape dictatorship.[2] Nurmi (1987) suggests another possible viewpoint, a viewpoint that we adopt in this paper and that, at the best of our knowledge, has not been deepened yet in the literature.

First of all, recall that a SCF fails strategy-proofness if there is an individual who exactly knows (or thinks to know) the preferences which the others decide to report and, on the basis of this information, she realizes that it is profitable to misreport her own preferences. The failure of strategy-proofness requires then the presence of an individual who has (or thinks to have) a complete knowledge of the others' preferences. Of course, it is possible to conceive situations where that might happen. For instance, when the members of a small committee discuss a lot about the candidates before voting, their preferences can get precisely revealed to the others. However, in most circumstances, individuals cannot draw a clear picture of the preferences of the others. In particular, when the number of alternatives or individuals is large, the level of knowledge of private information which an individual can expect to get is in general limited. In these cases violating strategy-proofness is not that serious since nobody can really decide to deviate on the basis of the exact knowledge of the others' preferences, since nobody can get that knowledge. Nurmi (1987) interestingly observes however that an individual might decide to deviate on the basis of a smaller amount of information about the others' preferences. Thus, if the information needed to make an individual realize that it is profitable for her to report false preferences is small enough and easy to get, then manipulability issues become much more significant.

In the framework of social choice correspondences, Nurmi introduces a concept of degree of vulnerability "which is intended to reflect the type of knowledge one *typically* needs in order to benefit from preference misrepresentation. The more detailed the knowledge of the preference profile one needs, the less vulnerable is the procedure to misrepresentation" (Nurmi, 1987, p.119). We have to say that such a concept is not rigorously defined and, as stressed by Kelly (1993), no precise measure of the amount of knowledge is given. Anyway, on the basis of a number of examples and observations, he proposes a misrepresentation hierarchy. In particular, Nurmi states that the plurality social choice correspondence has the highest level of manipulability as only the distribution of the first ranked alternatives is needed for successful misrepresentation while the Coombs, the Alternative Voting (Hare) and the plurality with runoff social choice correspondences typically need the knowledge of the entire preference profile. The author concludes then that "there are marked differences between procedures and that therefore is not of much consequence to discover that all

[1] A social choice correspondence is a procedure which associates a nonempty set of alternatives with each preference profile.

[2] There are other lines of research which try to overcome impossibility results of the Gibbard-Satterthwaite type without looking for weak versions of strategy-proofness. Among them it is worth mentioning the analysis of the so-called dictatorial domains (Aswal et al., 2003, Pramanik, 2015) as well as the study of the informatively richer framework where voters not only rank alternatives but also evaluate them as acceptable or inacceptable (Erdamar et al., 2017).

of them are manipulable. This finding conceals many important differences between the procedures and has thus very little practical bearing" (Nurmi, 1987, p.125).

As a consequence of Nurmi's observations, we believe it is very important in practical situations to carefully evaluate the type of information which individuals are expected to get and then design SCFs which cannot be manipulated by individuals having that type of information. Thus, we think it is crucial to deeply study the weak versions of strategy-proofness obtained by establishing

($a$) the information each individual is supposed to be able to get about the others' preferences,

($b$) the conditions under which an individual really has an incentive to misrepresent her preferences,

and then by imposing that, for every individual having the type of information described in ($a$), conditions described in ($b$) never occur. About the role of ($b$), note that an individual may realize that there are several combinations of preferences which the other individuals can report and which are consistent with the available information. When such an uncertain situation occurs, it is important to know the conditions which really cause the individual to misrepresent her preferences. Establishing ($b$) just serves to clarify this point. We finally stress that if a SCF satisfies a weak version of strategy-proofness associated with suitable ($a$) and ($b$), we cannot rule out that people are lying. However, if someone misreports her preferences, we are certain that it is due to the fact that she has obtained further information than the one described in ($a$).

Among the numerous possibilities to explore, in this paper we start focusing on the weak version of strategy-proofness which refers to the following qualifications of ($a$) and ($b$):

($a'$) each individual is supposed to be able to know, for every pair of alternatives, the number of individuals preferring the first alternative to the second one;

($b'$) an individual decides to report false preferences if, for every combination of preferences of the others consistent with her information, false preferences cannot make her worse off and, for at least one combination, make her better off.

Since individuals are assumed to have complete information about pairwise comparisons among alternatives, later on we refer to such a weak version of strategy-proofness as PC-strategy-proofness, where PC stands for pairwise comparison. Notice that Nurmi (1987) also focuses on this type of information. Assumption ($a'$) is certainly weaker than the one of complete knowledge of individuals' preferences implicitly used in the definition of strategy-proofness[3]. In fact, the possibility to have precise information about what people think of any pair of alternatives is definitely easier to get and, in our opinion, more natural to emerge during a debate than the whole preferences of individuals. Assumption ($b'$) refers instead to a sort of totally risk averse behaviour of individuals. If there is one single possibility that false preferences can make an individual worse off, the individual will report sincere preferences.

Let us explain better the concept of PC-strategy-proofness via an example. Assume that five individuals, denoted by $i_1$, $i_2$, $i_3$, $i_4$ and $i_5$, have to collectively select an alternative among the alternatives $a$, $b$ and $c$ using the Alternative Vote (AV)[4]. Assume that individual $i_1$ has preferences[5] $abc$ and that, considering the other four individuals, she knows that there are two individuals preferring $a$ to $b$, there is one individual preferring $a$ to $c$ and there are two individuals preferring $b$ to $c$. On the basis of this information, individual $i_1$ can deduce that the preferences of the other individuals are described by one of the following three lists[6]

$$\begin{aligned} \ell_1 &: acb,\ bca,\ bca,\ cab \\ \ell_2 &: abc,\ bca,\ cab,\ cba \\ \ell_3 &: bac,\ bca,\ cab,\ cab \end{aligned}$$

[3]We observe that if individuals are two, PC-strategy-proofness implies strategy-proofness. Moreover, every PC-strategy-proof SCF which is C2 in the sense of Fishburn (1977) is strategy-proof.

[4]As shown by Moulin (1983, pp.23-25), when five individuals and three alternatives are considered, AV is really a social choice function, that is, it always selects a unique outcome.

[5]The writing $xyz$ denotes the ranking where $x$ is preferred to $y$ and $y$ is preferred to $z$.

[6]Since AV is anonymous, we are not associating individual names to preferences. We also stress that the determination of such a set of lists (as well as all the other sets of lists presented in this introduction) from the information available by individual $i_1$ requires a bit of work but no sophisticated argument.

Observe now that if individual $i_1$ reported $acb$, $cab$ or $cba$, then, for any possible list, AV would select an alternative which is indifferent or worse for individual $i_1$ than the one obtained reporting her true preferences $abc$. Reporting $bca$ or $bac$ would certainly be profitable to individual $i_1$ if $\ell_2$ occurred, since in that case the outcome would be $b$ instead of $c$. However, if $\ell_1$ occurred, $bac$ and $bca$ would lead to $b$ while the true preferences would lead to the better outcome $a$. Thus, assumption $(b')$ on the behaviour of the individual allows to conclude that in this situation individual $i_1$ has no incentive to misreport her preferences.

Assume now that individual $i_1$ has preferences $abc$ and that, considering the other members of the society, she knows that there is one individual preferring $a$ to $b$, there is one individual preferring $a$ to $c$ and there are two individuals preferring $b$ to $c$. On the basis of this information, individual $i_1$ can deduce that the preferences of the other individuals are described by one of the following three lists

$$\begin{aligned}\ell_1 &: \ acb,\ bca,\ bca,\ cba\\ \ell_2 &: \ abc,\ bca,\ cba,\ cba\\ \ell_3 &: \ bac,\ bca,\ cab,\ cba\end{aligned}$$

It is easy to verify that reporting $bca$ cannot damage individual $i_1$ and, if $\ell_2$ occurs, it makes her better off. Then, individual $i_1$ has an incentive to report $bca$ instead of $abc$. That shows that, at least when five individuals and three alternatives are considered, AV does not fulfil PC-strategy-proofness. Thus, an individual can manipulate AV without having a perfect knowledge of the others' preferences but only knowing, for every pair of alternatives, how many individuals prefer the first alternative to the second one.

The fact that AV fails to be PC-strategy-proof is not a special situation. Indeed, the main result of the paper states that there is no PC-strategy-proof, Pareto optimal and anonymous SCF, provided there are at least three alternatives (Theorem 3). Thus, every anonymous and Pareto optimal SCF can be manipulated by an individual who knows, for every pair of alternatives, how many individuals prefer the first alternative to the second one (and possibly has some information more). In particular, that happens to any SCF which is obtained by a classical social choice correspondence, like the Borda, the Copeland, the Kemeny and the Simpson, endowed with an agenda (that is, an exogenously given ranking of the alternatives) for breaking the possible ties. It is also worth noticing that, according to assumption $(b')$, the conditions leading an individual to report false preferences are definitely very strict. Thus, different and reasonable qualifications of those conditions determine weak versions of strategy-proofness which certainly imply PC-strategy-proofness and then are still inconsistent with Pareto optimality and anonymity.

The proof of the main theorem is based on an induction argument on the number of voters and it is strongly inspired to the short proof of Gibbard-Satterthwaite theorem proposed by Lars-Gunnar and Reffgen (2014). As an intermediate step of the proof, we also prove that, when three alternatives are considered, every Pareto optimal and strategy-proof SCF defined on a special set of preference profiles is dictatorial (Proposition 10). Such a proposition does not seem to be a consequence of one of the known results about dictatorial domains (Aswal et al., 2003, Pramanik, 2015). We finally stress that we do not know yet whether every surjective and PC-strategy-proof SCF is dictatorial.

The last part of the paper is devoted to a description of a general framework for dealing with further weak versions of strategy-proofness, still based on the principle of limiting the amount of information individuals can get. In particular, we focus on the case where individuals only know that the number of individuals preferring an alternative to another belongs to a set of consecutive numbers having a given size $t$. Of course, if $t = 1$ we obtain again the PC-strategy-proofness and the larger is $t$ the weaker is the corresponding version of strategy-proofness. Interestingly, despite of further lack of information, manipulability may still hold true for $t \geq 2$. Indeed, consider again five individuals, denoted by $i_1$, $i_2$, $i_3$, $i_4$ and $i_5$, whose purpose is to collectively select an alternative among the alternatives $a$, $b$ and $c$ using AV. Assume that individual $i_1$ has preferences $abc$ and assume that she knows that the number of individuals preferring $a$ to $b$ belongs to the set $\{3, 4\}$, the number of individuals preferring $a$ to $c$ belongs to the set $\{0, 1\}$, the number of individuals preferring $b$ to $c$ belongs to the set $\{1, 2\}$[7]. On the basis of this information, individual $i_1$ can deduce that the

[7]For consistency reasons, we also assume that she knows that the number of individuals preferring $b$ to $a$ belongs to

preferences of the other individuals are described by one of the following six lists

$$\begin{array}{l} \ell_1: \ bca,\ cab,\ cab,\ cab \\ \ell_2: \ bac,\ cab,\ cab,\ cab \\ \ell_3: \ acb,\ bca,\ cab,\ cab \\ \ell_4: \ abc,\ cab,\ cab,\ cba \\ \ell_5: \ abc,\ bca,\ cab,\ cab \\ \ell_6: \ abc,\ cba,\ cba,\ cba \end{array}$$

A simple computation shows that reporting $bca$ never damages individual $i_1$ and, if $\ell_5$ occurs, it makes her better off. Then, individual $i_1$ has an incentive to report $bca$ instead of $abc$. That shows that, when five individuals and three alternatives are considered, an individual might realize that it is profitable to manipulate AV even if, for every pair of alternatives, she only knows that the number of individuals preferring the first alternative to the second one belongs to a set of two consecutive numbers.

Using the above described weak versions of strategy-proofness depending on the parameter $t$, we finally propose a special index for measuring the degree of manipulability of SCFs. Some simple remarks about that index complete the paper. We believe that its detailed analysis is an interesting topic of further research.

## 2 Preliminary definitions and notation

Given a finite set $X$, we denote by $|X|$ the size of $X$, by $\mathcal{P}_0(X)$ the set of the nonempty subsets of $X$ and by $X^2_*$ the set $\{(x, y) \in X^2 : x \neq y\}$.

Let $A$ be a finite set with $|A| \geq 2$. We denote by $\mathcal{L}(A)$ the set of complete, transitive and antisymmetric binary relations on $A$. Consider $q \in \mathcal{L}(A)$. If $x, y \in A$, we write $x \succeq_q y$ when $(x, y) \in q$ and we write $x \succ_q y$ when $(x, y) \in q$ and $(y, x) \notin q$. For every $k \in \{1, \ldots, |A|\}$, we write $r_k(q)$ to denote the element of $q$ which is ranked $k$-th in $q$. We denote by $q^r$ the element of $\mathcal{L}(A)$ such that, for every $x, y \in A$, $x \succeq_{q^r} y$ if and only if $y \succeq_q x$. If $q$ is such that $x_1 \succ_q \ldots \succ_q x_{|A|}$, where $A = \{x_1, \ldots, x_{|A|}\}$, we identify $q$ with the writing $x_1 \cdots x_{|A|}$. Note that if $q = x_1 \cdots x_{|A|}$, then $q^r = x_{|A|} \cdots x_1$ and that $q \neq q^r$.

Let $I$, $J$ and $K$ be nonempty and finite sets. The set of functions from $I$ to $\mathcal{L}(A)$ is denoted by $\mathcal{L}(A)^I$ and its elements are usually denoted by the letter $p$, possibly with suitable subscripts or superscripts. If $I$ and $J$ are disjoint, $p \in \mathcal{L}(A)^I$ and $p' \in \mathcal{L}(A)^J$, we denote by $(p, p')$ the element of $\mathcal{L}(A)^{I \cup J}$ defined, for every $i \in I \cup J$, by

$$(p, p')(i) = \begin{cases} p(i) & \text{if } i \in I \\ p'(i) & \text{if } i \in J \end{cases}$$

If $I$, $J$ and $K$ are pairwise disjoint, $p \in \mathcal{L}(A)^I$, $p' \in \mathcal{L}(A)^J$ and $p'' \in \mathcal{L}(A)^K$, we denote by $(p, p', p'')$ the element of $\mathcal{L}(A)^{I \cup J \cup K}$ defined, for every $i \in I \cup J \cup K$, by

$$(p, p', p'')(i) = \begin{cases} p(i) & \text{if } i \in I \\ p'(i) & \text{if } i \in J \\ p''(i) & \text{if } i \in K \end{cases}$$

If $p \in \mathcal{L}(A)^I$ and $J \subseteq I$, we denote by $p_{|J}$ the restriction of $p$ to $J$, that is, the element of $\mathcal{L}(A)^J$ defined, for every $i \in J$, by $p_{|J}(i) = p(i)$. Moreover, if $i \in I$ and $q \in \mathcal{L}(A)$, we denote by $q[i]$ the function from $\{i\}$ to $\mathcal{L}(A)$ such that $q[i](i) = q$. Thus, assuming $|I| \geq 2$ and $i \in I$, we have that, for every $q \in \mathcal{L}(A)$ and $p \in \mathcal{L}(A)^{I \setminus \{i\}}$, the writing $(q[i], p)$ represents the element of $\mathcal{L}(A)^I$ such that $(q[i], p)(i) = q$ and $(q[i], p)(j) = p(j)$ for all $j \in I \setminus \{i\}$.

the set $\{0, 1\}$, the number of individuals preferring $c$ to $a$ belongs to the set $\{3, 4\}$, the number of individuals preferring $c$ to $b$ belongs to the set $\{2, 3\}$.

If $p \in \mathcal{L}(A)^I$ and $(x, y) \in A^2_*$, we set

$$c_p(x, y) = |\{i \in I : x \succ_{p(i)} y\}|.$$

We also consider the equivalence relation $\sim$ on $\mathcal{L}(A)^I$ defined as follows: for every $p, p' \in \mathcal{L}(A)^I$, $p \sim p'$ if, for every $(x, y) \in A^2_*$, $c_p(x, y) = c_{p'}(x, y)$. Of course, $\sim$ depends on $A$ and $I$ but those symbols are omitted in the writing as they will be always clear from the context. Note that if $|I| = 1$ and $p, p' \in \mathcal{L}(A)^I$, then $p \sim p'$ if and only if $p = p'$. Given $p \in \mathcal{L}(A)^I$ and[8] $\varphi \in \mathrm{Sym}(I)$, we denote by $p^\varphi$ the element of $\mathcal{L}(A)^I$ defined, for every $i \in I$, as $p^\varphi(i) = p(\varphi^{-1}(i))$.

# 3 Main result

From now on, let $A$ and $I$ be finite sets such that $|A| \geq 2$ and $|I| \geq 2$. We interpret $A$ as the set of alternatives and $I$ as the set of individuals. Each $q \in \mathcal{L}(A)$ is interpreted as one of the possible individual preferences. Each element $p$ in $\mathcal{L}(A)^I$ is called preference profile and represents the complete description of individual preferences: for every $i \in I$, $p(i)$ is interpreted as the preferences of individual $i$.

A social choice function (SCF) is a function from $\mathcal{L}(A)^I$ to $A$. Given a SCF $F : \mathcal{L}(A)^I \to A$, we say that $F$ is Pareto optimal if, for every $p \in \mathcal{L}(A)^I$ and $x, y \in A$, if $x \succ_{p(i)} y$ for all $i \in I$, then $F(p) \neq y$; $F$ is dictatorial with dictator $i \in I$ if, for every $p \in \mathcal{L}(A)^I$, $F(p) = r_1(p(i))$; $F$ is dictatorial if there exists $i \in I$ such that $F$ is dictatorial with dictator $i$; $F$ is anonymous if, for every $p \in \mathcal{L}(A)^I$ and $\varphi \in \mathrm{Sym}(I)$, we have that $F(p^\varphi) = F(p)$; individual $i \in I$ is able to manipulate $F$ at $p \in \mathcal{L}(A)^I$ via $q \in \mathcal{L}(A)$ if $F(q[i], p_{|I\setminus\{i\}}) \succ_{p(i)} F(p)$; $F$ is strategy-proof if, for every $i \in I$, $p \in \mathcal{L}(A)^{I\setminus\{i\}}$ and $q, q' \in \mathcal{L}(A)$, $F(q[i], p) \succeq_q F(q'[i], p)$.

Of course, a SCF $F$ is strategy-proof if and only if, for every $i \in I$, $p \in \mathcal{L}(A)^I$ and $q \in \mathcal{L}(A)$, $i$ is not able to manipulate $F$ at $p$ via $q$. Moreover, Pareto optimality implies surjectivity as well as anonymity implies non-dictatorship. The next fundamental result holds true.

**Theorem 1** (Gibbard-Satterthwaite)**.** *Assume that $|A| \geq 3$ and let $F : \mathcal{L}(A)^I \to A$. If $F$ is surjective and strategy-proof, then $F$ is dictatorial.*

Recall that under the assumption $|A| = 2$ the simple majority with ties broken via an agenda is Pareto optimal, strategy-proof and anonymous (see Moulin, 1983, pp.62-65) so that Theorem 1 cannot be extended to the case of two alternatives. Let us introduce now the definition of PC-strategy-proofness[9].

**Definition 2.** *Let $F : \mathcal{L}(A)^I \to A$. We say that $F$ is PC-strategy-proof if, for every $i \in I$, $p \in \mathcal{L}(A)^{I\setminus\{i\}}$ and $q, q' \in \mathcal{L}(A)$, $F(q'[i], p) \succ_q F(q[i], p)$ implies that there exists $p' \in \mathcal{L}(A)^{I\setminus\{i\}}$ with $p' \sim p$ such that $F(q[i], p') \succ_q F(q'[i], p')$.*

Thus, a SCF is PC-strategy-proof if, every time an individual realizes that it is profitable for her to report false preferences when a certain combination of preferences of the others occurs, it can be found another combination of preferences of the others such that, for every pair of alternatives, the number of individuals preferring the first alternative to the second one is the same as before and such that the previously considered false preferences makes now the deviating individual worse off. As a consequence, if the information about the others' preferences is limited to the knowledge of the pairwise comparison among the alternatives, no individual in the society will have incentive to deviate when a PC-strategy-proof SCF is utilized.

It is immediate to verify that strategy-proofness implies PC-strategy-proofness and that the two concepts agree when $|I| = 2$ or when they are applied to C2 SCFs[10]. As a consequence, if $|A| \geq 3$ and $|I| = 2$, we have that, by Theorem 1, PC-strategy-proofness and surjectivity imply dictatorship.

[8]Given a nonempty set $I$, we denote by $\mathrm{Sym}(I)$ the set of bijective functions from $I$ to $I$.

[9]Equivalent definitions of PC-strategy-proofness are proposed in Sections 4 and 5.

[10]A SCF $F : \mathcal{L}(A)^I \to A$ is C2 in the sense of Fishburn (1977) if, for every $p, p' \in \mathcal{L}(A)^I$ with $p \sim p'$, $F(p) = F(p')$.

We stress that, at the moment, proving the existence of a PC-strategy-proof, surjective and non-dictatorial social choice function when $|A| \geq 3$ and $|I| \geq 3$ is an open problem. However, the next result shows that Pareto optimality, anonymity and PC-strategy-proofness are surely inconsistent with each other when at least three alternatives are considered.

**Theorem 3.** *Assume that $|A| \geq 3$ and let $F : \mathcal{L}(A)^I \to A$. If $F$ is Pareto optimal and PC-strategy-proof, then $F$ is not anonymous.*

Theorem 3 is the main result of this paper. Its technical proof is presented in the appendix. Note that, still considering the simple majority with ties broken via an agenda, Theorem 3 cannot be extended to the case of two alternatives.

# 4 A more general framework

In this section we propose a more general framework for dealing with further weak versions of strategy-proofness. Consider the set

$$\mathbf{\Omega} = \prod_{i \in I} \mathcal{P}_0 \left( \mathcal{P}_0 \left( \mathcal{L}(A)^{I \setminus \{i\}} \right) \right).$$

Given $\Omega \in \mathbf{\Omega}$, we denote the $i$-th component of $\Omega$ by $\Omega_i$. Note that $\Omega_i$ is simply a nonempty set whose elements are nonempty subsets of $\mathcal{L}(A)^{I\setminus\{i\}}$. Thus, if $i \in I$ and $\omega \in \Omega_i$, then $\omega$ is a nonempty subset of $\mathcal{L}(A)^{I\setminus\{i\}}$. In what follows, we are going to interpret $\Omega \in \mathbf{\Omega}$ as a complete description of the type of information individuals are allowed to get. More precisely, for every $i \in I$, individual $i$ is supposed to have the capability to know that the preferences of the others, described by an element of $\mathcal{L}(A)^{I\setminus\{i\}}$, belong to a suitable $\omega \in \Omega_i$.

The next definition adapts the concept of strategy-proofness to those situations where individuals' information about the preferences of the other members of the society can be described by an element of $\mathbf{\Omega}$.

**Definition 4.** *Let $F : \mathcal{L}(A)^I \to A$ and $\Omega \in \mathbf{\Omega}$. We say that $F$ is $\Omega$-strategy-proof if, for every $i \in I$, $\omega \in \Omega_i$, $p \in \omega$ and $q, q' \in \mathcal{L}(A)$, $F(q'[i], p) \succ_q F(q[i], p)$ implies that there exists $p' \in \omega$ such that $F(q[i], p') \succ_q F(q'[i], p')$.*

Note that $F : \mathcal{L}(A)^I \to A$ is not $\Omega$-strategy-proof if there exists $i^* \in I$, $\omega^* \in \Omega_{i^*}$, $p^* \in \omega^*$ and $q, q' \in \mathcal{L}(A)$ such that $F(q'[i^*], p^*) \succ_q F(q[i^*], p^*)$ and, for every $p \in \omega^*$, $F(q'[i^*], p) \succeq_q F(q[i^*], p)$. That implies that if individuals $i^*$ knows that the preferences of the others can be identified with an element of $\omega^*$, then she has an incentive to report $q'$ instead of her true preferences $q$.

It is immediate to show that if a SCF is strategy-proof, then it is $\Omega$-strategy-proof for all $\Omega \in \mathbf{\Omega}$. Moreover, strategy-proofness can be seen as a special case of $\Omega$-strategy-proofness. Indeed, considering $\Omega^{sp} \in \mathbf{\Omega}$ defined, for every $i \in I$, by

$$\Omega_i^{sp} = \left\{ \{p\} : p \in \mathcal{L}(A)^{I\setminus\{i\}} \right\},$$

we have that a SCF is $\Omega^{sp}$-strategy-proof if and only if it is strategy-proof. Of course, by Gibbard-Satterthwaite theorem, there is no surjective, non-dictatorial and $\Omega^{sp}$-strategy-proof SCF.

PC-strategy-proofness can be seen as a special case of $\Omega$-strategy-proofness, as well. Indeed, for every $i \in I$, let $\mathcal{C}_i$ be the set of functions $c$ from $A^2_*$ to $\{0, \dots, |I| - 1\}$ having the property that there exists $p \in \mathcal{L}(A)^{I\setminus\{i\}}$ such that, for every $(x, y) \in A^2_*$, $c_p(x, y) = c(x, y)$. Considering then $\Omega^{pc} \in \mathbf{\Omega}$ defined, for every $i \in I$, by

$$\Omega_i^{pc} = \left\{ \left\{ p \in \mathcal{L}(A)^{I\setminus\{i\}} : \forall (x, y) \in A^2_*, c_p(x, y) = c(x, y) \right\} : c \in \mathcal{C}_i \right\},$$

we have that a SCF is $\Omega^{pc}$-strategy-proof if and only if it is PC-strategy-proof. By Theorem 3, we know that there is no Pareto optimal, anonymous and $\Omega^{pc}$-strategy-proof SCF.

Another interesting example is given by $\Omega^w \in \mathbf{\Omega}$ defined, for every $i \in I$, by

$$\Omega_i^w = \left\{ \mathcal{L}(A)^{I \setminus \{i\}} \right\}.$$

The set $\Omega^w$ describes the extreme case where all individuals cannot get any information about the preferences of the others.

The next simple result shows that, in some cases, it is possible to compare different types of $\Omega$-strategy-proofness. In particular, it implies that $\Omega^w$-strategy-proofness is weaker than $\Omega^{pc}$-strategy-proofness.

**Proposition 5.** *Let $F : \mathcal{L}(A)^I \to A$ and $\Omega, \widehat{\Omega} \in \mathbf{\Omega}$. Assume that, for every $i \in I$ and $\omega \in \Omega_i$, there exist $k \in \mathbb{N}$ and $\widehat{\omega}_1, \ldots, \widehat{\omega}_k \in \widehat{\Omega}_i$ such that $\omega = \widehat{\omega}_1 \cup \ldots \cup \widehat{\omega}_k$. If $F$ is $\widehat{\Omega}$-strategy-proof, then it is $\Omega$-strategy-proof.*

*Proof.* Assume that $F$ is not $\Omega$-strategy-proof and prove that $F$ is not $\widehat{\Omega}$-strategy-proof. Indeed, we know that there exist $i^* \in I$, $\omega^* \in \Omega_{i^*}$, $p^* \in \omega^*$ and $q, q' \in \mathcal{L}(A)$ such that $F(q'[i^*], p^*) \succ_q F(q[i^*], p^*)$ and, for every $p \in \omega^*$, $F(q'[i^*], p) \succeq_q F(q[i^*], p)$. Consider then $k \in \mathbb{N}$ and $\widehat{\omega}_1^*, \ldots, \widehat{\omega}_k^* \in \widehat{\Omega}_i$ such that $\omega^* = \widehat{\omega}_1^* \cup \ldots \cup \widehat{\omega}_k^*$. Then there exists $r \in \{1, \ldots, k\}$ such that $p^* \in \widehat{\omega}_r^*$. Thus, in particular, we have that $F(q'[i^*], p^*) \succ_q F(q[i^*], p^*)$ and, for every $p \in \widehat{\omega}_r^*$, $F(q'[i^*], p) \succeq_q F(q[i^*], p)$. As a consequence, $F$ is not $\widehat{\Omega}$-strategy-proof. $\square$

# 5 An index for measuring the degree of manipulability

The evaluation of the degree of manipulability of social choice functions and correspondences is definitely an interesting problem and several indexes for measuring the degree of manipulability were proposed and studied. We only mention here the contributions by Nitzan (1985), Kelly (1988, 1993), Aleskerov and Kurbanov (1999) and Aleskerov et al. (2012). In this section we define a new index and propose some very preliminary observations about it.

Let us introduce first a special family of elements of $\mathbf{\Omega}$. For every $t \in \{1, \ldots, |I|\}$, let $S(t)$ be the set whose elements are the sets of $t$ consecutive numbers taken in $\{0, \ldots, |I|-1\}$. Thus, for instance, if $|I| = 5$ we have that

$$\begin{aligned}
S(1) &= \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}\} \\
S(2) &= \{\{0,1\}, \{1,2\}, \{2,3\}, \{3,4\}\} \\
S(3) &= \{\{0,1,2\}, \{1,2,3\}, \{2,3,4\}\} \\
S(4) &= \{\{0,1,2,3\}, \{1,2,3,4\}\} \\
S(5) &= \{\{0,1,2,3,4\}\}
\end{aligned}$$

For every $i \in I$ and $t \in \{1, \ldots, |I|\}$, let $\mathcal{V}_i(t)$ be the set of functions $v$ from $A_*^2$ to $S(t)$ such that

(*a*) for every $(x,y) \in A_*^2$, $v(y,x) = \big\{|I| - 1 - k : k \in v(x,y)\big\}$,

(*b*) there exists $p \in \mathcal{L}(A)^{I \setminus \{i\}}$ such that, for every $(x,y) \in A_*^2$, $c_p(x,y) \in v(x,y)$.

For every $t \in \{1, \ldots, |I|\}$, let $\Omega^t \in \mathbf{\Omega}$ be such that, for every $i \in I$,

$$\Omega_i^t = \left\{ \left\{ p \in \mathcal{L}(A)^{I \setminus \{i\}} : \forall (x,y) \in A_*^2, c_p(x,y) \in v(x,y) \right\} : v \in \mathcal{V}_i(t) \right\}.$$

The set $\Omega^t$ describes a situation where each individual is able to establish, for every pair of distinct alternatives, that the number of the other members of the society preferring the first alternative to the second one belongs to a suitable set of consecutive numbers having size $t$. In other words, for every $i \in I$, individual $i$ is able to determine a suitable function $v \in \mathcal{V}_i(t)$ having the property that, for every $(x,y) \in A_*^2$, the number of the other members of the society preferring $x$ to $y$ belongs to $v(x,y)$. We stress that condition $(a)$ requires that $v$ is consistent with the fact that individual preferences are linear orders so that if an individual does not prefer $x$ to $y$ then she prefers $y$ to $x$; condition

($b$) assures instead that $v$ is consistent with at least one combination of preferences of the others. It is immediate to show that PC-strategy-proofness is equivalent to $\Omega^1$-strategy-proofness as well as $\Omega^w$-strategy-proofness is equivalent to $\Omega^{|I|}$-strategy-proofness. Moreover, applying Proposition 5, we get that, for every $t \in \{1, \dots, |I|-1\}$, $\Omega^t$-strategy-proofness implies $\Omega^{t+1}$-strategy-proofness.

Given now a SCF $F$, let us associate with $F$ the number

$$\tau(F) = \max \{0\} \cup \big\{t \in \{1, \dots, |I|\} : F \text{ is not } \Omega^t\text{-strategy-proof}\big\}.$$

Observe first that $F$ is $\Omega^1$-strategy-proof if and only if $\tau(F) = 0$. Moreover, given $F_1$ and $F_2$ with $\tau(F_1) > \tau(F_2)$, we have that if $F_1$ is $\Omega^t$-strategy-proof for a certain $t$, then $F_2$ is $\Omega^t$-strategy-proof for the same $t$, and that $F_2$ is $\Omega^{\tau(F_1)}$-strategy-proof while $F_2$ is not. Thus, individuals can potentially manipulate $F_1$ more easily than $F_2$, namely using less information about pairwise comparison among the alternatives. As a consequence, $\tau(F)$ can be seen as an index which measures the degree of manipulability of $F$ with respect to the information related to the pairwise comparison among alternatives: the larger is $\tau(F)$, the easier is manipulating $F$. We stress that Theorem 3 implies that if $F$ is Pareto optimal and anonymous, then $\tau(F) \geq 1$.

Assume now that $|I| = 5$ and $A = \{a, b, c\}$ and consider the five SCFs given by the Alternative Vote (AV), the Borda (BOR), the Plurality (PLU), the Negative Plurality (NEG) and the Simpson (SIM) social choice correspondences[11] with ties alphabetically broken. With the help of a computer[12], it can be shown that

$$\tau(\text{AV}) = 2, \quad \tau(\text{BOR}) = 3, \quad \tau(\text{PLU}) = 2, \quad \tau(\text{NEG}) = 2, \quad \tau(\text{SIM}) = 3.$$

Thus, when five individuals and three alternatives are considered, BOR and SIM can be manipulated more easily then the other three SCFs. In particular, the index $\tau$ turns out to be able to discriminate among classical SCFs. Moreover, the fact that BOR and SIM are not $\Omega^3$-strategy-proof seems to us a strong and quite unexpected result as well as the fact that none of the considered SCFs is $\Omega^2$-strategy-proof. Those simple observations suggest interesting open problems about the index $\tau$ which, in our opinion, deserve to be investigated.

# A Proof of Theorem 3

## A.1 The three alternatives case

In this section we assume $|A| = 3$ and $A = \{a, b, c\}$. Thus, we have that

$$\mathcal{L}(A) = \{abc, acb, bac, bca, cab, cba\}.$$

**Proposition 6.** *Let $p, p' \in \mathcal{L}(A)^I$. Then the following conditions are equivalent:*

($a$) *there exists $\varphi \in \mathrm{Sym}(I)$ such that, $p' = p^\varphi$,*

($b$) *for every $q \in \mathcal{L}(A)$, $|\{i \in I : p(i) = q\}| = |\{i \in I : p'(i) = q\}|$.*

*Proof.* ($a$) $\Rightarrow$ ($b$) Assume that there exists $\varphi \in \mathrm{Sym}(I)$ such that, $p' = p^\varphi$ and fix $q \in \mathcal{L}(A)$. Then

$$\{i \in I : p'(i) = q\} = \{i \in I : p^\varphi(i) = q\} = \{i \in I : p(\varphi^{-1}(i)) = q\} = \varphi(\{i \in I : p(i) = q\}).$$

Since $\varphi$ is bijective, we get $|\{i \in I : p'(i) = q\}| = |\{i \in I : p(i) = q\}|$.

($b$) $\Rightarrow$ ($a$) Assume now that, for every $q \in \mathcal{L}(A)$, $|\{i \in I : p(i) = q\}| = |\{i \in I : p'(i) = q\}|$. For every $q \in \mathcal{L}(A)$, let us define $I(q) = \{i \in I : p(i) = q\}$ and $I'(q) = \{i \in I : p'(i) = q\}$. Observe that

$$I = \bigcup_{q \in \mathcal{L}(A)} I(q) = \bigcup_{q \in \mathcal{L}(A)} I'(q)$$

[11] The Simpson social choice correspondence is also known as the Maximin or the Condorcet social choice correspondence.

[12] The results are obtained using the CAS Maxima. The program is available upon request.

and that, for every $q, q' \in \mathcal{L}(A)$ with $q \neq q'$, $I(q) \cap I(q') = \varnothing$ and $I'(q) \cap I'(q') = \varnothing$. Moreover, for every $q \in \mathcal{L}(A)$, $|I(q)| = |I'(q)|$, so that there exists a bijective function $\varphi_q : I(q) \to I'(q)$. Let now consider $\varphi : I \to I$ be defined, for every $i \in I$, by $\varphi(i) = \varphi_q(i)$ if $i \in I(q)$. The function $\varphi$ is well-defined since, for every $i \in I$, there exists a unique $q \in \mathcal{L}(A)$, namely $q = p(i)$, such that $i \in I(q)$.

Let us prove that $\varphi \in \mathrm{Sym}(I)$. Since $I$ is finite it is sufficient to prove that $\varphi$ is injective. Consider $(i,j) \in I^2_*$. Assume first that there exists $q \in \mathcal{L}(A)$ such that $i, j \in I(q)$. Then $\varphi(i) = \varphi_q(i)$ and $\varphi(j) = \varphi_q(j)$. Since $\varphi_q : I(q) \to I'(q)$ is bijective, we get $\varphi_q(i) \neq \varphi_q(j)$ and then $\varphi(i) \neq \varphi(j)$. Assume now that there exist $q, q' \in \mathcal{L}(A)$ with $q \neq q'$ such that $i \in I(q)$ and $j \in I(q')$. Then $\varphi(i) = \varphi_q(i) \in I'(q)$ and $\varphi(j) = \varphi_{q'}(j) \in I'(q')$. Since $I'(q) \cap I'(q') = \varnothing$ we get $\varphi_q(i) \neq \varphi_q(j)$ and then $\varphi(i) \neq \varphi(j)$.

We complete the proof showing that $p' = p^\varphi$. Consider $i \in I$ and note that $i \in I(p(i))$. Since $\varphi(i) \in I'(p(i))$, we have that $p'(\varphi(i)) = p(i)$. As a consequence, for every $i \in I$, $p'(i) = p(\varphi^{-1}(i)) = p^\varphi(i)$, that is, $p' = p^\varphi$. ☐

**Proposition 7.** *Let $p \in \mathcal{L}(A)^I$. Then $\{p^\varphi \in \mathcal{L}(A)^I : \varphi \in \mathrm{Sym}(I)\} \subseteq \{p' \in \mathcal{L}(A)^I : p' \sim p\}$.*

*Proof.* Let $\varphi \in \mathrm{Sym}(I)$ and show that $p^\varphi \sim p$. Consider $(x,y) \in A^2_*$. Using the definition of $p^\varphi$ and the fact that $\varphi$ is bijective, we have that

$$c_{p^\varphi}(x,y) = |\{i \in I : x \succ_{(p^\varphi)(i)} y\}| = |\{i \in I : x \succ_{p(\varphi^{-1}(i))} y\}|$$

$$= |\varphi(\{i \in I : x \succ_{p(i)} y\})| = |\{i \in I : x \succ_{p(i)} y\}| = c_p(x,y).$$

Since the argument above does not depend on $x$ and $y$, we get that $p^\varphi \sim p$, as desired. ☐

**Theorem 8.** *Let $p \in \mathcal{L}(A)^I$. Then the following conditions are equivalent:*

(*a*) $\{p^\varphi \in \mathcal{L}(A)^I : \varphi \in \mathrm{Sym}(I)\} = \{p' \in \mathcal{L}(A)^I : p' \sim p\}$,

(*b*) *for every $i, j \in I$, $p(i) \neq p(j)^r$.*

*Proof.* (*a*) ⇒ (*b*) Assume that $\{p^\varphi \in \mathcal{L}(A)^I : \varphi \in \mathrm{Sym}(I)\} = \{p' \in \mathcal{L}(A)^I : p' \sim p\}$ and suppose by contradiction that there are $i^*, j^* \in I$ such that $p(i^*) = p(j^*)^r$. Thus, $p(i^*) \neq p(j^*)$ and, in particular, $i^* \neq j^*$. Since $|\mathcal{L}(A)| = 6$, there exists $q \in \mathcal{L}(A)$ such that $q \notin \{p(i^*), p(j^*)\}$. As a consequence, $\{q, q^r\} \cap \{p(i^*), p(j^*)\} = \varnothing$, since $q^r = p(i^*)$ leads to the contradiction $q = p(i^*)^r = p(j^*)$ and $q^r = p(j^*)$ leads to the contradiction $q = p(j^*)^r = p(i^*)$.

Consider now $p' \in \mathcal{L}(A)^I$ defined, for every $i \in I$, by

$$p'(i) = \begin{cases} p(i) & \text{if } i \neq i^* \text{ and } i \neq j^* \\ q & \text{if } i = i^* \\ q^r & \text{if } i = j^* \end{cases}$$

We claim that $p' \sim p$. Indeed, consider $(x,y) \in A^2_*$. Then

$$c_p(x,y) = |\{i \in I : x \succ_{p(i)} y\}| = |\{i \in I \setminus \{i^*, j^*\} : x \succ_{p(i)} y\}| + |\{i \in \{i^*, j^*\} : x \succ_{p(i)} y\}|$$

and

$$c_{p'}(x,y) = |\{i \in I : x \succ_{p'(i)} y\}| = |\{i \in I \setminus \{i^*, j^*\} : x \succ_{p'(i)} y\}| + |\{i \in \{i^*, j^*\} : x \succ_{p'(i)} y\}|.$$

Since, for every $i \in I \setminus \{i^*, j^*\}$, $p(i) = p'(i)$, we have that

$$|\{i \in I \setminus \{i^*, j^*\} : x \succ_{p(i)} y\}| = |\{i \in I \setminus \{i^*, j^*\} : x \succ_{p'(i)} y\}|.$$

Moreover, since $p(i^*) = p(j^*)^r$ and $p'(i^*) = p'(j^*)^r$, we have that

$$|\{i \in \{i^*, j^*\} : x \succ_{p(i)} y\}| = |\{i \in \{i^*, j^*\} : x \succ_{p'(i)} y\}| = 1.$$

Then we conclude that $c_p(x,y) = c_{p'}(x,y)$. As the argument does not depend on $x$ and $y$ we deduce that $p \sim p'$.

Observe now that $|\{i \in I : p'(i) = q\}| = |\{i \in I : p(i) = q\}| + 1$ so that, in particular, $|\{i \in I : p'(i) = q\}| \neq |\{i \in I : p(i) = q\}|$. Thus, by Proposition 6, we get that $p' \notin \{p^\varphi \in \mathcal{L}(A)^I : \varphi \in \mathrm{Sym}(I)\}$ and this fact contradicts $(a)$.

$(b) \Rightarrow (a)$ Assume now that, for every $i, j \in I$, $p(i) \neq p(j)^r$ and suppose by contradiction that $(a)$ does not hold true. Thus, by Proposition 7, there exists $p' \in \mathcal{L}(A)^I$ such that $p' \sim p$ and such that $p' \notin \{p^\varphi \in \mathcal{L}(A)^I : \varphi \in \mathrm{Sym}(I)\}$. Let us define

$$\mathcal{I} = \{K \subseteq I : \exists \rho : K \to I \text{ injective such that } p'(\rho(i)) = p(i) \text{ for all } i \in K\}.$$

Note that $\varnothing \in \mathcal{I}$, since $\rho : \varnothing \to I$ satisfies the required properties, and that $I \notin \mathcal{I}$, since otherwise we would have $p' = p^\rho$ for a suitable $\rho \in \mathrm{Sym}(I)$. Consider now $K^* \in \mathcal{I}$ such that $|K^*| \geq |K|$ for all $K \in \mathcal{I}$ and let $\rho^* : K^* \to I$ be injective and such that $p'(\rho^*(i)) = p(i)$ for all $i \in K^*$. Since $K^* \neq I$, we have that $I \setminus K^* \neq \varnothing$ as well as $I \setminus \rho^*(K^*) \neq \varnothing$.

We claim now that, for every $i \in I \setminus K^*$ and $j \in I \setminus \rho^*(K^*)$, $p(i) \neq p'(j)$. Indeed, if by contradiction there were $i^* \in I \setminus K^*$ and $j^* \in I \setminus \rho^*(K^*)$ such that $p(i^*) = p'(j^*)$, we could consider $\rho' : K^* \cup \{i^*\} \to I$ defined, for every $i \in K^* \cup \{i^*\}$, as

$$\rho'(i) = \begin{cases} \rho^*(i) & \text{if } i \in K^* \\ j^* & \text{if } i = i^* \end{cases}$$

This function is clearly injective and $p'(\rho'(i)) = p(i)$ for all $i \in K^* \cup \{i^*\}$. Thus, $K^* \cup \{i^*\} \in \mathcal{I}$ and since $|K^* \cup \{i^*\}| > |K^*|$ the contradiction is found.

Consider then the nonempty sets

$$U = \{p(i) \in \mathcal{L}(A) : i \in I \setminus K^*\}, \quad V = \{p'(j) \in \mathcal{L}(A) : j \in I \setminus \rho^*(K^*)\}.$$

By the previous claim we have that $U \cap V = \varnothing$. By $(b)$ we also know that if $q \in U$, then $q^r \notin U$ so that $|U| \leq \frac{1}{2}|\mathcal{L}(A)| = 3$. Define now, for every $(x,y) \in A^2_*$,

$$c_p^U(x,y) = |\{i \in I \setminus K^* : x \succ_{p(i)} y\}|, \quad c_{p'}^V(x,y) = |\{j \in I \setminus \rho^*(K^*) : x \succ_{p'(j)} y\}|.$$

We claim that $c_p^U(x,y) = c_{p'}^V(x,y)$. Indeed, since $p' \sim p$, we know that $c_p(x,y) = c_{p'}(x,y)$. Moreover

$$c_p(x,y) = |\{i \in K^* : x \succ_{p(i)} y\}| + c_p^U(x,y),$$

and

$$c_{p'}(x,y) = |\{j \in \rho^*(K^*) : x \succ_{p'(j)} y\}| + c_{p'}^V(x,y).$$

Since $\rho^*$ is a bijection from $K^*$ to $\rho^*(K^*)$, denoting by $(\rho^*)^{-1} : \rho^*(K^*) \to K^*$ its inverse, we also have that

$$\{i \in K^* : x \succ_{p(i)} y\} = \{i \in K^* : x \succ_{p'(\rho^*(i))} y\} = (\rho^*)^{-1}\left(\{j \in \rho^*(K^*) : x \succ_{p'(j)} y\}\right)$$

and then

$$|\{i \in K^* : x \succ_{p(i)} y\}| = |\{j \in \rho^*(K^*) : x \succ_{p'(j)} y\}|$$

which leads to $c_p^U(x,y) = c_{p'}^V(x,y)$.

Taking now into account the possible sizes of $U$, there are three cases to discuss. In what follows, we set $\tau = |I \setminus K^*| = |I \setminus \rho^*(K^*)| > 0$.

$(i)$ Assume first that $|U| = 1$ and let $xyz$ be its unique element. Then, $c_p^U(x,y) = c_p^U(y,z) = \tau$. Thus, $c_{p'}^V(x,y) = c_{p'}^V(y,z) = \tau$. Consider now $j \in I \setminus \rho^*(K^*)$. Then we necessarily have that $x \succ_{p'(j)} y$, $y \succ_{p'(j)} z$ so that $p'(j) = xyz \in U$. Since $p'(j) \in V$ and $U \cap V = \varnothing$, a contradiction is found.

(*ii*) Assume now that $|U| = 2$ and let $xyz$ be one of its elements. Then the other element of $U$ is one among $xzy$, $yxz$, $yzx$, $zxy$ since the reversal of $xyz$, that is $zyx$, cannot belong to $U$. Let us consider then all the possible cases.

(*ii*.1) Assume that $U = \{xyz, xzy\}$. We have $c_p^U(x,y) = c_p^U(x,z) = \tau$ and then $c_{p'}^V(x,y) = c_{p'}^V(x,z) = \tau$. Considering now $j \in I \setminus \rho^*(K^*)$, we necessarily have that $x \succ_{p'(j)} y$ and $x \succ_{p'(j)} z$, so that $p'(j) \in U$. Since $p'(j) \in V$ and $U \cap V = \varnothing$, a contradiction is found.

(*ii*.2) Assume that $U = \{xyz, yxz\}$. We have $c_p^U(x,z) = c_p^U(y,z) = \tau$ and then $c_{p'}^V(x,z) = c_{p'}^V(y,z) = \tau$. Considering now $j \in I \setminus \rho^*(K^*)$, we necessarily have that $x \succ_{p'(j)} z$ and $y \succ_{p'(j)} z$, so that $p'(j) \in U$. Since $p'(j) \in V$ and $U \cap V = \varnothing$, a contradiction is found.

(*ii*.3) Assume that $U = \{xyz, yzx\}$. Let

$$|\{i \in I \setminus K^* : p(i) = xyz\}| = s, \quad |\{i \in I \setminus K^* : p(i) = yzx\}| = s'.$$

Of course, $s, s' \geq 1$ and $s + s' = \tau$. We have $c_p^U(y,z) = \tau$ and then $c_{p'}^V(y,z) = \tau$. Considering $j \in I \setminus \rho^*(K^*)$, we necessarily have that $y \succ_{p'(j)} z$, so that $p'(j) \in \{xyz, yxz, yzx\}$. Thus, since $p'(j) \notin U$, $p'(j) = yxz$. As a consequence, $c_{p'}^V(y,x) = \tau$ while $c_p^U(y,x) = s' < \tau$ since $s \geq 1$ and $s + s' = \tau$. In particular, $c_p^U(y,x) \neq c_{p'}^V(y,x)$ and the contradiction is found.

(*ii*.4) Assume that $U = \{xyz, zxy\}$ and that

$$|\{i \in I \setminus K^* : p(i) = xyz\}| = s, \quad |\{i \in I \setminus K^* : p(i) = zxy\}| = s'.$$

Of course, $s, s' \geq 1$ and $s + s' = \tau$. We have $c_p^U(x,y) = \tau$ and then $c_{p'}^V(x,y) = \tau$. Considering $j \in I \setminus \rho^*(K^*)$, we necessarily have that $x \succ_{p'(j)} y$, so that $p'(j) \in \{xyz, xzy, zxy\}$. Thus, since $p'(j) \notin U$, $p'(j) = xzy$. As a consequence, $c_{p'}^V(x,z) = \tau$ while $c_p^U(x,z) = s < \tau$ since $s' \geq 1$ and $s + s' = \tau$. In particular, $c_p^U(x,z) \neq c_{p'}^V(x,z)$ and the contradiction is found.

(*iii*) Assume finally that $|U| = 3$ and that one of the elements of $U$ is $xyz$. Then the other two elements of $U$ are among $xzy$, $yxz$, $yzx$, $zxy$ since the reversal of $xyz$, that is $zyx$, cannot belong to $U$. Moreover, $U \setminus \{xyz\} \neq \{xzy, yzx\}$ and $U \setminus \{xyz\} \neq \{yxz, zxy\}$, since $U$ cannot contain a linear order and its reversal. There are then four cases to discuss.

(*iii*.1) Assume that $U = \{xyz, xzy, yxz\}$. Since $U \cap V = \varnothing$, we have that $V \subseteq \{yzx, zxy, zyx\}$. Since $c_p^U(x,z) = \tau$, $c_{p'}^V(x,z) = 0$ and $c_p^U(x,z) = c_{p'}^V(x,z)$, we get the contradiction $\tau = 0$.

(*iii*.2) Assume that $U = \{xyz, xzy, zxy\}$. Since $U \cap V = \varnothing$, we have that $V \subseteq \{yxz, yzx, zyx\}$. Since $c_p^U(x,y) = \tau$, $c_{p'}^V(x,y) = 0$ and $c_p^U(x,y) = c_{p'}^V(x,y)$, we get the contradiction $\tau = 0$.

(*iii*.3) Assume that $U = \{xyz, yxz, yzx\}$. Since $U \cap V = \varnothing$, we have that $V \subseteq \{xzy, zxy, zyx\}$. Since $c_p^U(y,z) = \tau$, $c_{p'}^V(y,z) = 0$ and $c_p^U(y,z) = c_{p'}^V(y,z) = 0$, we get the contradiction $\tau = 0$.

(*iii*.4) Assume that $U = \{xyz, yzx, zxy\}$. Since $U \cap V = \varnothing$, we have that $V \subseteq \{xzy, yxz, zyx\}$. Let us set

$$|\{i \in I \setminus K^* : p(i) = xyz\}| = s,$$
$$|\{i \in I \setminus K^* : p(i) = yzx\}| = s',$$
$$|\{i \in I \setminus K^* : p(i) = zxy\}| = s''$$

and

$$|\{j \in I \setminus \rho^*(K^*) : p'(j) = zyx\}| = t,$$
$$|\{j \in I \setminus \rho^*(K^*) : p'(j) = xzy\}| = t',$$
$$|\{j \in I \setminus \rho^*(K^*) : p'(j) = yxz\}| = t'',$$

where $s+s'+s''=t+t'+t''=\tau$ and $s,s',s''\geq 1$. A simple computation shows that $c_p^U(x,y)=s+s''$, $c_p^U(y,z)=s+s'$, $c_p^U(z,x)=s'+s''$, $c_{p'}^V(x,y)=t'$, $c_{p'}^V(y,z)=t''$, $c_{p'}^V(z,x)=t$. Since $c_p^U(x,y)=c_{p'}^V(x,y)$, $c_p^U(y,z)=c_{p'}^V(y,z)$ and $c_p^U(z,x)=c_{p'}^V(z,x)$, we get $s+s''=t'$, $s+s'=t''$ and $s'+s''=t$. It follows then that $2(s+s'+s'')=t+t'+t''$, that is $2\tau=\tau$. That leads to the contradiction $\tau=0$.

□

Define the set

$$\mathcal{Q}_I=\{p\in\mathcal{L}(A)^I:\forall i,j\in I,\ p(i)\neq p(j)^r\}.$$

Note that if $p\in\mathcal{Q}_I$ and $\varphi\in\mathrm{Sym}(I)$, then $p^\varphi\in\mathcal{Q}_I$. Consider now $F:\mathcal{Q}_I\to A$. We say that $F$ is Pareto optimal if, for every $p\in\mathcal{Q}_I$ and $x,y\in A$, if $x\succ_{p(i)} y$ for all $i\in I$, then $F(p)\neq y$; dictatorial with dictator $i\in I$ if, for every $p\in\mathcal{Q}_I$, $F(p)=r_1(p(i))$; dictatorial if there exists $i\in I$ such that $F$ is dictatorial with dictator $i$; strategy-proof if, for every $i\in I$, $p\in\mathcal{L}(A)^{I\setminus\{i\}}$ and $q,q'\in\mathcal{L}(A)$ with $(q[i],p)\in\mathcal{Q}_I$ and $(q'[i],p)\in\mathcal{Q}_I$, $F(q[i],p)\succeq_q F(q'[i],p)$. Moreover, we say that individual $i\in I$ is able to manipulate $F$ at $p\in\mathcal{Q}_I$ via $q\in\mathcal{L}(A)$ if $(q[i],p_{|I\setminus\{i\}})\in\mathcal{Q}_I$ and $F(q[i],p_{|I\setminus\{i\}})\succ_{p(i)} F(p)$. Of course, $F$ is strategy-proof if and only if, for every $i\in I$, $p\in\mathcal{Q}_I$ and $q\in\mathcal{L}(A)$, individual $i$ is not able to manipulate $F$ at $p$ via $q$, that is, if $(q[i],p_{|I\setminus\{i\}})\in\mathcal{Q}_I$ then $F(p)\succeq_{p(i)} F(q[i],p_{|I\setminus\{i\}})$.

The next proposition is inspired to Theorem 1 in Lars-Gunnar and Reffgen (2014).

**Proposition 9.** *Let $|I|=2$ and let $F:\mathcal{Q}_I\to A$. If $F$ is Pareto optimal and strategy-proof, then $F$ is dictatorial.*

*Proof.* Assume that $I=\{i_1,i_2\}$ and, given $p\in\mathcal{Q}_I$, identify $p$ with the vector $(p(i_1),p(i_2))$. Define then the sets

$$A_1=\{x\in A:\forall p\in\mathcal{Q}_I \text{ such that } r_1(p(i_1))=x, F(p)=x\},$$

$$A_2=\{x\in A:\forall p\in\mathcal{Q}_I \text{ such that } r_1(p(i_2))=x, F(p)=x\}.$$

Consider distinct $x,y,z\in A$ and $(xyz,yxz)\in\mathcal{Q}_I$ and note that, by Pareto optimality $F(xyz,yxz)\neq z$. We claim that

$$F(xyz,yxz)=x \text{ implies } x\in A_1, \tag{1}$$

$$F(xyz,yxz)=y \text{ implies } y\in A_2. \tag{2}$$

Let us first prove (1). Set $p^1=(xyz,yxz)$ and assume $F(p^1)=x$. Consider then the following elements of $\mathcal{Q}_I$

$$p^2=(xyz,yzx),\quad p^3=(xyz,zxy),\quad p^4=(xyz,xyz),\quad p^5=(xyz,xzy),\quad p^6=(xzy,xyz)$$

$$p^7=(xzy,xzy),\quad p^8=(xzy,yxz),\quad p^9=(xzy,zxy),\quad p^{10}=(xzy,zyx),$$

and note that $\{p\in\mathcal{Q}_I:r_1(p(i_1))=x\}=\{p^1,p^2,\ldots,p^{10}\}$. Thus, we get $x\in A_1$ proving that, for every $i\in\{2,\ldots,10\}$, $F(p^i)=x$. We proceed by a case-by-case analysis. By Pareto optimality $F(p^2)\neq z$. If $F(p^2)=y$ then individual $i_2$ would be able to manipulate $F$ at $p^1$ via $yzx$. As a consequence, $F(p^2)=x$. By Pareto optimality $F(p^3)\neq y$. If $F(p^3)=z$ then individual $i_2$ would be able to manipulate $F$ at $p^2$ via $zxy$. As a consequence, $F(p^3)=x$. By Pareto optimality, $F(p^4)=F(p^5)=F(p^6)=F(p^7)=x$. $F(p^8)=x$ otherwise individual $i_1$ would be able to manipulate $F$ at $p^8$ via $xyz$. $F(p^9)=x$ otherwise individual $i_1$ would be able to manipulate $F$ at $p^9$ via $xyz$. By Pareto optimality, $F(p^{10})\neq y$. If $F(p^{10})=z$ then individual $i_2$ would be able to manipulate $F$ at $p^9$ via $zyx$. As a consequence, $F(p^{10})=x$ and the proof of (1) is completed. The proof of (2) is similar and then omitted.

Consider now $p^*=(abc,bac)\in\mathcal{Q}_I$ and note that, by Pareto optimality, $F(p^*)\neq c$ so that $F(p^*)\in\{a,b\}$. We complete the proof of the theorem showing that $F(p^*)=a$ implies $A_1=A$, that is, $F$ is dictatorial with dictator $i_1$, and that $F(p^*)=b$ implies that $A_2=A$, that is, $F$ is dictatorial with dictator $i_2$.

Assume first that $F(p^*) = a$. By (1), we have that $a \in A_1$. Considering now $(bca, cba) \in \mathcal{Q}_I$, by (1) and (2), we get that $b \in A_1$ or $c \in A_2$. However, $c \notin A_2$, otherwise, given $(acb, cab) \in \mathcal{Q}_I$, we would have both $F(acb, cab) = a$ and $F(acb, cab) = c$, a contradiction. Then we get $b \in A_1$. Since $\{a, b\} \subseteq A_1$, we also have that $a \notin A_2$. Indeed, if $a \in A_2$, then considering $(bac, abc) \in \mathcal{Q}_I$ we should have both $F(bac, abc) = b$ and $F(bac, abc) = a$, a contradiction. Let us finally prove that $c \in A_1$. Indeed, consider $(cab, acb) \in \mathcal{Q}_I$. By (1) and (2), we have that $c \in A_1$ or $a \in A_2$. Since we know that $a \notin A_2$, we get $c \in A_1$.

The proof that $F(p^*) = b$ implies $A_2 = A$ is analogous and then omitted. □

The next result shows that any SCF which is Pareto optimal and strategy-proof on $\mathcal{Q}_I$ is dictatorial. It is inspired to Lemma 3 and Theorem 2 in Lars-Gunnar and Reffgen (2014). We stress that, at the best of our knowledge, it is not a consequence of any of the known results within the theory of dictatorial domains (Aswal et al., 2003, Pramanik, 2015).

**Theorem 10.** *Let $F : \mathcal{Q}_I \to A$. If $F$ is Pareto optimal and strategy-proof, then $F$ is dictatorial.*

*Proof.* Let us first rephrase the theorem as follows: for every $h \in \mathbb{N}$ with $h \geq 2$,

if $I$ is a finite set with $|I| = h$ and $F : \mathcal{Q}_I \to A$ is Pareto optimal and strategy-proof, then $F$ is dictatorial. (3)

We are going to prove the above result working by induction on $h$. By Proposition 9 we know that (3) is true when $h = 2$. In order to complete the proof of the theorem, assume then that (3) is true for $h \geq 2$ and show that (3) is true for $h + 1$ too.

Consider then $I$ such that $|I| = h + 1 \geq 3$ and $F : \mathcal{Q}_I \to A$ which is Pareto optimal and strategy-proof. For every $(i, j) \in I^2_*$, let us consider the function $\gamma_{(i,j)} : \mathcal{Q}_{I \setminus \{j\}} \to \mathcal{Q}_I$ such that, for every $p \in \mathcal{Q}_{I \setminus \{j\}}$,

$$\gamma_{(i,j)}(p) = (p, p(i)[j]).$$

Note that $\gamma_{(i,j)}$ is well defined, that is, if $p \in \mathcal{Q}_{I \setminus \{j\}}$, then $\gamma_{(i,j)}(p)$ is an element of $\mathcal{Q}_I$.

For every $(i, j) \in I^2_*$, let $G_{(i,j)} : \mathcal{Q}_{I \setminus \{j\}} \to A$ be defined, for every $p \in \mathcal{Q}_{I \setminus \{j\}}$, as

$$G_{(i,j)}(p) = F(\gamma_{(i,j)}(p)).$$

We are going to show that, for every $(i, j) \in I^2_*$, $G_{(i,j)}$ is dictatorial. Since $G_{(i,j)}$ refers to the set of individuals $I \setminus \{j\}$ whose size is $h$, we can get such a result by applying the inductive assumption, provided that $G_{(i,j)}$ is Pareto optimal and strategy proof.

Let us prove then that $G_{(i,j)}$ is Pareto optimal. Indeed, consider $p \in \mathcal{Q}_{I \setminus \{j\}}$ and $x, y \in A$ such that, for every $k \in I \setminus \{j\}$, $x \succ_{p(k)} y$. Then, for every $k \in I$, $x \succ_{(\gamma_{(i,j)}(p))(k)} y$. Since $F$ is Pareto optimal we get that $y \notin F(\gamma_{(i,j)}(p)) = G_{(i,j)}(p)$, and the Pareto optimality of $G_{(i,j)}$ is proved.

Let us prove now that $G_{(i,j)}$ is strategy-proof. Fix then $t \in I \setminus \{j\}$, $q, \hat{q} \in \mathcal{L}(A)$ and $p \in \mathcal{L}(A)^{I \setminus \{j,t\}}$ such that $(q[t], p), (\hat{q}[t], p) \in \mathcal{Q}_{I \setminus \{j\}}$ and show that $G_{(i,j)}(q[t], p) \succeq_q G_{(i,j)}(\hat{q}[t], p)$. Assume first that $t \neq i$. Using the fact that $F$ is strategy-proof, we get

$$G_{(i,j)}(q[t], p) = F(\gamma_{(i,j)}(q[t], p)) = F(q[t], p, p(i)[j])$$

$$\succeq_q F(\hat{q}[t], p, p(i)[j]) = F(\gamma_{(i,j)}(\hat{q}[t], p)) = G_{(i,j)}(\hat{q}[t], p),$$

as desired. Assume now that $t = i$. Using the fact that $F$ is strategy-proof, we get

$$G_{(i,j)}(q[i], p) = F(\gamma_{(i,j)}(q[i], p)) = F(q[i], p, q[j])$$

$$\succeq_q F(\hat{q}[i], p, q[j]) \succeq_q F(\hat{q}[i], p, \hat{q}[j]) = F(\gamma_{(i,j)}(\hat{q}[i], p)) = G_{(i,j)}(\hat{q}[i], p),$$

as desired. Thus, we conclude that $G_{(i,j)}$ is strategy-proof.

Let us prove now that $F$ is dictatorial. Assume first that there exist $(i^*, j^*) \in I^2_*$ and $t^* \in I \setminus \{j^*\}$ with $t^* \neq i^*$ such that $G_{(i^*,j^*)}$ is dictatorial with dictator $t^*$ and prove that $F$ is dictatorial with dictator $t^*$. Consider $p \in \mathcal{Q}_I$ and assume that $r_1(p(t^*)) = x$ and $F(p) = y$. Our purpose is then to show that $x = y$. Since $F$ is strategy-proof and since

$$F\left(p_{|I\setminus\{i^*,j^*\}}, p(i^*)[i^*], p(j^*)[j^*]\right) = F(p) = y \tag{4}$$

and[13]

$$F\left(p_{|I\setminus\{i^*,j^*\}}, p(j^*)[i^*], p(j^*)[j^*]\right) = G_{(i^*,j^*)}\left(p_{|I\setminus\{i^*,j^*\}}, p(j^*)[i^*]\right) = r_1(p(t^*)) = x,$$

we get $y \succeq_{p(i^*)} x$. Moreover, since $F$ is strategy-proof,

$$F\left(p_{|I\setminus\{i^*,j^*\}}, p(i^*)[i^*], p(i^*)[j^*]\right) = G_{(i^*,j^*)}\left(p_{|I\setminus\{i^*,j^*\}}, p(i^*)[i^*]\right) = r_1(p(t^*)) = x$$

and (4), we also get $x \succeq_{p(i^*)} y$. Since $p(i^*)$ is a linear order, from $y \succeq_{p(i^*)} x$ and $x \succeq_{p(i^*)} y$ we finally obtain $x = y$.

We complete the proof of the theorem showing that it cannot happen that, for every $(i,j) \in I^2_*$, $G_{(i,j)}$ is dictatorial with dictator $i$. Indeed, assume by contradiction otherwise. Then, for every $p \in \mathcal{Q}_I$ such $p(i) = p(j)$ for some $(i,j) \in I^2_*$, we have $F(p) = r_1(p(i))$ since

$$F(p) = F(p_{|I\setminus\{i,j\}}, p(i)[i], p(i)[j]) = G_{(i,j)}(p_{|I\setminus\{i,j\}}, p(i)[i]) = r_1(p(i)).$$

Thus, if $|I| = h + 1 = 3$ and $I = \{i_1, i_2, i_3\}$, consider $p \in \mathcal{Q}_I$ such that $p(i_1) = abc$, $p(i_2) = bca$, $p(i_3) = cab$. Note that if $F(p) = a$, then individual $i_2$ is able to manipulate $F$ at $p$ via $cab$; if $F(p) = b$, then individual $i_3$ is able to manipulate $F$ at $p$ via $abc$; if $F(p) = c$, then individual $i_1$ is able to manipulate $F$ at $p$ via $bca$[14]. Since $F$ is strategy-proof we get then a contradiction. If $|I| = h + 1 \geq 4$, consider $p \in \mathcal{Q}_I$ such that there are distinct $i_1, i_2, i_3, i_4 \in I$ such that $p(i_1) = p(i_2) = abc$, $p(i_3) = p(i_4) = bac$. Then both $F(p) = a$ and $F(p) = b$, a contradiction. □

**Theorem 11.** *Let $F : \mathcal{L}(A)^I \to A$. If $F$ is Pareto optimal and PC-strategy-proof, then $F$ is not anonymous.*

*Proof.* Assume by contradiction that there exists $F : \mathcal{L}(A)^I \to A$ which is Pareto optimal, PC-strategy-proof and anonymous. Let $G$ be the restriction of $F$ to the set $\mathcal{Q}_I$. We are going to prove that $G$ is dictatorial proving that it is Pareto optimal and strategy-proof and applying Theorem 10.

Let us first prove that $G$ is Pareto optimal. Consider $p \in \mathcal{Q}_I$ and $x, y \in A$ such that, for every $i \in I$, $x \succ_{p(i)} y$. Then $F(p) \neq y$ and since $G(p) = F(p)$ we also get $G(p) \neq y$.

Let us now prove that $G$ is strategy-proof. Consider $i \in I$, $p \in \mathcal{L}(A)^{I\setminus\{i\}}$ and $q, q' \in \mathcal{L}(A)$ with $(q[i], p), (q'[i], p) \in \mathcal{Q}_I$ and assume by contradiction that $G(q'[i], p) \succ_q G(q[i], p)$, that is,

$$F(q'[i], p) \succ_q F(q[i], p). \tag{5}$$

Since $F$ is PC-strategy-proof there exists $p' \in \mathcal{L}(A)^{I\setminus\{i\}}$ with $p' \sim p$ such that

$$F(q[i], p') \succ_q F(q'[i], p'). \tag{6}$$

Observe now that since $(q[i], p) \in \mathcal{Q}_I$, we have that $p \in \mathcal{Q}_{I\setminus\{i\}}$. Then, by Theorem 8, there exists $\varphi \in \mathrm{Sym}(I \setminus \{i\})$, such that $p' = p^\varphi$. Consider now $\hat{\varphi} \in \mathrm{Sym}(I)$ such that, for every $j \in I \setminus \{i\}$, $\hat{\varphi}(j) = \varphi(j)$ and $\hat{\varphi}(i) = i$. It is easily checked that

$$(q[i], p') = (q[i], p)^{\hat{\varphi}} \quad \text{and} \quad (q'[i], p') = (q'[i], p)^{\hat{\varphi}} \tag{7}$$

so that, in particular, both $(q[i], p')$ and $(q'[i], p')$ belong to $\mathcal{Q}_I$. By (7) and the fact that $F$ is anonymous we have that

$$F(q[i], p') = F\left((q[i], p)^{\hat{\varphi}}\right) = F(q[i], p) \tag{8}$$

[13] Note that $p \in \mathcal{Q}_I$ implies $\left(p_{|I\setminus\{i^*,j^*\}}, p(j^*)[i^*], p(j^*)[j^*]\right) \in \mathcal{Q}_I$ and $\left(p_{|I\setminus\{i^*,j^*\}}, p(i^*)[i^*], p(i^*)[j^*]\right) \in \mathcal{Q}_I$.

[14] Observe that in each of the three considered cases, the preference profile obtained after manipulating $F$ are in $\mathcal{Q}_I$.

and

$$F(q'[i],p') = F\left((q'[i],p)^{\hat{\varphi}}\right) = F(q'[i],p). \tag{9}$$

By (6), (8) and (9), we get $F(q[i],p) \succ_q F(q'[i],p)$ which contradicts (5).

We deduce then that $G$ is dictatorial, so that there exists $i^* \in I$ such that, for every $p \in \mathcal{Q}_I$, $G(p) = r_1(p(i^*))$. Consider now $j^* \in I \setminus \{i^*\}$, $p \in \mathcal{Q}_I$ such that $p(i^*) = abc$ and $p(j^*) = bac$ and $\varphi \in \mathrm{Sym}(I)$ such that $\varphi(i^*) = j^*$, $\varphi(j^*) = i^*$ and $\varphi(i) = i$ for all $i \in I \setminus \{i^*, j^*\}$. Then, we have that

$$F(p) = G(p) = r_1(p(i^*)) = r_1(abc) = a$$

and, since also $p^\varphi \in \mathcal{Q}_I$,

$$F(p^\varphi) = G(p^\varphi) = r_1(p^\varphi(i^*)) = r_1(p(\varphi^{-1}(i^*))) = r_1(p(j^*)) = r_1(bac) = b.$$

However, since $F$ is anonymous, we also have $F(p) = F(p^\varphi)$, a contradiction. □

## A.2 The general case

By Theorem 11, we know that the result is true when $|A| = 3$. Consider now $|A| = m \geq 4$ and assume by contradiction that there exists $F : \mathcal{L}(A)^I \to A$ which is Pareto optimal, PC-strategy-proof and anonymous.

Let us fix an ordering $x_1, \ldots, x_m$ of the elements of $A$ and let $\hat{A} = \{x_1, x_2, x_3\}$ and

$$U = \{(x_r, x_t) \in A^2 : r, t \geq 4, r \leq t\}.$$

Of course, $\hat{A} \subseteq A$, $|\hat{A}| = 3$ and $U \subseteq (A \setminus \hat{A})^2$. Let us consider now $\omega : \mathcal{L}(\hat{A}) \to \mathcal{L}(A)$ associating with every $\hat{q} \in \mathcal{L}(\hat{A})$ the element of $\mathcal{L}(A)$ given by

$$\omega(\hat{q}) = \hat{q} \cup \left(\hat{A} \times (A \setminus \hat{A})\right) \cup U.$$

In other words, if $\hat{q} = xyz$, where $x, y, z \in \hat{A}$, then $\omega(\hat{q}) = xyzx_4 \cdots x_m$. Given now $J \subseteq I$ with $J$ nonempty and $\hat{p} \in \mathcal{L}(\hat{A})^J$, we denote by $\omega(\hat{p})$ the the element of $\mathcal{L}(A)^J$ such that, for every $i \in J$, $\omega(\hat{p})(i) = \omega(\hat{p}(i))$. Finally let $G : \mathcal{L}(\hat{A})^I \to \hat{A}$ be defined, for every $\hat{p} \in \mathcal{L}(\hat{A})^I$, by $G(\hat{p}) = F(\omega(\hat{p}))$. Note that $G$ is well-defined, that is, for every $\hat{p} \in \mathcal{L}(\hat{A})^I$, $F(\omega(\hat{p})) \in \hat{A}$. Indeed, let $\hat{p} \in \mathcal{L}(\hat{A})^I$ and $y \in A \setminus \hat{A}$. Picking any $x \in \hat{A}$, we have that $x \succ_{\omega(\hat{p})(i)} y$ for all $i \in I$ and then, since $F$ is Pareto optimal, we get $F(\omega(\hat{p})) \neq y$. We complete the proof of the theorem showing that $G$ is Pareto optimal, PC-strategy-proof and anonymous. Indeed, since $|\hat{A}| = 3$, that contradicts Theorem 11.

Let us first prove that $G$ is Pareto optimal. Consider $\hat{p} \in \mathcal{L}(\hat{A})^I$ and $x, y \in \hat{A}$ and assume that, for every $i \in I$, $x \succ_{\hat{p}(i)} y$. Then, for every $i \in I$, $x \succ_{\omega(\hat{p}(i))} y$, that is, $x \succ_{\omega(\hat{p})(i)} y$. Since $F$ is Pareto optimal, $G(\hat{p}) = F(\omega(\hat{p})) \neq y$.

Let us now prove that $G$ is PC-strategy-proof. Consider $i \in I$, $\hat{p} \in \mathcal{L}(\hat{A})^{I \setminus \{i\}}$ and $\hat{q}, \hat{q}' \in \mathcal{L}(\hat{A})$ and assume that $G(\hat{q}'[i], \hat{p}) \succ_{\hat{q}} G(\hat{q}[i], \hat{p})$. We have to show that there exists $\hat{p}' \in \mathcal{L}(\hat{A})^{I \setminus \{i\}}$ with $\hat{p}' \sim \hat{p}$ such that $G(\hat{q}[i], \hat{p}') \succ_{\hat{q}} G(\hat{q}'[i], \hat{p}')$. Since $G(\hat{q}[i], \hat{p}) = F(\omega(\hat{q})[i], \omega(\hat{p})) \in \hat{A}$ and $G(\hat{q}'[i], \hat{p}) = F(\omega(\hat{q}')[i], \omega(\hat{p})) \in \hat{A}$, we get $F(\omega(\hat{q}')[i], \omega(\hat{p})) \succ_{\hat{q}} F(\omega(\hat{q})[i], \omega(\hat{p}))$, so that $F(\omega(\hat{q}')[i], \omega(\hat{p})) \succ_{\omega(\hat{q})} F(\omega(\hat{q})[i], \omega(\hat{p}))$. By PC-strategy-proofness of $F$, there exists $p' \in \mathcal{L}(A)^{I \setminus \{i\}}$ such that $p' \sim \omega(\hat{p})$ and

$$F(\omega(\hat{q})[i], p') \succ_{\omega(\hat{q})} F(\omega(\hat{q}')[i], p'). \tag{10}$$

Consider now $\hat{p}' \in \mathcal{L}(\hat{A})^{I \setminus \{i\}}$ defined, for every $j \in I \setminus \{i\}$, by

$$\hat{p}'(j) = \{(x, y) \in \hat{A}^2 : (x, y) \in p'(j)\}. \tag{11}$$

We claim that

$$\omega(\hat{p}') = p'. \tag{12}$$

In order to prove (12) it is sufficient to prove that, for every $j \in I \setminus \{i\}$, $\omega(\hat{p}'(j)) \subseteq p'(j)$. Indeed, since $\omega(\hat{p}'(j)), p'(j) \in \mathcal{L}(A)$, $\omega(\hat{p}'(j)) \subseteq p'(j)$ implies $\omega(\hat{p}'(j)) = p'(j)$. Fix then $j \in I \setminus \{i\}$ and

$$(x^*, y^*) \in \omega(\hat{p}'(j)) = \hat{p}'(j) \cup \left(\hat{A} \times (A \setminus \hat{A})\right) \cup U.$$

If $(x^*, y^*) \in \hat{p}'(j)$, then by (11), $(x^*, y^*) \in p'(j)$. Assume now that $(x^*, y^*) \in \hat{A} \times (A \setminus \hat{A})$. Note that, in particular, $x^* \neq y^*$and that $c_{\omega(\hat{p})}(x^*, y^*) = |I| - 1$. Since $p' \sim \omega(\hat{p})$, we know that $c_{p'}(x^*, y^*) = c_{\omega(\hat{p})}(x^*, y^*) = |I| - 1$ so that necessarily $(x^*, y^*) \in p'(j)$. Assume finally that $(x^*, y^*) \in U$. Then there exist $r, t \geq 4$ with $r \leq t$ such that $x^* = x_r$ and $y^* = x_t$. If $r = t$, then $x^* = y^*$ and $(x^*, y^*) \in p'(j)$ since $p'(j) \in \mathcal{L}(A)$ and so it is reflexive. If $r < t$ then $x^* \neq y^*$ and $c_{\omega(\hat{p})}(x^*, y^*) = |I| - 1$. Since $p' \sim \omega(\hat{p})$, we know that $c_{p'}(x^*, y^*) = c_{\omega(\hat{p})}(x^*, y^*) = |I| - 1$ so that necessarily $(x^*, y^*) \in p'(j)$. The claim is then proved.

From (10) and (12), we get $F(\omega(\hat{q})[i], \omega(\hat{p}')) \succ_{\omega(\hat{q})} F(\omega(\hat{q}')[i], \omega(\hat{p}'))$, that is, $G(\hat{q}[i], \hat{p}') \succ_{\omega(\hat{q})} G(\hat{q}'[i], \hat{p}')$. Thus, since $G(\hat{q}[i], \hat{p}') \in \hat{A}$ and $G(\hat{q}'[i], \hat{p}') \in \hat{A}$, we deduce $G(\hat{q}[i], \hat{p}') \succ_{\hat{q}} G(\hat{q}'[i], \hat{p}')$. We then conclude the proof of the PC-strategy-proofness of $G$ simply noticing that, by (12) and since $p' \sim \omega(\hat{p})$, for every $(x, y) \in \hat{A}^2_*$,

$$c_{\hat{p}'}(x, y) = c_{\omega(\hat{p}')}(x, y) = c_{p'}(x, y) = c_{\omega(\hat{p})}(x, y) = c_{\hat{p}}(x, y),$$

so that $\hat{p}' \sim \hat{p}$.

We are finally left with proving the anonymity of $G$. Consider $\hat{p} \in \mathcal{L}(\hat{A})^I$ and $\varphi \in \mathrm{Sym}(I)$. Then $\omega(\hat{p}^\varphi) = \omega(\hat{p})^\varphi$. Indeed, for every $i \in I$

$$\omega(\hat{p}^\varphi)(i) = \omega(\hat{p}^\varphi(i)) = \omega(\hat{p}(\varphi^{-1}(i))) = \omega(\hat{p})(\varphi^{-1}(i)) = \omega(\hat{p})^\varphi(i).$$

Then, using the definition of $G$ the the fact that $F$ is anonymous, we get

$$G(\hat{p}^\varphi) = F(\omega(\hat{p}^\varphi)) = F(\omega(\hat{p})^\varphi) = F(\omega(\hat{p})) = G(\hat{p}).$$

Thus, the anonymity of $G$ is proved.

# References


Aleskerov, F., Karabekyan, D., Sanver, M.R., Yakuba, V., 2012. On the manipulability of voting rules: The case of 4 and 5 alternatives. Mathematical Social Sciences 64, 67-73.

Aleskerov, F., Kurbanov, E., 1999. Degree of Manipulability of Social Choice Procedures, in: A. Alkan et.al. (eds.), Current Trends in Economics, Springer, Berlin.

Aswal, N., Chatterji, S., Sen, A., 2003. Dictatorial domains. Economic Theory 22, 45-62.

Campbell, D.E., Kelly, J.S., 2009. Gain from manipulating social choice rules. Economic Theory 40, 349-371.

Campbell, E.D., Kelly, J.S., Shaofang, Q., 2018. A stability property in social choice theory. International Journal of Economic Theory 14, 85-95.

Carrol, G., 2012. When are local incentive constraints sufficient? Econometrica 80, 661-686.

Erdamar, B., Sanver, M.R., Sato, S., 2017. Evaluationwise strategy-proofness. Games and economic Behavior 106, 227-238.

Fishburn, P.C., 1977. Condorcet social choice functions. SIAM Journal on Applied Mathematics 33, 469-489.

Gibbard, A., 1973. Manipulation of voting schemes: a general result. Econometrica 41, 587-601.

Kelly, J.S., 1988. Minimal manipulability and local strategy-proofness. Social Choice and Welfare 5, 81-85.

Kelly, J.S., 1993. Almost all social choice rules are highly manipulable, but a few aren't. Social Choice and Welfare 10, 161-175.

Lars-Gunnar, S., Reffgen, A., 2014. The proof of the Gibbard-Satterthwaite theorem revisited. Journal of Mathematical Economics 55, 11-14.

Moulin, H., 1983. The Strategy of Social Choice. In: Advanced Textbooks in Economics, North Holland Publishing Company.

Muto, N., Sato, S., 2017. An impossibility under bounded responce of social choice functions. Games and Economic Behaviour 106, 1-15.

Nitzan, S., 1985. The vulnerability of point-voting schemes to preference variation and strategic manipulation. Public Choice 47, 349-370.

Nurmi, H., 1987. Comparing voting systems. Reidel, Dordrecht.

Pramanik, A., 2015. Further results on dictatorial domains. Social Choice and Welfare 45, 379-398.

Reffgen, A., 2011. Generalizing the Gibbard-Satterthwaite theorem: partial preferences, the degree of manipulation and multi valuedness. Social Choice and Welfare 37, 39-59.

Sato, S., 2013. A sufficient condition for the equivalence of strategy-proofness and nonmanipulability by preferences adjacent to the sincere one. Journal of Economic Theory 148, 259-278.

Satterthwaite, M.A., 1975. Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. Journal of Economic Theory 10, 187-217.