Continuity and Robustness of Bayesian Equilibria in Tullock Contests*

Ezra Einy[†], Diego Moreno[‡], Aner Sela[§]

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Abstract

We study the continuity and robustness of the Bayesian equilibria of Tullock contests with incomplete information. We show that the Bayesian equilibrium correspondence is upper semicontinuous. We identify conditions under which the Bayesian equilibrium correspondence of Tullock contests with a unique equilibrium is also lower semicontinous. Furthermore, when the Bayesian equilibrium is unique, it is robust to small perturbations of the contest's attributes (the contest success function, and the players' information, value for the prize, and cost of effort).

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[†]Department of Economics, Ben-Gurion University of the Negev.

[‡]Departamento de Economía, Universidad Carlos III de Madrid.

[§]Department of Economics, Ben-Gurion University of the Negev.

1 Introduction

In a contest, a group of individuals compete for a prize by exerting effort. In a Tullock contest the probability that an individual wins the prize is a non-decreasing function of the effort he exerts (see Tullock 1980). In many economic environments, Tullock contests arise either naturally or by design. Baye and Hoppe (2003), for example, have identified conditions under which a variety of rent-seeking contests, innovation tournaments, and patent-race games are strategically equivalent to a Tullock contest. Most of the extensive literature studying the outcomes generated by Tullock contests focuses on the complete information case – see, for example, Nitzan (1994), Skaperdas (1996), Clark and Riis (1998), Konrad (2008), Fu and Lu (2012), and Fu *et al.* (2015). Recently, however, the literature has turned to study the equilibria of Tullock contests with incomplete information, as well as the impact of changes in the players' information endowments on equilibrium outcomes – see, for example, Wasser (2013), Einy, Moreno, and Shitovitz (2017), and Aiche *et al.* (2018 and 2019).

A Tullock contest is identified by the players' value for the prize, their cost of effort, and the impact of effort on the probability of winning the prize. When players have complete information about these *attributes*, a Tullock contests defines a complete information game. The Nash equilibria of this game are the equilibria of the contest. Szidarovszky and Okuguchi (1997), Cornes and Hartley (2005), Yamazaki (2008) and Chowdhury and Sheremeta (2009) have studied the existence and uniqueness of equilibria in Tullock contests with complete information. When players are uncertain about either of these attributes, a Tullock contests defines a Bayesian game. The Bayesian(-Nash) equilibria of this game are the Bayesian equilibria of the contest. Einy et al. (2015), Einy, Moreno, and Shitovitz (2017), and Ewerhart and Quartieri (2018) have studied the existence and uniqueness of Bayesian equilibrium in Tullock contests with incomplete information. The purpose of this paper is to study the continuity and robustness of the equilibrium correspondence of Tullock contests.

Following this literature, we restrict attention to pure strategy equilibria.

In a Tullock contests with incomplete information, we describe players' uncertainty by a probability space and represent the information of each player about the state by a σ-subfield of the field describing players' uncertainty. We study the behavior of the Bayesian equilibrium correspondence in response to perturbations on the players' information endowments, as well as their state-dependent values, costs of effort, and probabilities of winning the prize. We use the Boylan pseudometric to measure the distance between information fields. Einy *et al.* (2005 and 2008) have followed this approach to study the continuity properties of the core of an economy with differential information, and those of the value of zero-sum games under incomplete information, respectively.

It is well known that for general games with incomplete information the Bayesian equilibrium correspondence is usually not continuous. For example, Milgrom and Weber (1985) and Cotter (1991) have shown that it is not upper semicontinuous, whereas Monderer and Samet (1985) have shown that it is not lower semicontinuous. Moreover, even a strict Nash equilibrium of a game with complete information may not be approachable by equilibria of variations of the game in which the players have only minimal incomplete information – see Carlsson and Van Damme (1993). Further, Kaji and Morris (1997) have shown that even the unique strict Nash equilibrium of a complete information game may not be *robust* to incomplete information.

We show that the Bayesian equilibrium correspondence of Tullock contests is well behaved. Specifically, if a sequence of Tullock contests converges to another Tullock contest, then any limit point of a sequence formed by equilibria of this sequence of contests must be an equilibrium of the limiting contest, i.e., that the Bayesian equilibrium correspondence of Tullock contests is upper semicontinous. When the limiting contest has a unique equilibrium, we identify conditions that assure the existence of subsequences of equilibria of the sequence of contests converging to the unique equilibrium of the limiting contest, i.e., that the Bayesian equilibrium correspondence of

Tullock contests with a unique equilibrium is lower semicontinous. Finally, we demonstrate that if a Tullock contests has a unique Bayesian equilibrium, then any Tullock contest sufficiently close (i.e, whose attributes as sufficiently similar) must have a Bayesian equilibrium close to this unique equilibrium. In other words, we show that the unique Bayesian equilibrium in Tullock contests with incomplete information is necessarily robust.

2 Tullock Contests with Incomplete Information

In a Tullock contest, a group of players $N = \{1, ..., n\}$, with $n \geq 2$, compete for a prize by simultaneously choosing the effort they exert. Players are uncertain about the realized state of nature. This uncertainty is described by a probability space $(\Omega, \mathcal{F}, \mu)$, where Ω is a countable set of states of nature, $\mathcal{F}$ is a σ-field of subsets of Ω, and μ is a probability measure on $(\Omega, \mathcal{F})$ representing the players' common prior belief. The private information of each player $i \in N$ is described by a σ-subfield of $\mathcal{F}$, which we denote by $\mathcal{F}_i$. Since Ω is countable, every σ-subfield of $\mathcal{F}$ is generated by the atoms of a countable partition of Ω. Thus, player i observes the atom containing the realized state of nature of the partition of Ω that generates $\mathcal{F}_i$. The value for the prize of each player $i \in N$ is an $\mathcal{F}$-measurable random variable $V_i : \Omega \to \mathbb{R}_{++}$. The cost of effort of each player $i \in N$ is given by a state-dependent function $c_i : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$, such that for each $x \in \mathbb{R}_+$ the function $c_i(\cdot, x)$ is $\mathcal{F}$-measurable. The prize is awarded to the players in a probabilistic fashion, using a contest success function $\rho : \Omega \times \mathbb{R}^n_+ \to \triangle^n$, where $\triangle^n$ is the n-simplex, such that for each $x \in \mathbb{R}^n_+$ the function $\rho(\cdot, x)$ is $\mathcal{F}$-measurable. Thus, a *Tullock contest* with incomplete information is formally represented by a collection

$$T = (N, (\Omega, \mathcal{F}, \mu), \{\mathcal{F}_i\}_{i \in N}, \{V_i\}_{i \in N}, \{c_i\}_{i \in N}, \rho).$$

A Tullock contest T defines a Bayesian game in which the set of actions of each player $i \in N$ is $\mathbb{R}_+$ and his payoff for each $\omega \in \Omega$ and $x \in \mathbb{R}_+^n$ is

$$u_i(\omega, x) = \rho_i(\omega, x)V_i(\omega) - c_i(\omega, x_i).$$

In this game, a pure strategy for player $i \in N$ is an $\mathcal{F}_i$-measurable and integrable function $X_i : \Omega \to \mathbb{R}_+$ which describes i's choice of effort in each state of nature. (The measurability restriction implies that player i can condition his effort only on his private information.) We denote by S_i the set of strategies of player i, and by $S = \times_{i=1}^n S_i$ the set of strategy profiles. Given a strategy profile $X = (X_1, ..., X_n) \in S$ we denote by X_{-i} the profile obtained from X by suppressing the strategy of player i.

If Y is an $\mathcal{F}$-measurable random variable and $\mathcal{G}$ is a σ-subfield of $\mathcal{F}$, we denote by $E[Y \mid \mathcal{G}]$ a random variable which is (a version of) the conditional expectation of Y with respect to $\mathcal{G}$ – see, e.g., Borkar (1995) for a formal definition. Also, for any two random variables Y and Z, we write $Y = Z$, $Y > Z$, or $Y \geq Z$ when each of these relations hold almost everywhere on Ω.

A pure strategy *Bayesian equilibrium* of a Tullock contest T is a Bayesian Nash equilibrium of the Bayesian game defined by the contest; that is, it is a strategy profile $X = (X_1, ..., X_n)$ such that for every $i \in N$ and every $X_i' \in S_i$,

$$E[u_i(\cdot, X(\cdot))] \geq E[u_i(\cdot, X_{-i}(\cdot), X_i'(\cdot))], \tag{1}$$

or equivalently,

$$E[u_i(\cdot, X(\cdot)) \mid \mathcal{F}_i] \geq E[u_i(\cdot, X_{-i}(\cdot), X_i'(\cdot)) \mid \mathcal{F}_i] \tag{2}$$

almost everywhere on Ω. We restrict attention to pure strategies.

Throughout the paper we consider Tullock contests satisfying the following assumptions:

(A.1) For each $i \in N$, $V_i \in L_\infty(\Omega, \mathcal{F}, \mu)$.

(A.2) For each $i \in N$ *and* $\omega \in \Omega$, $c_i(\omega, \cdot)$ *is continuous, strictly increasing and convex on* $\mathbb{R}_+$, *and satisfies* $c_i(\omega, 0) = 0$. *Moreover, for each* $x \in \mathbb{R}_+$, *the functions* $c_i(\cdot, x)$ *and* $c_i^{-1}(\cdot, x)$ *are integrable on* $(\Omega, \mathcal{F}, \mu)$.

(A.3) For every $\omega \in \Omega$, $\rho(\omega, \cdot)$ *is continuous on* $\mathbb{R}_+^n \backslash \{0\}$, *and for each* $i \in N$ *and* $x \in \mathbb{R}_+^n$, $\rho_i(\omega, x_{-i}, \cdot)$ *is non-decreasing and concave, and satisfies* $\rho_i(\omega, 0, x_i) = 1$ *whenever* $x_i > 0$.

The assumptions on the players' cost of effort introduced in $(A.2)$ are standard. The properties of the function ρ assumed in $(A.3)$ are satisfied by the contest success functions studied in the literature on Tullock contests (see Skaperdas 1996, and Clark and Riis 1998). The following result which establishes the existence of equilibrium in every Tullock contest satisfying assumptions $(A.1)$-$(A.3)$ follows from the existence theorem of Einy et al. (2015).

Remark 1. *Every Tullock contest satisfying (A.1)-(A.3) has a Bayesian equilibrium.*

In order to precisely define convergence of contests with incomplete information, we use Boylan's (1971)'s pseudometric on the family $\mathcal{F}^*$ of σ-subfields of $\mathcal{F}$, given for $\mathcal{G}, \mathcal{H} \in \mathcal{F}^*$ by

$$D(\mathcal{G}, \mathcal{H}) = \sup_{A \in \mathcal{G}} \inf_{B \in \mathcal{H}} \mu(A \triangle B) + \sup_{B \in \mathcal{H}} \inf_{A \in \mathcal{G}} \mu(A \triangle B),$$

where $A \triangle B = (A \backslash B) \cup (B \backslash A)$ is the symmetric difference of A and B. This metric has been studied and used extensively in the literature.

Definition 1. *We say that the sequence of Tullock contests*

$$\{(N, (\Omega, \mathcal{F}, \mu), \{\mathcal{F}_i^k\}_{i \in N}, \{V_i^k\}_{i \in N}, \{c_i^k\}_{i \in N}, \rho^k)\}_{k=1}^{\infty}$$

converges to the Tullock contest $(N, (\Omega, \mathcal{F}, \mu), \{\mathcal{F}_i\}_{i \in N}, \{V_i\}_{i \in N}, \{c_i\}_{i \in N}, \rho)$ *if:*

$(C.1)$ *For all* $i \in N$, $\{\mathcal{F}_i^k\}_{k=1}^{\infty}$ *converges to* $\mathcal{F}_i$ *in the Boylan metric.*

$(C.2)$ *For all* $i \in N$, $\{V_i^k\}_{k=1}^{\infty}$ *converges uniformly to* V_i *on* Ω.

$(C.3)$ *For all* $i \in N$ *and* $\omega \in \Omega$, $\{c_i^k(\omega, \cdot)\}_{k=1}^{\infty}$ *converges uniformly to* $c_i(\omega, \cdot)$ *on every compact subset of* $\mathbb{R}_+$.

(C.4) For all $\omega \in \Omega$, $\{\rho^k(\omega, \cdot)\}_{k=1}^{\infty}$ *converges uniformly to* $\rho(\omega, \cdot)$ *on every compact subset of* $\mathbb{R}^n_+ \backslash \{0\}$.

Theorem 1 establishes that the Bayesian equilibrium correspondence is upper semicontinuous on the class Tullock contests satisfying assumptions $(A.1)$-$(A.3)$.

Theorem 1. *For each positive integer* k, *let*

$$T^k = (N, (\Omega, \mathcal{F}, \mu), \{\mathcal{F}_i^k\}_{i \in N}, \{V_i^k\}_{i \in N}, \{c_i^k\}_{i \in N}, \rho^k)$$

be a Tullock contest, and let X^k *be a Bayesian equilibrium of* T^k. *If* $\{T^k\}_{k=1}^{\infty}$ *converges to a Tullock contest*

$$T = (N, (\Omega, \mathcal{F}, \mu), \{\mathcal{F}_i\}_{i \in N}, \{V_i\}_{i \in N}, \{c_i\}_{i \in N}, \rho),$$

and $\{X^k\}_{k=1}^{\infty}$ *converges to* X *pointwise on* Ω, *then* X *is a Bayesian equilibrium of* T.

Proof. First we show that for all $i \in N$, X_i is $\mathcal{F}_i$-measurable. Since $V_i^k, V_i \in L_\infty(\Omega, F, \mu)$ for all (i, k),

$$0 < M_i^k := \sup_{\omega \in \Omega} V_i^k(\omega) < \infty,$$

and

$$0 < M_i := \sup_{\omega \in \Omega} V_i(\omega) < \infty.$$

We now show that for all (i, k),

$$c_i^k(\cdot, X_i^k(\cdot)) \leq M_i^k.$$

almost everywhere in Ω. Suppose, by way of contradiction, that there is (i, k) and an atom A of $\mathcal{F}$ such that

$$c_i^k(\omega, X_i^k(\omega)) > M_i^k \geq V_i^k(\omega) \tag{3}$$

holds for every $\omega \in A$. Since $\mathcal{F}_i^k \subseteq \mathcal{F}$ there exists an atom A_i^k of $\mathcal{F}_i^k$ such that $A \subset A_i^k$. Since X_i^k is $\mathcal{F}_i^k$-measurable, X_i^k is constant on A_i^k. Thus, (3) holds on A_i^k, and therefore

$$u_i(\omega, X^k(\omega)) \leq V_i^k(\omega) - c_i(\omega, X_i^k(\omega)) < 0$$

for all $\omega \in A_i^k$. Define $Y_i^k = 0 \cdot 1_{A_i^k} + X_i^k \cdot 1_{\Omega \backslash A_i^k}$. Then Y_i^k is $\mathcal{F}_i^k$-measurable, and

$$u_i(\omega, Y_i^k(\omega), X_{-i}^k(\omega)) = V_i^k(\omega) > u_i(\omega, X^k(\omega))$$

for all $\omega \in A_i^k$, whereas

$$u_i(\omega, Y_i^k(\omega), X_{-i}^k(\omega)) = u_i(\omega, X^k(\omega))$$

for all $\omega \in \Omega \backslash A_i^k$. Therefore

$$E[u_i(\cdot, Y_i^k(\cdot), X_{-i}^k(\cdot))] > E[u_i(\cdot, X^k(\cdot))],$$

which contradicts that X^k is a Bayesian equilibrium of T^k.

Now, for all $\omega \in \Omega$,

$$c_i^k(\omega, X_i^k(\omega)) \leq M_i^k$$

implies

$$X_i^k(\omega) \leq \left(c_i^k\right)^{-1}(\omega, M_i^k).$$

By $(C.2)$, for k sufficiently large $M_i^k \leq 1 + M_i$. Then

$$X_i^k(\omega) \leq \left(c_i^k\right)^{-1}(\omega, 1 + M_i)$$

for sufficiently large k and all $\omega \in \Omega$. Also, $\left\{c_i^k(\omega, \cdot)\right\}_{k=1}^{\infty}$ converges uniformly to $c_i(\omega, \cdot)$ on every compact interval of $\mathbb{R}_+$ by $(C.3)$, and since $c_i^k(\omega, \cdot)$ is continuous and increasing, by Theorem 1 in Barbinek *et al.* (1991), $\left\{(c_i^k)^{-1}(\omega, \cdot)\right\}_{k=1}^{\infty}$ converges uniformly to $(c_i)^{-1}(\omega, \cdot)$ on every compact interval of $\mathbb{R}_+$. In particular, $\left\{(c_i^k)^{-1}(\omega, 1 + M_i)\right\}_{k=1}^{\infty}$ converges to $(c_i)^{-1}(\omega, 1 + M_i)$. Since for all $\omega \in \Omega$,

$$X_i^k(\omega) \leq (c_i^k)^{-1}(\omega, 1 + M_i),$$

for sufficiently large k,

$$X_i^k(\omega) \leq 1 + (c_i)^{-1}(\omega, 1 + M_i) \tag{4}$$

Moreover, since the function $c_i^{-1}(\cdot, 1+M_i)$ is integrable and $\{X_i^k\}_{k=1}^{\infty}$ converges pointwise to X_i on Ω, by the dominated convergence theorem $\{X_i^k\}_{k=1}^{\infty}$ converges to X_i in the $L_1(\Omega, \mathcal{F}, \mu)$ norm – see, e.g., Theorem 12.2 of Schilling (2005). Hence, since X_i^k is $\mathcal{F}_i^k$-measurable, and $\{\mathcal{F}_i^k\}_{k=1}^{\infty}$ converges to $\mathcal{F}_i$ in the Boylan pseudometric, by Lemma 1 in Einy *et al.* (2005), X_i is $\mathcal{F}_i$-measurable.

We now show that X is a Bayesian equilibrium of T by distinguishing between two cases:

Case I. Assume that $X(\omega) \neq 0$ almost everywhere in Ω.

In this case $\rho(\omega, \cdot)$ is continuous at $X(\omega)$ for almost all $\omega \in \Omega$ by $(A.3)$. Now, for all $\omega \in \Omega$ and for sufficiently large k

$$\rho_i^k(\omega, X^k(\omega))V_i^k(\omega) \leq V_i^k(\omega) \leq M_i^k \leq 1 + M_i, \tag{5}$$

and

$$c_i^k(\omega, X_i^k(\omega)) \leq M_i^k \leq 1 + M_i. \tag{6}$$

Assume, by way of contradiction, that X is not a Bayesian equilibrium of T. In that case, there is $i \in N$, a $\mathcal{F}_i$-measurable and integrable random variable $Y_i > 0$, and an atom A of $\mathcal{F}$ such that

$$E[u_i(\cdot, Y_i(\cdot), X_{-i}(\cdot)) \mid A] > E[u_i(\cdot, X(\cdot)) \mid A].$$

For all k, let $Y_i^k = E[Y_i \mid \mathcal{F}_i^k]$. Then $Y_i^k \geq 0$ is a $\mathcal{F}_i^k$-measurable and $Y_i^k \in L_1(\Omega, \mathcal{F}, \mu)$. Since $\{\mathcal{F}_i^k\}_{k=1}^{\infty}$ converges to $\mathcal{F}_i$, by Theorem 4 in Boylan (1971), $\{Y_i^k\}_{k=1}^{\infty}$ converges in measure to $E[Y_i \mid \mathcal{F}_i] = Y_i$. Therefore, the sequence $\{Y_i^k\}_{k=1}^{\infty}$ has a subsequence that converges pointwise to Y_i on Ω.

Without loss of generality, assume that $\{Y_i^k\}_{k=1}^{\infty}$ converges to Y_i on Ω. Now, $(C.2) - (C.4)$, the inequalities (5) and (6), and the dominated convergence theorem together imply that

$$\lim_{k\to\infty} E[\rho_i^k(\cdot, X^k(\cdot))V_i^k(\cdot) \mid A] = E[\rho_i(\cdot, X(\cdot))V_i(\cdot) \mid A],$$

and

$$\lim_{k\to\infty} E[c_i^k(\cdot, X_i^k(\cdot)) \mid A] = E[c_i(\cdot, X_i(\cdot)) \mid A].$$

Hence

$$\lim_{k\to\infty} E[u_i^k(\cdot, X^k(\cdot)) \mid A] = E[u_i(\cdot, X(\cdot)) \mid A].$$

Since Y_i is constant on A (because Y_i is $\mathcal{F}$-measurable), i.e., $Y_i = \hat{y}_i \in \mathbb{R}_+$ for all $\omega \in A$, and $\{Y_i^k\}_{k=1}^{\infty}$ converges pointwise to Y_i on Ω, for sufficiently large k we have

$$Y_i^k(\omega) < 1 + \hat{y}_i$$

for all $\omega \in A$. And since $\{c_i^k(\omega, \cdot)\}_{k=1}^{\infty}$ converges uniformly to $c_i(\omega, \cdot)$ on every compact subset of $\mathbb{R}_+$ by $(C.3)$, for sufficiently large k we have

$$c_i^k(\omega, Y_i^k(\omega)) \leq c_i^k(\omega, 1 + \hat{y}_i) \leq 1 + c_i(\omega, 1 + \hat{y}_i).$$

for all $\omega \in A$. Since $c_i(\cdot, 1+\hat{y}_i)$ is integrable on Ω, $(C.3)$ and the dominated convergence theorem imply that

$$\lim_{k\to\infty} E[c_i^k(\cdot, Y_i^k(\cdot)) \mid A] = E[c_i(\cdot, Y_i(\cdot)) \mid A].$$

Now,

$$\rho_i^k(\omega, Y_i^k(\omega), X_{-i}^k(\omega))V_i^k(\omega) \leq 1 + M_i$$

for all k and all $\omega \in \Omega$. As $Y_i > 0$, then $(A.3)$, $(C.2)$, $(C.3)$, the last inequality, and the dominated convergence theorem imply that

$$\lim_{k\to\infty} E[\rho_i^k(\cdot, Y_i^k(\cdot), X_{-i}^k(\cdot))V_i^k(\cdot) \mid A] = E[\rho_i(\cdot, Y_i(\cdot), X_{-i}(\cdot))V_i(\cdot) \mid A].$$

Hence

$$\begin{aligned}\lim_{k\to\infty} E[u_i^k(\cdot, Y_i^k(\cdot), X_{-i}^k(\cdot)) \mid A] &= E[u_i(\cdot, Y_i(\cdot), X_{-i}(\cdot)) \mid A] \\ &> E[u_i(\cdot, X(\cdot)) \mid A] \\ &= \lim_{k\to\infty} E[u_i^k(\cdot, X^k(\cdot)) \mid A],\end{aligned}$$

which implies that there exists k such that

$$E[u_i^k(\cdot, Y_i^k(\cdot), X_{-i}^k(\cdot)) \mid A] > E[u_i^k(\cdot, X^k(\cdot)) \mid A],$$

contradicting that X^k is a Bayesian equilibrium of T^k.

Case II. Let $A := \{\omega \in \Omega \mid X(\omega) = 0\}$, and assume that $\mu(A) > 0$.

We show that this case contradicts the assumption that X^k is a Bayesian equilibrium of T^k for all k, and thus cannot be satisfied. Since $\{\rho^k(\cdot, X^k(\cdot))\} \subset (\triangle^n)^\Omega$ for all k, and $(\triangle^n)^\Omega$ is a compact and metrizable space with respect to the product topology (because Ω is countable), the sequence $\{\rho^k(\cdot, X^k(\cdot))\}_{k=1}^\infty$ has a subsequence that converges pointwise to an $\mathcal{F}$-measurable random variable $p \in (\triangle^n)^\Omega$. Without loss of generality, assume that $\{\rho^k(\cdot, X^k(\cdot))\}_{k=1}^\infty$ itself converges to p. For all $i \in N$ and $\omega \in \Omega$ let

$$\alpha_i(\omega) = p_i(\omega)V_i(\omega) - c_i(\omega, X_i(\omega)).$$

As in case I, the dominated convergence theorem implies that

$$\lim_{k\to\infty} E[u_i^k(\cdot, X^k(\cdot))] = E[\alpha_i].$$

Let $\omega^* \in A$. Since $\sum_{j\in N} p_j(\omega^*) = 1$, there exists $i \in N$ such that $p_i(\omega^*) < 1$. For every $\varepsilon > 0$, let $Y_i^\varepsilon := \max\{\varepsilon, X_i\}$. Then Y_i^ε is $\mathcal{F}_i$-measurable. Since $\{X_i^k\}_{k=1}^\infty$ converges pointwise to X_i on Ω, the inequality (4) established above implies that

$$Y_i^\varepsilon(\omega) \le \max\{\varepsilon, 1 + c_i^{-1}(\omega, 1 + M_i)\} \tag{7}$$

for all $\omega \in \Omega$. For all $\varepsilon > 0$ and $\omega \in \Omega$, $(Y_i^\varepsilon(\cdot), X_{-i}(\cdot)) \in \mathbb{R}_+^n\backslash\{0\}$, and since $\rho(\omega, \cdot)$ is continuous at $X(\omega)$ for all $\omega \in \Omega\backslash A$, we have

$$\lim_{\varepsilon\to 0^+} \rho_i(\omega, (Y_i^\varepsilon(\omega), X_{-i}(\omega))) = \rho_i(\omega, (X(\omega))) = \lim_{k\to\infty} \rho_i^k(\omega, \left(X^k(\omega)\right)) = p_i(\omega),$$

and thus

$$\begin{aligned}\lim_{\varepsilon\to 0^+} [\rho_i(\omega, (Y_i^\varepsilon(\omega), X_{-i}(\omega)))V_i(\omega) - c_i(\omega, Y_i^\varepsilon(\omega))] &= p_i(\omega)V_i(\omega) - c_i(\omega, X_i(\omega)) \\ &= \alpha_i(\omega).\end{aligned}$$

For all $\omega \in A$, since $X(\omega) = 0$, the assumption $(A.3)$ implies that

$$\rho_i(\omega, (Y_i^{\varepsilon}(\omega), X_{-i}(\omega))) = \rho_i(\omega, (Y_i^{\varepsilon}(\omega), 0)) = 1$$

for every $\varepsilon > 0$, and therefore

$$\begin{aligned}\lim_{\varepsilon \to 0^+} [\rho_i(\omega, (Y_i^{\varepsilon}(\omega), X_{-i}(\omega)))V_i(\omega) - c_i(\omega, Y_i^{\varepsilon}(\omega))] &= V_i(\omega) - c_i(\omega, 0) \\ &= V_i(\omega) \\ &\geq \alpha_i(\omega).\end{aligned}$$

Let $A(\omega^*)$ be the atom of $\mathcal{F}$ containing ω^*. Since X is $\mathcal{F}$-measurable and $X(\omega^*) = 0$, $A(\omega^*) \subseteq A$. Since for all $z \in \mathbb{R}_+^n \backslash \{0\}$, both $\rho_i(\cdot, z)$ and Y_i^{ε} are $\mathcal{F}$-measurable, $\rho_i(\cdot, (Y_i^{\varepsilon}(\cdot), 0))$ is constant on $A(\omega^*)$. As p is $\mathcal{F}$-measurable, for all $\omega \in A(\omega^*)$ we have

$$p_i(\omega) = p_i(\omega^*) < 1,$$

and hence $V_i > 0$ implies that

$$\lim_{\varepsilon \to 0^+} [\rho_i (\omega, (Y_i^{\varepsilon}(\omega), X_{-i}(\omega))) V_i(\omega) - c_i(\omega, Y_i^{\varepsilon}(\omega))] = V_i(\omega) > p_i(\omega)V_i(\omega) = \alpha_i(\omega)$$

for all $\omega \in A(\omega^*)$. Therefore, the dominated convergence theorem implies that

$$\lim_{\varepsilon \to 0^+} E[u_i(\cdot, (Y_i^{\varepsilon}(\cdot), X_{-i}(\cdot)))] > E[\alpha_i(\cdot)] = \lim_{k \to \infty} E[u_i^k(\cdot, X^k(\cdot))].$$

Hence, there exists $\bar{\varepsilon} > 0$ sufficiently small such that

$$E[u_i(\cdot, (Y_i^{\bar{\varepsilon}}(\cdot), X_{-i}(\cdot)))] > \lim_{k \to \infty} E[u_i^k(\cdot, X^k(\cdot))]$$

Let $\bar{Y}_i^k = E[Y_i^{\bar{\varepsilon}} \mid \mathcal{F}_i^k]$. Then $\bar{Y}_i^k$ is $\mathcal{F}_i^k$-measurable and, as in case I we may assume that $\{\bar{Y}_i^k\}_{k=1}^{\infty}$ converges pointwise to $E[Y_i^{\bar{\varepsilon}} \mid \mathcal{F}_i] = Y_i^{\bar{\varepsilon}}$ on Ω. The inequality (7) above, $(C.2) - (C.4)$, and the dominated convergence theorem imply that

$$\lim_{k \to \infty} E[u_i^k(\cdot, (\bar{Y}_i^k(\cdot), X_{-i}^k(\cdot)))] = E[u_i(\cdot, (Y_i^{\bar{\varepsilon}}(\cdot) X_{-i}(\cdot)))] > \lim_{k \to \infty} E[u_i^k(\cdot, X^k(\cdot))].$$

Hence, there exists k such that

$$E[u_i^k(\cdot, (\bar{Y}_i^k(\cdot), X_{-i}^k(\cdot)))] > E[u_i^k(\cdot, X^k(\cdot))],$$

contradicting that X^k is a Bayesian equilibrium of T^k. $\square$

Theorem 1 shows that the Bayesian equilibrium correspondence of Tullock contests with incomplete information is upper semicontinuous. The following proposition shows that if the players' costs of effort are state independent, then the Bayesian equilibrium correspondence is also lower semicontinuous at contests with a unique Bayesian equilibrium. Recall that by Remark 1, every Tullock contest has a Bayesian equilibrium.

Proposition 1. *For every positive integer k, let*

$$T^k = (N, (\Omega, \mathcal{F}, \mu), \{\mathcal{F}_i^k\}_{i\in N}, \{V_i^k\}_{i\in N}, \{c_i^k\}_{i\in N}, \rho^k)$$

be a Tullock contest in which the players' costs of effort are independent on the state of nature (i.e., for all $i \in N$ and $x \in \mathbb{R}_+$, $c_i^k(\cdot, x)$ is constant). If the sequence $\{T^k\}_{k=1}^{\infty}$ converges to a Tullock contest

$$T = (N, (\Omega, \mathcal{F}, \mu), \{\mathcal{F}_i\}_{i\in N}, \{V_i\}_{i\in N}, \{c_i\}_{i\in N}, \rho)$$

with a unique equilibrium X, then there exists a sequence $\{X^{k_r}\}_{r=1}^{\infty}$ such that for all r, X^{k_r} is a Bayesian equilibrium of T^{k_r}, and $\{X^{k_r}\}_{r=1}^{\infty}$ converges pointwise to X on Ω.

Proof. Assume that $\{T^k\}_{k=1}^{\infty}$ converges to a Tullock contest T with a unique equilibrium X, and let $\{X^k\}_{k=1}^{\infty}$ be a sequence such that each X^k is a Bayesian equilibrium of T^k. As shown in the proof of Theorem 1, for all $\omega \in \Omega$ we have

$$X_i^k(\omega) \le c_i^{-1}(1 + M_i),$$

where $M_i = \sup_{\omega\in\Omega} V_i(\omega)$. (Since c_i is state-independent, we omit ω from its argument.) Therefore for sufficiently large k, the sequence $\{X^k\}_{k=\bar{k}}^{\infty}$ is contained in $\bar{S} = \times_{i=1}^{n}[0, c_i^{-1}(1 + M_i)]^{\Omega}$, which is a compact and metrizable space in the product topology because Ω is countable. Consequently, $\{X^k\}_{k=1}^{\infty}$ has a subsequence $\{X^{k_r}\}_{r=1}^{\infty}$

that converges pointwise in Ω to some point in $\bar{S}$. By Theorem 1, this point is a Bayesian equilibrium of T, and therefore can only be X. $\Box$

The proof of Proposition 1 applies without change to Tullock contests in which the players' costs of effort are state dependent, but the effort that each individual can exert is uniformly bounded. We state this result in Proposition 2.

Proposition 2. *For every positive integer k, let*

$$T^k = (N, (\Omega, \mathcal{F}, \mu), \{\mathcal{F}_i^k\}_{i\in N}, \{V_i^k\}_{i\in N}, \{c_i^k\}_{i\in N}, \rho^k)$$

be a Tullock contest in which the set of pure strategies of each player is contained in $[0, M]^\Omega$ for some $M > 0$. If the sequence $\{T^k\}_{k=1}^\infty$ converges to a Tullock contest

$$T = (N, (\Omega, \mathcal{F}, \mu), \{\mathcal{F}_i\}_{i\in N}, \{V_i\}_{i\in N}, \{c_i\}_{i\in N}, \rho)$$

with a unique equilibrium X, then there exists a sequence $\{X^{k_r}\}_{r=1}^\infty$ such that for all r, X^{k_r} is a Bayesian equilibrium of T^{k_r}, and $\{X^{k_r}\}_{r=1}^\infty$ converges pointwise to X on Ω.

Next we show that when a Tullock contest has a unique equilibrium, the equilibrium is robust to small changes in the players' information, values, and costs of effort, as well as in the contest success function.

Definition 2. *The Tullock contests $T = (N, (\Omega, \mathcal{F}, \mu), \{\mathcal{F}_i\}_{i\in N}, \{V_i\}_{i\in N}, \{c_i\}_{i\in N}, \rho)$ and $\hat{T} = (N, (\Omega, \mathcal{F}, \mu), \{\hat{\mathcal{F}}_i\}_{i\in N}, \{\hat{V}_i\}_{i\in N}, \{\hat{c}_i\}_{i\in N}, \hat{\rho})$ are δ-neighbors, where $\delta > 0$, if for all $i \in N$:*

(2.1) *$D(\mathcal{F}_i, \hat{\mathcal{F}}_i) < \delta$, where D is the Boylan pseudometric;*

(2.2) $\sup_{\omega\in\Omega} \left| V_i(\omega) - \hat{V}_i(\omega) \right| < \delta$;

(2.3) *For all $\omega \in \Omega$, and every compact subset C of $\mathbb{R}_+$, $\sup_{t\in C} |c_i(\omega, t) - \hat{c}_i(\omega, t)| < \delta$; and*

(2.4) *For all $\omega \in \Omega$, and every compact subset C of $\mathbb{R}_+^n \backslash \{0\}$, $\sup_{t\in C} |\rho_i(\omega, t) - \hat{\rho}_i(\omega, t)| < \delta$.*

In a Tullock contest, the set of (pure) strategy profiles S is a subset of the space $(\mathbb{R}^n_+)^\Omega$. Given an enumeration $\{\omega_1, \omega_2, ...\}$ of Ω, the metric d defined for all $X, Y \in (\mathbb{R}^n_+)^\Omega$ by

$$d(X,Y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\|X(\omega_j) - Y(\omega_j)\|}{1 + \|X(\omega_j) - Y(\omega_j)\|},$$

where $\|\cdot\|$ is the Euclidian norm on $\mathbb{R}^n_+$, induces the product topology on $(\mathbb{R}^n_+)^\Omega$.

Proposition 3. *Let $T = (N, (\Omega, \mathcal{F}, \mu), \{\mathcal{F}_i\}_{i\in N}, \{V_i\}_{i\in N}, \{c_i\}_{i\in N}, \rho)$ be a Tullock contest with a unique equilibrium X. If the players' costs of effort are independent on the state of nature, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\hat{X}$ is a Bayesian equilibrium of a Tullock contest $\hat{T}$ that is a δ-neighbor of T, then $d(\hat{X}, X) < \varepsilon$.*

Proof. If Proposition 3 does not hold, then there exists $\varepsilon_0 > 0$ such that for each positive integer k there is a Tullock contest T^k that is a $(1/k)$-neighbor of T, and a Bayesian equilibrium X^k of T^k with $d(X^k, X) \geq \varepsilon_0$. Since the sequence $\{T^k\}_{k=1}^\infty$ converges to T, and since $\{c_i\}_{i\in N}$ do not depend on the state of nature, as in the proof of Proposition 1, the sequence $\{X^k\}_{k=1}^\infty$ is bounded. Therefore, the sequence $\{X^k\}_{k=1}^\infty$ has a subsequence $\{X^{k_r}\}_{r=1}^\infty$ converging pointwise to X on Ω. Hence there is $\bar{r}$ such that for all $r > \bar{r}$, $d(X^{k_r}, X) < \varepsilon_0$. However, by assumption $d(X^{k_r}, X) \geq \varepsilon_0$ for all r, which is a contradiction. □

The following proposition follows from Proposition 2.

Proposition 4. *Let $T = (N, (\Omega, \mathcal{F}, \mu), \{\mathcal{F}_i\}_{i\in N}, \{V_i\}_{i\in N}, \{c_i\}_{i\in N}, \rho)$ be a Tullock contest with a unique equilibrium X. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every Tullock contest $\hat{T}$ that is a δ-neighbor of T and in which the set of pure strategies of every player is contained in* $[0, M]^\Omega$ *for some $M > 0$ we have $d(\hat{X}, X) < \varepsilon$ for every Bayesian equilibrium $\hat{X}$* of $\hat{T}$.

Szidarovszky and Okuguchi (1997), Einy, Moreno, and Shitovitz (2017), and Ewerhart and Quartieri (2018) provide conditions implying the uniqueness of equilibrium in large classes of Tullock contests (see also Chowdhury and Sheremeta 2011). Propositions 1 to 3 imply that the unique Bayesian equilibrium of the Tullock contests in these classes are robust.

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