Dynamic Consistency, Valuable Information and Subjective Beliefs*

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Abstract

Ambiguity sensitive preferences must fail either Consequentialism or Dynamic Consistency (DC), two properties that are compatible with subjective expected utility and Bayesian updating, while forming the basis of backward induction and dynamic programming. We examine the connection between these properties in a general environment of convex preferences over monetary acts and find that, far from being incompatible, they are connected in an economically meaningful way. In single-agent decision problems, positive value of information characterises one direction of DC. We propose a weakening of DC and show that one direction is equivalent to weakly valuable information, whereas the other characterises the Bayesian updating of the subjective beliefs which are revealed by trading behavior. In financial markets, we characterize no speculative trade, without requiring any form of Consequentialism, and show that there is weakly negative value of public information in risk-sharing environments with no aggregate uncertainty.

JEL-Classifications: D81, D83, D91

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1 Introduction

In dynamic-choice problems under uncertainty, the decision maker updates his preferences and his beliefs as new information arrives, taking optimal actions in each period. Two are the most widely used constraints on how these preferences are updated. The

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first, *Dynamic Consistency* (DC), requires that an action plan is optimal when evaluated with the updated preferences of a later period if and only if it is optimal when evaluated with the preferences of an earlier period. DC ensures that an ex ante optimal action plan will remain optimal at every period and irrespective of how information is updated. The second, *Consequentialism*, requires that conditional preferences do not depend on past actions, foregone payoffs or unrealized events. These two properties form the basis of backward induction and dynamic programming.

The Subjective Expected Utility (SEU) model is consistent with both DC and Consequentialism, together with other attractive properties, such as positive value of information, recursive representation of preferences and Bayesian updating of beliefs. However, for other preferences this is not true in general. In particular, preferences which are ambiguity sensitive must either relax DC or Consequentialism (Siniscalchi [2009]).

The purpose of this paper is to examine the connection between these properties in a general environment of convex preferences over monetary acts. We find that, far from being incompatible, they are connected in an economically meaningful way. We break DC into two parts. Consider two acts f and g which specify the same monetary payoffs if event E occurs. The "if" direction of DC specifies that if the agent weakly prefers f over g ex ante, then he also prefers it when he learns that E did not occur. The "only if" direction specifies the converse.

We first show that the "only if" direction of DC is inconsistent with Ellsberg [1961] but equivalent to positive value of information, meaning that a single agent always prefers to receive more information to less. Intuitively, an agent who understands that his preferences will change if he learns an event might not want to receive this information, as this could lead to choices that he considers suboptimal in the ex ante stage. Although several authors have previously discussed this connection, to our knowledge this is the first paper that provides a formal characterization.

Al-Najjar and Weinstein [2009] have criticised the ambiguity aversion literature on the basis that aversion to information is not normatively appealing. However, we show that ambiguity averse preferences are consistent with a weaker version of valuable information. In particular, we say that information is *weakly* valuable if (ex ante) the agent always prefers mixing more information with less information, rather than receiving less information with certainty. Such an agent recognises that more information has at least some value, even if he does not think that more information with certainty is optimal. We then characterize weakly valuable information with respect to a weakening of the "only if" direction of DC. These results on the value of information do not require the full strength of Consequentialism (although they are still true when Consequentialism is assumed). In particular, we use a weaker axiom, Status Quo Bias. This property was first proposed in axiomatic work by Masatlioglu and Ok [2005], however it has been studied experimentally at least since Samuelson and Zeckhauser [1988], who provided evidence in a study concerning portfolio choices.

Second, we show that a weakening of the "if" direction of DC characterizes the Bayesian updating of the beliefs revealed by potential trading behavior (and are not necessarily part of the utility representation of preferences). They are called "subjective beliefs" by Rigotti et al. [2008] (RSS), who identify them for a wide variety of models with convex preferences over monetary acts, hence making our approach very general.

Subjective beliefs have economic content in static environments, as RSS show that they characterize efficient and full insurance allocations. Moreover, each subjective belief can be interpreted as state prices of Arrow-Debreu securities, for which the agent is not willing to trade the act f with which he is endowed.[1] If these prices prevail in the market, we say that f is *revealed preferred* to all the acts that are affordable for the agent. Our proposed weakening of the "if" direction of DC specifies that if f is ex ante revealed preferred (but not necessarily weakly preferred) to another act g then f is ex post weakly preferred to g. The weakening of the "only if" direction of DC specifies that if f is ex post weakly preferred to g, then it cannot be that g is ex ante revealed preferred (but could be strictly preferred) to f.[2]

We show that the economic content of subjective beliefs extends to dynamic and multi-agent environments, as their Bayesian updating is the minimum requirement which ensures that there is no speculative trade.[3] The absence of speculative trade is the familiar result of Milgrom and Stokey [1982], that starting from an ex ante efficient allocation, it cannot be common knowledge in the interim stage that there is another allocation which Pareto dominates it. Interestingly, the result does not require any form of Consequentialism, unlike other papers in the literature.

We also examine the value of information in a competitive risk-sharing environment without aggregate uncertainty, where agents trade state-contingent claims. In the SEU model with risk aversion, it is shown with an example by Hirshleifer [1971] and more generally by Schlee [2001] that public information makes everyone weakly worse off, as it destroys opportunities for mutual insurance. We generalize this result by showing that under the"if" direction of weak DC and Status Quo Bias, if information is (weakly) valuable for each agent, then public information is (weakly) not valuable.

Using the results of RSS, who characterise the subjective beliefs for several ambiguity averse preference models, we discuss some updating rules. We note that our domain of preferences, just like in RSS, is acts from states to monetary outcomes, a special case of Savage [1954]. This is the natural domain in order to interpret subjective beliefs as prices of Arrow-Debreu securities, motivate the weakening of DC using the notion of revealed preference and provide the applications on speculative trade and risk-sharing. However, many decision theoretic models use the more general domain of acts from states to lotteries over outcomes. We discuss in Section 8 how weak DC extends to this domain.

1.1 Our approach

We illustrate our approach by analyzing a dynamic Ellsberg's three-color problem, taken from Epstein and Schneider [2003] and Hanany and Klibanoff [2007]. We show

[1]An act can be identified by a convex combination of Arrow-Debreu securities, thus providing a mapping from states to consumption of a single good.

[2]As with DC, the axiom applies only for acts f and g which are identical outside of the event that is revealed ex post. Formal details are presented in Section 2.

[3]It is minimum in the sense that if at least one agent violates it, there are economies where there is speculative trade, whereas if everyone satisfies it there cannot be speculative trade.

that Ellsberg preferences, Consequentialism and an axiom we call Conditional Preference imply that DC and value of information are violated. We then explain our weakening of DC.

An urn contains 120 balls, 40 of which are known to be black (B), whereas the remaining 80 are somehow divided between red (R) and yellow (Y). The state space is $S = \{B, R, Y\}$. A bet is an act from S to $\mathbb{R}$, specifying a payoff at each state.

Consider acts $f_1 = (1, 0, 0)$, $f_2 = (0, 1, 0)$, $f_3 = (0, 1, 1)$ and $f_4 = (1, 0, 1)$, where, for example, f_1 specifies a payoff of 1 if the state is B and 0 otherwise. The agent has ex ante preferences $\succsim$ which conform to Ellsberg, hence $f_1 \succ f_2$ and $f_3 \succ f_4$.

We assume that the agent is endowed with preference relation $\succsim_{\{B,R\}}$ conditional on learning that event $\{B, R\}$ has occurred and $\succsim_{\{Y\}}$ conditional on learning that event $\{Y\}$ has occurred. This specification implicitly assumes Consequentialism, because the conditional preference only depends on the updated event and not on other parameters, such as the act that the agent chose in the previous period or the decision problem he faced.

Note that, conditional on $\{Y\}$, f_1 is identical to f_2 and f_3 is identical to f_4. The axiom Conditional Preference requires that the agent is indifferent between such acts if he learns that $\{Y\}$ has occurred, hence $f_1 \sim_{\{Y\}} f_2$ and $f_3 \sim_{\{Y\}} f_4$.[4] Similarly, because conditional on $\{B, R\}$ f_1 is identical with f_4 and f_2 is identical with f_3, Conditional Preference requires that $f_1 \sim_{\{B,R\}} f_4$ and $f_2 \sim_{\{B,R\}} f_3$.

We first show that these Ellsberg preferences imply that DC is violated and information is not valuable. We represent information by a partition Π of S. Let $\mathcal{A} = \{f_1, f_2\}$ be the set of feasible acts. Without loss of generality, suppose that $f_2 \succsim_{\{B,R\}} f_1$.[5] First, suppose that the agent has no information, so his partition is $\Pi_1 = \{S\}$. In other words, he never learns whether $\{Y\}$ has occurred or not, so $\succsim_{\{Y\}}$ and $\succsim_{\{B,R\}}$ are irrelevant. Then, he chooses f_1, because $f_1 \succ f_2$.

Consider now the more informative partition $\Pi_2 = \{\{B, R\}, Y\}$, meaning that he is informed whether Y has occurred or not, before making his choice. If he learns that $\{B, R\}$ has occurred, he chooses f_2 because $f_2 \succsim_{\{B,R\}} f_1$, whereas if he learns $\{Y\}$, he again chooses f_2 because $f_1 \sim_{\{Y\}} f_2$.

The agent understands that if his partition is Π_1 he will get f_1 in all states, whereas if his partition is Π_2 he will get f_2. He strictly prefers partition Π_1 to partition Π_2 ex ante because $f_1 \succ f_2$. Because Π_2 is finer than Π_1, information is not valuable. The "only if" direction of DC (Axiom 13) specifies that if the agent weakly prefers f_2 over f_1 given that he has learned $\{B, R\}$ and the acts are identical outside of $\{B, R\}$, then he also weakly prefers f_2 over f_1 before learning whether $\{B, R\}$ has occurred or not. This is violated here as we have both $f_2 \succsim_{\{B,R\}} f_1$ and $f_1 \succ f_2$. Violation of both Axiom 13 and valuable information is not particular to this example, as Proposition 1 shows that they are equivalent. In a risk-sharing environment with no aggregate uncertainty, Proposition 4 shows that if each agent values more information,

[4] Several authors call this property Consequentialism, implicitly assuming that the conditional preference only depends on the realised event. Following Epstein and Schneider [2003], we call this property Conditional Preference and formally define it in Section 2 as Axiom 5. We assume it throughout the paper.

[5] For the case where $f_1 \succsim_{\{B,R\}} f_2$, we can obtain negative value of information using a similar example, with feasible acts $\mathcal{A} = \{f_3, f_4\}$, as Consequentialism implies that $f_3 \succsim_{\{B,R\}} f_4$.

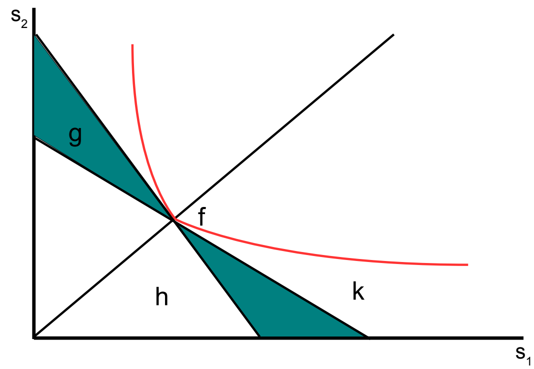

Figure 1: Revealed preference

then public information makes everyone weakly worse off, generalising the results of Hirshleifer [1971] and Schlee [2001].

The above results show that we cannot weaken the "only if" part of DC without losing positive value of information. Suppose now that $af_2 + (1-a)f_1 \succsim f_1$ for some $a \in (0, 1]$. Then, the agent recognises that more information has *some* value, because he would prefer to mix more with less information, instead of getting less information for sure.[6] When this happens, we say that information is *weakly* valuable. Proposition 1 characterizes this form of valuable information in terms of Axiom 15, which is weaker than Axiom 13, the "only if" part of DC.

We weaken DC by introducing the revealed preference relation, denoted $\succsim^*$. We say that f is revealed preferred to g if f is weakly preferred to all mixtures $af + (1-a)g$, for $a \in [0, 1]$. This is an incomplete but transitive relation. In Figure 1, there are two states, s_1 and s_2 and a point denotes an act. The depicted indifference curve has a kink at f and several supporting hyperplanes, given by the shaded area. Although f is strictly preferred to g, h and k, it is revealed preferred only to g and h. Even though f is not revealed preferred to k, convexity of preferences implies that f is strictly preferred to *some* convex combination of f and k. This convexity property is used extensively.

The normals of the hyperplanes that pass from f (normalized to be probability distributions) are called subjective beliefs at f by Rigotti et al. [2008] and denoted by $\pi(f)$. For convex preferences, each $p \in \pi(f)$ has the following property: if the expectation of f given p (denoted $\mathbb{E}_p f$) is greater or equal to the expectation of another act g given p, then f is weakly preferred to g. We can interpret p as a price vector, such that $p(s)$ is the price of the Arrow-Debreu security that pays 1 if state s occurs

[6] The interpretation of mixing depends on the range of acts. If the range is monetary outcomes (or more generally a convex set, a special case of Savage [1954]), then $af_2+(1-a)f_1$ provides, at state s, the monetary outcome $af_2(s)+(1-a)f_1(s)$. If acts are contingent lotteries, a special case of Anscombe and Aumann [1963], then $af_2(s) + (1-a)f_1(s)$ is a lottery that gives $f_2(s)$ with probability a and $f_1(s)$ with probability $1-a$. In the paper the range is monetary outcomes, but we discuss the extension to state contingent lotteries in Section 8.

and 0 otherwise. Then, the property says that if the agent's endowment is f, he would have zero net demand at each price vector $p \in \pi(f)$. Hence, we say that f is *revealed preferred* to g because the agent could afford g given prices p but chose his endowment, f.

Suppose that acts f and g are identical in terms of what they prescribe if event E does not occur. Then, the "if" part of DC is weakened by requiring that if f is revealed preferred (but not necessarily weakly preferred) to g ex ante, then f is weakly preferred to g conditional on E. The "only if" part of DC is weakened by requiring that if f is weakly preferred to g conditional on E, then g cannot be revealed preferred (but could be strictly preferred) to f ex ante.

The paper proceeds as follows. Section 2 presents the model, whereas Section 3 formalises the notions of (weak) valuable information and characterizes them with respect to the "only if" part of (weak) DC. In Section 4, we characterize the "if" part of weak DC with respect to Bayesian updating of subjective beliefs. We discuss various updating rules in Section 5. In Section 6, we show that Bayesian updating of subjective beliefs is the minimum requirement that precludes speculative trade. In Section 7, we show that if each agent individually considers information to be (weakly) valuable, then public information is not (weakly) valuable in competitive risk-sharing environments with no aggregate uncertainty. In Section 8, we discuss the related literature and provide a detailed comparison with Ghirardato et al. [2004] and Hanany and Klibanoff [2007, 2009], which are more closely related with our approach. All proofs are contained in the Appendix.

2 Model

2.1 Preliminaries

Fix a finite set of payoff relevant states S, with typical element s. The set of consequences is $\mathbb{R}_+$, interpreted as monetary payoffs. Let $\mathcal{F} = \mathbb{R}_+^S$ be the set of acts, with the natural topology. An act $f \in \mathcal{F}$ maps each state s to a monetary payoff. Given $x \in \mathbb{R}_+$, let $x \in \mathcal{F}$ be the constant act with payoff x at each state s. Let X be the set of constant acts. An act f is strictly positive if $f(s) > 0$ for all $s \in S$. Let $\mathcal{F}_+$ be the set of strictly positive acts.

For any two acts $f, g \in \mathcal{F}$ and event $E \subseteq S$, we denote by fEg the act h such that $h(s) = f(s)$ if $s \in E$ and $h(s) = g(s)$ if $s \notin E$. Define $f \geq_E g$ if $f(s) \geq g(s)$ for all $s \in E$, with strict inequality for some $s \in E$. Equality $f =_E g$ and strict inequality are similarly defined. Let E^c be the complement of E with respect to S.

Given events $E, F \subseteq S$ and probability measure $p \in \Delta E$, where $F \subseteq E$ and $p(F) > 0$, denote by $p_F \in \Delta F$ the measure obtained through Bayesian conditioning of p on F. Formally, for any event $G \subseteq S$, $p_F(G) = \frac{p(G \cap F)}{p(F)}$. We write $\mathbb{E}_p f := \sum\limits_{s \in E} p(s) f(s)$ for the expectation of f given p.

Let $\mathcal{E}$ be a collection of nonempty events $E \subseteq S$ which contains S. The decision maker is endowed with a collection of conditional preference relations, $\{\succsim_{E,h}\}_{E \in \mathcal{E}, h \in \mathcal{F}}$, one for each event $E \in \mathcal{E}$ and each act $h \in \mathcal{F}$. The interpretation is that in a previous period the agent had chosen act h and in the current period he learns that event E has occurred. His updated preference relation is then $\succsim_{E,h}$. The ex ante preference relation $\succsim_{S,h}$ does not depend on the act h and is denoted by $\succsim$.

A partition Π of S is a collection of mutually disjoint events, whose union is S. It is finer than another partition Π' if, for each $E' \in \Pi'$, there exists $E \in \Pi$ with $E \subseteq E'$. We then say that Π' is coarser than Π.

2.2 Revealed preference

Given preference relation $\succsim_{E,h}$, we say that act f is *revealed preferred* to act g, written $f \succsim^*_{E,h} g$, if $f \succsim_{E,h} ag + (1-a)f$ for all $a \in [0,1]$, so that f is weakly preferred to all convex combinations of f and g. Preference relation $\succsim^*_{E,h}$ is transitive but not necessarily complete.

The interpretation of $f \succsim^*_{E,h} g$ is that f is weakly preferred to g under $\succsim_{E,h}$ and g is inside a "budget set", which is constructed given f as the agent's endowment and some prices for the Arrow-Debreu securities, one for each state. If these prices were to prevail and the agent chose f, it would be revealed that the agent prefers f over g. Consider Figure 1. The indifference curve has a kink on act f and the two straight lines which define the shaded area are some of its supporting hyperplanes. Each such line defines a budget set, where f is affordable. Any act that is within this budget set, like g, is affordable but f is weakly preferred to g, hence we say that f is revealed preferred to g. On the contrary, although f is strictly preferred to k, it is not revealed preferred to it, because it is outside any of these budget sets.

2.3 Convex preferences

We consider the following axioms on preferences $\{\succsim_{E,h}\}_{E\in\mathcal{E},h\in\mathcal{F}}$, for all events $E \in \mathcal{E}$ and acts $h \in \mathcal{F}$.

Axiom 1. *(Preference).* $\succsim_{E,h}$ *is complete and transitive.*

Axiom 2. *(Continuity). For all* $f \in \mathcal{F}$, *the sets* $\{g \in \mathcal{F} : g \succsim_{E,h} f\}$ *and* $\{g \in \mathcal{F} : f \succsim_{E,h} g\}$ *are closed.*

Axiom 3. *(Monotonicity). For all* $f, g \in \mathcal{F}$, *if* $f >_E g$ *then* $f \succ_{E,h} g$.

Axiom 4. *(Convexity). For all* $f \in \mathcal{F}$, *the set* $\{g \in \mathcal{F} | g \succsim_{E,h} f\}$ *is convex.*

These four axioms are standard and imply that each $\succsim_{E,h}$ is represented by a continuous, increasing and quasiconcave function $U_{E,h} : \mathcal{F} \to \mathbb{R}$. The next axiom, which we require throughout the paper, specifies that if the agent knows that event E has occurred, his preferences depend only on what acts specify inside E.

Axiom 5. *(Conditional Preference) For all* $f, g \in \mathcal{F}$, *if* $f =_E g$ *then* $f \sim_{E,h} g$.

We say that preferences $\{\succsim_E\}_{E\in\mathcal{E},h\in\mathcal{F}}$ are convex if they satisfy Axioms 1 through 5. For our results on speculative trade, we use the strong versions of Monotonicity and Convexity.

Axiom 6. *(Strong Monotonicity). For all $f \neq_E g$, if $f \geq_E g$, then $f \succ_{E,h} g$.*

Axiom 7. *(Strict Convexity). For all $f \neq_E g$ and $\alpha \in (0,1)$, if $f \succsim_{E,h} g$, then $\alpha f + (1-\alpha)g \succ_{E,h} g$.*

The following axiom, No Flat Kinks, is weaker than Strict Convexity and specifies that if g is strictly preferred to f, then it is not the case that g is indifferent to all convex combinations of f and g for a closed interval of weights. We use this axiom in order to show the equivalence of weakly valuable information and one direction of weak DC. It allows for straight indifference curves but not those that have a "flat" kink.

Axiom 8. *(No Flat Kinks) If $g \succ_{E,h} f$ then there does not exist $\overline{a} \in (0,1]$ such that $af + (1-a)g \sim_{E,h} g$ for all $a \in [0, \overline{a}]$.*

An event $F \subseteq E$ is non $\succsim_{E,h}$-null if $g(s) > g'(s)$ for all $s \in F$ implies $gFf \succ_{E,h} g'Ff$, for all acts g, g', f. Hanany and Klibanoff [2007, 2009] restrict attention to non-null events.[7] However, we impose a weaker definition of non-nullity, which is motivated by the property that all subjective beliefs assign positive probability to not weakly null events, as we show in Lemma 2. Event F is *weakly* $\succsim_{E,h}$-null if there exists act g such that, for all acts g', f, $gFf \succsim_{E,h} g'Ff$. Because for every act f there exists another act g such that $g(s) > f(s)$ for all $s \in F$, we have that if F is non-null then it is also not weakly null. The following axiom requires that all events in $\mathcal{E}$ are not weakly null.

Axiom 9. *(Weak Full Support). For all events $E, F \in \mathcal{E}$, where $F \subseteq E$, F is not weakly $\succsim_{E,h}$-null.*

2.4 Consequentialism

Consequentialism requires that the agent's preferences depend only on the received information and not on the act that was chosen in the previous period.[8]

Axiom 10. *(Consequentialism) For all $f, g \in \mathcal{F}$ and events $E \in \mathcal{E}$, $\succsim_{E,f} = \succsim_{E,g}$.*

A weakening of Axiom 10 has been proposed in axiomatic work by Masatlioglu and Ok [2005], Sagi [2006] and Ortoleva [2010], where preference relation $\succsim_{E,h}$ depends on a "status quo" act (or frame) h. It specifies that if the agent ever prefers f over g (given some status quo h), then he would also prefer it if the status quo was f. In other words, the status quo exerts attraction towards itself.

Axiom 11. *(Status Quo Bias) For all $f, g, h \in \mathcal{F}$ and events $E \in \mathcal{E}$, if $f \succsim_{E,h} g$ then $f \succsim_{E,f} g$.*

As pointed by Masatlioglu and Ok [2005], Status Quo Bias is documented not only by experimental studies but also by empirical work in actual markets. For instance, Madrian and Shea [2001] examined how the default choice influenced participation in 401(k) saving plans, whereas Samuelson and Zeckhauser [1988] identified Status Quo Bias experimentally, in a study concerning portfolio choices.

[7] In their setting, F is non $\succsim_{E,h}$-null if $g(s) \succ_{E,h} g'(s)$ for all $s \in F$ implies $gFf \succ_{E,h} g'Ff$, for all acts g, g', f.

[8] Some papers refer to Consequentialism as the conjunction of Axioms 5 and 10.

2.5 Dynamic Consistency

DC provides restrictions on how two acts, which are identical outside of the conditioning event E, should be compared before and after E is known to have occurred. We break DC into two Axioms and adopt the names proposed by Ghirardato [2002].

Axiom 12. *(Consistency of Implementation) For all acts $f, g \in \mathcal{F}$ and events $E \in \mathcal{E}$, if $f \succsim g$ and $f =_{E^c} g$ then $f \succsim_{E,f} g$.*

Suppose that f and g specify the same payoff at each state not belonging to event E and that f is weakly preferred to g ex ante. Consistency of Implementation says that if the agent has chosen f ex ante and he is informed that event E has occurred (so that his preferences are $\succsim_{E,f}$), then in the interim stage f is still weakly preferred to g.

Axiom 13. *(Information is Valuable) For all acts $f, g \in \mathcal{F}$ and events $E \in \mathcal{E}$, if $f \succsim_{E,f} g$ and $f =_{E^c} g$ then $f \succsim g$.*

It is easier to interpret this axiom if we state the contrapositive, so that $g \succ f$ and $f =_{E^c} g$ imply $g \succ_{E,f} f$. Suppose that f and g specify the same payoff at each state not belonging to event E and that g is ex ante strictly preferred to f. Information is Valuable specifies that if the agent chose f ex ante and he is informed that event E has occurred (so that his preferences are $\succsim_{E,f}$), then, in the interim stage, g is still strictly preferred to f. But if g is ex ante strictly preferred to f, why should we put restrictions on the agent's preferences given that he has chosen f? The reason is that it may be that g was not feasible in the ex ante stage when f was chosen, hence his relevant conditional preferences are $\succsim_{E,f}$. Moreover, it could be that g becomes feasible in the interim stage.

2.6 Weak Dynamic Consistency

Using our notion of revealed preference, we provide a weakening of DC. Axiom 12 (Consistency of Implementation) is weakened by requiring that if ex ante f is revealed preferred (but not necessarily weakly preferred) to g, then f is weakly preferred to g, conditional on E. We also require that f is a strictly positive act.

Axiom 14. *(Weak Consistency of Implementation) For all acts $f \in \mathcal{F}_+$, $g \in \mathcal{F}$ and events $E \in \mathcal{E}$, if $f \succsim^* g$ and $f =_{E^c} g$ then $f \succsim_{E,f} g$.*

Suppose that there are three states $\{s_1, s_2, s_3\}$, the graph in Figure 1 depicts acts that are identical given s_3 but the indifference curve represents the ex ante preference relation $\succsim$, where $S = \{s_1, s_2, s_3\}$.[9] Act f is strictly preferred to g, h and k according to $\succsim$ but it is revealed preferred only to g and h. Hence, Axiom 14 requires that, given $E = \{s_1, s_2\}$, f is weakly preferred to g and h but not k. Proposition 2 shows that Axiom 14 characterizes Bayesian updating of subjective beliefs.

The other direction of DC, Axiom 13 (Information is Valuable), is weakened in a similar manner. Axiom 15 requires that if f is weakly preferred to g conditional on E and f but ex ante strictly preferred to f, then g is not revealed preferred to f.

[9] Axiom 14 has content if there are at least three states.

Axiom 15. *(Weak Information is Valuable) For all acts $f \in \mathcal{F}$, $g \in \mathcal{F}_+$ and events $E \in \mathcal{E}$, if $f \succsim_{E,f} g$, $f =_{E^c} g$ and $g \succ f$ then $g \not\succsim^* f$.*

To interpret this Axiom using the contrapositive, as with the stronger Axiom 13, suppose that f and g specify the same payoff at each state not belonging to event E and that g is ex ante both revealed preferred and strictly preferred to f. Weak Information is Valuable specifies that if the agent has chosen f ex ante and he is informed that event E has occurred (so that his preferences are $\succsim_{E,f}$), then, in the interim stage, g is still strictly preferred to f. Why impose restrictions on the agent's preferences given f, when g was strictly preferred ex ante? As with the justification of Axiom 13, this Axiom makes sense in the case where g was not feasible ex ante, so that the agent chose f and his preferences in the interim are $\succsim_{E,f}$.

In the previous example, if $k \succsim_{E,k} f$ and $k =_{E^c} f$, then Axiom 15 requires that $f \not\succsim^* k$, so that k lies strictly above the shaded area. This means that Axiom 15 allows for $f \succ k$, as is shown in Figure 1, unlike Axiom 13 which requires that $k \succsim f$ and therefore is stronger for convex preferences. Proposition 1 shows that Axiom 15 is equivalent to weakly valuable information.

What is the connection between the two parts of DC and weak DC? Under Strict Convexity and Status Quo Bias, (Weak) Information is Valuable implies (Weak) Consistency of Implementation. If we strengthen Status Quo Bias to Consequentialism, then the converse is also true.

Lemma 1. *Suppose convex preferences $\{\succsim_{E,h}\}_{E\in\mathcal{E},h\in\mathcal{F}}$ satisfy Axiom 7. Under Axiom 11, Axiom 13 (Axiom 15) implies Axiom 12 (Axiom 14). Under Axiom 10, the converse is also true.*

3 Valuable information

Several papers show with examples that failure of DC implies that information is not always valuable, under various settings (e.g. Wakker [1988], Epstein and Le Breton [1993]). In this section, we provide a formal treatment of this result and show that the converse is also true. In particular, we define information to be valuable if an agent always ex ante prefers having a finer than a coarser partition, given that he will choose an action in the interim stage, when a partition cell is revealed to him. We show that valuable information characterises the second part of DC (Axiom 13).[10]

However, even if the agent does not think that more information is always valuable, he may still recognise that it has *some* value. If the agent prefers mixing, for some $a \in (0, 1]$, between receiving the finer and the coarser partition, over receiving the coarser partition with certainty, we say that information is weakly valuable and show that it characterizes Axiom 15, which is the second part of weak DC.

In order to define (weakly) valuable information, we adopt the framework of Geanakoplos [1989], which assumed expected utility, to the present setting. There are two periods, 0 and 1. For simplicity, initially assume Axiom 10 (Consequentialism). An agent faces some uncertainty in period 0, represented by ex ante preferences $\succsim$ and a finite

[10] We are not aware of such a characterization previously shown in the literature.

state space S. His decision problem consists of a feasible set of acts, $\mathcal{A}$, and a partition $\Pi = \{E_1, \dots, E_n\}$ of S. The agent expects that, in period 1, he will be informed that a particular cell $E \in \Pi$ of his partition has occurred and will choose an "interim" act $f_E \in \mathcal{A}$, which is optimal according to his conditional preferences $\succsim_E$. By choosing an optimal interim act f_E for each partition cell $E \in \Pi$, he can generate an "ex-ante" optimal act h such that $h =_E f_E$, for each $E \in \Pi$. In other words, the ex ante optimal act h agrees with the interim acts $f_{E_1}, \dots f_{E_n}$, conditional on each element of Π, and $f_E \succsim_E g$ for all $g \in \mathcal{A}$.

Since we want to characterise valuable information also in the case where Consequentialism is not assumed, we adjust slightly the definition of ex ante optimality. In particular, we say that the ex ante optimal act h agrees with the interim acts $f_{E_1}, \dots f_{E_n}$, conditional on each element of Π, and $f_E \succsim_{E,h} g$ for all $g \in \mathcal{A}$.

The ex ante optimal act may not be unique, because there may be many interim optimal acts $f_E, f'_E \in \mathcal{A}$ given E. We therefore require, in order to say that partition Π is "more valuable" than partition Π' given $\mathcal{A}$, that for *every* ex ante optimal act $h^{\Pi'}$ for Π' there is an ex ante optimal act h^{Π} for Π such that $h^{\Pi} \succsim h^{\Pi'}$. It is weakly more valuable if $ah^{\Pi} + (1-a)h^{\Pi'} \succsim h^{\Pi'}$, for some $a \in (0, 1]$. We then say that information is (weakly) valuable if, for all $\mathcal{A}$, whenever Π is finer than Π' it is also (weakly) more valuable.

We now provide the formal treatment. Fix a collection of conditional preference relations $\{\succsim_{E,h}\}_{E \in \mathcal{E}, h \in \mathcal{F}}$, where $\succsim$ is the preference relation in period 0. A decision problem $\mathcal{D} = \{\Pi, \mathcal{A}\}$ consists of a partition $\Pi \subseteq \mathcal{E}$ of S representing the information in period 1 and a set of available acts $\mathcal{A} \subseteq \mathcal{F}$.

An act $f \in \mathcal{F}$ is feasible ex ante for decision problem $\mathcal{D}$ if for each $E \in \Pi$ there exists $g \in \mathcal{A}$ such that $f =_E g$, so that they provide the same payoffs given event E. Let $\mathcal{F}_{\mathcal{D}}$ be the set of acts which are feasible ex ante with respect to decision problem $\mathcal{D}$. Note that $\mathcal{A} \subseteq \mathcal{F}_{\mathcal{D}}$, whereas if $\Pi = \{S\}$ is the trivial partition, then $\mathcal{A} = \mathcal{F}_{\mathcal{D}}$. An act $f \in \mathcal{F}_{\mathcal{D}}$ which is feasible ex ante is optimal if, conditional on each element $E \in \Pi$ of the partition and f, it is weakly preferred to any act $g \in \mathcal{A}$.

Definition 1. *Act $f \in \mathcal{F}_{\mathcal{D}}$ is optimal for decision problem $\mathcal{D} = \{\Pi, \mathcal{A}\}$ if for all $E \in \Pi$, $f \succsim_{E,f} g$ for all $g \in \mathcal{A}$.*

Note that even if optimal $f \in \mathcal{F}_{\mathcal{D}}$ does not belong to $\mathcal{A}$, Axiom 5 and ex ante feasibility of f imply that there exists $g \in \mathcal{A}$ with $f =_E g$ and $f \sim_{E,f} g$, hence it is as if the agent picks f at each cell E. We compare two decision problems that differ only in terms of how they partition S, by comparing the optimal acts they generate, according to the ex ante preference relation $\succsim$. Decision problem $\mathcal{D}_1$ is more valuable than $\mathcal{D}_2$ if, for every optimal act for $\mathcal{D}_2$ there exists an optimal act for $\mathcal{D}_1$ that the agent weakly prefers. It is weakly more valuable if he weakly prefers a convex combination of the two.

Definition 2. *Decision problem $\mathcal{D}_1 = \{\Pi_1, \mathcal{A}\}$ is more valuable than decision problem $\mathcal{D}_2 = \{\Pi_2, \mathcal{A}\}$ if, whenever act $f \in \mathcal{F}_{\mathcal{D}_1}$ is optimal for $\mathcal{D}_1$ and act $g \in \mathcal{F}_{\mathcal{D}_2}$ is optimal for $\mathcal{D}_2$, we have $f \succsim g$. It is weakly more valuable if $af + (1-a)g \succsim g$ for some $a \in (0, 1]$.*

Recall that partition Π_1 is finer than partition Π_2 if for every element $E_2 \in \Pi_2$, there exists $E_1 \in \Pi_1$ with $E_1 \subseteq E_2$. Information is (weakly) valuable for $\{\succsim_{E,h}\}_{E\in\mathcal{E},h\in\mathcal{F}}$ if a decision problem generated by a finer partition is always (weakly) more valuable.[11]

Definition 3. *Information is (weakly) valuable for $\{\succsim_{E,h}\}_{E\in\mathcal{E},h\in\mathcal{F}}$ if for all $\mathcal{A} \subseteq \mathcal{F}$ ($\mathcal{F}_+$), whenever partition Π_1 is finer than partition Π_2, decision problem $\mathcal{D}_1 = \{\Pi_1, \mathcal{A}\}$ is (weakly) more valuable than decision problem $\mathcal{D}_2 = \{\Pi_2, \mathcal{A}\}$.*

We now show that valuable information is equivalent to Axiom 13, whereas under Axiom 8 weakly valuable information is equivalent to Axiom 15. Interestingly, we do not need to assume Consequentialism but Status Quo Bias, which is weaker.

Proposition 1. *Suppose Axiom 11 is satisfied. Then, information is valuable for convex preferences $\{\succsim_{E,h}\}_{E\in\mathcal{E},h\in\mathcal{F}}$ if and only if Axiom 13 is satisfied. Axiom 15 implies that information is weakly valuable. Conversely, under Axiom 8 weakly valuable information implies Axiom 15.*

4 Bayesian updating of subjective beliefs

RSS define the *subjective beliefs at an act* f and preference relation $\succsim_{E,h}$ to be the set of all normals (normalized to be probabilities) of the supporting hyperplanes of f,

$$\pi_{E,h}(f) = \{p \in \Delta S : \mathbb{E}_p g \geq \mathbb{E}_p f \text{ for all } g \succsim_{E,h} f\}.$$

In Figure 1, the indifference curve at f has a kink. All the supporting hyperplanes at f are in the shaded area. Set $\pi_{E,h}(f)$ contains their (normalized) normals.

RSS provide two alternative definitions for subjective beliefs and show that all three coincide for strictly positive acts. First, suppose that the agent's endowment is act f and we interpret a probability measure as a set of prices, one for each Arrow-Debreu security which pays 1 in a particular state and 0 otherwise. Given preference relation $\succsim_{E,h}$, the *subjective beliefs revealed by unwillingness to trade at* f contain the measures (prices) for which the agent would be unwilling to trade his endowment,

$$\pi^u_{E,h}(f) = \{p \in \Delta S : f \succsim_{E,h} g \text{ for all } g \text{ such that } \mathbb{E}_p g = \mathbb{E}_p f\}.$$

Second, let P be a set of measures (prices) such that whenever another act k is unaffordable for every $p \in P$, then there exists a mixture of k with endowment f that the agent would strictly prefer to his endowment. In Figure 1, there exist some points on the convex combination between f and k that are strictly preferred to f. The smallest such P of measures contains the *subjective beliefs revealed by willingness to trade at* f. Formally, let $\mathcal{P}_{E,h}(f)$ denote the collection of all compact, convex sets $P \subseteq \Delta S$ such that if $\mathbb{E}_p g > \mathbb{E}_p f$ for all $p \in P$, then $\epsilon g + (1-\epsilon)f \succ_{E,h} f$ for sufficiently small $\epsilon > 0$. Then, the subjective beliefs revealed by willingness to trade at f are

[11] The definition of weakly valuable information is restricted to feasible sets $\mathcal{A} \subseteq \mathcal{F}_+$ consisting of strictly positive acts. This simplifies the analysis (as in Hanany and Klibanoff [2009]), because it avoids the multiplicity of supporting hyperplanes at the boundary.

denoted by $\pi^w_{E,h}(f) = \bigcap \mathcal{P}_{E,h}(f)$. RSS show that for strictly positive acts f, $\pi_{E,h}(f) = \pi^u_{E,h}(f) = \pi^w_{E,h}(f)$.

We next define Bayesian updating of subjective beliefs. Note that even though we require that every subjective belief in $\pi(f)$ is updated when E occurs, we allow for the possibility that more subjective beliefs are included given E.

Definition 4. *The subjective beliefs of $\{\succsim_{E,h}\}_{E\in\mathcal{E},h\in\mathcal{F}}$ are updated using Bayes' rule if, for all $E \in \mathcal{E}$ and all $f \in \mathcal{F}_+$, if $p \in \pi(f)$ and $p(E) > 0$ then $p_E \in \pi_{E,f}(f)$.*

A convenient and often assumed property of dynamic models with ambiguity is mutual absolute continuity of the priors used to represent preferences. It says that all priors put positive probability on the same events. This facilitates the Bayesian updating of all the priors when new information arrives, without worrying how to update priors that assign zero probability on the event. Epstein and Marinacci [2007] characterize mutual absolute continuity in the multiple priors model, using a condition introduced by Kreps [1979]. In the following lemma we characterize mutual absolute continuity of the subjective beliefs, using the notion of a weakly null event.

Lemma 2. *If event $F \subseteq E \in \mathcal{E}$ is not weakly $\succsim_{E,h}$-null then $p \in \pi^u_{E,h}(f)$ implies $p(F) > 0$, for all acts $f \in \mathcal{F}$. Conversely, if, for all acts $f \in \mathcal{F}$, $p \in \pi^w_{E,h}(f)$ implies $p(F) > 0$, then F is not weakly $\succsim_{E,h}$-null.*

Using this lemma, Axiom 9 (which is implied by Strong Monotonicity) ensures that all subjective beliefs $p \in \pi^u_{E,h}(f)$ put positive probability at each event $F \in \mathcal{E}$, where $F \subseteq E$.

We now show that Axiom 14 (Weak Consistency of Implementation), which is weaker than the second part of DC (Axiom 12), is equivalent to Bayesian updating of subjective beliefs. For the "only if" direction we also need Axiom 9, which ensures that all subjective beliefs put positive probability on each conditioning event.

Proposition 2. *Suppose that convex preferences $\{\succsim_{E,h}\}_{E\in\mathcal{E},h\in\mathcal{F}}$ satisfy Axiom 9. Then, subjective beliefs are updated using Bayes' rule if and only if Axiom 14 is satisfied.*

5 Updating rules

In this section we discuss updating rules using the results of RSS, who characterise the subjective beliefs for various preference models with ambiguity aversion. We propose rules which satisfy Bayesian updating of subjective beliefs and therefore Axiom 14. As with the rest of the paper and RSS, the domain of preferences is acts from states to monetary outcomes. In Section 8, we discuss how Axiom 14 can be extended to a domain of state contingent lotteries. We restrict attention to strictly positive acts f, so that $\pi_h(f) = \pi^u_h(f) = \pi^w_h(f)$. We also assume that whenever u is defined, it is concave, increasing and continuously differentiable.

5.1 Variational preferences

Maccheroni et al. [2006a] axiomatize the variational preferences model, which contains as special cases the MEU model of Gilboa and Schmeidler [1989], the multiplier preferences model of Hansen and Sargent [2001] and the mean-variance preferences of Markowitz [1952] and Tobin [1958]. The representation of ex ante preferences is

$$U(f) = \min_{p \in \Delta S} \left\{ \int u(f) dp + c(p) \right\},$$

where $c : \Delta S \to [0, \infty]$ is a lower semicontinuous convex function, called ambiguity index, with $\min_{p \in \Delta S} c(p) = 0$ and u is increasing, concave and differentiable. For simplicity, we assume that if $c(p) \neq +\infty$, then $p(E) > 0$ for all $E \in \mathcal{E}$. RSS show that such preferences are convex.

Given event $E \in \mathcal{E}$, act $h \in \mathcal{F}$ and preference relation $\succsim_{E,h}$, the agent's utility at f is $U_{E,h}(f) = \min_{p \in \Delta E} \left\{ \int u(f) dp + c_{E,h}(p) \right\}$, where $c_{E,h}$ is the conditional ambiguity index with $\min_{p \in \Delta E} c_{E,h}(p) = 0$. Let $M_{E,h}(f) = \operatorname{argmin}_{p \in \Delta E} \left\{ \int u(f) dp + c_{E,h}(p) \right\}$ be the normals of the supporting hyperplanes at $U_{E,h}(f)$.

We define an updating rule by a tuple $\{D^f, n_{E,f}, k_{E,f}, c_{E,f}\}$, for each event $E \in \mathcal{E}$ and act f. First, $D^f \subseteq \Delta S$ is a convex set such that $M(f) \subseteq D^f$.[12] Let D^f_E be the prior by prior updating of D^f. Second, function $n_{E,f} : D^f_E \to \Delta S$ maps each $p \in D^f_E$ to an unconditional $q \in \Delta S$. Third, let

$$k_{E,f} = \min_{q \in D^f_E} \left\{ \int u(f) dn_{E,f}(q) + c(n_{E,f}(q)) - \int u(f) dq \right\}.$$

The ambiguity index $c_{E,f} : \Delta E \to [0, +\infty]$ is defined as

$$c_{E,f}(p) = \int u(f) dn_{E,f}(p) + c(n_{E,f}(p)) - \int u(f) dp - k_{E,f}, \tag{1}$$

if $p \in D^f_E$, otherwise $c_{E,f}(p) = +\infty$. Subtracting $k_{E,f}$ ensures that $c_{E,f}(p) \in [0, +\infty]$ and $c_{E,f}(p) = 0$ for some p. In order to ensure that $c_{E,f}$ is convex, continuous and subjective beliefs are updated, we consider two specific functions $n_{E,f}$.

For the first updating rule, fix measure $r \in M(f)$ with $r(E) > 0$ and let D^f such that $p \in D^f$ implies $c(p) \neq +\infty$.[13] Set $n_{E,f}(p) = r$ for all $p \in D^f_E$.

The second rule applies only in the case where $M(f) = \{r\}$ is a singleton and $r(E) > 0$. Set $D^f = \Delta S$ and let $n_{E,f}(p) = p \otimes^E r$ for all $p \in \Delta E$, where $p \otimes^E r \in \Delta S$ is such that $p \otimes^E r(F) = r(E)p(F) + r(F \cap E^c)$ for all events F. Then, in $p \otimes^E r$ the choice of p determines all probabilities given E whereas r determines all other probabilities.

We next show that these two rules generate variational preferences that satisfy Bayesian updating of subjective beliefs and Axiom 14, although Axioms 12 and 13 are

[12] Note that $M(f)$ is convex.

[13] Note that if there does not exist $r \in M(h)$ with $r(E) > 0$, Bayesian updating of subjective beliefs is trivially satisfied.

violated in general. Interestingly, for the first rule, if we set $D^f = D$ to be the set of all measures with $c(p) \neq +\infty$, for all $f \in \mathcal{F}$, so that it is independent of f, then Axiom 12 is satisfied. If $D^f = M(f)$ for all $f \in \mathcal{F}$, then Axiom 13 is satisfied. Hence, the largest possible set of prior by prior updating satisfies Consistency of Implementation, whereas the smallest possible set satisfies Information is Valuable. Any set which is in between, satisfies Weak Consistency of Implementation.

Lemma 3. *Suppose that $\succsim$ is a variational preference. Then, both updating rules described above generate variational preferences $\{\succsim_{E,f}\}_{E\in\mathcal{E},f\in\mathcal{F}}$ that satisfy Bayesian updating of subjective beliefs and Axiom 14. Moreover, in the first rule if $D^f = D$ for all $f \in \mathcal{F}$ and $c(p) \neq +\infty$ implies $p \in D$, then Axiom 12 is satisfied. If $D^f = M(f)$ for all $f \in \mathcal{F}$, then Axiom 13 is satisfied. However, in general both rules violate Axioms 12 and 13.*

5.2 Confidence preferences

Chateauneuf and Faro [2009] axiomatize a class of preferences where ambiguity is measured by a confidence function $\phi : \Delta S \to [0,1]$, with $\phi(p) = 1$ meaning full confidence in p. The set of full confidence measures is nonempty and ϕ is upper semicontinuous and quasiconcave. Preferences are represented by

$$U(f) = \min_{p\in L_a} \frac{1}{\phi(p)} \mathbb{E}_p u(f),$$

where $L_a = \{q \in \Delta S : \phi(q) \geq a\}$ is a set of measures with confidence of at least $a > 0$. Define $M(f) = \arg\min_{p\in L_a} \left\{\frac{1}{\phi(p)}\mathbb{E}_p u(f)\right\}$ for each $f \in \mathcal{F}$. For simplicity, we assume that if $p \in L_a$ then $p(E) > 0$ for all $E \in \mathcal{E}$. Hence, Axiom 9 is satisfied.

Consider the following updating rule. Fix measure $r \in M(h)$ with $r(E) > 0$ and convex set D^h such that $M(h) \subseteq D^h \subseteq L_a$. Let D^h_E be the prior by prior updating of D^h, given $E \in \mathcal{E}$. Let $k_{E,h} = \max_{q\in D_E} \phi(r)\frac{\mathbb{E}_q u(h)}{\mathbb{E}_r u(h)}$. For each $p \in \Delta E$, define

$$\phi_{E,h}(p) = \frac{\phi(r)}{k_{E,h}} \frac{\mathbb{E}_p u(h)}{\mathbb{E}_r u(h)} \tag{2}$$

if $p \in D_E$, otherwise $\phi_{E,h}(p) = 0$. Setting $a_{E,h} = \min_{p\in D_E} \phi_{E,h}(p)$, we have that $D^h_E = L_{a_{E,h}}$ and $U_{E,h}(f) = \min_{p\in L_{a_{E,h}}} \frac{1}{\phi_{E,h}(p)}\mathbb{E}_p u(f)$.

We now show that this updating rule satisfies Axiom 14. As with the first updating rule on variational preferences, the largest possible prior by prior updating set satisfies Axiom 12, whereas the smallest satisfies Axiom 13. Anything in between, satisfies Axiom 14. However, if D^h is a strict subset of L_a, then Axiom 12 may be violated.

Lemma 4. *Suppose that $\succsim$ is a confidence preference. Then, the updating rule described above generates confidence preferences $\{\succsim_{E,h}\}_{E\in\mathcal{E},h\in\mathcal{F}}$ that satisfy Bayesian updating of subjective beliefs and Axiom 14. Moreover, if $D^f = L_a$ for all $f \in \mathcal{F}$, then Axiom 12 is satisfied. If $D^f = M(f)$ for all $f \in \mathcal{F}$, then Axiom 13 is satisfied. If D^h is a strict subset of L_a, then Axiom 12 may be violated.*

6 Speculative trade

In this and the next section, we show that weak DC has economic content in financial markets. In particular, we show that Axiom 14, which is equivalent to Bayesian updating of subjective beliefs, is the minimum requirement which precludes speculative trade.

Consider an economy consisting of I agents, with $|I| = m$ and typical element i. Each agent's consumption set is the set of acts $\mathcal{F}$. He is endowed with a collection of convex preferences $\{\succsim^i_{E,h}\}_{E\in\mathcal{E},h\in\mathcal{F}}$.

An economy is a tuple $\langle\{\succsim^i_{E,h}\}_{i\in I,E\in\mathcal{E},h\in\mathcal{F}}, e, \{\Pi^i\}_{i\in I}\rangle$, where $e \in \mathbb{R}^S_{++}$ is the aggregate endowment and $\{\Pi^i\}_{i\in I}$ denotes the information structure, where each $\Pi^i \subseteq \mathcal{E}$ is a partition of S. If in period 0 the resulting allocation is f, then in period 1 and at state s, agent i considers states in $\Pi^i(s)$ to be possible and has conditional preferences $\succsim^i_{\Pi^i(s),f^i}$.

An allocation is a tuple $f = (f^1, \ldots, f^m) \in \mathcal{F}^m$. It is feasible if $\sum_{i=1}^m f^i = e$. It is interior if $f^i(s) > 0$ for all $s \in S$ and for all i. Given an event $E \subseteq S$, let $K^i(E) = \{s \in S : \Pi^i(s) \subseteq E\}$ be the set of states where i knows E. Event E is self evident if $E \subseteq K^i(E)$ for all $i \in I$. That is, an event is self evident if whenever it happens, everyone knows it. An event F is common knowledge at s if and only if there exists a self evident event E such that $s \in E \subseteq F$ (Aumann [1976]).

We say that there is speculative trade if an allocation is ex ante Pareto efficient (according to preferences $\{\succsim^i\}_{i\in I}$) but at some state $s \in S$ it is common knowledge that there exists a Pareto improvement.

Definition 5. *There is speculative trade in economy $\langle\{\succsim^i_{E,h}\}_{i\in I,E\in\mathcal{E},h\in\mathcal{F}}, e, \{\Pi^i\}_{i\in I}\rangle$ at an ex ante Pareto efficient allocation f if there is agent $j \in I$, state s' and feasible allocation g such that event $H = \{s \in S : g^i \succsim^i_{\Pi^i(s),f^i} f^i$ for all $i \in I$ and $g^j \succ^j_{\Pi^j(s),f^j} f^j\}$ is common knowledge at s'.*

We now show that under Strong Monotonicity, Axiom 14 is necessary and sufficient for preventing speculative trade. In particular, if all agents satisfy Axiom 14 then there is no speculative trade, whereas if at least one fails it, there are economies with speculative trade. Let $\mathbb{P}$ be the collection of convex preferences $\{\succsim_{E,h}\}_{E\in\mathcal{E},h\in\mathcal{F}}$ satisfying Axiom 6. Note that we do not require Consequentialism or Status Quo Bias.

Proposition 3. *If $\{\succsim^i_{E,h}\}_{i\in I,E\in\mathcal{E},h\in\mathcal{F}} \in \mathbb{P}^I$ satisfy Axiom 14 then there is no speculative trade in any economy $\langle\{\succsim^i_{E,h}\}_{i\in I,E\in\mathcal{E},h\in\mathcal{F}}, e, \{\Pi^i\}_{i\in I}\rangle$ and at any interior allocation f. Conversely, if $\{\succsim^k_{E,h}\}_{E\in\mathcal{E},h\in\mathcal{F}} \in \mathbb{P}$ fails Axiom 14 then there exist preferences $\{\succsim^i_{E,h}\}_{i\in I,E\in\mathcal{E},h\in\mathcal{F}} \in \mathbb{P}^I$ satisfying Axiom 14 and economy $\langle\{\succsim^i_{E,h}\}_{i\in I\cup\{k\},E\in\mathcal{E},h\in\mathcal{F}}, e, \{\Pi^i\}_{i\in I\cup\{k\}}\rangle$ such that there is speculative trade at an allocation f.*

To provide a sketch of the proof for one direction, suppose by contradiction that allocation $\{f^i\}_{i\in I}$ is ex ante Pareto efficient but in the interim it is common knowledge at some state s' that allocation $\{g^i\}_{i\in I}$ is a Pareto improvement. From Aumann [1976], there exists a self evident event F containing s', where $s \in F$ implies $g^i \succ_{\Pi^i(s),f^i} f^i$ for all $i \in I$.[14] Axiom 5 implies $g^i\Pi^i(s)f^i \succ_{\Pi^i(s),f^i} f^i$. Because each Π^i partitions F,

[14]We can assume strict preference for everyone due to Strong Monotonicity.

Axiom 14 implies $\mathbb{E}_p(g^i F f^i) > \mathbb{E}_p f^i$ for all $p \in \pi^i(f^i)$. By convexity of preferences, $\epsilon g^i F f^i + (1-\epsilon) f^i \succ^i f^i$ for small enough $\epsilon > 0$, which contradicts that $\{f^i\}_{i \in I}$ is ex ante Pareto efficient.

Following Kajii and Ui [2009], Martins-da-Rocha [2010] also shows that Bayesian updating of subjective beliefs precludes speculative trade. But both papers assume Consequentialism, which is not needed here. Halevy [2004] proves the absence of speculative trade by assuming DC but relaxing Consequentialism to the following two properties: Resolute Conditional preferences (which is similar to our Axiom 5) and Conditional Decomposability (which we do not require). Ma [2001] proves the result using DC and a weakening of Consequentialism, called piecewise monotonicity. Galanis [2018] examines speculative trade in an environment with unawareness, where DC is violated but Consequentialism is not.

7 The value of public information in competitive risk-sharing environments

Hirshleifer [1971] first argued with an example that if agents trade in order to mutually insure, then more public information could make everyone worse off. Schlee [2001] generalised this result, showing that the value of public information is negative in an expected utility model with a common prior, risk aversion and no aggregate uncertainty.[15]

For general convex preferences, the notion of a common prior is not necessarily well defined. However, we find that as long as Bayesian updating of subjective beliefs and Status Quo Bias are satisfied, if every agent (weakly) values information, then public information is (weakly) not valuable.[16] This result provides further evidence that Bayesian updating of subjective beliefs has economic content.

In order to rule out pure indifference to betting, we assume Strict Convexity and Strong Monotonicity. We also assume the following axiom, which is proposed by RSS.

Axiom 16. *(Translation Invariance at Certainty). For all acts $h \in \mathcal{F}$ and events $E \in \mathcal{E}$, for all $g \in \mathcal{F}$ and all constant bundles $x, x' > 0$, if $x + \lambda g \succsim_{E,h} x$ for some $\lambda > 0$, then there exists $\lambda' > 0$ such that $x' + \lambda' g \succsim_{E,h} x'$.*

RSS show that Axiom 16 is satisfied by most classes of ambiguity averse preferences and it implies that subjective beliefs do not change across constant acts: $\pi^i_{E,h}(x) = \pi^i_{E,h}(x')$ for all constant acts $x, x' > 0$. We henceforth write $\pi^i_{E,h}$ instead of $\pi^i_{E,h}(x)$ for all constant acts $x > 0$.

We also impose a slight variation of Axiom 14. First, it applies only to constant acts. Second, it should apply not only between the ex ante preference relation $\succsim^*$ and the interim $\succsim_{E,x}$, but also between $\succsim^*_{F,h}$ and $\succsim_{E,x}$, where $E \subseteq F$ and $F \in \mathcal{E}$. In other words, it is as if we consider a multi period model where the agent first learns F and

[15] Schlee [2001] uses the Blackwell [1951] criterion of more information. Moreover, he proves this result in two other cases, that we do not examine. First, there are some risk neutral agents who fully insure the risk averse ones. Second, all agents are risk averse and the economy has a representative agent.

[16] A similar result is shown by Galanis [2016], in an environment with unawareness, where DC is violated.

then E. Adapting Proposition 2, there is Bayesian updating of subjective beliefs at a constant act both when F occurs and when E occurs.[17]

Axiom 17. *(Multi Period Weak Consistency of Implementation) For all acts $x, g, h \in \mathcal{F}$, where $x > 0$ is constant, and events $F, E \in \mathcal{E}$ with $E \subseteq F$, if $x \succsim^*_{F,h} g$ and $x =_{E^c} g$ then $x \succsim_{E,x} g$.*

We adapt the setting of Section 3 for the case of finitely many agents. There are two periods, 0 and 1. In period 0, the agents' common information structure about period 1 is represented by partition Π of S. Hence, there is symmetric information among all agents. The initial allocation is $\{e^i\}_{i\in I}$, where $e^i \in \mathcal{F}_+$.The aggregate endowment is $\sum_{i\in I} e^i = e \in \mathbb{R}^S_{++}$. We assume that there is no aggregate uncertainty, so e is constant across all states in S. The economy in period 0 is a tuple $\langle S, \succsim^1, \ldots, \succsim^m, e\rangle$.

In period 1, all agents are informed that some event $E \in \Pi$ has occurred and trade, using their conditional preferences. Hence, information is symmetric. Trading at each $E \in \Pi$ generates an act for each agent, which is evaluated in period 0 using preference relation $\succsim^i$.

Given event $E \in \Pi$, an allocation for economy $\langle E, \succsim^1_{E,h^i}, \ldots, \succsim^m_{E,h^m}, e\rangle$ is a tuple $f_E = (f^1_E, \ldots, f^m_E) \in \mathcal{F}^m$.[18] It is feasible if $\sum_{i=1}^m f^i_E = e$. It is interior if $f^i_E(s) > 0$ for all $s \in E$ and for all i. A feasible allocation f_E is full insurance if each f^i_E is constant across all states in E. It is Pareto optimal if there is no feasible allocation g_E such that $g^i_E \succsim^i_{E,h^i} f^i_E$ for all $i \in I$ and $g^j_E \succ^j_{E,h^j} f^j_E$ for some $j \in I$.

Fix a collection of convex preferences $\{\succsim^i_{E,h}\}_{i\in I, E\in\mathcal{E}, h\in\mathcal{F}}$. Let $\mathcal{M} = \{\Pi, e\}$ be an aggregate decision problem, where $\Pi \subseteq \mathcal{E}$ is a partition of S and $e \in \mathbb{R}^S_{++}$ is the aggregate endowment, assumed constant across states.

Given event $E \in \Pi$ and economy $\langle E, \succsim^1_{E,h^i}, \ldots, \succsim^m_{E,h^m}, e\rangle$, define $f_E = \{f^i_E\}_{i\in I} \in \mathcal{F}^m$ to be an equilibrium allocation if it is feasible and there are prices $p \in \mathbb{R}^S_+$, with $p(s) = 0$ if $s \notin E$, such that, for each $i \in I$, $\mathbb{E}_p f^i_E \leq \mathbb{E}_p e^i$ and $f^i_E \succsim^i_{E,h^i} g$ for all g such that $\mathbb{E}_p g \leq \mathbb{E}_p e^i$.

We say that interior allocation $\{f^i\}_{i\in I}$ is admissible for aggregate decision problem $\mathcal{M} = \{\Pi, e\}$ if, for each agent $i \in I$, for each $E \in \Pi$, $f^i =_E f^i_E$, where $f_E = \{f^i_E\}_{i\in I}$ is an equilibrium allocation of economy $\langle E, \succsim^1_{E,f^1}, \ldots, \succsim^m_{E,f^m}, e\rangle$. In words, $\{f^i\}_{i\in I}$ is admissible if, under some equilibrium, agent i receives f^i_E at each $E \in \Pi$.

We compare aggregate decision problems by evaluating the admissible acts they generate.

Definition 6. *Aggregate decision problem $\mathcal{M}_1 = \{\Pi_1, e\}$ is not more valuable than $\mathcal{M}_2 = \{\Pi_2, e\}$ if whenever $\{g^i\}_{i\in I}$ is admissible for $\mathcal{M}_2$, there exists $\{f^i\}_{i\in I}$ which is admissible for $\mathcal{M}_1$ and $g^i \succsim^i f^i$, for all $i \in I$. It is weakly not more valuable if $ag^i + (1-a)f^i \succsim f^i$ for some $a \in (0,1]$.*

[17] Recall that Definition 4 requires that every subjective belief is updated, however it is allowed that more such beliefs are included. Hence, although Axiom 14 implies Bayesian updating of subjective beliefs between $\succsim^*$ and $\succsim_{E,f}$ and between $\succsim^*$ and $\succsim_{F,f}$, it does not imply Bayesian updating between $\succsim^*_{F,h}$ and $\succsim_{E,f}$.

[18] Note that we define the aggregate endowment of the economy as a map from S (rather than E) to $\mathbb{R}_{++}$. This is without loss of generality because, from Axiom 5, what the endowment prescribes outside of E is irrelevant. For consistency, we do the same for all subsequent acts.

We say that public information is (weakly) not valuable if an aggregate decision problem with a finer partition is always (weakly) not more valuable than a decision problem with a coarser partition.

Definition 7. *Public information is (weakly) not valuable for* $\{\succsim^i_{E,h}\}_{i\in I,E\in\mathcal{E},h\in\mathcal{F}}$ *if, for all endowments* e *and partitions* Π_1, Π_2 *of* S*,* Π_1 *finer than* Π_2*, aggregate decision problem* $\mathcal{M}_1 = \{\Pi_1, e\}$ *is (weakly) not more valuable than* $\mathcal{M}_2 = \{\Pi_2, e\}$*.*

The following Proposition shows that if information is (weakly) valuable for each agent, then public information is (weakly) not valuable.

Proposition 4. *Suppose convex preferences* $\{\succsim^i_{E,h}\}_{i\in I,E\in\mathcal{E},h\in\mathcal{F}}$ *satisfy Axioms 6, 7, 11, 16 and 17. If information is (weakly) valuable for all* $i \in I$ *then public information is (weakly) not valuable for* $\{\succsim^i_{E,h}\}_{i\in I,E\in\mathcal{E},h\in\mathcal{F}}$*.*

8 Discussion

Before discussing the literature, we compare our approach with Ghirardato et al. [2004] and Hanany and Klibanoff [2007, 2009], which are closely related. These papers study preferences over acts which are functions $f : S \to X$, where X is the set of simple lotteries over a set of consequences Z. In order to compare our approach with theirs, we assume that $Z = \mathbb{R}_+$. Let $\mathcal{F}_{\mathcal{AA}}$ be the set of such acts. Then, every $f \in \mathcal{F}$ can be identified as act $f_{\mathcal{AA}} \in \mathcal{F}_{\mathcal{AA}}$, where $f_{\mathcal{AA}}(s)$ is a degenerate lottery that puts probability 1 to outcome $f(s)$. Given acts $f, g \in \mathcal{F}_{\mathcal{AA}}$, denote by $a \otimes f + (1-a) \otimes g$ the act which gives, at each state s, $f(s)$ with probability a and $g(s)$ with probability $(1-a)$. Our notion of revealed preference in the bigger domain $\mathcal{F}_{\mathcal{AA}}$ is defined as follows. For all $f, g \in \mathcal{F}_{\mathcal{AA}}$, $f \succsim^*_{\mathcal{AA}} g$ if $f \succsim a \otimes g + (1-a) \otimes f$ for all $a \in [0,1]$.

8.1 Unambiguous preferences

Ghirardato et al. [2004] define f to be *unambiguously preferred* to g, written $f \succsim^*_{GMM} g$, if mixing with any other act h does not reverse preferences: $\alpha \otimes f + (1-\alpha) \otimes h \succsim \alpha \otimes g + (1-\alpha) \otimes h$ for all acts $f, g, h \in \mathcal{F}_{\mathcal{AA}}$ and $\alpha \in (0,1]$. Our notion of revealed preference (given Consequentialism) applies to a smaller domain $\mathcal{F}$ but requires mixing only with $h = f$, so that $f \succsim^* g$ if $f \succsim ag + (1-a)f$ for all $f, g, \in \mathcal{F}$ and $a \in [0,1]$. However, when our notion of revealed preference is applied in the bigger domain, it is weaker. That is, $f \succsim^*_{GMM} g$ implies $f \succsim^*_{\mathcal{AA}} g$ for all $f, g \in \mathcal{F}_{\mathcal{AA}}$.

Although their representation is in a setting which satisfies the Certainty Independence Axiom of Gilboa and Schmeidler [1989], Cerreia-Vioglio et al. [2011] show that it holds more generally for uncertainty averse preferences. In particular, they show that $f \succsim^*_{GMM} g$ if and only if $\int u(f)dp \geq \int u(g)dp$ for all $p \in C$, where $C = cl(co(\bigcup_{f\in\mathcal{F}} \pi^u_0(f)))$ and $\pi^u_0(f) = \{p \in \Delta S : f \succsim g \text{ for all } g \text{ such that } \mathbb{E}_p u(f) \geq \mathbb{E}_p u(g)\}$ is the set of subjective beliefs evaluated at $u(f)$, rather than at f, as is $\pi^u(f)$.

Ghirardato et al. [2008] provide a dynamic version of Ghirardato et al. [2004], assuming Axioms 5, 10 and DC on $\succsim^*_{GMM}$, so that $f \succsim^*_{GMM,E} g$ if and only if $f \succsim^*_{GMM}$

g, for all acts f, g with $f =_{E^c} g$. They show that Bayesian updating of all beliefs in C is equivalent to $\succsim^*_{GMM}$ satisfying DC.

8.2 Uncertainty averse preferences

Hanany and Klibanoff [2007] relax Consequentialism, allowing for preferences conditional on an event to also depend on the feasible set B of acts and on the act that was chosen ex ante. They show in the MEU model that a weakening of DC is equivalent to Bayesian updating of a *subset* of the unconditional beliefs. Moreover, their model allows for both Ellsberg type behavior and non-reversal of preferences when information is revealed, hence DC is satisfied. Hanany and Klibanoff [2009] generalise their approach to models that satisfy the uncertainty aversion axiom Schmeidler [1989], showing that their Dynamic Consistency Axiom is equivalent to having at least one measure which supports both the conditional indifference curve at the chosen act and its conditional optimality.

To explain their approach, let $\mathcal{Q}$ be the set of all quadruples $(\succsim, E, f, B)$, where non-null E is the conditioning event and act $f \in \mathcal{F}_{\mathcal{AA}}$ is optimally chosen from a convex feasible set $B \in \mathcal{B}$ ($f \succsim g$ for all $g \in B$), before the realization of E, where $\mathcal{B}$ is the set of all convex and compact sets. Let $\mathcal{D} \subseteq \mathcal{Q}$ be a domain if $(\succsim, E, f, B) \in \mathcal{D}$ implies $(\succsim, E, f', B') \in \mathcal{D}$ for each f', B' such that $(\succsim, E, f', B') \in \mathcal{Q}$ and $u \circ f'$ is the unique maximizer of V (which represents preferences) over $u \circ B'$. Their Dynamic Consistency axiom is the following.

Axiom 18. *(DC_{HK}) For any $(\succsim, E, f, B) \in \mathcal{D}$, if $g \in B$ with $f =_{E^c} g$, then $f \succsim_{E,f,B} g$.*

The main difference of their approach is that preferences depend not only on the conditioning event E and act f, but also on the feasible set B. Compared to theirs, the present paper additionally imposes that $\succsim_{E,f,B} = \succsim_{E,f,B'}$ for all convex feasible sets B and B'. Moreover, $\succsim$ is defined on the smaller domain $\mathcal{F}$, whereas their domain is $\mathcal{F}_{\mathcal{AA}}$.

To compare the two approaches, fix a preference relation $\succsim$ on $\mathcal{F}_{\mathcal{AA}}$ and consider the following axiom, recalling that every $f \in \mathcal{F}$ can be identified as act $f_{\mathcal{AA}} \in \mathcal{F}_{\mathcal{AA}}$, where $f_{\mathcal{AA}}(s)$ is a degenerate lottery that puts probability 1 to outcome $f(s)$. Slightly abusing notation, we denote $f_{\mathcal{AA}}$ by f.

Axiom 19. *For all $f, g \in \mathcal{F}$ and all $a \in [0, 1]$, $af + (1 - a)g \succsim a \otimes f + (1 - a) \otimes g$.*

This axiom specifies that, given any acts $f, g \in \mathcal{F}$, the agent always prefers an act which gives, at state s, the expected payoff $af(s) + (1-a)g(s) \in Z$ for sure rather than the act $a \otimes f + (1 - a) \otimes g$, which at s is the lottery that gives $f(s)$ with probability a and $g(s)$ with probability $1 - a$. It is satisfied in the case of variational and smooth convex preferences with concave u and ϕ, together with the usual monotonicity axiom: $h(s) \succsim k(s)$ for all $s \in S$ implies $h \succsim k$. The following lemma shows that under this axiom and given that conditional preferences do not depend on the feasible set B, DC_{HK} implies Axiom 14.

Lemma 5. *Suppose $\succsim_{E,f,B} = \succsim_{E,f,B'}$ for all convex feasible sets B and B'. Then, Axioms 18 and 19 imply Axiom 14.*

Axiom 14 applies to a more restrictive domain than that of Axiom 18, hence the converse cannot be true. However, if we restate Axiom 14 in the bigger $\mathcal{F}_{\mathcal{A}\mathcal{A}}$ domain, then the two axioms are equivalent. Recall that $f \succsim^*_{\mathcal{A}\mathcal{A}} g$ if $f \succsim a \otimes g + (1-a) \otimes f$ for all $a \in [0,1]$, where $f, g \in \mathcal{F}_{\mathcal{A}\mathcal{A}}$. For simplicity, we assume that all acts in $\mathcal{F}_{\mathcal{A}\mathcal{A}}$ are strictly positive in utility space (the interior acts in Hanany and Klibanoff [2009]).

Axiom 20. *For all acts $f, g \in \mathcal{F}_{\mathcal{A}\mathcal{A}}$ and events $E \in \mathcal{E}$, if $f \succsim^*_{\mathcal{A}\mathcal{A}} g$ and $f =_{E^c} g$ then $f \succsim_{E,f} g$.*

We now show that the two axioms are equivalent, given that conditional preferences do not depend on the feasible set B.

Lemma 6. *Suppose $\succsim_{E,f,B} = \succsim_{E,f,B'}$ for all convex feasible sets B and B'. Then, Axiom 18 is equivalent to Axiom 20.*

In general, however, where $\succsim_{E,f,B} \neq \succsim_{E,f,B'}$ is allowed, Axiom 20 is stronger than Axiom 18, as it requires that $\succsim_{F,f,B} = \succsim_{F,f,B'}$ for all B, B'. The characterizations in Hanany and Klibanoff [2007, 2009] are provided for this general case, except for the smooth ambiguity model, where there is a unique subjective belief at each f.

We now show that, in the case of variational preferences where subjective beliefs are not unique and $\succsim_{F,f,B} \neq \succsim_{F,f,B'}$, a rule proposed by Hanany and Klibanoff [2009] which satisfies Axiom 18 implies that at least one subjective belief is updated, but not all. This means that Bayesian updating of subjective beliefs and Axiom 14 fail.[19]

Fix $r \in M(f) \cap Q^{E,f,B}$, where $Q^{E,f,B}$ is the set of measures which, when conditioned on E, support the conditional optimality of f in B.[20] The ambiguity index of their proposed updating rule is

$$c_{E,f,B}(p) = \frac{1}{r(E)} \left[c(p \otimes^E r) - \min_{q \in \Delta E} c(q \otimes^E r) \right]. \tag{3}$$

Note that for all $p \in \Delta E$, $\int u(f)dp + c_{E,f,B}(p) \geq \int u(f)dr_E + c_{E,f,B}(r_E)$ if and only if $r(E)\int u(f)dp + c(p \otimes^E r) \geq r(E)\int u(f)dr_E + c(r)$, which is true because $\int u(f)dp \otimes^E r = r(E)\int u(f)dp + \int_{E^c} u(f)dr$ and $r \in M(f)$. Using the proof of Lemma 9, we can show that the subjective belief generated by r is updated.

If, however, $q \in M(f)$ and $q \neq r$, then it is not necessarily the case that $\int u(f)dp + c_{E,f,B}(p) \geq \int u(f)dq_E + c_{E,f,B}(q_E)$ for all $p \in \Delta E$, which means that the subjective belief generated by q is not updated. The reason is that we need $r(E)\int u(f)dp + c(p \otimes^E r) \geq r(E)\int u(f)dq_E + c(q_E \otimes^E r)$, which is equivalent to $\int u(f)dp \otimes^E r + c(p \otimes^E r) \geq \int u(f)dq_E \otimes^E r + c(q_E \otimes^E r)$, however if $q_E \otimes^E r \notin M(f)$ this may not be true. Hence, not all subjective beliefs are updated and Axiom 14 is violated.

To provide a numerical example, consider the setting of Section 1.1 with MEU ex ante preferences, $u(x) = x$ and C being the convex hull of the following three measures: $p_1 = (0.24, 0.33, 0.43), p_2 = (0.34, 0.2, 0.46)$ and $p_3 = (0.21, 0.68, 0.11)$. Let $f = x \in \mathbb{R}$ be a constant act and note that $M(f) = C$. Pick $r = p_1$ and define feasible set B

[19]Recall that Axiom 14 is the minimum requirement which ensures that there is no speculative trade, as we show in Proposition 3.

[20]The formal definition of $Q^{E,f,B}$ is in Hanany and Klibanoff [2009].

such that $f \in B$ if $\mathbb{E}_r x \geq \mathbb{E}_r f$. Then, $p_1 \in M(f) \cap Q^{E,f,B}$. If we pick p_1 to generate $c_{E,x,B}$, using (3), then we have that $p_{2E} \otimes^E p_1 = \{0.3589, 0.2111, 0.4300\} \notin C$. This means that $c_{E,x,B}(p_{2E}) = +\infty$ and $p_{2E} \notin M_{E,x}(x)$, hence not all subjective beliefs are updated.

Finally, we show that in the smooth ambiguity model (Maccheroni et al. [2006a]), where subjective beliefs are unique at each f, Hanany and Klibanoff [2009] propose a rule which satisfies Bayesian updating of subjective beliefs. An act f is weakly preferred to g if and only if $\mathbb{E}_\mu \phi(\mathbb{E}_p u \circ f) \geq \mathbb{E}_\mu \phi(\mathbb{E}_p u \circ g)$, where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing transformation called the ambiguity attitude and μ is a subjective probability over the set of probability measures $p \in \Delta S$ that the agent considers relevant for his problem. For simplicity, we assume that μ has finite support and that u and ϕ are concave, increasing and differentiable, so that preferences are convex. We also assume Axiom 9, which implies that for each $E \in p$, there exists $p \in \text{supp}(\mu)$ with $p(E) > 0$.

Hanany and Klibanoff [2009] propose the smooth rule, which specifies that

$$\mu_{E,f}(p) = \frac{\mu(p)p(E)\frac{\phi'(\mathbb{E}_p(u\circ f))}{\phi'(\mathbb{E}_{p_E}(u\circ f))}}{\sum_{\hat{p}\in\Delta S} \mu(\hat{p})\hat{p}(E)\frac{\phi'(\mathbb{E}_{\hat{p}}(u\circ f))}{\phi'(\mathbb{E}_{\hat{p}_E}(u\circ f))}}$$

if $p(E) > 0$ and 0 otherwise.

The smooth rule satisfies the following condition, which Hanany and Klibanoff [2009] show that it characterizes DC_{HK}:

$$\frac{\mathbb{E}_{\mu_{E,f}}[\phi'(\mathbb{E}_{p_E} u \circ f) p_E(s)]}{\mathbb{E}_{\mu_{E,f}}[\phi'(\mathbb{E}_{p_E} u \circ f)]} = \frac{\mathbb{E}_\mu[\phi'(\mathbb{E}_p u \circ f)p(s)]}{\mathbb{E}_\mu[\phi'(\mathbb{E}_p u \circ f)p(E)]} \tag{4}$$

for all $f \in \mathcal{F}$, $s \in E$ and $p \in \text{supp}(\mu)$ with $p(E) > 0$.[21] We show that (4) satisfies Bayesian updating of subjective beliefs.

Lemma 7. *There is Bayesian updating of subjective beliefs for any rule satisfying (4).*

8.3 Related literature

Cerreia-Vioglio et al. [2011] study general convex preferences which satisfy the uncertainty aversion axiom of Schmeidler [1989]. Epstein and Schneider [2003], Maccheroni et al. [2006b] and Klibanoff et al. [2009] use DC in order to provide recursive representations for the static models of Gilboa and Schmeidler [1989], Maccheroni et al. [2006a] and Klibanoff et al. [2005], respectively. Several other papers employ DC, such as Eichberger and Kelsey [1996], Eichberger et al. [2005], Takashi [2005] Sarin and Wakker [1998], Wang [2003] and Hayashi and Miao [2011].

Siniscalchi [2011] drops DC completely and replaces it with Consistent Planning, which specifies that the agent adjusts his actions today in order to restrict his future self's choices, because he recognises that his preferences will change tomorrow. This approach accommodates Ellsberg but information is not valuable, unless the agent can exogenously commit.

[21] Note that we state the condition on $\mathcal{F}$, whereas Hanany and Klibanoff [2009] state it on $\mathcal{F}_{\mathcal{AA}}$.

Bayesian updating of priors is suggested or characterized by Jaffray [1992, 1994], Fagin and Halpern [1991], Wasserman and Kadane [1990], Walley [1991], Epstein and Schneider [2003], Sarin and Wakker [1998], Pires [2002], Siniscalchi [2001], Wang [2003] and Faro and Lefort [2013]. Ghirardato et al. [2008] characterize the Bayesian updating of a set of beliefs which are used to represent the unambiguous preference relation (Ghirardato et al. [2004]), which is incomplete. Gilboa and Schmeidler [1993] analyze maximum likelihod updating, whereas Dempster [1968] and Shafer [1976] suggest the Dempster-Shafer updating rule. Epstein [2006] provides an axiomatic model of non-Bayesian updating.

Another property of Consequentialism (which we call Conditional Preference in this paper and assume it throughout) is that conditional on an event E, the agent only cares about what the act prescribes inside E. Dominiak et al. [2012] show experimentally that subjects are more prone to violate DC than this property.

RSS identify the subjective beliefs generated by a large number of models of ambiguity aversion, based on an idea of Yaari [1969], making our approach very general. These models are the convex Choquet model of Schmeidler [1989], the multiple priors model of Gilboa and Schmeidler [1989], the variational preferences model of Maccheroni et al. [2006a], the multiplier model of Hansen and Sargent [2001], the smooth second-order prior models of Klibanoff et al. [2005] and Nau [2006], the confidence preferences model of Chateauneuf and Faro [2009] and the second-order expected utility model of Ergin and Gul [2009].[22] Ghirardato and Siniscalchi [2018] extend the approach of RSS to non convex preferences in order to characterize betting in terms of disjoint beliefs.

In dynamic models, Strzalecki [2013] examines the relationship between ambiguity attitudes and attitudes about the timing of the resolution of uncertainty. Using the notion of concordancy, Strzalecki and Werner [2011] define (and identify for a large class of models of ambiguity aversion) the conditional probabilities induced by a partition and a set of subjective beliefs. Their focus is different from ours, as they study the properties of measurability and comonotonicity of Pareto efficient allocations, with respect to the aggregate endowment. Hanany et al. [2018] study dynamic games of incomplete information with players that might be ambiguity averse and update using the smooth rule. Ellis [2018] studies the effects of dynamic consistency and consequentialism on ambiguous games.

In a setting with preferences over lotteries, Wakker [1988] shows that if Independence is violated, the value of information is not always positive. Independence is related to DC and, under some conditions (e.g. English auctions), it is equivalent (Karni and Safra [1986]). Grant et al. [2000] provide necessary and sufficient conditions for a weakly dynamically consistent agent to always prefer more information. Our approach differs from theirs in two respects. First, they adopt the definition of "more information" suggested by Blackwell [1951], whereas we adopt the definition of a finer partition.[23] Second, they adopt a different weakening of DC, due to Machina [1989], which requires that an agent conforms to what he would have chosen ex ante only if he were able to commit. Snow [2010] examines the value of information in the special

[22]Note that RSS adopts a domain of preferences over monetary acts, whereas these models allow for more general domains.

[23]The two definitions are closely related, as shown by Green and Stokey [1978].

case where it either reduces or eliminates ambiguity, using the model of Klibanoff et al. [2005]. Li [2015] studies the link between ambiguity attitudes and aversion to receiving information when one is completely uninformed. Galanis [2015] studies the value of information in an environment with unawareness, where DC is violated. Jakobsen [2016] studies a revealed-preference model of information disclosure with SEU preferences. He shows that his primitives are sufficient to uniquely identify tastes and beliefs and determine whether beliefs are updated using Bayes' rule.

8.4 Concluding remarks

It has long been established in the literature that ambiguity sensitive agents must either fail DC or Consequentialism in some respect. Instead of proposing a particular combination of these properties, we explore what is implied by their relaxation, in terms of losing normatively appealing economic results. We first show that even if Consequentialism is relaxed to Status Quo Bias, one part of DC is equivalent to positive value of information in single agent decision problems. Hence, if the modeller is not willing to allow agents who reject free information, DC has to be assumed as well. However, we also show that a weakening of DC is equivalent to a suitable weakening of the value of information, which requires that the agent prefers mixing between a more informative and a less informative decision problem, rather than receiving the less informative decision problem for sure. Moreover, the other part of weak DC is equivalent to Bayesian updating of subjective beliefs.

Weak DC has economic content also in financial markets. First, subjective beliefs can be interpreted as prices for Arrow-Debreu securities which characterize efficient allocations. Their Bayesian updating ensures that an ex ante efficient allocation remains efficient in the interim stage and this is common knowledge, hence precluding speculative trade. Moreover, in risk-sharing environments without aggregate uncertainty, public information is weakly not valuable for all agents, as it destroys opportunities for them to mutually insure.

A Appendix

A.1 Revealed acts

In order to prove our results, we introduce the notion of revealed acts and connect it with preference relation $\succsim^*$. We have interpreted $\pi^u_{E,h}(f)$ as the set of (normalized) Arrow-Debreu prices for which the agent with preferences $\succsim_{E,h}$, endowed with f, would have zero net demand. The "dual" of $\pi^u_{E,h}(f)$ is the set of acts for which f is revealed preferred to them. In particular, if $\mathbb{E}_p f \geq \mathbb{E}_p g$ then act g is affordable given normalized price p and endowment f. If $p \in \pi^u_{E,h}(f)$, then from Axiom 3 we have $f \succsim_{E,h} g$, which means that f is revealed preferred to g. Formally, for act f and event $E \in \mathcal{E}$, let $\mathcal{R}^u_{E,h}(f)$ be the set of acts such that f is revealed preferred to them given preferences $\succsim_{E,h}$,

$$\mathcal{R}^u_{E,h}(f) = \{g \in \mathcal{F} : \mathbb{E}_p f \geq \mathbb{E}_p g \text{ for some } p \in \pi^u_{E,h}(f)\}.$$

If we use beliefs $\pi^w_{E,h}(f)$, instead of $\pi^u_{E,h}(f)$, we get

$$\mathcal{R}^w_{E,h}(f) = \{g \in \mathcal{F} : \mathbb{E}_p f \geq \mathbb{E}_p g \text{ for some } p \in \pi^w_{E,h}(f)\}.$$

The connection between $\mathcal{R}^w_{E,h}, \mathcal{R}^u_{E,h}$ and $\succsim^*_{E,h}$ is given by the following Lemma.

Lemma 8. *For all $f, g, h \in \mathcal{F}$, $f \succsim^*_{E,h} g$ implies $g \in \mathcal{R}^w_{E,h}(f)$ and $g \in \mathcal{R}^u_{E,h}(f)$ implies $f \succsim^*_{E,h} g$.*

Proof. Suppose $f \succsim^*_{E,h} g$, then $f \succsim_{E,h} ag + (1-a)f$ for all $a \in [0,1]$. Suppose $g \notin \mathcal{R}^w_{E,h}(f)$, so that $\mathbb{E}_p f < \mathbb{E}_p g$ for all $p \in \pi^w_{E,h}(f)$. Because $\pi^w_{E,h}(f)$ is a compact and convex set, by definition we have $\epsilon g + (1-\epsilon)f \succ_{E,h} f$ for sufficiently small $\epsilon > 0$, a contradiction. Conversely, suppose $g \in \mathcal{R}^u_{E,h}(f)$, so that $\mathbb{E}_p f \geq \mathbb{E}_p g$ for some $p \in \pi^u_{E,h}(f)$. This implies that for all $a \in [0,1]$, $\mathbb{E}_p f \geq a\mathbb{E}_p g + (1-a)\mathbb{E}_p f$. Axiom 3 and the definition of $\pi^u_{E,h}(f)$ imply $f \succsim_{E,h} ag + (1-a)f$, hence $f \succsim^*_{E,h} g$. □

Proposition 1 in RSS shows that $\pi_{E,h}(f) = \pi^u_{E,h}(f) = \pi^w_{E,h}(f)$ for all strictly positive acts $f \in \mathcal{F}_+$. Hence, $\mathcal{R}^u_{E,h}(f) = \mathcal{R}^w_{E,h}(f) \equiv \mathcal{R}_{E,h}(f)$. Consider the following two axioms, which are equivalent of the two axioms of weak DC, Axioms 14 and 15.

Axiom 21. *(Weak Consistency of Implementation) For all acts $f \in \mathcal{F}_+$, $g \in \mathcal{F}$ and events $E \in \mathcal{E}$, if $g \in \mathcal{R}^u(f)$ and $f =_{E^c} g$ then $f \succsim_{E,f} g$.*

Axiom 22. *(Weak Information is Valuable) For all acts $g \in \mathcal{F}_+$, $f \in \mathcal{F}$ and events $E \in \mathcal{E}$, if $f \succsim_{E,f} g$, $f =_{E^c} g$ and $g \succ f$ then $f \notin \mathcal{R}^w(g)$.*

We therefore have the following Corollary.

Corollary 1. *For all $f \in \mathcal{F}_+, g, h \in \mathcal{F}$, $f \succsim^*_{E,h} g$ if and only if $g \in \mathcal{R}_{E,h}(f)$. Hence, Axiom 14 is equivalent to Axiom 21 and Axiom 15 is equivalent to 22.*

A.2 Proofs

Proof of Lemma 1. Suppose Axioms 11, 13 and let $f \succsim g$ with $f =_{E^c} g$. From Axiom 7, for all $a \in (0,1)$, $af + (1-a)g \succ g$. Axiom 13 implies $af + (1-a)g \succ_{E,g} g$ for all $a \in (0,1)$. Because $\lim_{a \to 1}[af + (1-a)g] = f$, Axiom 2 implies $f \succsim_{E,g} g$. Axiom 11 implies $f \succsim_{E,f} g$.

Conversely, suppose Axioms 10, 12 and let $f \succsim_{E,f} g$ with $f =_{E^c} g$. From Axiom 7, for all $a \in (0,1)$, $af + (1-a)g \succ_{E,f} g$. Axiom 10 implies $af + (1-a)g \succ_{E,g} g$. Axiom 12 implies $af + (1-a)g \succ g$ for all $a \in (0,1)$. Because $\lim_{a \to 1}[af + (1-a)g] = f$, Axiom 2 implies $f \succsim g$.

For the second claim, using Corollary 1 suppose Axioms 11, 15, $g \in \mathcal{R}(f)$, $f =_{E^c} g$ and $g \succ_{E,f} f$. Axiom 11 implies $g \succsim_{E,g} f$. If $f \succ g$ then Axiom 15 and Corollary 1 imply $g \notin \mathcal{R}(f)$, a contradiction. Suppose $g \succsim f$. From Axiom 7 we have $af + (1-a)g \succ f$ for all $a \in (0,1)$. This implies that $\mathbb{E}_p(af + (1-a)g) > \mathbb{E}_p f$ for all $p \in \pi(f)$. But then $\mathbb{E}_p g > \mathbb{E}_p f$ for all $p \in \pi(f)$ and $g \notin \mathcal{R}(f)$, a contradiction.

Conversely, suppose Axioms 10, 14, $f \succsim_{E,f} g$, $f =_{E^c} g$ and $g \succ f$. From Axiom 7, we have that $af + (1-a)g \succ_{E,f} g$ for all $a \in (0,1)$. Axiom 10 implies $af + (1-a)g \succ_{E,g} g$.

Axiom 14 and Corollary 1 imply that $af+(1-a)g \notin \mathcal{R}(g)$, hence $\mathbb{E}_p(af+(1-a)g) > \mathbb{E}_p g$ for all $p \in \pi(g)$. But this implies that $\mathbb{E}_p f > \mathbb{E}_p g$ for all $p \in \pi(g)$, hence $f \notin \mathcal{R}(g)$ and $g \not\succsim^* f$.

□

Proof of Lemma 2. Suppose that for some $p \in \pi^u_{E,h}(f)$ we have $p(F) = 0$. Take any g such that $g =_{F^c} f$. Because $\mathbb{E}_p f = \mathbb{E}_p g$, we have $f \succsim_{E,h} g$, implying that F is weakly $\succsim_{E,h}$-null. Conversely, suppose that for all acts f, $p \in \pi^w_{E,h}(f)$ implies $p(F) > 0$. Suppose there exists act f such that for all g with $f =_{F^c} g$, $f \succsim_{E,h} g$. Let $k > 0$ and define act g such that $g(s) = f(s) + k$ if $s \in F$ and $g(s) = f(s)$ otherwise. Then, for all $p \in \pi^w_{E,h}(f)$, $\mathbb{E}_p g > \mathbb{E}_p f$. From the definition of $\pi^w_{E,h}(f)$, there exists small enough $\epsilon > 0$, such that $g' = \epsilon g + (1-\epsilon) f$ and $g' \succ f$. Because $f =_{F^c} g'$, we have a contradiction. □

Proof of Proposition 1. Recall that convex preferences satisfy Axioms 1 through 5. Suppose that for any event $E \in \mathcal{E}$ and all acts $f, g \in \mathcal{F}$, $f \succsim_{E,f} g$ and $f =_{E^c} g$ implies $f \succsim g$. Consider decision problems $\mathcal{D}_1 = \{\Pi_1, \mathcal{A}\}$ and $\mathcal{D}_2 = \{\Pi_2, \mathcal{A}\}$, where $\Pi_1, \Pi_2 \subseteq \mathcal{E}$ are partitions of S and Π_1 is finer than Π_2. Let act $f \in \mathcal{F}_{\mathcal{D}_1}$ be optimal for $\mathcal{D}_1$ and act $g \in \mathcal{F}_{\mathcal{D}_2}$ be optimal for $\mathcal{D}_2$. Since Π_2 is coarser than Π_1, $\mathcal{F}_{\mathcal{D}_2} \subseteq \mathcal{F}_{\mathcal{D}_1}$ and $g \in \mathcal{F}_{\mathcal{D}_1}$. This means that, for all $E \in \Pi_1$, $f \succsim_{E,f} g$.

Enumerate the partition cells of $\Pi_1 = \{E_1, \ldots, E_n\}$. If $n = 1$ then $\Pi_1 = \{S\}$ is the uninformative partition and the result is immediate, so suppose that $n \geq 2$. For cell $1 \leq k \leq n$ define act h_k as follows. Let $h_k(s) = f(s)$ if $s \in E_j$, where $1 \leq j \leq k$, and $h_k(s) = g(s)$ otherwise. Note that $h_n = f$ and let $h_0 = g$. From Axiom 5 we have that, for each $1 \leq k \leq n$, $f \succsim_{E_k,f} g$ implies $h_k \succsim_{E_k,f} h_{k-1}$. From Axiom 11 we have $h_k \succsim_{E_k,h_k} h_{k-1}$. Applying Axiom 13 we have $h_k \succsim h_{k-1}$, for each $1 \leq k \leq n$. Axiom 1 implies that $f \succsim g$.

Conversely, suppose that Axiom 13 is false, so that for some event $E \in \mathcal{E}$ and acts $f, g \in \mathcal{F}$, we have $f \succsim_{E,f} g$ and $f =_{E^c} g$ but $g \succ f$. Consider partitions $\Pi_1 = \{E, E^c\}$ and $\Pi_2 = \{S\}$. Let $\mathcal{A} = \{f, g\}$. Then, g is optimal for decision problem $\mathcal{D}_2 = \{\Pi_2, \mathcal{A}\}$. Because $f =_{E^c} g$, Axiom 5 implies that $f \succsim_{E^c,f} g$. Since $f \succsim_{E,f} g$, we have that f is optimal for decision problem $\mathcal{D}_1 = \{\Pi_1, \mathcal{A}\}$. Because $g \succ f$, $\mathcal{D}_1$ is not more valuable than $\mathcal{D}_2$, hence information is not valuable.

For the second claim, suppose Axiom 15, which from Corollary 1 implies Axiom 22. Consider the same decision problems as in the first paragraph of this proof (where now $\mathcal{A} \subseteq \mathcal{F}_+$) and let act $f \in \mathcal{F}_{\mathcal{D}_1}$ be optimal for $\mathcal{D}_1$ and act $g \in \mathcal{F}_{\mathcal{D}_2}$ be optimal for $\mathcal{D}_2$. Since Π_2 is coarser than Π_1, $\mathcal{F}_{\mathcal{D}_2} \subseteq \mathcal{F}_{\mathcal{D}_1}$ and $g \in \mathcal{F}_{\mathcal{D}_1}$. This means that, for all $E \in \Pi_1$, $f \succsim_{E,f} g$. If $f \succsim g$ then for $a = 1$ we have $af + (1-a)g \succsim g$ and information is weakly valuable.

Suppose $g \succ f$. For each $E \in \Pi_1$, Axiom 5 and $f \succsim_{E,f} g$ imply $fEg \succsim_{E,f} g$. Axiom 11 implies that $fEg \succsim_{E,fEg} g$. From Axiom 22, either $fEg \succsim g$ or $g \succ fEg$ and $fEg \notin \mathcal{R}(g)$, which implies that $\mathbb{E}_p fEg \geq \mathbb{E}_p g$ for all $p \in \pi(g)$. Because Π_1 is a partition of S, we have that $\mathbb{E}_p f \geq \mathbb{E}_p g$ for all $p \in \pi(g)$.

If it is not the case that $fEg \sim g$ for some $E \in \Pi_1$, then either $fEg \succ g$ or $fEg \notin \mathcal{R}(g)$, both implying $\mathbb{E}_p fEg > \mathbb{E}_p g$ for all $p \in \pi(g)$. Hence, if it is not the case that $fEg \sim g$ for some $E \in \Pi_1$, we have that $\mathbb{E}_p f > \mathbb{E}_p g$ for all $p \in \pi(g)$. By the definition of $\pi^w(g)$, there exists $a \in (0,1)$ such that $af + (1-a)g \succ g$, hence information is weakly valuable.

Suppose now that for all $E \in \Pi_1$, $fEg \sim g \succ f$. Note that $\sum_{E \in \Pi_1} \frac{1}{k} fEg = \frac{1}{k} f + \frac{k-1}{k} g$, where k is the number of Π_1's partition cells. Because $fEg \succsim g$ for each $E \in \Pi_1$, Axiom 4 implies $\frac{1}{k} f + \frac{k-1}{k} g \succsim g$. By setting $a = \frac{1}{k}$, information is weakly valuable.

Conversely, suppose that information is weakly valuable and Axiom 8 is satisfied. Using Corollary 1, we only need to show that Axiom 22 is satisfied. Suppose that for some event $E \in \mathcal{E}$ and acts $f \in \mathcal{F}$, $g \in \mathcal{F}_+$, we have $f \succsim_{E,f} g$, $f =_{E^c} g$, $g \succ f$ but $f \in \mathcal{R}(g)$. Suppose f is not strictly positive, so that for some set $A \subseteq E$ (because g is strictly positive and $f =_{E^c} g$), we have $f(s) = 0$ if and only if $s \in A$. If $\mathbb{E}_p g > \mathbb{E}_p f$ for some $p \in \pi(g)$, Axiom 2 implies that we can find strictly positive act $f' \in \mathcal{F}_+$, by infinitesimally increasing the payoff for all states in A, such that $f' \succsim_{E,f} g$, $f' =_{E^c} g$, $g \succ f'$ but $f' \in \mathcal{R}(g)$. If $\mathbb{E}_p g \leq \mathbb{E}_p f$ for all $p \in \pi(g)$, $f \in \mathcal{R}(g)$ implies $\mathbb{E}_p g = \mathbb{E}_p f$ for some $p \in \pi(g)$. By taking a convex combination of f and g, with large weight on f, Axioms 2, 4 and the definition of π imply that we can find a strictly positive f' such that $f' \succsim_{E,f} g$, $f' =_{E^c} g$, $g \succ f'$ but $f' \in \mathcal{R}(g)$.

From Axiom 11 we have $f' \succsim_{E,f'} g$, so wlog we set $f = f'$. Because $f \in \mathcal{R}(g)$, we have that $\mathbb{E}_p g \geq \mathbb{E}_p f$ for some $p \in \pi(g)$. Construct the same decision problems, $\mathcal{D}_1$ and $\mathcal{D}_2$, as in the third paragraph of the proof. Because information is weakly valuable and Π_1 is finer than Π_2, we have that $\mathcal{D}_1$ is weakly more valuable than $\mathcal{D}_2$. This implies that $af + (1-a)g \succsim g$ for some $a \in (0,1]$.

From Axiom 4 we have that, for all $b \in [0,1]$, $b(af + (1-a)g) + (1-b)g = abf + (1-ab)g \succsim g$, hence, for all $c \in (0,a]$, $cf + (1-c)g \succsim g$. If $cf + (1-c)g \succ g$ for some c, then $\mathbb{E}_p(cf + (1-c)g) > \mathbb{E}_p g$ for all $p \in \pi(g)$, contradicting $\mathbb{E}_p g \geq \mathbb{E}_p f$ for some $p \in \pi(g)$. We therefore have $cf + (1-c)g \sim g$ for all $c \in (0,a]$. Because $g \succ f$, Axiom 8 is contradicted.

□

Proof of Proposition 2. Suppose subjective beliefs are updated using Bayes' rule. Fix event $E \in \mathcal{E}$ and $f \in \mathcal{F}_+$. Suppose $f \succsim^* g$ and $f =_{E^c} g$. Using Corollary 1, we have $g \in \mathcal{R}(f)$. Then, $\mathbb{E}_p f \geq \mathbb{E}_p g$ for some $p \in \pi(f)$. Axiom 9 and Lemma 2 imply that $p(E) > 0$. Because subjective beliefs are updated using Bayes' rule and $f =_{E^c} g$, we have $\mathbb{E}_{p_E} f \geq \mathbb{E}_{p_E} g$ and $p_E \in \pi_{E,f}(f)$. Axioms 1 and 3 imply $f \succsim_{E,f} g$.

Conversely, suppose Axiom 14 and that there exist $f \in \mathcal{F}_+$, $p \in \pi(f)$ with $p(E) > 0$ and $p_E \notin \pi_{E,f}(f)$. Then, there exists act g such that $g \succ_{E,f} f$ and $\mathbb{E}_{p_E} g = \mathbb{E}_{p_E} f$. From Axiom 5 we have $gEf \succ_{E,f} f$. Because $\mathbb{E}_{p_E}(gEf) = \mathbb{E}_{p_E} f$ and $gEf =_{E^c} f$, we have that $\mathbb{E}_p(gEf) = \mathbb{E}_p f$, hence $gEf \in \mathcal{R}(f)$. From Corollary 1 Axiom 14 implies Axiom 21, hence $f \succsim_{E,f} gEf$, a contradiction.

□

The following is used in the proof of Lemmas 3 and 4.

Lemma 9. *Suppose that $\{\succsim_{E,h}\}_{E \in \mathcal{E}, h \in \mathcal{F}}$ are convex variational or confidence preferences and satisfy Axiom 9, where u is concave, increasing and differentiable. Suppose that for all f and all $r \in M(f)$, if $r(E) > 0$ then $r_E \in M_{E,f}(f)$. Then, Bayesian updating of subjective beliefs and Axiom 14 are satisfied.*

Proof. We show that there is Bayesian updating of subjective beliefs and then invoke Proposition 2, applied to strictly positive acts. Let $U : \mathbb{R}^S_+ \to \mathbb{R}^S$ be the function $U(f) = (u(f(1), \ldots, u(f(S))))$, giving ex post utilities in each state. For any $f \in \mathbb{R}^S_{++}$, let $DU(f)$ be the $S \times S$ diagonal matrix with diagonal given by the vector of ex post marginal utilities $(u'(f(1), \ldots, u'(f(S))))$. From Propositions 3 (for variational preferences) and 4 (for confidence preferences) in RSS, we have that $\pi(f) = \left\{ \frac{q}{\|q\|} : q = pDU(f) \text{ for some } p \in M(f) \right\}$.

Suppose $q \in \pi(f)$ and $q(E) > 0$ where, without loss of generality, $\|q\| = 1$ and $q = rDU(f)$ for $r \in M(f)$. This means that $q(s) = r(s)u'(f(s))$ for each $s \in S$. Because $r_E \in M_{E,f}(f)$, we have $q'_E \in \pi_{E,f}(f)$ where, for each $s \in E$, $q'_E(s) = \frac{r(s)u'(f(s))}{r(E)(\sum_{s' \in E} \frac{r(s')}{r(E)} u'(f(s')))} = \frac{r(s)u'(f(s))}{(\sum_{s' \in E} r(s')u'(f(s')))} = q_E(s)$, which implies $q'_E = q_E \in \pi_{E,f}(f)$.

□

If u is strictly increasing, then Axiom 9 is equivalent to requiring that the ambiguity index is finite only for beliefs that assign positive probability to all events in $\mathcal{E}$.

Lemma 10. *If u is strictly increasing, then Axiom 9 is equivalent to requiring that for all $E \in \mathcal{E}$ and all $p \in \Delta E$, $p(E) = 0$ implies $c_{E,h}(p) = +\infty$.*

Proof of Lemma 10. Fix $F, E \in \mathcal{E}$ with $F \subseteq E$ and $h \in \mathcal{F}$. We then need to show that the following are equivalent.

- $c_{E,h}(p) \neq +\infty$ for some $p \in \Delta E$ such that $p(F) = 0$,
- F is weakly $\succsim_{E,h}$-null.

Let $P_F \subseteq \Delta E$ be such that $p \in P_F$ implies $c_{E,h}(p) \neq +\infty$ and $p(F) = 0$. Suppose $P_F \neq \emptyset$. Let $f_{F,k}$ be the act such that $f_{F,k}(s) = k > 0$ if $s \in F$ and $f_{F,k}(s) = 0$ otherwise. For any act g and $p \in P_F$, note that $\int u(g)dp + c(p) = \int u(g + f_{F,k})dp + c(p)$ for all $k > 0$. This implies that for big enough k^*, $M_{E,h}(g + f_{F,k^*}) \subseteq P_F$. Moreover, for all $k > 0$, $g + (f_{F,k^*} + f_{F,k}) \sim_{E,h} g + f_{F,k^*}$. Suppose there exists g' with $g + f_{F,k^*} =_{F^c} g'$, such that $g' \succ_{E,h} g + f_{F,k^*}$. Set $k' = \max_{s \in F} g'(s)$. From Axiom 3, $g + (f_{F,k^*} + f_{F,k'}) \succsim_{E,h} g'$. But this implies $g + (f_{F,k^*} + f_{F,k'}) \succ_{E,h} g + f_{F,k^*}$, a contradiction. Therefore, F is weakly $\succsim_{E,h}$-null.

Conversely, suppose F is weakly $\succsim_{E,h}$-null. Then, there exists g such that, for all g' with $g =_{F^c} g'$, we have $g \succsim_{E,h} g'$. Suppose that for each $p \in \Delta E$, $p(F) = 0$ implies $c_{E,h}(p) = +\infty$. Then, $p \in M_{E,h}(f)$ implies $p(F) > 0$, for all $f \in \mathcal{F}$. Because $\int u(g)dp + c(p) < \int u(g + f_{F,k})dp + c(p)$ for all p with $p(F) > 0$ and some $k > 0$, we have that $g + f_{F,k} \succ_{E,h} g$, a contradiction.

□

Proof of Lemma 3. To show convexity of $c_{E,f}$, we need to establish that $c_{E,f}(ap + (1 - a)p') \leq ac_{E,f}(p) + (1 - a)c_{E,f}c(p')$ for all $p, p' \in \Delta E$. Using (1), we need to show that $\int u(f)dn_{E,f}(ap + (1 - a)p') + c(n_{E,f}(ap + (1 - a)p')) \leq a(\int u(f)dn_{E,f}(p) + c(n_{E,f}(p))) + (1 - a)(\int u(f)dn_{E,f}(p') + c(n_{E,f}(p')))$.

For the first rule, if $c_{E,f}(p), c_{E,f}(p') \neq +\infty$, then there exist $q, q' \in \Delta S$ such that $p \otimes^E q, p' \otimes^E q' \in D^f$ and $c(p \otimes^E q), c(p' \otimes^E q') \neq +\infty$. From Lemma 1 in Araujo et al. [2016] there exists b such that the Bayesian update of $bp \otimes^E q + (1-b)p' \otimes^E q'$ is $ap + (1-a)p'$. From the convexity of c, $c(bp \otimes^E q + (1-b)p' \otimes^E q') \neq +\infty$. From the convexity of D^f, $bp \otimes^E q + (1-b)p' \otimes^E q' \in D^f$ and from the definition of the updating rule, $ap+(1-a)p' \in D^f_E$ and $c_{E,f}(ap+(1-a)p') \neq +\infty$. But this means that $n_{E,f}(ap + (1-a)p') = n_{E,f}(p) = n_{E,f}(p')$, so the inequality is true. If $c_{E,f}(p) = +\infty$ or $c_{E,f}(p') + \infty$, then the inequality is trivially satisfied. For the second rule, again from Lemma 1 in Araujo et al. [2016] and because $p \otimes^E r(E) = p' \otimes^E r(E)$, we have that $n_{E,f}(ap + (1-a)p') = (ap + (1-a)p') \otimes^E r = ap \otimes^E r + (1-a)p' \otimes^E r$, so the inequality is true.

To show lower semi continuity, note that for all p with $c_{E,f}(p) \neq +\infty$, for the first updating rule the first two and the last term are constant, whereas the third term is a linear function of p. For the second rule, $c_{E,f}(p)$ is the sum of continuous functions of p.

To show Bayesian updating of subjective beliefs, suppose that $q \in M(f)$ and $q(E) > 0$. For both updating rules, $n_{E,f}(q_E) = r$ for some $r \in M(f)$ with $r(E) > 0$ and $r_E = q_E$. Let $p_E \in \Delta E$ and denote $n_{E,f}(p_E)$ by p. We then have $\int u(f)dr + c(r) \leq \int u(f)dp+c(p)$. By substituting from equation (1), we have that $\int u(f)dr_E+c_{E,f}(r_E)+k_{E,f} \leq \int u(f)dp_E+c_{E,f}(p_E)+k_{E,f}$, hence $\int u(f)dq_E+c_{E,f}(q_E) \leq \int u(f)dp_E+c_{E,f}(p_E)$ and $q_E \in M_{E,f}(f)$. From Lemma 9, Bayesian updating of subjective beliefs and Axiom 14 are satisfied.

To show that $D^f = D$ for all $f \in \mathcal{F}$ and $p \in D$ for all p with $c(p) \neq +\infty$, implies Axiom 12, suppose $f \succsim g$ and $f =_{E^c} g$. This implies that $\int u(f)dp + c(p) \geq \int u(g)dq + c(q)$, where $p \in M(f)$ and $q \in M(g)$. Suppose that $g \succ_{E,f} f$. Substituting from equation (1), we have that $U_{E,f}(f) = \int u(f)dr + c(r) - k_{E,f}$ for some fixed $r \in M(f)$ and $U_{E,f}(g) = \int u(f)dr + c(r) - k_{E,f} + \min_{t \in D_E} \{\int u(g)dt - \int u(f)dt\}$. Because $g \succ_{E,f} f$, we have that $\int u(g)dt > \int u(f)dt$, for all $t \in D_E$. Because $p(E) > 0$ for all p with $c(p) \neq +\infty$ and $q \in D$, we have $q_E \in D_E$ and $\int u(g)dq_E > \int u(f)dq_E$. From $f =_{E^c} g$, $\int u(g)dq_E > \int u(f)dq_E$ implies that $\int u(g)dq+c(q) > \int u(f)dq+c(q) \geq \int u(f)dp+c(p)$. Because $p \in M(f)$ and $q \in M(g)$ we have $g \succ f$, a contradiction.

To show that $D^f = M(f)$, for all $f \in \mathcal{F}$, implies Axiom 13, suppose that $f \succsim_{E,f} g$ and $f =_{E^c} g$. From the calculations of the previous paragraph, we have that $U_{E,f}(f) \geq U_{E,f}(g)$ if and only if $\min_{t \in D^f_E} \{\int u(g)dt - \int u(f)dt\} \leq 0$, which implies that $\int u(g)dt_E \leq \int u(f)dt_E$ for some $t_E \in D^f_E$. Since $D^f = M(f)$, there exists $t \in M(f)$ whose Bayesian update is t_E. Because $f =_{E^c} g$, we have that $U(f) = \int u(f)dt+c(t) \geq \int u(g)dt+c(t) \geq \int u(g)dq + c(q) = U(g)$, where $q \in M(g)$. This implies that $f \succsim g$.

To show that these updating rules do not imply Axiom 12 and therefore fail DC, consider the example of Section 1.1 with $S = \{s_1, s_2, s_3\}$ and $E = \{s_1, s_2\}$. The agent has MEU ex ante preferences with $u(x) = x$ and C being the convex hull of the following three measures: $p_1 = (0.24, 0.33, 0.43), p_2 = (0.34, 0.2, 0.46)$ and $p_3 = (0.21, 0.68, 0.11)$. In this example each $M(f_i)$ is a singleton, so that $M(f_1) = M(f_4) = \{p_3\}$ and $M(f_2) = M(f_3) = \{p_2\}$. The subjective beliefs consist of the normalized vectors $pDU(f_i)$, where $p \in M(f_i)$ and $DU(f_i)$ is a diagonal matrix, with a diagonal consisting of the ex post marginal utilities, in this case 1's. We then have that $\pi(f_1) = \pi(f_4) = \{p_3\}$ and $\pi(f_2) = \pi(f_3) = \{p_2\}$. Because u is linear, $f_1 \succ f_2$ and $M(f_1) = p_3$, where $f_1 = (1, 0, 0)$

and $f_2 = (0,1,0)$.

For the first updating rule, let D^{f_1} be the convex hull of p_1 and p_3, hence $M(f_1) \subseteq D^{f_1}$ and $D^{f_1}_E$ is the convex hull of p_{1E} and p_{3E}. We then have that $U_{E,f_1}(f_1) = \min_{p \in D^{f_1}_E} \{\int u(f_1)dp + c_{E,f_1}(p)\} = \int u(f)dp_3 + c(p_3) - k_{E,f_1}$ and $U_{E,f_1}(f_2) = \min_{p \in D^{f_1}_E} \{\int u(f_2)dp + c_{E,f_1}(p)\} = \int u(f)dp_3 + c(p_3) - k_{E,f_1} + \min_{p \in D^{f_1}_E} \{\int u(f_2)dp - \int u(f_1)dp\}$. The last term simplifies to $\min_{p \in D^{f_1}_E} \{u(1)p(s_2) - u(1)p(s_1)\}$. Because $p(s_2) > p(s_1)$ for all $p \in D^{f_1}_E$, we have $f_2 \succ_{E,f_1} f_1$.

To show that Axiom 13 is violated, it is enough to show that $f_2 \succ_{E,f_2} f_1$. Let D^{f_2} be the convex hull of p_2 and p_3. Noting that $M(f_2) = \{p_2\}$, we have $U_{E,f_2}(f_2) = \min_{p \in D^{f_2}_E} \{\int u(f_2)dp + c_{E,f_2}(p)\} = \int u(f)dp_2 + c(p_2) - k_{E,f_2}$ and $U_{E,f_2}(f_1) = \min_{p \in D^{f_2}_E} \{\int u(f_1)dp + c_{E,f_2}(p)\} = \int u(f)dp_2 + c(p_2) - k_{E,f_2} + \min_{p \in D^{f_2}_E} \{\int u(f_1)dp - \int u(f_2)dp\}$. The last term simplifies to $\min_{p \in D^{f_2}_E} \{u(1)p(s_1) - u(1)p(s_2)\} = \frac{u(1)}{p_3(E)}(p_3(s_1) - u(1)p_3(s_2)) < 0$, hence $f_2 \succ_{E,f_2} f_1$ but $f_1 \succ f_2$.

For the second updating rule we set $n_{E,f_1}(p) = p \otimes^E p_3$. Let C_E be the Bayesian updates of all elements of C, given E, consisting of the convex hull of p_{1E}, p_{2E} and p_{3E}. Note that if $p \notin C_E$, then $c_{E,f_1}(p) = +\infty$. Applying (1) and excluding constant k_{E,f_1} we have $U_{E,f_1}(f_1) = \min_{p \in C_E} \{p(s_1) + p(s_1)p_3(E) - p(s_1)\} = p_3(s_1) = 0.21$ and $U_{E,f_1}(f_2) = \min_{p \in C_E} \{p(s_2) + p(s_1)p_3(E) - p(s_1)\} = \min_{p \in C_E} \{1 - p(s_1)(1 + p_3(s_3))\} = 1 - p_{2E}(s_1)(1 + p_3(s_3)) = 0.3011$. Hence, $f_2 \succ_{E,f_1} f_1$.

□

Proof of Lemma 4. Note that $\phi_{E,h}(p)$ is either 0 or equal to $\frac{\phi(r)}{k_{E,h}} \frac{\mathbb{E}_p u(h)}{\mathbb{E}_r u(h)}$, where $\frac{\phi(r)}{k_{E,h} \mathbb{E}_r u(h)}$ is constant. Hence, quasiconcavity and upper semicontinuouty are satisfied. Proposition 4 in RSS shows that $\pi(h) = \left\{ \frac{q}{\|q\|} : q = pDU(h) \text{ for some } p \in M(h) \right\}$.

To show Bayesian updating of subjective beliefs, suppose that $q \in M(h)$ and $q(E) > 0$. By construction, $q_E \in D^h_E$. Moreover, for all $p \in D^h_E$, $\frac{\mathbb{E}_p u(h)}{\phi_{E,h}(p)} = k_{E,h} \frac{\mathbb{E}_r u(h)}{\phi(r)}$ for some fixed $r \in M(h)$ with $r(E) > 0$, so it is independent of p. Hence, $q_E \in M_{E,h}(h) = L_{a_{E,h}} = D^h_E$. From Lemma 9, Bayesian updating of subjective beliefs and Axiom 14 are satisfied.

Suppose now that $D^f = D = L_a$, for all $f \in \mathcal{F}$, and for all $p \in D$ we have $p(E) > 0$, $f \succsim g$ and $f =_{E^c} g$. This implies that $\frac{\mathbb{E}_p u(f)}{\phi(p)} \geq \frac{\mathbb{E}_q u(g)}{\phi(q)}$, where $p \in M(f)$ and $q \in M(g)$. Suppose that $g \succ_{E,f} f$. Substituting from (2) we have that $U_{E,f}(f) = \frac{k_{E,f}}{\phi(r)} \mathbb{E}_r u(f)$ for some $r \in M(f)$ and $U_{E,f}(g) = \frac{k_{E,f}}{\phi(r)} \mathbb{E}_r u(f) \min_{t \in D_E} \frac{\mathbb{E}_t u(g)}{\mathbb{E}_t u(f)}$. Because $g \succ_{E,f} f$, we have that $\mathbb{E}_t u(g) > \mathbb{E}_t u(f)$, for all $t \in D_E$. In particular, $\mathbb{E}_{q_E} u(g) > \mathbb{E}_{q_E} u(f)$ and, since $f =_{E^c} g$, we have $\frac{\mathbb{E}_q u(g)}{\phi(q)} > \frac{\mathbb{E}_q u(f)}{\phi(q)} \geq \frac{\mathbb{E}_p u(f)}{\phi(p)}$, a contradiction.

To show that $D^f = M(f)$, for all $f \in \mathcal{F}$, implies Axiom 13, suppose that $f \succsim_{E,f} g$ and $f =_{E^c} g$. From the calculations of the previous paragraph, we have that $U_{E,f}(f) \geq$

$U_{E,f}(g)$ if and only if $\min_{t \in D_E^f} \frac{\mathbb{E}_t u(g)}{\mathbb{E}_t u(f)} \leq 1$, which implies that $\mathbb{E}_{t_E} u(g) \leq \mathbb{E}_{t_E} u(f)$ for some $t_E \in D_E^f$. Since $D^f = M(f)$, there exists $t \in M(f)$ whose Bayesian update is t_E. Because $f =_{E^c} g$, we have that $U(f) = \frac{1}{\phi(t)} \mathbb{E}_t u(f) \geq \frac{1}{\phi(t)} \mathbb{E}_t u(g) \geq \frac{1}{\phi(q)} \mathbb{E}_q u(g) = U(g)$, where $q \in M(g)$. This implies that $f \succsim g$.

We now show that if D^h is a strict subset of L_a, Axiom 12 and therefore DC may be violated. Consider the example of Section 1.1, with $S = \{s_1, s_2, s_3\}$ and $E = \{s_1, s_2\}$. The agent has MEU ex ante preferences with $u(x) = x$ and L_a being the convex hull of the following three measures: $p_1 = (0.24, 0.33, 0.43), p_2 = (0.34, 0.2, 0.46)$ and $p_3 = (0.21, 0.68, 0.11)$. Note that MEU is a special case of confidence preferences, where $\phi(p) = 1$ if $p \in L_a$ and 0 otherwise, whereas $a = 1/2$. As we showed there, $f_1 \succ f_2$ and $M(f_1) = p_3$, where $f_1 = (1, 0, 0)$ and $f_2 = (0, 1, 0)$.

Let $D^{f_1} \subsetneq L_a$ be the convex hull of p_1 and p_3, hence $M(f_1) \subseteq D$ and $D_E^{f_1}$ is the convex hull of p_{1E} and p_{3E}. We then have that that if $p \in D_E$, $\phi_{E,f_1}(p) = \frac{\phi(p_3)}{k_{E,f_1}} \frac{\mathbb{E}_p u(f_1)}{\mathbb{E}_{p_3} u(f_1)}$, otherwise $\phi_{E,f_1}(p) = 0$. By construction, $L_{a_{E,f_1}} = D_E$.

We have $U_{E,f_1}(f_1) = \min_{p \in D_E} \frac{1}{\phi_{E,f_1}(p)} \mathbb{E}_p u(f_1) = \min_{p \in D_E} \frac{k_{E,f_1}}{\phi(p_3)} \mathbb{E}_{p_3} u(f_1) = \frac{k_{E,f_1}}{\phi(p_3)} \mathbb{E}_{p_3} u(f_1)$ and $U_{E,f_1}(f_2) = \min_{p \in D_E} \frac{1}{\phi_{E,f_1}(p)} \mathbb{E}_p u(f_2) = \min_{p \in D_E} \frac{k_{E,f_1}}{\phi(p_3)} \mathbb{E}_{p_3} u(f_1) \frac{\mathbb{E}_p u(f_2)}{\mathbb{E}_p u(f_1)} = \frac{k_{E,f_1}}{\phi(p_3)} \mathbb{E}_{p_3} u(f_1) \min_{p \in D_E} \frac{p(s_2)}{p(s_1)}$. Because $p(s_2) > p(s_1)$ for all $p \in D_E$, we have $f_2 \succ_{E,f_1} f_1$.

□

Proof of Lemma 5. Suppose Axioms 18, 19, $f \succsim^* g$, $f =_{E^c} g$ and $f, g \in \mathbb{R}_+^S$. This implies that $f \succsim af + (1-a)g$ for all $a \in [0,1]$, where $af + (1-a)g$ denotes an act from S to $\mathbb{R}_+$. Axioms 1, 19 imply that $f \succsim a \otimes f + (1-a) \otimes g$ for all $a \in [0,1]$. Let $B = \{h \in \mathcal{A} : h = a \otimes f + (1-a) \otimes g, a \in [0,1]\}$ be a convex feasible set. We then have that f is optimal in B. Axiom 18 implies $f \succsim_{E,f} g$, thus Axiom 14 is satisfied. □

Proof of Lemma 6. Suppose Axiom 18, $f \succsim^*_{\mathcal{AA}} g$ and $f =_{E^c} g$. This implies that $f \succsim a \otimes f + (1-a) \otimes g$ for all $a \in [0,1]$. Let $B = \{h \in \mathcal{A} : h = a \otimes f + (1-a) \otimes g, a \in [0,1]\}$ be a convex budget set. We then have that f is optimal in B. Applying Axiom 18, we have $f \succsim_{E,f} g$ and Axiom 20 is satisfied.

Conversely, suppose Axiom 20 and take $(\succsim, E, f, B) \in \mathcal{D}$ with $g \in B$ and $f =_{E^c} g$. Because $f, g \in B$ and B is convex, $a \otimes f + (1-a) \otimes g \in B$ for all $a \in [0,1]$. The optimality of f implies that $f \succsim a \otimes f + (1-a) \otimes g$. Hence, $f \succsim^*_{\mathcal{AA}} g$ and Axiom 20 implies $f \succsim_{E,f} g$, so that Axiom 18 is satisfied. □

Proof of Lemma 7. Let $U : \mathbb{R}_+^S \to \mathbb{R}^S$ be the function $U(f) = (u(f(1), \ldots, u(f(S))))$, giving ex post utilities in each state. For any $f \in \mathbb{R}_+^S$, let $DU(f)$ be the $S \times S$ diagonal matrix with diagonal given by the vector of ex post marginal utilities $(u'(f(1), \ldots, u'(f(S))))$. Let $DU(f)(s)$ be the s-th element of the diagonal. From Proposition 5 in RSS, the set of subjective beliefs is a singleton, given by $\pi(f) = \frac{\mathbb{E}_\mu[\phi'(\mathbb{E}_p u \circ f) p DU(f)]}{\|\mathbb{E}_\mu[\phi'(\mathbb{E}_p u \circ f) p DU(f)]\|}$, where $\|\mathbb{E}_\mu[\phi'(\mathbb{E}_p u \circ f) p DU(f)]\| = \sum_{s' \in S} \mathbb{E}_\mu[\phi'(\mathbb{E}_p u \circ f) p(s')] DU(f)(s')$.

Given event E and act f, the subjective belief becomes $\pi_E(f) = \frac{\mathbb{E}_{\mu_{E,f}}[\phi'(\mathbb{E}_{p_E} u \circ f) p_E DU(f)]}{\|\mathbb{E}_{\mu_{E,f}}[\phi'(\mathbb{E}_{p_E} u \circ f) p_E DU(f)]\|}$.

Let $q = \pi(f)$, with $q(E) > 0$ and $q' = \pi_{E,f}(f)$. We need to show that for each $s \in E$, $\frac{q(s)}{q(E)} = q'(s)$, or that

$$\frac{\mathbb{E}_{\mu_{E,f}}[\phi'(\mathbb{E}_{p_E} u \circ f) p_E(s)] DU(f)(s)}{\sum\limits_{s' \in E} \mathbb{E}_{\mu_{E,f}}[\phi'(\mathbb{E}_{p_E} u \circ f) p_E(s')] DU(f)(s')} = \frac{\mathbb{E}_{\mu}[\phi'(\mathbb{E}_{p} u \circ f) p(s)] DU(f)(s)}{\sum\limits_{s' \in E} \mathbb{E}_{\mu}[\phi'(\mathbb{E}_{p} u \circ f) p(s')] DU(f)(s')}. \quad (5)$$

Set $k(s) = \frac{\mathbb{E}_{\mu_{E,f}}[\phi'(\mathbb{E}_{p_E} u \circ f) p_E(s)]}{\mathbb{E}_{\mu_{E,f}}[\phi'(\mathbb{E}_{p_E} u \circ f)]} = \frac{\mathbb{E}_{\mu}[\phi'(\mathbb{E}_{p} u \circ f) p(s)]}{\mathbb{E}_{\mu}[\phi'(\mathbb{E}_{p} u \circ f)] p(E)}$ for all $s \in E$. By substituting the equalities $\mathbb{E}_{\mu_{E,f}}[\phi'(\mathbb{E}_{p_E} u \circ f) p_E(s)] = k(s)\mathbb{E}_{\mu_{E,f}}[\phi'(\mathbb{E}_{p_E} u \circ f)]$ and $\mathbb{E}_{\mu}[\phi'(\mathbb{E}_{p} u \circ f) p(s)] = k(s)\mathbb{E}_{\mu}[\phi'(\mathbb{E}_{p} u \circ f)] p(E)$ in (5), Bayesian updating of subjective beliefs is satisfied. $\square$

Proof of Proposition 3. Fix $j \in I$. From Axioms 2 and 6 we have that $H = G \equiv \{s \in S : g_0^i \succ^i_{\Pi^i(s), f^i} f^i \text{ for all } i \in I\}$ for some feasible allocation g_0, as we can always distribute a small enough portion of j's allocation to everyone else. We subsequently show that G cannot be common knowledge at any s, denoting g_0 by g.

Because f is an ex ante efficient allocation, there does not exist a feasible allocation h such that $h^i \succ^i f^i$ for all $i \in I$. Suppose that there exists feasible allocation g such that G is common knowledge at $s \in S$. Let F be a self evident event such that $s \in F \subseteq G$. Note that each Π^i partitions F. Then, we have that for each $i \in I$, for each $s' \in F$, $g^i \succ^i_{\Pi^i(s'), f^i} f^i$. From Axiom 5, $g^i \Pi^i(s') f^i \succ^i_{\Pi^i(s'), f^i} f^i$. From Corollary 1, Axiom 14 is equivalent to Axiom 21. Using Axiom 21 and noting that f^i is strictly positive, $g^i \Pi^i(s') f^i \succ^i_{\Pi^i(s'), f^i} f^i$ implies that $g^i \Pi^i(s') f^i \notin \mathcal{R}^i(f^i)$, hence $\mathbb{E}_p(g^i \Pi^i(s') f^i) > \mathbb{E}_p f^i$ for all $p \in \pi^i(f^i)$. Because this is true for all $s' \in S$ such that $\Pi^i(s') \subseteq F$ and Π^i partitions F, we have that $\mathbb{E}_p(g^i F f^i) > \mathbb{E}_p f^i$, for all $p \in \pi^i(f^i)$. Define $h^i = g^i F f^i$ and $h = \{h^i\}_{i \in I}$.

Allocation f is interior, hence $\pi^{ui}(f^i) = \pi^{wi}(f^i) = \pi^i(f^i)$. Because $\mathbb{E}_p h^i > \mathbb{E}_p f^i$ for all $p \in \pi^{wi}(f^i)$, we have that for small enough ϵ^i, $\epsilon^i h^i + (1 - \epsilon^i) f^i \succ^i f^i$. By taking $\epsilon < \epsilon^i$ for all $i \in I$, we have that $\epsilon h^i + (1 - \epsilon) f^i \succ^i f^i$ for all $i \in I$. Moreover, $\epsilon h + (1 - \epsilon) f$ is feasible because both f and h are feasible. Hence, f is not ex ante efficient, a contradiction.

Conversely, suppose that $\{\succsim^1_{E,h}\}_{E \in \mathcal{E}, h \in \mathcal{F}}$ fails Axiom 14 and therefore Axiom 21. This means that for some event $E \in \mathcal{E}$ and acts $f \in \mathcal{F}_+$, $g \in \mathcal{F}$, with $f =_{E^c} g$, we have $g \succ^1_{E,f} f$ and $\mathbb{E}_{p_0} g \leq \mathbb{E}_{p_0} f$ for some $p_0 \in \pi^{1u}(f)$. Consider an economy with two agents, 1 and 2. Their information structure is identical, so that $\Pi^1 = \Pi^2 = \Pi = \{E, E^c\}$. Let $e = f + g$. Agent 2 has preferences represented by expected utility. In particular, $h \sim^2 h'$ if and only if $\mathbb{E}_{p_0} h = \mathbb{E}_{p_0} h'$. His conditional preferences given E or E^c are given by updating p_0 using Bayes' rule. This is well defined because Axiom 6 implies Axiom 9, hence from Lemma 2 we have that $p(E), p(E^c) > 0$.

We next show that allocation $h = \{f, g\}$ is ex ante Pareto efficient. Suppose there exists allocation $\{x, y\}$ such that $x \succ^1 f$ and $y \succsim^2 g$. Because $p_0 \in \pi^{1u}(f)$, we have that $\mathbb{E}_{p_0} x > \mathbb{E}_{p_0} f$. Moreover, $\mathbb{E}_{p_0} y \geq \mathbb{E}_{p_0} g$. These inequalities imply that $\mathbb{E}_{p_0}(x + y) > \mathbb{E}_{p_0}(f + g) = \mathbb{E}_{p_0} e$, which implies that $x + y \neq e$, hence $\{x, y\}$ is not feasible. A similar argument applies if $x \succsim^1 f$ and $y \succ^2 g$.

Note that $f \succsim^2 g$ because $\mathbb{E}_{p_0} g \leq \mathbb{E}_{p_0} f$. Given E and since $f =_{E^c} g$, we have that $\mathbb{E}_{p_{0E}} g \leq \mathbb{E}_{p_{0E}} f$, which implies $f \succsim^2_{E,g} g$. Because $g \succ^1_E f$, at each $s \in E$ it is common knowledge that allocation $h' = \{g, f\}$ Pareto dominates $h = \{f, g\}$, hence there is speculative trade.

□

Proof of Proposition 4. First note that because Axiom 7 implies Axiom 8, Proposition 1 implies that if information is (weakly) valuable then $\{\succsim^i_{E,h}\}_{i\in I, E\in\mathcal{E}, h\in\mathcal{F}}$ satisfy Axiom 13 (Axiom 15), for each $i \in I$. Let e be the endowment and suppose partition Π_1 is finer than partition Π_2. Let $\{g^i\}_{i\in I}$ be admissible for $\mathcal{M}_2 = \{\Pi_2, e\}$, defined as follows. Let $\{g_{E_2}\}_{E_2\in\Pi_2}$ be a tuple where, for each $E_2 \in \Pi_2$, g_{E_2} is an equilibrium allocation for economy $\langle E_2, \succsim^1_{E_2,g^1}, \ldots, \succsim^m_{E_2,g^m}, e\rangle$ with (normalized) prices $p^{E_2} \in \Delta S$, such that $p(s) = 0$ if $s \notin E_2$. For each $E_2 \in \Pi_2$, let $g^i =_{E_2} g^i_{E_2}$. From the first welfare theorem, g_{E_2} is Pareto optimal. From Proposition 9 in RSS, g_{E_2} is a full insurance allocation. From Axiom 6, $p^{E_2} > 0$ for all $s \in E_2$ and $g^i_{E_2}(s) = \mathbb{E}_{p^{E_2}} e^i$, for all $s \in E_2$ and all $i \in I$. Hence, g_{E_2} is also an interior allocation. Proposition 9 in RSS implies that $p^{E_2} \in \bigcap_i \pi^i_{E_2,g^i}$.

Proposition 1 in RSS shows that $\pi_{E,g^i}(f) = \pi^{ui}_{E,g^i}(f)$ for all strictly positive acts. Moreover, Axiom 6 implies Axiom 9. Lemma 2 implies $p^{E_2}(E_1) > 0$. It is straightforward that we can use the proof of Proposition 2, but applying Axiom 17 instead of Axiom 14, to show that there is Bayesian updating of subjective beliefs at a constant act, between any events $F, E \in \mathcal{E}$, where $E \subseteq F$. We therefore have $p^{E_2}_{E_1} \in \bigcap_i \pi^i_{E_1}$, for each $E_1 \subseteq E_2$, where $E_1 \in \Pi_1$ and $p^{E_2}_{E_1}$ is the Bayesian update of p^{E_2} on E_1.

Define allocation $\{f^i\}_{i\in I}$ as follows. If $E_1 \subseteq E_2$, where $E_1 \in \Pi_1$ and $E_2 \in \Pi_2$, then $f^i =_{E_1} f^i_{E_1}$, where $f_{E_1} = (f^1_{E_1}, \ldots, f^m_{E_1})$ is such that, for each $i \in I$, $f^i_{E_1}(s) = \mathbb{E}_{p^{E_2}_{E_1}} e^i$ for all $s \in E_1$. Hence, each f_{E_1} is a full insurance allocation. Because $p^{E_2}_{E_1} \in \pi^i_{E_1}$, $f^i_{E_1}$ is weakly preferred to each act h that is affordable given prices $p^{E_2}_{E_1}$. Because f_{E_1} is feasible, it is an equilibrium allocation of economy $(E_1, \succsim^1_{E_1,f^1}, \ldots, \succsim^m_{E_1,f^m}, e)$.

By construction, $\mathbb{E}_{p^{E_2}} f^i = \mathbb{E}_{p^{E_2}} g^i_{E_2}$. Because $p^{E_2} \in \bigcap_i \pi^i_{E_2}$, we have that $g^i_{E_2} \succsim^i_{E_2,g^i} f^i$, for all $i \in I$ and all $E_2 \in \Pi_2$. Axiom 5 implies that $g^i \succsim^i_{E_2,g^i} f^i$ for each $E_2 \in \Pi_2$ and each $i \in I$.

Enumerate the partition cells of $\Pi_2 = \{E_1, \ldots, E_n\}$. If $n = 1$ then $\Pi_2 = \{S\}$ is the uninformative partition and $g^i \succsim^i f^i$ for each $i \in I$, so we are done. Suppose that $n \geq 2$. For cell $1 \leq k \leq n$ define act h^i_k as follows. Let $h^i_k(s) = g^i(s)$ if $s \in E_j$, where $1 \leq j \leq k$, and $h^i_k(s) = f^i(s)$ otherwise. Note that $h^i_n = g^i$ and let $h^i_0 = f^i$. For each $1 \leq k \leq n$, from Axiom 5, we have that $g^i \succsim^i_{E_k,g^i} f^i$ implies $h^i_k \succsim^i_{E_k,g^i} h^i_{k-1}$. Axiom 11 implies $h^i_k \succsim^i_{E_k,h^i_k} h^i_{k-1}$. Applying Axiom 13 we have $h^i_k \succsim^i h^i_{k-1}$. By Axiom 1, we have that $g^i \succsim^i f^i$, for each $i \in I$, which implies that $\mathcal{M}_1$ is not more valuable than $\mathcal{M}_2$. Therefore, public information is not valuable.

For the second claim, for each $E_2 \in \Pi_2$ define $h^i_{E_2} = g^i_{E_2} E_2 f^i$. From Axiom 5, $g^i_{E_2} \succsim^i_{E_2,g^i} f^i$ implies $h^i_{E_2} \succsim^i_{E_2,g^i} f^i$. Axiom 11 implies $h^i_{E_2} \succsim^i_{E_2,h^i_{E_2}} f^i$. From Axiom 15 and using Corollary 1, either $h^i_{E_2} \succsim^i f^i$ or $f^i \succ^i h^i_{E_2}$ and $h^i_{E_2} \notin \mathcal{R}^i(f^i)$. If $h^i_{E_2} \succsim^i f^i$, Axiom 7 implies that for all $a \in (0,1)$, $ah^i_{E_2} + (1-a)f^i \succ^i f^i$. This means that $\mathbb{E}_p(ah^i_{E_2} + (1-a)f^i) > \mathbb{E}_p f^i$ for all $p \in \pi^i(f^i)$, or that $\mathbb{E}_p h^i_{E_2} > \mathbb{E}_p f^i$. Hence, in both cases we have that $\mathbb{E}_p h^i_{E_2} > \mathbb{E}_p f^i$ for all $p \in \pi^i(f^i)$. Repeating this argument for all $E_2 \in \Pi_2$, we have that $\mathbb{E}_p g^i > \mathbb{E}_p f^i$ for all $p \in \pi^i(f^i)$. By definition, for small enough $\epsilon^i > 0$, $\epsilon^i g^i + (1-\epsilon^i) f^i \succ^i f^i$. By taking $\epsilon < \epsilon^i$ for all $i \in I$, we have that $\epsilon g^i + (1-\epsilon) f^i \succ^i f^i$ for all $i \in I$, hence public information is weakly not valuable.

□

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