Non-cooperative games with prospect theory players and dominated strategies

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Abstract

We investigate a framework for non-cooperative games in normal form where players have behavioral preferences following Prospect Theory (PT) or Cumulative Prospect Theory (CPT). On theoretical grounds CPT is usually considered to be the superior model, since it normally does not violate first order stochastic dominance in lottery choices. We find, however, that CPT when applied to games may select purely dominated strategies, while PT does not. For both models we also characterize the cases where mixed dominated strategies are preserved and where violations may occur.

Keywords: Prospect theory, framing, reference dependent utility, rank dependent probability weighting, Nash equilibrium, stochastic dominance, dominance of strategies

JEL classification: C70, C73, D81.

1 Introduction

To describe human behavior in decisions under risk, expected utility theory (EUT) has been complemented in recent years by more modern theories, in particular by prospect theory (PT) and cumulative prospect theory (CPT)[1]. In this article, we investigate the effect of these behavioral decision models on strategic choices in games. In particular we show that equilibria, similar to the classical Nash equilibria exist under certain conditions, but that at least under CPT, players might choose dominated actions. The results lay a theoretical foundation for the application of PT and CPT in the study of strategic choices, and contribute to the discussion on the appropriate selection of decision models.

When thinking about players with non-EUT preferences, at first it seems not at all clear why such a change of the decision model should lead to changes in the game theoretical framework. Are both theories (game theory and decision theory) not separated and can simply be analyzed independently?

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[1]Introduced in Kahneman & Tversky (1979) and Tversky & Kahneman (1992).

When we play a game, payoffs are often given in monetary (or similar) amounts. As von Neumann and Morgenstern noticed, game theory has therefore to take into account the players' preferences on sure and uncertain monetary outcomes of the game. This is why their famous work starts with a chapter on the notion of utility in which they present their formalization of EUT (v. Neumann & Morgenstern 1944). It is therefore not too surprising that this theory is somehow tailor-made for the application to game theory. In fact, the payoffs of a game are usually already defined to be given in utility units. Nevertheless it is an important and nontrivial feature that after a simple transformation of monetary outcomes into utility outcomes for each player, in the further analysis of the transformed game no additional considerations regarding the players' preferences have to be made.

This seamless interplay between EUT and game theory, however, obfuscates the fact that there is indeed something nontrivial in this connection, and that therefore changes might be necessary when using a different preference model in the study of games. This might explain why the problem of applying game theory in a PT setting has not been widely studied so far, although it seems very natural to substitute this successful behavioral model into the study of games. Shalev (2000) introduces one aspect of PT, reference dependent loss aversion, into game theory and provides some existence results.

The other central aspect of most behavioral decision models, probability weighting, is less popular in the game theoretic literature. A widely accepted perception is that as PT-agents who overweight small probabilities may prefer first order stochastically dominated lotteries (see Example 1 in section 3 or Quiggin 1982) and thus are 'leaving money on the table', it is regarded 'pointless to go through the trouble of modeling sophisticated strategic behavior'.[2]

CPT, introduced by Tversky & Kahneman (1992) and its predecessor RDU from (Quiggin 1982, 1993) do not violate first order stochastic dominance. Therefore it seems natural to study strategic choices of CPT-agents rather than PT-agents in normal form games.

Although nowadays CPT is mostly seen as the superior model – mostly because of the aforementioned violation of stochastic dominance for PT, but also because it can be axiomatized (Wakker 1989) – there are also some advantages of the older PT. For instance, Birnbaum & McIntosh (1996), Birnbaum & Martin (2003) and Birnbaum (2005) provide empirical evidence that in some situations choices of stochastically dominated lotteries regularly occur in human decision making. Such decisions can be modelled by PT, but not by CPT, see Rieger & Wang (2008). We therefore decided instead of only focusing on CPT, to study both theories in the context of strategic choices.

To incorporate CPT or PT into a game theoretical framework, at first glance looks straightforward: the naïve approach would be to transform the monetary outcomes via the value function and to transform actual probabilities for chance moves (i.e. moves of nature that occur with a predefined and known probability) into experienced probabilities by applying the probability weighting functions.[3] This procedure mimics the method one successfully applied when dealing with EUT. Nevertheless in the case of PT and CPT, there are two difficulties in this

[2] We are grateful to receive this and similar comments from anonymous referees.

[3] Here one has to be careful to transform the game into a form where the probabilities of the chance moves can be weighted separately for each player, since their probability weighting functions might differ.

approach which pose interesting challenges, namely:

1. If we believe that the reference point is not exogenously given, but also depends on endogenous factors such as the payoffs of the opponents, the reference point itself is endogenous and the transformation of the monetary outcomes via the value function is not as harmless as it seems, since the reference point has to be chosen first.

2. The probabilities of chance moves are not the only probabilities that ought to be transformed by the probability weighting function, since the existence of mixed strategy Nash equilibria complicates considerations: the objective probability with which one player chooses a particular strategy will be transformed to a (different) subjective probability by another player who will choose his own strategy according to this subjective probability, rather than to the objective probability. This leads to an interesting interplay between the weighting functions of the players.[4]

In this paper we offer an analytical framework for both concepts – PT and CPT – for strategic behavior in normal form games. We show that, in contrast to the results on stochastic dominance in lotteries, PT-agents do never select strategies that are strictly dominated (in monetary terms) by pure strategies. We show further, that if a PT-agent prefers a strategy which is strictly dominated by a mixed strategy, this is due to risk aversion, i.e. there also exists an EUT-agent who prefers the strictly dominated strategy. More surprisingly, in contrast to the perceived conceptual advantage of CPT we provide a simple example in which CPT-agents prefer a strategy which is strictly dominated by a pure strategy. We characterize the cases in which such preferences do not occur.

The remaining sections are structured as follows:

In Section 2 we give a quick review of PT and CPT, previous results on games where players have non-EUT preferences, and describe our model for PT- and CPT-equilibria, generalizing the concept of Nash equilibria. Moreover we prove existence of these equilibria for fixed reference points and give examples for non-existence when the reference point depends on other player's strategic choices. In Section 3 we study stochastic dominance and dominated strategies and derive in particular our results on the violations of dominance for CPT. Section 4 concludes.

2 Model

Our model has three "layers": a standard normal form game with monetary payoffs, a value game which we obtain by applying value functions to the game and finally the PT or CPT game, which we obtain from the value game through appropriate techniques of probability weighting.[5] We introduce these layers in

[4]To be more precise, one should not speak about subjective or weighted probabilities, since PT is not about misestimation of probabilities, but states that decisions are made *as if* the underlying probabilities were misestimated or weighted. Mathematically, however, this results in the same formula, and so we keep for simplicity the slight abuse of language and talk about subjective or weighted probabilities as if they had a real meaning in PT and were not only auxiliary quantities.

[5]We thank an anonymous referee for proposing an improvement of the structure of the model.

this section. PT and CPT are thereby two possible variants of the same "layer". Our approach in modelling non-cooperative games for CPT-agents is closest to Ritzberger (1996).

2.1 Monetary Games in Strategic Form

We consider finite games in normal form $(\mathcal{N}, S, x)$, where the three elements have the following meaning: $\mathcal{N} = \{1, \ldots, n\}$ is the finite set of players with $n \geq 2$, and $S = \times_{i=1}^{n} S_i$ is the finite set of pure strategy combinations with S_i being the set of pure strategies of player $i \in \mathcal{N}$. We denote the set of pure strategies of all players except i by $S_{-i} = \times_{j \neq i} S_j$ with typical element $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$. $x : S \to \mathbb{R}^n$ consists of n functions $x_i : S \to \mathbb{R}$, where $x_i(s)$ is the monetary payoff to player i given the pure strategy $s \in S$.[6] Denote by $\Delta_i = \left\{ \sigma_i \in \mathbb{R}_+^{|S_i|} \middle| \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \right\}$ the set of mixed strategies of player i, by $\Delta_{-i} = \times_{i \neq j} \Delta_j$ the set of mixed strategies of all players except i and by $\Delta = \times_{i=1}^{n} \Delta_i$ the set of mixed strategies of all players. Define further $\sigma_{-i}(A) = \sum\limits_{s_{-i} \in A} \prod\limits_{j \neq i} \sigma_j(s_j) \ \forall \ A \subseteq S_{-i}$ and $\sigma_{-i}(\emptyset) = 0$ and also $x_i(\sigma) = \sum\limits_{s \in S} \sigma(s) \cdot x_i(s)$.

2.2 Value Functions and Reference Points

We denote cardinal utility functions by small letters, where the curvature of the function captures the risk preferences of the agent. For EUT agents we use the term Bernoulli utility $u : \mathbb{R} \to \mathbb{R}$ and for PT and CPT agents we use the term value function $v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, where $v(x, r)$ is the value at some monetary payment $x \in \mathbb{R}$ given some reference point $r \in \mathbb{R}$. For any value function v we assume the following for all reference points $r \in \mathbb{R}$:

- $v(x, r)$ is continuous in x for all $x \in \mathbb{R}$
- $v(x, r) > v(y, r) \Leftrightarrow x > y$
- $v(x, r)$ is weakly concave in x for all $x \geq r$
- $v(x, r)$ is weakly convex in x for all $x < r$

A monetary outcome $x \geq r$ is considered as a *gain* and $x < r$ is considered as a *loss*. The set of assumptions on v imply that PT and CPT agents are risk averse in gains and risk loving in losses. We assume that the reference point is a continuous function $r_i : \Delta \to \mathbb{R}$ for each player i. Whenever the reference point is a constant, we refer to the context as 'fixed frames'.

The prototypical example has been given in Tversky & Kahneman (1992) for $\alpha, \beta \in (0, 1)$ and $\lambda > 1$:

$$v(x) := \begin{cases} x^\alpha & , \text{if } x \geq 0 \\ -\lambda \cdot (-x)^\beta & , \text{if } x < 0 \ , \end{cases} \tag{1}$$

where the reference point is normalized to zero.

[6] We assume that player i cares only for his own payment x_i, where i strictly prefers x_i to y_i whenever $x_i > y_i$.

We call the game a *game of losses*, if $x_i(\sigma) < r_i(\sigma)\ \forall\ \sigma \in \Delta$ and $i \in \mathcal{N}$. We say the game is a *game of gains*, if $x_i(\sigma) \geq r_i(\sigma)\ \forall\ \sigma \in \Delta$ and $i \in \mathcal{N}$. Otherwise, we call the game *regular*. Whenever necessary we include nature by adding some player with a set of 'strategies' which corresponds to the set of states and 'payoffs' which are all equal to zero.

2.3 Probability Weighting Functions

In the setup without mixed strategies, EUT, PT and CPT all induce the same predictions. One of the most interesting effects of PT and CPT on the analysis of games is the interplay between probability weighting and mixed strategies.

Consider a probability space $(P, \mathcal{P}, \pi)$, where $(P, \mathcal{P})$ is a measurable space and π is a probability measure. A probability weighting function $\omega : [0,1] \to [0,1]$ maps probabilities to real numbers. We assume that

- $\omega(0) = 0$, $\omega(1) = 1$,
- ω is continuous, strictly monotonic and differentiable,
- there exists a unique fixed point $\bar{p} \in (0,1)$,
- $\omega(p) > p$ for all $0 < p < \bar{p}$ and $\omega(p) < p$ for all $1 > p > \bar{p}$.

The original example of Tversky & Kahneman (1992) is given by

$$w(p) := \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{\frac{1}{\gamma}}} \tag{2}$$

with $\gamma < 1$. As this probability weighting function is decreasing in p whenever $p^\gamma \cdot (\gamma - 1) + (1-p)^\gamma \cdot (\gamma + \frac{p}{1-p}) < 0$, we assume $\gamma > 0.3$ (see Rieger & Wang 2006).

The ex-ante utility of player i for the stochastic monetary payment induced by the mixed strategy $(\sigma_i, \sigma_{-i}) \in \Delta_i \times \Delta_{-i}$ depends on the underlying decision model. In the case of EUT the reference point is irrelevant and this utility becomes

$$EU_i(\sigma_i, \sigma_{-i}) = \sum_{s \in S} u_i(x_i(s)) \cdot \sigma_i(s_i) \cdot \sigma_{-i}(s_{-i}) \ .$$

Using PT and CPT, we need to decide whether the probability weighting occurs on the individual or on the joint strategies of the opponents.

In the case of PT we allow for two alternative variants of probability weighting, our results apply for both of them. In the first variant agent i weights the individual probabilities with which the other players choose their mixed strategies. Player i perceives the probability of state s_{-i} as $\prod_{j \neq i} \omega_i(\sigma_j(s_j))$. In the second variant we assume that the agent weights the joint probability over the outcomes in s_{-i}, hence the probability is perceived as $\omega_i\left(\prod_{j \neq i} \sigma_j(s_j)\right)$. As the results do not depend on the probability-variant we write $\omega_i(\sigma_{-i}(s_{-i}))$. In the case of CPT we exclusively use the second variant.

2.3.1 Prospect Theory

We define the ex-ante utility of a PT-agent as

$$V_i^{PT}(\sigma_i, \sigma_{-i}, r_i) = \sum_{s_i \in S_i} \sigma_i(s_i) \cdot \sum_{s_{-i} \in S_{-i}} v_i(x_i(s_i, s_{-i}), r_i(\sigma)) \cdot w_i\left(\sigma_{-i}(s_{-i})\right) \ , \quad (3)$$

where we follow the straightforward generalization of PT to arbitrarily many outcomes that has been suggested, e.g., by Lopes & Oden (1999).

2.3.2 Cumulative Prospect Theory

In the case of CPT the reference point does not only influence the value attached to the monetary payoff but also the weight which is associated with this value. If x_i is a gain, the agent weights the probability with which *at least* x_i is obtained while if x_i is a loss, the agent weights the probability with which *at most* x_i realizes. The weight associated with the value function consists of the marginal contribution to this probability. We firstly need to rank the possible outcomes, before we can compute the probability weighting. For each player $i \in \mathcal{N}$ define the function $l_i : S \to \{1, \ldots, |S_{-i}|\}$ such that for each $s_i \in S_i$:

$$l_i(s_i, s_{-i}) < l_i(s_i, \tilde{s}_{-i}) \Rightarrow x_i(s_i, s_{-i}) \leq x_i(s_i, \tilde{s}_{-i})$$

and $l_i(s_i, \cdot)$ is a bijection for each $s_i \in S_i$.

Given the index function l define for each $s \in S$ and $i \in \mathcal{N}$ the set $S_{-i}^R(s) = \{\tilde{s}_{-i} \in S_{-i} : l_i(s_i, \tilde{s}_{-i}) \ R \ l_i(s_i, s_{-i})\}$, where $R \in \{<, \leq, \geq, >\}$. Now define for given mixed strategy $\sigma \in \Delta$ and reference point $r_i(\sigma) \in \mathbb{R}$ the perceived probability function $\psi_i(\cdot | r_i, \sigma_{-i}) : S \to \mathbb{R}_+$ as

$$\psi_i(s | r_i, \sigma_{-i}) = \begin{cases} \omega_i\left(\sigma_{-i}\left(S_{-i}^{\leq}(s)\right)\right) - \omega_i\left(\sigma_{-i}\left(S_{-i}^{<}(s)\right)\right) & \text{if } x_i(s) < r_i(\sigma) \\ \omega_i\left(\sigma_{-i}\left(S_{-i}^{\geq}(s)\right)\right) - \omega_i\left(\sigma_{-i}\left(S_{-i}^{>}(s)\right)\right) & \text{if } x_i(s) \geq r_i(\sigma) \end{cases} \quad (4)$$

Finally, define the ex-ante utility of a CPT agent as

$$V_i^{CPT}(\sigma) = \sum_{s \in S} \sigma_i(s_i) \cdot \psi_i(s | r_i(\sigma), \sigma_{-i}) \cdot v_i(x_i(s), r_i(\sigma)) \quad (5)$$

Example 5 in the Appenix illustrates how the CPT-model operates.

2.4 Equilibrium

A Nash equilibrium (Nash 1950, 1951) requires that all players choose their strategies optimally given their beliefs and that all beliefs are consistent with the choices of the opponents. It is the latter point which is violated if players non-linearly weight the probabilities. For this reason we provide the following definitions for PT and CPT:

Definition 1 (PT- & CPT-equilibrium)
We call a strategy $\hat{\sigma} \in \Delta$ a PT-equilibrium given reference point $r \in \mathbb{R}^n$ if for all $i = 1, \dots, n$ and all $\sigma_i \in \Delta_i$ we have $V_i^{PT}(\hat{\sigma}, r_i) \geq V_i^{PT}(\sigma_i, \hat{\sigma}_{-i}, r_i)$.
Analogously, we say that $\hat{\sigma} \in \Delta$ is a CPT-equilibrium given reference point $r \in \mathbb{R}^n$ if for $i = 1, \dots, n$ and all $\sigma_i \in \Delta_i$ we have $V_i^{CPT}(\hat{\sigma}, r_i) \geq V_i^{CPT}(\sigma_i, \hat{\sigma}_{-i}, r_i)$.

As we point out in the introduction and underpin with Examples 2 to 4 in the Appendix the simple model which we provide here fills a gap in the literature. Furthermore, as the own probabilities enter the ex-ante utility function linearly, the existence proof uses standard arguments.

Proposition 1 (existence) *Every finite monetary game with a continuous reference point function admits a PT- and a CPT-equilibrium.*

Proof : The arguments for PT and CPT are identical and we suppress the superscripts PT and CPT for this proof. For any given continuous function $r_i : \Delta \to \mathbb{R}$ and for each $i \in \mathcal{N}$ and $\sigma_{-i} \in \Delta_{-i}$ the ex-ante utility function $V_i(\sigma_i, \sigma_{-i}, r_i(\sigma))$ is continuous in $\sigma_i(s_i)$ for each $s_i \in S_i$ and Δ_i is compact, hence the set $BR_i(\sigma_{-i}, r_i) := \{\sigma_i \in \Delta_i : V_i(\sigma_i, \sigma_{-i}, r_i(\sigma)) \geq V_i(\tilde{\sigma}_i, \sigma_{-i}, r_i(\sigma))\}$ is nonempty, compact and convex valued. We assume in section 2.3 that $\omega_i(p)$ is continuous in p, therefore $V_i(\sigma_i, \sigma_{-i}, r_i)$ is continuous in σ_{-i} for each $\sigma_{-i} \in \Delta_{-i}$. To see this, note that σ_{-i} enters $V_i^{PT}(\cdot)$ via $\omega(\cdot)$ in the form of $\sigma_j(s_j)$ or $\sigma_{-i}(s_{-i}) = \prod_{j \neq i} \sigma_j(s_j)$. For CPT observe that σ_{-i} enters V_i^{CPT} via $\omega(\cdot)$ in the form of $\sigma_{-i}(A) = \sum_{s_{-i} \in A} \prod_{j \neq i} \sigma_j(s_j)$ for given subsets $A \subseteq S_{-i}$. The set Δ_{-i} is compact. By Berge's Maximum Theorem the correspondence $BR_i(\sigma_{-i}, r_i)$ is upper hemicontinuous and by Kakutani's Fixed Point Theorem there exists some $\hat{\sigma} \in \Delta$ such that $\hat{\sigma}_i \in BR_i(\hat{\sigma}_{-i}, r_i)$ for all $i \in \mathcal{N}$. □

2.5 Effects of Framing

Keskin (2016) provides results on a particular endogenous reference point function, where – given some mixed strategy profile – the reference point is roughly the conditional ex-ante value obtained by the mixed strategy profile. Hereby, the reference point is uniquely defined for each profile of mixed strategies. Keskin uses this fact to show equilibrium existence. On the other hand there is evidence that suggests that in the same real-life decision different people might select different frames ("self-framing"), see Wang & Fischbeck (2004). In the case of games, the choice of the frame might depend on the situation, e.g. on the payoff of other players. But shouldn't all these considerations be irrelevant to obtain existence at least in simple normal form games? It is interesting to see that this is not the case: even in the simplest possible setting of a unilateral decision problem the framing effect can play a decisive role, as Example 6 in the Appendix demonstrates, which also shows that Proposition 1 can in fact not be generalized to discontinuous reference point functions.

Considering this, we will concentrate in the remainder of this article on situations where the reference point is fixed, and therefore existence of equilibria is guaranteed.

3 Dominated Strategies and Stochastic Dominance

In this section we compare and characterize the choices of EUT-, PT- and CPT-agents of dominated strategies. As the main difference between the three concepts becomes only visible if the decision problem is stochastic, we can apply the standard notion of dominance in game theory (see Rapoport 1966) to monetary games in strategic form with any of the three concepts EUT, PT and CPT. In a monetary game, any tuple of pure strategies $s \in S$ induces a monetary payment $x_i(s) \in \mathbb{R}$ for each player $i \in \mathcal{N}$.

Definition 2 (Dominated Strategy) *Given a strictly monotonic function $f : \mathbb{R} \to \mathbb{R}$ we say that strategy $\hat{s}_i$ is dominated by strategy $\sigma_i \in \Delta_i$ for player $i \in \mathcal{N}$, if f is i's utility function and if for all $s_{-i} \in S_{-i}$ we have*

$$f(x_i(\sigma_i, s_{-i})) = \sum_{s_i \in S_i} \sigma_i(s_i) \cdot f(x_i(s_i, s_{-i})) > f(x_i(\hat{s}_i, s_{-i})) \ .$$

If f is the identity function with $f(x) = x$ for all $x \in \mathbb{R}$, then $\hat{s}_i$ is dominated in monetary terms. *If f is a Bernoulli utility function with $f(x) = u_i(x)$ for all $x \in \mathbb{R}$, then $\hat{s}_i$ is dominated* with respect to u *and if f is a value function for given reference point $r_i \in \mathbb{R}$ with $f(x) = v_i(x, r_i)$ for all $x \in \mathbb{R}$, then $\hat{s}_i$ is dominated* with respect to $v_i(\cdot, r_i)$.

Note that if a strategy is dominated by a pure strategy, all concepts are equivalent and we do not need to specify a reference to some strictly increasing function f. The next definition of dominance refers to general lotteries over monetary outcomes and is due to Quirk & Saposnik (1962).

Definition 3 (First Order Stochastic Dominance) *Given two distribution funtions $F : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$, $F \neq G$, F* first order stochastically dominates *G if for all $x \in \mathbb{R}$, $F(x) \leq G(x)$.*

Intuitively, the probability of observing an outcome of at least x is higher under lottery A than under lottery B for any value of x. To our knowledge, Bawa (1975) is the first who uses first order stochastic dominance as a selection criterion for decisions under uncertainty. Bawa (1975) shows that any decision maker whose utility function is increasing and differentiable and who assesses the uncertainty correctly chooses the first order stochastically dominant lottery. While in our setting all agents have increasing utility functions, PT- & CPT-agents fail to have an unbiased perception of stochastic environments. Hence, an agent that uses PT does not necessarily prefer a stochastically dominant lottery A over lottery B, as the following example which we adapt from Quiggin (1982) illustrates:

Example 1 (stochastic dominance and PT) For lottery A let the valuation of outcomes be $a_i = 1 - i \cdot \epsilon$ and the probabilities be $\alpha_i = \frac{1}{n}$ for $i = 1, \dots, n$ and let B be associated with the outcome $b_1 = 1$ which occurs with probability 1. Let the reference point be equal to zero. Then

$$V^{PT}(A) = \sum_{i=1}^{n} (1 - i \cdot \epsilon) \cdot w\left(\frac{1}{n}\right) = n \cdot w\left(\frac{1}{n}\right) \cdot \left(1 - \epsilon \cdot \frac{n+1}{2}\right)$$

and

$$V^{PT}(B) = 1 \ .$$

For any $\epsilon > 0$, lottery B stochastically dominates A. For ϵ small enough, $V^{PT}(A) > V^{PT}(B)$ as small probabilities are over-weighted: $w\left(\frac{1}{n}\right) > \frac{1}{n}$.

This seemingly artificial result is implied by the inability of a PT-agent to categorize a class of similar outcomes and evaluate the probability of the whole category.[7]

For each choice s_i of player i the opponent's mixed strategy $\sigma_{-i} \in \Delta_{-i}$ induces a lottery on the set of outcomes $\{s_i\} \times S_{-i}$. Note that given σ_{-i} two different lotteries, one for strategy $\hat{s}_i$ and one for strategy $\tilde{s}_i$ say, have the same number of outcomes and have equal probability distributions. When choosing among these two different lotteries, the agent faces the same distribution on the set of outcomes while he may attach different values to the outcomes. This is the reason why PT-agents do not choose dominated strategies, as the following proposition states:

Proposition 2 *$\sigma_i \in \Delta_i$ dominates $\hat{s}_i \in S_i$ with respect to $v_i(\cdot, r_i)$ given some reference point $r_i \in \mathbb{R}$, if and only if*

$$V_i^{PT}(\sigma_i, \sigma_{-i}) > V_i^{PT}(\hat{s}_i, \sigma_{-i}) \ \forall \ \sigma_{-i} \in \Delta_{-i} \ .$$

PROOF : Suppose the pure strategy $\hat{s}_i$ is dominated by mixed strategy $\sigma_i \in \Delta_i$ with respect to the value function v_i for some given reference point $r_i \in \mathbb{R}$. Then

$$\sum_{s_i \in S_i} \sigma_i(s_i) \cdot v_i(x_i(s_i, s_{-i}), r_i) \quad > \quad v_i(x_i(\hat{s}_i, s_{-i}), r_i) \ \forall \ s_{-i} \in S_{-i}$$

As the probability weighting function $\omega_i(\sigma_{-i}(s_{-i})) > 0 \Leftrightarrow \sigma_{-i}(s_{-i}) > 0$ we have that

$$\begin{aligned} V_i^{PT}(\sigma_i, \sigma_{-i}) &= \sum_{s_{-i} \in S_{-i}} \omega_i(\sigma_{-i}(s_{-i})) \sum_{s_i \in S_i} \sigma_i(s_i) \cdot v_i(x_i(s_i, s_{-i}), r_i) \\ &> \sum_{s_{-i} \in S_{-i}} \omega_i(\sigma_{-i}(s_{-i})) \cdot v_i(x_i(\hat{s}_i, s_{-i}), r_i) = V_i^{PT}(\hat{s}_i, \sigma_{-i}) \end{aligned}$$

for all $\sigma_{-i} \in \Delta_{-i}$. Suppose now that

$$V_i^{PT}(\sigma_i, \sigma_{-i}) > V_i^{PT}(\hat{s}_i, \sigma_{-i}) \ \forall \ \sigma_{-i} \in \Delta_{-i} \ .$$

In particular, as the vertices of Δ_{-i} correspond to the pure strategies $s_{-i} \in S_{-i}$ and as $\omega_i(1) = 1$, we have

$$\begin{aligned} \sum_{s_i \in S_i} \sigma_i(s_i) \cdot v_i(x_i(s_i, s_{-i}), r_i) &= V_i^{PT}(\sigma_i, s_{-i}) \\ &> V_i^{PT}(\hat{s}_i, s_{-i}) = v_i(x_i(\hat{s}_i, s_{-i}), r_i) \ \forall \ s_{-i} \in S_{-i} \ , \end{aligned}$$

therefore $\hat{s}_i$ is dominated by σ_i with respect to v_i with reference point r_i. $\square$

The following results disentangle the effects due to risk aversion, which also occur in EUT, and the effects due to probability weighting.

[7]For further discussion of stochastic dominance in the original formulation of PT and in the variant by Karmarkar (1978) we refer to Rieger & Wang (2008).

Proposition 3 (PT: pure monetary domination)
$s_i \in S_i$ dominates $\hat{s}_i \in S_i$ in monetary terms, if and only if

$$V_i^{PT}(s_i, \sigma_{-i}) > V_i^{PT}(\hat{s}_i, \sigma_{-i}) \;\forall\; \sigma_{-i} \in \Delta_{-i} \;.$$

Proof : Suppose $\hat{s}_i \in S_i$ is dominated by $s_i \in S_i$ in monetary terms. Then

$$x_i(s_i, s_{-i}) > x_i(\hat{s}_i, s_{-i}) \;\forall\; s_{-i} \in S_{-i} \;.$$

As $v_i(x, r_i)$ is strictly monotonic in $x \in \mathbb{R}$ for any reference point $r_i \in \mathbb{R}$, this is equivalent to

$$v_i(x_i(s_i, s_{-i}), r_i) > v_i(x_i(\hat{s}_i, s_{-i}), r_i) \;\forall\; s_{-i} \in S_{-i} \;.$$

Therefore $\hat{s}_i \in S_i$ is dominated by $s_i \in S_i$ with respect to v_i for any given reference point $r_i \in \mathbb{R}$ if and only if $\hat{s}_i$ is dominated by s_i in monetary terms. The statement of the proposition follows now by Proposition 2. □

If the choice σ_{-i} of the other players induces two monetary lotteries which are associated with s_i and $\hat{s}_i$, one first order stochastically dominating the other, we have shown that there is no value function $v(\cdot)$, probability weighting function $\omega(\cdot)$ and reference point r as defined in section 2 such that a PT-agent prefers the dominated lottery, if the lotteries are generated via a monetary game. Note that choosing $\hat{s}_i$ would imply to relinquish some amount of money *with certainty*.

If $\hat{s}_i \in S_i$ is dominated by $\sigma_i \in \Delta_i$ in monetary terms, agent i looses money *in expectation* but not necessarily in each possible outcome $s_{-i} \in S_{-i}$. In this case agent i may prefer $\hat{s}_i$ over σ_i, as Example 7 in the appendix illustrates.

The following proposition implies that the preference for a dominated strategy generally is not caused by the probability weighting function but by the risk preferences of the agents – which of course is not due to PT as EUT also allows for risk aversion.

Proposition 4 (PT: mixed monetary domination) *If $\sigma_i \in \Delta_i$ strictly dominates $\hat{s}_i \in S_i$ in monetary terms and, given reference point $r_i \in \mathbb{R}$, the value function $v_i(\cdot, r_i)$ satisfies*

$$\begin{aligned} v_i(\lambda \cdot x_i(s') + (1-\lambda) \cdot x_i(s''), r_i) \quad &< \quad \lambda \cdot v_i(x_i(s'), r_i) + (1-\lambda) \cdot v_i(x_i(s''), r_i) \\ &+ \quad \min_{\tilde{s}_{-i}} v_i(x_i(\sigma_i, \tilde{s}_{-i}), r_i) - v_i(x_i(\hat{s}_i, \tilde{s}_{-i}), r_i) \end{aligned}$$

for all $\lambda \in (0,1)$ and all $s', s'' \in S$, then

$$V_i^{PT}(\sigma_i, \sigma_{-i}) > V_i^{PT}(\hat{s}_i, \sigma_{-i}) \;\forall\; \sigma_{-i} \in \Delta_{-i} \;.$$

Proof : For all $\sigma_{-i} \in \Delta_{-i}$ we have

$$\begin{aligned} V_i^{PT}(\sigma_i, \sigma_{-i}) \quad &> \quad \sum_{s_{-i} \in S_{-i}} \omega_i(\sigma_{-i}(s_{-i})) \cdot (v_i\left(x_i(\sigma_i, s_{-i}), r_i\right) - \bar{\epsilon}) \\ &\geq \quad \sum_{s_{-i} \in S_{-i}} \omega_i(\sigma_{-i}(s_{-i})) \cdot v_i\left(x_i(\hat{s}_i, s_{-i}), r_i\right) = V_i^{PT}(\hat{s}_i, \sigma_{-i}) \;, \end{aligned}$$

where $\bar{\epsilon} := \min_{\tilde{s}_{-i}} v_i(x_i(\sigma_i, \tilde{s}_{-i}), r_i) - v_i(x_i(\hat{s}_i, \tilde{s}_{-i}), r_i)$. □

Note that $\bar{\epsilon}$ is positive for any game and value function and that therefore the condition of the proposition on the value function is trivially satisfied if v_i is convex over all monetary payoffs of the game. The condition also allows for payoffs under which v_i is concave but it must not be too concave. To see this, consider Example 7 and see that a PT-agent would reject to choose the dominated strategy, iff $\alpha > \ln 2/\ln 3$ (with $v_i(x_i, 0) = x_i^\alpha \ \forall\ x_i \geq 0$). To sum up, Proposition 4 states that a PT-agent does not forgo expected money, if the agent exhibits few enough risk-aversion. The following corollary exploits the fact that PT-agents are risk-loving in losses:

Corollary 1 (PT: mixed monetary domination in losses)
If $x_i(s) < r_i \ \forall\ s \in S$ and σ_i strictly dominates $\hat{s}_i$ in monetary terms, then

$$V_i^{PT}(\sigma_i, \sigma_{-i}) > V_i^{PT}(\hat{s}_i, \sigma_{-i}) \ \forall\ \sigma_{-i} \in \Delta_{-i} \ .$$

Proof : As $v_i(\cdot, r_i)$ is convex for all $x < r_i$ it holds that $v_i(\lambda \cdot x_i(s') + (1-\lambda) \cdot x_i(s''), r_i) < \lambda \cdot v_i(x_i(s'), r_i) + (1-\lambda) \cdot v_i(x_i(s''), r_i) \ \forall\ \lambda \in (0,1)$ and $s', s'' \in S$ and Proposition 4 applies. □

In any regular game which has both gains and losses or any game of gains there are ample possibilities that PT-agents choose a pure strategy which is strictly dominated by a mixed strategy in monetary terms. Is this fact sufficient to reject the application of PT-preferences in non-cooperative game theory? We believe that this stance is unsustainable because it would also argue against EUT as Proposition 5 points out:

Proposition 5 (Mixed monetary domination, PT and EUT) *Let $\hat{s}_i \in S_i$ be dominated by $\sigma_i \in \Delta_i$ in monetary terms and consider a PT-agent with reference point r_i and value function v_i and a EUT-agent with utility function $u_i = v_i(\cdot, r_i)$. If for a given mixed strategy $\sigma_{-i} \in \Delta_{-i}$ the PT-agent prefers $\hat{s}_i$ to σ_i, then there exists some pure strategy $\tilde{s}_{-i} \in S_{-i}$ such that the EUT-agent prefers $\hat{s}_i$ to σ_i given $\tilde{s}_{-i}$. Analogously, if for given σ_{-i} the EUT-agent prefers $\hat{s}_i$ to σ_i, there exists a pure strategy $\bar{s}_{-i} \in S_{-i}$ such that the PT-agent prefers $\hat{s}_i$ to σ_i given $\bar{s}_{-i}$.*

Proof : We have

$$\begin{aligned} V_i^{PT}(\sigma_i, \sigma_{-i}) &= \sum_{s_{-i} \in S_{-i}} \omega_i(\sigma_{-i}(s_{-i})) \sum_{s_i \in S_i} \sigma_i(s_i) \cdot v_i(x_i(s_i, s_{-i}), r_i) \\ &< \sum_{s_{-i} \in S_{-i}} \omega_i(\sigma_{-i}(s_{-i})) \cdot v_i(x_i(\hat{s}_i, s_{-i}), r_i) \ = V_i^{PT}(\hat{s}_i, \sigma_{-i}) \ . \end{aligned}$$

As $\omega_i(\sigma_{-i}(s_{-i})) \geq 0 \ \forall\ s_{-i} \in S_{-i}$, there exists some $\tilde{s}_{-i} \in S_{-i}$ with $\sigma_{-i}(\tilde{s}_{-i}) > 0$ and $EU_i(\sigma_i, \tilde{s}_{-i}) = \sum_{s_i \in S_i} \sigma_i(s_i) \cdot v_i(x_i(s_i, \tilde{s}_{-i}), r_i) < v_i(x_i(\hat{s}_i, \tilde{s}_{-i}), r_i) = EU_i(\hat{s}_i, \tilde{s}_{-i})$.
Analogously, if

$$\begin{aligned} EU_i(\sigma_i, \sigma_{-i}) &= \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) \sum_{s_i \in S_i} \sigma_i(s_i) \cdot v_i(x_i(s_i, s_{-i}), r_i) \\ &< \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) \cdot v_i(x_i(\hat{s}_i, s_{-i}), r_i) = EU_i(\hat{s}_i, \sigma_{-i}) \end{aligned}$$

then there exists some pure strategy $\bar{s}_{-i} \in S_{-i}$ such that $V_i^{PT}(\sigma_i, \bar{s}_{-i}) = \sum_{s_i \in S_i} \sigma_i(s_i) \cdot v_i(x_i(s_i, \bar{s}_{-i}), r_i) < v_i(x_i(\hat{s}_i, \bar{s}_{-i}), r_i) = V_i^{PT}(\hat{s}_i, \bar{s}_{-i})$. $\square$

How does strict dominance of strategies relate to stochastic dominance of a lottery? As strict dominance of strategies is a dominance relation restricted to lotteries that are induced by the strategy choice in a monetary game it implies 'state-wise dominance' which is stronger than stochastic dominance. Hence strict dominance implies stochastic dominance but not vice versa.

We turn now to the analysis of the choices of CPT-agents. Let us firstly consider two pure strategies $s_i, \hat{s}_i \in S_i$ for which s_i strictly dominates $\hat{s}_i$ in monetary terms, which implies by monotonicity of v_i that $v_i(x_i(s_i, s_{-i}), r_i) > v_i(x_i(\hat{s}_i, s_{-i}), r_i) \; \forall \; s_{-i} \in S_{-i}$. The argument why the CPT-agent i prefers s_i over $\hat{s}_i$ if at least one other player mixes is non-trivial because, given reference point $r_i \in \mathbb{R}$ and mixed strategy $\sigma_{-i} \in \Delta_{-i}$, the perceived probability $\psi_i(s_i, s_{-i}|r_i, \sigma_{-i})$ that (s_i, s_{-i}) occurs is rank dependent and does not need to coincide with the rank dependent perceived probability $\psi_i(\hat{s}_i, s_{-i}|r_i, \sigma_{-i})$ that $(\hat{s}_i, s_{-i})$ occurs. Firstly, we rewrite (5) to show that CPT-agents do not prefer first order stochastically dominated *perceived* lotteries. Secondly, we show that if a pure strategy strictly dominates another pure strategy and the game is not regular, then the perceived lotteries can be ordered via first order stochastic dominance.

Given a mixed strategy $\sigma \in \Delta$ use $\psi(\cdot|r_i, \sigma_{-i})$ from (4) to define the cumulative distribution function $F(\cdot|\sigma) : \mathbb{R} \to [0,1]$, where for $x \in \mathbb{R}$

$$F(x|\sigma) = \sum_{s \in S : x_i(s) \leq x} \sigma_i(s_i) \cdot \psi(s|r_i, \sigma_{-i}) \Big/ \sum_{s \in S} \sigma_i(s_i) \cdot \psi(s|r_i, \sigma_{-i}) \; .$$

We can now use (5) to derive

$$V_i^{CPT}(\sigma) = \left(\sum_{s \in S} \sigma_i(s_i) \cdot \psi(s|r_i, \sigma_{-i}) \right) \cdot \int v_i(x, r_i) dF(x|\sigma) \; . \tag{6}$$

Note that $\psi_i(s|r_i, \sigma_{-i})$ as defined in (4) does not need to sum up to one, if the monetary game is regular:

$$\Psi_i(s_i, \sigma_{-i}|r_i) := \sum_{s_{-i} \in S_{-i}} \psi_i(s_i, s_{-i}|r_i, \sigma_{-i}) =$$
$$\begin{cases} \omega_i\left(\sigma_{-i}(S_{-i})\right) - \omega_i\left(\sigma_{-i}(\emptyset)\right) = 1 & \text{if } x_i(s_i, s_{-i}) < r_i \; \forall \; s_{-i} \in S_{-i} \\ \omega_i\left(\sigma_{-i}(\{s_{-i} \in S_{-i} : x_i(s) < r_i\}\right) & \\ \quad +\omega_i\left(\sigma_{-i}(\{s_{-i} \in S_{-i} : x_i(s) \geq r_i\}\right) & \text{otherwise} \\ \omega_i\left(\sigma_{-i}(S_{-i})\right) - \omega_i\left(\sigma_{-i}(\emptyset)\right) = 1 & \text{if } x_i(s_i, s_{-i}) \geq r_i \; \forall \; s_{-i} \in S_{-i} \end{cases}$$

Note that if $\omega_i(\cdot)$ is defined according to (2), then $\omega_i(p) + \omega_i(1-p) \; < 1$ for all $\gamma < 1$ and $p \in (0,1)$. Nevertheless, for $x_i(s) < r_i \; \forall \; s \in S$ or $x_i(s) \geq r_i \; \forall \; s \in S$ we state the following corollary given that $v_i(x, r_i)$ is strictly increasing and finite in x for any $x, r_i \in \mathbb{R}$:

Corollary 2 (Bawa (1975), Theorem 1) *If the monetary game is a game of losses or a game of gains, a CPT-agent prefers s_i over $\hat{s}_i$ whenever $F(x|s_i, \sigma_{-i}) \leq F(x|\hat{s}_i, \sigma_{-i}) \ \forall\ x \in \mathbb{R}$ and $<$ for some $x \in \mathbb{R}$.*

The proof is the by now well known application of Theorem 1 in Bawa (1975) given that $\Psi_i(s_i, \sigma_{-i}|r_i)$ and $\Psi_i(\hat{s}_i, \sigma_{-i}|r_i)$ are equal to one. The corollary is extremely useful in the light of the next statement:

Proposition 6 (CPT: pure dominance in monetary games) *Suppose that either*

- $x_i(\hat{s}_i, s_{-i}) \geq r_i \ \forall\ s_{-i} \in S_{-i}$ *or*
- $x_i(\tilde{s}_i, s_{-i}) < r_i \forall\ s_{-i} \in S_{-i}$ *or*
- $\Psi(\hat{s}_i, \sigma_{-i}|r_i)$ *and* $\Psi(\tilde{s}_i, \sigma_{-i}|r_i)$ *are close to 1 for all* $\sigma_{-i} \in \Delta_{-i}$.

$\tilde{s}_i \in S_i$ weakly dominates $\hat{s}_i \in S_i$ with respect to $v_i(\cdot, r_i)$ given reference point $r_i \in \mathbb{R}$ if and only if $F(\cdot|\tilde{s}_i, \sigma_{-i})$ first order stochastically dominates $F(\cdot|\hat{s}_i, \sigma_{-i})$ for all $\sigma_{-i} \in \Delta_{-i}$.

Proof : For convenience define $S^R_{-i}(s_i, x) = \{s_{-i} \in S_{-i} : x_i(s_i, s_{-i}) R x\}$ for $R \in \{\leq, <, >, \geq\}$, $\underline{\hat{x}}_i = \min_{s_{-i} \in S_{-i}} x_i(\hat{s}_i, s_{-i})$, $\bar{\hat{x}}_i = \max_{s_{-i} \in S_{-i}} x_i(\hat{s}_i, s_{-i})$, $\underline{\tilde{x}}_i = \min_{s_{-i} \in S_{-i}} x_i(\tilde{s}_i, s_{-i})$ and $\bar{\tilde{x}}_i = \max_{s_{-i} \in S_{-i}} x_i(\tilde{s}_i, s_{-i})$. For $x \in \mathbb{R}$ and σ_i representing a pure strategy $s_i \in S_i$ we have

$$F(x|s_i, \sigma_{-i}) = \begin{cases} \omega_i\left(\sigma_{-i}\left(S^{\leq}_{-i}(s_i, x)\right)\right) \Big/ \Psi_i(s_i, \sigma_{-i}|r_i) & \text{if } x < r_i \\ 1 - \omega_i\left(\sigma_{-i}\left(S^{>}_{-i}(s_i, x)\right)\right) \Big/ \Psi_i(s_i, \sigma_{-i}|r_i) & \text{if } x \geq r_i \ . \end{cases}$$

Assume that $\hat{s}_i$ is weakly dominated by $\tilde{s}_i$ with respect to v_i. Then $v_i(x_i(\hat{s}_i, s_{-i}), r_i) \leq v_i(x_i(\tilde{s}_i, s_{-i}), r_i) \ \forall\ s_{-i} \in S_{-i}$ with a strict inequality for some s_{-i} and by monotonicity of v_i we have $x_i(\hat{s}_i, s_{-i}) \leq x_i(\tilde{s}_i, s_{-i}) \ \forall\ s_{-i} \in S_{-i}$ and a strict inequality for some s_{-i}. Hence $S^{\leq}_{-i}(\hat{s}_i, x) \supseteq S^{\leq}_{-i}(\tilde{s}_i, x)$ and $S^{>}_{-i}(\hat{s}_i, x) \subseteq S^{>}_{-i}(\tilde{s}_i, x) \ \forall\ x \in \mathbb{R}$ with strict inclusions for some x. Suppose now that $x_i(\hat{s}_i, s_{-i}) \geq r_i \ \forall\ s_{-i} \in S_{-i}$, which implies $x_i(\tilde{s}_i, s_{-i}) > r_i \ \forall s_{-i} \in S_{-i}$ or suppose $x_i(\tilde{s}_i, s_{-i}) < r_i \ \forall\ s_{-i} \in S_{-i}$ which implies $x_i(\hat{s}_i, s_{-i}) < r_i \ \forall\ s_{-i} \in S_{-i}$. In both cases $\Psi_i(s_i, \sigma_{-i}|r_i) = 1 \ \forall\ \sigma_{-i} \in \Delta_{-i}$ and $s_i \in \{\hat{s}_i, \tilde{s}_i\}$. If the first two assumptions of the proposition fail to hold then by the third assumption we have $\Psi_i(s_i, \sigma_{-i}|r_i) \approx 1 \ \forall\ \sigma_{-i} \in \Delta_{-i}$. With the monotonicity of ω_i it follows $F(x|\hat{s}_i, \sigma_{-i}) \geq F(x|\tilde{s}_i, \sigma_{-i}) \ \forall\ x \in \mathbb{R}, \sigma_{-i} \in \Delta_{-i}$ with strict inequalities for some x.

Assume that $F(\cdot|\tilde{s}_i, \sigma_{-i})$ first order stochastically dominates $F(\cdot|\hat{s}_i, \sigma_{-i})$ for $\sigma_{-i} \in \Delta_{-i}$. Then $F(x|\hat{s}_i, \sigma_{-i}) \geq F(x|\tilde{s}_i, \sigma_{-i})$ for all $\sigma_{-i} \in \Delta_{-i}$ and $x \in \mathbb{R}$ with a strict inequality for some $x \in \mathbb{R}$. We then have

$$\begin{aligned} \omega_i(\sigma_{-i}(S^{\leq}_{-i}(\hat{s}_i, x)))/\Psi(\hat{s}_i, \sigma_{-i}|r_i) &\geq \omega_i(\sigma_{-i}(S^{\leq}_{-i}(\tilde{s}_i, x)))/\Psi(\tilde{s}_i, \sigma_{-i}|r_i) \ \forall\ x < r_i \\ \omega_i(\sigma_{-i}(S^{>}_{-i}(\hat{s}_i, x)))/\Psi(\hat{s}_i, \sigma_{-i}|r_i) &\leq \omega_i(\sigma_{-i}(S^{>}_{-i}(\tilde{s}_i, x)))/\Psi(\tilde{s}_i, \sigma_{-i}|r_i) \ \forall\ x \geq r_i \ . \end{aligned}$$

Suppose now that $x_i(\hat{s}_i, s_{-i}) \geq r_i \ \forall\ s_{-i} \in S_{-i}$ (or $x_i(\tilde{s}_i, s_{-i}) < r_i \ \forall\ s_{-i} \in S_{-i}$) which implies that $\Psi(\hat{s}_i, \sigma_{-i}|r_i) = 1 \ \forall\ \sigma_{-i} \in \Delta_{-i}$ (or $\Psi(\tilde{s}_i, \sigma_{-i}|r_i) = 1 \ \forall\ \sigma_{-i} \in \Delta_{-i}$). If $\Psi(\tilde{s}_i, \sigma_{-i}|r_i) < 1$ (or $\Psi(\hat{s}_i, \sigma_{-i}|r_i) > 1$) for some σ_{-i}

then $\omega_i(\sigma_{-i}(S^{\leq}_{-i}(\hat{s}_i, x))) > \omega_i(\sigma_{-i}(S^{\leq}_{-i}(\tilde{s}_i, x))) \ \forall \ x < r_i$. But for x small enough $S^{\leq}_{-i}(\hat{s}_i, x) = S^{\leq}_{-i}(\tilde{s}_i, x) = S_{-i}$ and the inequality is violated. Hence $\Psi(\tilde{s}_i, \sigma_{-i}|r_i) \geq 1 \, \forall \, \sigma_{-i} \in \Delta_{-i}$ (or $\Psi(\hat{s}_i, \sigma_{-i}|r_i) \leq 1 \, \forall \, \sigma_{-i} \in \Delta_{-i}$). If $\Psi(\tilde{s}_i, \sigma_{-i}|r_i) > 1$ (or $\Psi(\hat{s}_i, \sigma_{-i}|r_i) < 1$) for some $\sigma_{-i} \in \Delta_{-i}$ then $\omega_i(\sigma_{-i}(S^{>}_{-i}(\hat{s}_i, x))) < \omega_i(\sigma_{-i}(S^{>}_{-i}(\tilde{s}_i, x)))$ for all $x \geq r_i$. Note that for any x large enough we have $S^{>}_{-i}(\hat{s}_i, x) = S^{>}_{-i}(\tilde{s}_i, x) = \emptyset$ and the strict inequality is violated. Hence $\Psi(\tilde{s}_i, \sigma_{-i}|r_i) = 1 \ \forall \ \sigma_{-i} \in \Delta_{-i}$ (or $\Psi(\hat{s}_i, \sigma_{-i}|r_i) = 1 \ \forall \ \sigma_{-i} \in \Delta_{-i}$). If the first two assumptions of the proposition fail to hold then by the third assumption we have $\Psi_i(s_i, \sigma_{-i}|r_i) \approx 1 \ \forall \ \sigma_{-i} \in \Delta_{-i}$. With $S^{>}_{-i}(s_i, x) = S_{-i} \setminus S^{\leq}_{-i}(s_i, x)$ we have $\sigma_{-i}(S^{>}_{-i}(s_i, x)) = 1 - \sigma_{-i}(S^{\leq}_{-i}(s_i, x))$ and by the strict monotonicity of ω_i we therefore have $\sigma_{-i}(S^{\leq}_{-i}(\hat{s}_i, x)) \geq \sigma_{-i}(S^{\leq}_{-i}(\tilde{s}_i, x)) \ \forall \ x \in \mathbb{R}, \sigma_{-i} \in \Delta_{-i}$ with a strict inequality for some x. Applying this inequality to the vertices of Δ_{-i} yields $S^{\leq}_{-i}(\tilde{s}_i, x) \subseteq S^{\leq}_{-i}(\hat{s}_i, x)$ for all $x \in \mathbb{R}$ with a strict inclusion for some x. This implies $x_i(\hat{s}_i, s_{-i}) \leq x_i(\tilde{s}_i, s_{-i}) \ \forall \ s_{-i} \in S_{-i}$ with a strict inequality for some s_{-i}. By monotonicity of v_i we have that $\hat{s}_i$ is weakly dominated by $\tilde{s}_i$ with respect to v_i. $\square$

Proposition 6 (together with corollary 2) gives us the clear result that any CPT-agent prefers pure strategy s_i over pure strategy $\hat{s}_i$ if s_i strictly dominates $\hat{s}_i$ in monetary terms, if the game is a game of losses or a game of gains, or if the perceived probabilities $\Psi_i(s_i, \sigma_{-i}|r_i)$ and $\Psi_i(\hat{s}_i, \sigma_{-i}|r_i)$ sum up sufficiently close to one.

In Proposition 2 we showed that if a mixed strategy σ_i strictly dominates a pure strategy $\hat{s}_i$ in terms of $v_i(\cdot, r_i)$, then PT-agents prefer σ_i over $\hat{s}_i$ for all mixed strategies σ_{-i}. In Proposition 4 we showed the stronger result that if a mixed strategy σ_i strictly dominates a pure strategy $\hat{s}_i$ in monetary terms, then a PT-agent who exhibits sufficiently few risk aversion prefers σ_i over $\hat{s}_i$. Surprisingly, not even the weaker of the two results holds for CPT-agents! See Example 8 in the appendix.

This counterexample reveals that the assumption of rank dependent probability weighting may imply irrational choices when it comes to mixed domination in strategic games. In the light of this example and propositions 2 and 4 it seems that PT has a conceptual advantage over CPT in the study of strategic interactions. The arguments against CPT become even stronger with Example 9 in the appendix where a strategy is dominated by a pure strategy but for some mixed strategies of the opponent, the CPT-agents prefers the dominated strategy.

4 Conclusions

We analyze decisions of agents who use Prospect Theory or Cumulative Prospect Theory (Kahneman & Tversky 1979, Tversky & Kahneman 1992) when they face strategic interaction. These agents differ from traditional expected utility maximizers with respect to two dimensions of irrationality: Firstly, they are risk averse in gains and risk loving in losses. Secondly, those agents overestimate small probabilities and underestimate large probabilities.

(Cumulative) Prospect Theory describes behavior under uncertainty which usually is modeled as a choice among various exogenous lotteries. In our strategic setting – normal form games – a lottery arises from the potentially mixed expectation on the choice of the strategic opponents. If not the choice itself, its expectation certainly fails to be independent of the own decision. Hence, when analyzing solution concepts of game theory we have to admit endogenous lotteries. In this setting, we analyze irrational choices in the sense of Expected Utility Theory and identify the effects caused by probability misestimation.

An immediate finding is that pure best replies are equivalent for EUT-and (C)PT-agents. This implies that purely dominant strategies or pure Nash Equilibria are invariant with respect to any monotone value function or probability weighting function. When it comes to mixed strategies, less properties carry over from Expected Utility Theory to (Cumulative) Prospect Theory. We give examples in which the set of best replies to some beliefs is not invariant with respect to the probability misestimation. While a dominated strategy is dominated for agents who maximize according to Prospect Theory in any case, this does not need to be the case for agents who apply Cumulative Prospect Theory, if the dominating strategy is mixed. This is a striking result, since previously CPT was thought to be the "mathematically" superior theory as compared to PT, since CPT does not violate first order stochastic dominance, but PT does. Our analysis demonstrates that, when only considering games, PT might be the mathematically preferable theory.

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A Examples

In the literature equilibrium existence has been shown under various necessary conditions. Examples 2, 3 and 4 illustrate that equilibrium existence can be sustained without these conditions. Example 2 violates concavity of probability weighting functions, as required in Ritzberger (1996). Example 3 violates quasi-convexity of utility functions as required in Chen & Neilson (1999). Example 4 violates convexity of the capacity implied by the probability weighting function as required by Eichberger & Kelsey (2000).

Example 2 *Consider the doubly symmetric 2×2-game with* $S_i = \{a, b\}$, $x_i(a, a) = x_i(b, b) = 0$, $x_i(a, b) = x_i(b, a) = 1$, *reference points* $r_i = 0$, $v_i(\cdot)$ *as defined in (1) and* $w_i(\cdot)$ *as defined in (2) for* $i = 1, 2$. *We have* $V_i^{CPT}(a, \sigma_{-i}) = w_i(\sigma_{-i}(b))$

and $V_i^{CPT}(b, \sigma_{-i}) = w_i(\sigma_{-i}(a))$*, hence* $\hat{\sigma}_i = \left(\frac{1}{2}, \frac{1}{2}\right)$ *is a CPT-equilibrium and* $w_i(\pi)$ *is locally convex at* $\pi = \frac{1}{2}$.

Example 3 *Player 2 may choose between left and right inducing a loss -3 and a gain -2. Given a mixed strategy* σ_2 *and weighting function* $\omega_1(\cdot)$ *the ex-ante utility is* $V_1^{CPT}(s_1, \sigma_2) = \omega_1(\sigma_2(L)) \cdot (-3) + \omega(\sigma_2(R)) \cdot (-2)$*. For* $\omega(\cdot)$ *defined as in (2) with* $\gamma = \frac{1}{2}$ *the reader can verify that* $V_1^{CPT}\left(s_1, \left(\frac{1}{2}, \frac{1}{2}\right)\right) = -\frac{5}{2\sqrt{2}} > -2 = V_1^{CPT}(s_1, R) > -3 = V_1^{CPT}(s_1, L)$.

Example 4 *Clearly, the function* $\omega_i \circ \sigma_{-i} : S_{-i} \to \mathbb{R}$ *is a capacity:* $\omega_i(\sigma_{-i}(\emptyset)) = 0$, $\omega_i(\sigma_{-i}(S_{-i})) = 1$ *and* $A \subseteq B \subseteq S_{-i} \Rightarrow \omega_i(\sigma_{-i}(A)) \leq \omega(\sigma_{-i}(B))$*. Consider two non-empty sets* $A, B \subset S_{-i}$, $A \cap B = \emptyset$ *with* $0 < \sigma_{-i}(A), \sigma_{-i}(B) < \bar{p} \leq \sigma_{-i}(A) + \sigma_{-i}(B)$*, where* $\bar{p} = \omega_i(\bar{p})$*. Non-convexity follows by* $\sigma_{-i}(A) + \sigma_{-i}(B) = \sigma_{-i}(A \cup B)$ *and* $\sigma_{-i}(A \cap B) = 0$*:*
$\omega_i(\sigma_{-i}(A)) + \omega_i(\sigma_{-i}(B)) > \sigma_{-i}(A) + \sigma_{-i}(B) > \omega_i(\sigma_{-i}(A \cup B)) + \omega_i(\sigma_{-i}(A \cap B))$.

The next example illustrates how the CPT-model operates. It is also used in Example 9.

Example 5 (rank dependent probability) *Tom chooses between* L *and* R *with the probabilities* $\sigma_2(L)$ *and* $\sigma_2(R)$ *and Sally chooses between* T*,* M *and* B*. The table below lists Sally's monetary payoffs, her reference point is* $r = 7$*:*

	L	R
T	5	5
M	8	7
B	6	7

The payoffs implied by M *induce the index function* l *with values* $l_1(M, R) = 1$ *and* $l_1(M, L) = 2$*. As both outcomes induced by* M *are* gains*, she calculates the probabilities that she receives* at least as much as *and* strictly more than *the respective gain. Her ex-ante utility derived from the choice* M *is given by*

$$\begin{aligned} V_1^{CPT}(M, \sigma_2, r) &= \left(\overbrace{\omega(\underbrace{\sigma_2(L) + \sigma_2(R)}_{prob\{l_1(M,s_2) \geq 1\}}) - \omega(\underbrace{\sigma_2(L)}_{prob\{l_1(M,s_2) > 1\}})}^{perceived\ prob\{l_1(M,s_2)=1\}} \right) \cdot v_1(7, r) \\ &+ \left(\overbrace{\omega(\underbrace{\sigma_2(L)}_{prob\{l_1(M,s_2) \geq 2\}}) - \omega(\underbrace{\sigma_2(\emptyset)}_{prob\{l_1(M,s_2) > 2\}})}^{perceived\ prob\{(l_1(M,s_2)=2\}} \right) \cdot v_1(8, r) . \end{aligned}$$

The index function l *which is associated with* T *may have the values* $l_1(T, L) = 1$ *and* $l_1(T, R) = 2$*. Sally considers these payoffs as* losses *and therefore calculates the probabilities that she receives* at most as much as *and* strictly less than *the respective loss. Her ex-ante utility induced by* T *is*

$$\begin{aligned} V_1^{CPT}(T, \sigma_2) &= \left(\overbrace{\omega(\underbrace{\sigma_2(L)}_{prob\{l_1(T,s_2) \leq 1\}}) - \omega(\underbrace{\sigma_2(\emptyset)}_{prob\{l_1(T,s_2) < 1\}})}^{perceived\ prob\{l_1(T,s_2)=1\}} \right) \cdot v_1(5, r) \\ &+ \left(\overbrace{\omega(\underbrace{\sigma_2(L) + \sigma_2(R)}_{prob\{l_1(T,s_2) \leq 2\}}) - \omega(\underbrace{\sigma_2(L)}_{prob\{l_1(T,s_2) < 2\}})}^{perceived\ prob\{l_1(T,s_2)=2\}} \right) \cdot v_1(5, r) \end{aligned}$$

.

The pair (B, L) induces a loss and the pair (B, R) induces a gain. The values of the associated index function are given by $l_1(B, L) = 1$ and $l_1(B, R) = 2$. Therefore, Sally's ex-ante utility is given by

$$\begin{aligned} V_1^{CPT}(B, \sigma_2) &= \left(\overbrace{\omega(\underbrace{\sigma_2(L)}_{prob\{l_1(B,s_2)\le 1\}}) - \omega(\underbrace{\sigma_2(\emptyset)}_{prob\{l_1(B,s_2)<1\}})}^{perceived\ prob\{l_1(B,s_2)=1\}} \right) \cdot v_1(6, r) \\ &+ \left(\overbrace{\omega(\underbrace{\sigma_2(R)}_{prob\{l_1(B,s_2)\ge 2\}}) - \omega(\underbrace{\sigma_2(\emptyset)}_{prob\{l_1(B,s_2)>2\}})}^{perceived\ prob\{l_1(B,s_2)=2\}} \right) v_1(7, r) \end{aligned}$$

.

Example 6 (non-existence with discontinuous reference point) *Consider two pure strategies s_1 and s_2 with associated monetary payoffs 1 and 2. Suppose that the reference point r is a function of the mixed strategy σ and specify r and v such that*

$$r(\sigma) = \begin{cases} 1 & \text{if } \sigma(s_1) > 0 \\ 2 & \text{if } \sigma(s_1) = 0 \end{cases} \qquad \text{and} \qquad v(x, r) = \begin{cases} \sqrt{x - r} & \text{if } x \ge r \\ -2\sqrt{r - x} & \text{if } x < r \end{cases}.$$

Then

$$V^{PT}(\sigma) = \begin{cases} 1 - \sigma(s_1) & \text{if } \sigma(s_1) > 0 \\ -2 & \text{if } \sigma_1 = 0 \end{cases}$$

and there does not exist a σ which maximizes V^{PT}.

Example 7 (PT: preference for dominated strategy) Consider the following regular monetary game ($r_1 = 1$, the payoffs of player 2 are not relevant), the value function as defined in (1) with $\alpha = \frac{1}{2}$ and any admissable parameters β, λ and probability weighting function as defined in section 2.3.

	L	R
T	3	0
M	1	1
B	0	3

The mixed strategy $\sigma_1 = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$ strictly dominates the pure strategy M but $V_1^{PT}(\sigma_1, \sigma_2) = \frac{1}{2} \cdot \sqrt{3} \cdot (\omega_1(\sigma_2(L)) + \omega_1(\sigma_2(R))) < \omega_1(\sigma_2(L)) + \omega_1(\sigma_2(R)) = V_1^{PT}(M, \sigma_2)$.

Example 8 (CPT: preference for dominated strategy continued) Consider the monetary game of Example 7 with $r_i = 0$ and either consider an almost risk neutral CPT-agent or suppose that the payoff matrix reflects valuations v_i rather than monetary payoffs x_i. Clearly, the mixed strategy $\sigma_1 = (\frac{1}{2}, 0, \frac{1}{2})$ strictly dominates the pure strategy $\hat{s}_1 = M$, that is $\frac{1}{2} \cdot \sigma_2(L) \cdot 3 + \frac{1}{2} \cdot \sigma_2(R) \cdot 3 = \frac{3}{2} > 1 \ \forall \ \sigma_2 \in \Delta_2$. The mixed strategy $\tilde{\sigma} \in \Delta$ induces the following cumulative distribution functions (see Figure 1):

$F(x|\tilde{\sigma}) =$

$$\begin{cases} 0 & \text{if } x < 0 \\ \tilde{\sigma}_1(T) \cdot (1 - \omega_1(\tilde{\sigma}_2(L))) + \tilde{\sigma}_1(B) \cdot (1 - \omega_1(\tilde{\sigma}_2(R))) & \text{if } 0 \leq x < 1 \\ 1 - \tilde{\sigma}_1(T) \cdot \omega_1(\tilde{\sigma}_2(L)) - \tilde{\sigma}_1(B) \cdot \omega_1(\tilde{\sigma}_2(R)) & \text{if } 1 \leq x < 3 \\ 1 & \text{if } 3 \leq x \end{cases}$$

As $\omega_1(\tilde{\sigma}_2(L)) + \omega_1(\tilde{\sigma}_2(R)) < 2$ for any $\tilde{\sigma}_2$ we have that $F(x|\hat{s}_1, \tilde{\sigma}_2)$ is not stochas-

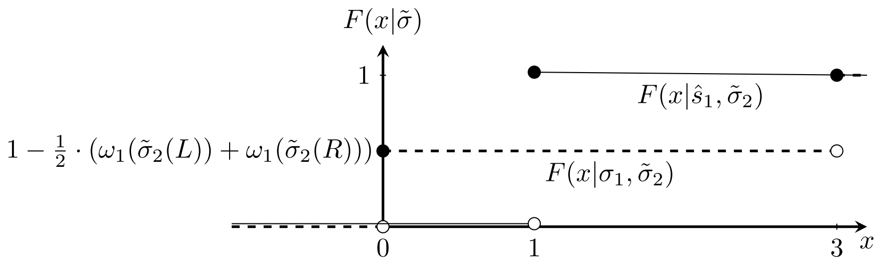

Figure 1: cumulative distribution functions induced by σ_i and $\hat{s}_i$.

tically dominated by $F(x|\sigma_1, \tilde{\sigma}_2)$ for any $\tilde{\sigma}_2 \in \Delta_2$ and we cannot apply corollary 2 to infer that the player prefers σ_1 over $\hat{s}_1$. In fact, the opposite can be true:

$$\begin{aligned} V_1^{CPT}(\sigma_1, \tilde{\sigma}_2) &= \frac{3}{2} \cdot (\omega_1(\tilde{\sigma}_2(L)) + \omega_1(\tilde{\sigma}_2(R))) \\ V_1^{CPT}(\hat{s}_1, \tilde{\sigma}_2) &= 1 \end{aligned}$$

Hence whenever $\omega_1(\tilde{\sigma}_2(L)) + \omega_1(\tilde{\sigma}_2(R)) < \frac{2}{3}$, player 1 prefers $\hat{s}_1$ over σ_1! If $\omega_1(\cdot)$ is defined according to (2) and $\tilde{\sigma}_2(L) = \tilde{\sigma}_2(R) = \frac{1}{2}$ and $\gamma = \frac{1}{3}$ then $\omega_1(\tilde{\sigma}_2(L)) + \omega_1(\tilde{\sigma}_2(R)) = \left(\frac{1}{2}\right)^{\frac{1}{3}} < \frac{2}{3}$. Note that this surprising result is not due to risk aversion but due the way in which CPT-agents rank probabilities.

Example 9 (CPT: pure domination in regular games) Suppose Sally from Example 5 has the weighting function $w(p) = \frac{\sqrt{p}}{\left(\sqrt{p}+\sqrt{1-p}\right)^2}$. Strategy B strictly dominates strategy T (in terms of $v(\cdot)$ and in monetary terms). With $v_1(x, r) = x$ we know from Example 5 that

$$\begin{aligned} V_1^{CPT}(T, \sigma_2) &= 5 \\ V_1^{CPT}(B, \sigma_2) &= \frac{\sqrt{\sigma_2(L)}}{\left(\sqrt{\sigma_2(L)} + \sqrt{\sigma_2(R)}\right)^2} \cdot 6 + \frac{\sqrt{\sigma_2(R)}}{\left(\sqrt{\sigma_2(L)} + \sqrt{\sigma_2(R)}\right)^2} \cdot 7 \end{aligned}$$

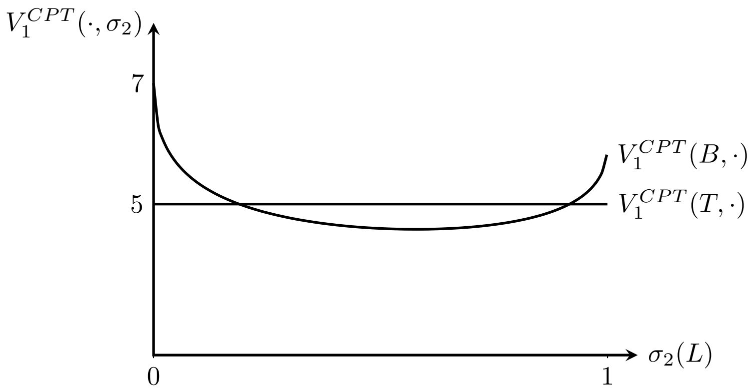

Figure 2: The ex-ante CPT utility of player 1

In particular, for $\sigma_2(L) = \sigma_2(R) = \frac{1}{2}$ we have $V^{CPT}\left(B, \left(\frac{1}{2}, \frac{1}{2}\right)\right) = \frac{13}{2\cdot\sqrt{2}} < 5$.

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