BARGAINING OVER HETEROGENEOUS GOOD WITH STRUCTURAL UNCERTAINTY

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Abstract. We study a war-of-attrition bargaining over a pie with heterogeneous parts, where players have incomplete information over their opponent preferences as well as their behavioral types. Each player demands that the opponent chooses from a menu of offers. The menu may consist of a single offer. If the bargaining position is exogenously fixed, we show that the equilibrium behavior can be simply characterized by comparing appropriately defined strengths of the two players. The equilibirum is unique with one-sided uncertainty, but not necessarily with two-sided uncertainty about preferences equilibria. We also consider the menu choice game prior to the bargaining. Being able to commit to a menu instead of a single-offer removes a certain first-mover disadvantage. When the preferences of one player are known, in equilibrium, this player proposes a menu of all allocations that give him a half of the pie; the opponent chooses optimally from such a menu.

1. Introduction

A typical bargaining situation involves some kind of uncertainty about the preferences of their opponents.[1] In this paper, we study bargaining over heterogeneous pie with a *structural uncertainty* about the relative value of the different elements of the pie. Such an uncertainty is a common feature of complicated negotiations over multiple issues at the same time. For instance, the UE officials likely began the Brexit talks without fully understanding the relative value for their British counterparts of the Irish border issue, the access to the common market, or fishing rights (if for no

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PRELIMINARY AND INCOMPLETE. And with typos.

[1]There is a large literature that studies various types of uncertainty in bargaining. Axiomatic solutions in general environment has been proposed in Harsanyi and Selten 1972 and Myerson 1984. The strategic literature either focuses on one-dimensional or two type cases, including the uncertainty about values (Gul et al. 1986), the discount factor and time preferences (Rubinstein 1985, Abreu et al. 2015), bargaining postures (Myerson 1991, Abreu and Gul 2000, Fanning 2016) among many others.

other reason than the British had yet to figure out their own preferences). An employer negotiating wage and/or employment reduction may not know which of those two is more acceptable for a labor union. On one hand, such an uncertainty typically conceals the exact value of the pie. In the same time, the information revealed during the bargaining process may help to find previously unexpected deals (Jackson et al. 2018). The negotiators may try to screen the opponents by offering menus of acceptable offers instead of a single proposal.[2]

There are many natural questions to ask in such an environment. Is the uncertainty advantageous in bargaining and, if so, for whom? Do the parties want to reveal their preferences, possibly in order to find a mutually beneficial deals? Is there any value of using menus instead of single offers? What is the outcome of the bargaining? Is it efficient? How does the behavior look like?

We offer partial answers to these questions. We show that, when preferences of one of the sides are known, the other side benefits from incomplete information. The players won't typically completely reveal their preferences. The player with known preferences strictly benefits from being able to commit to a menu. In equilibrium, he proposes a menu of all allocation that give him the payoff of at least $\frac{1}{2}$ of the whole pie. The opponent chooses her optimal allocation from the menu, which typically leaves her with a payoff more than $\frac{1}{2}$, and, sometimes, more than the complete information Nash outcome. The outcome is *ex post* efficient. We discuss the behavior below

We analyze two games. In both games, the players want to divide an heterogeneous (i.e., N-dimensional with $N \geq 2$) pie. In the war-of-attrition bargaining game, two players begin with exogenously given bargaining demands. The demands take a form of

[2]The author of this study participated in 2017-18 in the bargaining over a pension plan reform between three Ontario universities and faculty associations and labor unions representing staff. In one of the stages of the process, the parties negotiated the benefits of a new pension plan. Among others, the parties needed to decide the scope of the spousal benefit, early retirement options, inflation indexation, etc. It was understood that the universities care only about the total actuarial cost. The preferences of the labor side were uncertain, mostly due to the heterogeneity of the labor side (for instance, the staff valued the early retirement more than the spousal benefit; the faculty preferences were reversed). The negotiations were preceded by months of meetings and consultations. The bargaining itself was very fast and it took a weekend in a hotel in downtown Toronto. The university proposed a menu of options that were acceptable to them. The labor side chose an option from the menu.

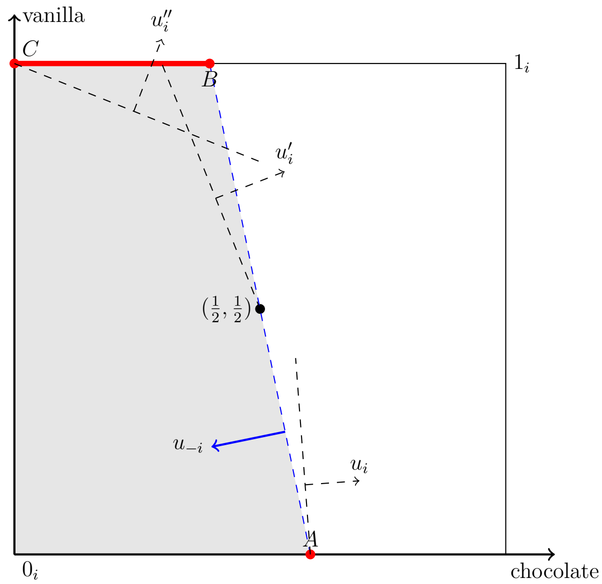


FIGURE 1.1. Nash outcomes and menu $m^{1/2}$.

a menu of allocations. Each player chooses when to concede. When a player concedes first, she chooses an allocation from the menu demanded by her opponent. The players face uncertainty about preferences over multi-dimensional allocations, as well as the behavioral type of the opponent. The behavioral type never concedes. We are interested in the case, where the probability of the behavioral type is always positive, but very small. We consider both one-sided and two-sided incomplete information about preferences, but our results are stronger in the former case. Second, in the menu choice, the players sequentially announce their demands and learn their behavioral type before the war of attrition commences. Importantly, once chosen, the players do not have an opportunity to revise their demands. We believe that this is a reasonable assumption in situations when the object of bargaining is very complicated, preparing an offer takes significant resources (time, lawyers, internal negotiations), and the bargaining process itself is fast.

When the both player preferences are known, the dynamics are very similar toAbreu and Gul (2000) (from now on, AG). AG defines a strength of a player as a ratio of the payoff from winning (i.e., the payoff received when the opponent concedes) and the concession payoff. In the limit equilibrium, the stronger type concedes faster throughout the game, and to make up for it, the weaker player must concede with a large probability (that is arbitrarily close to 1 in the rational limit) in the initial period of the game. In the menu choice game, the players choose and accept the Nash outcome. See Figure 1.1 for the illustration of Nash outcomes when the players divide a pie with $N = 2$ parts: chocolate and vanilla. Player $-i$ (with preferences u_{-i}) prefers chocolate, but he also likes vanilla. The Nash outcome depends on the preferences of player i. If i prefers chocolate more than $-i$ does (preference u_i), the Nash outcome is the allocation A, which gives payoff $\frac{1}{2}$ to $-i$, and, more than $\frac{1}{2}$ to i. If i prefers chocolate more than vanilla but she likes chocolate less than than $-i$ (preference u_i'), then, the outcome is an allocation between B and C that gives her the payoff of $\frac{1}{2}$. Finally, if i prefers vanilla, then both players receive their favorite part. The minimum Nash payoff of player i is $\frac{1}{2}$.

When preferences are not known, and each player demand consists of a single offer, we can define a strength of a player's type as a ratio between the winning and the concession payoffs. The strength will typically vary across types. In equilibrium, the weaker types concede first. We also define the player's strength as the strength of the strongest type in the support of the type distribution. Generically, it is one of the extreme types; under the full support, it is either the type who only likes chocolate, or only vanilla. When the probability of the behavioral type converges to 0, we show that the weaker player concedes with a probability arbitrarily close to 1, in one of the initial periods of the game. The argument relies on the AG logic combined with the following observation: most of the time before the final concession of strategic types is spent in the *late game*, where all types are close to the strongest types. The concession rates during the late game are determined by the strongest types; any concession behavior before that is swamped by the length of the late game.

With single-offer demands, the menu choice game equilibrium payoff may fall below the minimum Nash payoff of $\frac{1}{2}$. To see it, recall that the Nash outcome depends on the preferences of the opponent, in particular, whether the opponent likes chocolate more or less than the player. The first mover cannot avoid the possibility of committing to an offer that is very unattractive to some of the types of the second player; against whom, she would be very weak. This disadvantage would disappear if the first mover was able to offer a menu of Nash outcomes.

When the players can demand general menus, and the preferences of one of the players, say $-i$, are known (we refer to such a player as *uninformed*), we can still define the strength of his opponent, player i, as the ratio of winning to concession payoffs. Contrary to the single-offer case, the strongest type of player i is typically in the interior of the support of the type distribution. The strength of player $-i$ is not well-defined as the "winning payoff" depends on the allocation chosen by the conceding type of the opponent. However, in the late game, all the remaining types of informed player are close to her strongest type. Hence, we define the strength of player $-i$ as if she faces the stronger type of player i. We show that with so-defined strengths, the weaker player concedes early with a probability that is arbitrarily close to 1.

The last result allows us to show that, under the full support distribution of types of player i, the equilibrium outcomes of the menu choice game are as if player $-i$ proposes a menu of all allocations that ensure him a payoff of at least $\frac{1}{2}$, and the opponent chooses optimally from such a menu. On Figure 1.1, the equilibrium menu $m^{1/2}$is depicted with a gray color; a generic type of the opponent chooses one of the allocations A or B. The equilibrium payoffs are *ex post* efficient. Player $-i$ receives his worst and each of his opponent types receives her best payoff across all possible Nash outcomes. In the proof, we show that if player i chooses any Nash outcome (including A or B) as its demand, then, unless player $-i$ proposes a menu that includes such an outcome, she becomes a stronger player, and which ensures her the winning payoff.

With two-sided incomplete information about preferences, the winning payoffs of both players and any notion of strength depend on the concession strategy of the opponent. It follows that there is no natural exogenous sorting. In fact, we show on an example types, that the war-of-attrition game can have multiple equilibria. The example has two types, with two types conceding in a different order in each equilibrium.

On the other hand, we show that when $N = 2$ and each player demands the opponent to choose from a linear menu, the preferences have a continuous density, and that they are sufficiently separate (i.e., each of the types prefers winning that conceding, regardless of the opponent's choice), there is an unique equilibrium. We define the strength of a player as the winning/concession ratio under the restriction that, when conceding, the player must choose an allocation that belongs to the diagonal. Because of linearity of preferences, so defined strength does not depend on the player's type. In the equilibrium, the weaker player concedes in early periods of the game with a probability arbitrarily close to 1. The proof relies on the fact that with linear menus, we have a partial sorting of types that make the same concession choice.

AG study a generalized protocol of alternating offer bargaining (Rubinstein 1982) with a two-sided possibility of behavioral types. The behavior in the game looks like the war of attrition that ends when one of the player reveals herself to be rational. When that happens, an earlier result by Myerson 1991 shows that the revealed player will concede fast in any equilibrium. A closer model is Kambe 1999, where players learn their commitment type after the initially chosen menu and the strategic types are not able to revise their offers upwards. The main difference with our model is that we assume that players cannot revise their offers. In particular, the players must concede when they reveal themselves to be rational. The assumption seems appropriate for situations when the object of bargaining is very complicated, preparing an offer takes significant resources (time, lawyers, internal negotiations), and the bargaining process itself is fast. It also allows us to focus on the new aspects of the model, the structural uncertainty, and menus, and how they affect the well-known dynamics of AG. We do not know whether a version of Myerson's result holds in our context. We discuss some of the difficulties in the paper.

The solution to the one-sided case is reminiscent of the Coasean bargaining literature that originated with Gul et al. 1986. More specifically, Strulovici 2017 considers bargaining in multi-dimensional environment where only one sides makes offers and any accepted offer becomes a status quo for future bargaining. He shows that the uninformed player is unable to offer an inefficient payoff to type u_1' in order to screen out the more extreme type u_1''. The argument relies on the Coasean dynamics of frequent offer revisions. Instead, our result is reputational, and it relies on the comparison of the commitment strengths across players and types.

TBA. The role of menus in bargaining Wang 1998, Sen 2000, Inderst 2003, Yildiz 2003. Multi-issue legislative bargaining.

Section 2 describes the model. Section 3 discusses the case of singleton menus. Section 4 is devoted to the one-sided incomplete information. Two-sided uncertainty is discussed in Section 5 The last section conludes and discusses some open questions.

2. Model

2.1. **War-of-attrition bargaining.** Two players, $i = 1, 2$, bargain over a heterogeneous pie with $N \geq 2$ parts. Depending on the context, we refer to player i, j, or player 1 using the female pronoun, and to player $-i, -j$,or 2 using the male pronoun.

An allocation is defined as $x \in X := [0,1]^N$. Each player has a linear preference over allocations $u_i \in \mathcal{U} := \left\{u \in \mathbb{R}_+^N : \sum u_n = 1\right\}$. (The normalization is w.l.o.g.) The payoff from allocation x is equal to $u_i \cdot x$.

The bargaining takes form of a war of attrition. In alternating periods (starting with player 1 in period 1), player i either continues or concedes. If she continues, the game moves to the next period and the other player. If she concedes, she must choose an allocation x from a (closed) menu of allocations $m_{-i} \subseteq X$, in which case the other player receives the complementary allocation $\mathbf{1} - x = (1 - x_n)_n$. We refer to m_{-i} as the bargaining position of player $-i$.

The i's preference is drawn from distribution $\pi_i \in \Delta\mathcal{U}$, and it is known to player i but not her opponent. Additionally, and independently from the type distribution, each player is either strategic with probability $1 - \lambda$ or behavioral with a strictly positive probability $\lambda \in (0,1)$. The behavioral player never concedes. The role of the behavioral types is to pin down the equilibrium; it is well known that, without them, the war-of-attrition games have a continuum of equilibria. The players are expected utility maximizers. They discount future with a common discount factor $\mathrm{e}^{-\Delta}$, where Δ represents the length between two subsequent decision points, and the interest rate is normalized to 1.

Let T_i be the set of periods in which player i makes decision in the war-of-attrition. A strategy of the (strategic type of) player i is a pair $\sigma_i = \left(\sigma_i^T, \sigma_i^M\right)$ of measurable stopping time $\sigma_i^T : \mathcal{U} \to \Delta T_i$ and a choice $\sigma_i^M : \mathcal{U} \to \Delta m_{-i}$. A belief of player $-i$ is a pair of mappings $\lambda_i : T_i \to [0,1]$ and $\mu_i : T_i \to \Delta A_i$, with the interpretation that $\lambda_i(t)$ is the probability at the beginning of the period that player i is behavioral, and $\mu_i(.|t)$ is the probability distribution over the (strategic) types of player i who yield in period $t \in T_i$. Let $U_i^\sigma(u_i)$ denote the expected payoff of player i type $u_i \in \mathcal{U}_i$.

A *(Perfect Bayesian) equilibrium* is a profile of (mixed) strategies and beliefs such that (a) players best respond at each point in time and (b) beliefs are updated through Bayes formula whenever possible. It is easy to see that if $\lambda > 0$, each equilibrium is sequential, and, in fact, the specification of beliefs at 0 probability events does not matter. We are interested in the equilibrium payoffs as the game approximates continuous time, $\Delta \to 0$, and players become fully rational, $\lambda \to 0$.

2.2. **Menu choice.** In a menu choice game, players choose their bargaining positions. We assume that each player has only partial information about their preference type before they make the choice of bargaining position. Formally, each player i privately observes a signal s_i drawn from a distribution $\rho_i \in \Delta\mathcal{U}$. Next, player 1 followed by player 2 announce their bargaining positions. Player i chooses m_i from a finite set $M_{-i} \subseteq \mathcal{C}_X$, where $\mathcal{C}_X$ is a collection of all closed subsets of X that contain at least one strictly positive allocation. After the bargaining positions are chosen, player i learns her preference type drawn from distribution $\pi_i(s) \in \Delta\mathcal{U}$. At the same time, independently, the player learns with probability λ that she is the behavioral type (see Kambe 1999 for a similar approach to the behavioral types). Finally, the war of attrition game commences.

Assumption 1. *For each player i, there exists a closed subset $\mathcal{U}_i \subseteq \mathcal{U}$ such that for each s_i, $\mathcal{U}_i = supp\pi_i(s_i)$.*

The assumption ensures the support of the posterior beliefs at the beginning of the war-of-attrition does not depend on the chosen menu. We say that the preferences of player i are known if $\mathcal{U}_i$ is a singleton.

A strategy in the menu choice game is a measurable mapping $m_i : \mathcal{U} \to M_i$. After the menu choices, players form beliefs $\mu_1 : M_1 \to \Delta\mathcal{U}, \mu_2 : M_1 \times M_2 \to \Delta\mathcal{U}$ about signals and use them, together with π_i, to form beliefs about preferences. A Perfect Bayesian equilibrium is a profile of menu choice strategies, beliefs, as well as a continuation PBE in the war-of-attrition such that players best respond to each other and the beliefs are derived from strategies through the Bayes formula whenever possible. The definition and the Assumption 1 ensure that the beliefs have a full support at the beginning of the war of attrition.

We are interested in two limits:

- the finite set of menus M_i approximates certain (closed) sub-collection of menus $M_i^* \subseteq \mathcal{C}_X$, and we write $M_i \to M_i^*$, where the convergense is in the sense of the induced Hausdorff distance.
- the initial information becomes approximately perfect: $\pi(s) \to \delta_s$weakly, for each $s \in \mathcal{U}_i$. We write $\pi \to \delta$.

3. Singleton menus

In this section, we consider the special case of the model with each menu being a singleton.

3.1. **War of attrition.** Suppose that $m_i = \{x_{-i}\}$ for each player i. Because the players payoff from winning or losing the war of attrition are fixed, the strength ratio does neither depend on time nor the strategies of the opponent. To avoid dealing with trivial cases, we assume that $x_{i,n} > 0$ for each i, n.

For each type $u_i \in \mathcal{U}_i$, define

$$S_i(u_i) = \frac{u_i \cdot (\mathbf{1} - x_{-i})}{u_i \cdot x_i},$$

We refer to $S_i(u_i)$ as the strength of type u_i of player i. As in AG, the strength is equal to the ratio of the payoff from winning the war of attrition (and getting the allocation $\mathbf{1} - x_{-i}$) and the concession payoff (i.e., allocation x_i). Let the strength of player i be defined as

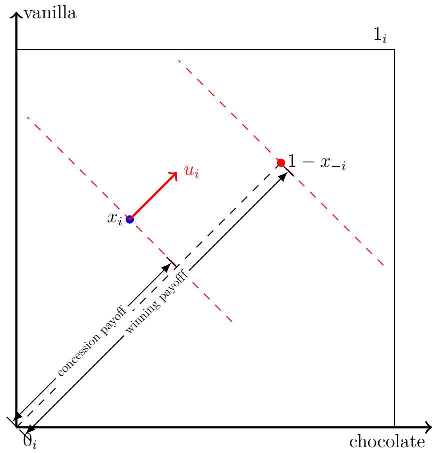
$$S_i^* = \max_{u_i \in \mathcal{U}_i} S_i(u_i),$$

i.e., the strength of the strongest type in the support of the belief distribution. Figure 3.1a illustrates a geometric interpretation of the strength of a type as the ratio of the length of the ray that connects allocations $\mathbf{0}_i$ and $\mathbf{1} - x_{-i}$ to the distance between the allocation $\mathbf{0}_i$ and the indifference curve that passes through x_i along the ray. By rotating the indifference curves, we can see that the strongest type is typically one of the most extreme types of the support.

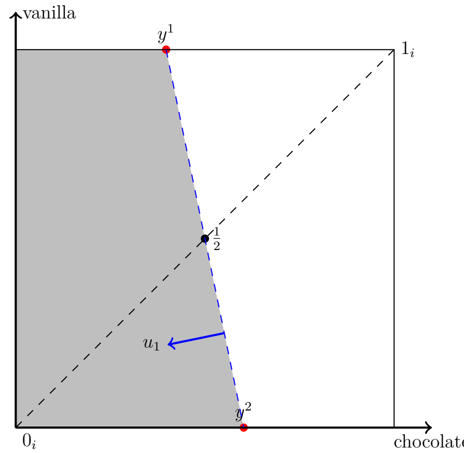
Theorem 1. *Suppose that $S_i^* > S_{-i}^*$, i.e., player i is stronger. For each $\delta > 0$, there exist $\lambda^*, \Delta^* > 0$ such that if $\lambda \leq \lambda^*$ and $\Delta \leq \Delta^*$, then there is $T < \infty$ such that $\mathrm{e}^{-\Delta T} > 1 - \delta$ and, in any equilibrium, player $-i$ concedes with probability at least $1 - \delta$ before the end of period T.*

By the Theorem 1, the weaker type concedes with a probability close to one, in one of the initial stages of the game. The limit equilibrium behavior depends on the comparison between the strengths of the strongest types of each player. In particular, the behavior depends only the support of the type distribution.

We use Theorem 1 to derive the limit payoffs. If the stronger player moves first, i.e., $i = 1$, then any of her types u_i can ensure the maximum of her concession or



(A) Single offers and the strength of the type.



(B) Single offers and the strength of the type.

winning payoffs (multiplied by $\mathrm{e}^{-\Delta T}$), either by conceding immediately, or wating for T periods. Because player i cannot get any higher payoff then any of these two, the limit of payoffs of player i is equal to

$$U_i\left(u_i\right) \to \max\left(u_i \cdot x_i, u_i \cdot \left(\mathbf{1} - x_{-i}\right)\right).$$

Player $-i$'s payoff depends on whether i concedes immediately (which happens with probability converging to $\pi^C = \pi_i\left(\left\{u_i : S_i\left(u_i\right) \leq 1\right\}\right)$) or waits. The asymptotic limit is equal to

$$U_{-i}\left(u_{-i}\right) \to \pi^C\left(u_{-i} \cdot x_{-i}\right) + \left(1 - \pi^C\right) u_{-i} \cdot \left(\mathbf{1} - x_{-i}\right).$$

If the stronger player moves second, $i = 2$, then the above derivation is additionally complicated by the fact that some types of player $-i$ may prefer to concede immediately rather than wait for the possible concession of player i.

3.1.1. *Proof intuition.* The proof relies on the familiar logic of the AG model as well as the equilibrium sorting property (a closely related argument can be found in Abreu et al. 2015). Let $p_{-j}\left(t\right)$ be the concession rate, i.e., the probability of conceding in period $t \in T_{-j}$ conditionally on reach t. The gain for player j type u_j from waiting from period $t-1$ to t is equal to

$$\begin{aligned}&\mathrm{e}^{-\Delta}p_{-j}\left(t\right)\left(u_j\cdot\left(\mathbf{1}-x_j\right)\right)+\mathrm{e}^{-2\Delta}\left(1-p_{-j}\left(t\right)\right)u_j\cdot x_j-u_j\cdot x_j\\=&u_j\cdot x_j\left[\mathrm{e}^{-\Delta}p_{-j}\left(t\right)\left(S_j\left(u_j\right)-\mathrm{e}^{-\Delta}\right)-\left(1-\mathrm{e}^{-2\Delta}\right)\right].\end{aligned}\tag{3.1}$$

Because the gain is increasing in strength, it must be that in equilibrium, the weaker types concede before the stronger ones. Let $S_j\left(t\right)$ be the strength of the strongest type who has not conceded before and including period t. In equilibrium, the concession rate $p_{-j}\left(t\right)$ must be such that the strongest type $S_j\left(t\right)$ is indifferent between conceding now and waiting till the next opportunity.

Next, for each $\eta>0$, let $T_i^{\eta}=\min\left\{t:S_i\left(t\right)\geq S_i^*-\eta\right\}$ be the first period after which all the types of player i are η-close to the strongest type. We refer to the time after $T^{\eta}=\max_j T_j^{\eta}$ as the *late game*. In the late game, each player i concedes at the rate that is approximately constant and equal to

$$p_j\left(t\right)\approx\frac{1}{S_{-j}^*-1}2\Delta=\gamma_j^*2\Delta.$$

As in AG, the stronger player i concedes faster in the late game. It follows that it must be that $T^{\eta}=T_i^{\eta}$, and by the standard arguments, the mass of types that survives till the beginning of the late game is approximately equal to

$$\pi_i\left(u_i:S_i\left(u_i\right)\geq S_i^*-\eta\right)\simeq\lambda\mathrm{e}^{\gamma_i^*\left(T^{\varepsilon}-T^*\right)\Delta}.$$

We have two observations. First, as $\lambda\to0$, the late game becomes arbitrarily long, i.e., $\left(T^*-T^{\eta}\right)\Delta$ is arbitrarily large. Second, the mass of types of player $-i$ that do not concede before the beginning of the game is approximately equal to

$$\lambda\mathrm{e}^{\gamma_{-i}^*\left(T^{\varepsilon}-T^*\right)\Delta}\simeq\mathrm{e}^{\left(\gamma_i^*-\gamma_{-i}^*\right)\left(T^{\varepsilon}-T^*\right)\Delta}\pi_i\left(u_i:S_i\left(u_i\right)\geq S_i^*-\eta\right).$$

In particular, the ratio of the prior probability of types of player i that concede in the late game to the analoguous probability of types of player $-i$ is arbitrarily large.

Finally, we bound the concession rates of the two players before the beginning of the late game: a lower bound for player i, and an upper bound for player $-i$. The bounds show that the ratio of no-concession probabilities in the initial periods of the game (after, perhaps, T periods) remains arbitrarily high as $\lambda\to0$. To make up for the missing probability, player $-i$ must concede in the first T periods of the game with a probability that is arbitrarily close to 1. The formal argument proof is complicated by the need to deal with the possibility that some types of the stronger player i may prefer the allocation x_{-i} rather than the x_i, in which case they are going to try to concede as early as possible. The details of the proof can be found in Appendix B.1.

3.2. **Menu choice.** Although the characterization of Lemma 1 is straightforward, the general solution to the menu choice game is complicated by the following difficulty. Contrary to the properties of the AG model, given a choice of player 1, player 2 may strictly prefer to choose an allocation that makes him weaker in the continuation war of attrition. In particular, it is not enough to consider the allocations that make player 2 stronger when finding his best response.

To see a simple example, suppose that $N = 2$ and player 1's choice is $m_i = \left\{\mathbf{\frac{1}{2}}\right\}$, where $\mathbf{\frac{1}{2}} = \left(\frac{1}{2}, \frac{1}{2}\right)$. It is possible to show that if both players types have full support, $\mathcal{U}_i = \mathcal{U}$,then, the choice of player 2 $m_2 = \{x_1\}$ makes him (weakly) stronger in the continuation game iff $\mathbf{1} - x_1 \leq \mathbf{\frac{1}{2}}$ (in the vector order sense). The best payoff that player 2 can ensure by chooing such an allocation is not higher than $\frac{1}{2}$. However, generically, player 2 can do better. Suppose that player 2 prefers the first part of the pie, $u_2^{(1)} > \frac{1}{2} > u_2^{(2)}$. Consider an offer $m_2' = \{x_1'\}$ where $x_1' = (0, 1)$, i.e., player 2 offers the less attractive part to player 1. There are two possible outcomes of the war of attrition. If player 1 prefers the first part of the pie, player 2 is going to be forced to concede and accept the payoff $\frac{1}{2}$. But, if player 1 prefers to second part of the pie, she will concede early, in which case player 2 receives a payoff strictly higher than $\frac{1}{2}$. Notice that in the second outcome, the losing allocation x_1' is a Pareto-improvement over the equal division of the pie.

Although the solution to the menu choice game is possible to obtain, we do not find it illuminating. Instead, we show that when player 1's preferences are known, her equilibrium payoff might be strictly lower than $\frac{1}{2}$, which, as it is easy to check, is lower than her worst payoff if the type of the opponent is known. To keep the argument simple, assume that $N = 2$.

Theorem 2. *Suppose that $\mathcal{U}_1 = \{u_1\}$ and the preferences of player 2 have full support, $supp \rho = \mathcal{U}_2 = \mathcal{U}$. Let M^* be the collection of all single-element menus. Let $E_1\left(\Delta, \lambda, (M_i)_i, (\mathcal{U}_2, \pi_2)\right)$ be the set of equilibrium payoffs of type u_i of player i. Then,*

$$\lim sup_{\pi_2 \to \delta_2, M_2 \to M^*} \lim \sup_{\lambda, \Delta \to 0} \sup E_1\left(\Delta, \lambda, (M_i), (\mathcal{U}_2, \pi_2)\right) < \frac{1}{2}. \tag{3.2}$$

The idea is very simple. Because the preferences of player 1 are known, for each offer $m_1 = \{x_2\}$ of player 1, player 2 can find a counter-offer that Pareto improves on x_2, i.e., it gives the same payoff to player 1, but a strictly higher payoff to player 2. For example, if x_2 is too stingy on the side $k = 1, 2$ of the pie, the types of player 2 who like the side k can counter-offer with an allocation that lies on the same 1's indifference curve as $\mathbf{1} - x_2$ and give a larger part of k to player 2. Any such a counter-offer makes player 2 strictly stronger and, more importantly, give him a strictly higher payoff what is offered by player 1. In fact, player 2 can further increase her payoff by choosing an allocation with a payoff to player 1 that is below $\frac{1}{2}$.

In short, player 1's problem is that he cannot simultaneously choose an allocation that is satisfactory for different types of player 2. Such a problem would be solved if player 1 could offer a menu of options for player 2. We discuss this in the next section.

4. One-sided incomplete information

In this section, we assume that player $-i$'s preferences are known, $\mathcal{U}_{-i} = \left\{u^*_{-i}\right\}$. We also make the following addition to Assumption 1.

Assumption 2. *Set $\mathcal{U}_i$ is convex, $u^*_{-i} \in \mathcal{U}_i$, and and $\pi_i(s)$ has a Lipschitz continuous density with respect to the Lebesgue measure on $\mathcal{U}_i$.*[3]

4.1. **War of attrition.** As a first approximation, it is convenient to assume that the menu from which the uninformed player $-i$ chooses consists of a single element, $m_i = \{x_{-i}\}$. The assumption is not without a loss of generality, because even if player $-i$ is indifferent between multiple allocations in menu m_i, player i might not be, and the different choices may have different consequences for player i's behavior.

Given the assumption, we define the strength of the types u_i of player i as the ratio of the payoff from winning versus the payoff from losing:

$$S_i(u_i) = \frac{u_i \cdot (\mathbf{1} - x_{-i})}{\max_{x \in m_{-i}} u_i \cdot x} \text{ and } S_i^* = \max_{u_i \in \mathcal{U}_i} S_i(u_i).$$

As in the singleton menu case, player i's strength is defined as the stregth of the strongest type in the support. It is useful to describe a geometric intuition how to

[3]For each convex subset $U \subseteq \mathbb{R}^N$, one can find its affine hull, i.e. the intersection of all affine spaces that contain U. The Lebesgue measure on the affine hull assigns positive mass to U. Whenever we mention "the Lebesgue measure on U", we refer to the restriction of such a measure to set U.

find the strongest type. See the left side of Figure 4.1. The dashed ray connects allocations $\mathbf{0}$ and the winning allocation $\mathbf{1}-x_{-i}$. The winning/loss strength ratio is equal to the ratio of the length of the ray and the distance between allocations $\mathbf{0}$ and the intersection of player i's losing indifference line with the ray. By moving the indifference curve around the menu, we can see that the ratiop is maximized when the indifference curve touches the menu exactly at the ray. More formally, let

$$m_{-i}^{*}=\bigcap_{u_i\in\mathcal{U}_{-i}}\left\{x:u_i\cdot x\leq\max_{x'\in m_{-i}}u_i\cdot x\right\} \tag{4.1}$$

be the largest menu that gives each type of player i exactly the same utility as menu m_{-i}. We say that $m_{-i}^{*}$is a *completion* of m_{-i}. Further, let

$$\begin{aligned}\kappa_i^{*}&=\sup\left\{\kappa\in[0,1]:\kappa\left(\mathbf{1}-x_{-i}\right)\in m_{-i}^{*}\right\},\text{ and}\\ x_i^{*}&=\kappa_i^{*}\left(\mathbf{1}-x_{-i}\right).\end{aligned} \tag{4.2}$$

Allocation x_i^{*} is the best allocation in menu m_{-i} that belongs to the ray connecting allocations $\mathbf{0}$ and $\mathbf{1}-x_{-i}$ (see the right panel of Figure 4.1). The strength of player i types is equal to

$$S_i\left(u_i\right)=\frac{u_i\cdot(\mathbf{1}-x_{-i})}{\max_{x\in m_{-i}^{*}}u_i\cdot x}=\frac{\frac{1}{\kappa_i^{*}}u_i\cdot x_i^{*}}{\max_{x\in m_{-i}^{*}}u_i\cdot x}=\frac{1}{\kappa_i^{*}}\frac{1}{\frac{\max_{x\in m_{-i}^{*}}u_i\cdot\left(x-x_i^{*}\right)}{u_i\cdot x_i^{*}}+1}.$$

The last expression is maximized by any type $u_i^{*}\in\mathcal{U}_i$ such that $x_i^{*}\in\arg\max_{x\in m_{-i}^{*}}u_i^{*}\cdot x$ (such a type exists, due to the menu m_{-i}^{*} being complete). Thus, x_i^{*} is a (possibly, one of many) allocation of the (possibly, one of many) strongest type.

Define the strength of player $-i$ as the ratio of the payoff from allocation $\mathbf{1}-x_i^{*}$ (i.e., the winning allocation against the strongest type) and the losing payoff:

$$S_{-i}^{*}=\frac{u_{-i}^{*}\left(\mathbf{1}-x_i^{*}\right)}{u_{-i}^{*}\cdot x_{-i}}. \tag{4.3}$$

Lemma 1. *Assume 2. Additionally, suppose that* $m_i=\{x_{-i}\}$, $\mathbf{1}-x_i\notin m_{-i}^{*}$, *and* $\inf_{u\in\mathcal{U}_i}u\cdot(\mathbf{1}-x_{-i})>0$. *If* $S_i^{*}>S_{-i}^{*}$ *then there exist* $\lambda^{*},\Delta^{*}>0$ *such that if* $\lambda\leq\lambda^{*}$ *and* $\Delta\leq\Delta^{*}$, *then there is* $T<\infty$ *such that* $\mathrm{e}^{-r\Delta T}>1-\delta$ *and, in any equilibrium, player* $-i$ *concedes with probability at least* $1-\delta$ *before the end of period* T.

The Lemma says that if player i is stronger, player $-i$ concedes with a large probability in the initial periods of the game.

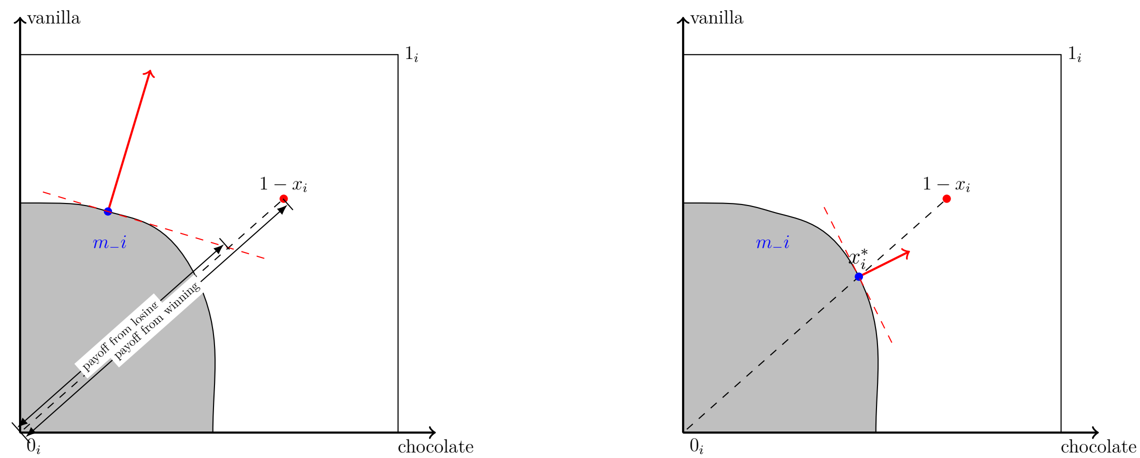


Figure 4.1. Player's strenght and general menus.

In the special case when the strongest type and her optimal allocation are unique, the intuition is similar to the one developed in Section 3. The concessions of i's types are sorted by the strength. The late game types of player i choose allocations that are arbitrarily close to x_i^*, which implies that the late game winning payoff of player $-i$ is approximately equal to $u_{-i}^* \cdot (\mathbf{1} - x_i^*)$. This determines the concession rates, and the rest of the argument follows the same logic as in the case of Lemma 1.

There are two complications in the general case. First, if the menu is linear in the neighborhood of x_i^*, the optimal allocation of all but the strongest type is significantly far away from x_i^*, even during the late game. In such a case, the average allocation conditional on concession converges to x_i^*. The intuition is described on the left panel of Figure 4.2. Almost all types of player i pick one of two y^1, y^2 optimal allocations. The dotted lines represent the indifference curves of the strongest types of player i that concede in a given period t. The intersection of the indifference curves belongs to the dashed ray because, as we explained above, the strength of any type can be parameterized by the distance between the zero allocation and the intersection of the ray with an indifference curve. In the late game only the types with strength close to S_i^* survive. Due to the continuity of the density function, in the late game, the conditional probabilities of the two optimal allocations are proportional to the angles

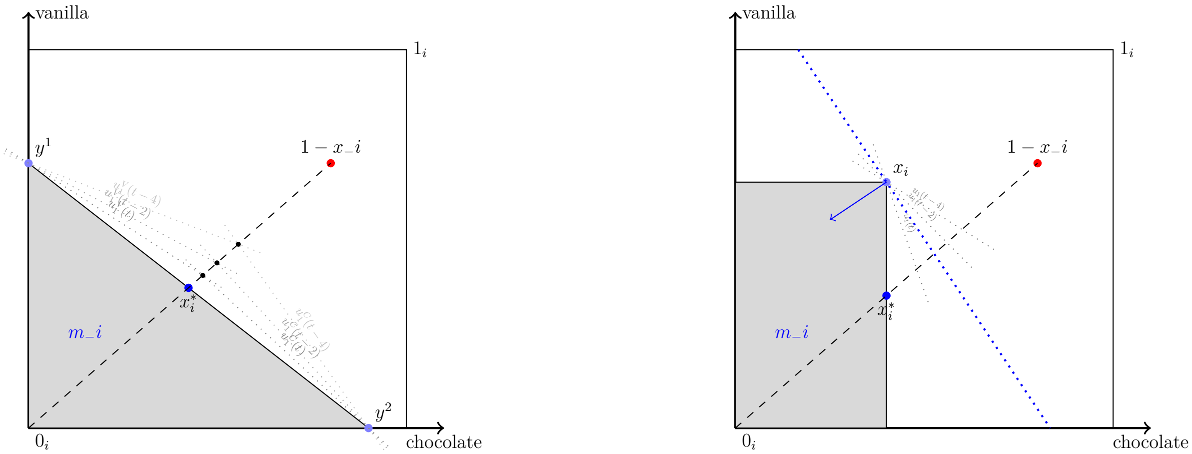


FIGURE 4.2. Special cases.

between the two consecutive indifference curve. A simple geometric argument shows that, in such a case, the weighted average concession allocation is close to x_i^*.

The second complication occurs when the strongest type is in the boundary of the type distribution $\mathcal{U}_i$. See the right panel of Figure 4.2, where player i chooses from a single-element menu $\{x_i\}$. In such a case, the optimal choice of each type of player i is x_i, and not x_i^*. Nevertheless, the thesis of the Lemma holds. The reason is that the optimal choice induces the same strength for player i as allocation x_i^*; it follows that the late game concession rate of player $-i$ is as if the late game choice was close to x_i^*. On the other hand, player $-i$'s strength becomes smaller if allocation x_i^* is replaced by x_i, and S_{-i}^* is a lower bound on the strength of player $-i$ in the late game.

Because of the second complication, the converse version of the Lemma is not true without any further assumptions. However, we can show converse when player $-i$ chooses menu $m_{-i} = m_{-i}^{1/2} = \left\{x_i : u_{-i}^* \cdot (\mathbf{1} - x_i) = \frac{1}{2}\right\}$, i.e. the menu of player i's allocations that give payoff $\frac{1}{2}$ to player $-i$.

Lemma 2. *Suppose that Assumptions 1 and 2 hold. For any $\eta > 0$, there is $\varepsilon > 0$ such that if*

$$\max_{x \in m_i} u_{-i}^* \cdot x < \frac{1}{2} - \eta \text{ and } d_H\left(m_{-i}, m_{-i}^{1/2}\right) \leq \varepsilon,$$

then for each $\delta > 0$, there exist $\lambda^*, \Delta^* > 0$ such that if $\lambda \leq \lambda^*$ and $\Delta \leq \Delta^*$, then player $-i$ concedes with probability at least $1 - \delta$ in his first period of the game.

4.2. **Menu choice game.** The above results lead to a straightforward characterization of the limit payoffs in the menu choice game.

Theorem 3. *Suppose that* $\mathcal{U}_{-i} = \left\{u^*_{-i}\right\}$ *and that Assumptions 1 and 2 hold. Then, for each* $u_i \in \mathcal{U}_i$*,*

$$E_i\left(u_i\right) = \max_{x: u^*_{-i}\cdot(\mathbf{1}-x)\geq \frac{1}{2}} u_i \cdot x \text{ and } E_{-i}\left(u^*_{-i}\right) = \frac{1}{2}. \tag{4.4}$$

The Theorem says that player i receives her optimal subject to the constraint that player $-i$ receives at least $\frac{1}{2}$. As we discuss in the introduction, this means that player $-i$ receives his worst and each of his opponent types receives her best payoff across all possible Nash outcomes. The equilibrium payoffs are *ex post* efficient.

Although the statement focuses on the payoffs, note that the only way to obtain such payoffs is when player $-i$ is able to pick an allocation from menu $m^{1/2}_{-i}$. The proof makes clear that, in the equilibrium (as the various limits become close), player $-i$ offers menu $m^{1/2}$, and the other player (roughly) accepts.

Comparing to Theorem 3.1b, player $-i$ receives a higher payoff. Thus, being able to commit to a menu against an opponent with unknown preferences is beneficial.

Proof intuition. We describe the main idea of the proof when $i = 2$, i.e., when player $-i$ is the first to propose a menu. Suppose that player i proposes a (completed) menu that does not contain $m^{1/2}$. We are going to show that player i can guarantee herself the payoff of at least (4.4), and possibly more. See Figure 4.3. Suppose that player i's counteroffer is $m_i = \{x_{-i}\}$, where $x_{-i} \in m^{1/2}$ and $\mathbf{1} - x_{-i} \notin m_{-i}$. Let u^*_i be the strongest type of player i in the continuation war of attrition. As we explain above, the optimal choice u^*_i belongs to the ray between the zero allocation and $\mathbf{1} - x_{-i}$. The strength of u^*_i is equal to the ratio of the distance between the zero allocation and $\mathbf{1} - x_{-i}$ to the distance between the zero allocation and the optimal allocation x^*_i. By The Thales theorem, it is also equal to the ratio of the distance between the zero and the half allocations, and the zero and allocation A, where A is chosen as an allocation on the diagonal that makes player $-i$ indifferent between A and x^*_i. Thus,

$$S^*_i = \frac{\left|\mathbf{0}\tfrac{\mathbf{1}}{\mathbf{2}}\right|}{|\mathbf{0}A|}.$$

On the other hand, the strength of the uniformed player is equal to the ratio of the utilities associated with the indifference curves passing by x^*_i and passing by $\mathbf{1} - x_{-i}$;

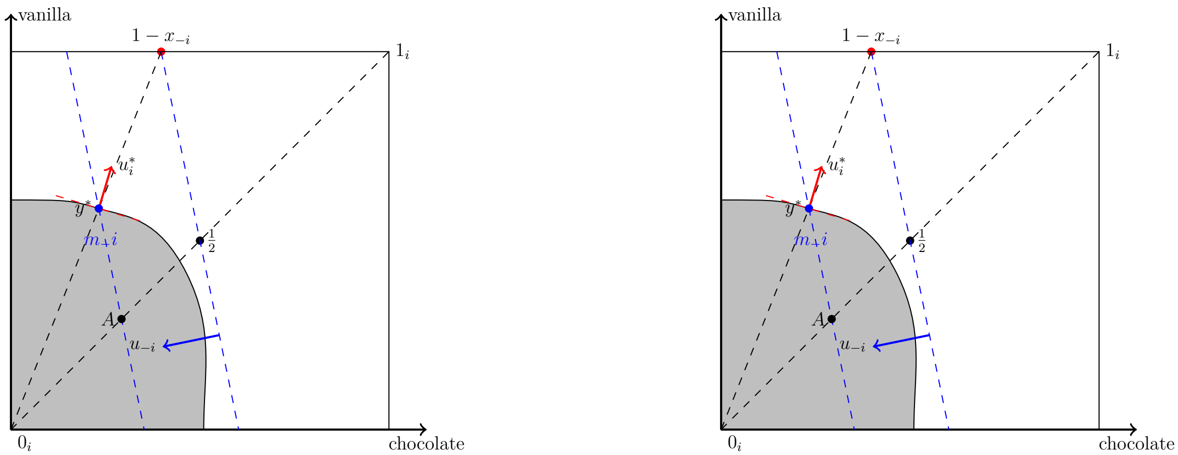


FIGURE 4.3. Menu choice with one-sided incomplete information.

hence it is equal to

$$S^*_{-i} = \frac{\left|\mathbf{1}\tfrac{\mathbf{1}}{\mathbf{2}}\right|}{|\mathbf{1}A|}.$$

A simple algebra shows that the latter is strictly smaller than the former. By Lemma 1, if faced with counteroffer $m_{-i} = \{x_{-i}\}$, player $-i$ concedes quickly. The optimal choice of $x_{-i} \in m^{1/2}$ ensures payoffs (4.4). In fact, because the argument shows that player i is strictly stronger, player i can win the war of attrition with an allocation that leads to strictly higher payoffs for her and strictly lower payoffs for player $-i$. Given that player i can ensure payoffs $\frac{1}{2}$, m_{-i} cannot be a best response choice.

4.3. **Bargaining with revisions.** Our model assumes that the players cannot revise their offers after the initial stage. Here, we informally discuss some of the issues that arise if revisions were allowed. TBA

5. Two-sided incomplete information

In this section, we study the war of attrition when there is an incomplete information about preferences of each player. We start with using an example to show that the two-sided version of the model can have multiple equilibria. Next, we show that the multiplicity will disappear if the types are continuum rather than discreet. In this section, we assume $N = 2$.

5.1. **Two-type example.** Fix two constants a, b such that

$$\frac{a}{a+1} < b < a < \frac{1}{2}. \tag{5.1}$$

Each player i, let

$$m_{-i} = \{(a, 0), (0, b)\}$$

be the menu of choices when player i concedes. Each player has two types $u^c = (1, 0)$ and $u^v = (0, 1)$ and both types have a positive probability. The assumptions (5.1) imply that each player's type prefers to win regardless of the choice of the other player. See Figure 5.1a. The allocation $x_i = (a, b)$ is defined as the unique allocation such that the two types of player i are indifferent between their optimal concession allocation from menu m_{-i} and x_i. We refer to x_i as the *indifference point*.

Proposition 1. *There exists $\pi^* \in (0, 1)$ such that for each i, if $\pi\left(u^v_{-i}\right) \geq \pi^*$, then there is a sequence of equilibria of the above game as $\lambda \to 0, \Delta \to 1$ such that player i concedes with a probability arbitrarily close to 1 in his first period of action.*

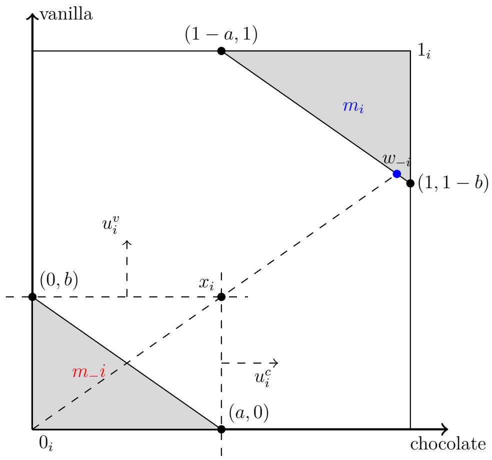
The short reason for the existence of multiple equilibria is that there is no natural sorting that decides which types concede first. In the equilibrium that we describe, the last types to concede are u^v_{-i} and u^c_i; if the roles i and $-i$ are exchanged, a different pair of types ends the game.

We briefly describe the construction. The equilibrium has three phases:

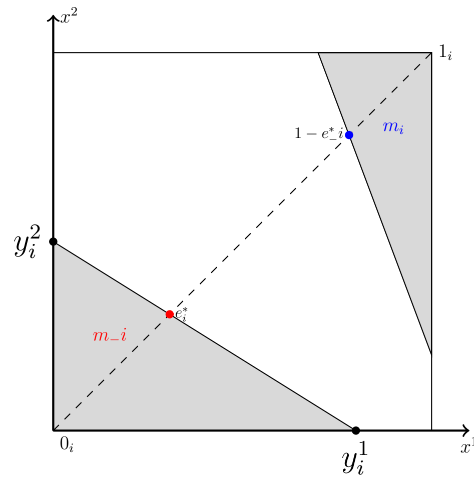
(1) *Atom concession.* In its first period of action, each type u of player i concedes with a positive probability (that is arbitrarily close to 1). If player i moves second, then player $-i$ does not concede in her first period. For each subsequent period after the initial concession, the expected continuation payoff of each type of each player is equal to her immediate concession payoff.

(2) *War of attrition with both sides active.* In the intermediate phase, each type $u = u^c, u^v$ concedes with a positive probability. The rates are chosen so that each type is indifferent between waiting and conceding. In order to satisfy the indifference condition, the average winning allocation of player j conditionally on $-j$ concession must lie on the ray that connects the 0 payoff and the indifference point:

$$w_j = \gamma x_j = \alpha (1, 1 - b) + (1 - \alpha)(1 - a, 1)$$



(A) Example with two types for each player.



(B) Theorem 4.

for some $\gamma > 1$ and $\alpha \in (0,1)$ (see Figure 5.1a). The end date of the phase is chosen as the last period when types u_i^v for player i and u_{-i}^c for player $-i$ concede with a positive probability. (To make sure that it is possible, we need to assume that the initial probability of type u_{-i}^v is sufficiently high.)

(3) *War of attrition with one sides active.* For each of the remaining periods the two remaining types u_{-i}^v and u_i^c concede at constant rates that make the opponent type indifferent between conceding and waiting. The concession rate of player $-i$ is higher. (To see that, notice that is the winning payoff of type u_{-i}^v facing u_i^c is equal to 1 and the concession payoff is b. Hence, her strength is equal to $\frac{1}{b}$. Analogously, $\frac{1}{a}$ is the strength of type u_i^c facing u_{-i}^v.) Importantly, the two concession rates are sufficiently slow so that the other two types that fully conceded before the beginning of the last phase (u_{-i}^c and u_i^v) do not want to deviate and wait till this phase. (To see why, notice that the strength of type u_i^v facing u_{-i}^v is equal to $\frac{1-b}{b}$, which is less than the strength $\frac{1}{b}$ of type u_i^c facing u_{-i}^v. Because player $-i$ concedes at the right chosen to make the stronger type indifferent, the weaker type u_i^v wants to concede early.) The phase ends when the strategic types fully reveal themselves.

The structure of equilibrium and the above comments ensure that none of the players wants to deviate. We verify that the probabilities add up in Appendix D.

5.2. **Continuum types.** Next, we assume that players bargaining positions take form of linear menus: $m_{-i} = \{x : \psi_i \cdot x \leq v_i\}$ for some vector $\psi_i \in \mathcal{U}$ and $v_i > 0$ and each i. (The completions of the menus from the example are linear. We discuss that case of general menus below.) Additionally, we make two assumptions. The first assumption ensure that the beliefs about i's types are sufficiently regular in the neighborhood of vectors β_i.

Assumption 3. *(Regularity) For each player i, $\mathcal{U}_i = supp\pi_i$ has a nonempty interior in $\mathcal{U}$, $\psi_i \in int\mathcal{U}_i$, and π_i has a strictly positive Lipschitz continuous density with respect to the Lebesgue measure on $\mathcal{U}$.*

Recall that the payoff from winning the war of attrition depends on the choice made by the other player when conceding. The next assumption says that, no matter what is the choice, all types of player i would rather win than lose.

Assumption 4. *(Large Gap). For each $u \in \mathcal{U}_i$, for each $x_i \in m_i$ and each $y_i \in m_{-i}$,* $\inf_{x \in m_i} u \cdot (1 - x) > u \cdot \sup_{y \in m_{-i}} u \cdot y$.

Let

$$\alpha_i^* = \sup \{\alpha : \alpha \mathbf{1}_i + (1 - \alpha) \mathbf{0}_i \in m_{-i}\},$$
$$e_i^* = \alpha_i^* \mathbf{1}_i + (1 - \alpha_i^*) \mathbf{0}_i.$$

Here, $e_i^*$is the unique allocation that lies in the intersection of the diagonal and the boundary of menu m_{-i}. Let

$$S_i^* = \frac{1 - \alpha_{-i}^*}{\alpha_i^*} = \frac{u_i \cdot \left(1 - e_{-i}^*\right)}{u_i \cdot e_i^*},$$

where the last equality holds for arbitrary preference type $u_i \in \mathcal{U}$. Thus, S_i^* is the strength of player i defined as the winning/concession ratio under the restriction that, when conceding, the player must choose an allocation that belongs to the diagonal. Because of linearity of preferences, so defined strength does not depend on the player's type. The main result of this section shows that the strength characterizes the behavior in the war of attrition.

Theorem 4. *Suppose that Assumptions 3 and 4 hold. Suppose that $S_i^* > S_{-i}^*$. For each $\delta > 0$, there exist $\lambda^*, \Delta^* > 0$ such that if $\lambda \leq \lambda^*$ and $\Delta \leq \Delta^*$, then there is $T < \infty$*

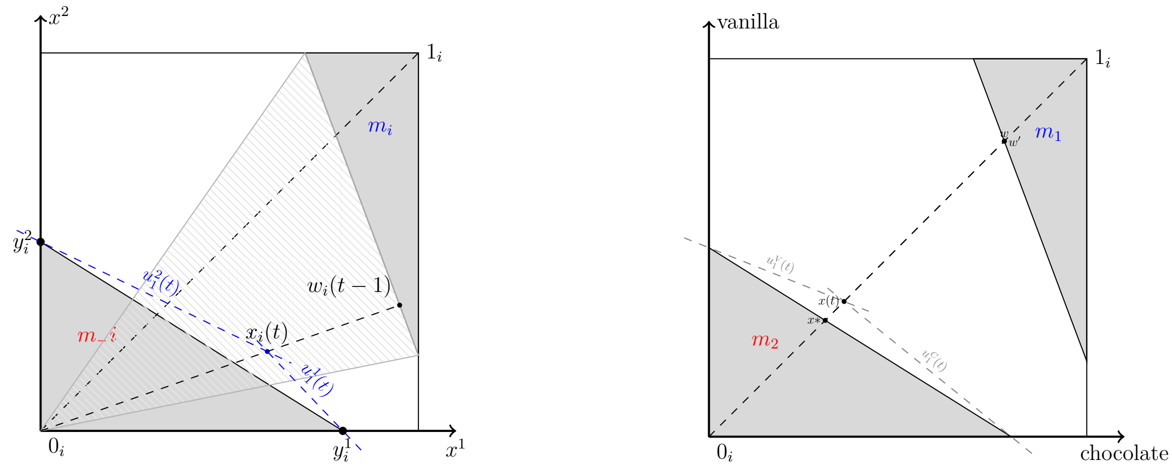


FIGURE 5.1. Illustration of the proof .

such that $\mathrm{e}^{-r\Delta T} > 1-\delta$ *and, in any equilibrium, player* $-i$ *concedes with probability at least* $1-\delta$ *before the end of period* T.

Compared to the above example, Theorem 4 shows that when the type distribution is continuous, there is an unique equilibrium. The theorem says that the equilibrium concession behavior is the same *as if* the players choices were restricted to the diagonal. Of course, in the equilibrium the probability mass 1 types chooses one of the extreme points in menu. We explain below that, in the late game, the ratios with which the extreme points are chosen balance so that their average lies on the diagonal. The assumption about separate preferences ensures that the concession rates in the early game are bounded; because the late game is arbitrarily long, it means that the late game effects dominate over anything that happens in the early game.

Proof intuition. We describe the intuition behind the proof in few steps. As in the rest of the paper, the argument relies on the analysis of the late game. The goal is to show that after sufficiently many periods, the players behave *as if* they conceded with outcomes e_i^* for each i. Then, their concession behavior is determined by strengths S_i^*. Because $S_j^* > S_{-j}^*$, player j concedes significantly faster than her opponent. The rest of the argument proceeds in the same way as in the case of Lemma 1.

Sorting. The main difficulty with two-sided incomplete information is that there is no natural sorting. When the menus are linear, we show that there is a partial sorting.

Let $y_i^1, y_i^2 \in m_{-i}$ be two extreme points of menu m_{-i}. (See the left panel on Figure 5.1.) Let $\mathcal{U}_i^k$ be the subset of types of player i who strictly prefer allocation y_i^k to allocation y_i^{-k}, i.e., the types who care about issue k relatively more than the type ψ_i, and,, as follows, than all types in $\mathcal{U}_i^{-k}$. We say that such types are on side k. Take any two types $u, u' \in \mathcal{U}_i^k$ and suppose that $u^k > u'^k > \psi_i^k$. Using a similar argument as in the previous sections, we can show that for any allocation $y \notin m_{-i}$, we have

$$\frac{u \cdot y}{u \cdot y^k} < \frac{u' \cdot y}{u' \cdot y^k}.$$

In other words, type u cares relatively less about winning and obtaining y rather than losing than type u'. This implies that type u is going to concede before type u' in the war of attrition. From now on, we rank player i types according to their distance to the last type ψ_i.

Let $u_i^k(t)$ denote the largest type on side k who survives till period t. (See the left panel of Figure 5.1.) We say that player i *is active on side* k in period t if $u_i^k(t) \neq u_i^k(t+2)$, i.e, if outcome y_i^k is chosen with strictly positive probability in period t. Because of the general properties of the war-of-attrition games, each player must be active on at least one side in each period before the final concession of the strategic player.

Indifference condition. If the player is active on side k in two consecutive periods $t-2$ and t, then types $u_i^k(t)$ must be indifferent between conceding in those two periods. There is a simple geometric characterization of this indifference. For each $t \in T_i$, let $p_{-i}(t-1)$ be the concession rate, i.e. the probability of $-i$ conceding conditionally on reaching period $t-1$ and let

$$w_i(t-1) = \sum_k \text{Prob}\left(-i \text{ chooses } y_{-i}^k | -i \text{ concedes at } t\right)\left(\mathbf{1} - y_{-i}^k\right)$$

be the average allocation left to her by player $-i$ conditionally on him conceding. (We take a convention that allocations indexed with i, like $w_i(t-1)$, are stated from the point of vew of player i; the average allocation chosen by player $-i$ is the complementary allocation $\mathbf{1} - w_i(t-1)$). Then, type $u_i^k(t)$ is indifferent if

$$\begin{aligned} & u_i^k(t) \cdot y_i^k, \\ =& p_{-i}(t-1)\, \mathrm{e}^{-\Delta} u_i^k(t) \cdot w_i(t-1) + (1 - p_{-i}(t-1))\left(1 - \mathrm{e}^{-2\Delta}\right)\left(u_i^k(t) \cdot y_i^k\right) \end{aligned}$$

or, if allocation

$$q_i(t-1) = \frac{p_{-i}(t-1)\,\mathrm{e}^{-\Delta}}{\mathrm{e}^{-2\Delta} + p_{-i}(t-1)\left(1-\mathrm{e}^{-2\Delta}\right)} w_i(t-1)$$

belongs to the indifference curve of type $u_i^k(t)$ that passes through her optimal choice in the menu. We refer to $w_i(t-1)$ as the *win outcome* and to $q_i(t-1)$ as the *virtual payoff* in period $t-1$. The latter belongs to the ray between the win outcome and the allocation $\mathbf{0}$.

If the player is active on both sides, then the virtual payoff must be equal to the *indifference point* $x_i(t)$, i.e., the unique allocation such that each type $u_i^k(t)$ is indifferent between $x_i(t)$ and her optimal concession allocation y_i^k. For future reference, note that this is only possible if the indifference point belongs to the convex hull spanned by the allocations $\mathbf{1}-y_{-i}^1, \mathbf{1}-y_{-i}^2$ and $\mathbf{0}$ (the dashed area of Figure 5.1).

Structure of the late game. We show in the proof that the players must be active on both sides in each period of the late game, i.e., when the remaining types are sufficiently close to the lowest type ψ_i. There are two steps to the argument. First, we show that the indifference point must remain in the convex hull of $\mathbf{1}-y_{-i}^1, \mathbf{1}-y_{-i}^2$ and $\mathbf{0}$ (the dashed area of Figure 5.1). Otherwise, say if at some t the indifference point leaves the convex hull one the side k, then, we show using the indifference condition that the player must be only active on side k for each $t' < t$. But that leads to the contradiction as there must be a substantial revelation of types on side $-k$ before the late game is reached. TBA

The diagonal. Finally, we can show that the late behavior must remain close to the diagonal. We can estimate the late game rate of movement of the indifference point by the distance between $x_i(t)$ and the win outcome $\mathbf{1}-w_{-i}(t)$:

$$\Delta x_i(t) = x_i(t) - x_i(t+2) \approx c_i(t)\left[\mathbf{1} - w_{-i}(t) - x_i(t+2)\right], \tag{5.2}$$

where the proportionality constant $c_i(t)$ depends on the concession rate, etc. The idea is simple: if player i chooses y_i^k with a relatively high probability in period t, then the gap between types $u_i^k(t+2)$ and $u_i^k(t)$ is relatively large. But it also means that the indifference point is moving towards side k. A careful calculation that relies on the Lipschitzness of the density in the neighborhood of ψ_i shows that the indifference point does not change (much) only if the win outcome is very close.

Suppose that in the late game, the indifference points $x_i(t)$ remain in the close neighborhood of some constant x_i^*. In such a case, (5.2) implies that $\mathbf{1} - w_{-i}(t) \approx x_i^*$ for both players i. A the same time, the indifference condition implies that $x_i(t)$ is a convex combination of allocations $\mathbf{0}$ and $w_i(t-1) \approx \mathbf{1} - x_{-i}^*$. Putiing those two conditions together, we obtain that x_i^* must lie on a diagonal for each i (see the right panel of Figure 5.1).

6. Conclusions and open questions

In this paper, we analyze the war of attrition bargaining and the menu choice game under the structural uncertainty. The difficulty of the analysis depends on whether one can establish an a priori sorting among the types. We show that the equilibrium behavior in various settings is unique and it is determined by the behavior in the late game, when only the strongest types survive. With one-sided incomplete information about the preferences, the player with known preferences offers a menu of all allocations that give her at least half of the pie; the other player chooses an optimal allocation from the menu. Being able to offer menu is beneficial to the player. The outcome is efficient.

The paper leaves many open questions. First, what is the relation between the current results and the model in which players are allowed to revise their offers? In particular, what is the analogue of the Myerson result under the structural uncertainty? Second, what is the equilibrium characterization under general menus with two-sided uncertainty? Finally, one may want to further generalize the offers that players can make. For instance, players may offer menus of menus like like "I divide and you choose," or more general mechanisms. An important and interesting question is whether it is possible to provide strategic foundations for one of the axiomatic solutions proposed in papers like Harsanyi and Selten 1972 and Myerson 1984. In words, is it possible to extend the Nash program to incomplete information? We leave such questions for future research.

Appendix A. Equilibrium analysis

In this part of the appendix, we conduct a preliminary analysis of the equilibrium.

A.1. **Notations.** For each player $i = 1, 2$, let $t_i^0 = i$ be the first decision period for player i.

For each player i and each $t \in T_i$, each measurable set $U \subseteq \mathcal{U}_i$, define the probability that player i with preferences in U yield in period t as

$$f_i^\sigma(U|t) = (1-\lambda)\int_U \sigma_i^T(t|u)\, d\pi_i(u).$$

We also write $f_i^\sigma(t) = f_i^\sigma(\mathcal{U}|t)$. Let

$$F_i^\sigma(t) = \lambda + \sum_{s \in T_i : s \geq t} f_i^\sigma(t), \text{ and}$$
$$p_i^\sigma(t) = \frac{1}{F_i^\sigma(t)} f_i^\sigma(t),$$

be the probability that player i has not conceded before period t and the concession rate in period t.

For each $t \in T_{-i}$, let

$$w_i^\sigma(t) = \int \left(\mathbf{1} - \sigma_{-i}^M(u_{-i})\right) \frac{1}{f^\sigma(t)} df^\sigma(u_{-i}|t) \in X,$$
$$y_i^\sigma(t) = \frac{\mathrm{e}^{-\Delta} p_{-i}^\sigma(t)}{\mathrm{e}^{-2\Delta} p_{-i}^\sigma(t) + (1-\mathrm{e}^{-2\Delta})} w_i^\sigma(t) \in X.$$

Here, $w_i^\sigma(t)$ denotes the allocation that player i obtains in period t, conditionally on the opponent's concession in that period t; $y_i^\sigma(t)$ is the winning allocation weighted by the concession probability. Further, for each type $u \in \mathcal{U}_i$ of player i, let

$$L_i(u) = \max_{x \in m_i} u \cdot x, \text{ and } S_i^\sigma(u,t) = \frac{u \cdot w_i^\sigma(u,t)}{L_i(u)}.$$

Here, $L_i(u)$ is the payoff received upon concession, and $S_i^\sigma(u,t)$ is the strength ratio.

The superscripts σ in the above notation denotes dependence on the strategy profile σ; the subscript i, on the player i. We drop the superscripts and/or the subscripts from the above notation whenever it does not lead to confusion.

A.2. **Best response characterization.** The expected payoff of player i type u from yielding in period $t \in T_i$ given opponent strategies (σ) is equal to

$$U_i^\sigma(u,t) = \sum_{s: s<t, s \in T_{-i}} \mathrm{e}^{-s\Delta} f_{-i}^\sigma(s)\left(u \cdot w_i^\sigma(u,s)\right) + \mathrm{e}^{-t\Delta} F_{-i}^\sigma(t+1)\, L_i(u).$$

For each $t \in T_i$, we have

$$\begin{aligned} & \mathrm{e}^{t\Delta}\left[U_i^\sigma(u,t+2) - U_i^\sigma(u,t)\right] \qquad (A.1) \\ =& \mathrm{e}^{-\Delta} f_{-i}^\sigma(t+1)\left(u \cdot w_i^\sigma(u,t+1)\right) + \left[\mathrm{e}^{-2\Delta}\left(F_{-i}^\sigma(t+1) - f_{-i}^\sigma(t+1)\right) - F_{-i}^\sigma(t+1)\right] L_i(u) \\ =& F_{-i}^\sigma(t+1)\left[\mathrm{e}^{-\Delta} p_{-i}^\sigma(t+1)\left(u \cdot w_i^\sigma(u,t+1)\right) - \left(\mathrm{e}^{-2\Delta} p_{-i}^\sigma(t+1) + 1 - \mathrm{e}^{-2\Delta}\right) L_i(u)\right] \\ =& \left(f_{-i}^\sigma(t+1) + \left(1 - \mathrm{e}^{-2\Delta}\right) F_{-i}^\sigma(t+3)\right)\left[u \cdot y_i(t+1) - L_i(u)\right]. \end{aligned}$$

We have the following corollary to the above calculations and definitions.

Lemma 3. *For each type u, each $t \in T_i$, $U_{-i}^\sigma(u,t+1) \geq (\leq) U_{-i}^\sigma(u,t-1)$ if and only if*

$$u \cdot y_{-i}^\sigma(t) \geq (\leq) L_{-i}(u), \text{ or } p_i^\sigma(t) \geq (\leq) \left(\mathrm{e}^{\Delta} - \mathrm{e}^{-\Delta}\right) \frac{1}{S_{-i}^\sigma(u,t) - \mathrm{e}^{-\Delta}}.$$

A.3. **End of the war of attrition.** Let $T_i^{*,\sigma} = \max\{t \in T_i : f_i^\sigma(t) > 0\}$ be the last period in which the strategic types yield. We have the following standard result.

Lemma 4. *Suppose that σ is an equilibrium.*

(1) *For each $t \leq T_i^{*,\sigma}$, $f^\sigma(t) > 0$. Also, $|T_i^{*,\sigma} - T_{-i}^{*,\sigma}| = 1$.*
(2) *For each $t < T_i^{*,\sigma}$, $y_i^\sigma(t) \notin intm_i$.*
(3) *For each i, $T_i^{*,\sigma} < \infty$, and $F_i^\sigma(T_i^{*,\sigma} + 2) = \lambda$.*

Proof. By Lemma 3, if$f(t) = 0$ for some $t \in T_{-i}$, then it is a strictly better response for (almost any type u of player i to yield in period $t-1$ rather than to wait to period $t+1$. It follows that $f_i^\sigma(t+1) = 0$. An induction implies that $f_i^\sigma(t') > 0$ for each $t' > t$. The second claim follows from the same argument.

If $t < T_i^{*\sigma}$, then the part 1 of Lemma 4 implies that there is a type of player i for whom period $t+1$ is a best response. By Lemma 3, $u \cdot y_i^\sigma(t) < L_i(u)$. However, the latter inequality cannot be satisfied if $y_i^\sigma(t) \in \mathrm{int} m_i$.

For each i, let $L_i^{\min} = \inf_{u_i \in \mathcal{U}_i} L_i(u_i)$. Because $f_i^\sigma(t) > 0$ for each $t \leq T_i^{*,\sigma}$, it must be that for each $t \in T_i$, if $t < T_i^{*,\sigma}$, there is a type $u \in \mathcal{U}_{-i}$ of player $-i$ such that $U_{-i}^\sigma(u,t-1) \leq U_{-i}^\sigma(u,t+1)$. It follows from Lemma 3 that for each $t < T_i^{*,\sigma}$,

$$p_i^\sigma(t) \geq \left(1 - \mathrm{e}^{-\Delta}\right) \frac{1 + \mathrm{e}^{-\Delta}}{\mathrm{e}^{-\Delta}} \frac{1}{\max_{u \in A_{-i}} S_{-i}^\sigma(u,t) - \mathrm{e}^{-\Delta}} \geq \left(1 - \mathrm{e}^{-\Delta}\right) L_{-i}^{\min} > 0,$$

which implies for each $t \leq T_i^{*,\sigma}$

$$F_i^\sigma(t) = (1 - p_i^\sigma(t-2)) F_i^\sigma(t-2) \leq \left(1 - \left(1 - \mathrm{e}^{-\Delta}\right) L_{-i}^{\min}\right) F_i^\sigma(t-2)$$
$$\leq \left(1 - \left(1 - \mathrm{e}^{-\Delta}\right) L_{-i}^{\min}\right)^{\left(t - t_i^0\right)/2}.$$

Because $F_i^\sigma(t) \geq \lambda$, it must be that $T_i^{*,\sigma} - t_i^0 \leq \frac{\log \lambda}{\log\left(1 - \left(1 - \mathrm{e}^{-\Delta}\right) L_{-i}^{\min}\right)}$. □

A.4. **Monotonicity.** Recall that for $A, B \subseteq \mathbb{R}$,$A$ is strongly dominated by B, we write $A \leq_S B$ if for each $a, \in A, b \in B$, $\min(a,b) \in A$ and $\max(a,b) \in B$.

Lemma 5. *(Monotonicity) Take two types $u, u' \in A_i$, and suppose that $S_i^\sigma(u,s) \leq S_i^\sigma(u',s)$ for each $s \in T_{-i}$ such that $s < T_{-i}^{*,\sigma}$. Then,* $\arg\max U_i^\sigma(u,.) \leq_S \arg\max U_i^\sigma(u',.)$*. If $S_i^\sigma(u,s) < S_i^\sigma(u',s)$ for each $s \in T_{-i}$ such that $s < T_{-i}^{*,\sigma}$, then, if $U_i^\sigma(u,t) \leq U_i^\sigma(u,t')$* for some $t < t'$, then $U_i^\sigma(u',t) < U_i^\sigma(u',t')$.

Proof. Notice that

$$\frac{1}{L_i(u)}\left(U_i^\sigma(u,t') - U_i^\sigma(u,t)\right)$$
$$= \sum_{s:t<s<t', s\in T_{-i}} \mathrm{e}^{-s\Delta} f^\sigma(s) S_i^\sigma(u,s) + \mathrm{e}^{-t'\Delta}\left(1 - \sum_{s:s<t', s\in T_{-i}} f^\sigma(s)\right) - \mathrm{e}^{-T\Delta}\left(1 - \sum_{s:s<t, s\in T_{-i}} f^\sigma(s)\right)$$
$$= \frac{1}{L_i(u')}\left(U_i^\sigma(u',t') - U_i^\sigma(u',t)\right) - \sum_{s:t<s<t', s\in T_{-i}} \mathrm{e}^{-s\Delta} f^\sigma(s)\left[S_i^\sigma(u',s) - S_i^\sigma(u,s)\right].$$

Thus, function $U_i^0(u,t) = \frac{1}{L_i(u)} U_i^\sigma(u,t)$ has increasing differences in the strength ratio and time. The result follows from the Topkis Theorem. □

A.5. **Early game.** The next result discusses the early game, where a player may still have very weak types.

Lemma 6. *For each $\delta > 0$, there exists $\varepsilon > 0$ and $\Delta^* > 0$ such that if $\Delta \leq \Delta^*$, then there exists T_0 such that $\mathrm{e}^{-r\Delta T_0} \geq 1 - 2\delta$ and for each equilibrium σ, either (a) $F_{-i}^\sigma(T_0) \leq \delta$, or (b) $\sigma^{T_0}(u) \leq T_0$ for all u st. $\sup_t S_i^\sigma(u,t) \leq 1 + \varepsilon$.*

Proof. Let $k^* = \lceil -\log_2 \delta \rceil \leq -\log_2 \delta + 1$. Find $\varepsilon > 0$ such that $(1 - 2\varepsilon) \geq (1 - \delta)^{\frac{1}{k^*}}$. Let n^* be the smallest even integer such that $\mathrm{e}^{-r\Delta n^*} \leq 1 - 2\varepsilon$. Then, $\mathrm{e}^{-r\Delta n^*} \geq (1 - 2\varepsilon)\mathrm{e}^{-2r\Delta}$. Take $T_0 = k^* n^*$. Find $\Delta^* > 0$ so that $2r\Delta(1 - \log_2 \delta) \leq \log \frac{1-\delta}{1-2\delta}$. Then,

$$\mathrm{e}^{-rT_0\Delta} \geq (1 - 2\varepsilon)^{k^*} \mathrm{e}^{-2r\Delta k^*} \geq (1 - \delta)\mathrm{e}^{-2r\Delta(1 - \log_2 \delta)} \geq 1 - 2\delta.$$

Suppose that there is u such that $u_i \cdot y_0 \leq (1+\varepsilon) L_i(u)$ and suppose that $T \geq T_0$ is a best response stopping time for such u. Because T is a best response for u, it must be that for each $t \in T_i$,$t < T$, player i type u prefers to continue waiting till period T rather than stopping in period t:

$$F_i^\sigma(t) L_i(u) \leq \sum_{t<s<T: s\in T_{-i}} f_{-i}^\sigma(s) \mathrm{e}^{-(s-t)\Delta} S_i(u,s) L_i(u) + F_i^\sigma(T) \mathrm{e}^{-(T-t)\Delta} L_i(u).$$

After some algebra, and taking into account that $S_i(u,s) \leq 1+\varepsilon$, we get

$$0 \leq \sum_{s>t: s\in T_{-i}} f_{-i}^\sigma(s) \left(\mathrm{e}^{-(s-t)\Delta}(1+\varepsilon) - 1\right).$$

If $t + n^* \leq T$, then due to the choice of n^*, the above is not smaller than

$$\begin{aligned}
&\leq \sum_{t<s<t+n^*: s\in T_{-i}} f(s)\varepsilon + \sum_{s>t+n^*: s\in T_{-i}} f(s)\left(\mathrm{e}^{-n^*\Delta}(1+\varepsilon) - 1\right) \\
&\leq \varepsilon\left(\sum_{t<s<t+n^*: s\in T_{-i}} f_{-i}^\sigma(s) - \sum_{s>t+n^*: s\in T_{-i}} (s)\right).
\end{aligned}$$

which implies that

$$\sum_{t<s<t+n^*: s\in T_{-i}} f(s) \geq \frac{1}{2}\left(\sum_{t<s<t+n^*: s\in T_{-i}} f_{-i}^\sigma(s) + \sum_{s>t+n^*: s\in T_{-i}} f_{-i}^\sigma(s)\right) = \frac{1}{2}\sum_{t<s<T: s\in T_{-i}} f_{-i}^\sigma(s).$$

It follows that

$$1 - F_{-i}^\sigma(s) = \sum_{s<T_0: s\in T_{-i}} f_{-i}^\sigma(s) \geq \sum_{l=1}^{k^*} \frac{1}{2^l} = 1 - \frac{1}{2^{k^*}} \geq 1 - \delta.$$

□

Appendix B. Proofs of Section 3

B.1. **Proof of Theorem 1.** Notice that the Theorem is trivially satisfied if $S_{-i}^* \leq 1$, as in such a case, no type of player $-i$ wants ever to wait. From now on, we assume that $S_i^* > S_{-i}^* > 1$. Let $\eta = \frac{1}{2}\left(S_i^* - S_{-i}^*\right)$. Let $q_i = P_i(S_i^* - \eta) > 0$. Let

$$x = \frac{S_{-i}^* - 1}{S_i^* - \eta - 1} < 1.$$

As we describe in the text, the equilibrium behavior can be sorted by the strength. For each t, let

$$S_i(t) = \sup\{S_i(u) : u \in \mathcal{U}_i, \sigma_i(t|u) > 0\}$$

be the maximum strength among all types that concede with positive probability in period t. Then, Lemma 5 implies that all types with strength not higher than $S_i(t)$ concede either before or in period t; similarly, all types with strength higher than $S_i(t)$ concede after period t. The continuity implies that each type with strength $S_i(t)$ is indifferent between conceding immediately and waiting for the next opportunity. Let $T_i^\eta = \min\{t : S_i(t) \geq S_i^* - \eta\}$.

In the rest of the proof, we divide the time of the game into three zones:

- *Early game*: By Lemma 6, for each $\delta > 0$, there exists $\varepsilon > 0$, $\Delta_0 > 0$, and T_0 such that either (a) $F_{-i}^\sigma(t) \leq \delta$, or (b) $\sigma_i^T(u) \leq T_0$ for all types $u \in \mathcal{U}_i$ such that $S_i(u) \leq 1+\varepsilon$. If (a), the thesis of the Lemma holds. On the contrary, from now on, assume (b) and $F_{-i}^\sigma(T_0) \leq \delta$. Let
$$y = \max\left(1, 2\frac{S_{-i}^* - 1}{\varepsilon}\right),$$
and we find $\Delta^* \leq \Delta_0$ such that $\Delta \leq \Delta^*$, we have
$$1 - \frac{1-\mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}}\frac{1}{\varepsilon} \geq 1 - y\frac{1}{2}\frac{S_{-i}^* - \mathrm{e}^{-\Delta}}{S_{-i}^* - 1}\frac{1-\mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}}\frac{1}{S_{-i}^* - \mathrm{e}^{-\Delta}} \geq \left(1 - \frac{1-\mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}}\frac{1}{S_{-i}^* - \mathrm{e}^{-\Delta}}\right)^y. \tag{B.1}$$
(Such Δ^* exists, as for all sufficiently small $c > 0$, $1 - \frac{1}{2}yc \geq (1-c)^y$.)
- *Middle game*: $T_0 \leq t < T_i^\eta$. In the middle game, $S_i(t) \geq 1+\varepsilon$ for $t \in T_i$ and $S_{-i}(t) \leq S_{-i}^*$ for $t \in T_{-i}$. By Lemma 3, we have
$$p_i^\sigma(t) \geq \frac{1-\mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}}\frac{1}{S_{-i}^* - \mathrm{e}^{-\Delta}}, \text{ and } p_{-i}^\sigma(t) \leq \frac{1-\mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}}\frac{1}{\varepsilon}.$$
Then, inequality (B.1) implies that
$$1 - p_{-i}^\sigma(t) \geq \left(1 - \frac{1-\mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}}\frac{1}{S_{-i}^* - \mathrm{e}^{-\Delta}}\right)^y. \tag{B.2}$$
- *Late game*: $T_i^\eta \leq t < T^*$. By Lemma 3 and because $S_i(t) \geq S_i^* - \eta$ for $t \in T_i$ and $S_{-i}(t) \leq S_{-i}^*$ for $t \in T_{-i}$, we have
$$p_i^\sigma(t) \geq \frac{1-\mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}}\frac{1}{S_{-i}^* - \mathrm{e}^{-\Delta}}, \text{ and } p_{-i}^\sigma(t) \leq \frac{1-\mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}}\frac{1}{S_i^* - \eta - \mathrm{e}^{-\Delta}}.$$
The choice of x and the fact that $x < 1$ implies that
$$1 - p_{-i}^\sigma(t) \geq 1 - \frac{S_{-i}^* - \mathrm{e}^{-\Delta}}{S_i^* - \eta - \mathrm{e}^{-\Delta}} p_i^\sigma(t) \geq 1 - x p_i^\sigma(t) \geq (1 - p_i^\sigma(t))^x. \tag{B.3}$$

where the last inequality holds for sufficiently small Δ (hence, sufficiently small $1-\mathrm{e}^{-2\Delta}$).

Notice that for each player l and each t, $\lambda = F_l^\sigma(T^*) = F_l^\sigma(t)\prod_{s\in T_l:t\leq s\leq T^*}(1-p_l^\sigma(s))$. The late game estimates (B.3) imply that

$$F_{-i}^\sigma(T_i^\eta) = \frac{\lambda}{\prod_{s\in T_{-i}:T_i^\eta\leq s\leq T^*}(1-p_{-i}^\sigma(s))} \leq \frac{\lambda}{\prod_{s\in T_{-i}:T_i^\eta\leq s\leq T^*}(1-p_i^\sigma(s))^x} = \lambda^{1-x}(F_i^\sigma(T_i^\eta))^x.$$

Further,

$$\begin{aligned}\frac{F_{-i}^\sigma(T_0)}{F_{-i}^\sigma(T_i^\eta)} &= \frac{1}{\prod_{s\in T_{-i}:T_0\leq s\leq T}(1-p_{-i}^\sigma(s))} \leq \prod_{s\in T_{-i}:T_0\leq s\leq T}\left(1-\frac{1-\mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}}\frac{1}{S_{-i}^*-\mathrm{e}^{-\Delta}}\right)^{-y}\\ &\leq \left(\frac{F_i^\sigma(T_0)}{F_i^\sigma(T_i^\eta)}\right)^y \leq (F_i^\sigma(T_i^\eta))^{-y},\end{aligned}$$

Together, we obtain,

$$F_{-i}^\sigma(T_0) \leq \lambda^{1-x}(F_i^\sigma(T_i^\eta))^{x-y} \leq \lambda^{1-x}q^{x-y}$$

where the last inequality comes from the fact that $F_i^\sigma(T_i^\eta)\geq q$. If λ is sufficiently small, we obtain the contradiction with $F_{-i}^\sigma(T_0)\geq\delta$.

B.2. **Proof of Lemma 2.** (Sketch). Let $m^{1/2} = \left\{x : u_1\cdot(\mathbf{1}-x)\geq\frac{1}{2}\right\}$ be the menu of allocations of player 2 that give at least $\frac{1}{2}$ to player 1. Let y^1, y^2 denote the two extreme points of the menu. (See Figure ???). Let $U_i(x_2,u_2)$ denote the limit ("$\limsup$" for player $i=1$ and "$\liminf$" for player $i=2$) of the equilibrium payoffs of player i conditionally on signal realization $s=u_2$ (with all the limits from the statement (3.2)) and if player 2 chooses allocation x_2.

Claim 1. If $U_2(x_2,u_2)\geq\max_k u_2\cdot y^k+\varepsilon$, then $U_1(x_2,u_2)\leq\frac{1}{2}-C(u_2)\varepsilon$ for each $\varepsilon>0$ and some constant $C(u_2)>0$. If the first inequality is strict, the second is strict as well.

Proof. Let $x_i(u_2)$ be an equilibrium expected and discounted allocation from the point of view of player i given preferences u_2. Then, $x_1(u_2)+x_2(u_2)\leq\mathbf{1}$. Moreover, if $U_i^*(x_2,u_2)$ denote the expected equilibrium payoffs of player i given preferences u_2, then $U_i^*(x_2,u_2)=u_2\cdot x_2(u_2)$,and $U_1^*(x_2,u_2)=u_1\cdot x_2$. If $U_2^*(x_2,u_2)\geq\max_k u_2\cdot y^k-\varepsilon$, then $U_1^*(x_2,u_2)\leq\frac{1}{2}+C\varepsilon$ for some constant $C<\infty$. Finally, note that the limits of

U_i^* ("lim sup" for player $i=1$ and "lim inf" for player $i=2$) are equal to U_i due to the convergence $\pi(u_2) \to \delta_{u_2}$. $\square$

For each $x_2 \in X$, , define sets

$$\begin{aligned} W_0(x_2, u_2) &= \{x : (u_1 \cdot (\mathbf{1} - x_2))(u_2 \cdot x_2) \leq (u_1 \cdot (1 - x))(u_2 \cdot x)\} \\ P(x_2) &= \{x : u_1 \cdot (\mathbf{1} - x) \geq u_1 \cdot (\mathbf{1} - x_2)\} \\ W(x_2) &= \bigcup_{u_2 \in \mathcal{U}_2} W_0(x_2, u_2) \cup P(x_2). \end{aligned}$$

$W_0(x_2, u_2)$ is the set of counteroffers of player 2 (expressed as allocations of player 2) that are either winning (i.e., lead to higher strength) against player 1's offer x_2; $P(x_2)$ is the set of counteroffers such that their complements are preferred by player 1 than the complement of x_2.

Claim 2. For each x_2, $m^{1/2} \subseteq W(x_2)$. Moreover, there exists $\varepsilon > 0$ such that if $\left\|y^k - x_2\right\| \geq \frac{1}{4}$, then $B\left(y^k, \varepsilon\right) \subseteq W(x_2)$.

Proof. Let $\upsilon(x_2) = \max(u_1 \cdot (\mathbf{1} - x_2), 1 - u_1 \cdot (\mathbf{1} - x_2))$. For the first claim, notice that

$$m^{\upsilon(x_2)} \subseteq W(x_2, u_1) \cup P(x_2) \text{ for each } x_2. \tag{B.4}$$

For the second claim, due to the continuity of set $W(x_2)$ with respect to x_2 and compactness, it is enough to show that for each x^2 such that $\left\|y^k - x_2\right\| \geq \frac{1}{4}$, there is $\varepsilon > 0$ such that $B\left(y^k, \varepsilon\right) \subseteq W(x_2)$. If $u_1 \cdot (1 - x_2) \neq \frac{1}{2}$, then, the claim follows from the fact that $\upsilon(x_2) > \frac{1}{2}$ and (B.4). If $u_1 \cdot (1 - x_2) = \frac{1}{2}$, then for each $u_2 \in U^k$

$$(u_1 \cdot (\mathbf{1} - x_2))(u_2 \cdot x_2) = \frac{1}{2}(u_2 \cdot x_2) < \frac{1}{2}\left(u_2 \cdot y^k\right) = \left(u_1 \cdot \left(1 - y^k\right)\right)\left(u_2 \cdot y^k\right).$$

The claim follows from the continuity. $\square$

Let $\overline{u}(u_2) = \int u\pi(du|u_2)$ be the expected preference of a player 2 with signal $s = u_2$.

Claim 3. For each u_2, $U_2(x_2, u_2) \geq \max_{x \in W(x_2)} x_2 \cdot \overline{u}(u_2)$.

Proof. This comes from the fact that due to Lemma 1, for each $x \in \mathrm{int}W(x_2)$, an offer $x_1 = \mathbf{1} - x$ would be accepted in equilibrium (as the various limits converge). $\square$

Finally, the claims imply that for each x_2, the limit of the expected payoffs of player 2 who chooses offer x_2 is not larger than

$$\leq \frac{1}{2} - \varepsilon \min_k \int_{u \in U^k} C(u_2)\, \rho(du) < \frac{1}{2}.$$

Appendix C. Proofs of Section 4

C.1. **Proof of Lemma 1.** The proof is divided into the following parts. First, we re-normalize the space of the preference types of player i, which is going to allow to tie the strength of player i to the behavior of certain convex function. Next, we show that the concession behavior of player i is sorted by the strength. Third, we establish late game bounds. Finally, we conclude the proof of the Lemma.

C.1.1. *Re-normalization.* For each $u \in \mathcal{U}$, $\rho(u) = \frac{1}{u \cdot x_i^*} u$. Function ρ projects $\mathcal{U}$ onto the affine plane $\mathcal{U}' = \rho(x_i^*) + I_0$, where $I_0 = \left\{ u \in \mathbb{R}^N : x_i^* \cdot u = 0 \right\}$. Because $\inf_{u \in \mathcal{U}_i} u \cdot x_i^* > 0$, ρ is an homeomorphism between $\mathcal{U}_i$ and $\mathcal{U}_i' = \rho(\mathcal{U}_i) \subseteq \mathcal{U}'$.

For each $u' \in \mathcal{U}'$,let $\pi'(u') = \pi(\rho^{-1}(u')) \frac{d\gamma}{du}(\rho^{-1}(u'))$ let be the induced density on $\mathcal{U}_i{}'$. Notice that that π' is Lipschitz on $\mathcal{U}_I{}'$. Let

$$\pi_{\min} := \inf_{u \in \mathcal{U}_i{}'} \pi'(u) > 0,$$
$$\pi_{\max} := \sup_{u \in \mathcal{U}_i{}_i'} \pi'(u) < \infty.$$

For each $u' \in \mathcal{U}_i$, let

$$h'(u') = \max_{x \in m_{-i}^*} u' \cdot (x - x_i^*),$$
$$x_{u'} = \arg\max_{x \in m_{-i}^*} u' \cdot x = \arg\max_{x \in m_{-i}^*} \rho^{-1}(u') \cdot x$$

We have the following result:

Lemma 7. *Function h' is convex, hence it is continuous and has a derivative $Dh'(u) : I_0 \to \mathbb{R}$ for almost all $u \in \mathcal{U}'$. Moreover, for almost all $u \in \mathcal{U}_i'$* and $v \in \mathcal{U}'$,

$$v \cdot (x_u' - x_0') = h'(u) + Dh'(u) \cdot (v - u).$$

For each $u \in \mathcal{U}_i$, each $t \in T_{-i}$, the strength of i 's types does not depend on t, and

$$S_i^\sigma(u, t) = \frac{1}{\kappa_i^*} \frac{1}{h'(\rho(u)) + 1}.$$

Proof. Define function $h^*:\mathbb{R}^N\to\mathbb{R}$by $h^*(u)=\max_{x\in m^*_{-i}} u\cdot(x-x_i^*)$. Then $h^*|_{\mathcal{U}_i'}=h'$ and $Dh^*(u)|_{I_0}=Dh'(u)$. Function h^*, hence h', are convex by standard arguments. Because h^* is homogeneous of degree 1, for each $u\in\mathcal{U}_I$, we have

$$(Dh^*(u))\cdot u=h^*(u)=h'(u).$$

By the Envelope Theorem, we have $x_u'-x_i^*=Dh^*(u)$. Hence,

$$\begin{aligned}v\cdot(x_u'-x_i^*)&=v\cdot(Dh^*(u))\\&=(Dh^*(u))\cdot u+(Dh^*(u))\cdot(v-u)\\&=h'(u)+(Dh(u))\cdot(v-u),\end{aligned}$$

where the last equality follows from the fact that $u,v\in\mathcal{U}'$. □

For each $u\in\mathcal{U}_i$, notice that

$$\begin{aligned}S_i^\sigma(u,t)&=\frac{u\cdot(1-x_{-i})}{\max_{x\in m^*_{-i}} u\cdot x}=\frac{1}{\kappa_i^*}\frac{u\cdot x_i^*}{\max_{x\in m^*_{-i}} u\cdot x}=\frac{1}{\kappa_i^*}\frac{1}{\max_{x\in m^*_{-i}}\rho(u)\cdot x}\\&=\frac{1}{\kappa_i^*}\frac{1}{h'(\rho(u'))+1},\end{aligned}$$

hence, the strength of the re-nomoralized type can be factorized through the value of function h'.

In the rest of the proof of Lemma 1, we only use the re-normalized preferences. To save on notational clutter, from now on, we drop the primes from all subsequent notation.

In what follows, we use the following two facts about the convex functions. Let λ be the Lebesgue measure on $\mathcal{U}$

Lemma 8. *For each $\eta>0$, we have*

$$\begin{aligned}&\lambda\{u:h(u)=\eta\}=0,\\&\lambda\{u:h(u)\le\eta\}\le 2^{N-1}\lambda\left\{u:h(u)\le\frac{1}{2}\eta\right\}.\end{aligned}$$

C.1.2. *Sorting.* Lemma 5 implies that the stopping time of different types can be ordered by the strength in the following sense: $\sigma_i(u)<\sigma_i(u')$ implies that $S_i(u)<S_i(u')$. Here, we show that the equilibrium is essentially unique.

Lemma 9. *Suppose that $\mathcal{U}_i$ is convex and that the measure π_i has a density on $\mathcal{U}_i$ with respect to the Lebesgue measure. Then, there exists a strictly decreasing sequence $\eta_{t_i^0} > \eta_{t_i^0+2} \ldots > \eta_{T_i^0} = 0$ such that in each equilibrium σ, $h_i(u) \in (\eta_{t-2}, \eta_t)$ iff $\sigma_i(u) = t$, and if $h(u) = 0$ then $\sigma_i(u) \geq T_i^0$. (Here, we take $\eta_{t_i^0-2} = \infty$.). Moreover, if $\mu(h^{-1}(0)) = 0$, then $T_i^0 = T_i^*$.*

Proof. By Lemma 5, if t is a best response of type u, and t' is a best response of type u' such that $h(u) > h(u')$, then $t \leq t'$. Thus, for any two types u, u' such that $h(u), h(u') > 0$ if the two types share $t \neq t'$as their best responses, it must be that $h(u) = h(u')$. However, by Lemma 8, the set of types with the same value of function h index has a measure 0. Let $\eta_t = \sup\{h(u) : t \in \sigma(u)\}$. For each u st. $h(u) > \eta_t$, t is not a best response. The claim follows. □

Using the notation from section A, we have

$$f_i^\sigma(t) = \mu\left(h^{-1}[\eta_t, \eta_{t-2}]\right), \text{ and } p_i^\sigma(t) = \frac{\mu(h^{-1}[\eta_t, \eta_{t-2}])}{\mu(h^{-1}[n_{t-2}, 0])}.$$

We take $\eta_t = 0$ for each $t > T_i^0$ and $t \in T_i$. The above proof implies that types $u \in h^{-1}(\eta_t)$ are indifferent between stopping in period t and $t+2$. By Lemma 3, for each $t \in T_{-i}$, and $t_i^0 < t < T_i^*$

$$p_i^\sigma(t) = \frac{1 - \mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}} \kappa_i^* \frac{1}{\frac{1}{\eta_t + 1} - \kappa_i^* \mathrm{e}^{-\Delta}}.$$

C.1.3. *Late game bounds I.* Let $T_i^\eta = \min\{t \in T_i : \eta_t \leq \eta\}$. Let $|x|_+ = \max(x, 0)$. Let $\zeta_1 > 0$ be small enough so that

$$\alpha_{\min} := \inf_{u: h(u) \leq \zeta_1} \left\| u_{-i}^* - u \right\| > 0.$$

Lemma 10. *Suppose that $h\left(u_{-i}^*\right) > 0$. There exists constant $C_0 < \infty$ and $\zeta > 0$ such that for each $\eta \leq \zeta$, all $t_1, t_2 \in T_i$ such that $T_\eta \leq t_1 < t_2 < T_{-i}^{*,\sigma}$,*

$$\sum_{t \in T_i : t_1 < t \leq t_2} \left| u_{-i}^* \cdot \left(w_{-i}^\sigma(t) - (1 - x_i^*) \right) \right|_+ f^\sigma(t) \leq C_0 \eta \left(F^\sigma(t) - \lambda \right).$$

The Lemma has two separate proofs depending on the value of $h\left(u_{-i}^*\right)$.

Case $h\left(u_{-i}^{*}\right)>0$.

Proof. By Lemma 7, for each $t\in T_{i}$, we have

$$\begin{aligned}
& u_{-i}^{*}\cdot\left(w_{-i}^{\sigma}(t)-(1-x_{i}^{*})\right)f^{\sigma}(t) \\
= & \int\limits_{u:\eta_{t}\leq h(u)<\eta_{t-2}} u_{-i}^{*}\cdot(x_{i}^{*}-x_{u})\,\pi(u)\,du \\
= & -\int\limits_{u:\eta_{t}\leq h(u)<\eta_{t-2}} h(u)\,f(u)\,du-\int\limits_{u:\eta_{t}\leq h(u)<\eta_{t-2}} Dh(u)\cdot\left(u_{-i}^{*}-u\right)\pi(u)\,du \\
\leq & \int\limits_{u:\eta_{t}\leq h(u)<\eta_{t-2}} Dh(u)\cdot\left(u-u_{-i}^{*}\right)\pi(u)\,du. \qquad (C.1)
\end{aligned}$$

To calculate the integral in the last line of (C.1), we switch to polar coordinates with center at u_{-i}^{*}. Let $S_{N-2}=\{x\in I_{0}:x\cdot x=1\}$ be the subset of the unit vectors (i.e., the $(N-2)$-dimensional sphere). For each $\eta>0$ and $x\in S_{N-2}$, if $\inf_{\alpha}\left(u_{-i}^{*}+\alpha x\right)<\eta_{t}$, let

$$\begin{aligned}
\alpha^{\max}(x,t) &= \sup\left\{\alpha:h\left(u_{-i}^{*}+\alpha x\right)\leq\eta_{t}\right\}, \\
\alpha^{\min}(x,t) &= \inf\left\{\alpha:h\left(u_{-i}^{*}+\alpha x\right)\leq\eta_{t}\right\}.
\end{aligned}$$

Otherwise, let $\alpha^{\mathrm{m..}}(x,t)=a^{\mathrm{m..}}(x,t-2)$. Finally, let

$$\alpha(x,t)=\alpha^{\max}(x,t)-\alpha^{\min}(x,t).$$

Then, due to the convexity of function h, we have

$$\begin{aligned}
& \{u:\eta_{t-2}\leq h(u)<\eta_{t}\} \\
= & \bigcup_{x\in S_{N-2}}\left\{u_{-i}^{*}+\alpha x:\alpha^{\min}(x,t-2)\geq\alpha\geq\alpha^{\min}(x,t)\ \text{ or }\alpha^{\max}(x,t)\leq\alpha\leq\alpha^{\min}(x,t-2)\right\}.
\end{aligned}$$

Let $|S_{N-2}|$be the Lebesgue measure of the $N-2$-dimensional sphere D_{N-1}. Then, the integral in the last line of (C.1) is equal to

$$\begin{aligned}
& \int\limits_{u:\eta_{t}\leq h(u)<\eta_{t-2}}\left((Dh(u))\left(u-u_{-i}^{*}\right)\right)\pi(u)\,du \qquad (C.2) \\
= & \frac{1}{|S_{N-2}|}\int\limits_{S_{N-2}}\begin{pmatrix} \int_{\alpha^{\min}(x,t-2)}^{\alpha^{\min}(x,t)}\left(Dh\left(u_{-i}^{*}+\alpha x\right)\cdot\alpha x\right)\pi\left(u_{-i}^{*}+\alpha x\right)\alpha^{N-2}d\alpha \\ +\int_{\alpha^{\max}(x,t)}^{\alpha^{\max}(x,t-2)}\left(Dh\left(u_{-i}^{*}+\alpha x\right)\cdot\alpha x\right)\pi\left(u_{-i}^{*}+\alpha x\right)\alpha^{N-2}d\alpha \end{pmatrix}dx.
\end{aligned}$$

Let $g(\alpha, x) = \alpha^{N-1}\pi\left(u_{-i}^{*} + \alpha x\right)$. Then, function g is Lipschitz in α with constant K_0. The above is bounded by

$$\leq \frac{1}{|S_{N-2}|}\int\limits_{x\in D^1} g\left(\alpha^{\max}(x, t-2), x\right)\begin{pmatrix} \int_{\alpha^{\min}(x,t-2)}^{\alpha^{\min}(x,t)}\left(Dh\left(u_{-i}^{*}+\alpha x\right)\cdot x\right)d\alpha \\ +\int_{\alpha^{\max}(x,t)}^{\alpha^{\max}(x,t-2)}\left(Dh\left(u_{-i}^{*}+\alpha x\right)\cdot x\right)d\alpha\end{pmatrix}dx \tag{C.3}$$

$$+\frac{1}{|S_{N-2}|}\int\limits_{x\in D^1} K_0\alpha(x, t-2)\begin{pmatrix} \int_{\alpha^{\min}(x,t-2)}^{\alpha^{\min}(x,t)}\left|Dh\left(u_{-i}^{*}+\alpha x\right)\cdot x\right|d\alpha \\ +\int_{\alpha^{\max}(x,t)}^{\alpha^{\max}(x,t-2)}\left|Dh\left(u_{-i}^{*}+\alpha x\right)\cdot x\right|d\alpha\end{pmatrix}dx.$$

We show that the first term is smaller or equal to 0. Notice that if $\alpha^{\max}(x, \eta) > 0$, then

$$h\left(u_{-i}^{*} + \alpha^{\max}(x, t)\, x\right) \leq \eta_t,$$

and if the inequality is strict, then $u_{-i}^{*} + \alpha^{\max}(x, t)\, x \in \mathrm{bd}\mathcal{U}_i$, and

$$h\left(u_{-i}^{*} + \alpha^{\max}(x, t)\, x\right) = h\left(u_{-i}^{*} + \alpha^{\max}(x, t-2)\, x\right) < \eta_t.$$

In particular,

$$h\left(u_{-i}^{*} + \alpha^{\max}(x, t-2)\, x\right) - h\left(u_{-i}^{*} + \alpha^{\max}(x, t)\, x\right) \leq \eta_{t-2} - \eta_t.$$

On the other hand, because h is convex (hence, continuous) and $h\left(u_{-i}^{*}\right) \geq \eta_{t-2}$, the continuity implies that

$$h\left(u_{-i}^{*} + \alpha^{\min}(x, t-2)\, x\right) = \eta_{t-2}.$$

Hence, the integral in the first line of (C.3) is equal to

$$= \int\limits_{x\in D^1} g\left(\alpha^{\max}(x, t-2), x\right)\begin{pmatrix} h\left(u_{-i}^{*}+\alpha^{\min}(x,t)\, x\right) - h\left(u_{-i}^{*}+\alpha^{\min}(x,t-2)\, x\right) \\ +h\left(u_{-i}^{*}+\alpha^{\max}(x,t-2)\, x\right) - h\left(u_{-i}^{*}+\alpha^{\max}(x,t)\, x\right)\end{pmatrix}dx$$

$$\leq \int\limits_{x\in D^1} g\left(\alpha^{\max}(x, t-2), x\right)\left(\eta_t - \eta_{t-2} + \eta_{t-2} - \eta_t\right)dx = 0.$$

Summing over the bounds (C.2) and (C.3), we obtain

$$\begin{aligned}
&\sum_{t\in T_i:t_1<t\leq t_2}\left|u_{-i}^*\cdot\left(w_{-i}^{\sigma}(t)-(1-x_i^*)\right)\right|_+ f^{\sigma}(t)\\
\leq&\frac{K_0}{|S_{N-2}|}\int\limits_{x\in D^1}\alpha(x,t_1)\sum_{t\in T_i:t_1<t\leq t_2}\left(\begin{array}{c}\int_{\alpha^{\min}(x,t-2)}^{\alpha^{\min}(x,t)}\left|Dh\left(u_{-i}^*+\alpha x\right)\cdot x\right|d\alpha\\+\int_{\alpha^{\max}(x,t)}^{\alpha^{\max}(x,t-2)}\left|Dh\left(u_{-i}^*+\alpha x\right)\cdot x\right|d\alpha\end{array}\right)dx\\
\leq&\frac{K_0}{|S_{N-2}|}\int\limits_{x\in D^1}\alpha(x,t_1)\left(\int\limits_{\alpha^{\min}(x,t_1)}^{\alpha^{\max}(x,t_1)}\left|Dh\left(u_{-i}^*+\alpha x\right)\cdot x\right|d\alpha\right)dx.
\end{aligned}$$

Because function h is convex and that $0\leq h\left(u_{-i}^*+\alpha x\right)\leq\eta_{t_1}$ for each $\alpha\in[\alpha^{\min}(x,t_1),\alpha^{\max}(x,t_1)]$, we have the integral in the brackets is not larger than $2\eta_{t_1}$. Hence, the above is bounded by

$$\begin{aligned}
\leq&\frac{2K_0}{|S_{N-2}|}\eta_{t_1}\int\limits_{x\in D^1}\alpha(x,\eta_{t_1})\,dx\\
\leq&\frac{2K_0}{|S_{N-2}|}\frac{1}{\pi_{\min}\alpha_{\min}^{N-2}}\eta_{t_1}\int\limits_{x\in D^1}\left(\int\limits_{\alpha^{\min}(x,\eta)}^{\alpha^{\max}(x,\eta)}\alpha^{N-2}\pi\left(u_{-i}^*+\alpha x\right)d\alpha\right)dx\\
=&\frac{2K_0}{|S_{N-2}|}\frac{1}{\pi_{\min}\alpha_{\min}^{N-2}}\eta_{t_1}\int\limits_{h(u)<\eta_t}\pi(u)\,du\\
\leq&\frac{2K_0}{|S_{N-2}|}\frac{1}{\pi_{\min}\alpha_{\min}^{N-2}}\eta_{t_1}\left(F^{\sigma}(t_1)-\lambda\right).
\end{aligned}$$

□

Case $h\left(u_{-i}^*\right)>0$.

Proof. Let $\alpha^{\max}=\sup\left\{\alpha:u_{-i}^*+\alpha x\in\mathcal{U}_i\text{ for some }x\in S_{N-2}\right\}$. As in the other case, we show that for each $t\in T_i$,

$$\begin{aligned}
&u_{-i}^*\cdot\left(w_{-i}^{\sigma}(t)-(1-x_i^*)\right)f^{\sigma}(t)\\
\leq&\frac{1}{|S_{N-2}|}\int\limits_{S_{N-2}}\left(\int_{\alpha(x,t)}^{\alpha(x,t-2)}\left(Dh\left(u_{-i}^*+\alpha x\right)\cdot\alpha x\right)\pi\left(u_{-i}^*+\alpha x\right)\alpha^{N-2}d\alpha\right)dx, \qquad \text{(C.4)}
\end{aligned}$$

where

$$\alpha(x,t)=\sup\left\{\alpha:h\left(u_{-i}^*+\alpha x\right)\leq\eta_t\right\},$$

and equal to α^* if the set is empty. Because h is positive, convex and $h\left(u^*_{-i}\right)=0$, the expression inside the integral (C.4) is always positive. Hence,

$$\sum_{t\in T_i:t_1<t\leq t_2}\left|u^*_{-i}\cdot\left(w^\sigma_{-i}(t)-(1-x^*_i)\right)\right|_+ f^\sigma(t) \tag{C.5}$$

$$=\frac{1}{|S_{N-2}|}\int\limits_{S_{N-2}}\left(\int_0^{\alpha(x,t_1)}\left(Dh\left(u^*_{-i}+\alpha x\right)\cdot x\right)\pi\left(u^*_{-i}+\alpha x\right)\alpha^{N-1}d\alpha\right)dx$$

$$\leq\frac{1}{|S_{N-2}|}\left(\pi\left(u^*_{-i}\right)+K_0\alpha^{\max}\right)\int\limits_{S_{N-2}}\left(\int_0^{\alpha(x,t_1)}\alpha^{N-1}\left(Dh\left(u^*_{-i}+\alpha x\right)\cdot x\right)d\alpha\right)dx.$$

where K_0 is the Lipschitz constant associated with density $\pi(.)$. By integration by parts,

$$\begin{aligned}&\int_0^{\alpha(x,t_1)}\alpha^{N-1}\left(Dh\left(u^*_{-i}+\alpha x\right)\cdot x\right)d\alpha\\ \leq\ &(\alpha(x,t_1))^{N-1}\eta-(N-1)\int_0^{\alpha(x,t_1)}\alpha^{N-2}h\left(u^*_{-i}+\alpha x\right)d\alpha\\ \leq&\eta(\alpha(x,t_1))^{N-1}=\eta(N-1)\int\limits_0^{\alpha(x,t_1)}\alpha^{N-2}d\alpha.\end{aligned}$$

Hence, (C.5) is not larger than

$$\leq\frac{1}{|S_{N-2}|}\frac{\pi\left(u^*_{-i}\right)+K_0\alpha^{\max}}{\pi_{\min}}(N-1)\eta\int\limits_{S_{N-2}}\left(\int\limits_0^{\alpha(x,t_1)}\pi\left(u^*_{-i}+\alpha x\right)\alpha^{N-2}d\alpha\right)dx$$

$$=\frac{1}{|S_{N-2}|}\frac{\pi\left(u^*_{-i}\right)+K_0\alpha^{\max}}{\pi_{\min}}(N-1)\eta\left(F^\sigma(t_1)-\lambda\right).$$

□

C.1.4. *Late game bounds II.*

Lemma 11. *For each $\delta>0$, there exists $\zeta_\delta>0$ such that for each $\eta\leq\zeta_\delta$,*

$$\sum_{t\in T_i:T_i^\eta<t\leq T_i^{*,\sigma}}p^\sigma(t)\left|u^*_{-i}\cdot\left(w^\sigma_{-i}(t)-(1-x^*_i)\right)\right|_+\leq\delta.$$

Proof. Inductively, define a sequence: $t_0=T_i^\eta$, and for each l,

$$t_t=\max\left(t>t_{l-1}:t\in T_i,\eta_t>\frac{1}{2}\eta_{t_{l-1}}\right).$$

The definition implies that $\eta_{t_{l+1}} < \eta_{t_l}$, and that $\eta_{t_{l+2}} \leq \frac{1}{2}\eta_{t_l}$. Hence, $\sum_{l\geq 1}\eta_{t_{l-1}} \leq 2\eta_{t_0} \leq 2\eta$.

By Lemma 8, the Lebesgue mass of set $\left\{u : h(u) \leq \frac{1}{2}\eta_{t_{l-1}}\right\}$ is at least $2^{-(N-1)}$ of the Lebesgue mass of set $\left\{u : h(u) \leq \eta_{t_{l-1}}\right\}$. It follows that

$$\frac{F^{\sigma}(t_{l-1})}{F^{\sigma}(t_l)} \leq \frac{\pi_{\max}\left|\left\{u : h(u) \leq \frac{1}{2}\eta_{t_{l-1}}\right\}\right|}{\pi_{\min}\left|\left\{u : h(u) \leq \eta_{t_{l-1}}\right\}\right|} \leq \frac{\pi_{\min}}{2^{N-1}\pi_{\max}}.$$

Using the above bound and Lemma 10, we obtain for each $l \geq 1$,

$$\begin{aligned}
&\sum_{t\in T_i: t_{l-1}<t\leq t_l} p^{\sigma}(t)\left|u^*_{-i}\cdot\left(w^{\sigma}_{-i}(t)-(1-x^*_i)\right)\right|_+ \\
\leq&\frac{1}{F(t_{l-1})}\sum_{t\in T_i: t_{l-1}<t\leq t_l}\frac{F^{\sigma}(t_{l-1})}{F^{\sigma}(t)}\left|u^*_{-i}\cdot\left(w^{\sigma}_{-i}(t)-(1-x^*_i)\right)\right|_+ f^{\sigma}(t) \\
\leq&\frac{2^{N-1}\pi_{\max}}{\pi_{\min}}\frac{1}{F(t_{l-1})}\sum_{t\in T_i: t_{l-1}<t\leq t_l}\left|u^*_{-i}\cdot\left(w^{\sigma}_{-i}(t)-(1-x^*_i)\right)\right|_+ f^{\sigma}(t) \\
\leq&\frac{2^{N}\pi_{\max}}{\pi_{\min}}C_0{}'\eta_{t_{l-1}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{t\in T_i: T_\eta<t\leq T_i^{*,\sigma}}\left|u^*_{-i}\cdot\left(w^{\sigma}_{-i}(t)-(1-x^*_i)\right)\right|_+ \\
\leq&\sum_{l\geq 1}\sum_{t\in T_i: t_{l-1}<t\leq t_l}\left|u^*_{-i}\cdot\left(w^{\sigma}_{-i}(t)-(1-x^*_i)\right)\right|_+ \\
\leq&\frac{2^{N}\pi_{\max}}{\pi_{\min}}C_0{}'\sum_{l\geq 1}\eta_{t_{l-1}} \leq \frac{2^{N+1}\pi_{\max}}{\pi_{\min}}C_0{}'\eta.
\end{aligned}$$

Take $\zeta_\delta = \left(\frac{2^{N+1}\pi_{\max}}{\pi_{\min}}C_0{}'\right)^{-1}\delta$. $\square$

C.1.5. *Late game bounds III.* Notice that the assumptions imply that

$$D_0 = u^*_{-i}\cdot(\mathbf{1}-x^*_i) - u^*_{-i}\cdot x_{-i} > 0.$$

Then,

$$p^*_i = \frac{1-\mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}}\frac{1}{\frac{u^*_{-i}\cdot(\mathbf{1}-x^*_i)}{u^*_{-i}\cdot x_{-i}}-\mathrm{e}^{-\Delta}} > 0. \tag{C.6}$$

Lemma 12. *For each $\delta > 0$, there exists $\zeta_\delta > 0$ such that for each $\eta \leq \zeta_\delta$,*

$$\prod_{t \in T_i : T_i^\eta < t \leq T_i^{*,\sigma}} \left(1 - p^\sigma(t)\right) \leq \left(1 - p_i^*\right)^{\frac{1}{2}\left(T_i^{*,\sigma} - T_i^\eta\right)} \exp(\delta).$$

Proof. If $t \in T_i$ and $t < T_{-i}^{*,\sigma}$, player $-i$ must be indifferent between yielding in periods $t-1$ and $t+1$ (see Lemma 4). By Lemma 3,

$$\begin{aligned} p^\sigma(t) =& \frac{1 - \mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}} \frac{1}{S_{-i}\left(u_{-i}^*, t\right) - \mathrm{e}^{-\Delta}} \\ =& \frac{1 - \mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}} \frac{1}{\frac{u_{-i}^* \cdot w_{-i}^\sigma(t)}{u_{-i}^* \cdot x_{-i}} - \mathrm{e}^{-\Delta}} \\ \geq& \frac{1 - \mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}} \frac{1}{\frac{u_{-i}^* \cdot \left(\mathbf{1} - x_i^*\right)}{u_{-i}^* \cdot x_{-i}} - \mathrm{e}^{-\Delta} + \frac{\left|u_{-i}^* \cdot \left(w_{-i}^\sigma(t) - \left(1 - x_i^*\right)\right)\right|_+}{u_{-i}^* \cdot x_{-i}}}. \end{aligned}$$

Using the definition of p_i^* from (C.6), after some calculations, we obtain

$$\begin{aligned} \frac{1 - p^\sigma(t)}{1 - p^*} &\leq 1 + \frac{1}{D_0 - \left(\mathrm{e}^\Delta - 1\right) u_{-i}^* \cdot x_{-i}} p^\sigma(t) \left|u_{-i}^* \cdot \left(w_{-i}^\sigma(t) - \left(1 - x_i^*\right)\right)\right|_+ \\ &\leq 1 + \frac{2}{D_0} p^\sigma(t) \left|u_{-i}^* \cdot \left(w_{-i}^\sigma(t) - \left(1 - x_i^*\right)\right)\right|_+ \end{aligned}$$

for $\Delta > 0$ small enough.

Thus,

$$\prod_{t \in T_i : T_i^\eta < t \leq T_i^{*,\sigma}} \left(1 - p^\sigma(t)\right) \leq \left(1 - p_i^*\right)^{\frac{1}{2}\left(T_i^{*,\sigma} - T_i^\eta\right)} \prod_{t \in T_i : T_i^\eta < t \leq T_i^{*,\sigma}} \frac{1 - p^\sigma(t)}{1 - p_i^*}.$$

By Lemma 11, if $\eta \leq \zeta_{\frac{D_0}{2}\delta}$,

$$\begin{aligned} \prod_{t \in T_i : T_i^\eta < t \leq T_i^{*,\sigma}} \frac{1 - p^\sigma(t)}{1 - p_i^*} &\leq \prod_{t \in T_i : T_i^\eta < t \leq T_i^{*,\sigma}} \left(1 + \frac{2}{D_0} p^\sigma(t) \left|u_{-i}^* \cdot \left(w_{-i}^\sigma(t) - \left(1 - x_i^*\right)\right)\right|_+\right) \\ &\leq \exp\left(\frac{2}{D_0} \sum_{t \in T_i : T_i^\eta < t \leq T_i^{*,\sigma}} p^\sigma(t) \left|u_{-i}^* \cdot \left(w_{-i}^\sigma(t) - \left(1 - x_i^*\right)\right)\right|_+\right) \\ &\leq \exp(\delta). \end{aligned}$$

$\square$

C.1.6. *Proof of Lemma 1.* The proof follows the same lines as the proof of Lemma 1. The assumptions imply that that $S_i^* > S_{-i}^* > 1$. Let $\zeta_0 = \frac{1}{2}\left(S_i^* - S_{-i}^*\right)$. For each type $u \in \mathcal{U}_{-i}$, each $t \in T_i$ st. $t < T_{-i}^{*,\sigma}$, we have

$$S_{-i}^{\sigma}(u,t) \leq S_{-i}^{\max} := \frac{\sup_{u\in\mathcal{U}_{-i}} u \cdot \mathbf{1}}{\inf_{u\in\mathcal{U}_{-i}} u \cdot (1-y_0)}.$$

(Note that the numerator is not necessarily equal to 1 because of the re-normalization.) Assume that $\eta \leq \min(\zeta_0, \zeta_1)$, where ζ_1 comes from Lemma 12. Let $q_i = P_i(S_i^* - \eta) > 0$. Let

$$x = \frac{S_{-i}^* - 1}{S_i^* - \eta - 1} < 1.$$

In the rest of the proof, we divide the time of the game into three zones:

- *Early game*: By Lemma 6, for each $\delta > 0$, there exists $\varepsilon > 0$, $\Delta_0 > 0$, and T_0 such that either (a) $F_{-i}^{\sigma}(t) \leq \delta$, or (b) $\sigma_i^T(u) \leq T_0$ for all types $u \in \mathcal{U}_i$ such that $S_i(u) \leq 1 + \varepsilon$. If (a), the thesis of the Lemma holds. On the contrary, from now on, assume (b) and $F_{-i}^{\sigma}(T_0) \leq \delta$. Let

$$y = \max\left(1, 2\frac{S_{-i}^* - 1}{\varepsilon}\right),$$

and we find $\Delta^* \leq \Delta_0$ such that $\Delta \leq \Delta^*$, we have

$$1 - \frac{1-\mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}}\frac{1}{\varepsilon} = 1 - \frac{1}{2}\frac{S_{-i}^* - 1}{S_{-i}^* - \mathrm{e}^{-\Delta}} y \frac{1-\mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}}\frac{1}{S_{-i}^* - \mathrm{e}^{-\Delta}} \geq \left(1 - \frac{1-\mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}}\frac{1}{S_{-i}^* - \mathrm{e}^{-\Delta}}\right)^y. \tag{C.7}$$

- *Middle game*: $T_0 \leq t < T_i^{\eta}$. In the middle game, $S_i(t) \geq 1 + \varepsilon$ for $t \in T_i$ and $S_{-i}(t) \leq S_{-i}^{\max}$ for $t \in T_{-i}$. By Lemma 3, we have

$$p_i^{\sigma}(t) \geq \frac{1-\mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}}\frac{1}{S_{-i}^{\max} - \mathrm{e}^{-\Delta}}, \text{ and } p_{-i}^{\sigma}(t) \leq \frac{1-\mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}}\frac{1}{\varepsilon}.$$

Then, inequality (C.7) implies that

$$1 - p_{-i}^{\sigma}(t) \geq \left(1 - \frac{1-\mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}}\frac{1}{S_{-i}^* - \mathrm{e}^{-\Delta}}\right)^y. \tag{C.8}$$

- *Late game*: $T_i^{\eta} \leq t < T^*$. By Lemma 12,

$$\prod_{t\in T_i: T_i^{\eta} < t \leq T_i^{*,\sigma}} \left(1 - p_i^{\sigma}(t)\right) \leq \prod_{t\in T_i: T_i^{\eta} < t \leq T_i^{*,\sigma}} \left(1 - p_i^*\right)\mathrm{e}.$$

Moreover, the choice of x and the fact that $x<1$ implies that

$$1-p^{\sigma}_{-i}(t)\geq 1-\frac{S^{*}_{-i}-\mathrm{e}^{-\Delta}}{S^{*}_{i}-\eta-\mathrm{e}^{-\Delta}}p^{*}_{i}\geq 1-xp^{*}_{i}\geq\left(1-p^{*}_{i}\right)^{x}.$$

where the last inequality holds for sufficiently small Δ (hence, sufficiently small $1-\mathrm{e}^{-2\Delta}$).

Notice that for each player l and each t, $\lambda=F^{\sigma}_{l}(T^{*})=F^{\sigma}_{l}(t)\prod_{s\in T_l:t\leq s\leq T^{*}}\left(1-p^{\sigma}_{l}(s)\right)$. The late game estimates imply that

$$F^{\sigma}_{-i}(T^{\eta}_{i})=\frac{\lambda}{\prod_{s\in T_{-i}:T^{\eta}_{i}\leq s\leq T^{*}}\left(1-p^{\sigma}_{-i}(s)\right)}\leq\frac{\lambda}{\prod_{s\in T_{-i}:T^{\eta}_{i}\leq s\leq T^{*}}\left(1-p^{*}_{i}\right)^{x}}=\lambda^{1-x}\left(F^{\sigma}_{i}(T^{\eta}_{i})\right)^{x}\mathrm{e}^{x}.$$

Further,

$$\begin{aligned}\frac{F^{\sigma}_{-i}(T_0)}{F^{\sigma}_{-i}(T^{\eta}_{i})}&=\frac{1}{\prod_{s\in T_{-i}:T_0\leq s\leq T}\left(1-p^{\sigma}_{-i}(s)\right)}\leq\prod_{s\in T_{-i}:T_0\leq s\leq T}\left(1-\frac{1-\mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}}\frac{1}{S^{*}_{-i}-\mathrm{e}^{-\Delta}}\right)^{-y}\\&\leq\left(\frac{F^{\sigma}_{i}(T_0)}{F^{\sigma}_{i}(T^{\eta}_{i})}\right)^{y}\leq\left(F^{\sigma}_{i}(T^{\eta}_{i})\right)^{-y},\end{aligned}$$

Together, we obtain,

$$F^{\sigma}_{-i}(T_0)\leq\lambda^{1-x}\left(F^{\sigma}_{i}(T^{\eta}_{i})\right)^{x-y}\leq\lambda^{1-x}q^{x-y}\mathrm{e}^{x}$$

where the last inequality comes from the fact that $F^{\sigma}_{i}(T^{\eta}_{i})\geq q$. If λ is sufficiently small, we obtain the contradiction with $F^{\sigma}_{-i}(T_0)\geq\delta$.

C.2. **Proof of Lemma 2.** Let $X_{-i}=\arg\max_{x\in m_i}u^{*}_{-i}\cdot x_{-i}$ be the set of optimal choices of player $-i$. Let $\upsilon_{-i}=\max_{x\in m_i}u^{*}_{-i}\cdot x_{-i}\leq\frac{1}{2}-\eta$ be $-i$'s optimal payoff from concession. Assume that ε is small enough that $\frac{1}{2}\left(\frac{1}{2}-\varepsilon\right)>\left(\frac{1}{2}-\eta\right)\left(\frac{1}{2}+\eta\right)$.

For each v, let $X(v)=\left\{x\in X:u^{*}_{-i}\cdot x\leq 1-v\right\}$ be the set of player i's allocations of player i that leave $-i$'s payoff at least v. (To see this, recall that if x is allocation of i, then the payoff of $-i$ is $u^{*}_{-i}\cdot(\mathbf{1}-x)=1-u^{*}_{-i}\cdot x$.) Notice that $\mathbf{1}-X_{-i}\subseteq X(\upsilon_{-i})\subseteq$

$\frac{1-\upsilon_{-i}}{1/2} X\left(\frac{1}{2}\right)$. Then, for each $u_i \in \mathcal{U}_i$

$$\begin{aligned}\max_{x\in X_{-i}} u_i\cdot(\mathbf{1}-x_i) &\leq \max_{x\in X_{-i}(\upsilon_{-i})} u_i\cdot x \leq \max_{x\in \frac{1-\upsilon_{-i}}{1/2} X_{-i}\left(\frac{1}{2}\right)} u_i\cdot x\\ &= \frac{1-\upsilon_{-i}}{1/2} \max_{x\in X_{-i}\left(\frac{1}{2}\right)} u_i\cdot x\\ &= \frac{1-\upsilon_{-i}}{1/2} \max_{x\in m_{-i}^{1/2}} u_i\cdot x.\end{aligned}$$

The inequality implies that for each $t\in T_{-i}$, and each $u_i\in\mathcal{U}_i$,

$$S_i\left(u_i\right) \leq \frac{\max_{x\in X_{-i}} u_i\cdot(\mathbf{1}-x)}{\max_{x\in m_i} u^*_{-i}\cdot x} \leq \frac{1-\upsilon_{-i}}{1/2}.$$

Additionally, for each $t\in T_i$

$$S^\sigma_{-i}\left(u^*_i,t\right) \geq \min_{u_i\in\mathcal{U}_i} \frac{\min_{x_i\in\arg\max_{x\in m_{-i}} u_i\cdot x} u_i\cdot(\mathbf{1}-x_i)}{\max_{x\in m_i} u^*_{-i}\cdot x} \geq \frac{\frac{1}{2}-\varepsilon}{\upsilon_{-i}} > \frac{1-\upsilon_{-i}}{1/2},$$

where the last inequality comes from By Lemma 3, in each period, the concession rate of player i is strictly smaller than the concession rate of player $-i$

$$\begin{aligned}p^\sigma_i\left(t\right) \leq p^\Delta_i &= \frac{1-\mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}} \frac{1}{\frac{\frac{1}{2}-\varepsilon}{\upsilon_{-i}}-\mathrm{e}^{-\Delta}}\\ &< p^\Delta_{-i} = \frac{1-\mathrm{e}^{-2\Delta}}{\mathrm{e}^{-\Delta}} \frac{1}{\frac{\frac{1}{2}}{\upsilon_{-i}}-\mathrm{e}^{-\Delta}} \leq p^\sigma_{-i}\left(t\right).\end{aligned}$$

It follows from the standard arguments (see for instance the proof of Lemma 1) that

$$\frac{F^\sigma_i\left(t^0_i+2\right)}{F^\sigma_{-i}\left(t^0_{-i}+2\right)} = \frac{\lambda/\prod_{t\in T_i:t^0_i\leq t\leq T^*_i}\left(1-p^\sigma_i\left(t\right)\right)}{\lambda/\prod_{t\in T_{-i}:t^0_{-i}\leq t\leq T^*_{-i}}\left(1-p^\sigma_{-i}\left(t\right)\right)} < \left(\frac{1-p^\Delta_{-i}}{1-p^\Delta_i}\right)^{\frac{1}{2}T^*} \to 0,$$

as $T^*\to\infty$. Similar arguments to those used in the proof of Lemma 1 show that T^* is arbitrarily large if Δ and λ are small. Thus, for sufficiently small Δ and λ, $f^\sigma_i\left(t_i\right)$ is arbitrarily close to 1.

C.3. **Proof of Theorem 3.** Consider first the case when $i=2$. Suppose first that player $-i$ proposes a menu $m^{1/2}$. Then, by Lemma 2, if player i counteroffer leaves player $-i$ with payoffs significantly less than $\frac{1}{2}$, then player i concedes quickly, which leads to payoffs (4.4). Otherwise, if player $-i$ is counter-offered at least $\frac{1}{2}$, any such offer cannot lead to higher payoffs than $\max_{x \in m^{1/2}} u_i \cdot x$ for player i. Hence, payoffs (4.4) are the only outcome of the subgame.

Further, if player $-i$ proposes a menu that strictly includes $m^{1/2}$, then, as long as it makes a difference, i.e., if any type of player i picks an allocation from the menu that does not belong to $m^{1/2}$, such an allocation leads to payoffs that are lower than $\frac{1}{2}$ for player $-i$.

Finally, the case when player $-i$ proposes a completed menu that does not contain $m^{1/2}$ is discussed in the text.

The case $i = 2$ is described in the main body of the paper. Here, we discuss the case $i = 1$. Observe first that by choosing menu $m_i = \{x_{-i}\}$ such that $u^*_{-i} \cdot x_{-i} \geq \frac{1}{2}$,player i with signal u_i ensures payoff $u_i \cdot (\mathbf{1} - x_{-i})$. This is because the same argument as in the main body of the text imply that any counteroffer menu of player $-i$ that does not include $\mathbf{1} - x_{-i}$ is going to make player $-i$ weaker, and the loser in the war of attrition (by Lemma 1). Hence, player i can ensure the payoffs at least (4.4).

Alternatively, any menu m_i that is strictly separated from $m^{1/2}$ can be counter-offered with $m^{1/2}$. In such a case, Lemma 2 shows that player i is going to concede in the continuation game, which leads to payoffs (4.4).To sum up, player $-i$ can ensure the payoff of $\frac{1}{2}$, and player i, of $\max_{x \in m^{1/2}} u_i \cdot x$. This concludes the proof of the Theorem.

Appendix D. Proof of Proposition 1

As we discuss in the main body of the paper, the equilibrium has three phases. We start the discussion from the last phase. Let $F^k_j(t)$ denote the probability that type u^k_j survives till period t.

(1) *War of attrition with one sides active.* For each period $t \in T_j$ such that $t^1_i < t < T^*_j$, the two remaining types u^v_{-i} and u^c_i concede at constant rates that make the opponent type indifferent between conceding and waiting. One can calculate using Lemma 3 that the concession rates are equal to ,

$$p^2_i = \left(\mathrm{e}^{\Delta} - \mathrm{e}^{-\Delta}\right) \frac{1}{\frac{1}{b} - \mathrm{e}^{-\Delta}} \text{ and } p^2_{-i} = \left(\mathrm{e}^{\Delta} - \mathrm{e}^{-\Delta}\right) \frac{1}{\frac{1}{a} - \mathrm{e}^{-\Delta}}.$$

where the approximation is when $\Delta \to 0$. Here, $\frac{1}{b}$ is the strength of type u^v_{-i} facing u^c_i (winning payoff is 1 and the concession payoff is b); analogously, $\frac{1}{a}$ is the strength of type u^c_i facing u^v_{-i}. The concession rate of player $-i$ is higher.

The phase ends when the strategic types fully reveal themselves.

Importantly, the two concession rates are too slow for the other two types (u_i^v and u_{-i}^c)

$$p_i^2 < \left(\mathrm{e}^{\Delta} - \mathrm{e}^{-\Delta}\right) \frac{1}{\frac{1-a}{a} - \mathrm{e}^{-\Delta}} \text{ and } p_{-i}^2 < \left(\mathrm{e}^{\Delta} - \mathrm{e}^{-\Delta}\right) \frac{1}{\frac{1-b}{b} - \mathrm{e}^{-\Delta}};$$

each of them would prefer to concede immediately. (To see it, notice that type u_{-i}^c winning payoff against u_i^c is equal to $1-a$. Hence, the strength of u_{-i}^c is equal to $\frac{1-a}{a}$.) This ensures that none of those two types has a profitable deviation to reach the third phase. We have

$$F_{-i}\left(t_{-i}^1+1\right) = F_{-i}^v\left(t_{-i}^1+1\right) = \left(1-p_{-i}^2\right)^{-\frac{1}{2}\left(T^*-t_i^1\right)}\lambda - \lambda,$$
$$F_i\left(t_i^1+1\right) = F_i^c\left(t_i^1+1\right) = \left(1-p_i^2\right)^{-\frac{1}{2}\left(T^*-t_i^1\right)}\lambda - \lambda,$$

When $\Delta \to 0$, this is approximately equal to

$$F_j\left(t_j^1+1\right) + \lambda \approx \mathrm{e}^{-\gamma_j^2\left(T^*-t_j^1\right)\Delta}\lambda v.$$

where $\gamma_i^2 = \frac{b}{1-b} < \frac{a}{a-1} = \gamma_{-i}^2$.

(2) *War of attrition with both sides active.* For each period $t \in T_j$ such that $t_i^0 < t < t_j^1$, the average concession allocation of player j conditionally on the concession is equal to

$$\begin{aligned} w_j &= \alpha\left(1, 1-b\right) + \left(1-\alpha\right)\left(1-a, 1\right) \\ &= \left(\frac{1}{2} + \frac{1}{2}a\frac{1-b}{b}, \frac{1}{2} + \frac{1}{2}b\frac{1-a}{a}\right), \text{ where} \\ \alpha &= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{b} - \frac{1}{a}\right). \end{aligned}$$

By Lemma 3,

$$p^1 = \left(\mathrm{e}^{\Delta} - \mathrm{e}^{-\Delta}\right) \frac{1}{\frac{1}{2a} + \frac{1}{2b} - \frac{1}{2} - \mathrm{e}^{-\Delta}}.$$

(Note that $\frac{1}{2a} + \frac{1}{2b} - \frac{1}{2} = \frac{w_j^{(c)}}{a} = \frac{w_j^{(v)}}{b}$ is equal to the strength of type u_j^c and/or u_j^v who wins with allocation w_j.) In order to ensure that the average concession allocation is equal to w_j, the types must concede with probability $p_j^1\left(u^k\right) =$

$\alpha^k p^1$, where

$$\alpha^k = \begin{cases} \alpha, & \text{if } k = v, \\ 1-\alpha, & \text{if } k = c. \end{cases}$$

Hence,

$$\begin{aligned} F_j\left(t_j^0+1\right)+\lambda &= \left(1-p^1\right)^{-\frac{1}{2}\left(t_j^1-t_j^0\right)} \lambda \left(F_j\left(t_j^0+1\right)+\lambda\right) \\ &\approx \mathrm{e}^{-\frac{1}{2}\gamma^1\left(t_j^1-t_j^0\right)\Delta-\frac{1}{2}\gamma_j^2\left(T^*-t_j^1\right)\Delta}\lambda. \end{aligned} \tag{D.1}$$

Moreover, because $F_j^k(t) = F_j^k(t-2) - \alpha^k p^1 F_j(t)$, it is easy to check that for each $t > t_j^0$,

$$\frac{F_j^k(t-2)}{F_j(t-2)} - \alpha^k\left(1-p^1\right) = \frac{1}{1-p^1}\left(\frac{F_j^k(t)}{F_j(t)} - \alpha^k\left(1-p^1\right)\right).$$

Hence,

$$\begin{aligned} \frac{F_j^k\left(t_i^0+1\right)}{F_j\left(t_i^0+1\right)} =& \alpha^k\left(1-p^1\right)\left(1-\left(1-p^1\right)^{-\frac{1}{2}\left(t_j^1-t_j^0\right)-O(1)}\right) \\ &+ \left(1-p^1\right)^{-\frac{1}{2}\left(t_j^1-t_j^0\right)-O(1)}\frac{F_j^k\left(t_j^1\right)}{F_j\left(t_j^1\right)}, \end{aligned}$$

where $O(1) \leq 1$. If we take $\gamma^1 = \frac{2}{\frac{1}{a}+\frac{1}{b}-3}$, the latter is approximately equal to

$$\frac{F_j^k\left(t_i^0+1\right)}{F_j\left(t_i^0+1\right)} \approx \alpha^k\left(1-\mathrm{e}^{-\gamma^1\left(t_j^1-t_j^0\right)\Delta}\right) + \mathrm{e}^{-\gamma^1\left(t_j^1-t_j^0\right)\Delta}\mathbf{1}_{(k,j)\in\{(c,i),(v,-i)\}}. \tag{D.2}$$

(3) *Atom concession.* In its first period t_i^0, each type u_i^k of player i concedes with probability $1 - F_i^k\left(t_i^0+1\right)$. If $t_i^0 = 2$, then player $-i$ does not concede in her first period $t_{-i}^0 = 1$.

Let $X^1 = \left(t_j^1 - t_j^0\right)\Delta$ and $X^2 = \left(T^* - t_j^1\right)\Delta$ for some j (neither of the two quantities depends on j but for more that $O(\Delta)$). It follows from (D.2) implies that

$$\pi\left(u_{-i}^c\right) \approx \alpha^c\left(1-\mathrm{e}^{-X^2}\right).$$

Hence, (D.2) allows to determine X^2 if $\pi\left(u_{-i}^c\right) \leq 1-\pi^*$ and $\pi^* > \alpha$. Further, (D.1) implies that

$$X^1 \approx -2\frac{1}{\gamma^1}\log\lambda - \frac{1}{\gamma^1}\gamma_{-i}^2 X^2.$$

This makes sure that the probabilities add up for player $-i$. For player $-i$, let $\rho = \gamma_i^2/\gamma_{-i}^2 < 1$. Then, (D.1) implies that

$$\begin{aligned} F_i\left(t_i^0+1\right)+\lambda &\approx \mathrm{e}^{-\frac{1}{2}\gamma^1 X^1}\mathrm{e}^{-\frac{1}{2}\gamma_i^2 X^2}\lambda \\ &\approx \mathrm{e}^{-\frac{1}{2}\gamma^1 X^1}\left(\mathrm{e}^{-\frac{1}{2}\gamma_{-i}^2 X^2}\lambda\right)^{\rho}\lambda^{1-\rho} \\ &\approx \mathrm{e}^{-\frac{1}{2}\gamma^1 X^1(1-\rho)}\lambda^{1-\rho}, \end{aligned} \tag{D.3}$$

where in the last equality we used the fact that $\mathrm{e}^{-\frac{1}{2}\gamma^1 X^1}\mathrm{e}^{-\frac{1}{2}\gamma_{-i}^2 X^2}\lambda = 1$. Hence, for appropriately small λ, $F_i\left(t_i^0+1\right) < \min_k \pi_i\left(u_i^k\right)$. This verifies that the probabilities add up for player i as well.

Appendix E. Proof of Theorem 4

E.1. **Outline of the argument.** In this subsection, we describe the main structure of the argument, with notation and key steps. The proofs of the key lemmas can be found in the rest of the section.

E.1.1. *Menus.* In the first step, we describe the notation related to the linear menus. For each player i, define two extreme allocations in menu $-i$: for each $k = 1, 2$, let

$$y_i^k = \begin{cases} \left(\frac{v_{-i}}{\psi_i}{}_{k\text{th coordinate}}, 0_{-k\text{th coordinate}}\right), & \text{if } \psi_i \geq v_{-i}, \\ \left(1_{k\text{th coordinate}}, \frac{v_{-i}-\psi_i}{\psi_i}{}_{-k\text{th coordinate}}\right) & \text{otherwise.} \end{cases}$$

Then, $y_i^k \in m_{-i}$ for each i and k and

$$\mathrm{bd}m_{-i} = \mathrm{con}\left\{y_i^1, y_i^2\right\}$$

be the outer boundary of menu m_{-i}. Let

$$\mathcal{U}_i^k = \left\{u \in \mathcal{U}_i : \arg\max_{x\in m_{-i}} u\cdot x = \left\{y_i^k\right\}\right\}$$

be the set of player i types for whom y_i^k is their optimal choice. Then, $\mathcal{U}_i = \mathcal{U}_i^1 \cup \{\psi_i\} \cup \mathcal{U}_i^2$.

For each allocation, we define projections on the menu boundary. For each player i, each side k, and each allocation $x \neq \mathbf{0}$, let $P_i^k x, R_{-i}^k x \geq 0$ be uniquely defined by

$$\sum_k \left(P_i^k x\right) = 1 \text{ and } \sum_k \left(P_i^k x\right) y_i^k = \alpha x \text{ for some } \alpha > 0,$$
$$\sum_k \left(R_{-i}^k x\right) = 1 \text{ and } \sum_k \left(R_{-i}^k x\right)\left(\mathbf{1} - y_{-i}^k\right) = \alpha x \text{ for some } \alpha > 0.$$

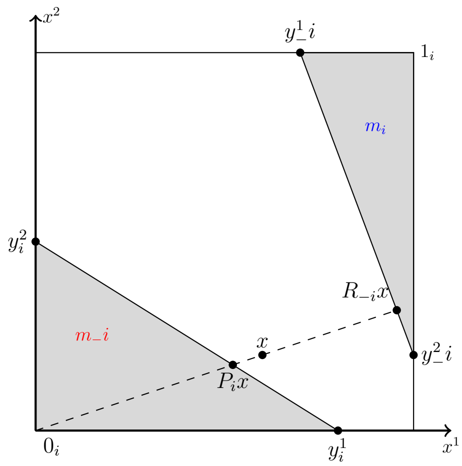


FIGURE E.1

Let

$$P_ix = \sum_k \left(P_i^k x\right) y_i^k \text{ and } R_{-i}x = \sum_k \left(R_{-i}^k x\right)\left(\mathbf{1} - y_{-i}^k\right).$$

Then, $P_i^k x$ is the "k " th coordinate of the projection of x on the boundary of menu m_{-i}; $R_i^k x$ is the "k " th coordinate of the projection of x on the line containing the boundary of menu m_i (expressed as allocations of player i). See the left panel of Figure E.1

E.1.2. *Sorting.* Next, we show that the equilibrium types can be partially sorted.

Lemma 13. *For each equilibrium σ', there exists an equilibrium σ with exactly the same payoffs, $T_i^{*,\sigma} = T_i^{*,\sigma'}$ for each i, and such that for each player i and each $k = 1, 2$, there exists monotonic sequences $(-1)^k \eta_i^k(t) \geq (-1)^k \eta_i^k(t+2), t \in T_i$, such that $\eta_i^k(T_i^{*,\sigma}) = \psi_i^{(1)}$ and for each u,*

$$\sigma(u) = t \text{ and } \sigma_i^M = y_i^k \text{ and iff } \left(u^{(2)} - \psi_i^{(2)}\right) \in \left[\eta_i^k(t+2), \eta_i^k(t)\right).$$

From now on, we assume that the equilibrium satisfies the thesis of the Lemma.

Define a vector

$$\gamma = (-1, 1) \in \mathbb{R}^2. \tag{E.1}$$

Then, $u_i^k(t) := \psi_i + \eta_i^k(t)\gamma \in \mathcal{U}_i^k$ is the unique type u such that $\eta_i^k(t) = u^{(2)} - \psi_i^{(2)}$. By the Lemma, $u_i^k(t)$ is the "highest" type to yield in period t among the types who choose y_i^k. We also take $\eta_i^k(t) = 0$ for each $t > T^*$.

E.1.3. *Notation.* For each function $f : T_i \to \mathbb{R}$,we write

$$\Delta f(t) = f(t) - f(t+2).$$

We use the sorting properties to rewrite the definitions from Appendix A.1. For each η, let

$$F(\eta) = \sum_k \pi\left\{u : (-1)^k \left(u^{(2)} - \psi_i^{(2)}\right) \geq (-1)^k \eta_i^k\right\}$$

be the mass of the types that are“higher” than η. By assumptions, F is differentiable, and its derivative is Lipschitz continuous with constant $K < \infty$. Let

$$f^* = \frac{dF}{d\eta}(0) > 0.$$

For each player i, each $t \in T_i, t \leq T_i^{*,\sigma}$, each k, let

$$F_i^k(t) = (-1)^k \left(F\left(\eta_i^k(t)\right) - F(0)\right) \text{ and } Q_i^k(t) = \frac{\Delta F_i^k(t)}{\sum_l \Delta F_i^l(t)}.$$

Using this notation, we can rewrite the conditional probability of yielding in period t as

$$p^\sigma(t) = \frac{\sum_k \Delta F_i^k(t)}{\sum_k F_i^k(t) + \lambda},$$

and the win outcomes of player $-i$ as

$$w_{-i}(t) = \sum_k Q_i^k(t)\left(\mathbf{1} - y_i^k\right), \text{ and}$$

$$y_{-i}(t) = \frac{\mathrm{e}^{-\Delta} p^\sigma(t)}{\mathrm{e}^{-2\Delta} p^\sigma(t) + (1 - \mathrm{e}^{-2\Delta})} \sum_k Q_i^k(t)\left(\mathbf{1} - y_i^k\right).$$

E.1.4. *Best response properties.* Next, we provide a characterization of the best response concession thresholds. For each $t \in T_i, t \leq T^{*,\sigma}$ define $x_i(t) \in X$ to be the unique allocation such that, if offered in period t,it would make each of the types $u_i^k(t)$ indifferent to yielding:

$$u_i^k(t) \cdot \left(x_i(t) - y_i^k\right) = 0 \text{ for each } k. \tag{E.2}$$

We refer to $x_i(t)$ as the *indifference point* of player i.

We say that *player i is active on side k in period $t \in T_i$* if $\eta_i^k(t) > \eta_i^k(t+2)$. Because of Lemma 4, in each period before T^*(with a possible exception of the first one), each player must be active on at least one side.

Lemma 14. *If player i is active on side k in period t, then, it must be that*

$$u_i^k(t+2)\cdot\left(y_i(t+1)-y_i^k\right)\leq 0,\ \textit{and}$$
$$u_i^k(t)\cdot\left(y_i(t-1)-y_i^k\right)\geq 0.$$

If player i is active on both sides in periods t and $t-2$, then

$$y_i(t-1)=x_i(t). \tag{E.3}$$

Proof. A straightforward corollary to Lemma 3. □

E.1.5. *Late game: Estimates.* We being the analysis of the late game. For each i and each η, define

$$T_i^\eta=\max\left\{t:\sum_k\eta_i^k(t)\geq\eta\right\}$$

and let $T^\eta=\max_i T_i^\eta$. We refer to periods $t>T$ as the *late game.* If η is small, all the remaining types in the late game are very close to ψ_i. The equilibrium behavior has many natural approximations. For each $t\in T_i$, let

$$\overline{P_i^k}(t)=\frac{\eta_i^k(t)}{\sum_l\eta_i^l(t)}\text{ and }\overline{Q_i^k}(t)=\frac{\Delta\eta_i^k(t)}{\sum_l\Delta\eta_i^l(t)}.$$

We have a simple observation: for each t,

$$\Delta\overline{P_i^k}(t)=\frac{\sum_l\Delta\eta_i^l(t)}{\sum_l\eta_i^l(t)}\left(\overline{Q_i^k}(t)-\overline{P_i^k}(t+2)\right). \tag{E.4}$$

Additionally, we have the following approximations:

Lemma 15. *There exists constant C that is independent from λ and β such that if $\eta\leq\frac{1}{4K}$, then for each i,k, each $t\in T_i$ and $t>T^\eta$,*

$$\left|P_i^kx_i(t)-\overline{P_i^k}(t)\right|\leq C\left(\sum_l F_i^l(t)\right),$$

$$\left|\Delta P_i^kx_i(t)-\Delta\overline{P_i^k}(t)\right|\leq C\left(\sum_l\Delta F_i^l(t)\right),$$

$$\left|Q_i^k(t)-\overline{Q_i^k}(t)\right|\leq C\left(\sum_l F_i^l(t)\right),$$

$$\left|\frac{\sum_l\Delta\eta_i^l(t)}{\sum_l\eta_i^l(t)}-\frac{\sum_l\Delta F_i^l(t)}{\sum_l F_i^l(t)}\right|\leq C\left(\sum_l\Delta F_i^l(t)\right).$$

E.1.6. *Late game: both sides active.* We use the above estimates to establish the key technical property of the late game:

Lemma 16. *There exists $n > 0$ that is independent from λ and Δ and such that, if $\lambda < \frac{1}{4}n$, then, for each $t > T^n$, each player is active on each side.*

Together with Lemma 14, the result implies that for $\eta > 0$ sufficiently small, each $t > T^\eta$, each player i such that $t \in T_i$, (E.3) holds.

E.1.7. *Late game: diagonal.* Finally, we show that the win outcome must remain close to the diagonal. Recall that $\gamma = (-1, 1)$ is defined in (E.1). Then, $|\gamma \cdot x|$ measures the distance of allocation x from the diagonal.

Lemma 17. *There exists $\Delta^*, \lambda^* > 0$ such that for each $\delta > 0$, there exists $\eta_\delta \leq n$ such that for each $\Delta \leq \Delta^*$, $\lambda \leq \lambda^*$ for each $t > T^{\eta_\delta}, t \in T_i$,*

$$\|w_i(t) - (\mathbf{1} - e_i(t))\| \leq \delta.$$

E.1.8. *Proof of Theorem 4.* Let $\xi = \frac{1}{3}\left(S_j^* - S_{-j}^*\right) > 0$ and let

$$x = \frac{S_{-j}^* + \xi - 1}{S_j^* - \xi - 1} < 1.$$

As in the proof of Lemma 1, let $S_i(t)$ be defined as the maximum strength of the type conceding in period t. Then, for each player j, $t > T^{\eta_\delta}, t \in T_{-i}$, we have

$$\begin{aligned} S_i(t) &= \max_k \max_{\eta \in \left[\eta_i^k(t+2), \eta_i^k(t)\right)} \frac{(\psi_i + \eta\gamma) \cdot w_i(t)}{(\psi_i + \eta\gamma) \cdot y_i^k} \\ &= \max_k \max_{\eta \in \left[\eta_i^k(t+2), \eta_i^k(t)\right)} \frac{1 - \alpha_{-i}^* + \left((\psi_i + \eta\gamma) \cdot \left(w_i(t) - \left(\mathbf{1} - e_{-i}^*\right)\right)\right)}{\alpha_i^* + \eta\gamma \cdot y_i^k}. \end{aligned}$$

Hence, by Lemma 17, there exists $\eta^*, \lambda^*, \Delta^* > 0$ such that for each $\eta \leq \eta^*, \eta \leq \lambda^*, \Delta \leq \Delta^*$, and each $t \in T_i$we have

$$\begin{aligned} S_j(t) &\geq S_j^* - \xi, \\ S_{-j}(t) &\leq S_{-j}^* + \xi. \end{aligned}$$

The rest of the proof follows the same three-zone strategy as the proof of Lemma 1. We omit the details.

E.2. **Proof of Lemma 13.**

Lemma 18. *For each k, each allocation x, if $u^{(k)} > v^{(k)}$ for $u, v \in \mathcal{U}$,and either (a) $\left(y_i^k\right)^{(-k)} = 0$, or (b) $1 = \left(y_i^k\right)^{(-k)}$ and $x^{(-k)} \geq \left(y_i^k\right)^{(-k)} \geq 0$, we have*

$$\frac{u \cdot x}{u \cdot y_i^k} \leq \frac{v \cdot x}{v \cdot y_i^k}.$$

Proof. Consider case (a).Then

$$\frac{u \cdot x}{u \cdot y_l^k} = \frac{\left(1 - u^{(-k)}\right) x^{(k)} + u^{(-k)} x^{(-k)}}{\left(1 - u^{(-k)}\right) \left(y_i^k\right)^{-k}} = \left(y_i^k\right)^{-k} \left[x^{(k)} + \frac{\alpha^{-k}}{1 - \alpha^{-k}} y^{-k}\right]$$

The above expression is increasing in $u^{(-k)}$. Next, suppose that $y_l^k = \left(1, \frac{u_l - \beta_l^k}{\beta_l^{-k}}\right)$. If $y \in Y_l$, then $y^k \leq 1$and $y^{-k} \geq y_l^{-k}$. We have

$$\frac{\alpha \cdot y}{\alpha \cdot y_l^k} = \frac{y^k - \alpha^{-k}\left(y^k - y^{-k}\right)}{1 - \alpha^{-k}\left(1 - y_l^{-k}\right)}.$$

□

Corollary 1. *For each equilibrium σ, any two types $\psi_i + \eta\gamma, \psi_i + \eta'\gamma \in \mathcal{U}$ and such that $(-1)^k \eta' \leq (-1)^k \eta$, we have*

$$S_i^\sigma\left(\psi_i + \eta\gamma, t\right) \leq S_i^\sigma\left(\psi_i + \eta'\gamma, t\right).$$

Proof. The result follows from Lemma 18, and the fact that, because $y_i(t) \notin \mathrm{int} m_{-i}$, we have $\left(y_i(t)\right)^{(-k)} \geq \left(y_i^k\right)^{(-k)}$. □

We check that the derivative of the above expression with respect to α^{-k} is equal to

$$= \frac{\alpha^{-k}\left[y^{-k} - y_l^{-k} y^k\right]}{\left(1 - \alpha^{-k}\left(1 - y_l^{-k}\right)\right)^2} \geq 0.$$

Fix an equilibrium σ. For each player i, each k, choose a monotonic sequence $(-1)^k \eta_i^k(t) \geq (-1)^k \eta_i^k(t+2), t \in T_i$, such that for each $t \in T_i$,

$$\pi\left\{\beta + \gamma\eta : (-1)^k \eta \in \left[\eta_i^k(t+2), \eta_i^k(t)\right)\right\} = \int_{\mathcal{U}_i^k} \sigma(u|t)\, d\pi(u).$$

is equal to the probability that a type in $\mathcal{U}_i^k$ stops in period t in equilibrium σ. Consider a strategy

$$\sigma'(u) = t \text{ and } \sigma_i^M = y_i^k \text{ and } \text{ iff } \left(u^{(2)} - \psi_i^{(2)}\right) \in \left[\eta_i^k(t+2), \eta_i^k(t)\right).$$

We going to show that $\left(\sigma', \sigma^M\right)$ is an equilibrium with the same payoffs. First, notice that the strategy σ_i' of player i leads to the same probabilities of yielding by player i as well as the same outcomes. It follows that the payoffs of player $-i$ are not affected by the modification.

Second, we are going to show that t is a best response for each type $u = \beta + \gamma\eta$ such that $(-1)^k \eta \in \left(\eta_i^k(t+2), \eta_i^k(t)\right)$. On the contrary, suppose that t is not a best response for u. Notice that if the interval is not empty, t is played with strictly positive probability under strategy σ. Hence, there is some type $u' = \beta + (-1)^k \eta'\gamma \in \mathcal{U}_i^k$ for which t is a best response, $u' \neq u$. Suppose that $(-1)^k \eta' > (-1)^k \eta$. By Corollary 1 and Lemma 5, the best response of all types $v = \beta + \eta''\gamma$ such that $(-1)^k \eta'' \leq (-1)^k \eta$ is strictly larger than t. But this implies that

$$\begin{aligned}
\sum_{s\in T_i: s>t} \int_{\mathcal{U}_i^k} \sigma(u|s)\, d\pi(u) &\geq \pi\left\{\beta + \gamma\eta'' : (-1)^k \eta'' \leq (-1)^k \eta\right\} \\
&= \sum_{s\in T_i: s>t} \pi\left\{\beta + \gamma\eta'' : (-1)^k \eta'' (-1)^k \eta \in \left[\eta_i^k(s+2), \eta_i^k(s)\right)\right\} \\
&+ \pi\left\{\beta + \gamma\eta'' : (-1)^k \eta'' (-1)^k \eta \in \left[\eta_i^k(t+2), \eta\right)\right\} \\
&\geq \sum_{s\in T_i: s>t} \int_{\mathcal{U}_i^k} \sigma(u|t)\, d\pi(u) + \pi\left\{\beta + \gamma\eta'' : (-1)^k \eta'' (-1)^k \eta \in \left[\eta_i^k(t+2), \eta\right)\right\}. \\
&> \sum_{s\in T_i: s>t} \int_{\mathcal{U}_i^k} \sigma(u|t)\, d\pi(u).
\end{aligned}$$

But this leads to a contradiction. A similar contradiction can be found when $(-1)^k \eta' < (-1)^k \eta$. This concludes the proof of the Lemma.

E.3. **Proof of Lemma 15.** Let $\alpha_i^k(t)$ be such that

$$x_i(t) = \sum_l \alpha_i^l(t)\, y_i^l.$$

Lemma 19. *There exists constants $c_i, d_i^k > 0$ such that $\sum_l d_i^l = 1$,* such that if $\overline{\alpha}_i^{-k}(t) = \alpha_i^{-k} - d_i^{-k}\left(\sum_l \alpha_i^l(t) - 1\right)$, then

$$(-1)^k \eta_i^k(t) = \frac{\upsilon_{-i}\left(\sum_l \alpha_i^l(t) - 1\right)}{c_i \overline{\alpha}_i^{-k}(t)},$$
$$P_i^k x_i(t) = \overline{\alpha}_i^{-k}(t).$$

Proof. For each k, we have

$$\begin{aligned}0 &= u_i^k(t)\left(y_i^k - x_i(t)\right) = \left(\psi_i + \eta_i^k(t)\gamma\right)\cdot\left(\left(\alpha_i^k(t) - 1\right)y_i^k + \alpha_i^{-k} y_i^{-k}\right)\\ &= \left(\sum_l \alpha_i^l(t) - 1\right)\upsilon_{-i} - (-1)^k \eta_i^k(t)\left(\alpha_i^{-k}(t)\left(\gamma\cdot\left(y_i^2 - y_i^1\right)\right) - (-1)^k\left(\sum_l \alpha_i^l(t) - 1\right)\left(\gamma\cdot y_i^k\right)\right)\end{aligned}$$

Take $c_i = \gamma\cdot(y_i^2 - y_i^1) > 0$, and $d_i^{-k} = \frac{(-1)^k\left(\gamma\cdot y_i^k\right)}{c_i}$. To see that the latter constants are positive, notice that $(y_i^2)^{(2)} > (y_i^2)^{(1)}$ and that $(y_i^1)^{(2)} < (y_i^1)^{(1)}$. The last equality comes from the fact that

$$\overline{P_i^k} x_i(t) = \frac{(-1)^k \eta_i^k(t)}{\sum_l (-1)^l \eta_i^l(t)} = \frac{\overline{\alpha}_i^k(t)}{\sum_l \overline{\alpha}_i^l(t)},$$

and the sum in the denominator of the last expression is equal to $\sum_l \overline{\alpha}_i^l(t) = \sum_l \alpha_i^l(t) + \left(\sum_l d_i^l\right)\left(\sum_l \alpha_i^l(t) - 1\right) = 1.$ ∎

Lemma 20. *For each i, k, t, $\sum_l \Delta\alpha_i^l(t) \geq 0$, and there is a constant $\alpha^* > 0$ such that $\alpha_i^k(t), \overline{\alpha}_i^k(t) \leq \alpha^*$, and*

$$\left|\sum_l \alpha_i^l(t) - 1\right|\left|\Delta\overline{\alpha}_i^k(t)\right| \leq \overline{\alpha}_i^k(t)\sum_l \Delta\alpha_i^l(t).$$

Proof. Because space X is compact, there is a constant $\alpha^* > 0$ such that $\alpha_i^k(t) \leq \alpha^*$ for each k, t. It follows that $\overline{\alpha}_i^k(t) \leq \alpha_i^k(t) \leq \alpha^*$.

Using Lemma 19, we show that

$$\sum_l \frac{1}{(-1)^l \eta_i^l(t)} = \frac{c}{\upsilon_{-i}} \frac{\sum \alpha_i^l(t) - \left(\sum_l d_i^l\right)\left(\sum \alpha_i^l(t) - 1\right)}{\sum \alpha_i^l(t) - 1} = \frac{c}{\upsilon_{-i}}\left(\frac{\sum \alpha_i^l(t)}{\sum \alpha_i^l(t) - 1} - 1\right).$$

Because $(-1)^{-k}\eta_i^k(t)$ is increasing with t for each k, the left hand side is decreasing with t, which implies that the right hand side is decreasing with t, or that $\sum \alpha_i^l(t)$ is increasing in t.

By the first claim, for each i and t, there is k such that $\left|\Delta\overline{\alpha}_i^{-k}(t)\right| \leq \Delta\overline{\alpha}_i^k(t)$. Because $(-1)^{-k}\eta_i^{-k}(t) = \frac{v_{-i}}{c_i}\frac{\sum_l \alpha_i^l(t)-1}{\overline{\alpha}_i^{-k}(t)}$ is increasing in t, we have

$$\frac{\sum_l \Delta\alpha_i^l(t)}{\Delta\overline{\alpha}_i^k(t)} \geq \frac{\sum_l \alpha_i^l(t)-1}{\overline{\alpha}_i^k(t+2)},$$

which implies that

$$\left|\Delta\overline{\alpha}_i^{-k}(t)\right|\left|\sum_l \alpha_i^l(t)-1\right| \leq \left(\Delta\overline{\alpha}_i^k(t)\right)\left|\sum_l \alpha_i^l(t)-1\right| \leq \overline{\alpha}_i^k(t)\sum_l \Delta\alpha_i^l(t)$$

□

For the next result, recall that $f^* = \frac{dF}{d\eta}(0)$.

Lemma 21. *There exist constants $C<\infty$ (all independent of β and λ) such that for each $i,k,t\in T^i$ and $t>T^\eta$,*

$$\left|\sum_l \Delta\alpha_i^l(t)-1\right| \leq C\left|\sum_l (-1)^l\Delta\eta_i^l(t)\right| \leq C^2\left|\sum_l \Delta F_i^l(t)\right|,$$

$$\left|\sum_l \alpha_i^l(t)-1\right| \leq C\left|\sum_l (-1)^l\eta_i^l(t)\right| \leq C^2\left|\sum_l F_i^l(t)\right|,$$

$$\left|\frac{f^*(-1)^k\Delta\eta_i^k(t)}{\Delta F_i^k(t)}-1\right|, \left|\frac{f^*(-1)^k\eta_i^k(t)}{F_i^k(t)}-1\right| \leq C\left|\sum_l F_i^l(t)\right|,$$

$$\left|\frac{f^*\sum_l(-1)^l\Delta\eta_i^l(t)}{\sum_l \Delta F_i^l(t)}-1\right|, \left|\frac{f^*\sum_l(-1)^l\eta_i^l(t)}{\sum_l F_i^l(t)}-1\right| \leq C\left|\sum_l F_i^l(t)\right|.$$

Proof. By Lemma 19,

$$\frac{v_{-i}}{c_i}\left(\sum_l \alpha_i^l(t)-1\right) = \frac{c_i}{v_{-i}}\left(\sum_l\left((-1)^l\eta_i^l(t)\right)^{-1}\right)^{-1}.$$

Hence,

$$\begin{aligned}\frac{v_{-i}}{c_i}\left|\sum_l \Delta\alpha_i^l(t)\right| &= \left|\sum_l\left((-1)^l\eta_i^l(t)\right)^{-1}\right|^{-1}\left|\sum_l\left((-1)^l\eta_i^l(t+2)\right)^{-1}\right|^{-1}\left|\sum_l(-1)^l\frac{\Delta\eta_i^l(t)}{\eta_i^l(t)\,\eta_i^l(t+2)}\right| \\ &= \sum_l(-1)^l\overline{\alpha}_i^l(t)\,\overline{\alpha}_i^l(t+2)\,\Delta\eta_i^l(t) \leq \sum_l(-1)^l\Delta\eta_i^l(t),\end{aligned}$$

where the last inequality comes from the fact that $\overline{\alpha}_i^l(t)\leq 1$ for each $i,,l,t$. This shows the first inequality in the first line.

Because the increments in F are always positive, and $\sum_l \eta_i^l \left(T_i^{*,\sigma}+2\right)=0$, the first inequality in the second line inequality follows from the above.

The first inequality in the third line follows from the fact that the derivative is Lipschitz. All the remaining inequalities follow from the first. □

We can proceed with the proof of Lemma 15. The definition of $\alpha_i^k(t)$ as well as 19 Lemma imply that

$$P_i^k x_i(t)=\frac{\alpha_i^k}{\sum_l \alpha_i^l} \text{ and } \overline{P_i^k}(t)=\overline{\alpha_i^k}.$$

Taking into account that $\alpha_i^k(t)=\overline{\alpha}_i^k(t)+d_i^k(t)\left(\sum_l \alpha_i^l-1\right)$, we have

$$P_i^k x_i(t)-\overline{P_i^k}(t)=\frac{\alpha_i^k(t)}{\sum_l \alpha_i^l(t)}-\overline{\alpha_i^k}(t)=-\left(\overline{\alpha_i^k}(t)-d_i^k\right)\frac{\sum_l \alpha_i^l-1}{\sum_l \alpha_i^l}.$$

An application of Lemma 21 demonstrates the first estimate in the thesis of Lemma 15.

Second, by Lemma 20,

$$\begin{aligned}\left|\Delta\left(P_i^k x_i(t)-\overline{P_i^k}(t)\right)\right| \leq & \left|\Delta\overline{\alpha_i^k}(t)\right|\frac{\sum_l \alpha_i^l(t)-1}{\sum_l \alpha_i^l(t)}+\left|\overline{\alpha_i^k}(t)-d_i^k\right|\frac{\left|\sum_l \Delta\alpha_i^l(t)\right|}{\sum_l \alpha_i^l(t)}\\ & +\left|\overline{\alpha_i^k}(t)-d_i^k\right|\frac{\left|\sum_l \alpha_i^l(t)-1\right|\left|\sum_l \Delta\alpha_i^l\right|}{\left(\sum_l \alpha_i^l(t)\right)\left(\sum_l \alpha_i^l(t+2)\right)}\\ \leq & 3\alpha^*\left|\sum_l \Delta\alpha_i^l(t)\right|.\end{aligned}$$

Another application of Lemma 21 shows the second estimate in the thesis of Lemma 15.

Third, observe that due to Lemma 21,

$$\begin{aligned}\left|\frac{\overline{Q_i^k}(t)}{Q_i^k(t)}-1\right| &= \left|\frac{\Delta F_i^k(t)}{f^*(-1)^k\Delta\eta_i^k(t)}\frac{f^*\sum_l \eta_i^l(t)}{\sum_l \Delta F_i^l(t)}-1\right| \leq \left|\frac{1+K\sum_l(-1)^l\eta_i^l(t)}{1-K\sum_l(-1)^l\eta_i^l(t)}-1\right|\\ &\leq 8K\left|\sum_l(-1)^l\eta_i^l(t)\right| \leq C\left|\sum_l F_i^l(t)\right|\end{aligned}$$

for appropriately defined constant C.

The same calculations show that

$$\left|\frac{\sum_l \Delta\eta_i^l(t)}{\sum_l \eta_i^l(t)} / \left(\frac{\sum_l \Delta F_i^l(t)}{\sum_l F_i^l(t)}\right) - 1\right| = \left|\frac{f^* \sum_l \Delta\eta_i^l(t)}{\sum_l \Delta F_i^l(t)} \frac{\sum_l F_i^l(t)}{f^* \sum_l \eta_i^l(t)} - 1\right| \leq C \left|\sum_l F_i^l(t)\right|.$$

E.4. **Proof of Lemma 16.** Define

$$T_i^O = \max\left\{t \in T_i, t \leq T_i^{*,\sigma} : x_i(t) \in \mathrm{int} O_i^k \text{ for some } k\right\},$$
$$T_i^\eta = \max\left\{t \in T_i, t \leq T_i^{*,\sigma} : \eta_i^k(t) \geq \eta \text{ for some } k\right\} \text{ for each } \eta,$$
$$T_i^k = \max\left\{t \in T_i, t \leq T_i^{*,\sigma} : \text{player } i \text{ is only active on side } k\right\}.$$

For each $x = 0, \eta, k$,let $T^x = \max T_i^x$.

E.4.1. *Geometry.* For each player i, and each k, define

$$Y_i = \left\{x \in X \backslash \mathrm{int} m_{-i} : P_i^k x \leq P_i^k \left(\mathbf{1} - y_{-i}^{-k}\right) \text{ for each } k\right\} = \mathrm{con}\left\{\mathbf{1} - y_{-i}^1, \mathbf{1} - y_{-i}^2, \mathbf{0}\right\} \backslash \mathrm{int} m_{-i},$$
$$Y_i^k = \left\{x \in Y_i : P_i^k x = P_i^k \left(\mathbf{1} - y_{-i}^{-k}\right)\right\} = B_i \cap \mathrm{con}\left\{\mathbf{1} - y_{-i}^{-k}, \mathbf{0}\right\} \backslash \mathrm{int} m_{-i},$$
$$O_i^k = \left\{x : P_i^k x \geq P_i^k \left(\mathbf{1} - y_{-i}^{-k}\right)\right\}.$$

To interpret the above sets, it is helpful to notice that $y(t) \in Y_i$ for each $t \in T_{-i}$. Additionally, the definition (**??**) implies that $y(t)$ belongs to the convex hull spanned by the allocation obtained from the optimal choices of the other player and the 0 allocation.) Thus, set Y_i contains all possible (weighted) outcomes obtained if player $-i$ yields. Its subset Y_i^k contains only those outcomes that are obtained if player $-i$ yields and chooses y_{-i}^{-k}. (The reason for the notation is that $-k$ for player $-i$ faces side k for player i.) Sets O_i^k contain allocations that cannot be obtained as win outcomes.

We say that *side k of player i is regular* if $\mathrm{int} O_i^k \neq \emptyset$.

Lemma 22. *For each $x \notin m_{-i}$,*

$$0 < P_{-i}^k \left(\sum_p \left(P_i^p x\right)\left(\mathbf{1} - y_i^p\right)\right) < 1.$$

Proof. The projection of projection. □

E.4.2. *Best response properties.* The subsequent claims are illustrated on Figure E.2.

Lemma 23. *For each k, l, each $t \in T_i, t \leq T_i^{*,\sigma}$, if $x_i(t+2) \notin \mathrm{int} O_i^{-k}$, and player i is only active on side k in period t, then $x_i(t) \notin O_i^{-k}$.* If $x_i(t+2) \in O_i^k$, and *player i is only active on side k in period t, then $x_i(t) \in \mathrm{int} O_i^k$.*

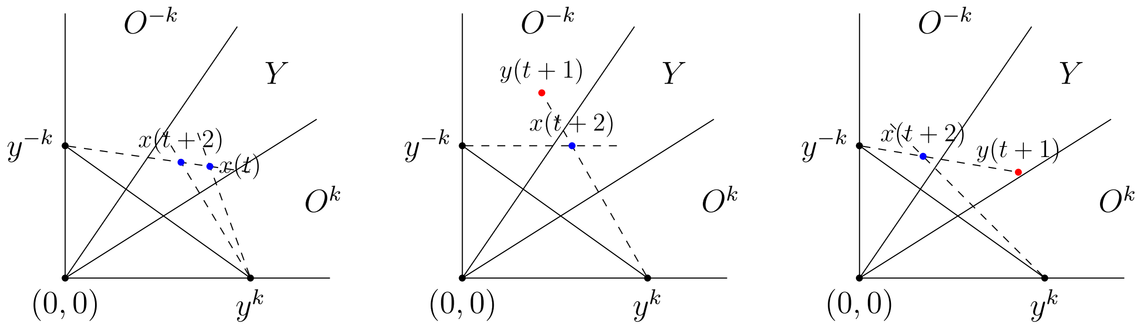


FIGURE E.2. Illustration of Lemma 23

Proof. TBA □

Lemma 24. *For each k, l, each $t \in T_i, t \leq T_i^{*,\sigma}$, if $x_i(t+2) \notin O_{-i}^{-k}$, player i is active on side k in period $t+2$, and player $-i$ is only active on side k in period $t+1$, then player i is only active on k in period t.*

Proof. TBA □

Lemma 25. *For each k, l, each $t \in T_i, t \leq T_i^{*,\sigma}$, if $x_i(t+2) \in intO_i^{-k}$, and player i is active on side $-k$ in period $t+2$, then player i is only active on $-k$ in period t.*

Proof. TBA □

Lemma 26. *For each k, l, each $t \in T_i, t \leq T_i^{*,\sigma}$, if $P_i^k x_i(t+2) > P_i^k y_i(t+1)$, and player i is active on side k in period $t+2$, then the player is active only on side k in period t.*

Proof. TBA □

E.4.3. *Approximations.* For each x, let $G_i^k(x) = \pi \left\{ u \in \mathcal{U}_i^k : u \cdot x \leq u \cdot y_i^k \right\}$. For each player i, l, k, let

$$Q_{i,l}^{*,k} = P_i^k \left(\mathbf{1} - y_{-i}^l \right).$$

Lemma 27. *There exists a constant $C < \infty$ and $\delta > 0$ such that for each $x \in X \backslash intm_{-i}$, if $\sum_l G_i^k(x) < \delta$, then*

$$\left| \frac{G_i^k(x)}{\sum G_i^l(x)} - P_i^k x \right| \leq C \sum_l G_i^k(x)$$

Proof. Let $\alpha^k(x)$ be defined by $x = \sum_l \alpha_i^l(x)\, y_i^l$. Let $\overline{\alpha}_i^k(x) = \alpha_i^k(x) - d_i^k \left(\sum_l \alpha_i^l(x) - 1\right)$, where constants d_i^k are defined in Lemma 19. By the same argument as in Lemma ???, we show that if

$$P_i^k x = \overline{\alpha}_i^k(x), \text{ and}$$
$$G_i^k(x) = F\left(\frac{v_{-i}}{c_i} \frac{\sum_l \alpha_i^l(x) - 1}{\overline{\alpha}_i^{-k}(x)}\right).$$

Using the same arguments as in the proof of Lemma 21, we can show that there exists a constant $C < \infty$, such that

$$\left| \frac{G_i^k(x)}{f^* \frac{v_{-i}}{c_i} \frac{\sum_l \alpha_i^l(x) - 1}{\overline{\alpha}_i^{-k}(x)}} \frac{f^* \sum_l \frac{v_{-i}}{c_i} \frac{\sum_l \alpha_i^l(x) - 1}{\overline{\alpha}_i^l(x)}}{\sum_l G_i^l(x)} - 1 \right| \leq C \sum_l G_i^k(x).$$

But

$$\frac{f^* \frac{v_{-i}}{c_i} \frac{\sum_l \alpha_i^l(x) - 1}{\overline{\alpha}_i^{-k}(x)}}{f^* \sum_l \frac{v_{-i}}{c_i} \frac{\sum_l \alpha_i^l(x) - 1}{\overline{\alpha}_i^l(x)}} = \frac{\frac{1}{\overline{\alpha}_i^{-k}(x)}}{\sum_l \frac{1}{\overline{\alpha}_i^{-l}(x)}} = \frac{\overline{\alpha}_i^k(x)}{\sum_l \overline{\alpha}_i^l(x)} = \overline{\alpha}_i^k(x) = P_i^k x.$$

□

Lemma 28. *There exist constants $C < \infty$ and $\zeta_2 > 0, \zeta_2 \leq \zeta_1$ (that do not depend on λ and β) such that for each $t \in T_i$* st. $T^{\zeta_2} < t < T^{*\sigma}$, if $x_i(t+2) \in Y_i^k$, then

$$Q_i^k(t) - C\left(\sum_k \Delta F_i^k(t)\right) \leq Q_{i,k}^{*k} \leq Q_i^k(t+2) + C\left(\sum_k \Delta F_i^k(t)\right).$$

Proof. We only show the first inequality; the proof of the second one is analogous. Assume that $x_i(t) \in Y_i$ and that $x_i(t+2) \in Y_i^k$.

By assumption, there exists $\alpha > 0$ such that $x_i(t+2) = \alpha\left(\mathbf{1} - y_{-i}^{-k}\right) =: x \in Y_i^k$. Because $x_i(t) \notin \mathrm{int} O_i^k$, we can find $x' = \alpha'\left(\mathbf{1} - y_{-i}^{-k}\right)$ such that

$$\sum_l G_i^l(x_i(t)) - G_i^l(x_i(t+2)) = \sum_l G_i^l(x') - G_i^l(x), \text{ and}$$
$$G_i^k(x_i(t+2)) - G_i^k(x_1) \leq G_i^k(x') - G_i^k(x)$$

Then,

$$Q_i^k(t) = \frac{G_i^k(x_i(t)) - G_i^k(x_i(t+2))}{\sum_l G_i^l(x_i(t)) - G_i^l(x_i(t+2))} \leq \frac{G_i^k(x') - G_i^k(x)}{\sum_l G_i^l(x') - G_i^l(x)}. \tag{E.5}$$

For each $\alpha \geq 0$, let $H_i^k(\alpha) := G_i^k\left(\alpha\left(\mathbf{1} - y_{-i}^{-k}\right)\right)$. Let α^* be such that $\alpha^*\left(\mathbf{1} - y_{-i}^{-k}\right) = \sum_l \left(P_i^l\left(\mathbf{1} - y_{-i}^{-k}\right)\right) y_i^l$. The assumptions on π imply that H_i^k has a Lipschitz continuous derivative h_i^k with a Lipschitz constant K. (To see it, notice that $\frac{u \cdot y_i^k}{u \cdot \left(\mathbf{1} - y_{-i}^{-k}\right)}$ is a continuous function of $u \in \mathcal{U}_i^k$.) Let $h^k = h_i^k(\alpha^*)$. As $\alpha \to \alpha^*$, $H_i^k(\alpha) \to 0$ and, by the L'Hospital's rule,

$$\frac{H_i^k(\alpha)}{\sum_l H_i^l(\alpha)} \to \frac{h^k}{\sum_l h^l}.$$

At the same time, Lemma 27 implies that

$$\frac{H_i^k(\alpha)}{\sum_l H_i^l(\alpha)} \to Q_{i,-k}^k.$$

Hence, the two limits are equal.

Then, for appropriately small η, $\alpha' \leq \frac{1}{2K} \sum_l h^l$, and the expression (E.5) is not larger than

$$\begin{aligned}
&\leq \frac{h^k(\alpha' - \alpha) + K(\alpha' - \alpha)^2}{\sum_l h^l(\alpha' - \alpha) - K(\alpha' - \alpha)^2} - Q_{i,-k}^k = \frac{h^k + K(\alpha' - \alpha)}{\sum_l h^l - K(\alpha' - \alpha)} - \frac{h^k}{\sum_l h^l} \\
&= K \frac{\left(h^k + \sum_l h^l\right)(\alpha' - \alpha)}{\left(\sum_l h^l - K(\alpha' - \alpha)\right)\left(\sum_l h^l\right)} \leq \frac{8K}{\left(\sum_l h^l\right)^2}\left[H_i^k(\alpha') - H_i^k(\alpha)\right] \\
&\leq C\left[\sum_l G_i^l(x_i(t)) - G_i^l(x_i(t+2))\right] = C\left(\sum_k \Delta F_i^k(t)\right)
\end{aligned}$$

for constant $C = 8K\left(\sum_l g^l\right)^{-2}$. □

Lemma 29. *There exist constants $\zeta_2' > 0$ and $C_0 > 0$ ((that does not depend on λ and β) such that for for each $t \in T_i$* st. $T^{\zeta_2'{}'} < t < T^{*\sigma}$, if $x_i(t+2), x_i(t) \in Y_i^k$ and player $-i$ is active on side $-k$ in periods $t+1$ and $t-1$, then

$$\beta_i \cdot \left(y_{-i}(t) - y_{-i}^k\right) \geq C_0 (-1)^{-k} \eta_{-i}^{-k}(t+1).$$

Proof. Let $Q^* = \sum_l Q_{i,k}^{*l}\left(\mathbf{1} - y_i^l\right)$, and let $q^* = \sum_p P_{-i}^p Q^*$. By definition, q^* belongs to the line that connects y_{-i}^k and y_{-i}^{-k}. Moreover, by Lemma 22, $P_{-i}^k q^* > 0$, and, because $(-1)^{-k} \gamma \cdot \left(y_{-i}^{-k} - y_{-i}^k\right) > 0$, we have

$$C_0 := \frac{1}{2}(-1)^{-k} \gamma \cdot \left(y_{-i}^{-k} - q^*\right) = \frac{1}{2}\left(P_{-i}^k q^*\right)(-1)^{-k} \gamma \cdot \left(y_{-i}^{-k} - y_{-i}^k\right) > 0.$$

Let $C < \infty$ be as in Lemma 28. Let $\zeta_2$be as in Lemma 28. We are going to fix $\zeta_2' \leq \zeta_2$ later. From now on assume that $t \geq T^{\zeta_2}$.

Because $-i$ is active on side $-k$ in period $t+1$ and $t-1$, Lemma 14 implies that

$$u_{-i}^{-k}(t+1)\cdot\left(y_{-i}(t)-y_{-i}^{-k}\right)=0.$$

Recall that $u_{-i}^{-k}(t+1)=\beta_i+\eta_{-i}^{-k}(t+1)\gamma$. Because $\beta_i\cdot y_{-i}^{k}=\beta_i\cdot y_{-i}^{-k}$, we have

$$\begin{aligned}0&=\left(\beta_i+\eta_{-i}^{-k}(t+1)\gamma\right)\cdot\left(y_{-i}(t)-y_{-i}^{-k}\right)\\&=\beta_i\cdot\left(y_{-i}(t)-y_{-i}^{k}\right)+\eta_{-i}^{-k}(t+1)\gamma\cdot\left(y_{-i}(t)-y_{-i}^{-k}\right),\end{aligned}\tag{E.6}$$

and

$$\begin{aligned}\beta_i\cdot\left(y_{-i}(t)-y_{-i}^{k}\right)&=(-1)^{-k}\eta_{-i}^{-k}(t+1)\left[(-1)^{-k}\gamma\cdot\left(y_{-i}^{-k}-y_{-i}(t)\right)\right]\\&=(-1)^{-k}\eta_{-i}^{-k}(t+1)\,2C_0\\&\quad+(-1)^{-k}\eta_{-i}^{-k}(t+1)\left[(-1)^{-k}\gamma\cdot\left(q^*-y_{-i}(t)\right)\right].\end{aligned}$$

We are going to show that the term in the square brackets of the last line is smaller than C_0. First, notice that

$$y_{-i}(t)=\alpha w_{-i}^{\sigma}(t)=\alpha^* w_{-i}^{\sigma}(t)+(\alpha-\alpha^*)\,w_{-i}^{\sigma}(t),$$

where we denoted $\alpha=\frac{\beta p^{\sigma}(t)}{\mathrm{e}^{-2\Delta}p^{\sigma}(t)+\left(1-\mathrm{e}^{-2\Delta}\right)}$ and α_0 is chosen so that $\alpha_0\beta_i\cdot w_{-i}^{\sigma}(t)=v_i$. By (E.6),

$$(\alpha-\alpha_0)\,\beta_i\cdot\left(w_{-i}^{\sigma}(t)-y_{-i}^{k}\right)=-\eta_{-i}^{-k}(t+1)\gamma\cdot\left(y_{-i}(t)-y_{-i}^{-k}\right),$$

and, using Lemma 21, we can find a constant $C'<\infty$ such that

$$|\alpha-\alpha_0|\le C'\left|\sum_l F_{-i}^{l}(t)\right|\le C'm_2'.$$

Additionally, notice that $w_{-i}^{\sigma}(t)=\sum_l Q_i^l(t)\left(\mathbf{1}-y_i^k\right)$, and, by Lemma 28,

$$Q_{i,k}^{*k}-C\left(\sum_k\Delta F_i^k(t)\right)\le Q_i^k(t)\le Q_{i,k}^{*k}+C\left(\sum_k\Delta F_i^k(t)\right).\tag{E.7}$$

Hence,

$$\left\|\alpha_0 w_{-i}^{\sigma}(t)-q^*\right\|\le\left\|w_{-i}^{\sigma}(t)-Q^*\right\|\le C''\left(\sum_k\Delta F_i^k(t)\right)\le Cm_2'.$$

Thus,

$$\begin{aligned}&\left[(-1)^{-k}\gamma\cdot(q^*-y_{-i}(t))\right]\\ \leq&2\left(\left\|(\alpha-\alpha_0)\,w_{-i}^{\sigma}(t)\right\|+\left\|\alpha_0 w_{-i}^{\sigma}(t)-q^*\right\|\right)\\ \leq&(4C'+2C)\,m_2'.\end{aligned}$$

Pick $\zeta_2'\leq\zeta_2$ such that $(4C'+2C)\,m_2'\leq\frac{1}{2}C_0$. □

We have the following useful bounds on the yielding probability.

Lemma 30. *Suppose that $\beta\geq\beta_0$. There are constants $0<p\min\leq p_{\max}<\infty$ such that for each equilibrium, each $t\in T_i$ st. $t_i^0<t\leq T_i^{*,\sigma}$,*

$$(1-\beta)\,p_{\min}\leq p^{\sigma}(t)\leq(1-\beta)\,p_{\max}.$$

Proof. By Lemma 4, for each each $t\in T_i$ st. $t_i^0<t\leq T_i^{*,\sigma}$, there are types $u,u'\in\mathcal{U}_{-i}$ such that $t-1$ is a best response for type u and $t+1$ is a best response for type u'. By Lemma 3 ,

$$(1-\beta)\frac{1+\beta}{\beta}\frac{1}{S_{-i}^{\sigma}(u,t)-\beta}\leq p^{\sigma}(t)\leq(1-\beta)\frac{1+\beta}{\beta}\frac{1}{S_{-i}^{\sigma}(u',t)-\beta}.$$

The claim follows from the fact that $S_{-i}^{\sigma}(u,t)\leq\frac{1}{v_i}=:S_{\max}$ and t $S_{-i}^{\sigma}(u,t)\geq\frac{\min_{x\in m_i}u\cdot(1-x)}{\max_{x\in m_{-i}}u\cdot x}=:S_{\min}>1$, where the last inequality comes from Assumption 4. □

For each player i,define $T_i^F(\eta)=\max\{t:\sum_l F_i(t)+\lambda\geq\eta\}$.

Lemma 31. *There exist constants $a\geq a'>0$, such that for each $\beta\geq\beta_0$, each $\eta\in[0,1]$,*

$$\eta^a\leq\beta^{T_i^F(\eta)}\leq\eta^{a'},\ \textit{and}\ \eta^{\frac{a}{a'}}\leq\lambda+\sum_l F_{-i}^l\left(T_i^F(\eta)\right)\leq\eta^{\frac{a'}{a}}.$$

Proof. Notice that

$$\sum_l F_i(t)+\lambda=\prod_{s\in T_i:s<t}(1-p^{\sigma}(t)).$$

Due to Lemma 30, and the choice of $\beta \geq \beta_0$ (which implies $\mathrm{e}^{-2(1-\beta)} \leq 1-(1-\beta) \leq \mathrm{e}^{-(1-\beta)}$), we have

$$\begin{aligned}
&\left(\mathrm{e}^{-(1-\beta)T_i^F(\eta)}\right)^{p_{\max}} \leq (1-(1-\beta)\,p_{\min})^{T_i^F(\eta)/2}. \\
\leq &\eta = \prod_{s\in T_i : s<T_i^F(\eta)} (1-p^{\sigma}(t)) \leq \\
\leq &(1-(1-\beta)\,p_{\min})^{T_i^F(\eta)/2} \leq \left(\mathrm{e}^{-(1-\beta)T_i^F(\eta)}\right)^{\frac{1}{2}p_{\min}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\eta^{\frac{2}{p_{\min}}} \leq \left(\mathrm{e}^{-(1-\beta)T_i^F(\eta)}\right)^2 \\
\leq &\beta^{T_i^F(\eta)} = (1-(1-\beta))^{T_i^F(\eta)} \\
\leq &\mathrm{e}^{-(1-\beta)T_i^F(\eta)} \leq \eta^{\frac{1}{p_{\max}}}.
\end{aligned}$$

Take $a = \frac{2}{p\min}$ and $a' = \frac{1}{p_{\max}}$. The second claim follows from the first. $\square$

E.4.4. *Late game properties.* Let $\zeta_1 = \min_{i,k}\max_{u\in\mathcal{U}_i^k}(-1)^k\left(u^{(2)}-\psi_i^{(2)}\right)$.

Lemma 32. *If $\eta \leq \zeta_1$, then $T^\eta \geq T^O$.*

Proof. Suppose that $T^O = T_i^O < T^\eta$. By definition there is k, such that $T_i^O = \max\left\{t\in T_i, t\leq T_i^{*,\sigma} : x_i(t)\in \mathrm{int}O_i^k\right\}$. By Lemma 23, it must be that player i is active on side k in period t. By Lemma 24, player i is only active on side k in period t. By another application of the first part, $x_i(t-2)\notin O_i^k$. A repetition of the same argument shows that player i is active only on side k for each $t<T^O, t\in T_i$. But this implies that $\eta_i^{-k}(t_i^0) = \eta_i^{-k}\left(T^O\right) < \eta \leq \zeta_1$, which contradicts the choice of $\eta\leq\zeta_1$. $\square$

Lemma 33. *If $\eta \leq \zeta_1$, then, for each $t > T^\eta$* st. *$T^\eta < t < T^{*\sigma}$, if player i is active on side $-k$ in period $t+1$, and player $-i$ is only active on side k in period t, then $x_i(t+1)\in Y_i^{-k}$.*

Proof. Suppose that there is $t > T^\eta$ such that player i is active on side $-k$ in period $t+1$, and player $-i$ is only active on side k in period t. Lemma 32 implies that $x_i(t+1)\notin \mathrm{int}O_i^k$ and $x_{-i}(t+2)\notin\mathrm{int}O_{-i}^{-k}$. Suppose that $x_i(t+1)\notin Y_i^{-k}$, which implies that $x_i(t+1)\notin O_i^{-k}$. By Lemma 24, player i is only active on side k in period $t-1$. Another application of Lemma 23 shows that $x_{-i}(t)\notin O_{-i}^{-k}$ and player $-i$ is only active on side k in period t. A repetition of the same argument shows that each player is active only on side k for each $t' < t$. But this implies that $\eta_i^{-k}(t_i^0) = \eta_i^{-k}\left(T^O\right) < \eta \le \zeta_1$, which contradicts the choice of ζ_1. □

Lemma 34. *If $T^\eta < T^{*,\sigma} - 2$, then, for each k,$T^k < T^{*,\sigma}$.*

Proof. On the contrary, suppose that $T_i^k = T_i^{*,\sigma}$ for some i. Let $\overline{T}_i = \max\{t :$ player i is active on side $-$ T_i^k. Because type ψ_i is the limit of types in $\mathcal{U}_i^{-k}$, the proof of Lemma 13 implies that type with preferences ψ_i must be indifferent between yielding in any period $t \in T_i$ and $T_i^{*,\sigma} \le t \le \overline{T}_i$. The calculations in Lemma 3 show that it must be that $\psi_i \cdot y_i(t) = L_i(\psi_i) = v_{-i}$, which implies that $y_i(t) \in \mathrm{bd}m_{-i}$. Because there are types in $\mathcal{U}_i^k$ who weakly prefer to wait and yield only in period $T_i^{*,\delta}$, a similar argument shows that it must be $y_i(t) = y_i^k$ for each $t \in T_{-i}, T_i^{*,\sigma} \le t \le \overline{T}_i$. However, because $y_i(t)$ is a convex combination of $\mathbf{1} - y_{-i}^l$ for $l = 1, 2$ and the zero allocation, it must be that for each $t \in T_{-i}, T_i^{*,\sigma} \le t \le \overline{T}_i$, player $-i$ is only active on $-k$ side and that side k of player i is not regular. Note that it follows that side $-k$ of player $-i$ is regular.

If player i is the last player, i.e., $T_i^{*,\sigma} = T^{*,\sigma}$, then $x_{-i}(T^{*,\sigma} - 1) = y_{-i}^{-k} \in \mathrm{int}O_{-i}^{-k}$. If player $-i$ is the last player, then Lemma 33 implies that it must be that $x_{-i}(T^{*,\sigma}) \in Y_{-i}^{-k}$. But because player $-i$ is only active on side $-k$ in period $T^{*,\sigma} - 2$, then Lemma 23 implies that $x_{-i}(T^{*,\sigma} - 2) \in \mathrm{int}O_{-i}^{-k}$. In any case, we obtain a contradiction with Lemma 32. □

Lemma 35. *There is $m_3 > 0, \eta^3 \le \zeta_2, \zeta_1$ such that if $\eta \le m_3$, then, for each $t > T^\eta$* st. *$T^\eta + 2 < t < T^{*\sigma}$, if player i is active on side $-k$ in period $t+1$, $y_{-i}(t+1) \in U_{-i}^l(t+2)$ for each l, and player $-i$ is only active on side k in period t, then, $y_{-i}(t-1) \in U_{-i}^l(t)$ for each l, player i is active on side $-k$ in period $t-1$, and player $-i$ is only active on side k in period $t-2$.*

Proof. Take period $t \in T_{-i}$ such that $T^\eta + 2 < t < T^{*\sigma}$, and such that player i is active on side $-k$ in period $t+1$, and player $-i$ is only active on side k in period t. By Lemma 33, $x_i(t+1) \in Y_i^{-k}$, and by Lemma 32, $x_i(t+3), x_i(t-1) \in Y_i$.

First, we are going to show that player $-i$ is only active on side k in period $t-2$. By Lemma 26, it is enough to show that

$$P_{-i}^k x_{-i}(t) > P_{-i}^k y_{-i}(t-1) \tag{E.8}$$

Because $y_{-i}(t+1) \in U^l_{-i}(t+2)$ for each l, and player $-i$ is active on side k in period t, it must be that $y_{-i}(t+1) \in I^k_{-i}(t+2)$ and that

$$P^k_{-i} x_{-i}(t+2) \geq P^k_{-i} y_{-i}(t+1). \tag{E.9}$$

By Lemma 28,

$$\begin{aligned}
Q^{*-k}_{i,-k} &\leq Q^{-k}_i(t+1) + C\left(\sum_l \Delta F^l_i(t)\right), \text{ and} \\
Q^{*-k}_{i,-k} &\geq Q^{-k}_i(t-1) - C\left(\sum_k \Delta F^k_i(t)\right),
\end{aligned}$$

where $Q^{*l}_{i,-k} = P^l_i\left(\mathbf{1} - y^k_{-i}\right)$. Because $P^k_{-i}\left(\mathbf{1} - y^{-k}_i\right) > P^k_{-i}\left(\mathbf{1} - y^k_i\right)$ (due to the side $-k$ of player i facing the side k of player $-i$), inequality (E.9) implies that

$$\begin{aligned}
P^k_{-i} x_{-i}(t+2) &\geq P^k_{-i} y_{-i}(t+1) \\
&= \sum_l Q^l_i(t+1) P^k_{-i}\left(\mathbf{1} - y^l_i\right) \\
&= P^k_{-i}\left(\mathbf{1} - y^k_i\right) + Q^{-k}_i(t+1)\left[P^k_{-i}\left(\mathbf{1} - y^{-k}_i\right) - P^k_{-i}\left(\mathbf{1} - y^k_i\right)\right] \\
&\geq P^k_{-i}\left(\mathbf{1} - y^k_i\right) + Q^{*-k}_{i,-k}\left[P^k_{-i}\left(\mathbf{1} - y^{-k}_i\right) - P^k_{-i}\left(\mathbf{1} - y^k_i\right)\right] - C\left(\sum_l \Delta F^l_i(t)\right) \\
&= A - C\left(\sum_l \Delta F^l_i(t)\right),
\end{aligned}$$

where we denoted $A = P^k_{-i}\left(\mathbf{1} - y^k_i\right) + Q^{*-k}_{i,-k}\left[P^k_{-i}\left(\mathbf{1} - y^{-k}_i\right) - P^k_{-i}\left(\mathbf{1} - y^k_i\right)\right]$. On the other hand, we have

$$P^k_{-i} y_{-i}(t-1) \leq A + C\left(\sum_l \Delta F^l_i(t)\right). \tag{E.10}$$

Because player $-i$ is only active on side k in period t, we have $\overline{Q^k_{-i}}(t) = 1$. By the equation (E.4) and Lemma 15, we have

$$\begin{aligned}
P^k_{-i}x_{-i}(t) &\geq P^k_{-i}x_{-i}(t+2) + \frac{\sum_l \Delta F^l_{-i}(t)}{\sum_l F^l_{-i}(t)}\left(1 - P^k_{-i}(t+2)\right) - C\sum_l \Delta F^l_i(t) \\
&\geq P^k_{-i}x_{-i}(t+2)\left(1 - \frac{\sum_l \Delta F^l_{-i}(t)}{\sum_l F^l_{-i}(t)}\right) + \frac{\sum_l \Delta F^l_{-i}(t)}{\sum_l F^l_{-i}(t)} - C\sum_l \Delta F^l_i(t) \\
&\geq A\left(1 - \frac{\sum_l \Delta F^l_{-i}(t)}{\sum_l F^l_{-i}(t)}\right) + \frac{\sum_l \Delta F^l_{-i}(t)}{\sum_l F^l_{-i}(t)} - 2C\left(\sum_l \Delta F^l_i(t)\right) \\
&= A + \frac{\sum_l \Delta F^l_{-i}(t)}{\sum_l F^l_{-i}(t)}(1-A) - 2C\left(\sum_l \Delta F^l_i(t)\right). \qquad \text{(E.11)}
\end{aligned}$$

Note that $A < 1$. Find $m_3 \leq \zeta_1, \zeta_2$ small enough so that

$$m_3 \leq \frac{1-A}{12f^*C}.$$

Then, for each $\eta \leq m_3$ $t > T^\eta$, Lemma 15 implies that

$$\sum_l F^l_{-i}(t) \leq 2f^* \sum_l (-1)^l \eta^l_{-i}(t) \leq 4f^*\eta \leq \frac{1-A}{3C}.$$

The inequality (E.8) follows from the above bound, as well as the inequalities (E.10) and (E.11). ☐

E.4.5. *Proof of Lemma 16.* Let m_3 be as in Lemma 35.Let $a \geq a' > 0$ be constants from Lemma 31. Let

$$m_4 = \left(\frac{1}{2}\right)^{\frac{a}{a'}} (m_3)^{\left(\frac{a}{a'}\right)^2}.$$

Let $n > 0, n < m_3$ be a very small number to be fixed later and such that

$$n \leq \frac{1}{4} m_4^{\frac{a}{a'}}.$$

Suppose that $\lambda < \frac{1}{4}n$.

On the contrary, suppose that there is an equilibrium such that $T^k > T^n$. Using Lemma 31, we can show that $T^n > T^{*,\sigma} + 2$. By Lemma 34, $T^k < T^{*,\sigma}_i$ for each i. Thus, if $T^k_{-i} = T^k$, then player $-i$ is active on both sides in period $T^k + 2$. In particular, by Lemma 14,$y_{-i}\left(T^k + 1\right) \in U^l_{-i}\left(T^k + 2\right)$ for each l. Moreover, player i is active on both sides (including side $-k$) in period $T^k + 1$. In such a situation, a repeated application of Lemma 35 shows that player $-i$ is active only on side k in each period t such that $T^k \leq t < T^{m_3}$.

Then, by Lemma 31, we have

$$\lambda + \sum_l F_i^l\left(T^{m_3}\right) \geq m_3^{\frac{a}{a'}},$$
$$\lambda + \sum_l F_i^l\left(T^{m_4}\right) \leq \left(m_4\right)^{\frac{a'}{a}} = \frac{1}{2} m_3^{\frac{a}{a'}},$$
$$\beta^{T^{m_4}} \geq m_4^a = \left(\frac{1}{2}\right)^{a^2(a')^{-1}} m_3^{a^3(a')^{-2}}.$$

Using the first two inequalities, we conclude that

$$\sum_l \left(F_i^l\left(T^{m_3}\right) - F_i^l\left(T^{m_4}\right)\right) \geq m_3^{\frac{a}{a'}} - \left(m_4\right)^{\frac{a'}{a}} = \frac{1}{2} m_3^{\frac{a}{a'}}.$$

Let $n_4 = \eta_{-i}^k\left(T^{m_4}\right)$. Then, an application of Lemmas 21 and 31 shows that

$$n_4 \geq \frac{1}{2} f^* F_{-i}^k\left(T^{m_4}\right) \geq \frac{1}{2} f^*\left(m_4^{\frac{a}{a'}} - \lambda - n\right) \geq \frac{1}{4} f^*\left(\frac{1}{2}\right)^{a^2(a')^{-2}} m_3^{a^3(a')^{-3}}.$$

Let $n_0 = \eta_{-i}^{-k}\left(T^k + 2\right)$, where $(-1)^{-k} n_0 \leq n$ be the last type on side $-k$ to yield in the "late game". Let $T_0 = \max\left\{t < T^k : \text{player } -i \text{ is active on side } -k \text{ in period } t\right\}$. By the above, $T_0 \leq T^{m_3}$.Moreover, the continuity implies that the type n_0 must be indifferent between yielding in period $T^k + 2$ and T^0 and weakly prefer it to yielding in any period in-between. However, we are going to show that if n is sufficiently small, then type n_0 strictly prefers to yield in period T^{m_4} rather than in period T_0. This will yield a contradiction, and conclude the proof of the Lemma.

For this purpose, let $u_0 = \beta_i + n_0\gamma$ be the type that corresponds to n_0. Notice that formula (A.1) implies that

$$\begin{aligned}
&U_{-i}^{\sigma}\left(u_0 T^{m_4}\right) - U_{-i}^{\sigma}\left(u_0, T_0\right) \\
&= \sum_{t \in T_i : T_0 < t \leq T^{m_4}} \mathrm{e}^{-s\Delta}\left(f^{\sigma}(t+1) + \left(1 - \mathrm{e}^{-2\Delta}\right)\left(\sum_{s : s > t+1, z \in T_i} f^{\sigma}(z)\right)\right)\beta_i \cdot \left[y_{-i}(t+1) - y_{-i}^{-k}\right] \\
&\quad + n_0 \sum_{t \in T_i : T_0 < t \leq T^{m_4}} \mathrm{e}^{-s\Delta}\left(f^{\sigma}(t+1) + \left(1 - \mathrm{e}^{-2\Delta}\right)\left(\sum_{s : s > t+1, z \in T_i} f^{\sigma}(z)\right)\right)\gamma \cdot \left[y_{-i}(t+1) - y_{-i}^{-k}\right].
\end{aligned} \tag{E.12}$$

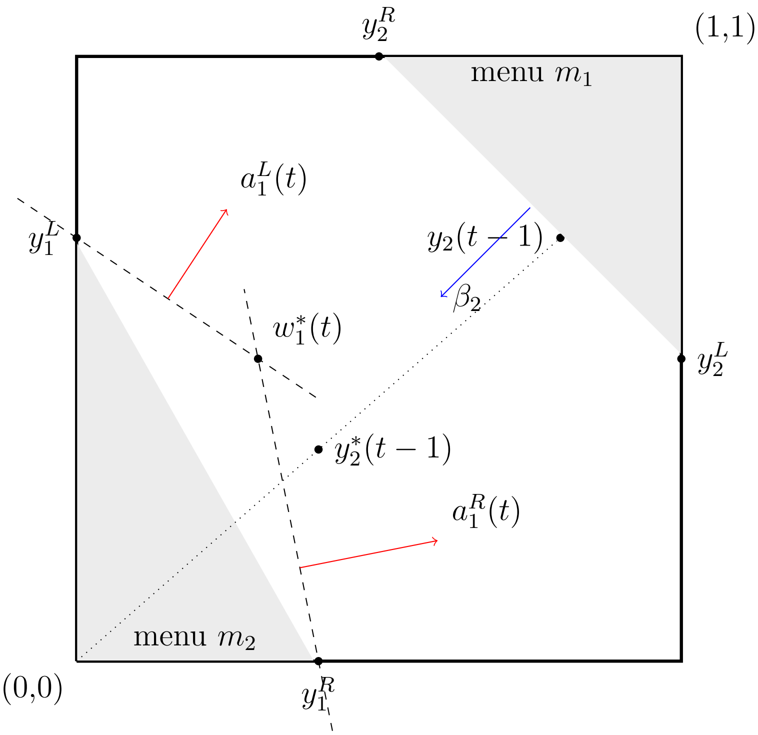


FIGURE E.3. TBA

Because $y_{-i}(t+1) \in X \backslash \mathrm{int} m_i$, we have $\beta_i \cdot \left[y_{-i}(t+1) - y_{-i}^{-k}\right] \geq 0$ for each t. Moreover, by Lemma 29, for each $t \geq T^{m_3}$,

$$\beta_i \cdot \left(y_{-i}(t) - y_{-i}^k\right) \geq C_0 (-1)^k \eta_{-i}^k (t+1) \geq C_0 n_4.$$

Hence, the first term of (E.12) is not smaller than

$$\begin{aligned} &\geq \sum_{t \in T_i : T^0 < t \leq T^{m_4}} \mathrm{e}^{-s\Delta} \left(f^\sigma (t+1)\right) \beta_i \cdot \left[y_{-i}(t+1) - y_{-i}^{-k}\right] \\ &\geq \beta^{T^{m_4}} \sum_l \left(F_i^l \left(T^{m_3}\right) - F_i^l \left(T^{m_4}\right)\right) C_0 n_4 \\ &\geq \frac{1}{4} C_0 f^* \left(\frac{1}{2}\right)^{a^2(a')^{-2}} m_3^{a^3(a')^{-3}} \frac{1}{2} m_3^{a(a')^{-1}} \left(\frac{1}{2}\right)^{a^2(a')^{-1}} m_3^{a^3(a')^{-2}} =: c_0. \end{aligned}$$

Let $x^* = \max_{x \in X} \left|\gamma \cdot \left(x - y_{-i}^{-k}\right)\right|$ Then, the second term of (E.12) is not smaller than

$$\geq -n x^*.$$

The lemma is concluded by picking $n < \frac{c_0}{x}$.

E.5. **Proof of Lemma 17.**

Lemma 36. *For each x, if $y \in bdm_i$ and $P_i x = P_i y$, then $R_{-i} x = y$ and $P_{-i} y = y$. Moreover, there exist constants $A_i > 1$ for each i such that for each i, each $x \in bdm_i$,*

$$\gamma \cdot (\mathbf{1} - R_{-i} x) = -A_i (\gamma \cdot x) .$$

Proof. See Figure E.1. $\square$

Let n be as in Lemma 16. Let

$$\overline{p_i} (t) = \gamma \cdot \left(\sum_k \overline{P_i^k} w_i (t) \, y_i^k \right),$$

$$p_i (t) = \gamma \cdot \left(\sum_k P_i^k x_i (t) \, y_i^k \right),$$

$$q_i (t) = \gamma \cdot \left(\sum_l Q_i^l (t) \, y_{-i}^l \right),$$

$$\overline{q_i} (t) = \gamma \cdot \left(\sum_l \overline{Q_i^l} (t) \, y_{-i}^l \right).$$

Using the projection notation from Section E.1.1, we show (**??**) is equivalent to

$$P_i x_i (t) = P_i y_i (t-1) = P_i \left(\mathbf{1} - \sum_l Q_{-i}^l (t-1) \, y_{-i}^l \right).$$

Because $\sum_l Q_{-i}^l (t-1) \, y_{-i}^l \in \mathrm{bd} m_i$, Lemma 36 implies that

$$\gamma \cdot \sum_l Q_{-i}^l (t-1) \, y_{-i}^l = A_{-i} (\gamma \cdot P_i x_i (t)) , \text{ or} \tag{E.13}$$

$$q_{-i} (t-1) = A_{-i} p_i (t) .$$

Further, (E.4) implies that

$$\Delta \overline{p_i} (t) = c (t) \left(\overline{q_i} (t) - \overline{p_i} (t+2) \right), \tag{E.14}$$

for $c(t) = \frac{\sum_l \Delta \eta_i^l (t)}{\sum_l \eta_i^l (t)} \in [0, 1]$.

Let $C < \infty$ be the constant from Lemma 15. Let $C = C' \max A_i$. Then, Lemma 15 implies that

$$\Delta \overline{p_i} (t-2) = c (t-2) \left(A_{-i} \overline{p_{-i}} (t-1) - \overline{p_i} (t) \pm C' \eta \right), \tag{E.15}$$

where $\pm C \eta$ is a bound on the error term of the expression in the brackets.

Lemma 37. *For each $\eta \leq n$, we have that for each $t > T^{\eta}$ and $t \in T_i$, either (a) $\overline{p_i}(t) \cdot \overline{p_{-i}}(t-1) > 0$, or (b) $\left|\overline{p_i}(t) - \overline{p_{-i}}(t-1)_i\right| \leq C'\eta$.*

Proof. We prove the Lemma by induction on $t \geq T^*$. If $t = T^*$, then $c(T^*-1) = 1$, and $\overline{p_i}(T^*-1) = \overline{q_i}(T^*-1) = A_i\overline{p}_i \pm C\eta$, where $\pm C\eta$ is a bound on the error term. Thus, the claim holds for $t = T^*$.

Suppose that the claim holds for some $t > T^{\eta}$ and $t \in T_i$. Suppose that $\overline{p_{-i}}(t-1) > 0$ (the proof in the other case is analogous). The inductive claim implies that $\overline{p_i}(t) \geq \overline{p_{-i}}(t-1) - C\eta$, and we need to show that $\overline{p_i}(t-2) \geq p_{-i}(t-1) - C\eta$. By Lemma 16 and the above discussion, equation (E.15) holds. Because $A_{-i} > 1$,

$$A_{-i}\overline{p_{-i}}(t-1) \pm C'\eta \geq \overline{p_{-i}}(t-1) - C'\eta,$$

and

$$\begin{aligned}\overline{p_i}(t-2) - \overline{p_{-i}}(t-1) &= \Delta\overline{p_i}(t-2) + \overline{p_i}(t) - \overline{p_{-i}}(t-1) \\ &\geq (1 - c(t-2))\left(\overline{p_i}(t) - \overline{p_{-i}}(t-1)\right) - c(t)\,C'\eta \\ &\geq -C'\eta.\end{aligned}$$

□

Lemma 38. *For each $\delta > 0$, there is $c_0 > 0$ and $\eta_\delta \leq n$, such that for each $t > T^{\eta_\delta}$ and $t \in T_i$, if $\overline{p_i}(t), \overline{p_{-i}}(t-1) \geq \delta$, then $p_{-i}(t-2) \geq \delta + c(t-2)\,c_0\delta$. (An analogous claim holds when $\overline{p_i}(t), \overline{p_{-i}}(t-1) \leq -\delta$.)*

Proof. Choose η_δ so that $(\min_i A_i - 1)\,\delta \geq 2C'\eta_\delta$. Let $c_0 = \frac{1}{2}(\min_i A_i - 1)$. By formula (E.15)

$$\begin{aligned}\overline{p_i}(t-2) - \delta \geq& \Delta\overline{p_i}(t-2) + \overline{p_i}(t) - \delta \\ \geq& c(t-2)\left((A_{-i} - 1)\,\overline{p_{-i}}(t-1) - \delta\right) \\ &+ c(t-2)\left(\overline{p_{-i}}(t-1) - \delta\right) \\ &+ (1 - c(t-2))\left(\overline{p_i}(t) - \delta\right) \\ \geq& c(t-2)\left((A_{-i} - 1)\,\delta - C'\eta\right) \geq c(t-2)\,C\delta.\end{aligned}$$

□

Lemma 39. *There exists a* $D<\infty$ *such that for each* $\delta>0$*, there is* $\eta_\delta \leq n$*, such that if* $\overline{p_i}(t) \geq D\delta$ *for some* t*, then for each* $t' > T^{\eta_\delta}$*,*$t'<t$*,* $t \in T_j'$*,* $\overline{p_j}(t') \geq \delta$*. (An analogous claim holds when* $p_i(t) \leq -D\delta$*.)*

Proof. Let $D = 2\max_i A_i$. Let η_δ' be the constant from Lemma 38. Let $\eta_\delta \leq \eta_\delta'$ be such that $C\eta_\delta \leq \frac{1}{2}D\delta$. By Lemma 38, it is enough to show that if $t_0 = \max\{t : \overline{p_i}(t) \geq D\delta \text{ for } i \text{ st. } t \in T_i\}$, then $\overline{p_{-i}}(t_0+1) \geq \delta$. To see it, notice that $\overline{p_i}(t_0+2) \leq D\delta \leq \overline{p_i}(t_0)$. Hence, formula (E.15) and the fact that $c(t_0) \leq 1$ imply that

$$\begin{aligned}
0 \leq D\delta - \overline{p_i}(t_0+2) \leq & A_{-i}\overline{p_{-i}}(t_0+1) - \overline{p_i}(t_0+2) + C'\eta \\
\leq & A_{-i}\overline{p_{-i}}(t_0+1) + C'\eta - D\delta - (\overline{p_i}(t_0+2) - D\delta) \\
\leq & A_{-i}\overline{p_{-i}}(t_0+1) - \frac{1}{2}D\delta.
\end{aligned}$$

The claim follows from the choice of constant D. □

Lemma 40. *There exists* $\Delta^*, \eta^* > 0$ *such that for each integer* $A>0$*, for each* $\eta \leq \frac{1}{2^{8A}}\eta^*$*, there exists* $\lambda^*>0$ *such that if* $\Delta \leq \Delta^*, \lambda \leq \lambda^*$*, then for each player* i*,*

$$\sum_{t \in T_i : T^\eta \leq t \leq T^{2^{8(A+1)}\eta}} \frac{\sum_l \Delta\eta_i^l(t)}{\sum_l \eta_i^l(t)} \geq A.$$

Proof. If $\Delta>0$ is sufficiently small, then Assumption 4 implies that $p_i^\sigma(t) \leq \frac{1}{4}$ for each $t \in T_i$ and $t>1$. Then, Lemma 15 implies that there exists $\eta^*>0$ such that for each $\eta \leq \eta^*$, there exists $\lambda^*>0$ such that

$$\frac{\sum_l \Delta\eta_i^l(t)}{\sum_l \eta_i^l(t)} \leq \frac{1}{2} \text{for each t} \in T_i \text{ and } T_i^\eta \leq \text{t} \leq T_i^{\eta^*}.$$

Hence, for each k, there exist t_k^1, t_k^2 such that $8^k\eta \leq \sum_l \eta_i^l(t_k^1) \leq 2\cdot 8^k$ and $4\cdot 8^k\eta \leq \sum_l \eta_i^l(t_k^2) \leq 8^{k+1}$. Then, we have

$$\begin{aligned}
\sum_{t \in T_i : T^\eta \leq t \leq T^{8(A+1)\eta}} \frac{\sum_l \Delta\eta_i^l(t)}{\sum_l \eta_i^l(t)} &\geq \sum_{k=0}^{8A} \frac{1}{8^{k+1}\eta} \sum_{t \in T_i : T^{8^k\eta} \leq t < T^{8^{k+1}\eta}} \sum_l \Delta\eta_i^l(t) \\
&\geq \sum_{k=0}^{8A} \frac{1}{8^{k+1}\eta}\left(\sum_l \eta_i^l\left(t_k^2\right) - \sum_l \eta_i^l\left(t_k^1\right)\right) \geq \sum_{k=0}^{8A} \frac{1}{8^{k+1}\eta}\left(2\cdot 8^k\eta\right) \\
&\geq 8A \cdot \frac{1}{4} > A.
\end{aligned}$$

□

E.5.1. *Proof of Lemma 17.*

Proof. Let $P_{\max} = \left\lceil Dc_0^{-1} \max_{x\in X} |\gamma \cdot x| \right\rceil$, where C is the constant from Lemma 38. Let η'_δ be the constant from Lemma 39 (in the proof, we choose it so that it satisfies also Lemma 38). Let $\eta_\delta = \frac{1}{8^{2(P_{\max}+1)}} \eta'_{\frac{1}{D}\frac{1}{\max_i A_i}\delta}$, where D is the constant from Lemma 39 .

On the contrary, suppose that $|p_i(t)| \leq \delta$ for some $t > T^{\eta_\delta}, t \in T_i$. W.l.o.g. we assume that $\overline{p_i}(t) > 0$. Then, Lemma 39 implies that $\overline{p_{-i}}(t+1) \geq \frac{1}{D}\frac{1}{\max_i A_i}\delta$. By Lemma 38, for each $t \geq T^{\eta'_{D^{-1}\delta}}$,

$$p_{-i}(t-2) \geq \delta + c_0 D^{-1} \frac{1}{\max_i A_i} \delta c(t-2).$$

Hence, using the definition of $c(.)$, we have

$$\frac{1}{\delta} P_{\max} \geq \sum_{t \in T_i : T^{\eta_\delta} \leq t \leq T^{4^{\left\lceil \log \frac{1}{\delta} P_{\max} \right\rceil} \eta_\delta}} \frac{\sum_l \Delta \eta_i^l(t)}{\sum_l \eta_i^l(t)}.$$

It follows from Lemma 40 that

$$|\overline{p_i}(t)| \leq \frac{1}{\max_i A_i} \delta.$$

Note that by Lemma 16 and equation (E.3), we have

$$w_i(t) = R_{-i} y_i(t) = R_{-i} x_i(t).$$

Hence, by Lemma 36,

$$|\gamma \cdot w_i(t)| \leq \delta.$$

The result follows from the fact that $w_i(t)$ belongs to the boundary of menu m_i, and that $\mathbf{1} - e^*_{-i}$ is the intersection of the boundary with the diagonal $\{x : \gamma \cdot x = 0\}$. ☐

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