A Geometric Approach to the Complexity of Mechanisms

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Abstract

In this paper, we study the minimum complexity of mechanisms required to achieve a certain vector of objectives under the assumption that the map between the set of mechanisms and the vector of objectives is linear. Our measure of complexity translates in most of the applications into the minimum number of separating contracts required by the mechanism. Our main result is that this minimum number is equal to the dimension of the vector of objectives space plus one. We provide four applications, bilateral trade with either one-sided or both-sided asymmetric information, constrained efficiency in Akerlof’s model and optimal taxation.

Key Words: Mechanism design.

JEL-Classification: G21, D45.
1 Introduction

In this paper, we provide lower bounds to the complexity of mechanisms necessary to implement a vector of objectives. To do so, we use that the typical set of mechanisms is a closed convex subset of a vectorial space and hence, each mechanism can be written as a convex combination of the set of extreme points of the set of mechanisms. We provide lower bounds on the number of elements of the convex combinations of extreme points required to achieve a certain set of vector of objectives.

In our setting, a payoff function maps a mechanism into a set of vector of objectives. A vector of objectives is a vector of all the variables of interest. In the typical problem these variables are the vector of objectives to be maximized, for instance expected social surplus, and some constraints, for instance expected budget surplus.

An important assumption is that we restrict attention to maps from the set of mechanisms to the set of vector of objectives that are linear. Under this assumption, our main result is that any vector of objectives can be implemented with a convex combination of \( n + 1 \) extreme point mechanisms, where \( n \) is the dimensionality of the space of feasible vector of objectives. In the particular case, in which we are only interested in vector of objectives that lie at the boundary of the feasible set, only \( n \) extreme points are needed.

For instance, our result implies that one can restrict to extreme point mechanisms if one is interested in maximizing expected revenue in the standard monopoly problem because the only vector of objectives is the expected revenue of the monopolist. Since extreme point mechanisms are, in this example, deterministic mechanisms our result implies that one can restrict to deterministic mechanisms.

Another example is the case of bilateral trade with two-side asymmetric information. In this case, the vector of objectives in the standard problem are two: expected social surplus and expected budget surplus. Our result thus implies that one can fully characterized the set of vector of objectives restricting to mechanisms that are the convex combination of at most three extreme point mechanisms, these are mechanisms in which the probability of trade function is a step function with no more than two steps. Again, if we are interested in vector of objectives at the boundary (for instance, because we are interested in mechanisms that maximize expected social surplus subject to budget balance), one can restrict attention to probability of trade functions that are step functions with only one interior step.

To derive our results we use some basic tools from linear algebra. Under our linearity assumption,
the set of vector of objectives is convex. Since it lies in \( n \) dimensional space any vector of vector of objectives can be written as the convex combination of no more than \( n + 1 \) extreme points of the set of vector of objectives by application of Caratheodory’s Theorem. Linearity also means that any extreme point of the set of vector of objectives is attainable with an extreme point mechanism. Our main result thus follows from another implication of linearity, the vector of objectives of a convex combination of mechanisms is equal to the same convex combination of the vector of objectives associated to the mechanisms. The result for vector of objectives in the boundary is a consequence that points in the boundary lie in the intersection of the set with a hyperplane and this intersection has dimension \( n - 1 \) at most.

We complement these results studying under which conditions, one can restrict to extreme point mechanisms. This leads to two set of conditions. The first condition is that the cyclical monotonicity condition necessary and sufficient for incentive compatibility must not be binding. This condition has been called in previous work the *regularity condition*. The second one is the condition that the vector of vector of objectives associated to each vector of types must be linearly independent almost surely with respect to the distribution of types. This condition seems to be satisfied generically when the distribution is atomless but not otherwise.

Finally, we provide tighter bounds on the number of extreme points required for the particular case in which cyclical monotonicity is binding.

## 2 General Results

The main primitives of our analysis are a type space, denoted by \((\Theta, \mathcal{B})\), a set of alternatives, denoted by \(Q\) and a set of mechanisms, denoted by \(C\). Formally, \((\Theta, \mathcal{B})\) is a measurable space on a compact set \(\Theta\), \(Q\) is a compact and convex subset of an Euclidean space and \(C\) is a compact and convex subset of the set of measurable maps from \(\Theta\) to \(Q\), denoted by \(L(\Theta, Q)\).\(^1\) Note that \(Q\) bounded means that \(Q\) bounded means that\(^2\)

\(^1\)These conditions can be relaxed and our main result (Lemma 1) still applies. Indeed, this result only needs of the conditions of Lemma I in Artstein (1980) and compactness of \(C\). The conditions we provide are, however, met by the mechanism design problems we are interested in and simplify our notation.

\(^2\)\(L^\infty(\Theta)\) denotes the collection of equivalent classes \([f]\) for which \(f: \Theta \to \mathbb{R}\) is essentially bounded, i.e. there exists an \(M \in \mathbb{R}\) for which \(|f| \leq M\) a.e., see Royden and Fitzpatrick (2010), page 395, for a detailed definition. We also adopt the convention of writing the elements of \(L^\infty(\Theta)\) as functions rather than equivalent classes. Thus, for any two elements \(f, g \in L^\infty(\Theta)\), one must interpret the statement \(f = g\) as \(f(\theta) = g(\theta)\) almost everywhere in \(\Theta\), and the statement \(f \in L^\infty\) that \(f\) belongs to an equivalent class that it is essentially bounded.
We are interested in continuous affine maps $A : C \to \mathbb{R}^n$ that associate to each mechanism a vector in an $n$-dimensional Euclidean space. In applications, each of the dimensions correspond to an endogenous variable with respect to which we either evaluate the mechanism (e.g. expected utility for each of the agents, expected revenue generated by the mechanism,...) or define a constraint (non positive expected budget deficit, a participation constraint for an uninformed agent,...) Let $A(C)$ be the range of $A$ and $\partial A(C)$ be the boundary of $A(C)$. We let $m \leq n$ denote the number of algebraic dimensions of $A(C)$.

**Definition:** We say that $x$ is an extreme point of a convex set $S$ if there are no two distinct elements, $y$ and $z$, in $S$ and $\lambda \in (0, 1)$ such that $x = \lambda y + (1 - \lambda)z$.

Thus, an extreme point of $S$ is an element of $S$ that it is not contained in any segment connecting two other elements of $S$. We denote the set of extreme points of $S$ by $\text{ext}(S)$. Our first main result provides an upper bound to the number of extreme points required to achieve a given point in the range of $A$.

**Lemma 1.**

(i) For any $y \in A(C)$, there exists $\{q_i\}_{i=1}^{m+1} \subset \text{ext}(C)$ such that:

$$A \left( \sum_{i=1}^{m+1} \lambda_i q_i \right) = y, \text{ for some } \lambda \in \Delta(m + 1).$$

(ii) For any $y \in \partial A(C)$, there exists $\{q_i\}_{i=1}^{\min\{m+1,n\}} \subset \text{ext}(C)$ such that:

$$A \left( \sum_{i=1}^{\min\{m+1,n\}} \lambda_i q_i \right) = y, \text{ for some } \lambda \in \Delta(\min\{m+1,n\}).$$

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3 In particular, Royden and Fitzpatrick (2010) shows that: $L^\infty(\Theta)$ is weakly* compact by Alaoglu theorem, page 299; the weak* topology in $L^\infty$ is metrizable since $L^\infty(\Theta)$ is the dual of $L^1(\Theta)$ that it is separable, Theorem 11, page 152, and Corollary 11, page 306; close subsets of compact metric spaces are compact, Proposition 15, page 234; and finally, in a metric space compactness is equivalent to sequential compactness, Theorem 16, page 199. Thus, to prove compactness of $C$ we only need to show that the weak* limit of a convergent sequence in $C$ belongs to $C$. This can be derived in all our applications from an adaptation of the arguments in Manelli and Vincent (2010).

4 Aliprantis and Border (2006) defines: (page 166) a function $f : X \to Y$ between two vector spaces is linear if it satisfies $f(\alpha x + \beta z) = \alpha f(x) + \beta f(z)$ for every $x, z \in X$ and $\alpha, \beta \in \mathbb{R}$; and (page 256) a function $f : X \to \mathbb{R}$ on a vector space is affine if it is of the form $f(x) = x^*(x) + c$ for some linear function $x^* \in X^*$ and some $c \in \mathbb{R}$.

5 We denote by $\Delta(\tau) = \{ \lambda \in \mathbb{R}^\tau : \sum_{i=1}^{\tau} \lambda_i = 1 \}$ the $(\tau - 1)$-dimensional simplex.
Proof. To prove (i)\(^6\) note that since \(A(C)\) is compact and convex (because \(C\) is compact and convex and \(A\) is continuous and affine) Krein-Milman theorem, see Aliprantis et al (2006), page 297, implies that \(A(C)\) is the convex hull of the extreme points of \(A(C)\). Thus, the application of Caratheodory Convexity Theorem, see Aliprantis et al (2006), page 184, implies that \(y = \sum_{i=1}^{m+1} \lambda_i y_i\) for some \(m+1\) extreme points \(\{y_i\}_{i=1}^{m+1}\) of \(A(C)\). Lemma 1 in Artstein (1980) (it applies because \(C\) is compact, and hence it is closed, and Remark 2.1(a) in Arstein applies) implies that for any extreme point \(y_i\) of \(A(C)\), there exists an extreme point \(q_i\) of \(C\) such that \(A(q_i) = y_i\). Thus, we can use the affinity of \(A\) to prove that \(A(\sum_{i=1}^{m+1} \lambda_i q_i) = \sum_{i=1}^{m+1} \lambda_i A(q_i) = \sum_{i=1}^{n+1} \lambda_i y_i = y\) as desired.

To prove (ii). We shall argue here that \(y \in \partial A(C)\) lies in a compact and convex set \(F\) that has dimension at most \(\min\{m, n-1\}\) and whose extreme points are extreme points of \(A(C)\). The rest of the argument is as in the paragraph above. We define \(F \equiv A(C) \cap H_y\), where \(H_y\) is a supporting hyperplane of \(A(C)\) that contains \(y\). That \(H_y\) exists, and so \(F\), is a consequence of the supporting hyperplane theorem. \(F\) is closed and convex because both \(A(C)\) and \(H_y\) are closed and convex. Since \(F \equiv A(C) \cap H_y\) and the dimensionalities of \(A(C)\) and \(H_y\) are, respectively, \(m\) and \(n-1\), the dimensionality of \(F\) is at most \(\min\{m, n-1\}\) as desired. Finally, the extreme points of \(F\) are extreme points of \(A(C)\) because \(F\) is a face of \(A(C)\), see Rockafellar (1972), pp. 162-3. \(\blacksquare\)

The lemma says that any point in \(A(C)\) or \(\partial A(C)\) can be achieved with a convex combination of \(m+1\) or \(\min\{m+1, n\}\) extreme points of \(C\), respectively. However, it could be that the case that the same point could be achieved with a convex combination of a smaller number of extreme points.

Next lemma gives conditions under which this is generically not the case.

**Lemma 2.** If \(\text{ext}(C)\) is countable, then:

- (1) implies \(\lambda_i > 0\) for all \(i = 1, \ldots, m+1\) except in a meagre subset of \(A(C)\).
- (2) implies \(\lambda_i > 0\) for all \(i = 1, \ldots, \min\{m+1, n\}\) except in a meagre subset of \(\partial A(C)\).

\(^6\)The proof of (i) can be shown to follow from Lemma 1 in Artstein (1980) and Caratheodory Convexity Theorem without appealing to Krein-Milman theorem. Our proof, however, uses an alternative argument whose direct generalization implies the second claim.
union of meagre subsets is also a meagre subset. To show (i) note that \(\text{ext}(C)\) countable means that the cardinality of the set of subsets of \(\text{ext}(C)\) with \(m\) elements is weakly less than the cardinality of \(\mathbb{N}^m\), which is a countable set. To show (ii) note that \(A\) affine means that the image set of the convex hull of \(m\) elements of \(\text{ext}(C)\) is equal to the convex hull of the images of the \(m\) elements of \(\text{ext}(C)\) and the geometric dimension of the convex hull of \(m\) points is at most \(m - 1\). Finally, (iii) follows from the definition of a meager set, see for instance Royden and Fitzpatrick (2010), page 214.

To prove the second claim, we distinguish two cases, when \(m < n\) and when \(m = n\). In the former case, \(A(C)\) has an empty interior, see Ok (2007) Section G.1.4, and thus \(\partial A(C) = A(C)\) since \(A(C)\) is closed. Consequently, the second claim follows from the first claim. Suppose now that \(m = n\). Then, the interior of \(A(C)\) is non-empty, again see Ok (2007) Section G.1.4, and thus \(\partial A(C)\) is homomorphic to the \(n - 1\) dimensional sphere \(\{u \in \mathbb{R}^n : ||u|| = 1\}\) (see, for instance, Proposition 4.26 in Lee (2010)) and thus a manifold of dimension \(n - 1\). We can now repeat the arguments of the proof of the first claim replacing \(A(C)\) by \(\partial A(C)\) and \(m\) by \(n - 1\).

Typically, the set of extreme points is countable when the distribution of types is discrete, i.e. \(\Theta\) finite, see for instance Proposition 4. This is because \(Q\) is usually a finite dimensional simplex, and the set of mechanisms is defined by finitely many linear inequalities (one per each incentive compatibility constraint), over the set \(\{q : \Theta \to Q\} \subset \mathbb{R}^T\), where \(T\) is equal to the cardinality of \(\Theta\) times the number of dimensions of \(Q\). Thus, the set of mechanisms is a polytope with a finite number of extreme points.

Lemma 2 provides conditions under which the bounds on the number of extreme points required in Lemma 1 are tight. Next, we study the opposite case, when we can restrict to one single extreme point rather than the convex combinations of several extreme points. In this case, it is required that \(A\) admits an integral representation and it is regular:

**Definition:** We say that \(A\) admits an integral representation if:

\[
A(q) = \int (A(\theta) \cdot q(\theta)) d\mu(\theta),
\]

where \((\Theta, \mathcal{B}, \mu)\) is a complete non-atomic \(\sigma\)-finite measurable space and \(A : \Theta \to \mathbb{R}^{s \times n}\) is measurable.

**Definition:** An \(A\) that admits an integral representation is regular at \(y \in \partial A(C)\) if: (i) \(y \in \partial A(C)\) and (ii) \(A\) admits an integral representation.

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\(^7\)We use \(\cdot\) to denote the matrix product.
\[ \partial A(L(\Theta, Q)) \]; and (ii) \( \arg \max_{x \in Q} \{ \lambda' \cdot A(\theta) \cdot x \} \) is a singleton a.e. for \( \lambda \in \mathbb{R}^n \) a vector that defines a supporting hyperplane of \( A(L(\Theta, Q)) \) at \( y \).

The interpretation of (i) is clear in the context of mechanism design. In the typical application \( y \) comes as the result of choosing a mechanism in \( C \) that gives an optimal point in \( A(C) \). Thus, (i) says that the restriction to mechanisms in \( C \) rather than in \( L(\Theta, Q) \) is not binding. In the applications, the difference between \( C \) and \( L(\Theta, Q) \) is that the former is the subset of the latter that satisfies the cyclical monotonicity conditions, see ?, that characterize incentive compatibility for each agent. Thus, the interpretation of (i) is the cyclical monotonicity conditions are not binding. In this sense our notion of regularity generalizes the regular case described by Myerson (1981). (ii) is a technical condition that we expect to be met in most of the applications. To see why, note that \( Q \) is a simplex in the typical mechanism design application, and in this case (ii) is equivalent to require that the components of the vector \( \lambda' \cdot A(\theta) \) have one single maximum a.e.

Next lemma shows that regularity is sufficient to guarantee that there is no loss in restricting to extreme point mechanisms.

**Lemma 3.** If \( A \) admits an integral representation and it is regular at \( y \in \partial A(C) \), then there exists an extreme point \( q \in \text{ext}(C) \) for which \( A(q) = y \).

**Proof.** First, we argue that there exists a \( q^e \in \text{ext}(L(\Theta, Q)) \) such that \( A(q^e) = y \). This follows by application of Theorem 3.2 in Artstein (1980) for \( \Omega(s) \equiv Q, \forall s \in S \equiv \Theta, U_{\Omega} \equiv L(\Theta, Q) \) and \( A : L(\Theta, Q) \rightarrow \mathbb{R}^n \). The conditions of the theorem are met because by assumption \( Q \) is closed, convex, and bounded, and thus \( \Omega \) is \( p \)-integrally bounded \( (1 \leq p \leq \infty) \), which implies that \( L(\Theta, Q) \) is weak\(^*\) compact, see Artstein discussion preceding its Theorem 3.2. Finally, \( A : L(\Theta, Q) \rightarrow \mathbb{R}^n \) is weak\(^*\) continuous by definition of the weak\(^*\) topology.

Next, we argue that \( q \in L(\Theta, Q) \) and \( A(q) = y \) implies that \( q = q^e \). This is because by definition of a supporting hyperplane,

\[
y = \max_{x \in A(L(\Theta, Q))} \{ \lambda' \cdot x \} = \max_{q \in L(\Theta, Q)} \{ \lambda' \cdot A(q) \},
\]

and by the second condition of regularity,

\[
\arg \max_{q \in L(\Theta, Q)} \{ \lambda' \cdot A(q) \} = \arg \max_{q \in L(\Theta, Q)} \left\{ \int \lambda' \cdot A(\theta) \cdot q(\theta) d\mu(\theta) \right\}
\]

is a singleton.
Note that the last claim and $C \subset L(\Theta, Q)$ means that $q^e \in C$. Thus, the lemma follows from the fact that the extreme points of $L(\Theta, Q)$ contained in $C$ are also extreme points of $C$. ■

3 Applications

In this section, we show how Lemma 1 can be used to simplify some mechanism design problems with an interesting economic interpretation.

3.1 Samuelson’s (1984) Model of Bilateral Trade

In this section, we revisit the model of Samuelson (1984). The purpose of this section is to show how their central results can be deduced from our Lemma 1 and illustrate in a simple setting the role of the elements of Section 2 in applications. We first study a version of the model with a continuum of types and then a version with a discrete distribution of types.

3.1.1 A Continuum Distribution

Samuelson (1984) studies the bargaining problem between two parties, a buyer and a seller with quasilinear preferences in money. The seller’s monetary (opportunity) cost of selling the good is equal to $\theta$ and the buyer’s monetary value of obtaining the good is equal to $w(\theta)$, where $w(\theta)$ is bounded. The seller knows privately $\theta \in [\underline{\theta}, \bar{\theta}]$ that has a distribution $\mu$ with density $\mu' \in (\eta, 1/\eta)$ for some $\eta > 0$. We adopt the convention that $\mu(x)$ denotes the probability that the seller’s type is less than $x$.

A direct mechanism is a measurable map from $[\underline{\theta}, \bar{\theta}]$ to a probability of trade $q \in [0, 1]$ and a transfer $t \in \mathbb{R}$ from the buyer to the seller. The application of standard mechanism design tools means here that a direct mechanism is incentive compatible if and only if the probability of trade function $q$ belongs to the set:

$C' \equiv \{q \in L([\underline{\theta}, \bar{\theta}], [0, 1]) : q \text{ decreasing}\}$

$\text{Note}^{8}$ Recall that $L([\underline{\theta}, \bar{\theta}], [0, 1])$ denotes the subset of elements of $L^\infty([\underline{\theta}, \bar{\theta}])$ with range $[0, 1]$ and that we simplify and that the elements of $L^\infty([\underline{\theta}, \bar{\theta}])$ are equivalent classes although for simplicity we describe as functions, see Footnote 2. Thus, $q \in L([\underline{\theta}, \bar{\theta}], [0, 1]) : q \text{ decreasing}$ must be interpreted that there exists an essentially bounded and increasing function $\tilde{q}$ such that $q = \tilde{q}$ a.e. Similarly, $q \in C$ must be interpreted that there exists an incentive compatible mechanism with probability of trade $\tilde{q}$ such that $q = \tilde{q}$ a.e.

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and the transfer satisfies:

\[ t(\theta) = q(\theta)\theta + \int_{\theta}^{\bar{\theta}} q(x)dx + u_S, \]

for some \( u_S \in \mathbb{R} \) that corresponds to the expected utility of a seller with type \( \bar{\theta} \). Thus, one can deduce that the seller’s expected utility conditional on her type is a decreasing function of her type and that her unconditional expected utility is equal to:

\[ \int \frac{\mu(\theta)}{\mu'(\theta)} q(\theta) d\mu(\theta) + u_S, \]

while the buyer’s expected utility is equal to:

\[ \int \left( w(\theta) - \theta - \frac{\mu(\theta)}{\mu'(\theta)} \right) q(\theta) d\mu(\theta) - u_S. \]

It is easy to see that maximizing the buyer’s expected utility subject to all the seller’s types making non-negative expected utility requires fixing \( u_S = 0 \) and choosing a point in the upper bound of the range of the continuous linear map:

\[ A(q) = \int \left( w(\theta) - \theta - \frac{\mu(\theta)}{\mu'(\theta)} \right) q(\theta) d\mu(\theta), \text{ for any } q \in C. \]

Thus, Lemma 1 implies that one can restrict to extreme point mechanisms without loss of generality. Since the set of extreme points of the set \( C \) are its deterministic functions, i.e. decreasing step functions with range in \( \{0, 1\} \), one can deduce the following result (and no additional proof is required):

**Proposition 1.** The maximum expected utility of the buyer subject to non-negative expected utility of the seller can be achieved with a deterministic mechanism, i.e. a mechanism whose allocation function \( q \in C \) is a step function with range in \( \{0, 1\} \).

Now, consider instead the problem of maximizing the seller’s expected utility subject to the buyer getting non negative expected utility. Again, the solution corresponds to a point at the boundary of the range of the continuous linear map:

\[ A(q) = \left( \int \frac{\mu(\theta)}{\mu'(\theta)} q(\theta) d\mu(\theta) \right), \text{ for any } q \in C. \]

Thus, Lemma 1 implies that one can restrict to convex combinations of two extreme point mechanisms without loss of generality, which implies the following proposition:

\[ ^9 \text{See, for instance, Börgers (2015), Lemma 2.7 page 17.} \]
Proposition 2. The maximum expected utility of the seller subject to non-negative expected utility of the buyer can be achieved with a mechanism with one-random step, i.e. a mechanism characterized by a decreasing step function $q$ with range in $\{0, \lambda, 1\}$ for some $\lambda \in (0, 1)$.

Proof. In light of Lemma 1 and since the set of extreme points of $C$ is equal to the set of decreasing step functions with range in $\{0, 1\}$, we only need to show that the convex combination of any two of these step functions satisfies the conditions of the proposition. Clearly, the convex combination of two decreasing functions is a decreasing function. Note next that any decreasing step functions with range in $\{0, 1\}$ satisfies that it is equal to one up to the point at which it jumps to zero. Consider the convex combination of two such step functions with jumping points $c_1$ and $c_2$, where $c_1 \leq c_2$ without loss of generality. This convex combination takes value one for $\theta < c_1$, value zero for $\theta > c_2$ and for intermediate values the parameter of the convex combination as desired. 

Note that all types that trade with the same probability also trade at the same expected price, and thus the number of points in the range of the function $q$ correspond to the minimum number of semipooling intervals required in an indirect mechanism that implements $q$. For instance, the mechanism that correspond to Proposition 2 can be implemented with an indirect mechanism that offers only two trade contracts to the seller: one that trades with probability $\lambda$ and another one that trades with probability one.

Note that the optimization of the buyer’s problem subject to the seller’s participation constraint correspond to maximize the first component of the vector function $A(q)$ in (4) subject to the second component achieving a certain value. The solutions to this problem maximize the Lagrangian:

$$\int \left( \frac{\mu(\theta)}{\mu(\theta)} + \lambda \left( w(\theta) - \theta - \frac{\mu(\theta)}{\mu(\theta)} \right) \right) q(\theta) d\mu(\theta),$$

where $\lambda \geq 0$ is the Lagrange multiplier. Thus, a sufficient condition for the the optimal $q \in L([\theta, \bar{\theta}], [0, 1])$ of (5) to be decreasing is that both $\frac{\mu(\theta)}{\mu(\theta)}$ and $w(\theta) - \theta - \frac{\mu(\theta)}{\mu(\theta)}$ are decreasing in $\theta$. This means that the monotonicity constraint in the definition of $C$ is not binding and thus optimazing in $C$ or in $L([\theta, \bar{\theta}], [0, 1])$ gives the same results. As a consequence, we can apply Lemma 3 to conclude (and no further proof is necessary).

Proposition 3. If both $\frac{\mu(\theta)}{\mu(\theta)}$ and $w(\theta) - \theta - \frac{\mu(\theta)}{\mu(\theta)}$ are decreasing in $\theta$, the solution to the problem in Proposition 2 is a deterministic mechanism.
3.1.2 A Discrete Distribution

We now look at the discrete case. We assume the same model as in Section 3.1.1 with one difference, we assume that the probability distribution \( \mu \) has a finite support \( \Theta = \{ \theta^1, \ldots, \theta^K \} \) with mass \( \mu'(\theta^i) \) at each point \( \theta^i, i = 1, \ldots, K \).

A mechanism is a map from \( \Theta = \{ \theta^1, \ldots, \theta^K \} \) to a probability of trade \( q \in [0, 1] \) and a transfer \( t \in \mathbb{R} \) from the buyer to the seller. The application of standard mechanism design tools means here that a direct mechanism is incentive compatible if and only if the probability of trade function \( q \) belongs to the set

\[
C \equiv \{ q \in L(\{\theta^1, \ldots, \theta^K\}, [0, 1]) : q \text{ decreasing} \}
\]

and the transfer satisfies:

\[
t(\theta^i) = q(\theta^i)\theta^i + \sum_{j=i}^{K}(\theta^{j+1} - \theta^j) \left( q(\theta^{j+1}) + \gamma(\theta^{j+1}) \right) + u_S,
\]

for \( \theta^{K+1} = \theta^K \), and for some \( u_S \in \mathbb{R} \) that corresponds to the expected utility of a seller with type \( \theta^K \) and some \( \gamma(\theta^{j+1}) \in [0, q(\theta^j) - q(\theta^{j+1})] \). Thus, one can deduce that the seller’s expected utility conditional on her type \( \theta^i \) is equal to:

\[
\sum_{j=i}^{K}(\theta^{j+1} - \theta^j) \left( q(\theta^{j+1}) + \gamma(\theta^{j+1}) \right) + u_S,
\]

the expected social surplus is equal to:

\[
\sum_{i=1}^{K} \left( w(\theta^i) - \theta^i \right) q(\theta^i)\mu'(\theta^i),
\]

and the buyer’s expected utility is equal to:

\[
\sum_{j=1}^{K} \left( w(\theta^j) - \theta^j - \frac{\mu(\theta^{j-1})}{\mu'(\theta^j)}(\theta^j - \theta^{j-1}) \right) q(\theta^j)\mu'(\theta^j) - \sum_{j=1}^{K}(\theta^j - \theta^{j-1})\gamma(\theta^j)\mu(\theta^{j-1}) - u_B + u_S,
\]

for \( \theta^0 \equiv \theta^1 \).

We are interested in the generic properties of the mechanisms that maximize the social expected surplus subject to the buyer and seller’s participation constrains. For simplicity, we model this question considering a family of problems indexed by the value of the buyer’s and seller’s outside option, \( u_B \) and \( u_S \) respectively. The former simply requires that (8) greater than \( u_B \), whereas the latter requires that \( u_S \geq u_S \) since the first term of (6) is equal to zero at \( \theta^i = \theta^K \) and positive, otherwise.
Obviously, for values of \( u_B + u_S \) very high no solution exists, while for values of \( u_B + u_S \) very low the participation constraints are not binding and thus can be ignored. Thus, we focus in an open set of values of \( (u_B, u_S) \) for which there exists a solution and the participation constraints are binding. Such solution corresponds to a boundary problem of the linear map:

\[
A(q) = \left( \frac{\sum_{i=1}^{K} (w(\theta^i) - \theta^i) q(\theta)\mu'(\theta)}{\sum_{j=1}^{K} (w(\theta^j) - \theta^j - \frac{\mu(\theta^{j-1})}{\mu'(\theta^j)}(\theta^j - \theta^{j-1})) q(\theta^j)\mu'(\theta^j)} \right), \quad q \in C.
\]  

(9)

in which the second component is equal to \( u_B + u_S \). Note that we have dropped the term \( \sum_{j=1}^{K} (\theta^j - \theta^{j-1})\gamma(\theta^j)\mu(\theta^{j-1}) \) as binding participation constraints make it optimal to \( \gamma(\theta^j) = 0 \) to alleviate them.

We can now apply Lemma 2 and use that in this case the set of extreme points of \( C \) is finite to prove the next proposition (and no additional proof is required).

**Proposition 4.** Suppose an open set of values of \( (u_B, u_S) \) for which there exists a solution to the maximum social’s expected surplus subject to the buyer’s and seller’s participation constraints and in which the participation constraints are binding. This solution is generically a mechanism with one-random step, i.e. a mechanism characterized by a decreasing step function \( q \) with range in \( \{0, \lambda, 1\} \) for some \( \lambda \in (0, 1) \).

### 3.2 A Two-Sided Asymmetric Information Version of Samuelson’s (1984)

In our second application we extend the model of Samuelson (1984) and study a version in which both the buyer and the seller have private information. In particular, we assume that the seller’s monetary (opportunity) cost is equal to \( \theta_1 + \omega_1(\theta_2) \) and the buyer’s value to \( \theta_2 + \omega_2(\theta_1) \) where \( \theta_1 \) and \( \theta_2 \) are, respectively, the seller’s and the buyer’s private informations. We assume independency between the distribution of \( \theta_1 \) and \( \theta_2 \) and denote them by \( \mu_i \) with support \([\underline{\theta}_i, \overline{\theta}_i]\) and density \( \mu'_i \in \left( \eta, \frac{1}{\eta} \right) \) for some \( \eta > 0 \) and \( i = 1, 2 \). We denote the joint distribution by \( \mu \) and the joint support by \( \Theta \).

A direct mechanism maps \( \theta \in \Theta \) into a probability of trade \( q(\theta) \in [0, 1] \) and a payment from the buyer to the seller \( t(\theta) \in \mathbb{R} \). In this case, we restrict attention to direct mechanism that are Bayesian incentive compatible and satisfy interim participation constraints. We refer to them simply as mechanisms. The usual mechanism design tools applied here imply that the set of mechanisms is characterized by the functions \((q, t)\) such that \( q \) belongs to:

\[
\bar{C} \equiv \{ q \in L(\Theta, [0, 1]) : \int q(\theta)\mu_2(d\theta_2) \text{ dec. in } \theta_1 \text{ and } \int q(\theta)\mu_1(d\theta_1) \text{ inc. in } \theta_2 \},
\]  

(10)
and the interim transfers satisfy for some $u_1, u_2 \geq 0$, 
\[ \int t(\theta) \mu_2(d\theta_2) = \int \left( (\theta_1 + \omega_1(\theta_2)) q(\theta) + \int_{\theta_1}^{\theta_2} q(x, \theta_2) \, dx \right) \mu_2(d\theta_2) + u_1, \]  
(11)
for any $\theta_1 \in [\theta_1, \overline{\theta}_1]$, and:
\[ \int t(\theta) \mu_1(d\theta_1) = \int \left( (\theta_2 + \omega_2(\theta_1)) q(\theta) - \int_{\theta_2}^{\theta_1} q(\theta_1, x) \, dx \right) \mu_1(d\theta_1) - u_2, \]  
(12)
for any $\theta_2 \in [\theta_2, \overline{\theta}_2]$.

In this case, the seller’s expected utility is equal to:
\[ \int \left( \frac{\mu_1(\theta_1)}{\mu'_1(\theta_1)} \right) q(\theta) \mu(d\theta) + u_1, \]  
(13)
and the buyer’s expected utility is equal to:
\[ \int \left( \frac{1 - \mu_2(\theta_2)}{\mu'_2(\theta_2)} \right) q(\theta) \mu(d\theta) + u_2, \]  
(14)

The next lemma makes use of the argument that one can transform an ex post budget balance constraint into an ex ante budget balance constraint in the setting of this section.

**Lemma 4.** There exists a function $t : \Theta \to \mathbb{R}$ that satisfies (11) and (12) if and only if:
\[ \int \left( \theta_2 + \omega_2(\theta_1) - \theta_1 - \omega_1(\theta_2) - \frac{\mu_1(\theta_1)}{\mu'_1(\theta_1)} - \frac{1 - \mu_2(\theta_2)}{\mu'_2(\theta_2)} \right) q(\theta) \mu(d\theta) = u_1 + u_2. \]  
(15)

**Proof.** First, note that (15) can be transformed into:
\[ \int \left( (\theta_2 + \omega_2(\theta_1) - \theta_1 - \omega_1(\theta_2)) q(\theta) - \int_{\theta_2}^{\theta_1} q(\theta_1, x) \, dx - \int_{\theta_1}^{\theta_2} q(x, \theta_2) \, dx \right) \mu(d\theta) = u_1 + u_2, \]  
(16)
using that,
\[ \int \int_{\theta_1}^{\theta_2} q(x, \theta_2) \, dx \mu_1(d\theta_1) = \int \frac{\mu_1(\theta_1)}{\mu'_1(\theta_1)} q(\theta) \mu_1(d\theta_1), \]  
(17)
and that,
\[ \int \int_{\theta_2}^{\theta_1} q(\theta_1, x) \, dx \mu_2(d\theta_2) = \int \frac{1 - \mu_2(\theta_2)}{\mu'_2(\theta_2)} q(\theta) \mu_2(d\theta_2). \]  
(18)

Now, that (11) and (12) imply (16) can be easily shown taken expectations in (11) and (12) with respect to $\mu_1(.)$ and $\mu_2(.)$, respectively, and equalising the resulting right hand sides.

To prove the “if”-part, we assume (16) and show that,
\[ t(\theta) = (\theta_1 + \omega_1(\theta_2)) q(\theta) + \int_{\theta_1}^{\theta_2} q(x, \theta_2) \, dx + u_1 + \int D(\theta) \mu_1(d\theta_1), \]  
(19)
where,

\[ D(\theta) \equiv (\theta_2 + \omega_2(\theta_1) - \theta_1 - \omega_1(\theta_2)) q(\theta) - \int_{\theta_1}^{\theta_1} q(x, \theta_2) \, dx - \int_{\theta_2}^{\theta_2} q(\theta_1, x) \, dx - (u_1 + u_2), \]

satisfies (11) and (12). The former is a straightforward implication of the fact that (16) implies that \( \int D(\theta) \mu(d\theta) = 0 \). To see that \( t \) in (19) also satisfies (12), note that using the definition of \( D(\theta) \), one can show that:

\[ t(\theta) = \left( \theta_2 + \omega_2(\theta_1) \right) q(\theta) - u_2 - \int_{\theta_2}^{\theta_2} q(\theta_1, x) \, dx - D(\theta) + \int D(\theta) \mu_1(d\theta_1). \]

Note that (13), (14) and (15) only depend on \( q \) up to the interim probabilities \( \int q(\theta) \mu_i(d\theta_i), \) \( i = 1, 2 \). Next lemma shows that any interim probabilities associated to a \( q \in \bar{C} \) can be implemented with a function \( q \) decreasing in \( \theta_1 \) and increasing in \( \theta_2 \).

**Lemma 5.** For any \( \bar{q} \in \bar{C} \) there exists an element of:

\[ C \equiv \{ q \in L(\Theta, [0, 1]) : \text{dec. in } \theta_1 \text{ and inc. in } \theta_2 \}, \]

that induces the same interim probabilities \( \int q(\theta) \mu_i(d\theta_i), \) \( i = 1, 2 \).

**Proof.** The proof follows from the application of Lemmas 1-3 by Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013). To see why their results apply here, note that one can apply the optimization problem in their display (1) for \( K = \{1, 2\}, q = q^1, v_i(x) = q^1(x), V_i(x)_i = \int q(x, \theta_j) \mu_j(d\theta_j) \) and \( E_x(f) = \int f(\theta) \mu(d\theta). \)

Consequently, to compute a mechanism that maximizes the seller’s expected utility is the same as to choose a point in the boundary of the range of the continuous linear map:

\[ A(q) = \begin{pmatrix} \int \left( \frac{\mu_1(\theta_1)}{\mu_1(\theta_1)} \right) q(\theta) \mu(d\theta) \\ \int \left( \theta_2 + \omega_2(\theta_1) - \theta_1 - \omega_1(\theta_2) - \frac{\mu_1(\theta_1)}{\mu_1(\theta_1)} - \frac{1-\mu_1(\theta_1)}{\mu_1(\theta_1)} \right) q(\theta) \mu(d\theta) \end{pmatrix}, \quad q \in C. \]  

A similar expression characterizes the buyer’s expected utility maximization. Besides, it is also easy to show with a similar argument as in the previous application that the set of extreme points of \( C \) is composed by deterministic mechanisms. In this case, however, note that the convex combination of two deterministic mechanisms can give rise to up to random steps in the allocation function \( q \), see Figure 3.2.

Consequently, we can apply Lemma 1 to conclude (and no additional proof is necessary):
Figure 1: A mechanism in which $q = \lambda q_1 + (1 - \lambda)q_2$, where $\lambda \in (0,1)$ and $q_i$ is equal to one to the left of the corresponding plotted line and equal to zero to the right.

**Proposition 5.** The maximum expected utility of the seller (buyer) subject to non-negative expected utility of the buyer (seller) can be achieved with a mechanism characterized by a step function $q$ whose range lies in $\{0, \lambda, 1 - \lambda, 1\}$ for some $\lambda \in (0,1)$.

Another interesting problem is the characterization of the ex ante Pareto frontier. This can be described with the boundary of the map:

$$A(q) = \begin{pmatrix} \int \left( \frac{\mu_1(\theta_1)}{\mu_1(\theta_1)} \right) q(\theta) \mu(d\theta) \\ \int \left( \frac{1 - \mu_2(\theta_2)}{\mu_2(\theta_2)} \right) q(\theta) \mu(d\theta) \\ \int \left( \theta_2 + \omega_2(\theta_1) - \theta_1 - \omega_1(\theta_2) - \frac{\mu_1(\theta_1)}{\mu_1(\theta_1)} - \frac{1 - \mu_2(\theta_2)}{\mu_2(\theta_2)} \right) q(\theta) \mu(d\theta) \end{pmatrix}, \quad q \in C. \quad (21)$$

In this case, Lemma 1 implies that one can restrict attention without loss of generality to convex combinations of three extreme point mechanisms. As Figure 3.2 shows this implies that one can restrict to step functions whose range has no more than 6 points in $(0,1)$. 

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Figure 2: A mechanism in which \( q = \lambda_1 q_1 + \lambda_2 q_2 + \lambda_3 q_3 \), where \( \lambda_1 + \lambda_2 + \lambda_3 = 1 \) and \( q_i \) is equal to one to the left of the corresponding plotted line and equal to zero to the right.

Consequently, one can derive the following proposition (no additional proof is required):

**Proposition 6.** The ex ante Pareto frontier can be computed restricting attention to mechanisms characterized by a step function \( q \) whose range lies in \( \{0, \lambda_1, \lambda_2, \lambda_3, \lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3, 1\} \) for some \((\lambda_1, \lambda_2, \lambda_3) \in \Delta(3)\).

The same analysis and results can be extended to other formally similar settings. For instance, to the case of the socially optimal provision of an indivisible public good in a setting with \( n \) agents with interdependent valuations additively separable in each of the agents’ types.

### 3.3 Constraint Efficiency of the Competitive Equilibrium in Akerlof’s Model

In this section, we provide a version of the model of previous section in which instead of two agents we assume a continuum of buyers with measure normalized to 1 and a continuum of sellers with measure also 1.\(^\text{10}\) We shall use our results of Section 2 to provide necessary and sufficient conditions for the constraint efficient outcome being implementable in a competitive market.

\(^{10}\)The assumption that the measure of buyers and bidders is the same simplifies a little bit the notation but plays no role otherwise. It can be relaxed without upsetting the results.
There is a continuum of sellers and buyers with quasilinear preferences in money and an indivisible good. Each seller has one unit of the good and each buyer puts positive value in at most one unit of the good. If a seller with type $\theta_1$ gives its unit to a buyer with type $\theta_2$, the seller incurs a monetary (opportunity) cost $\theta_1 + w_1(\theta_2)$ and the seller gets a monetary value $\theta_2 + w_2(\theta_1)$, where we assume that $w_1$ and $w_2$ are continuous increasing functions. Each seller (buyer, respectively) knows privately its type $\theta_1$ ($\theta_2$). The measure of $\theta_i$’s agents is described by a finite measure space $([\theta_i, \bar{\theta}_i], \mathcal{B}_i, \mu_i)$, $i = 1, 2$, with density $\mu'_i \in (\eta, 1/\eta)$ for some $\eta > 0$, and whose product measure space is denoted by $(\Theta, \mathcal{B}, \mu)$, where $\Theta \equiv \Theta_1 \times \Theta_2$ and $\Theta_i = [\theta_i, \bar{\theta}_i]$.

The role of our two last assumptions is to avoid some tedious case differentiation:

$$\bar{\theta}_2 + w_2(\theta_1) > \theta_1 + w_1(\bar{\theta}_2).$$

$$\theta_2 + \int_{\theta_1}^{\bar{\theta}_1} w_2(\theta_1) \mu_1(d\theta_1) < \bar{\theta}_1 + \int_{\theta_2}^{\bar{\theta}_2} w_1(\theta_2) \mu_2(d\theta_2).$$

These assumptions guarantee that the competitive equilibria, defined later, are all interior, i.e. the quantity traded lies in $(0, 1)$.

An allocation is described by a measure $\nu$ on the measure space $(\Theta, \mathcal{B})$ where $\nu(E), E \in \mathcal{B}$, denotes the measure of trades between agents with types in $E$ and thus, the marginals $\nu_i(E_i) \equiv \nu(E_i \times \Theta_j)$ defined on $(\Theta_i, \mathcal{B}_i)$ must satisfy $\nu_i(E_i) \leq \mu_i(E_i)$ for any $E_i \in \mathcal{B}_i$ and $\nu_1(\Theta_1) = \nu(\Theta) = \nu_2(\Theta_2)$. Thus, $\nu$ is absolute continuous with respect to $\nu$ and has a Radon-Nikodym derivative with respect to $\mu$ that we denote by $q$.

A direct mechanism is characterized by an allocation $\nu$ and two measurable transfer functions $t_i : \Theta_i \rightarrow \mathbb{R}, i = 1, 2$, where $t_1(\theta_1)$ is the transfer that a seller $\theta_1$ receives and $t_2(\theta_2)$ is the transfer that a buyer $\theta_2$ pays.

To write the expected payoffs we find convenient to define conditional probabilities of a matching. To do so, we define for any $i \in \{1, 2\}$ and $\theta_j \in \Theta_j, j \neq i$, the measure $q_i(A_i|\theta_j) \equiv \int_{A_i} q(\theta) \mu_i(d\theta_i)$ for any $A_i \in \mathcal{B}_i$. The measure $q_i$ gives us the conditional probability that type $\theta_j$ is matched with a type $\theta_i$. The role of the assumption that $w_i$ are increasing is to guarantee that the constraint efficient mechanism does not run a surplus.
in $A_i$.\(^{12}\) Thus, the seller’s incentive compatible constraint requires that:

$$t_1(\theta_1) - \int_{\Theta_2} (\theta_1 + w_1(\theta_2)) q_2(d\theta_2|\theta_1) \geq t_1(\tilde{\theta}_1) - \int_{\Theta_2} (\theta_1 + w_1(\theta_2)) q_2(d\theta_2|\tilde{\theta}_1),$$

for any $\theta_1, \tilde{\theta}_1 \in \Theta_1$ and the buyer’s:

$$\int_{\Theta_2} (\theta_2 + w_2(\theta_1)) q_1(d\theta_1|\theta_2) - t_2(\theta_2) \geq \int_{\Theta_1} (\theta_2 + w_2(\tilde{\theta}_1)) q(d\theta_1|\tilde{\theta}_2) - t_2(\tilde{\theta}_2),$$

for any $\theta_2, \tilde{\theta}_2 \in \Theta_2$, while the sellers' and buyer's individual rationality constraints require that the left hand side of the two last expressions are non negative, respectively.

Let,

$$C_1 \equiv \{ \tilde{Q}_1 \in L(\Theta_1, [0, 1]) : \tilde{Q}_1 \text{ dec.} \},$$

$$C_2 \equiv \{ \tilde{Q}_2 \in L(\Theta_2, [0, 1]) : \tilde{Q}_2 \text{ inc.} \},$$

and $C \equiv C_1 \times C_2$.

The application of the usual tools of mechanism design imply that a direct mechanism $(\nu, t_1, t_2)$ is incentive compatible and individually rational if and only if each of the interim probability functions $Q_i(\theta_i) \equiv q_j(\Theta_j|\theta_i), i = 1, 2$, belongs to the respective set $C_i$ and satisfy the feasibility constraint,

$$\int_{\Theta_1} Q_1(\theta_1) d\mu_1(\theta_1) = \int_{\Theta_2} Q_2(\theta_2) d\mu_2(\theta_2)$$

and the transfers satisfy:

$$t_1(\theta_1) = \int_{\Theta_2} (\theta_1 + w_1(\theta_2)) q_2(d\theta_2|\theta_1) + \int_{\theta_1} \tilde{\theta}_1 Q_1(x) dx + u_1,$$

and,

$$t_2(\theta_2) = \int_{\Theta_1} (\theta_2 + w_2(\theta_1)) q_1(d\theta_1|\theta_2) - \int_{\theta_2} \tilde{\theta}_2 Q_2(x) dx - u_2,$$

for some $u_1, u_2 \geq 0$.

Finally, we are interested in mechanisms that do not require of external funds, thus we impose the constraint:

$$\int_{\Theta_2} t_2(\theta_2) \mu_2(d\theta_2) - \int_{\Theta_1} t_1(\theta_1) \mu_1(d\theta_1) \geq 0.$$

\(^{12}\)To see why it is indeed a conditional probability function note that for any $A_j \in \mathcal{B}_j$:

$$\int_{A_j} q_i(A_i|\theta_j) \mu_j(d\theta_j) = \int_{A_i} \int_{A_j} q(\theta) \mu_i(d\theta_i) \mu_j(d\theta_j) = \nu(\theta).$$
The following lemma characterizes the solutions to (27), (28) and (29) in terms of what has been called the budget incentive constraint.

**Lemma 6.** There exists measurable \( t_i : \Theta_i \to \mathbb{R}, i = 1, 2, \) that satisfy (27), (28) and (29) for some \( u_1, u_2 \in \mathbb{R}_+ \), if and only if:

\[
\int_{\Theta_2} \left( \theta_2 - w_1(\theta_2) - \frac{1 - \mu_2(\theta_2)}{\mu'_2(\theta_2)} \right) Q_2(\theta_2) \mu_2(d\theta_2) - \int_{\Theta_1} \left( \theta_1 - w_2(\theta_1) + \frac{\mu_1(\theta_1)}{\mu'_1(\theta_1)} \right) Q_1(\theta_1) \mu_1(d\theta_1) \geq 0.
\]

(30)

**Proof.** As a preliminary step, we show that (30) is equivalent to:

\[
\int_{\Theta_2} \left( \int_{\Theta_1} (\theta_2 + w_2(\theta_1)) q_1(d\theta_1|\theta_2) - \int_{\Theta_2} Q_2(x) dx \right) \mu_2(d\theta_2) - \int_{\Theta_1} \left( \int_{\Theta_2} (\theta_1 + w_1(\theta_2)) q_2(d\theta_2|\theta_1) + \Theta_i \right) Q_1(x) dx \mu_1(d\theta_1) \geq 0.
\]

(31)

This can be shown using that by definition \( Q_i(\theta_i) = \int_{\Theta_j} q_j(d\theta_j|\theta_i) \), and the following equalities:

\[
\int_{\Theta_i} \int_{\Theta_j} Q_1(x) dx \mu_1(d\theta_1) = \int_{\Theta_i} \frac{\mu_1(\theta_1)}{\mu'_1(\theta_1)} Q_1(\theta_1) \mu_1(d\theta_1),
\]

(32)

and,

\[
\int_{\Theta_2} \int_{\Theta_2} Q_2(x) dx \mu_2(d\theta_2) = \int_{\Theta_2} \frac{1 - \mu_2(\theta_2)}{\mu'_2(\theta_2)} Q_2(\theta_2) \mu_2(d\theta_2),
\]

(33)

that follow from standard arguments in mechanism design, and,

\[
\int_{\Theta_i} w_j(\theta_i) Q_i(\theta_i) \mu_i(d\theta_i) = \int_{\Theta_i} w_j(\theta_i) \left( \int_{\Theta_j} q_j(d\theta_j|\theta_i) \right) \mu_i(d\theta_i)
\]

\[
= \int_{\Theta_i} w_j(\theta_i) q_j(\Theta_j|\theta_i) \mu_i(d\theta_i)
\]

\[
= \int_{\Theta_i} w_j(\theta_i) \left( \int_{\Theta_j} q(\theta) \mu_j(d\theta_j) \right) \mu_i(d\theta_i)
\]

\[
= \int_{\Theta} w_j(\theta_i) q(\theta) \mu(d\theta)
\]

\[
= \int_{\Theta} \left( \int_{\Theta_i} w_j(\theta_i) q(\theta) \right) \mu_j(d\theta_j).
\]

To prove the “only if” part note that (27) for \( t_1 = t \) and (28) for \( t_2 = t \) imply (31): take expectations in (27) and (28) with respect to \( \mu_1(.) \) and \( \mu_2(.) \), respectively, equalise the resulting right hand sides and use that \( u_1, u_2 \in \mathbb{R}_+ \).
To prove the “if”-part, we assume (31) and show that:

\[
\begin{align*}
t_1(\theta_1) & \equiv \int_{\Theta_2} t(\theta) \mu_2(\theta_2) \\
t_2(\theta_2) & \equiv \int_{\Theta_1} t(\theta) \mu_1(\theta_1)
\end{align*}
\]

where:

\[
\begin{align*}
t(\theta) & \equiv \int_{\Theta_2} (\theta_1 + w_1(\theta_2)) q_2(d\theta_2|\theta_1) + \int_{\Theta_1} \tilde{D}_1 \cdot Q_1(x) dx + \int_{\Theta_1} D(\theta) \mu_1(d\theta_1) \\
D(\theta) & \equiv \int_{\Theta_1} (\theta_2 + w_2(\theta_1)) q_1(d\theta_1|\theta_2) - \int_{\Theta_2} Q_2(x) dx - \int_{\Theta_2} (\theta_1 + w_1(\tilde{\theta}_2)) q_2(d\tilde{\theta}_2|\theta_1) - \int_{\Theta_1} \tilde{D}_1 \cdot Q_1(x) dx,
\end{align*}
\]

satisfies (27) for \(u_1 \equiv \int_{\Theta} D(\theta) \mu(d\theta)\) and \(u_2 \geq 0, (28)\) for \(u_2 = 0,\) and (29). That (27) is satisfied is direct while \(u_1 \geq 0\) follows from (31). That (28) is satisfied can be shown noting that:

\[
t(\theta) = \int_{\Theta_1} (\theta_2 + w_2(\theta_1)) q_1(d\theta_1|\theta_2) - \int_{\Theta_2} Q_2(x) dx - D(\theta) + \int_{\Theta_1} D(\theta) \mu_1(d\theta_1).
\]

Thus, taking expectations with respect to \(\mu_1\) gives the desired result. Finally, (29) follows directly from the definition of \(t_1\) and \(t_2\) and the fact that \(u_1, u_2 \geq 0. \) ■

Finally, a simple computation shows that the expected social surplus can also be written in terms of functions \((Q_1, Q_2)\):

\[
\int_{\Theta} (\theta_2 + w_2(\theta_1) - \theta_1 - w_1(\theta_2)) \nu(d\theta) = \\
\int_{\Theta_2} (\theta_2 - w_1(\theta_2)) Q_2(\theta_2) d\mu_2(\theta_2) - \int_{\Theta_1} (\theta_1 - w_2(\theta_1)) Q_1(\theta_1) d\mu_1(\theta_1).
\]

Consequently, we can define the incentive constraint efficient mechanism as follows:

**Definition:** We say that a mechanism is incentive constraint efficient if it is a maximizer of

\[
\max_{Q \in C} \quad A_1(Q)
\]

subject to

\[
\begin{align*}
A_2(Q) & = 0 \\
A_3(Q) & \geq 0,
\end{align*}
\]

where \(A : C \to \mathbb{R}^2\), for \(C \equiv C_1 \times C_2\), is defined as follows:

\[
A(Q) \equiv \left( \begin{array}{c}
\int_{\Theta_2} (\theta_2 - w_1(\theta_2)) Q_2(\theta_2) d\mu_2(\theta_2) - \int_{\Theta_1} (\theta_1 - w_2(\theta_1)) Q_1(\theta_1) d\mu_1(\theta_1) \\
\int_{\Theta_2} Q_2(\theta_2) d\mu_2(\theta_2) - \int_{\Theta_1} Q_1(\theta_1) d\mu_1(\theta_1) \\
\int_{\Theta_2} (\theta_2 - w_1(\theta_2) - \frac{\mu_2(\theta_2) - \mu_2(\tilde{\theta}_2)}{\mu_2(\tilde{\theta}_2)}) Q_2(\theta_2) d\mu_2(\theta_2) - \int_{\Theta_1} (\theta_1 - w_2(\theta_1) + \frac{\mu_1(\theta_1)}{\mu_1(\tilde{\theta}_1)}) Q_1(\theta_1) d\mu_1(\theta_1)
\end{array} \right).
\]
Besides, the set of extreme points of $C$ has a trivial characterization (and no proof is provided).

**Lemma 7.** The set of extreme points of $C$ is equal to the subset of deterministic functions, i.e. those $Q \in C$ with range in $\{0, 1\}^2$.

Thus, we can apply Lemma 1 and conclude the following (no proof is provided):

**Proposition 7.** The constraint efficient mechanism can always be implemented with a $Q$ with range $\{0, \lambda_1, \lambda_2, 1\}$ for some $\lambda_1, \lambda_2 \in [0, 1]$.

Next, we provide an economic interpretation of extreme point mechanisms. Note, first, that any extreme point mechanism $Q$ is determined by the cut-off points at which $Q_1$ and $Q_2$ jump between zero and one, say $\theta^*_1$ and $\theta^*_2$ respectively, and $A_2(Q)$ and $A_3(Q)$ can be written, respectively, as functions of these cutoffs:

$$A_2(Q) = \mu_2(\theta_2) - \mu_2(\theta_2^*), \quad \text{Equation (36)}$$

$$A_3(Q) = \theta_2^*(\mu(\theta_2) - \mu_2(\theta_2^*)) + \int_{\theta_1}^{\theta_2^*} w_2(\theta_1)\mu_1(d\theta_1) - \theta_1^*\mu_1(\theta_1^*) - \int_{\theta_2}^{\theta_2^*} w_1(\theta_2)\mu_2(d\theta_2). \quad \text{Equation (37)}$$

We use these expressions to deduce that any extreme point mechanism that satisfies $A_2(Q) = A_3(Q) = 0$ is also implementable as a competitive equilibrium.

**Definition:** We call a competitive equilibrium a vector $(q, p) \in \mathbb{R}_+ \times \mathbb{R}$ that satisfies:

- **Buyer’s optimality:** $\theta_2 + \int_{\theta_1}^{\theta_2} w_2(\theta_1)\frac{\mu_1(d\theta_1)}{\mu(\theta_1)} - p \geq 0$ if and only if $\theta_2 \geq \theta_2^*$.

- **Seller’s optimality:** $p - \theta_1 - \int_{\theta_2}^{\theta_2^*} w_1(\theta_2)\frac{\mu_2(d\theta_2)}{\mu(\theta_2) - \mu_2(\theta_2^*)} \geq 0$ if and only if $\theta_1 \leq \theta_1^*$.

- **Market clearing:** $q = \mu_1(\theta_1^*) = \mu(\theta_2) - \mu_2(\theta_2^*)$.

A competitive equilibrium defines implicitly an extreme point mechanism with cut-offs $(\theta_1^*, \theta_2^*)$, and any extreme point mechanism with cut-offs $(\theta_1^*, \theta_2^*)$ that satisfies $A_2(Q) = A_3(Q) = 0$ defines the

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The case of $A_2(Q)$ is obvious. For the case of $A_3(Q)$, we use that (32) and (33) imply:

$$\int_{\theta_1}^{\theta_2} \left( \theta_1 + \frac{\mu_1(\theta_1)}{\mu(\theta_1)} \right) d\mu_1(\theta_1) = \theta_1^*\mu_1(\theta_1^*),$$

and:

$$\int_{\theta_2}^{\theta_2^*} \left( \theta_2 - \frac{\mu_2(\theta_2) - \mu_2(\theta_2^*)}{\mu_2(\theta_2)} \right) d\mu_2(\theta_2) = \theta_2^*(\mu(\theta_2) - \mu_2(\theta_2^*)).$$

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competitive equilibrium \( q = \mu_1(\theta_1^*) - \mu_2(\theta_2^*) \), and,

\[
p = \theta_1^* + \int_{\theta_2^*}^{\theta_1^*} w_1(\theta_2) \frac{\mu_2(d\theta_2)}{\mu(\theta_2) - \mu_2(\theta_2^*)} = \theta_2^* + \int_{\theta_1^*}^{\theta_2^*} w_2(\theta_1) \frac{\mu_1(d\theta_1)}{\mu(\theta_1) - \mu_1(\theta_1^*)}
\]

Thus, the following proposition is straightforward and no proof is provided.

**Proposition 8.** Any extreme point mechanism \( Q \in \text{ext}(C) \) that satisfies \( A_2(Q) = 0 \) and \( A_3(Q) = 0 \) can be implemented with a competitive equilibrium.

Our next result is to show that in the regular case, the constraint \( A_3(Q) \geq 0 \) is binding and thus, we can derive the following result from Lemma 3.

**Proposition 9.** If \( A \) is regular at the constraint efficient mechanism, then the constraint efficient mechanism can be implemented with a competitive equilibrium.

**Proof.** By application of Lemma 3, the constraint efficient mechanism can be implemented with an extreme point mechanism. Obviously, this extreme point mechanism is also constraint efficient within the set of extreme point mechanisms that satisfy \( A_2(Q) = 0 \). In our prove, we shall use that each of these mechanisms is characterized by the quantity traded \( q \). This quantity determines the thresholds \( \theta_1^* \) and \( \theta_2^* \) at which the functions \( Q_1 \) and \( Q_2 \) jump between zero and one: \( \mu_1(\theta_1^*) = q \) and \( \mu_2(\theta_2^*) = 1 - q \). Thus, we can study the constraint efficient within the set of extreme point mechanisms that satisfy \( A_2(Q) = 0 \) with the problem:

\[
\max_{q \in [0,1]} \quad \hat{A_1}(q) \\
\text{s.t. } \hat{A_3}(q) \geq 0.
\]

where:

\[
\hat{A}_1(q) \equiv \int_{\mu_2^{-1}(1-q)}^{\theta_2} (\theta_2 - w_1(\theta_2)) \frac{d\mu_2(\theta_2)}{\mu(\theta_2) - \mu_2(\theta_2)} - \int_{\theta_1}^{\mu_1^{-1}(q)} (\theta_1 - w_2(\theta_1)) \frac{d\mu_1(\theta_1)}{\mu(\theta_1) - \mu_1(\theta_1)}
\]

\[
\hat{A}_3(q) \equiv q \left( \mu_2^{-1}(1-q) - \int_{\mu_2^{-1}(1-q)}^{\theta_2} \frac{w_1(\theta_2)d\mu_2(\theta_2)}{\mu(\theta_2) - \mu_2(\theta_2)} \right) - \left( \mu_1^{-1}(q) - \int_{\theta_1}^{\mu_1^{-1}(q)} \frac{w_2(\theta_1)d\mu_1(\theta_1)}{\mu(\theta_1) - \mu_1(\theta_1)} \right).
\]

The assumptions (22) and (23) imply that \( \hat{A}_1'(0) > 0 \) and \( \hat{A}_1'(1) < 0 \), whereas one can deduce that \( \hat{A}_1'(q) \geq \frac{\hat{A}_3(q)}{q} \) from,

\[
\int_{\mu_2^{-1}(1-q)}^{\theta_2} \frac{w_1(\theta_2)d\mu_2(\theta_2)}{\mu(\theta_2) - \mu_2(\theta_2)} \geq w_1(\mu_2^{-1}(1-q)) \tag{39}
\]

\(^{14}\)It is straightforward that \( \hat{A}_1(0) = \hat{A}_2(0) = 0.\)
\[ \int_{\tilde{g}_i}^{\mu_1^{-1}(q)} w_2(\theta_1) d\mu_1(\theta_1) \leq w_2(\mu_1^{-1}(q)), \]  

which are a consequence of \( w_1 \) and \( w_2 \) increasing. Finally, \( A_3(0) = 0 \). Thus, the optimum \( q \in [0,1) \) and satisfies \( \hat{A}_3(q) = 0 \) as desired. \( \blacksquare \)

Outside the regular case, Lemma 1 implies that constraint efficient mechanism is implemented with convex combinations of two extreme point mechanisms and thus there is no guarantee that can be implemented with a competitive equilibrium. The next definitions and proposition provide conditions under which this is the case.

### 3.4 Optimal Taxation

Taxes: linear utilities with objectives that are composition of linear functions give tax schedules that have a finite number of consumption levels which seems unrealistic.

In our last application, we study optimal redistributive-taxes in the sense of maximizing a function of the average income and its Gini coefficient, in a model in which agents have linear preferences and technology, see for instance Sheshinski et al. (1972) for a motivation. The purpose of this section is to show how our analysis applies to situations in which the original map of interest is non-linear but can be written as the finite composition of linear maps.

Suppose a economy with a measure one of agents each with linear preferences consumption-leisure \( c - l \cdot \theta \), and where \( c \) is consumption, \( l \in [0,\bar{l}] \) (number of hours of) work, and \( \theta \) the marginal rate of substitution between consumption and leisure. \( \theta \) is assumed to be privately known by the agent and denote the population distribution by \( \mu \), with density \( \mu' \) and support \( \Theta \equiv [\underline{\theta}, \bar{\theta}] \). We assume a linear technology that converts work into the consumption good at a rate of 1. Thus, we call \( l(\theta) \) the agent’s gross income. In this setting, a tax schedule maps the agent’s gross income \( l \) into a tax (possibly negative) to determines the agent’s net income, which we assume to be fully consumed by the agent.

In our study of the optimal tax schedule, one can restrict without loss of generality to direct incentive compatible mechanisms. In this setting, a direct mechanism is defined by two measurable functions, a net income (consumption) function \( c : \Theta \rightarrow \mathbb{R}_+ \) and a gross income (work) function \( l : \Theta \rightarrow [0,\bar{l}] \), that satisfy the budget constraint:

\[ \int c(\theta) \mu(d\theta) = \int l(\theta) \mu(d\theta). \]  

(41)
The usual mechanism design tools imply that a direct mechanism is incentive compatible if $l$ belongs to the set:

$$C \equiv \{ l \in L(\Theta, [0, \bar{l}]), \text{ } l \text{ decreasing} \}$$ (42)

and $c$ satisfies:

$$c(\theta) = l(\theta) \cdot \theta + \int_{\theta}^{\bar{\theta}} l(x) \, dx.$$ (43)

Consequently, the budget constraint becomes after some algebraic transformations:

$$\int \left( 1 - \theta - \frac{\mu(\theta)}{\mu'(\theta)} \right) l(\theta) \mu(d\theta) = u$$ (44)

In this case, the average net income (i.e. consumption) is equal to the average gross income, see (41), and the Gini coefficient of the distribution of net income is equal to:

$$\frac{\int \int |c(\theta) - c(\hat{\theta})| \, d\mu(\hat{\theta}) \, d\mu(\theta)}{2 \int c(\theta) \, d\mu(\theta)} = \frac{\int \int_{\theta}^{\bar{\theta}} \left( c(\theta) - c(\hat{\theta}) \right) \, d\mu(\hat{\theta}) \, d\mu(\theta)}{\int c(\theta) \, d\mu(\theta) + u}$$ (45)

$$= \frac{\int \int_{\theta}^{\bar{\theta}} \left( l(\theta) - l(\hat{\theta}) + \int_{\theta}^{\hat{\theta}} l(x) \, dx \right) \, d\mu(\hat{\theta}) \, d\mu(\theta)}{\int l(\theta) \, d\mu(\theta) + u}$$ (46)

$$= \frac{\int \int_{\theta}^{\bar{\theta}} \left( l(\theta) - \left( 1 - \frac{1 - \mu(\hat{\theta})}{\mu'(\hat{\theta})} \right) \, l(\hat{\theta}) \right) \, d\mu(\hat{\theta}) \, d\mu(\theta)}{\int l(\theta) \, d\mu(\theta) + u}$$ (47)

where the first step follows from the fact that $c$ is decreasing, since $l$ is decreasing and $c$ and $l$ are related by (43), and that

$$\int \int_{\theta}^{\bar{\theta}} \left( c(\theta) - c(\hat{\theta}) \right) \, d\mu(\hat{\theta}) \, d\mu(\theta) + \int \int_{\theta}^{\hat{\theta}} \left( c(\hat{\theta}) - c(\theta) \right) \, d\mu(\hat{\theta}) \, d\mu(\theta) = 2 \int \int_{\theta}^{\bar{\theta}} \left( c(\theta) - c(\hat{\theta}) \right) \, d\mu(\hat{\theta}) \, d\mu(\theta),$$

the second step uses (41) and (43), and the last step uses that

$$\int_{\theta}^{\bar{\theta}} \int_{\theta}^{\hat{\theta}} l(x) \, dx \, d\mu(\hat{\theta}) = \int_{\theta}^{\bar{\theta}} \left( 1 - \mu(x) \right) l(x) \, dx.$$

\[^{15}\]A similar analysis can also be conducted using the distribution of utility rather than the distribution of net income. In this case, we would use that the average utility is equal to:

$$\int \frac{\mu(\theta)}{\mu'(\theta)} l(\theta) \mu(d\theta),$$

and the Gini coefficient of the distribution of utility is:

$$\frac{\int \int_{\theta}^{\bar{\theta}} l(x) \, dx \, \mu(d\hat{\theta}) \, \mu(d\theta)}{\int \frac{\mu(\theta)}{\mu'(\theta)} l(\theta) \mu(d\theta)}. $$
Hence, the study of the tax schedule that minimizes the Gini coefficient of the distribution of net income corresponds to finding a particular boundary point in the range of the continuous linear map:

\[ A(l) \equiv \begin{pmatrix}
\int \int_\theta (l(\theta) - \left(1 - \frac{1 - \mu(\theta)}{\mu'(\theta)}\right) l(\hat{\theta})) d\mu(\theta) d\mu(\hat{\theta}) \\
\int l(\theta) d\mu(\theta) \\
\int \left(1 - \theta - \frac{\mu(\theta)}{\mu'(\theta)}\right) l(\theta) \mu(d\theta)
\end{pmatrix}. \quad (48)

Thus, by application of Lemma 1, one can restrict attention without loss of generality to mechanisms in which \( l \) is the convex combination of no more than three extreme points of \( C \), i.e. \( l \) step functions whose range has no more than two interior points in \((0, l)\) which implies the following result (and no additional proof is required).

**Proposition 10.** The minimization of inequality measured by the Gini coefficient can be achieved with a tax schedule that induces no more than four gross income levels, i.e. functions \( l \) whose range \( \{0, l_1, l_2, l\} \) for some \( l_1, l_2 \in (0, l) \).

This result together with (43) implies that the range of the net income has no more than four points.\(^{16}\)

\(^{16}\)One may find this prediction unreasonable. In this sense, one can interpret the analysis of this section as a method to easily disregard the current model as unrealistic.
References


