Bayesian Comparative Statics*

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Abstract

We study how changes to the informativeness of signals in Bayesian games (and single-agent decision problems) affect the distribution of equilibrium actions. A more precise private signal about an unknown state of the world leads to a mean-preserving spread of an agent's beliefs. Focusing on supermodular environments, we provide conditions under which a more precise private signal for at least one agent also leads to an increasing-mean spread or a decreasing-mean spread of equilibrium actions for all agents. We apply our comparative statics to sender-receiver persuasion games and derive sufficient conditions on the primitive payoffs that lead to extremal disclosure of information. Additionally, we study information acquisition in oligopolies and characterize the differences between covert and overt information acquisition.

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1. Introduction

Economists have long been interested in how equilibrium actions and welfare in Bayesian games vary with the quality of private or public information. For example, Morris and Shin (2002) find that an informative public signal in coordination games can sometimes have a negative welfare effect, while Ganuza and Penalva (2010) show that a more precise private signal yields a more efficient allocation in a second-price auction. More recently, the welfare effects of information in Bayesian games have been studied through the key concept of *informational externalities* (Angeletos and Pavan, 2007; Bergemann and Morris, 2013): first, analyze how changes to the players' information affect the distribution of Bayesian Nash equilibrium actions, and then study how changes to the distribution of equilibrium actions impact welfare.

For the most part, the literature on informational externalities has focused on linear-quadratic games, i.e., games with quadratic utility functions, normally distributed prior beliefs, and normally distributed signals. The tractability of linear-quadratic games makes it possible to show that an increase in the quality of private information (a decrease in the noise of a player's private signal) leads to a mean-preserving spread in the distribution of actions. Hence, whether private information has positive or negative welfare effects depends on whether mean-preserving spreads of actions are socially desirable.

Beyond linear-quadratic settings however, characterizing the effect of information on the distribution of equilibrium actions has proven difficult. For example, the elegant tools from monotone comparative statics (Milgrom and Shannon, 1994) are not applicable because most stochastic orders, such as mean-preserving spread or second-order stochastic dominance, over joint probability distributions of actions and signals (or actions and beliefs) lack a lattice structure (Müller and Scarsini, 2006). Furthermore, Roux and Sobel (2015) show that desirable comparative statics such as "higher quality of information leads to a more dispersed distribution of equilibrium actions" cannot always be established even when players use monotone equilibrium strategies.

In this paper, we identify two classes of Bayesian games/decision problems for which a higher quality of private information unambiguously leads to more dispersed distribution of equilibrium actions along with monotone changes to the mean action. In particular, we consider Bayesian settings in which players have supermodular utility functions with either (i) supermodular and convex marginal utility, or (ii) submodular and concave marginal utility. Supermodular payoffs imply that there are complementarities between a player's action and the state as well as strategic complementarities between two players. Supermodular and convex (resp., submodular and concave) marginal utilities further imply that these complementarities are getting stronger (resp., weaker) as a player's action increases.

Games with quadratic utility functions lie at the intersection of these two classes of payoffs. More generally, these two classes of payoffs contain ultramodular and inframodular functions, which have been applied to study cooperative games with transferable utility (Marinacci and Montrucchio, 2005b,a) and multivariate risk aversion (Müller and Scarsini, 2012).[1]

Our main contribution is to develop a theory of *Bayesian comparative statics* connecting the dispersion of actions to the quality of private information for the two classes of payoffs we consider. To formally describe the comparative statics, we must first discuss an order over distributions of actions that captures changes in the mean and dispersion, and an order over information structures that captures quality.

Each information structure ρ induces a joint distribution $H(\rho)$ over the players' equilibrium actions.[2] Given two information structures ρ and ρ', we say the players are more *responsive with a higher mean* under ρ than ρ' if each player i's marginal distribution of equilibrium actions $H_i(\rho)$ dominates $H_i(\rho')$ in the increasing convex order. Loosely speaking, players become more responsive with a higher mean when each player's equilibrium actions undergo an "increasing-mean spread." Alternatively, we say that players are more *responsive with a lower mean* under ρ than ρ' if, for each player i, $H_i(\rho')$ second-order stochastically dominates $H_i(\rho)$.

To compare the quality of information, we use the *supermodular stochastic order*: We first restrict attention to monotone information structures, i.e., higher signal realizations lead to first-order stochastic shifts in posterior beliefs. Given two information structures ρ and ρ', we say ρ dominates ρ' in the supermodular stochastic order if ρ exhibits more interdependence between signals and the state than ρ'. Within the class of monotone information structures, the supermodular order is more general than the Blackwell order (Blackwell, 1951, 1953) and the Lehmann order (Lehmann, 1988). Athey and Levin (2017) show that an individual decision-maker with a supermodular payoff function values more information if, and only if, information quality is increasing in the supermodular stochastic order.

Our main result shows that for the subclass of supermodular utility functions with supermodular and convex marginal utility, players are more responsive with a higher mean under ρ than under ρ' whenever ρ dominates ρ' in the supermodular stochastic order. Furthermore, if players are more responsive with a higher mean under ρ than ρ' for all supermodular utility functions with supermodular and convex marginal utility, then ρ necessarily

[1] Ultramodular and inframodular functions, also known as directionally convex and directionally concave functions, capture notions of multidimensional convexity and complementarity.

[2] While our discussion here ignores the possibility of multiple equilibrium outcomes for a given information structure, we do so only for ease of exposition. Our formal definition allows for multiplicity.

dominates ρ' in the supermodular stochastic order. We also present symmetric results linking responsiveness with a lower mean to the subclass of supermodular utility functions with submodular and concave marginal utility.

As an application of our main result in the single-agent setting, we consider a sender-receiver Bayesian persuasion framework (Kamenica and Gentzkow, 2011). We depart from common restrictions in the literature of binary states and actions, or assumptions that the sender's preferences depend only on the posterior mean.[3] Instead, we restrict the preferences of the receiver to the two classes of payoffs and we characterize the minimal and maximal levels of conflict between a sender and a receiver, conditions under which extremal disclosure of information is optimal.

Finally, we consider an application of oligopolistic competition between an incumbent firm, which can invest to acquire a higher quality of information, and a rival entrant. We show that the analysis is formally equivalent to a novel comparison between two different games of information acquisition: one in which information acquisition is a covert activity (the entrant cannot observe the quality of information the incumbent acquires) and another in which information acquisition is overt. The difference between overt and covert information acquisition is the indirect effect of information on the incumbent's profit through the induced behavior of the entrant, an effect we call *the value of transparency*. We characterize the value of transparency depending on the entrant's responsiveness to the incumbent's information and the sign of the externality imposed on the incumbent by the entrant's responsiveness. We also connect our analysis of overt versus covert information acquisition games to the strategic effects of investment in sequential versus simultaneous games of complete information in Fudenberg and Tirole (1984) and Bulow, Geanakoplos, and Klemperer (1985).

1.1. Related Literature

The closest paper to ours is Jensen (2018). He also studies how the distribution of individual decisions and equilibrium outcomes vary with changes in the distribution of some economic parameters. Given a parameter μ and action a, Jensen shows that if a utility function $U(\mu, a)$ satisfies a *quasi-convex differences* condition,[4] then a more dispersed distribution of the parameter μ would lead to a more dispersed distribution of optimal actions. In our setting, the economic parameters are the posterior beliefs over some state of the world θ. However, such an approach would require imposing the quasi-convex differences condition

[3] For example, these conditions are used in Rayo and Segal (2010), Gentzkow and Kamenica (2016), Kolotilin et al. (2017), Dworczak and Martini (2019).

[4] $U(\mu, a)$ satisfies a quasi-convex differences if for all small $\delta > 0$, $U(\mu, a) - U(\mu, a - \delta)$ is quasi-convex.

on the interim utility function

$$U(\mu, a) = \int_{\Theta} u(\theta, a) d\mu(\theta)$$

instead of the primitive $u(\theta, a)$. Since quasi-covexity is not preserved under integration, we are left with the open question of what conditions on the primitives lead to quasi-convex differences in the interim utility function. This is similar to the question of what conditions on primitives lead to the single-crossing condition on the interim utility function, a question that was answered by Quah and Strulovici (2012). We show that the class of games we consider do in fact lead to interim utility functions that satisfy quasi-convex differences, with the added benefit that all our assumptions are only on the primitives. We provide a more formal comparison in Section 4.3.

Another paper that is related to ours is Amir and Lazzati (2016) who show that for a class of games with supermodular payoff functions, the value of information is increasing and convex in the supermodular stochastic order. Additionally, they show that the distance between a player's highest and lowest equilibrium actions increases as information quality increases in the supermodular order. Our main result shows that such "dispersion" in the players' equilibrium behavior extends beyond the highest/lowest actions; it holds for the entire distribution of equilibrium actions.

In a single-agent setting, our work is also connected to Lu (2016). He studies how information affects menu choice when an agent can observe a signal about the value of menus. He shows that the probability of choosing any given menu over an outside option (a "test menu" with ex-ante known value) becomes more dispersed as the agent's signal becomes Blackwell more informative. In contrast, we show that the choice of action from *within* a given menu becomes more dispersed as the quality of information increases. In other words, Lu characterizes the dispersion in the interim value of a menu while we characterize the dispersion of actions chosen from within a menu. To make the distinction more apparent, notice that the interim value of a singleton menu, and therefore the probability of choosing the singleton menu over an outside option, is affected by the informativeness of the agent's signal. However, there cannot be any meaningful dispersion of choice from within the singleton menu.

Methodologically, this paper contributes to the literature on the theory of monotone comparative statics. Specifically focusing on Bayesian single-agent decision problems, Athey (2002) and Quah and Strulovici (2009) show that optimal actions are a monotone function of beliefs (for beliefs ordered by stochastic dominance). Similarly, in Bayesian games, Athey (2001) and Van Zandt and Vives (2007) show that a player's Bayesian Nash equilibrium action is a monotone function of the player's beliefs. We add to this literature by showing that the *distribution* of optimal/equilibrium actions is monotone (in the increasing/decreasing convex order) as a function of the distribution of beliefs (for distributions of beliefs ordered consistently with the supermodular order).

The remainder of the paper is structured as follows: Section 2 presents our model, and Section 3 introduces the orders over distributions of actions and information structures. Our main result is then presented in Section 4, followed by applications in Section 5. Section 6 concludes. All proofs that are not in the main text or in the Appendix, along with additional examples, can be found in the online supplement.

2. Model

2.1. Preliminary Definitions and Notation

Let X_i be a compact subset of $\mathbb{R}$ for $i = 1, \ldots, m$, and define $X = \times_{i=1}^{m} X_i$ and $X_{-i} = \times_{j \neq i} X_j$. We equip X with the coordinatewise order $\geq$, i.e., for $x'', x' \in X$, $x'' \geq x'$ if $x''_i \geq x'_i$ for all $i = 1, 2, \ldots, m$. We also equip X_{-i} with the same coordinatewise order.

We say a function $g : X \to \mathbb{R}$ has increasing (resp., decreasing, or constant) differences in $(x_{-i}; x_i)$ if $g(x_i, x''_{-i}) - g(x_i, x'_{-i})$ is increasing (resp., decreasing, or constant) in x_i for all $x''_{-i}, x'_{-i} \in X_{-i}$ with $x''_{-i} \geq x'_{-i}$.

For a twice differentiable function $g : X \to \mathbb{R}$, we write g_{x_i} as a shorthand for $\frac{\partial g(x)}{\partial x_i}$ and $g_{x_i x_j}$ for $\frac{\partial^2 g(x)}{\partial x_i x_j}$. If g is twice differentiable and has increasing (resp., decreasing, or constant) differences in $(x_{-i}; x_i)$, then $g_{x_i x_j} \geq 0$ (resp., $g_{x_i x_j} \leq 0$, or $g_{x_i x_j} = 0$) for each $j \neq i$.

All references to "increasing/decreasing," "increasing/decreasing differences," and "concave/convex" are in the weak sense.

2.2. Setup

There are n players with $N = \{1, 2, \ldots, n\}$ denoting the set of players. While our exposition highlights games with $n > 1$, we emphasize that our setup and results also apply to single-agent decision problems with $n = 1$.

Each player $i \in N$ has a finite or absolutely continuous random state variable (or type) $\tilde{\theta}_i$ with support in $\Theta_i = [\underline{\theta}_i, \bar{\theta}_i]$. Define $\Theta = \times_{i \in N} \Theta_i$ and $\Theta_{-i} = \times_{j \neq i} \Theta_j$. To distinguish random variables from their realizations, we denote the random state variables by $\tilde{\theta} = (\tilde{\theta}_i, \tilde{\theta}_{-i})$ and the realized states by $\theta = (\theta_i, \theta_{-i})$.

The players hold a common prior given by the joint distribution $F : \Theta \to [0, 1]$. Let F_{Θ_i} be the marginal distribution of $\tilde{\theta}_i$ induced by F. Similarly, let $F_{\Theta_{-i}}(\cdot|\theta_i)$ be the joint distribution of $\tilde{\theta}_{-i}$ conditional on $\tilde{\theta}_i = \theta_i$. We assume that the mapping $\theta_i \mapsto F_{\Theta_{-i}}(\cdot|\theta_i)$ is measurable.

Assumption 1
For all $i \in N$ and $\theta_i > \theta_i'$, $F_{\Theta_{-i}}(\cdot|\theta_i)$ first-order stochastically dominates $F_{\Theta_{-i}}(\cdot|\theta_i')$. We adopt the notation $F_{\Theta_{-i}}(\cdot|\theta_i) \succeq_{FOSD} F_{\Theta_{-i}}(\cdot|\theta_i')$.

Assumption 1 is weaker than assuming the random state variables satisfy the monotone likelihood ratio property (i.e., affiliation, Milgrom and Weber (1982)).

Let $A_i = [\underline{a}_i, \bar{a}_i]$ be the player i's action space, $A = \times_{i \in N} A_i$, and $A_{-i} = \times_{j \neq i} A_j$. The payoff for each player $i \in N$ is given by a utility function $u^i : \Theta \times A \to \mathbb{R}$ such that

Assumption 2

a. $u^i(\theta, a)$ is uniformly bounded, measurable in θ, and twice differentiable in a_i,

b. for all $(\theta, a_{-i}) \in \Theta \times A_{-i}$, $u^i(\theta, a_{-i}, \cdot)$ is strictly concave in a_i,

c. for all $(\theta, a_{-i}) \in \Theta \times A_{-i}$, there exists an action $a_i \in A_i$ such that $u^i_{a_i}(\theta, a_{-i}, a_i) = 0$, and

d. $u^i(\theta, a)$ has increasing differences in $(\theta, a_{-i}; a_i)$.

Assumption 2.a-c imply that agents have unique interior best responses. Assumption 2.d implies that there are complementarities between the state of the world and a player's action. Additionally, there are strategic complementarities between the players' actions. Thus, when the state θ is high or when player j takes a high action, player i wants to do the same.

2.3. Information Structures

Each player $i \in N$ observes a (possibly noisy) signal $\tilde{s}_i$ from an information structure $\Sigma_{\rho_i} = \langle S_i, G(\cdot, \cdot; \rho_i) \rangle$ where $S_i \subseteq \mathbb{R}$ is the signal space, $G(\cdot, \cdot; \rho_i) : \Theta_i \times S_i \to [0, 1]$ is a joint probability distribution of $(\tilde{\theta}_i, \tilde{s}_i)$, and ρ_i is an index.[5] Let $G_{\Theta_i}(\cdot; \rho_i)$ and $G_{S_i}(\cdot; \rho_i)$ be the marginal distribution of $\tilde{\theta}_i$ and $\tilde{s}_i$ respectively. Let $G_{\Theta_i}(\cdot|s_i; \rho_i)$ be player i's posterior distribution conditional on $\tilde{s}_i = s_i$, and let $G_{S_i}(\cdot|\theta_i; \rho_i)$ be the distribution of signals conditional on $\tilde{\theta}_i = \theta_i$. We assume that the mappings $s_i \mapsto G_{\Theta_i}(\cdot|s_i; \rho_i)$ and $\theta_i \mapsto G_{S_i}(\cdot|\theta_i; \rho_i)$ are measurable. Additionally,

Assumption 3 *for all $i \in N$,*

a. $G_{\Theta_i}(\cdot; \rho_i) = F_{\Theta_i}(\cdot)$,

[5] There is an implicit assumption in the setup that player i can directly learn only about $\tilde{\theta}_i$. We make this assumption explicit in Assumption 4.

b. $G_{S_i}(\cdot;\rho_i) = G_{S_i}(\cdot)$,

c. $G_{\Theta_i}(\cdot|s_i;\rho_i) \succeq_{FOSD} G_{\Theta_i}(\cdot|s_i';\rho_i)$ *whenever* $s_i > s_i'$*, and*

d. $G_{S_i}(\cdot|\theta_i;\rho_i) \succeq_{FOSD} G_{S_i}(\cdot|\theta_i';\rho_i)$ *whenever* $\theta_i > \theta_i'$*.*

Assumption 3.a implies that posterior beliefs satisfy Bayes plausibility (Kamenica and Gentzkow, 2011). Assumption 3.b, which holds without loss of generality, states that all information structures induce the same marginal distribution on $\tilde{s}_i$.[6] Assumption 3.c implies that higher states are more likely when the signal realization is high while Assumption 3.d implies that higher signal realizations are more likely when the state is high. If $(\tilde{\theta}_i, \tilde{s}_i)$ are affiliated, then Assumption 3.c-d are jointly satisfied.

Let $S = \times_{i\in N} S_i$. We denote the profile of information structures by $\Sigma_\rho = (\Sigma_{\rho_1}, \ldots, \Sigma_{\rho_n})$. A profile Σ_ρ induces a joint distribution $\boldsymbol{G}(\cdot,\cdot;\rho) : \Theta \times S \to [0,1]$ over $(\tilde{\theta}, \tilde{s})$. We assume that player i cannot directly learn about $(\tilde{\theta}_{-i}, \tilde{s}_{-i})$. Formally,

Assumption 4 *for all* $(\theta, s) \in \Theta \times S$,

$$\boldsymbol{G}(s|\theta;\rho) = \prod_{i\in N} G_{S_i}(s_i|\theta_i;\rho_i).$$

2.4. Equilibrium Outcomes

Following the terminology of Gossner (2000), we decompose a Bayesian game into a basic game and a profile of information structures. The basic game $\Gamma = \langle N, \{A_i, u^i\}_{i\in N}, F\rangle$ is comprised of (i) a set of players N, (ii) for each player $i \in N$, an action space A_i along with a payoff function $u^i : \Theta \times A \to \mathbb{R}$ that satisfies Assumption 2, and (iii) a common prior F that satisfies Assumption 1. The profile of information structures Σ_ρ satisfies Assumption 3 and Assumption 4. Both Γ and Σ_ρ are common knowledge. The full Bayesian game is given by $\mathcal{G}_\rho = (\Sigma_\rho, \Gamma)$. The setting is general enough to accommodate private and common values as well as independence and affiliation.[7]

Each player $i \in N$ first privately observes a signal realization $s_i \in S_i$ generated from Σ_{ρ_i}. Then the players participate in the basic game Γ by simultaneously choosing an action. A pure strategy for player $i \in N$ is given by the measurable function $\alpha_i : S_i \to A_i$. Let

[6] The assumption is without loss of generality because we can apply the integral probability transform to any random signal $\tilde{s}_i$ and create a new signal which is uniformly distributed on the unit interval. As noted by Lehmann (1988), the integral probability transform is applicable even when the CDF of $\tilde{s}_i$ has a discontinuity.

[7] For example, the IPV case is given by $\tilde{\theta}_i \perp \tilde{\theta}_j$ for all $j \neq i$ and $u^i(\theta_i, \theta_{-i}, a) = u^i(\theta_i, \theta'_{-i}, a)$ for all $\theta_{-i} \neq \theta'_{-i}$. The pure common values case is given by $\tilde{\theta}_i = \tilde{\theta}_j$ for all $j \neq i$.

$\alpha = (\alpha_i, \alpha_{-i})$ be a pure strategy profile. In a Bayesian game $\mathcal{G}_\rho$, player i's interim utility when taking action $a_i \in A_i$, given a signal realization s_i and a profile of opponent's strategies α_{-i}, is

$$U^i(a_i, \alpha_{-i}; s_i, \rho) = \int\limits_{\Theta \times S_{-i}} u^i\big(\theta, \alpha_{-i}(s_{-i}), a_i\big) d\boldsymbol{G}(\theta, s_{-i}|s_i; \rho).$$

Momentarily ignoring existence issues, let $a^\star(\rho) = \big(a_1^\star(\rho), a_2^\star(\rho), \ldots, a_n^\star(\rho)\big)$ be a profile of pure strategy actions that constitute a Bayesian Nash equilibrium (BNE) of the game $\mathcal{G}_\rho$. For each player $i \in N$ and each $s_i \in S_i$,

$$a_i^\star(s_i; \rho) = \arg\max_{a_i \in A_i} U^i(a_i, a_{-i}^\star(\rho); s_i, \rho).$$

We restrict our attention to monotone BNEs, i.e., each player's equilibrium strategy, $a_i^\star(s_i; \rho)$ is increasing in the signal realization s_i. The existence of monotone pure strategy BNE has long been established by the literature on supermodular Bayesian games (games that satisfy Assumption 1-Assumption 3). In particular, the existence result of Van Zandt and Vives (2007) is noteworthy in our setting because their existence result does not require players to have atomless posterior beliefs when they participate in the basic game. While restricting attention to monotone BNEs may be with loss of generality, extremal equilibria are nonetheless monotone. Specifically, the least and the greatest BNEs of a supermodular Bayesian game are in monotone pure strategies (Milgrom and Roberts, 1990; Van Zandt and Vives, 2007).

Our goal in this paper is to characterize a comparative statics of $a^\star(\rho)$ as the information structure Σ_ρ changes while holding the underlying basic game Γ fixed. To do so, we will first introduce the relevant orders over actions and information structures on which our comparative statics is based.

3. Orders

3.1. Order over Distributions of Actions

From an interim perspective, each player $i \in N$ first observes a signal realization $s_i \in S_i$ and then takes some action $\alpha_i(s_i) \in A_i$. From an ex-ante perspective, the signal realizations are yet to be observed. Therefore, $\alpha_i(\tilde{s}_i)$ is a random variable that is distributed according to the CDF $H(\cdot; \alpha_i) : \mathbb{R} \to [0, 1]$ given by

$$H(z; \alpha_i) = \int\limits_{S_i} \mathbf{1}_{[\alpha_i(s_i) \leq z]} dG_{S_i}(s_i)$$

for $z \in \mathbb{R}$, where $\mathbf{1}_{[\cdot]}$ is the indicator function.

Given two pure strategies α_i and α_i', we say that α_i dominates α_i' in the *increasing convex order* if for any measurable, covex, and increasing function $\psi : \mathbb{R} \to \mathbb{R}$,

$$\int \psi(z) dH(z;\alpha_i) \geq \int \psi(z) dH(z;\alpha_i').$$

Loosely, α_i dominates α_i' in the increasing order if α_i is more variable and larger on average than α_i'. We write $\alpha_i \succeq_{icx} \alpha_i'$ when α_i dominates α_i' in the increasing convex order. Given a profile of pure strategies $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\alpha' = (\alpha_1', \ldots, \alpha_n')$, we say that $\alpha \succeq_{icx} \alpha'$ if $\alpha_i \succeq_{icx} \alpha_i'$ for all $i \in N$.

Similarly, given two pure strategies α_i and α_i', we say that α_i dominates α_i' in the *decreasing convex order* if for any measurable, covex, and decreasing function $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\int \varphi(z) dH(z;\alpha_i) \geq \int \varphi(z) dH(z;\alpha_i').$$

Notice that α_i dominates α_i' in the decreasing convex order if, and only if, α_i' second-order stochastically dominates α_i. Additionally, α_i dominates α_i' in both the increasing and decreasing convex order if, and only if, α_i is a mean-preserving spread of α_i'. We write $\alpha_i \succeq_{dcx} \alpha_i'$ when α_i dominates α_i' in the decreasing convex order. Given a profile of pure strategies $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\alpha' = (\alpha_1', \ldots, \alpha_n')$, we say that $\alpha \succeq_{dcx} \alpha'$ if $\alpha_i \succeq_{dcx} \alpha_i'$ for all $i \in N$.

Example 1 Let the marginal distribution of player i's signal be the uniform distribution over the unit interval. Consider two monotone strategies α_i and α_i' given by

$$\alpha_i(s_i) = \begin{cases} 1 & \text{if} \quad s_i \leq \frac{1}{2} \\ 5 & \text{if} \quad s_i > \frac{1}{2} \end{cases} \qquad \alpha_i'(s_i) = \begin{cases} 1 & \text{if} \quad s_i \leq \frac{1}{3} \\ 2 & \text{if} \quad s_i \in \left(\frac{1}{3}, \frac{2}{3}\right] \\ 5 & \text{if} \quad s_i > \frac{2}{3} \end{cases}.$$

For any convex and increasing function $\psi : \mathbb{R} \to \mathbb{R}$,

$$\begin{aligned} \int \psi(z) \Big[dH(z;\alpha_i) - dH(z;\alpha_i') \Big] &= \frac{1}{3} \left(\frac{1}{2}\psi(1) + \frac{1}{2}\psi(5) - \psi(2) \right) \\ &\geq \frac{1}{3} \left(\psi\left(\frac{1}{2}1 + \frac{1}{2}5 \right) - \psi(2) \right) \geq 0. \end{aligned}$$

Thus, $\alpha_i \succeq_{icx} \alpha_i'$. However, $\alpha_i \not\succeq_{dcx} \alpha_i'$, which we can show by taking $\varphi(z) = -z$ (a decreasing and convex function) so that $\int \varphi(z) \Big[dH(z;\alpha_i) - dH(z;\alpha_i') \Big] = -\frac{1}{3}$. Similarly, we can show that $\alpha_i' \not\succeq_{dcx} \alpha_i$ by taking the decreasing and convex function

$$\varphi(z) = \begin{cases} 4 - 2z & \text{if} \quad z \leq 2 \\ 0 & \text{if} \quad z > 2 \end{cases}$$

so that $\int \varphi(z) \Big[dH(z; \alpha_i') - dH(z; \alpha_i) \Big] = -\frac{1}{3}$.

Definition 1 (Responsiveness)
Given a basic game Γ, we say players are ***more responsive with a higher mean*** *under Σ_ρ than $\Sigma_{\rho'}$ if*

- *for each monotone BNE $a^\star(\rho')$ of $\mathcal{G}_{\rho'} = (\Sigma_{\rho'}, \Gamma)$, there exists a monotone BNE $a^\star(\rho)$ of $\mathcal{G}_\rho = (\Sigma_\rho, \Gamma)$ such that $a^\star(\rho) \succeq_{icx} a^\star(\rho')$, and*
- *for each monotone BNE $a^\star(\rho)$ of $\mathcal{G}_\rho$, there exists a monotone BNE $a^\star(\rho')$ of $\mathcal{G}_{\rho'}$ such that $a^\star(\rho) \succeq_{icx} a^\star(\rho')$.*

If $\succeq_{icx}$ is replaced by $\succeq_{dcx}$, we say players are ***more responsive with a lower mean****.*

The definition for responsiveness takes into account the possibility of multiple BNE outcomes. However, whenever the players are more responsive with a higher (resp., lower) mean, the least and greatest BNE outcomes of $\mathcal{G}_\rho$ dominate the respective least and greatest BNE outcomes of $\mathcal{G}_{\rho'}$ in the increasing (resp., decreasing) convex order. Additionally, if $n = 1$, there is no multiplicity because of Assumption 2.

3.2. Order over Information Structures

The next step is to determine an appropriate way to compare different information structures. Recall that we focus, without loss of generality, on information structures that induces the same marginals on $\tilde{\theta}_i$ and $\tilde{s}_i$ (Assumption 3a-b).

Definition 2 (Supermodular Stochastic Order)
Given two information structures Σ_{ρ_i} and $\Sigma_{\rho_i'}$, we say that Σ_{ρ_i} dominates $\Sigma_{\rho_i'}$ in the ***supermodular stochastic order*** *if for all $(\theta_i, s_i) \in \Theta_i \times S_i$,*

$$G(\theta_i, s_i; \rho_i) \geq G(\theta_i, s_i; \rho_i').$$

We use the notation $\rho_i \succeq_{spm} \rho'_i$ whenever Σ_{ρ_i} dominates $\Sigma_{\rho'_i}$ in the supermodular stochastic order.[8] Given two profiles of information structures $\Sigma_\rho = (\Sigma_{\rho_1}, \ldots, \Sigma_{\rho_n})$ and $\Sigma_{\rho'} = (\Sigma_{\rho'_1}, \ldots, \Sigma_{\rho'_n})$, we write $\rho \succeq_{spm} \rho'$ if $\rho_i \succeq_{spm} \rho'_i$ for all $i \in N$.

In words, $\rho_i \succeq_{spm} \rho'_i$ if $\tilde{\theta}_i$ and $\tilde{s}_i$ are more positively correlated under Σ_{ρ_i}. Formally, by Assumption 3.c, low signal realizations are evidence of low states. When $\rho_i \succeq_{spm} \rho'_i$, a signal $\tilde{s}_i \leq s_i$ from Σ_{ρ_i} presents a stronger evidence of a low state than the same signal from $\Sigma_{\rho'_i}$. Thus, $\mathbb{P}(\tilde{\theta}_i \leq \theta_i | \tilde{s}_i \leq s_i; \rho_i) \geq \mathbb{P}(\tilde{\theta}_i \leq \theta_i | \tilde{s}_i \leq s_i; \rho')$, and

$$G(\theta_i, s_i; \rho_i) = \mathbb{P}(\tilde{\theta}_i \leq \theta_i | \tilde{s}_i \leq s_i; \rho_i) G_{S_i}(s_i) \geq \mathbb{P}(\tilde{\theta}_i \leq \theta_i | \tilde{s}_i \leq s_i; \rho'_i) G_{S_i}(s_i) = G(\theta, s; \rho'_i).$$

The supermodular stochastic order has been studied extensively by Tchen (1980), Epstein and Tanny (1980), and Meyer and Strulovici (2012, 2015). In economics, Athey and Levin (2017) show that all single-agent decision makers with supermodular preferences value a higher quality of information if, and only if, information quality is ranked by the supermodular order. Amir and Lazzati (2016) extend Athey and Levin's result to supermodular Bayesian games and show that the value of information is increasing and convex in the supermodular stochastic order.

Within the class of information structures that satisfy Assumption 3, the supermodular order is more general than Blackwell informativeness (Blackwell, 1951, 1953) and the Lehmann/accuracy order (Lehmann, 1988).[9] In particular,

$$\text{Blackwell order} \implies \text{Lehmann order} \implies \text{supermodular stochastic order.}$$

The converse however is not true; we provide an example of information structures that cannot be ranked by the Blackwell or Lehmann order but are ranked by the supermodular order in Appendix D. Nonetheless, all the above information orders coincide when the relevant state space is binary (i.e., the support of the prior is $\{\theta_1, \theta_2\} \subset \Theta$).

As an example, let Σ_{ρ_i} be a truth-or-noise information structure such that with probability $\rho_i \in [0, 1]$, the signal reveals the state ($\tilde{s}_i = \tilde{\theta}_i$), and with probability $1 - \rho_i$, the signal and the state are iid. For two truth-or-noise information structures Σ_{ρ_i} and $\Sigma_{\rho'_i}$, Σ_{ρ_i} is Blackwell more informative than $\Sigma_{\rho'_i}$ (and therefore $\rho_i \succeq_{spm} \rho'_i$) if $\rho_i > \rho'_i$.

[8] Since each player's information structure is a bi-variate random variable, $(\tilde{\theta}_i, \tilde{s}_i)$, the supermodular stochastic order is equivalent to the Positive Quadrant Dependence order. This equivalence no longer holds for random vectors with more than two variables.

[9] See Persico (2000) and Jewitt (2007) for a more complete analysis of the Lehmann ordering.

4. Preferences and Main Result

The main contribution of this paper is to identify a class of payoffs for which players become more responsive when information quality increases according to the supermodular stochastic order.

Let $\boldsymbol{\Gamma_{icx}}$ be the class of basic games $\Gamma = \langle N, \{A_i, u^i\}_{i\in N}, F\rangle$ such that for all $i \in N$, the payoff function $u^i : \Theta \times A \to \mathbb{R}$ satisfies Assumption 2 and has a marginal utility $u^i_{a_i}(\theta, a)$ that for all $j \in N$,

(i) is convex in a_j for all $(\theta, a_{-j}) \in \Theta \times A_{-j}$, and

(ii) has increasing differences in $(\theta, a_{-j}; a_j)$.

Below, we show $\boldsymbol{\Gamma_{icx}}$ is linked to responsiveness with a higher mean (hence the subscript "icx" for increasing convex order).

Similarly, let $\boldsymbol{\Gamma_{dcx}}$ be the class of basic games $\Gamma = \langle N, \{A_i, u^i\}_{i\in N}, F\rangle$ such that for all $i \in N$, the payoff function satisfies Assumption 2 and has a marginal utility $u^i_{a_i}(\theta, a)$ that for all $j \in N$,

(i) is concave in a_j for all $(\theta, a_{-j}) \in \Theta \times A_{-j}$, and

(ii) has decreasing differences in $(\theta, a_{-j}; a_j)$.

Below, we show $\boldsymbol{\Gamma_{dcx}}$ is linked to responsiveness with a lower mean (hence the subscript "dcx" for decreasing convex order).

A basic game $\Gamma \in \boldsymbol{\Gamma_{icx}} \cap \boldsymbol{\Gamma_{dcx}}$ if the payoff function for all $i \in N$ satisfies Assumption 2 and has a marginal utility $u^i_{a_i}(\theta, a)$ that for all $j \in N$,

(i) is linear in a_j for all $(\theta, a_{-j}) \in \Theta \times A_{-j}$, and

(ii) has constant differences in $(\theta, a_{-j}; a_j)$.

Beauty contests (Keynes, 1936; Morris and Shin, 2002) and quadratic games with strategic complementarities (Angeletos and Pavan, 2007; Bergemann and Morris, 2013) fall into the class of games in $\boldsymbol{\Gamma_{icx}} \cap \boldsymbol{\Gamma_{dcx}}$. For example, suppose each player i has a payoff

$$u^i(\theta, a) = -\frac{\beta_i}{2}\Big(t_i(\theta) - a_i\Big)^2 - \frac{(1-\beta_i)}{2}\left(\frac{1}{n-1}\sum_{k\neq i} a_k - a_i\right)^2.$$

When $\beta_i \in (0,1)$ and $t_i : \Theta \to \mathbb{R}$ is an increasing function, then $u^i(\theta, a)$ satisfies increasing differences in $(\theta, a_{-i}; a_i)$. The marginal utility is

$$u^i_{a_i}(\theta, a) = \beta_i t_i(\theta) + \frac{1-\beta_i}{n-1} \sum_{k \neq i} a_k - a_i,$$

which is linear in a_j and satisfies constant differences in $(\theta, a_{-j}; a_j)$ for all $j \in N$. Below, we show $\mathbf{\Gamma}_{icx} \cap \mathbf{\Gamma}_{dcx}$ is linked to mean-preserving spreads in the distribution of actions as information quality increases. We provide some additional examples in the online supplement.

Consider two profiles of information structures Σ_ρ and $\Sigma_{\rho'}$. Theorem 1 below states that when the basic game Γ belongs to the class of games in $\mathbf{\Gamma}_{icx}$, the BNE action of each player is more dispersed and higher on average in $\mathcal{G}_\rho = (\Sigma_\rho, \Gamma)$ than in $\mathcal{G}_{\rho'} = (\Sigma_{\rho'}, \Gamma)$ whenever Σ_ρ dominates $\Sigma_{\rho'}$ in the supermodular stochastic order. Moreover, if Σ_ρ does NOT dominate $\Sigma_{\rho'}$ in the supermodular stochastic order, then there is a game in which the players' preferences belong to $\mathbf{\Gamma}_{icx}$ but the players are NOT more responsive with a higher mean under Σ_ρ. The theorem also establishes a similar result relating $\mathbf{\Gamma}_{dcx}$ and responsiveness with a lower mean.

Theorem 1 *Players are more responsive with a higher (resp., lower) mean under Σ_ρ than $\Sigma_{\rho'}$ for any basic game $\Gamma \in \mathbf{\Gamma}_{icx}$ (resp., $\Gamma \in \mathbf{\Gamma}_{dcx}$) if, and only if, Σ_ρ dominates $\Sigma_{\rho'}$ in the supermodular stochastic order.*

We defer the proof of Theorem 1 until Appendix B. Here, we provide some intuition starting with the single-agent case. Readers who are interested in applications may wish to skip ahead to Section 5.

4.1. Responsiveness: Single-agent

We start with the case of a single-agent ($n = 1$) and drop the player-index "i" for now. The mechanism behind Theorem 1 is best understood through Proposition 1 which shows that when $\Gamma \in \mathbf{\Gamma}_{icx}$ (resp., $\Gamma \in \mathbf{\Gamma}_{dcx}$), optimal actions are "convex" (resp., "concave") in the agent's posterior belief. To state the proposition formally, let $\mu \in \Delta(\Theta)$ be a probability measure representing an arbitrary belief the agent may hold. Define

$$a^*(\mu) = \arg\max_{a \in A} \; U(\mu, a) = \int_\Theta u(\theta, a)\mu(d\theta).$$

Since $u(\theta, a)$ has increasing differences in $(\theta; a)$, we know that $a^*(\cdot)$ is an increasing function in the sense that $a^*(\mu_2) \geq a^*(\mu_1)$ whenever $\mu_2 \succeq_{FOSD} \mu_1$ (Athey, 2002).

Proposition 1 *Let $\mu_1, \mu_2 \in \Delta(\Theta)$ be any two beliefs with $\mu_2 \succeq_{FOSD} \mu_1$. If $\Gamma \in \boldsymbol{\Gamma_{icx}}$,*

$$a^*\big(\lambda\mu_1 + (1-\lambda)\mu_2\big) \leq \lambda a^*(\mu_1) + (1-\lambda)a^*(\mu_2)$$

for all $\lambda \in [0,1]$. If $\Gamma \in \boldsymbol{\Gamma_{dcx}}$, the opposite inequality holds.

Notice that the convexity in Proposition 1 is established only for the case when posteriors are ordered by stochastic dominance. Henceforth, we focus on $\boldsymbol{\Gamma_{icx}}$ but the intuition we provide can be symmetrically applied to $\boldsymbol{\Gamma_{dcx}}$.

Proof. Consider a basic game $\Gamma \in \boldsymbol{\Gamma_{icx}}$. Let $a_k^* = a^*(\mu_k)$ for $k = 1, 2$, $a_\lambda = \lambda a_1^* + (1-\lambda)a_2^*$, and $\mu_\lambda = \lambda\mu_1 + (1-\lambda)\mu_2$. By the first order condition, we have that $U_a(\mu_k, a_k^*) = 0$.

$$\begin{aligned}
U_a(\mu_\lambda, a_\lambda) &\leq \lambda^2 U_a(\mu_1, a_1^*) + (1-\lambda)^2 U_a(\mu_2, a_2^*) + \lambda(1-\lambda)\Big(U_a(\mu_2, a_1^*) + U_a(\mu_1, a_2^*)\Big) \\
&= \lambda(1-\lambda)\int_\Theta \left[u_a(\theta, a_1^*) - u_a(\theta, a_2^*)\right](\mu_2(d\theta) - \mu_1(d\theta)) \\
&\leq 0
\end{aligned}$$

where the first inequality follows from the convexity of U_a in a and linearity in μ. Increasing differences (ID) of the utility $u(\theta, a)$ in $(\theta; a)$ along with $\mu_2 \succeq_{FOSD} \mu_1$ implies $a_2 \geq a_1$. By ID of the marginal utility u_a in $(\theta; a)$, we have $u_a(\theta, a_1) - u_a(\theta, a_2)$ is a decreasing function of θ. The last inequality then follows from the definition of first-order stochastic dominance. Since the marginal value of a_λ is non-positive at μ_λ, we must have $a^*(\mu_\lambda) \leq a_\lambda$. A symmetric argument establishes that if $\Gamma \in \boldsymbol{\Gamma_{dcx}}$, then $a^*(\mu_\lambda) \geq a_\lambda$. ■

To see how the "convexity" of the optimal action is related to responsiveness, notice that each information structure that satisfies Assumption 3 induces a distribution over first-order ranked posterior beliefs. The distribution over optimal actions is a composition of the induced distribution over beliefs with $a^*(\cdot)$, the optimal action function.

Consider two information structures Σ_ρ and $\Sigma_{\rho'}$. If Σ_ρ is Blackwell more informative than $\Sigma_{\rho'}$. The distribution over posteriors induced by Σ_ρ is a mean-preserving spread of the distribution over posteriors induced by $\Sigma_{\rho'}$. A convex and increasing $a^*(\cdot)$ maps the mean-preserving spread in the distribution over posterior beliefs to a shift in the increasing convex order for the distribution over actions (Theorem 4.A.8, Shaked and Shanthikumar (2007)). By Proposition 1, whenever $\Gamma \in \boldsymbol{\Gamma_{icx}}$, the optimal action $a^*(\cdot)$ is indeed an increasing and convex function of posterior beliefs. Thus, the agent is more responsive with an higher mean under Σ_ρ than under $\Sigma_{\rho'}$. The proof for Theorem 1 generalizes this intuition from the Blackwell order to the more general supermodular order.

Corollary 1 *Let Σ_ρ be an information structure that satisfies Assumption 3. Let $\Sigma_{\rho'}$ be any* garbling *of Σ_ρ. If an agent has utility $\Gamma \in \boldsymbol{\Gamma_{icx}}$ $\big($resp., $\Gamma \in \boldsymbol{\Gamma_{dcx}}\big)$, then the agent is more responsive with a higher (resp., lower) mean under Σ_ρ than under $\Sigma_{\rho'}$.*

Proposition 1 directly implies Corollary 1, which shows that the agent becomes more responsive when information quality increases in the Blackwell order. While the result appears to be an implication of Theorem 1, there is a subtle difference—the garbling $\Sigma_{\rho'}$ does NOT have to satisfy Assumption 3.c-d.

Since $\Theta \subset \mathbb{R}$ is an ordered set, the full information structure induces posteriors that trivially satisfy Assumption 3. Furthermore, any other information structure is a garbling of the full information structure. Hence, when $\Gamma \in \boldsymbol{\Gamma_{icx}}$ (resp., $\Gamma \in \boldsymbol{\Gamma_{dcx}}$), Corollary 1 implies that the agent's actions are the most dispersed with the highest (resp., lowest) mean under the full information structure.

Remark 1 *Whenever $\Gamma \notin \boldsymbol{\Gamma_{icx}}$, there exist beliefs $\mu_1, \mu_2 \in \Delta(\Theta)$ with $\mu_2 \succeq_{FOSD} \mu_1$, and a constant $\lambda \in (0,1)$ for which Proposition 1 is violated. Consequently, we can find a prior $\mu_o = \lambda\mu_1 + (1-\lambda)\mu_2$, an uninformative structure $\Sigma_{\rho'}$, and a more informative structure Σ_ρ that induces posteriors μ_1 and μ_2 with probabilities λ and $1-\lambda$ respectively such that $\rho \succeq_{spm} \rho'$ but the agent is NOT more responsive with a higher mean under Σ_ρ. In this sense, the class of preferences $\boldsymbol{\Gamma_{icx}}$ is not only sufficient but also necessary for responsiveness with a higher mean. We present such an example in Section Appendix C.*

4.2. Responsiveness: Bayesian Games

Let us now consider the case with $n > 1$. Suppose $\Gamma \in \boldsymbol{\Gamma_{icx}}$, and consider a profile of information structures Σ_ρ and $\Sigma_{\rho'}$. Fix a player $i \in N$. The proof proceeds in four steps.

1. Holding all else fixed, $\rho_i \succeq_{spm} \rho'_i$ implies that the induced distribution over player i's best-responses shift in the increasing convex order (Lemma B.1). The intuition is an extension of the intuition for the single-agent setting.

2. Holding all else fixed, $\rho_j \succeq_{spm} \rho'_j$ for some $j \neq i$ implies that the induced distribution over player i's best-responses shift in the increasing convex order (Lemma B.2). As player j's information quality increases, the signals $\tilde{s}_i$ and $\tilde{s}_j$ become indirectly (weakly) more correlated.[10] Hence, player i can better predict player j's random action and match it.

[10]We allow for $\tilde{s}_i$ to be independent of $\tilde{s}_j$, e.g., IPV setting.

3. Holding all else fixed, a pure strategy $\alpha_j \succeq_{icx} \alpha'_j$ for player $j \neq i$ implies that the induced distribution over player i's best-responses shift in the increasing convex order (Lemma B.3). It is of similar spirit to the result that strategic complementarities between (a_j, a_i) imply that player i's best-reply is in monotone strategies whenever player j uses a monotone strategy. Except, here, player i's best-reply becomes more dispersed whenever player j chooses a more dispersed strategy.

4. Finally, we show that the combination of the three aforementioned effects is that each player's distribution of BNE outcomes becomes more dispersed if at least one player gets a higher quality of information.

4.3. Connection to Jensen

As mentioned in the literature review, the closest paper to ours is Jensen (2018); he also studies how the distribution of individual decisions and equilibrium outcomes vary with changes in the distribution of some economic parameter. As the connection is most clear in the single-agent setting, we will focus our discussion to the case when $n = 1$.[11]

In our setting, each information structure induces a distribution of μ—the parameter—which in turn induces a distribution of $a^*(\mu)$—the agent's optimal decision. Thus, the problem we are studying can be equivalently formulated in Jensen's model. Jensen shows that if if optimal solutions are not corner solutions, and if $U(\mu, a) - U(\mu, a - \delta)$ is quasi-convex (resp., quasi-concave) for all $\delta > 0$ small enough, $\mu \in \Delta(\Theta)$, and $a \in A$, then $a^*(\mu)$ is a convex (resp., concave) function of μ. Consequently, the agent's optimal actions become more dispersed with a higher (resp., lower) mean as the information structure becomes Blackwell more informative.

While the quasi-convexity condition is useful to answer the questions we are interested in, it is not known what conditions on $u(\theta, a)$ would yield the quasi-convexity conditions on the interim utility $U(\mu, a)$. In particular, quasi-convexity of $u(\theta, a) - u(\theta, a - \delta)$ does not imply quasi-convexity of $U(\mu, a) - U(\mu, a - \delta)$, as quasi-convexity is not closed under integration.

Below, we show that our class of utility functions are sufficient conditions to establish Jensen's quasi-convexity conditions on the interim utility for posteriors that are ranked by first-order stochastic dominance. Since we are considering differentiable functions, Jensen's conditions are equivalent to quasi-convexity/quasi-concavity of $U_a(\mu, a)$.

[11] The connection between the two papers in the Bayesian game setting is further complicated because Jensen only considers the case of IPV while we allow for interdependence in payoffs and correlation in the states.

Proposition 2 *Let $n = 1$. Consider any two beliefs $\mu_1, \mu_2 \in \Delta(\Theta)$ with $\mu_2 \succeq_{FOSD} \mu_1$, and any $a_1, a_2 \in A$. If $\Gamma \in \boldsymbol{\Gamma_{icx}}$,*

$$U_a\big(\lambda\mu_1 + (1-\lambda)\mu_2, \lambda a_1 + (1-\lambda)a_2\big) \leq \max\big\{U_a(\mu_1, a_1), U_a(\mu_2, a_2)\big\}$$

for all $\lambda \in [0,1]$. Similarly, if $\Gamma \in \boldsymbol{\Gamma_{dcx}}$,

$$U_a\big(\lambda\mu_1 + (1-\lambda)\mu_2, \lambda a_1 + (1-\lambda)a_2\big) \geq \min\big\{U_a(\mu_1, a_1), U_a(\mu_2, a_2)\big\}$$

for all $\lambda \in [0,1]$.

Proof. We prove the case for $\Gamma \in \boldsymbol{\Gamma_{icx}}$; a symmetric argument proves the case for $\Gamma \in \boldsymbol{\Gamma_{dcx}}$. Define $a_\lambda = \lambda a_1 + (1-\lambda)a_2$ and $\mu_\lambda = \lambda\mu_1 + (1-\lambda)\mu_2$. By convexity of U_a in a,

$$U_a(\mu_\lambda, a_\lambda) \leq \lambda^2 U_a(\mu_1, a_1) + (1-\lambda)^2 U_a(\mu_2, a_2) + \lambda(1-\lambda)\Big(U_a(\mu_1, a_2) + U_a(\mu_2, a_1)\Big).$$

There are two cases to consider:

1. $\boldsymbol{a_1 \leq a_2}$. Since u_a has ID in $(\theta; a)$, so does U_a. As $\mu_2 \succeq_{FOSD} \mu_1$, we have

$$\begin{aligned} U_a(\mu_\lambda, a_\lambda) \leq& \lambda^2 U_a(\mu_1, a_1) + (1-\lambda)^2 U_a(\mu_2, a_2) + \lambda(1-\lambda)\Big(U_a(\mu_1, a_2) + U_a(\mu_2, a_1)\Big) \\ \leq& \lambda U_a(\mu_1, a_1) + (1-\lambda) U_a(\mu_2, a_2) \\ \leq& \max\{U_a(\mu_1, a_1), U_a(\mu_2, a_2)\}, \end{aligned}$$

   where the first equality follows from $U_a(\mu_1, a_2) + U_a(\mu_2, a_1) \leq U_a(\mu_1, a_1) + U_a(\mu_2, a_2)$.

2. $\boldsymbol{a_1 > a_2}$. Since u is concave in a, so is U. Therefore, $U_a(\mu, a_1) \leq U_a(\mu, a_2)$ for any $\mu \in \Delta(\Theta)$. Additionally, since u has ID in $(\theta; a)$, so does U. As $\mu_2 \succeq_{FOSD} \mu_1$, we have $U_a(\mu_1, a_2) \leq U_a(\mu_2, a_2)$. In other words, $\max\{U_a(\mu_1, a_1), U_a(\mu_2, a_2)\} = U_a(\mu_2, a_2)$. We can then conclude that

$$\begin{aligned} U_a(\mu_\lambda, a_\lambda) \leq& \lambda^2 U_a(\mu_1, a_1) + (1-\lambda)^2 U_a(\mu_2, a_2) + \lambda(1-\lambda)\Big(U_a(\mu_1, a_2) + U_a(\mu_2, a_1)\Big). \\ \leq& U_a(\mu_2, a_2) \\ =& \max\{U_a(\mu_1, a_1), U_a(\mu_2, a_2)\}. \end{aligned}$$

In both cases, we get the desired quasi-convexity condition. ■

5. Applications

We consider two application of our main result, one in the single-agent setting and another in Bayesian games.

5.1. Single-agent application: Information Disclosure

In the information disclosure game of Rayo and Segal (2010) and the Bayesian persuasion game of Kamenica and Gentzkow (2011), a sender (he) has full flexibility in what information to disclose to a receiver (she) in order to persuade the receiver to take an action that is desirable to the sender. Kamenica and Gentzkow provide a tool to solve the sender's problem: first, characterize the sender's interim value as a function of the receiver's posterior belief, and then take the concave closure of the sender's interim value function.

In practice, deriving the sender's interim value function from the primitives of a persuasion problem is a non-trivial task which may require a closed form solution to the receiver's optimization strategy. The literature has mostly focused on tractable cases when either the receiver's action set is binary or when the optimal strategy of the receiver depends only on the posterior mean. Additionally, concavifying the sender's interim value function is often computationally complex, especially when the state space is a continuum (which would make the belief space infinite dimensional).

We depart from that approach and restrict the receiver's preferences to the class of payoffs that allows unambiguous Bayesian comparative statics (Theorem 1). We then characterize the conditions on the preferences of the sender that give maximal or minimal disclosure in two cases: when the sender is restricted to signals that satisfy Assumption 3 (information structures that generate posteriors that are first-order stochastically ordered), and when we allow for complete flexibility in disclosure policies.

Let the sender's payoff be given by $v : \Theta \times A \to \mathbb{R}$ which is continuous in a for all $\theta \in \Theta$. The receiver's payoff is given by $u : \Theta \times A \to \mathbb{R}$. Given a choice of information structure $\Sigma_\rho = \langle S, G(\cdot, \cdot; \rho)\rangle$, the receiver's optimal strategy is a mapping $a^\star(\rho) : S \to A$ given by $a^\star(s; \rho) = \arg\max_{a \in A} \int_\Theta u(\theta, a) dG_\Theta(\theta|s; \rho)$ for each $s \in S$. Thus, the sender's problem is given by

$$\max_{\Sigma_\rho} \; V(\rho) = \int\limits_{\Theta \times S} v(\theta, a^\star(s; \rho)) dG(\theta, s; \rho).$$

For the next result, we assume that the sender is restricted to $\mathcal{P}$, a set of information structures that satisfy Assumption 3. The no-information structure trivially satisfies

Assumption 3 and is therefore in $\mathcal{P}$. Additionally, since the state space $\Theta \subset \mathbb{R}$ is an ordered set, the full-information structure also satisfies Assumption 3 and is contained in $\mathcal{P}$.

Proposition 3 *Assume $v(\theta, a)$ has increasing differences (resp., decreasing differences) in $(\theta; a)$, and one of the following holds:*

i) $\Gamma \in \mathbf{\Gamma_{icx}}$ *and $v(\theta, a)$ is increasing and convex (resp., decreasing and concave) in a,*

ii) $\Gamma \in \mathbf{\Gamma_{dcx}}$ *and $v(\theta, a)$ is decreasing and convex (resp., increasing and concave) in a, or*

iii) $\Gamma \in \mathbf{\Gamma_{icx}} \cap \mathbf{\Gamma_{dcx}}$ *and $v(\theta, a)$ is convex (resp., concave) in a.*

If $\Sigma_\rho, \Sigma_{\rho'} \in \mathcal{P}$ with $\rho \succeq_{spm} \rho'$, then $V(\rho) \geq V(\rho')$ $\Big($resp., $V(\rho) \leq V(\rho')\Big)$.

Proof. Here, we provide a short proof when the sender's preferences are state-independent, i.e., $v(\theta, a) = \tilde{v}(a)$ for all $\theta \in \Theta$, where $\tilde{v}$ is a convex (resp., concave) function. The proof for state-dependent preferences is provided in the online appendix.

Given $\Sigma_\rho \in \mathcal{P}$, the sender's ex-ante utility is

$$V(\rho) = \int_S \tilde{v}(a^\star(s; \rho)) dG_S(s) = \int_{-\infty}^{\infty} \tilde{v}(z) dH(z; a^\star(\rho)).$$

The conclusions of Proposition 3 then follow by the definition of increasing/decreasing convex order and Theorem 1. ■

Proposition 3 provides sufficient conditions under which there is minimal and maximal conflict between a sender and a receiver: if their desire to correlate actions and states goes in the same (opposite) direction and the sender likes (dislikes) dispersion of the actions there will be maximal (minimal) disclosure.

We are not the first to study conditions under which there is maximal or minimal information disclosure in a persuasion game. In their seminal paper, Kamenica and Gentzkow (2011) show that the sender will disclose all (resp., no) information if the sender's interim value function is convex (resp., concave) in the receiver's posterior beliefs. Kolotilin (2018) and Dworczak and Martini (2019) use duality theory to derive conditions for maximal and minimal information disclosure when the sender's interim utility depends only on the posterior mean.[12] Mensch (2019) derives novel single-crossing conditions on what he calls the sender's "virtual utility" to characterize maximal and minimal information disclosure in environments with complementarities. While these conditions are general (these papers place

[12] Dworczak and Martini (2019) have an extension to the case where the sender's interim value function depends on the posterior beliefs, not just the posterior mean.

no restrictions on information structures), it is unclear what conditions on the primitives of a persuasion problem would imply the conditions on the sender's interim value function or the sender's virtual utility, which are endogenous functions. In contrast, our conditions are directly on the primitives.

Naturally, Assumption 3 places a restriction on the sender; with full flexibility, the sender could choose information structures that do not induce first-order ranked posteriors. Nonetheless, in the special case with only two states of the world, Proposition 3 implies full or no disclosure. The reason is that Assumption 3 is always satisfied when there are only two possible states, and any information structure is both dominated by the full-information structure and dominates the no-information structure in the supermodular stochastic order.

The next result holds over all possible information structures.

Theorem 2 *Assume $v(\theta, a)$ satisfies increasing differences in $(\theta; a)$, and suppose one of the following holds:*

i) $\Gamma \in \boldsymbol{\Gamma_{icx}}$ *and $v(\theta, a)$ is increasing and convex in a,*

ii) $\Gamma \in \boldsymbol{\Gamma_{dcx}}$ *and $v(\theta, a)$ is decreasing and convex in a, or*

iii) $\Gamma \in \boldsymbol{\Gamma_{icx}} \cap \boldsymbol{\Gamma_{dcx}}$ *and $v(\theta, a)$ is convex in a.*

Then full-information revelation is the optimal disclosure policy among all possible signals.

Theorem 2 follows from a similar reasoning as Corollary 1: the full information structure is Blackwell more informative than any other signal and trivially induces posteriors that satisfy Assumption 3 (because Θ is an ordered set). Thus, when the sender can use any information structure, Corollary 1 and the conditions in Theorem 2 imply that there is minimal conflict between the sender and the receiver, establishing the optimality of full disclosure.

Example 2 (Portfolio Agency Problem)
To illustrate the value in Proposition 3, consider the portfolio management problem in Rothschild and Stiglitz (1971). There is a risk-averse investor (the receiver) with a Bernoulli utility $\vartheta : \mathbb{R} \to \mathbb{R}$ which is continuous, strictly increasing, and strictly concave. There are two assets: money that yields a zero rate of return and stocks that yield a random rate of return of $\tilde{x}$. The random rate of return $\tilde{x}$ is distributed according to the CDF P_θ with support in $[\underline{x}, \bar{x}]$ where $\underline{x} < 0 < \bar{x}$. The state of the world θ captures the riskiness of stocks. In particular, for $\theta'' > \theta'$,

$$\int_{\underline{x}}^{z} x \big[dP_{\theta''}(x) - dP_{\theta'}(x)\big] \geq 0, \tag{RS}$$

for all $z \in [\underline{x}, \bar{x}]$, with equality when $z = \bar{x}$. Rothschild and Stiglitz show that all risk-averse agents invest more in a risky asset distributed according to $P_{\theta''}$ than $P_{\theta'}$ if, and only if, (RS) holds.

We augment this problem by adding a risk-neutral financial adviser (the sender). Ex-ante, neither the adviser nor the investor know the value of θ. The financial adviser chooses an information structure for investor, so that she can learn about θ before choosing how much of her wealth to invest in stocks. In exchange, the adviser gets a share $\pi \in (0, 1)$ of the investor's return on stocks. Hence, if the investor places a fraction $a \in [0, 1]$ of her wealth $W > 0$ in the risky asset, her ex-post payoff is

$$u(\theta, a) = \int_{\underline{x}}^{\bar{x}} \vartheta\Big(W\big(1 + ax(1 - \pi)\big)\Big) dP_\theta(x),$$

whereas the financial adviser's ex-post payoff is given by

$$v(\theta, a) = aW\pi \int_{\underline{x}}^{\bar{x}} x dP_\theta(x) = aW\pi\mathbb{E}[\tilde{x}].$$

What is the financial adviser's optimal disclosure policy? It is unlikely that the investor's optimal strategy is only a function of her posterior mean; it could depend on her moments of her posterior beliefs in rather complex ways. Thus, the example does not fit the simplifying assumptions often made in the persuasion literature.

Nonetheless, in our portfolio management example, (RS) implies that $u(\theta, a)$ has increasing differences in $(\theta; a)$, and that the financial adviser has a payoff $v(\theta, a)$ which is state-independent, linear, and increasing in a. Additionally, if the investor's Bernoulli utility satisfies the relative prudence condition

$$-\frac{\vartheta'''(x)}{\vartheta''(x)} x \geq 1,$$

then $\Gamma \in \boldsymbol{\Gamma_{icx}}$. Thus, by Proposition 3, the financial adviser prefers information structures that are ranked higher by the supermodular order. Additionally, by Theorem 2, full information revelation is the optimal persuasion policy over all information structures.

5.2. Games application: Information Acquisition and the Value of Transparency

Oligopolists are affected by many variables they cannot observe or estimate precisely: cost parameters, demand elasticity, etc. We can model the firms' process of gathering and learning about these pieces of information as a game of information acquisition.

In this section, we consider a game of information acquisition between an incumbent firm and a rival entrant. The entrant has access only to an exogenous "rudimentary" information structure whereas the incumbent can acquire more informative structures, possibly at a cost. We illustrate how investing in information differs from other types of classical investments, such as capacity, learning by doing, advertising, etc. (Bulow et al., 1985; Tirole, 1988).

We focus our analysis on entry accommodation.[13] When the incumbent acquires a higher quality of information, there are two effects: the first (direct) effect stems from the incumbent's ability to make better decisions using the acquired information, and the second (indirect) effect stems from the entrant's strategic response to the incumbent's information acquisition. The direct effect always plays a role and always increases the incumbent's profits. In contrast, the indirect effect plays a role only when information acquisition is overt, i.e., when the entrant observes the quality of information the incumbent acquires. As such, we call the second effect *the value of transparency*. We show that the value of transparency can either increase or decrease the incumbent's profits depending on (i) the responsiveness of the entrant, and (ii) the externality imposed on the incumbent by the entrant's responsiveness. More generally, we show that the value of transparency is useful in characterizing the differences in the value and the demand of information between overt and covert information acquisition games.

5.2.1. Setup

We consider a two-player Bayesian game composed of two stages: an information acquisition stage followed by a basic game $\Gamma = \langle N, \{A_i, u^i\}_{i\in N}, F\rangle$ where $N = \{1, 2\}$, the payoff u^i satisfies Assumption 2 for all $i \in N$, and the common prior F satisfies Assumption 1.

In the information acquisition stage, player 2 (the entrant) has an exogenously given information structure Σ_{ρ_2}. On the other hand, player 1 (the incumbent) is allowed to choose an information structure from a set $\mathcal{P}_1$ such that for any $\Sigma_{\rho_1} \in \mathcal{P}_1$, $\Sigma_\rho = (\Sigma_{\rho_1}, \Sigma_{\rho_2})$ satisfies Assumption 3. Additionally, we assume that for any two information structures $\Sigma_{\rho_1''}, \Sigma_{\rho_1'} \in \mathcal{P}_1$, either $\rho_1'' \succeq_{spm} \rho_1'$ or $\rho_1' \succeq_{spm} \rho_1''$. Let $\kappa : \mathcal{P}_1 \to \mathbb{R}$ be player 1's cost function with $\kappa(\rho_1)$ denoting the cost of acquiring Σ_{ρ_1}.

Throughout this section, we only consider information acquisition in pure strategies in the first stage.[14] We also assume that players coordinate on the maximal pure-strategy monotone BNE in the second stage.

[13] In the face of an entry threat, three kinds of behavior by the incumbent will be possible: entry might be blockaded, deterred or accommodated. See Tirole (1988) textbook.

[14] For overt information acquisition, this is without loss of generality as player 1 randomizes only when indifferent.

5.2.2. Covert versus Overt Information Acquisition

To better understand the difference between overt and covert information acquisition, suppose initially that player 1 is endowed with information structure $\Sigma_{\hat{\rho}_1}$ and this is common knowledge, i.e., both players know the Bayesian game is $\mathcal{G}_{\hat{\rho}} = (\Sigma_{\hat{\rho}_1}, \Sigma_{\rho_2}, \Gamma)$. Let $\big(a_1^\star(\hat{\rho}), a_2^\star(\hat{\rho})\big)$ be the resulting maximal BNE of $\mathcal{G}_{\hat{\rho}}$. Consider the following two scenarios as a thought experiment.

In the first scenario, player 1 switches from $\Sigma_{\hat{\rho}_1}$ to Σ_{ρ_1} and Player 2 observes the switch. This scenario corresponds to overt information acquisition game. The game changes from $\mathcal{G}_{\hat{\rho}}$ to $\mathcal{G}_\rho = (\Sigma_{\rho_1}, \Sigma_{\rho_2}, \Gamma)$ and the resulting maximal BNE is $\big(a_1^\star(\rho), a_2^\star(\rho)\big)$.

In the second scenario, player 1 again switches from $\Sigma_{\hat{\rho}_1}$ to Σ_{ρ_1} but player 2 is not aware that player 1 has switched. This scenario corresponds to covert information acquisition game. Player 2 naively believes that the game is still $\mathcal{G}_{\hat{\rho}}$ and continues to play $a_2^\star(\hat{\rho})$. On the other hand, player 1 best-replies to $a_2^\star(\hat{\rho})$ by playing the strategy $a_1^{BR}(a_2^\star(\hat{\rho}), \rho)$.

Since we wish to distinguish between player 1's actual choice of information and player 2's beliefs, we denote the actual outcome of the information acquisition stage by $\rho = (\rho_1, \rho_2)$ and player 2's belief of the outcome of the information acquisition stage by $\hat{\rho} = (\hat{\rho}_1, \rho_2)$. We say player 2 has correct beliefs when $\hat{\rho} = \rho$ (which must be the case in any equilibrium).

Given actual first stage outcome ρ and player 2's belief $\hat{\rho}$, let player 1's ex-ante payoff in the covert game (second scenario) be $U_1(\rho; \hat{\rho}) - \kappa(\rho_1)$ where

$$U_1(\rho; \hat{\rho}) = \int\limits_{\Theta \times S} u^1\big(\theta, a_1^{BR}(s_1; a_2^\star(\hat{\rho}), \rho), a_2^\star(s_2; \hat{\rho})\big) d\boldsymbol{G}(\theta, s; \rho).$$

In the overt game (first scenario), player 2 has correct beliefs. Hence, given actual first stage outcome ρ, player 1's payoff in the overt game is $U_1(\rho; \rho) - \kappa(\rho_1)$ with

$$\begin{aligned} U_1(\rho; \rho) &= \int\limits_{\Theta \times S} u^1\big(\theta, a_1^{BR}(s_1; a_2^\star(\rho), \rho), a_2^\star(s_2; \rho)\big) d\boldsymbol{G}(\theta, s; \rho) \\ &= \int\limits_{\Theta \times S} u^1\big(\theta, a_1^\star(s_1; \rho), a_2^\star(s_2; \rho)\big) d\boldsymbol{G}(\theta, s; \rho), \end{aligned}$$

where the equality follows from $a_1^{BR}(a_2^\star(\rho), \rho) = a_1^\star(\rho)$ by the definition of a BNE.

Definition 3 (Value of Transparency)
Given actual first stage outcome ρ and player 2's belief $\hat{\rho}$, the value of transparency is defined as $VT(\rho;\hat{\rho}) = U_1(\rho;\rho) - U_1(\rho;\hat{\rho})$.

In words, $VT(\rho;\hat{\rho})$ represents the gain/loss to player 1 from disclosing to player 2 her actual first stage choice, Σ_{ρ_1}, instead of letting player 2 believe that the first stage choice is $\Sigma_{\hat{\rho}_1}$. The value of transparency does not capture any direct substantive advantages of information; player 1's chosen information structure in both cases is Σ_{ρ_1}. Instead, it captures the indirect effects of information stemming from a change in player 2's beliefs and, therefore, her strategic response.[15]

5.2.3. Value and Demand for Information

Before we discuss how to characterize the value of transparency, we present why it is an interesting economic concept. In particular, we show that the value of transparency is helpful in answering the following questions: When is it beneficial for player 1 to *overtly* acquire a more informative structure at no cost? Does player 1 acquire a more informative structure when information acquisition is overt or when it is covert?

In covert games, information only has a direct effect, i.e., more informative structures allow player 1 to make better decisions in the second stage. Therefore, the value of costless information is never negative (Neyman, 1991).

While information has the same beneficial direct effect in overt games, there are also strategic effects; player 2 observes the quality of information acquired by player 1 and responds to it in the second stage. If player 2 finds it optimal to choose an unfavorable action (punish player 1) in the equilibrium of the second stage whenever player 1 acquires more information, then the value of information in overt games may be negative (Kamien et al., 1990). Nonetheless, we show that the value of *overt* information cannot be negative if player 1 benefits from disclosing to player 2 that a higher quality of information has been acquired.

Proposition 4 *For any two information structures $\Sigma_{\rho_1}, \Sigma_{\hat{\rho}_1} \in \mathcal{P}_1$, suppose $\rho_1 \succeq_{spm} \hat{\rho}_1$ implies $VT(\rho;\hat{\rho}) \geq 0$. Then $U_1(\rho;\rho) \geq U_1(\hat{\rho};\hat{\rho})$.*

[15] The value of transparency is connected to the *expectations conformity* conditions in Tirole (2015). Expectations conformity implies that player 1 is more willing to acquire Σ_{ρ_1} instead of $\Sigma_{\hat{\rho}_1}$ when player 2 believes that player 1 will acquire Σ_{ρ_1} instead of $\Sigma_{\hat{\rho}_1}$. Expectations conformity is equivalent to $VT(\rho;\hat{\rho}) + VT(\hat{\rho};\rho) \geq 0$.

Proof. For two information structures $\Sigma_{\rho_1}, \Sigma_{\hat{\rho}_1} \in \mathcal{P}_1$, we can write

$$U_1(\rho;\rho) - U_1(\hat{\rho};\hat{\rho}) = \underbrace{U_1(\rho;\rho) - U_1(\rho;\hat{\rho})}_{=VT(\rho;\hat{\rho})} + \underbrace{U_1(\rho;\hat{\rho}) - U_1(\hat{\rho};\hat{\rho})}_{\text{value of covert information}}.$$

Amir and Lazzati (2016) (Proposition 7) show that the second term is non-negative when $\rho_1 \succeq_{spm} \hat{\rho}_1$, i.e., the value of covert information is non-negative when quality of information increases in the supermodular order. Hence, if $VT(\rho;\hat{\rho}) \geq 0$, we can conclude that the value of overt information is also non-negative when quality of information increases. ■

To answer the question about the demand of information, let $\mathcal{P}_1^c, \mathcal{P}_1^o \subseteq \mathcal{P}_1$ denote the subsets of information structures player 1 acquires in a pure strategy Nash equilibrium (PSNE) of covert and overt games respectively.[16] Specifically, $\Sigma_{\rho_1^c} \in \mathcal{P}_1^c$ is a fixed point solution to

$$\max_{\Sigma_{\rho_1} \in \mathcal{P}_1} U_1(\rho;\rho^c) - \kappa(\rho_1).$$

In other words, given player 2 believes player 1 chooses $\Sigma_{\rho_1^c}$ in equilibrium, it is indeed optimal for player 1 to choose $\Sigma_{\rho_1^c}$.

In contrast, $\Sigma_{\rho_1^o} \in \mathcal{P}_1^o$ solves the optimization problem

$$\max_{\Sigma_{\rho_1} \in \mathcal{P}_1} U_1(\rho;\rho) - \kappa(\rho_1).$$

In other words, $\Sigma_{\rho_1^o}$ is optimal for player 1 after taking into account that player 2 will observe the chosen information structure in the first stage and will respond to it in the second stage.

We show that whenever the value of transparency is non-negative, player 1 acquires a higher quality of information in overt games than in covert games, regardless of the cost function.

Proposition 5 *For any two information structures $\Sigma_{\rho_1}, \Sigma_{\hat{\rho}_1} \in \mathcal{P}_1$, assume $VT(\rho;\hat{\rho}) \geq 0$ if, and only if, $\rho_1 \succeq_{spm} \hat{\rho}_1$. Then for any $\Sigma_{\rho_1^o} \in \mathcal{P}_1^o$ and $\Sigma_{\rho_1^c} \in \mathcal{P}_1^c$, $\rho_1^o \succeq_{spm} \rho_1^c$.*

[16] By definition, $\mathcal{P}_1$ endowed with $\succeq_{spm}$ relation is a lattice. A PSNE exists for the overt game as equilibrium is characterized by a straightforward optimization problem for player 1 over $\mathcal{P}_1$. A sufficient condition for the existence of PSNE for the covert game is that $U_1(\rho;\hat{\rho}) - \kappa(\rho_1)$ satisfies single-crossing in $(\rho_1;\hat{\rho}_1)$, i.e., given $\rho_1'' \succeq_{spm} \rho_1'$ and $\hat{\rho}_1'' \succeq_{spm} \hat{\rho}_1'$, $U_1(\rho'';\hat{\rho}') - \kappa(\rho_1'') \geq U_1(\rho';\hat{\rho}') - \kappa(\rho_1') \implies U_1(\rho'';\hat{\rho}'') - \kappa(\rho_1'') \geq U_1(\rho';\hat{\rho}'') - \kappa(\rho_1')$. For example, when κ is a constant function, the single-crossing condition is satisfied, and player 1 acquires the most informative structure in $\mathcal{P}_1$.

Proof. By definition, for any $\Sigma_{\rho_1^o} \in \mathcal{P}_1^o$ and $\Sigma_{\rho_1^c} \in \mathcal{P}_1^c$,

$$U_1(\rho^c;\rho^c) - \kappa(\rho_1^c) \geq U_1(\rho^o;\rho^c) - \kappa(\rho_1^o)$$
$$U_1(\rho^o;\rho^o) - \kappa(\rho_1^o) \geq U_1(\rho^c;\rho^c) - \kappa(\rho_1^c).$$

Combining the inequalities, we get $U_1(\rho^o;\rho^o) - U_1(\rho^o;\rho^c) = VT(\rho^o;\rho^c) \geq 0 \Leftrightarrow \rho_1^o \succeq_{spm} \rho_1^c$. ■

5.2.4. Characterizing the Value of Transparency

We now characterize the value of transparency which depends on the responsiveness of player 2 and the externality player 2's responsiveness imposes on player 1.

Theorem 3 *Suppose either the basic game Γ is one of independent private values, or $u^1(\theta, a)$ has increasing differences in $(\theta, a_1; a_2)$. Additionally, suppose one of the following holds:*

i. $\Gamma \in \boldsymbol{\Gamma_{icx}}$ *and* u^1 *is increasing and convex in* a_2*,*

ii. $\Gamma \in \boldsymbol{\Gamma_{dcx}}$ *and* u^1 *is decreasing and convex in* a_2*, or*

iii. $\Gamma \in \boldsymbol{\Gamma_{icx}} \cap \boldsymbol{\Gamma_{dcx}}$ *and* u^1 *is convex in* a_2*.*

For any two information structures $\Sigma_{\rho_1}, \Sigma_{\hat{\rho}_1} \in \mathcal{P}_1$, $VT(\rho;\hat{\rho}) \geq 0$ if, and only if, $\rho_1 \succeq_{spm} \hat{\rho}_1$.

For example, the canonical model of differentiated Bertrand competition with linear demand (Raith, 1996) satisfies the conditions of Theorem 3. Hence, applying Proposition 5, we can conclude that the demand for information is higher when information acquisition is overt.

To gain some intuition for Theorem 3, let us consider the simpler case with IPV. We can expand $VT(\rho;\hat{\rho}) = U_1(\rho;\rho) - U_1(\rho;\hat{\rho})$ into the expression

$$\int_{\Theta_1 \times S} \Big[u^1\big(\theta_1, a_1^\star(s_1;\rho), a_2^\star(s_2;\rho)\big) - u^1\big(\theta_1, a_1^{BR}(s_1; a_2^\star(\hat{\rho}), \rho), a_2^\star(s_2;\hat{\rho})\big)\Big] d\boldsymbol{G}(\theta_1, s; \rho)$$

$$= \int_{\Theta_1 \times S} \Big[u^1\big(\theta_1, a_1^\star(s_1;\rho), a_2^\star(s_2;\rho)\big) - u^1\big(\theta_1, a_1^{BR}(s_1; a_2^\star(\hat{\rho}), \rho), a_2^\star(s_2;\rho)\big)\Big] d\boldsymbol{G}(\theta_1, s; \rho)$$

$$+ \int_{\Theta_1 \times S} \Big[u^1\big(\theta_1, a_1^{BR}(s_1; a_2^\star(\hat{\rho}), \rho), a_2^\star(s_2;\rho)\big) - u^1\big(\theta_1, a_1^{BR}(s_1; a_2^\star(\hat{\rho}), \rho), a_2^\star(s_2;\hat{\rho})\big)\Big] d\boldsymbol{G}(\theta_1, s; \rho).$$

The first term is non-negative since $(a_1^\star(\rho), a_2^\star(\rho))$ is a BNE of $\mathcal{G}_\rho$. Using the IPV assumption, the second term can be expressed as

$$\int_{-\infty}^{\infty} \underbrace{\int_{\Theta_1 \times S_1} u^1\big(\theta_1, a_1^{BR}(s_1; a_2^\star(\hat{\rho}), \rho), z\big) dG(\theta_1, s_1; \rho_1)}_{\triangleq \zeta(z)} \Big[dH_2(z; a_2^\star(\rho)) - dH_2(z; a_2^\star(\hat{\rho}))\Big].$$

If $\Gamma \in \boldsymbol{\Gamma_{icx}}$ and $\rho_1 \succeq_{spm} \hat{\rho}_1$, we can conclude from Theorem 1 that $a_1^\star(\rho) \succeq_{icx} a_2^\star(\hat{\rho})$. Additionally, if u^1 is convex and increasing in a_2, so is $\zeta(\cdot)$. Thus, the second term is also non-negative (by the definition of the increasing convex order). Consequently, $VT(\rho; \hat{\rho}) \geq 0$ whenever $\rho_1 \succeq_{spm} \hat{\rho}_1$. Furthermore, if $\rho_1 \not\succeq_{spm} \hat{\rho}_1$, then by the definition of $\mathcal{P}_1$, $\hat{\rho}_1 \succeq_{spm} \rho_1$. Following a similar argument as above establishes that $VT(\hat{\rho}; \rho) \geq 0$.

More generally, the conditions in Theorem 3 connect the sign for the value of transparency to player 2's responsiveness ($a_2^\star(\rho)$ v.s. $a_2^\star(\hat{\rho})$), the type of externality player 2's action imposes on player 1 (the sign of $u^1_{a_2}$), and player 1's "risk" attitude towards player 2's action (the sign of $u^1_{a_2 a_2}$). For the independent private values case, Theorem 3 can be generalized into the taxonomy provided in Table 1. The first two columns describe how player 2 responds when the information structure changes from $\Sigma_{\hat{\rho}}$ to Σ_ρ. The next two columns are assumptions placed on player 1's utility function. The last column presents the resulting sign on the value of transparency. The first, third, and fifth rows of Table 1 correspond to condition *i*, *ii*, and *iii* of Theorem 3 respectively. For instance, the fifth row of Table 1 states that if a change from $\Sigma_{\hat{\rho}_1}$ to Σ_{ρ_1} leads to a mean-preserving spread in player 2's actions (*cst* stands for constant mean), and if player 1's utility is convex in a_2 (without any more restrictions on sign of $u^1_{a_2}$), then the value of transparency $VT(\rho; \hat{\rho})$ is non-negative.

5.2.5. Relation to Strategic Effects of Investment in Firm Competition

The characterization of the value of transparency is related to the taxonomy of strategic behavior in firm competition studied by Fudenberg and Tirole (1984), and Bulow et al. (1985). Here we follow the textbook treatment of Tirole (1988) and only consider the case of entry accommodation in a duopoly under complete information.

There are two periods: in the first period, the incumbent chooses an investment $K_1 \in \mathbb{R}$, which the entrant observes.[17] In the second period, both firms compete either in quantities (strategic substitutes) or prices (strategic complements). Let $\big(a_1^\star(K_1), a_2^\star(K_1)\big)$ be the resulting Nash equilibrium of the second period after the incumbent chose K_1 in

[17] The term investment is used in a broad sense and can represent, for example, investment in R&D that lowers the incumbent's marginal costs or advertising that captures a share of the market.

	$a_2(\rho)$ v.s. $a_2(\hat{\rho})$ responsiveness	mean	Externality sign($u^1_{a_2}$)	sign($u^1_{a_2 a_2}$)	Transparency $VT(\rho;\hat{\rho})$
i	$\nearrow$	$\nearrow$	$+$	$+$	$+$
ii	$\nearrow$	$\nearrow$	$-$	$-$	$-$
iii	$\nearrow$	$\searrow$	$-$	$+$	$+$
iv	$\nearrow$	$\searrow$	$+$	$-$	$-$
v	$\nearrow$	cst	$\cdot$	$+$	$+$
vi	$\nearrow$	cst	$\cdot$	$-$	$-$
vii	$\searrow$	$\nearrow$	$+$	$-$	$+$
$viii$	$\searrow$	$\nearrow$	$-$	$+$	$-$
ix	$\searrow$	$\searrow$	$-$	$-$	$+$
x	$\searrow$	$\searrow$	$+$	$+$	$-$
xi	$\searrow$	cst	$\cdot$	$-$	$+$
xii	$\searrow$	cst	$\cdot$	$+$	$-$

Table 1: A taxonomy of the value of transparency for independent private values.

the first period. The incumbent's payoff from choosing an investment level K_1 is given by $U_1\big(K_1, a_1^\star(K_1), a_2^\star(K_1)\big)$.

Fudenberg and Tirole (1984) show that the total marginal effect on the incumbent's payoff from increasing investment can be decomposed into

$$\frac{dU_1}{dK_1} = \underbrace{\underbrace{\frac{\partial U_1}{\partial K_1}}_{\text{direct effect}} + \underbrace{\frac{\partial U_1}{\partial a_1}\frac{da_1^\star}{dK_1}}_{\substack{=0\\ \text{by Envelope theorem}}}}_{\text{value of ``covert'' investment}} + \underbrace{\frac{\partial U_1}{\partial a_2}\frac{da_2^\star}{dK_1}}_{\text{strategic effect}}.$$

Increasing the level of investment has a direct effect on the incumbent's payoff, for example, by reducing the marginal cost. If the entrant was unable to observe the incumbent's investment choice, the direct effect would have been the only marginal effect. However, since the entrant observes the incumbent's choice of investment, there are also strategic effects stemming from the entrant's production/pricing decision as a function of K_1. This strategic effect depends on the entrant's equilibrium response, $\frac{da_2^\star}{dK_1}$, and the externality the entrant's actions impose on the incumbent's payoff, $\frac{\partial U^1}{\partial a_2}$.

In our model, the game is one of incomplete information and the investment corresponds to the quality of the incumbent's information structure ρ_1. The total effect of overtly increasing investment in information from $\Sigma_{\hat{\rho}_1}$ to Σ_{ρ_1} can be similarly decomposed into

$$U_1(\rho;\rho) - U_1(\hat{\rho};\hat{\rho}) = \underbrace{U_1(\rho;\hat{\rho}) - U_1(\hat{\rho};\hat{\rho})}_{\text{value of covert investment}} + \underbrace{U_1(\rho;\rho) - U_1(\rho;\hat{\rho})}_{\text{strategic effect}}.$$

The value of covert investment (value of covert information) captures how the incumbent's payoff increases with her ability to make better informed decisions while holding entrant's actions fixed. The strategic effect in our model corresponds to the value of transparency. It captures how the incumbent's payoff changes when the entrant's actions are allowed to depend on the incumbent's information quality. We have shown that the strategic effect of information depends on the entrant's responsiveness, the externality the entrant's action imposes on the incumbent, and additionally, the incumbent's "risk" attitude towards the entrant's actions. Our characterization of the value of transparency can hence be thought of as a stochastic extension of the strategic effects of investment by Fudenberg and Tirole (1984).

6. Conclusion

We provide a general framework to study how the distribution of equilibrium outcomes in Bayesian games and decision problems change when the quality of private information increases. Our theory of Bayesian comparative statics is comprised of three key components: an order over information structures (the supermodular order), a stochastic ordering of actions (the increasing/decreasing convex order), and a class of supermodular utility functions with supermodular/submodular and convex/concave marginal utilities. Our main theorem proves that for the class of utility functions we consider, there is a duality between the order of actions and the information order: equilibrium outcomes become more dispersed in the increasing/decreasing convex order if, and only if, the quality of information structures increases in the supermodular order.

The theory of Bayesian comparative statics could prove useful to generalize the insights from quadratic games to a richer class of payoffs. One avenue for future research is to study the efficient and equilibrium use of information in non linear-quadratic environments.[18] More generally, the framework can be applied in information design,[19] for example, studying the comparative statics of welfare and equilibrium outcomes with respect to the quality of public information, exogenous changes to the prior distribution of market fundamentals, and changes in attitudes towards risk or temporal resolution of uncertainty.

[18] Angeletos and Pavan (2007) study the efficient and equilibrium use of information in quadratic economies.
[19] See Bergemann and Morris (2018) for a recent survey on information design.

Appendix A. Preliminary Lemmas

This section contains results that we use to prove our main theorem. These are not our results!

Given a pure strategy $\alpha_i : S_i \to A_i$ for player $i \in N$, we defined the ex-ante distribution of actions by

$$H(z; \alpha_i) = \int_{S_i} \mathbf{1}_{[\alpha_i(s_i) \leq z]} dG_{S_i}(s_i)$$

for $z \in \mathbb{R}$. Define the quantile function $\hat{a}(\cdot; \alpha_i) : [0, 1] \to \mathbb{R}$ by

$$\hat{a}(q; \alpha_i) = \inf\{z : q \leq H(z; \alpha_i)\}$$

for $q \in (0, 1)$.

Lemma A.1 *[Theorem 4.A.2-A.3 of Shaked and Shanthikumar (2007)]*
Given two pure strategies α_i and α_i', the following are equivalent:

(*i*) $\alpha_i \succeq_{icx} \alpha_i'$.

(*ii*) *For all $x \in \mathbb{R}$,*

$$\int_x^\infty H(z; \alpha_i) dz \leq \int_x^\infty H(z; \alpha_i') dz.$$

(*iii*) *For all $t \in [0, 1]$,*

$$\int_t^1 \hat{a}(q; \alpha_i) dq \geq \int_t^1 \hat{a}(q; \alpha_i') dq.$$

Similarly, the following are equivalent:

(*iv*) $\alpha_i \succeq_{dcx} \alpha_i'$.

(*v*) *For all $x \in \mathbb{R}$,*

$$\int_{-\infty}^x H(z; \alpha_i) dz \geq \int_{-\infty}^x H(z; \alpha_i') dz.$$

(*vi*) *For all $t \in [0, 1]$,*

$$\int_0^t \hat{a}(q; \alpha_i) dq \leq \int_0^t \hat{a}(q; \alpha_i') dq.$$

Lemma A.2 *[Theorem 3.8.2 of Müller and Stoyan (2002) or Tchen (1980)]*
Given two information structures Σ_{ρ_i} and $\Sigma_{\rho'_i}$, $\rho_i \succeq_{spm} \rho'_i$ if, and only if, for all integrable functions $\psi : \Theta_i \times S_i \to \mathbb{R}$ that satisfy increasing differences (ID) in $(\theta_i; s_i)$,

$$\int_{\Theta_i \times S_i} \psi(\theta_i, s_i) dG(\theta_i, s_i; \rho_i) \geq \int_{\Theta_i \times S_i} \psi(\theta_i, s_i) dG(\theta_i, s_i; \rho'_i).$$

Lemma A.3 *[Quah and Strulovici (2009)]*
Let $g : [\underline{x}, \bar{x}] \to \mathbb{R}$ and $h : [\underline{x}, \bar{x}] \to \mathbb{R}$ be integrable functions.

1. *If g is increasing and $\int_z^{\bar{x}} h(x)dx \geq 0$ for all $z \in [\underline{x}, \bar{x}]$, then $\int_{\underline{x}}^{\bar{x}} g(x)h(x)dx \geq g(\underline{x}) \int_{\underline{x}}^{\bar{x}} h(x)dx$.*
2. *If g is decreasing and $\int_{\underline{x}}^{z} h(x)dx \geq 0$ for all $z \in [\underline{x}, \bar{x}]$, then $\int_{\underline{x}}^{\bar{x}} g(x)h(x)dx \geq g(\bar{x}) \int_{\underline{x}}^{\bar{x}} h(x)dx$.*

Appendix B. Proof of Theorem 1

Proof. ($\Longrightarrow$) We only prove the case for $\boldsymbol{\Gamma_{icx}}$. A symmetric argument establishes the result for the case for $\boldsymbol{\Gamma_{dcx}}$. Without loss of generality, let the marginal G_{S_i} be the uniform distribution on the unit interval for each $i \in N$ (see Footnote 6).

Fix a basic game $\Gamma \in \boldsymbol{\Gamma_{icx}}$. For each player $i \in N$, let $\alpha_i : S_i \to A_i$ be an arbitrary measurable and monotone strategy. Since α_i is monotone, it is almost everywhere equal to its quantile function, i.e., $\alpha_i(s_i) = \hat{a}(s_i; \alpha_i)$ for almost all $s_i \in [0, 1] = S_i$. Given two monotone strategies α_i and α'_i, from Lemma A.1, $\alpha_i \succeq_{icx} \alpha'_i$ if, and only if,

$$\int_t^1 \alpha_i(s) ds_i \geq \int_t^1 \alpha'_i(s_i) ds_i$$

for all $t \in [0, 1]$.

Let $\mathcal{A}_i$ be the set of all monotone and measurable strategies and let $\mathcal{A} = \times_{i \in N} \mathcal{A}_i$. Given a profile of information structures $\Sigma_\rho = (\Sigma_{\rho_1}, \ldots, \Sigma_{\rho_n})$ and opponents' strategies $\alpha_{-i} \in \mathcal{A}_{-i}$, let $a_i^{BR}(\cdot; \alpha_{-i}, \rho) : S_i \to A_i$ be player i's best response strategy. Specifically, for all $s_i \in S_i$,

$$a_i^{BR}(s_i; \alpha_{-i}, \rho) = \underset{a_i \in A_i}{\arg\max} \int_{\Theta \times S_{-i}} u^i\Big(\theta, \alpha_{-i}(s_{-i}), a_i\Big) d\boldsymbol{G}(\theta, s_{-i} | s_i; \rho).$$

Using Assumption 1-Assumption 4 and monotone comparative statics of Bayesian supermodular games (Van Zandt and Vives, 2007), $a_i^{BR}(\cdot; \alpha_{-i}, \rho) \in \mathcal{A}_i$ for all $i \in N$.

For any given profile of monotone strategies $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{A}$, denote the profile of best-response strategies by $a^{BR}(\alpha, \rho) = \big(a_1^{BR}(\cdot; \alpha_{-1}, \rho), \ldots, a_n^{BR}(\cdot; \alpha_{-n}, \rho)\big)$. Then, a monotone BNE $a^{\star}(\rho)$ of a Bayesian game $\mathcal{G}_\rho = (\Sigma_\rho, \Gamma)$ is given by the fixed point $a^{BR}(a^{\star}(\rho), \rho) = a^{\star}(\rho)$.

The proof proceeds in four steps:

1. Player i's best response strategy increases in the increasing convex order when player i's information quality increases in the supermodular order (Lemma B.1). This concludes the proof if $n = 1$.

2. For all $j \in N \backslash \{i\}$, player i's best response strategy increases in the increasing convex order when player j's information quality increases in the supermodular order (Lemma B.2).

3. For all $j \in N \backslash \{i\}$, player i's best response strategy increases in the increasing convex order when player j's strategy increases in the increasing convex order (Lemma B.3).

4. Given 1-3, apply comparative statics on fixed points to get desired result.

Lemma B.1 *Fix some $i \in N$ and some monotone strategy $\alpha_{-i} \in \mathcal{A}_{-i}$. Take two profiles of information structures $\Sigma_{\rho''} = (\Sigma_{\rho_i''}, \Sigma_{\rho_{-i}})$ and $\Sigma_{\rho'} = (\Sigma_{\rho_i'}, \Sigma_{\rho_{-i}})$. If $\rho_i'' \succeq_{spm} \rho_i'$, then $a_i^{BR}(\cdot; \alpha_{-i}, \rho'') \succeq_{icx} a_i^{BR}(\cdot; \alpha_{-i}, \rho_i')$.*

Proof. To economize on notation, we suppress the dependence of $a_i^{BR}(\cdot; \alpha_{-i}, \rho_i')$ on α_{-i}. For any signal realization $s_i \in S_i$, the first order conditions imply that

$$\int_{\Theta \times S_{-i}} \Big[u^i_{a_i}\big(\theta, \alpha_{-i}(s_{-i}), a_i^{BR}(s_i; \rho'')\big) - u^i_{a_i}\big(\theta, \alpha_{-i}(s_{-i}), a_i^{BR}(s_i; \rho')\big)\Big] d\boldsymbol{G}(\theta, s_{-i}|s_i; \rho'')$$

$$+ \int_{\Theta \times S_{-i}} u^i_{a_i}\big(\theta, \alpha_{-i}(s_{-i}), a_i^{BR}(s_i; \rho')\big) \Big[d\boldsymbol{G}(\theta, s_{-i}|s_i; \rho'') - d\boldsymbol{G}(\theta, s_{-i}|s_i; \rho')\Big] = 0$$

As $\Gamma \in \boldsymbol{\Gamma_{icx}}$, $u^i_{a_i}(\theta, a)$ is convex in a_i for all $(\theta, a_{-i}) \in \Theta \times A_{-i}$. Thus,

$$\begin{aligned} &u^i_{a_i}\big(\theta, \alpha_{-i}(s_{-i}), a_i^{BR}(s_i; \rho'')\big) - u^i_{a_i}\big(\theta, \alpha_{-i}(s_{-i}), a_i^{BR}(s_i; \rho')\big) \\ \geq &u^i_{a_i a_i}\big(\theta, \alpha_{-i}(s_{-i}), a_i^{BR}(s_i; \rho')\big)\Big(a_i^{BR}(s_i; \rho'') - a_i^{BR}(s_i; \rho')\Big), \end{aligned}$$

and for each $t \in [0,1]$,

$$\int_t^1 \Big(a_i^{BR}(s_i;\rho') - a_i^{BR}(s_i;\rho'')\Big) ds_i$$
$$\leq \int_t^1 B(s_i) \int\limits_{\Theta\times S_{-i}} u^i_{a_i}\big(\theta, \alpha_{-i}(s_{-i}), a_i^{BR}(s_i;\rho')\big) \Big[d\boldsymbol{G}(\theta, s_{-i}|s_i;\rho') - d\boldsymbol{G}(\theta, s_{-i}|s_i;\rho'')\Big] ds_i$$
$$= \int\limits_{\Theta_i\times S_i} \mathbf{1}_{[s_i\geq t]} B(s_i) \underbrace{\int\limits_{\Theta_{-i}\times S_{-i}} u^i_{a_i}\big(\theta, \alpha_{-i}(s_{-i}), a_i^{BR}(s_i;\rho')\big) d\boldsymbol{G}(\theta_{-i}, s_{-i}|\theta_i;\rho_{-i})}_{\triangleq D(\theta_i, s_i)} \Big[dG(\theta_i, s_i;\rho_i') - dG(\theta_i, s_i;\rho_i'')\Big]$$

where the last equality follows from Assumption 4, and

$$B(s_i) = \left(- \int_{\Theta\times S_{-i}} u^i_{a_i a_i}\big(\theta, \alpha_{-i}(s_{-i}), a_i^{BR}(s_i;\rho')\big) d\boldsymbol{G}(\theta, s_{-i}|s_i,\rho'') \right)^{-1}.$$

Note that $B(s_i) > 0$ by the concavity of u^i in a_i. Additionally, it is an increasing function because $-u^i_{a_i}$ satisfies decreasing differences in $(\theta, a_{-i}; a_i)$, is concave in a_i, and $\boldsymbol{G}(\tilde{\theta}, \tilde{s}_{-i}|s_i'';\rho'') \succeq_{FOSD} \boldsymbol{G}(\tilde{\theta}, \tilde{s}_{-i}|s_i';\rho'')$ for $s_i'' > s_i'$ by Assumption 1, Assumption 3, and Assumption 4.

For any $\theta_i'' > \theta_i'$,

$$D(\theta_i'', s_i) = \int\limits_{\Theta_{-i}\times S_{-i}} u^i_{a_i}\big(\theta_i'', \theta_{-i}, \alpha_{-i}(s_{-i}), a_i^{BR}(s_i;\rho')\big) d\boldsymbol{G}(\theta_{-i}, s_{-i}|\theta_i'';\rho_{-i})$$
$$\geq \int\limits_{\Theta_{-i}\times S_{-i}} u^i_{a_i}\big(\theta_i', \theta_{-i}, \alpha_{-i}(s_{-i}), a_i^{BR}(s_i;\rho')\big) d\boldsymbol{G}(\theta_{-i}, s_{-i}|\theta_i'';\rho_{-i})$$
$$\geq \int\limits_{\Theta_{-i}\times S_{-i}} u^i_{a_i}\big(\theta_i', \theta_{-i}, \alpha_{-i}(s_{-i}), a_i^{BR}(s_i;\rho')\big) d\boldsymbol{G}(\theta_{-i}, s_{-i}|\theta_i';\rho_{-i}) = D(\theta_i', s_i),$$

where the first inequality follows because u^i has ID in $(\theta_i; a_i)$, and the second inequality follows because u^i has ID in $(\theta_{-i}, a_{-i}; a_i)$ and $\boldsymbol{G}(\tilde{\theta}_{-i}, \tilde{s}_{-i}|\theta_i'';\rho_{-i}) \succeq_{FOSD} \boldsymbol{G}(\tilde{\theta}_{-i}, \tilde{s}_{-i}|\theta_i';\rho_{-i})$

by Assumption 1, Assumption 3, and Assumption 4. Moreover,

$$\begin{aligned}
&D(\theta_i'', s_i) - D(\theta_i', s_i) \\
&= \int\limits_{\Theta_{-i}\times S_{-i}} \Big[u^i_{a_i}\big(\theta_i'', \theta_{-i}, \alpha_{-i}(s_{-i}), a_i^{BR}(s_i;\rho')\big) - u^i_{a_i}\big(\theta_i', \theta_{-i}, \alpha_{-i}(s_{-i}), a_i^{BR}(s_i;\rho')\big)\Big] d\boldsymbol{G}(\theta_{-i}, s_{-i}|\theta_i''; \rho_{-i}) \\
&+ \int\limits_{\Theta_{-i}\times S_{-i}} u^i_{a_i}\big(\theta_i', \theta_{-i}, \alpha_{-i}(s_{-i}), a_i^{BR}(s_i;\rho')\big)\Big[d\boldsymbol{G}(\theta_{-i}, s_{-i}|\theta_i''; \rho_{-i}) - d\boldsymbol{G}(\theta_{-i}, s_{-i}|\theta_i'; \rho_{-i})\Big]
\end{aligned}$$

is an increasing function of s_i. To see this, notice that the first term is increasing in s_i as $u^i_{a_i}$ has ID in $(\theta_i; a_i)$ and a_i^{BR} is monotone in s_i. The second term is also increasing in s_i as $u^i_{a_i}$ satisfies ID in $(\theta_{-i}, s_{-i}; a_i)$, a_i^{BR} is monotone in s_i, and $\boldsymbol{G}(\tilde{\theta}_{-i}, \tilde{s}_{-i}|\theta_i''; \rho_{-i}) \succeq_{FOSD} \boldsymbol{G}(\tilde{\theta}_{-i}, \tilde{s}_{-i}|\theta_i'; \rho_{-i})$.

We can therefore conclude that the function $\psi(\theta_i, s_i; t) = \mathbf{1}_{[s_i \geq t]} B(s_i) D(\theta_i, s_i)$ satisfies ID in $(\theta_i; s_i)$ for any $t \in [0,1]$. Hence, for each $t \in [0,1]$,

$$\int_t^1 \Big(a_i^{BR}(s_i;\rho') - a_i^{BR}(s_i;\rho'')\Big) ds_i \leq \int\limits_{\Theta_i\times S_i} \psi(\theta_i, s_i; t)\Big[dG(\theta_i, s_i; \rho_i') - dG(\theta_i, s_i; \rho_i'')\Big] \leq 0$$

where the last inequality follows from Lemma A.2. Thus, $a_i^{BR}(\cdot;\alpha_{-i},\rho'') \succeq_{icx} a_i^{BR}(\cdot;\alpha_{-i},\rho')$.
■

Lemma B.2 *Fix some $i \in N$ and some monotone strategy $\alpha_{-i} \in \mathcal{A}_{-i}$. Take two profiles of information structures $\Sigma_{\rho''} = (\Sigma_{\rho_j''}, \Sigma_{\rho_{-j}})$ and $\Sigma_{\rho'} = (\Sigma_{\rho_j'}, \Sigma_{\rho_{-j}})$ for some $j \neq i$. If $\rho_j'' \succeq_{spm} \rho_j'$, then $a_i^{BR}(\cdot;\alpha_{-i},\rho'') \succeq_{icx} a_i^{BR}(\cdot;\alpha_{-i},\rho')$.*

Proof. Once again, we suppress the dependence of $a_i^{BR}(\cdot;\alpha_{-i},\rho_i')$ on α_{-i} to economize on notation. Following the same first order condition argument we used in the proof of Lemma B.1, for each $t \in [0,1]$,

$$\int_t^1 \Big(a_i^{BR}(s_i;\rho') - a_i^{BR}(s_i;\rho'')\Big) ds_i$$

$$\leq \int\limits_{\Theta\times S} \mathbf{1}_{[s_i\geq t]} B(s_i) u^i_{a_i}\big(\theta, \alpha_{-i}(s_{-i}), a_i^{BR}(s_i;\rho')\big)\Big[d\boldsymbol{G}(\theta,s;\rho') - d\boldsymbol{G}(\theta,s;\rho'')\Big]$$

$$= \int\limits_{\Theta_j\times S_j} \underbrace{\int\limits_{\Theta_{-j}\times S_{-j}} \mathbf{1}_{[s_i\geq t]} B(s_i) u^i_{a_i}\big(\theta, \alpha_{-i}(s_{-i}), a_i^{BR}(s_i;\rho')\big) d\boldsymbol{G}(\theta_{-j}, s_{-j}|\theta_j;\rho_{-j})}_{\triangleq \hat{\psi}(\theta_j,s_j;t)} \Big[dG(\theta_j,s_j;\rho'_j) - dG(\theta_j,s_j;\rho''_j)\Big]$$

where $B(s_i)$ is as defined in the proof of Lemma B.1.

Given any $t \in [0,1]$ and any $s''_j > s'_j$, $\hat{\psi}(\theta_j, s''_j; t) - \hat{\psi}(\theta_j, s'_j; t)$

$$= \int\limits_{\Theta_{-j}\times S_{-j}} \Big\{ u^i_{a_i}\big(\theta, \alpha_{-i,j}(s_{-i,j}), \alpha_j(s''_j), a_i^{BR}(s_i;\rho')\big) - u^i_{a_i}\big(\theta, \alpha_{-i,j}(s_{-i,j}), \alpha_j(s'_j), a_i^{BR}(s_i;\rho')\big)\Big\}$$

$$\times \mathbf{1}_{[s_i\geq t]} B(s_i) d\boldsymbol{G}(\theta_{-j}, s_{-j}|\theta_j;\rho_{-j}),$$

is an increasing function of θ_j. To see why, notice that the integrand is increasing in (θ_{-j}, s_{-j}) because $u^i_{a_i}$ has ID in $(\theta_{-j}, a_{-j}; a_j)$, $\boldsymbol{G}(\tilde{\theta}_{-j}, \tilde{s}_{-j}|\theta''_j;\rho_{-j}) \succeq_{FOSD} \boldsymbol{G}(\tilde{\theta}_{-j}, \tilde{s}_{-j}|\theta'_j;\rho_{-j})$ whenever $\theta''_j > \theta'_j$ (by Assumption 1, Assumption 3, and Assumption 4), and actions are monotone in signal realizations. Additionally,

$$u^i_{a_i}\big(\theta, \alpha_{-i,j}(s_{-i,j}), \alpha_j(s''_j), a_i^{BR}(s_i;\rho')\big) - u^i_{a_i}\big(\theta, \alpha_{-i,j}(s_{-i,j}), \alpha_j(s'_j), a_i^{BR}(s_i;\rho')\big)$$

is increasing in θ_j because $u^i_{a_i}$ has ID in $(\theta_j; a_j)$ and α_j is monotone in s_j.

Thus, for any $t \in [0,1]$, $\hat{\psi}(\theta_j, s_j; t)$ has ID in $(\theta_j; s_j)$. By Lemma A.2, $\rho''_j \succeq_{spm} \rho'_j$ implies

$$\int_t^1 \Big(a_i^{BR}(s_i;\rho') - a_i^{BR}(s_i;\rho'')\Big) ds_i \leq \int_{\Theta_j\times S_j} \hat{\psi}(\theta_j,s_j;t)\Big[dG(\theta_j,s_j;\rho'_j) - dG(\theta_j,s_j;\rho''_j)\Big] \leq 0$$

for all $t \in [0,1]$. Thus, $a_i^{BR}(\cdot;\alpha_{-i},\rho'') \succeq_{icx} a_i^{BR}(\cdot;\alpha_{-i},\rho')$. ■

Lemma B.3 *Fix $i, j \in N$ with $j \neq i$, a monotone strategy $\alpha_{-i,j} \in \mathcal{A}_{-i,j}$, and an information structures Σ_ρ. For $\alpha''_j, \alpha'_j \in \mathcal{A}_j$ such that $\alpha''_j \succeq_{icx} \alpha'_j$, $a_i^{BR}(\cdot;\alpha''_{-i},\rho) \succeq_{icx} a_i^{BR}(\cdot;\alpha'_{-i},\rho)$.*

Proof. We suppress the dependence on Σ_ρ as it is held fixed. For any $t \in [0,1]$, we use the first order conditions argument to get the expression

$$\int_t^1 \big(a_i^{BR}(s_i;\alpha'_{-i}) - a_i^{BR}(s_i;\alpha''_{-i})\big)ds_i$$

$$\leq \int\limits_{\Theta\times S} \mathbf{1}_{[s_i\geq t]}\tilde{B}_i(s_i)\left[u^i_{a_i}\Big(\theta,\alpha'_{-i}(s_{-i}),a_i^{BR}(s_i;\alpha'_{-i})\Big) - u^i_{a_i}\Big(\theta,\alpha''_{-i}(s_{-i}),a_i^{BR}(s_i;\alpha'_{-i})\Big)\right]d\boldsymbol{G}(\theta,s),$$

where

$$\tilde{B}(s_i) = \left(-\int\limits_{\Theta\times S_{-i}} u^i_{a_ia_i}\Big(\theta,\alpha''_{-i}(s_{-i}),a_i^{BR}(s_i;\alpha'_{-i})\Big)d\boldsymbol{G}(\theta,s_{-i}|s_i)\right)^{-1}.$$

By convexity of $u^i_{a_i}$ in a_j,

$$u^i_{a_i}\Big(\theta,\alpha'_{-i}(s_{-i}),a_i^{BR}(s_i;\alpha'_{-i})\Big) - u^i_{a_i}\Big(\theta,\alpha''_{-i}(s_{-i}),a_i^{BR}(s_i;\alpha'_{-i})\Big)$$

$$\leq u^i_{a_ia_j}\Big(\theta,\alpha'_{-i}(s_{-i}),a_i^{BR}(s_i;\alpha'_{-i})\Big)\big(\alpha'_j(s_j)-\alpha''_j(s_j)\big).$$

Thus,

$$\int_t^1 \big(a_i^{BR}(s_i;\alpha'_{-i}) - a_i^{BR}(s_i;\alpha''_{-i})\big)ds_i$$

$$\leq \int\limits_{S_j}\big(\alpha'_j(s_j)-\alpha''_j(s_j)\big)\int\limits_{\Theta\times S_{-j}} \mathbf{1}_{[s_i\geq t]}\tilde{B}(s_i)u^i_{a_ia_j}\Big(\theta,\alpha'_{-i}(s_{-i}),a_i^{BR}(s_i;\alpha'_{-i})\Big)d\boldsymbol{G}(\theta,s_{-j}|s_j)ds_j.$$

As $\alpha''_j,\alpha'_j \in \mathcal{A}_j$, $\alpha''_j \succeq_{icx} \alpha'_j$ if, and only if,

$$\int_t^1 \big(\alpha'_j(s_j)-\alpha''_j(s_j)\big)ds_j \leq 0,\ \ \forall t\in[0,1].$$

Furthermore, for each $s_j \in [0,1]$

$$\mathbf{1}_{[s_i\geq t]}\tilde{B}(s_i)u^i_{a_ia_j}\Big(\theta,\alpha'_{-i}(s_{-i}),a_i^{BR}(s_i;\alpha'_{-i})\Big) \geq 0,\ \forall(\theta,s_{-j})\in\Theta\times S_{-j}$$

as u^i has ID in $(a_j;a_i)$. It is also increasing in (θ,s_{-j}) because $u^i_{a_i}$ has ID in $(\theta,a_{-j};a_j)$ and $\tilde{B}(s_i)$ is positive and increasing in s_i for the same reason as $B(s_i)$ in Lemma B.1. Since $\boldsymbol{G}(\theta,s_{-j}|s''_j) \succeq_{FOSD} \boldsymbol{G}(\theta,s_{-j}|s'_j)$ whenever $s''_j > s'_j$ (by Assumption 1, Assumption 3, and

Assumption 4),

$$\int_{\Theta\times S_{-j}} \mathbf{1}_{[s_i\geq t]}\tilde{B}(s_i)u^i_{a_ia_j}\Big(\theta,\alpha'_{-i}(s_{-i}),a_i^{BR}(s_i;\alpha'_{-i})\Big)d\boldsymbol{G}(\theta,s_{-j}|s_j)$$

is an increasing function of s_j. Applying Lemma A.3, we have

$$\begin{aligned}
&\int_t^1 \big(a_i^{BR}(s_i;\alpha'_{-i}) - a_i^{BR}(s_i;\alpha''_{-i})\big)ds_i \\
\leq &\int_{S_j} \big(\alpha'_j(s_j)-\alpha''_j(s_j)\big) \int_{\Theta\times S_{-j}} \mathbf{1}_{[s_i\geq t]}\tilde{B}(s_i)u^i_{a_ia_j}\Big(\theta,\alpha'_{-i}(s_{-i}),a_i^{BR}(s_i;\alpha'_{-i})\Big)d\boldsymbol{G}(\theta,s_{-j}|s_j)ds_j \\
\leq &\underbrace{\int_{S_j} \big(\alpha'_j(s_j)-\alpha''_j(s_j)\big)ds_j}_{\leq 0} \int_{\Theta\times S_{-j}} \underbrace{\mathbf{1}_{[s_i\geq t]}\tilde{B}(s_i)u^i_{a_ia_j}\Big(\theta,\alpha'_{-i}(s_{-i}),a_i^{BR}(s_i;\alpha'_{-i})\Big)}_{\geq 0}d\boldsymbol{G}(\theta,s_{-j}|s_j=0) \\
\leq &0
\end{aligned}$$

for each $t\in[0,1]$. Thus, $a_i^{BR}(\cdot;\alpha''_{-i},\rho)\succeq_{icx} a_i^{BR}(\cdot;\alpha'_{-i},\rho)$. ■

We now tackle the last step in the "if" part of the proof: comparative statics of the BNEs. We apply the comparative statics of fixed points from Villas-Boas (1997). To do so, we will need the following definition.

Definition B.1 (Contractible Space) *Let X be a topological space. We say that X is a contractible space if there exists some $x^*\in X$ and a map $\Phi: X\times[0,1]\to X$ such that*

1. *$\Phi(\cdot,\lambda)$ is continuous in λ, and*
2. *For all $x\in X$, $\Phi(x,0)=x$ and $\Phi(x,1)=x^*$.*

Intuitively, X is a contractible space if it can be continuously shrunk into a point inside itself.

Theorem 6 & 7; Villas-Boas (1997) Let X be a compact subset of a Banach space. Consider continuous mappings $T_1 : X \to X$ and $T_2 : X \to X$, and a transitive and reflexive order $\succeq$ on X. For all $x \in X$, let the upper-sets $\mathcal{U}(x) = \{x' \in X : x' \succeq x\}$ and lower-sets $\mathcal{L}(x) = \{x' \in X : x \succeq x'\}$ be compact and contractible subsets of X. Let both T_1 and T_2 have a fixed point on X.

A. Suppose $x' \succeq x \Rightarrow T_1(x') \succeq T_1(x)$, and suppose $T_1(x) \succeq T_2(x)$ for all $x \in X$. Then for every fixed point $x_2^\star$ of T_2, there is a fixed point $x_1^\star$ of T_1 such that $x_1^\star \succeq x_2^\star$.

B. Suppose $x' \succeq x \Rightarrow T_2(x') \succeq T_2(x)$, and suppose $T_1(x) \succeq T_2(x)$ for all $x \in X$. Then for every fixed point $x_1^\star$ of T_1, there is a fixed point $x_2^\star$ of T_2 such that $x_1^\star \succeq x_2^\star$.

The remaining few steps prove that our setting satisfies the assumptions needed to apply the Villas-Boas result.

Let $BV([0,1],\mathbb{R})$ be the space of functions of bounded variation from $[0,1]$ to $\mathbb{R}$. Given a function $g \in BV([0,1],\mathbb{R})$, let $V(g)$ be the total variation of g given by

$$V(g) = \sup_{p \in P} \sum_{i=0}^{n_p - 1} |g(x_{i+1}) - g(x_i)|$$

where P is the set of all partitions $p = \{x_0, x_1, \ldots, x_{n_p}\}$ on $[0,1]$. Define the bounded variation norm by $||g||_{BV} = \int_0^1 |g(s)|ds + V(g)$. The space $BV([0,1],\mathbb{R})$ equipped with the $||\cdot||_{BV}$ norm is a Banach space.

Lemma B.4 *For each $i \in N$, $\mathcal{A}_i$ is a compact subset of $\big(BV([0,1],\mathbb{R}), ||\cdot||_{BV}\big)$.*

Proof. Any $\alpha_i \in \mathcal{A}_i$ is of bounded variation as it is an increasing function. Therefore, $\mathcal{A}_i$ is a subset of $BV([0,1],\mathbb{R})$. To show that $\mathcal{A}_i$ is a compact subset $BV([0,1],\mathbb{R})$, take a sequence $\{\tilde{\alpha}_{i,k}\}_{k=1}^{\infty} \in \mathcal{A}_i$. The sequence is uniformly bounded as the image of each $\alpha_{i,k}$ is a subset of the compact interval A_i. By Helly's Selection Theorem, the sequence converges to an increasing function $\tilde{\alpha}_i \in BV([0,1],\mathbb{R})$.

Furthermore, as $\underline{a}_i \leq \tilde{\alpha}_{i,k}(0)$ for all k, the limit also satisfies $\underline{a}_i \leq \tilde{\alpha}_i(0)$. Similarly, as $\bar{a}_i \geq \tilde{\alpha}_{i,k}(1)$ for all k, the limit also satisfies $\bar{a}_i \geq \tilde{\alpha}_i(1)$. Finally, the point-wise limit of measurable functions is measurable (Corollary 8.9, Measure, Integrals, and Martingales, Schilling (2005)). As $\tilde{\alpha}_i$ is a monotone and measurable function that maps from $[0,1]$ to A_i, $\tilde{a}_i \in \mathcal{A}_i$. Thus, $\mathcal{A}_i$ is sequentially compact for each $i \in N$. ■

Let $\mathcal{U}(\alpha_i) = \{\alpha_i' \in \mathcal{A}_i : \alpha_i' \succeq_{icx} \alpha_i\}$ and $\mathcal{L}(\alpha_i) = \{\alpha_i' \in \mathcal{A}_i : \alpha_i \succeq_{icx} \alpha_i'\}$ be the upper and lower-sets of $\mathcal{A}_i$ respectively.

Lemma B.5 *For each $i \in N$ and for any $\alpha_i \in \mathcal{A}_i$, $\mathcal{U}(\alpha_i)$ and $\mathcal{L}(\alpha_i)$ are compact and contractible.*

Proof. For a given $\alpha_i \in \mathcal{A}_i$, $\mathcal{U}(\alpha_i)$ and $\mathcal{L}(\alpha_i)$ are closed subsets of $\mathcal{A}_i$ (follows from the dominated convergence Theorem). Hence, they are compact. Let $\alpha_i^u : [0,1] \to A_i$ be a constant function with $\alpha_i^u(s_i) = \bar{a}_i$ for all $s_i \in [0,1]$. Note that $\alpha_i^u \in \mathcal{A}_i$. Furthermore, $\alpha_i^u(s_i) \geq \alpha_i(s_i)$, $\forall s_i \in [0,1]$ which implies $\alpha_i^u \succeq_{icx} \alpha_i \Rightarrow \alpha_i^u \in \mathcal{U}(\alpha_i)$.

For each $\alpha_i \in \mathcal{A}_i$, define the mapping $\Phi^u : \mathcal{U}(\alpha_i) \times [0,1] \to \mathcal{U}(\alpha_i)$ such that

$$\Phi^u(\alpha_i', \lambda) = (1-\lambda)\alpha_i' + \lambda \alpha_i^u.$$

$\Phi^u(\cdot, \lambda)$ is continuous in λ. As λ increases from 0 to 1, Φ^u continuously deforms any strategy in $\mathcal{U}(\alpha_i)$ to the constant strategy α_i^u, which is itself in $\mathcal{U}(\alpha_i)$. Therefore, $\mathcal{U}(\alpha_i)$ is contractible.

Similarly, let $\alpha_i^\ell : [0,1] \to A_i$ be a constant function with $\alpha_i^\ell(s_i) = \underline{a}_i$ for all $s_i \in [0,1]$. Again, $\alpha_i^\ell \in \mathcal{A}_i$. Furthermore, $\alpha_i^\ell(s_i) \leq \alpha_i(s_i)$, $\forall s_i \in [0,1]$ which implies $\alpha_i \succeq_{icx} \alpha_i^\ell \Rightarrow \alpha_i^\ell \in \mathcal{L}(\alpha_i)$. Then for each $\alpha_i \in \mathcal{A}_i$, define the mapping $\Phi^\ell : \mathcal{L}(\alpha_i) \times [0,1] \to \mathcal{L}(\alpha_i)$ such that

$$\Phi^\ell(\alpha_i', \lambda) = (1-\lambda)\alpha_i' + \lambda \alpha_i^\ell.$$

$\Phi^\ell(\cdot, \lambda)$ is continuous in λ. As λ increases from 0 to 1, Φ^ℓ continuously deforms any strategy in $\mathcal{L}(\alpha_i)$ to the constant strategy α_i^ℓ, which is itself in $\mathcal{L}(\alpha_i)$. Therefore, $\mathcal{L}(\alpha_i)$ is contractible. ■

Thus far, we have an order $\succeq_{icx}$ on $\mathcal{A}_i$ that generates compact and contractible upper and lower-sets. We extend these properties to $\mathcal{A} = \times_{i \in N} \mathcal{A}_i$ by the product order: given $\alpha'', \alpha' \in \mathcal{A}$, $\alpha'' \succeq_{icx} \alpha'$ if, and only if, $\alpha_i'' \succeq_{icx} \alpha_i'$ for each $i \in N$. Along with the product topology, $\succeq_{icx}$ is a partial order on $\mathcal{A}$ that generates compact and contractible upper and lower-sets.[20]

For a Bayesian game $\mathcal{G}_\rho = (\Sigma_{\rho 1}, \ldots, \Sigma_{\rho n}, \Gamma)$, define an operator $T_\rho : \mathcal{A} \to \mathcal{A}$ with

$$T_\rho(\alpha) = \big(a_1^{BR}(\cdot; \alpha_{-1}, \rho), \ldots, a_n^{BR}(\cdot; \alpha_{-n}, \rho)\big).$$

[20] $\mathcal{A}$ is a subset of a Banach space equipped with the metric $d(\alpha', \alpha) = \sum_i ||\alpha_i' - \alpha_i||_{BV}$.

T_ρ is continuous in α as utility functions are continuous in actions. A monotone BNE of $\mathcal{G}_\rho$, $a^\star(\rho)$, is a fixed point of T_ρ. We know such a fixed point exists (Van Zandt and Vives, 2007).

Consider two different games, $\mathcal{G}_{\rho''} = (\Sigma_{\rho''}, \Gamma)$ and $\mathcal{G}_{\rho'} = (\Sigma_{\rho'}, \Gamma)$, with $\rho''_i \succeq_{spm} \rho'_i$ for all $i \in N$. For all $\alpha \in \mathcal{A}$,

$$\rho''_i \succeq_{spm} \rho'_i, \forall i \underbrace{\Rightarrow}_{\substack{\text{by Lemma B.1} \\ \text{and Lemma B.2}}} a_i^{BR}(\alpha_{-i}, \rho'') \succeq_{icx} a_i^{BR}(\alpha_{-i}, \rho'), \forall i \Leftrightarrow T_{\rho''}(\alpha) \succeq_{icx} T_{\rho'}(\alpha).$$

Furthermore,

$$\alpha'' \succeq_{icx} \alpha' \Leftrightarrow \alpha''_i \succeq_{icx} \alpha'_i, \forall i \underbrace{\Rightarrow}_{\text{by Lemma B.3}} a_i^{BR}(\alpha''_{-i}, \rho) \succeq_{icx} a_i^{BR}(\alpha'_{-i}, \rho), \forall i \Leftrightarrow T_\rho(\alpha'') \succeq_{icx} T_\rho(\alpha').$$

We can now directly apply Theorem 6 and 7 of Villas-Boas (1997) to conclude that, for every fixed point $a^\star(\rho')$ of $T_{\rho'}$, there is a fixed point $a^\star(\rho'')$ of $T_{\rho''}$ such that $a^\star(\rho'') \succeq_{icx} a^\star(\rho')$, and for every fixed point $a^\star(\rho'')$ of $T_{\rho''}$, there is a fixed point $a^\star(\rho')$ of $T_{\rho'}$ such that $a^\star(\rho'') \succeq_{icx} a^\star(\rho')$. Hence, players are more responsive with a higher mean under $\Sigma_{\rho''}$ than under $\Sigma_{\rho'}$.

($\Longleftarrow$) Given two profiles of information structures $\Sigma_{\rho''}$ and $\Sigma_{\rho'}$, $\rho'' \not\succeq_{spm} \rho'$ if there exists a player $i^* \in N$ such that $\rho''_{i^*} \not\succeq_{spm} \rho'_{i^*}$. From Lemma A.2, $\rho''_{i^*} \not\succeq_{spm} \rho'_{i^*}$ implies there exist a $(\theta^*_{i^*}, s^*_{i^*}) \in \Theta_{i^*} \times S_{i^*}$ such that

$$G(\theta^*_{i^*}, s^*_{i^*}; \rho''_{i^*}) < G(\theta^*_{i^*}, s^*_{i^*}; \rho'_{i^*}).$$

Consider a basic game $\Gamma = \langle \{A_i, u^i\}_{i\in N}, F \rangle$ such that $\tilde{\theta}_i = \tilde{\theta}_j$ for all $i \neq j$, and $u^i : \Theta \times A \to \mathbb{R}$ is given by

$$u^i(\theta, a) = -\frac{1}{2}\Big(\bar{a}_i - \mathbf{1}_{[\theta_i \le \theta^*_{i^*}]}(\bar{a}_i - \underline{a}_i) - a_i\Big)^2$$

for all $i \in N$. In other words, this is a common value setting in which each player's payoffs depend only on her own action and the common state. Thus, each player acts as a single decision maker.

For all $i \in N$, $u^i(\theta, a)$ satisfies Assumption 2: It is continuous, twice differentiable, and strictly concave in a_i. It satisfies ID in $(\theta, a_{-i}; a_i)$. For each $(\theta, a_{-i}) \in \Theta \times A_{-i}$, the optimal action under complete information is $\underline{a}_i$ if $\theta_i \le \theta^*_{i^*}$ and $\bar{a}_i$ otherwise. Furthermore, the marginal utility $u^i_{a_i}(\theta, a) = \bar{a}_i - \mathbf{1}_{[\theta_i \le \theta^*_{i^*}]}(\bar{a}_i - \underline{a}_i) - a_i$ is

(*i*) linear in a_j for all $(\theta, a_{-j}) \in \Theta \times A_{-j}$, and (*ii*) has constant differences in $(\theta, a_{-j}; a_j)$.

Therefore, $\Gamma \in \boldsymbol{\Gamma_{icx}} \bigcap \boldsymbol{\Gamma_{dcx}}$. For any given Σ_ρ, there is a unique BNE $a^\star(\rho)$ where[21]

$$a_i^\star(s_i;\rho) = \bar{a}_i - (\bar{a}_i - \underline{a}_i)G(\theta_{i^*}^*|s_i;\rho_i).$$

Now consider player i^*; Given $\Sigma_{\rho'}$ and $\Sigma_{\rho''}$,

$$\begin{aligned}&\int_0^{s_{i^*}^*} \Big(a_{i^*}^\star(s_{i^*};\rho'') - a_{i^*}^\star(s_{i^*};\rho')\Big)dG_{S_{i^*}}(s_{i^*})\\ =&(\bar{a}_{i^*} - \underline{a}_{i^*})\Big(G(\theta_{i^*}^*, s_{i^*}^*;\rho_{i^*}') - G(\theta_{i^*}^*, s_{i^*}^*;\rho_{i^*}'')\Big) > 0,\end{aligned}$$

which implies $a_{i^*}^\star(\rho'') \not\succeq_{dcx} a_{i^*}^\star(\rho')$ (by Lemma A.1). By definition, the players are therefore not more responsive with a lower mean under $\Sigma_{\rho''}$ than $\Sigma_{\rho'}$.

Notice that for any Σ_ρ,

$$\mathbb{E}[a_{i^*}^\star(\rho)] = \bar{a}_{i^*} - (\bar{a}_{i^*} - \underline{a}_{i^*})\int_{S_{i^*}} G(\theta_{i^*}^*|s_i;\rho_i)dG_{S_{i^*}}(s_{i^*}) = \bar{a}_{i^*} - (\bar{a}_{i^*} - \underline{a}_{i^*})F_{\Theta_{i^*}}(\theta_{i^*}^*),$$

which is independent of ρ. Thus,

$$\begin{aligned}&\int_{s_{i^*}^*}^1 \Big(a_{i^*}^\star(s_{i^*};\rho'') - a^\star(s_{i^*};\rho')\Big)dG_{S_{i^*}}(s_{i^*})\\ =&\underbrace{\int_{S_{i^*}} \Big(a_{i^*}^\star(s_{i^*};\rho'') - a_{i^*}^\star(s_{i^*};\rho')\Big)dG_{S_{i^*}}(s_{i^*})}_{\substack{=\mathbb{E}[a_{i^*}^\star(\rho'')]-E[a_{i^*}^\star(\rho')]\\ =0}} - \left(\underbrace{\int_0^{s_{i^*}^*} \Big(a_{i^*}^\star(s_{i^*};\rho'') - a_{i^*}^\star(s;\rho')\Big)dG_{S_{i^*}}(s_{i^*})}_{>0}\right) < 0,\end{aligned}$$

which implies $a_{i^*}^\star(\rho'') \not\succeq_{icx} a_{i^*}^\star(\rho')$ (by Lemma A.1). By definition, the players are therefore not more responsive with a higher mean under $\Sigma_{\rho''}$ than $\Sigma_{\rho'}$. ■

Appendix C. When Responsiveness Fails

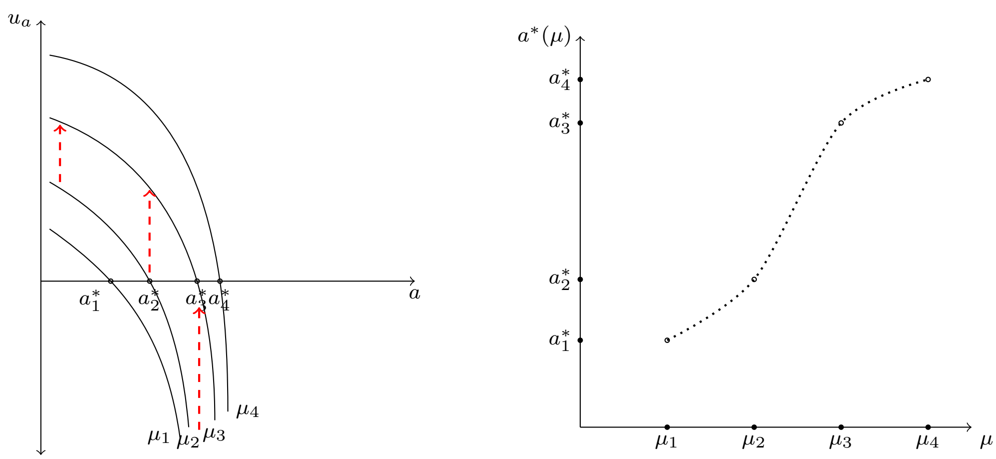
In this section, we explore why a higher quality of information may not lead to more dispersed optimal actions when $u \notin \boldsymbol{\Gamma_{icx}} \cup \boldsymbol{\Gamma_{dcx}}$. It suffices to consider a single-agent setting ($n = 1$). We therefore suppress the index "i".

[21] As each player acts as a single-decision maker, the unique BNE is just a profile of each player's optimal choice.

Consider a simple binary-states setting in which the agent's prior places mass on only two points $\{\underline{\theta}, \bar{\theta}\} \subset \Theta$ with $\bar{\theta} > \underline{\theta}$. Let $\mu = \mathbb{P}(\tilde{\theta} = \bar{\theta}) \in [0, 1]$ represent some posterior belief the agent holds. Suppose the agent's payoff $u(\theta, a)$ satisfies Assumption 2.

Take four different beliefs $\{\mu_n\}_{n=1,2,3,4}$ such that $\mu_n = n\delta$ for some $\delta \in (0, 1/4)$. Beliefs are ordered by first-order stochastic dominance with $\mu_4 \succeq_{FOSD} \mu_3 \succeq_{FOSD} \mu_2 \succeq_{FOSD} \mu_1$. Define $a_n^* = a^*(\mu_n) = \arg\max_{a \in A} \mu_n u(\bar{\theta}, a) + (1 - \mu_n) u(\underline{\theta}, a)$. As u satisfies ID in $(\theta; a)$, we can conclude that $a_4^* \geq a_3^* \geq a_2^* \geq a_1^*$.

In Figure 1a, we plot the expected marginal utilities. Since u satisfies ID in $(\theta; a)$, the expected marginal utility of μ_{n+1} lies above the expected marginal utility of μ_n. Assume that u_a also satisfies ID in $(\theta; a)$—in the figure, the height of the dashed arrows increases left to right. However, notice that the marginal utilities are concave in a which implies $u \notin \mathbf{\Gamma_{icx}} \cup \mathbf{\Gamma_{dcx}}$. Notice that $a_4^* - a_3^* < a_3^* - a_2^*$ whereas $a_3^* - a_2^* > a_2^* - a_1^*$. Figure 1b depicts this non-convexity of the optimal action as a function of beliefs.



(a) Marginal utilities (b) Optimal action

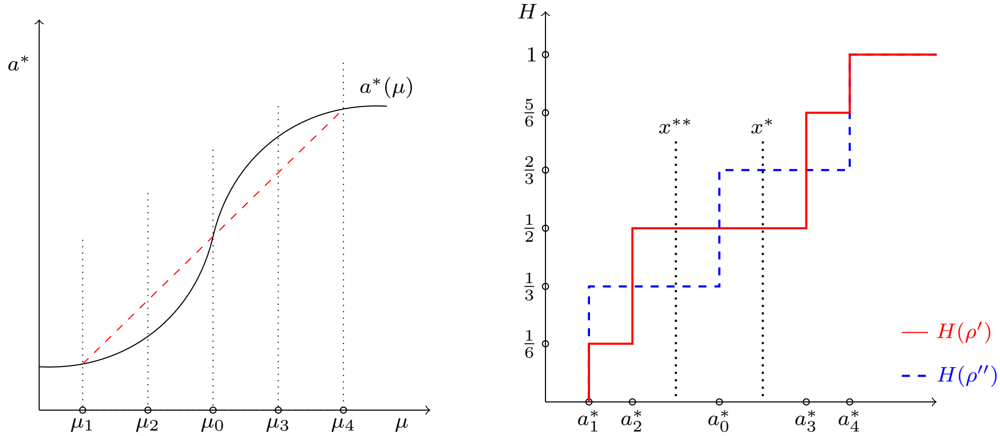
Figure 1: Non-convexity for $u \notin \mathbf{\Gamma_{icx}} \cup \mathbf{\Gamma_{dcx}}$

Figure 2 illustrates why the agent may not be responsive to an increase in the quality of information when the optimal action is neither convex nor concave, as in Figure 1b. Let $\mu_0 \in (0, 1)$ be the agent's prior belief that the state is $\bar{\theta}$.

Let $\Sigma_{\rho'}$ induce posteriors $\{\mu_1, \mu_2, \mu_3, \mu_4\}$ with probability $\{1/6, 1/3, 1/3, 1/6\}$ such that $\mu_4 \succeq_{FOSD} \mu_3 \succeq_{FOSD} \mu_0 \succeq_{FOSD} \mu_2 \succeq_{FOSD} \mu_1$. Additionally, let $\mu_2 = 0.5\mu_1 + 0.5\mu_0$ and $\mu_3 = 0.5\mu_4 + 0.5\mu_0$.

Let $\Sigma_{\rho''}$ be an information structure that induces three posteriors $\{\mu_1, \mu_0, \mu_4\}$ with probabilities $\{1/3, 1/3, 1/3\}$ Notice that $\Sigma_{\rho'}$ is a equivalent to getting information from $\Sigma_{\rho''}$ with probability 0.5 and no information with probability 0.5. Thus, $\Sigma_{\rho''}$ is Blackwell more informative than $\Sigma_{\rho'}$, which implies $\rho'' \succeq_{spm} \rho'$.

Let $a^*(\mu)$ be neither convex nor concave and let the average action under $\Sigma_{\rho''}$ equal the average action under $\Sigma_{\rho'}$. In Figure 2a, this corresponds to the point of intersection of the dashed line and the solid curved line at μ_0. Figure 2b maps the distribution over optimal actions: $\Sigma_{\rho''}$ induces the dashed line while $\Sigma_{\rho'}$ induces the solid line.



(a) Optimal action

(b) Induced distribution

Figure 2: Non-convexity/concavity and non-responsiveness

If we start integrating from the right, then $\int_x^\infty H(z;\rho'') - H(z;\rho')dz \leq 0$ for all $x > a_3^*$ but the sign changes at some point $x^* \in (a_0^*, a_3^*)$. Thus, the agent is not more responsive with a higher mean under $\Sigma_{\rho''}$. If we instead integrate from the left, then $\int_{-\infty}^x H(z;\rho'') - H(z;\rho')dz \geq 0$ for all $x < a_2^*$ but the sign changes at some point $x^{**} \in (a_2^*, a_0)$. Thus, the agent is not more responsive with a lower mean under $\Sigma_{\rho''}$.

In fact, as the average action under $\Sigma_{\rho''}$ equals the average action under $\Sigma_{\rho'}$, we can conclude that $a^\star(\rho'')$ and $a^\star(\rho')$ cannot be ordered by most univariate stochastic variability orders such as second-order stochastic dominance, mean-preserving spreads, Lorenz order, dilation order, and dispersive order.[22]

Another reason why a higher quality of information may not lead to more responsive behavior is when the interior solution assumption, Assumption 2c, is violated. Suppose the upper limit on the action space, $\bar{a}$, is a binding constraint for the prior, i.e., $a^*(\mu_0) = \bar{a}$. Let $\Sigma_{\rho'}$ be a completely uninformative information structure. Then $\Sigma_{\rho'}$ induces $\bar{a}$ with probability one, thereby first-order stochastically dominating the distribution over actions induced by any other information structure $\Sigma_{\rho''}$, even if $\rho'' \succeq_{spm} \rho'$.

[22] Shaked and Shanthikumar (2007) provide a thorough treatment of these orders.

Appendix D. Blackwell, Lehmann, and Supermodular Order

In this section, we present an example of information structures that can be ordered using the supermodular stochastic order but not the Lehmann order or the Blackwell order.

Let $n = 1$. For this section only, we consider information structures $\Sigma_\rho = \langle S, G(\cdot, \cdot; \rho)\rangle$ such that for each $\theta \in \Theta$, $G_S(\cdot|\theta;\rho)$ has a density function $g_S(\cdot|\theta;\rho)$ which satisfies the MLRP property, i.e., for any $s < s'$, the likelihood function

$$\frac{g_S(s'|\theta;\rho)}{g_S(s|\theta;\rho)}$$

is non-decreasing in θ.[23]

Lehmann (Accuracy) Order: $\Sigma_{\rho''}$ dominates $\Sigma_{\rho'}$ in the Lehmann order, denoted $\rho'' \succeq_L \rho'$, if for all $s \in S$,

$$G_S^{-1}\Big(G_S(s|\theta;\rho')\Big|\theta;\rho''\Big)$$

is increasing in θ.

Example: Let $\theta \in \{\theta_l, \theta_m, \theta_h\}$ with $\theta_l < \theta_m < \theta_h$. Let $f(\theta)$ be the prior mass at θ with $f(\theta_l) = f(\theta_m) = \frac{2}{5}$ and $f(\theta_h) = \frac{1}{5}$. Consider two information structure $\Sigma_{\rho'}$ and $\Sigma_{\rho''}$ such that the signal space S is the unit interval for both structures and $G_S(\cdot|\theta;\rho')$ is given by

	$0 \le s < 1/2$	$1/2 \le s \le 1$
θ_l	$s\frac{3}{2}$	$\frac{1+s}{2}$
θ_m	s	s
θ_h	0	$2s-1$

while $G_S(\cdot|\theta;\rho'')$ is given by

	$0 \le s < 1/2$	$1/2 \le s \le 1$
θ_l	$2s$	1
θ_m	$\frac{s}{2}$	$\frac{3s-1}{2}$
θ_h	0	$2s-1$

For both information structures, the marginal on the signal is simply the uniform distribution on the unit interval. Furthermore, both structures satisfy the MLRP property.

[23]This is a more restrictive assumption on signal structures than Assumption 3.

We first show that $\rho' \not\succeq_L \rho''$ and $\rho'' \not\succeq_L \rho'$. If $\rho' \succeq_L \rho''$, then

$$G_S^{-1}\Big(G_S(s|\theta;\rho'')\big|\theta;\rho'\Big)$$

must be increasing in θ for every $s \in [0,1]$. For all $s \in [0,1]$,

$$G_S^{-1}\Big(G_S(s|\theta_l;\rho'')\big|\theta_l;\rho'\Big) = \begin{cases} \frac{4s}{3} & \text{if} \quad s \in [0,\frac{3}{8}) \\ 4s-1 & \text{if} \quad s \in [\frac{3}{8},\frac{1}{2}) \\ 1 & \text{if} \quad s \in [\frac{1}{2},1] \end{cases},$$

and

$$G_S^{-1}\Big(G_S(s|\theta_m;\rho'')\big|\theta_m;\rho'\Big) = \begin{cases} \frac{s}{2} & \text{if} \quad s \in [0,\frac{1}{2}) \\ \frac{3s-1}{2} & \text{if} \quad s \in [\frac{1}{2},1] \end{cases},$$

and

$$G_S^{-1}\Big(G_S(s|\theta_h;\rho'')\big|\theta_h;\rho'\Big) = \begin{cases} 0 & \text{if} \quad s \in [0,\frac{1}{2}) \\ s & \text{if} \quad s \in [\frac{1}{2},1] \end{cases}$$

Altogether, we have

$$G_S^{-1}\Big(G_S(\cdot|\theta_m;\rho'')\big|\theta_m;\rho'\Big) < G_S^{-1}\Big(G_S(\cdot|\theta_h;\rho'')\big|\theta_h;\rho'\Big) < G_S^{-1}\Big(G_S(\cdot|\theta_l;\rho'')\big|\theta_l;\rho'\Big)$$

for all $s \in [\frac{1}{2},1)$, violating the Lehmann monotonicity condition. Thus, $\rho' \not\succeq_L \rho''$.

We now show that $\rho'' \not\succeq_L \rho'$. If $\rho'' \succeq_L \rho'$, then

$$G_S^{-1}\Big(G_S(s|\theta;\rho')\big|\theta;\rho''\Big)$$

must be increasing in θ for every $s \in [0,1]$. For all $s \in [0,1]$,

$$G_S^{-1}\Big(G_S(s|\theta_l;\rho')\big|\theta_l;\rho''\Big) = \begin{cases} \frac{3s}{4} & \text{if} \quad s \in [0,\frac{1}{2}) \\ \frac{1+s}{4} & \text{if} \quad s \in [\frac{1}{2},1] \end{cases},$$

and

$$G_S^{-1}\Big(G_S(s|\theta_m;\rho')\big|\theta_m;\rho''\Big) = \begin{cases} 2s & \text{if} \quad s \in [0,\frac{1}{4}) \\ \frac{2s+1}{3} & \text{if} \quad s \in [\frac{1}{4},1] \end{cases},$$

and

$$G_S^{-1}\Big(G_S(s|\theta_h;\rho')\big|\theta_h;\rho''\Big) = \begin{cases} 0 & \text{if} \quad s \in [0, \frac{1}{2}) \\ s & \text{if} \quad s \in [\frac{1}{2}, 1] \end{cases}$$

Altogether, we have

$$G_S^{-1}\Big(G_S(\cdot|\theta_m;\rho')\big|\theta_m;\rho''\Big) > G_S^{-1}\Big(G_S(\cdot|\theta_h;\rho')\big|\theta_h;\rho''\Big) > G_S^{-1}\Big(G_S(\cdot|\theta_l;\rho')\big|\theta_l;\rho''\Big)$$

for all $s \in [\frac{1}{2}, 1)$, violating the Lehmann monotonicity condition. Thus, $\rho'' \not\succeq_L \rho'$. Furthermore, $\Sigma_{\rho''}$ and $\Sigma_{\rho'}$ are also not Blackwell ordered since Blackwell ordering implies Lehmann ordering (within the class of information structures with MLRP property).

Finally, we show that $\rho'' \succeq_{spm} \rho'$ by noting that $G(\theta, s; \rho'') - G\theta, s; \rho') \geq 0$ for all (θ, s). Notice that for all $s \in [0, 1]$,

$$G(\theta_l, s; \rho'') - G(\theta_l, s; \rho') = f(\theta_l)\Big(G_S(s|\theta_l;\rho'') - G_S(s|\theta_l;\rho')\Big) \geq 0,$$

with a strict inequality for all $s \in (0, 1)$. Furthermore, for all $s \in [0, 1]$,

$$\begin{aligned} G(\theta_m, s; \rho'') - G(\theta_m, s; \rho') =& f(\theta_l)\Big(G_S(s|\theta_l;\rho'') - G_S(s|\theta_l;\rho')\Big) + f(\theta_m)\Big(G_S(s|\theta_m;\rho'') - G_S(s|\theta_m;\rho')\Big) \\ =& \frac{2}{5}\Big(G_S(s|\theta_l;\rho'') + G_S(s|\theta_m;\rho'') - G_S(s|\theta_l;\rho') - G_S(s|\theta_m;\rho')\Big) = 0. \end{aligned}$$

Finally, $G(\theta_h, s; \rho'') - G(\theta_h, s; \rho') = G_S(s; \rho'') - G_S(s; \rho') = 0$. Hence, $\rho'' \succeq_{spm} \rho'$.

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