Stable Coalition Structures of Patent Licensing Games∗

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Abstract

We consider what coalition structures are stable in a Cournot oligopoly market with homogenous goods. Each firm’s payoff is affected by a coalition to which it belongs as well as other firms’ coalitions, so that we consider a coalition formation game with externalities to deal with the problem. For stability concepts, we consider three types of the stylized core of coalition structures: the projective, optimistic, and pessimistic core. First, as a benchmark, we show that the pessimistic core is always non-empty, whereas the projective and optimistic core are always empty unless it is a duopoly market. Next, we consider an opportunity to make a contract with an external patent holder of a new technology, which can be used to reduce each firm’s production cost, and its effects on the stability of coalition structures. We show that, under certain conditions, the projective core can be non-empty and a coalition structure can be in the projective core if and only if some firms contract with the patent holder to use the technology and other firms do not make any cooperation, which is a common assumption in the patent licensing games. We also show that the pessimistic core can support more varieties of coalition structures and even the optimistic core is non-empty if market size is not large. Our results provide a first theoretical foundation of the common assumption in the patent licensing games and clarifies when this assumption is plausible.

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1 Introduction

A patent is a common scheme to enhance technological innovation by an inventor as well as to disseminate the innovation to the third party. It provides the inventor with a monopoly right over the innovation for a certain period of times. Suppose that an inventor is not a producer in an underlying market and the patented technology can reduce a production cost of firms in the market.\(^1\) During the period, on the one hand, a licensing of the patented technology can be a great source of revenue for the inventor. For licensee firms, on the other hand, they can use the technology in an exclusive way, so that they can have competitive advantages against other non-licensee firms. Hence, contracting with the patent holder is a useful competitive strategy for firms in market competition.\(^2\)

Based on cooperative and non-cooperative game approaches, the theoretical literature of the patent licensing of the cost reduction technology have mainly investigated the following two questions. The first question concerns what type of a contract, i.e., fee vs. loyalty, is optimal for the patent holder (e.g., Kamien and Tauman, 1986; Muto, 1993; Wang, 1998, 2002; Sen, 2005; Sen and Tauman, 2007; San Martín and Saracho, 2010; Kishimoto and Muto, 2012; Colombo and Filippini, 2016; Llobet and Padilla, 2016). The second question concerns what payoff distribution of the firms in the underlying market is stable (e.g., Tauman and Watanabe, 2007; Watanabe and Muto, 2008; Kishimoto et.al, 2011; Kishimoto, 2013; Hirai and Watanabe, 2018; Hirai et.al, 2019). A common assumption made in the studies of both questions is that each firm is prohibited to communicate and/or cooperate with each other. This assumption is problematic because, in particular, non-licensee firms would have incentive to cooperate with each other to mitigate competition against a cost disadvantage.\(^3\) Moreover, since the revenue of the patent holder depends on the number of licensee firms and their profits, the optimal contract also depends on the market structure itself. This observation raises the following third question: what coalition structures of the firms is stable in the underlying market?

To answer the question, we consider a coalition formation game in a Cournot oligopoly market. The game is proceeded as follows. There are ex-ante symmetric firms and an external patent holder which does not have any production facility. In the first stage, each firm and the patent holder make a coalition structure. If some firms make a contract with the patent holder to use the patented technology, these firms enjoy it in an exclusive way to reduce the production cost and, in return, the patentee earns predetermined rate

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\(^1\)This type of patent holder is called an external licensor. Research laboratories, national research and development agency, and some departments in university are typical examples.

\(^2\)See Kamien (1992) for a survey on the literature.

\(^3\)Of course, antitrust law prohibits firms from making a cartel/coalition, but it does not put a ban on all cartels.
of licensee firms’ total profit (i.e., ad valorem royalty). Other non-licensee firms suffer from a cost disadvantage against licensee firms but, firms in each coalition can cooperate with each other. In the second stage, each coalition made in the first stage can act as a single firm (e.g., M&A) and competes in the market. We assume that each firm in a coalition equally shares the equilibrium profit between them because each firm is ex-ante symmetric. Regarding this shared profit as a firm’s payoff from each coalition structure, we analyze stability of a coalition structure in the first stage. Since the coalition structure affects the payoff of each firm, our coalition formation games fall into the class of partition function form games or hedonic games with externalities.

In the presence of externalities, upon deviating from the current coalition structure, firms must consider other firms’ reaction. This aspect allows us to define a variety of stability concepts. Following Abe (2018), we consider three core concepts: the projective, optimistic, and pessimistic core. These different concepts reflect deviating firms’ different view of other firms’ reaction. For the projective core, each firm anticipates that other coalitions do not react, that is, all firms other than members of a deviating coalition remains in the same coalitions. In contrast, for the optimistic/pessimistic core, each firm anticipates other firms react against deviating firms by making a best/worst coalition structure for deviating firms, respectively.

First, as a benchmark, we consider the case that there is no patent licensing opportunity. We show that both the projective and optimistic core are always empty unless it is a duopoly market. In contrast, the pessimistic core is always non-empty and, in particular, the grand coalition is always stable in this sense, which is a consequence of positive externalities: for each coalition of firms, merging other coalitions is beneficial because the market becomes smaller. Moreover, the pessimistic core is singleton if the number of the firms is small, but other coalition structures can be also the pessimistic core outcomes as the number of firms increases. We characterize the condition for the pessimistic core outcomes.

Second, we show that, under certain conditions, the projective core is non-empty and provide the following characterization; a coalition structure can be in the projective core if and only if some firms contract with the patent holder and other firms do not make any cooperation. Moreover, if the market size is not large, even the optimistic core can be also non-empty. By enlarging the projective core to the pessimistic core, we show that such a coalition structure can be in the pessimistic core if and only if the loyalty to the patent holder is in a certain range and the size of licensee firms is bigger than one-half of the market size. In particular, the sufficiently large cost reduction makes it stable easier. The intuition behind this case is as follows. Upon deviating from the current coalition structure, the worst case for a firm in the deviating coalition is a severe market competition.
where other firms react by making a singleton coalition. If the cost reduction is sufficiently large and the loyalty is not large enough, the most profitable deviation for each licensee firm is the deviation with the patent holder. Hence, to do so, the patent holder must agree with a deviation. For the patent holder, given the pessimistic view, a deviation is profitable if and only if the market size becomes smaller by doing so to increase the total payoff of the licensee firms. Since the size of licensee firms are large enough, it is possible only if some licensee and non-licensee firms must join the deviation. Although a non-licensee firm is willing to join such a deviation, licensee firms are not willing to join the deviation because licensee firm’s per-profit after the deviation becomes smaller than that of the current situation. Hence, by the above arguments, since there is no agreeable deviations between licensee firms and the patent holder, neither licensee firms nor patent holder join any deviation. However, other non-licensee firms may have incentive to deviate from the coalition structure by making a coalition by themselves to mitigate the competition. If the size of licensee firms is large enough, this is not the case. Indeed, under the pessimistic view, each non-licensee firm anticipates that the number of rivals in the market increases after the deviation, so that the deviation is not profitable. Therefore, summarizing the above arguments, the coalition structure is stable under such conditions. Our main result characterizes the general condition for the stability of such coalition structures. Since such coalition structures are commonly assumed in the literature of patent licensing games, our results clarify when such an assumption is plausible in terms of the stability of the coalition structure.

The rest of the paper is organized as follows. In Section 2, we introduce our model. In Section 3, we define our stability concepts and provide some preliminary results. In Section 4, we provide our main result. Section 5 is the conclusion of the paper. Some proofs are relegated to Appendix.

2 Model

Let \( N = \{1, 2, \cdots, n\} \) be the set of firms which compete in an oligopolistic market. We assume that \( n \geq 2 \) and each firm engages in the Cournot competition with homogenous goods. Let \( \Pi \) be the set of all partitions on \( N \). We denote by \( |\mathcal{P}| \) the number of coalitions included in \( \mathcal{P} \in \Pi \). For each \( \mathcal{P} \in \Pi \), let \( T_i \in \mathcal{P} \) be a coalition such that \( i \in T_i \). We also write \( \mathcal{P}|_T \) be the partition which restricts \( \mathcal{P} \) on \( T \subseteq N \). Given a partition \( \mathcal{P} \in \Pi \), each firm in the same coalition can cooperate via business alliance or, such a coalition is formed via M&. Hence, we can regard competition in this market as competition by \( |\mathcal{P}| \) firms by identifying each coalition \( T \in \mathcal{P} \) with a single firm. In particular, let \( \mathcal{P}^* = \{N\} \) be the partition by the grand coalition, so that underlying market is monopolized. Following
Ray and Vohra (1997), we assume that the total profit for each coalition is shared equally by its members.

The inverse demand function of the market is given by $p = \alpha - \beta \left(\sum_{i \in N} q_i\right)$ where $q_i \in \mathbb{R}_+$ is the quantity of the supply by firm $i \in N$ and $\alpha, \beta > 0$. Let $c > 0$ be the marginal cost of the production for each firm. We assume that $\alpha - c > 0$. In this setup, it is easy to see that each firm’s equilibrium payoff $\pi$ in the symmetric Cournot competition is given as follows:

$$\pi = \frac{1}{(n+1)^2} M,$$

where

$$M = \frac{(\alpha - c)^2}{\beta}.$$ 

For each partition $\mathcal{P} \in \Pi$, and each coalition $T \in \mathcal{P}$, let $\pi_T(\mathcal{P})$ be the total payoff of coalition $T \in \mathcal{P}$. By the above discussion, we have

$$\pi_T(\mathcal{P}) = \frac{1}{(|\mathcal{P}| + 1)^2} M.$$

By our assumption, each firm’s payoff function over $\Pi$ is also given by

$$\pi_i(\mathcal{P}) = \frac{1}{|T_i|} \pi_{T_i}(\mathcal{P}) = \frac{1}{|T_i|(|\mathcal{P}| + 1)^2} M$$

for each $i \in N$. Finally, by this preparation, we can define an induced hedonic game with externalities $(N, (\succ_i)_{i \in N})$ (Abe, 2018) where each firm $i \in N$ has a preference relation $\succ_i$ over $\Pi$ defined as follows: for each $\mathcal{P}, \mathcal{P}' \in \Pi$,

$$\mathcal{P}' \succ_i \mathcal{P} \Leftrightarrow \pi_i(\mathcal{P}') \geq \pi_i(\mathcal{P}).$$

For any $\mathcal{P}, \mathcal{P}' \in \Pi$ and $S \subseteq N$, we also define $\succ_S$ by $\mathcal{P}' \succ_S \mathcal{P} \Leftrightarrow \mathcal{P}' \succ_i \mathcal{P}$ for any $i \in S$.

**Remark.** The function defined as $v(\mathcal{P}, T) = \pi_T(\mathcal{P})$ for all $\mathcal{P} \in \Pi$ and $T \in \mathcal{P}$ can be considered as a partition function form games or transferable utility games with externalities (Thrall and Lucas, 1963; Myerson, 1977). Hence, our definition of each firm’s payoff can be considered as an allocation for the partition function form games. Similar (but, a different) concept is an allocation of the transferable utility games with a priori coalition structure. For this formulation, see Aumann and Dréze, (1974), Hart and Kurz (1984), Casajus (2009), Abe (2018, 2019), and references therein.

### 3 Stable coalition structures

We consider what kinds of partitions are stable as a solution of the induced hedonic game. For each firm, if other firms’ coalitions do not affect it’s payoff, we can use the standard
definition of the core partition as a stability of the coalition structures (Banerjee et al., 2001; Bogomolnaia and Jackson, 2002). However, if other firms’ coalitions do affect its payoff, we must consider how other firms react against a deviation by a coalition. To deal with this issue, some core concepts are considered (Funaki and Yamato, 1999; Köczy, 2007; Abe, 2018). In our study, we consider the following three core concepts.

**Definition 1.** A partition \( P \) is in the projective core, \( C^{pro} \), if there is no \( S \subseteq N \) such that \( S \notin P \) and \( P' \succ_S P \) where \( P' = \{S\} \cup (P|_{N \setminus S}) \).

**Definition 2.** A partition \( P \) is in the optimistic core, \( C^{opt} \), if there is no \( S \subseteq N \) such that \( S \notin P \) and for some \( P' \in \Pi \) with \( S \in P' \), \( P' \succ_S P \).

**Definition 3.** A partition \( P \) is in the pessimistic core, \( C^{pes} \), if there is no \( S \subseteq N \) such that \( S \notin P \) and for all \( P' \in \Pi \) with \( S \in P' \), \( P' \succ_S P \).

In above definitions, we call a coalition \( S \) satisfying each property as a blocking coalition. For the projective core, we assume that other coalitions do not react against a blocking coalition, so that all firms other than members of a blocking coalition remains in the same coalitions. In contrast, for the optimistic and pessimistic core, we assume that other firms react against a blocking coalition. As the name suggests, in the former case, other firms react to make a best partition for a blocking coalition, whereas in the latter case, a worst partition for a blocking coalition. These assumptions reflect the belief or expectation of the blocking coalition for other firms. By definitions, note that \( C^{opt} \subseteq C^{pro} \subseteq C^{pes} \). We first show that both \( C^{pro} \) and \( C^{opt} \) are empty unless it is a duopoly market.

**Proposition 1.** For \( n \geq 3 \), \( C^{pro} = \emptyset \).

*Proof.* Take any partition \( P \in \Pi \). By the definition of \( C^{pro} \), player \( i \in N \) blocks \( P \) by \( S = \{i\} \) if and only if

\[
\frac{1}{|T_i|} \left( \frac{1}{|P|+1} \right)^2 < \frac{1}{(|P|+2)^2} \iff \left( 1 + \frac{1}{|P|+1} \right)^2 < |T_i|.
\]

First, consider the case \( |P| = 1 \) and thus \( |T_i| = n \). By \( (1 + \frac{1}{1+1})^2 = \frac{9}{4} < 3 \), \( P \notin C^{pro} \) as \( n \geq 3 \).

Next consider the case \( |P| \geq 2 \). Since \( (1 + \frac{1}{|P|+1})^2 \leq (1 + \frac{1}{2+1})^2 = \frac{16}{9} < 2 \), \( |T_i| \geq 2 \) for some \( i \in N \) implies that \( P \notin C^{pro} \).

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4Yi (1998) also considers coalition formation games with externalities and he treats the Cournot competition as a special case. The difference from his study is that he considers a Nash equilibrium as a stability concept.

5In each concept, we do not consider such a reaction really occurs, but a firm in the blocking coalition forms such belief. See Bloch and Van den Nouweland (2014) about a foundation about this issue.
Finally, if \( P = \{i\}_{i \in N} \) then \( \pi_i(P) = \frac{1}{(n+1)^2}M < \frac{1}{(n+1)^2 - (n-1)^2}M = \frac{1}{4n}M = \pi_i(\{N\}) \) for any \( i \in N \) as \( n \geq 2 \), which implies that \( P \notin C^{pro} \). Therefore, by above arguments, for \( n \geq 3 \), \( C^{pro} = \emptyset \).

\[ \text{Corollary 1. For } n \geq 3, C^{opt} = \emptyset. \]

Note that, for \( n = 2 \), it is easily seen that \( C^{opt} = C^{pro} = \{P^*\} \). Proposition 1 shows that stable partitions do not exist in that sense. Hence, we then consider \( C^{pes} \). In contrast to the previous cases, we show that some partitions can be supported by \( C^{pes} \). Observe that, for each coalition \( S \in \mathcal{P} \), merging other coalition \( T, T' \in \mathcal{P}\backslash \{S\} \) is beneficial because this merger virtually mitigates market competition by reducing the number of rival firms. In this sense, there is only a positive externality. This observation implies the following.

\[ \text{Lemma 1. For each } P \in \Pi \text{ and } S \notin P, \]

\[ \min_{P' \in \Pi; S \notin P'} \pi_i(P') = \frac{1}{|S|(n-|S| + 2)^2}M. \]

Thus, the worst partition for \( S \) is \( P' = \{S, \{j\}_{j \notin S}\} \).

Hence, the most and worst partition for a coalition \( T \subseteq N \) are \( \{T, N \backslash T\} \) and \( \{T, \{i\}_{i \in N \backslash T}\} \), respectively. When \( N = T \), we can show that any coalition \( S \not
subseteq N \) prefers \( P^* \) to \( \{S, \{j\}_{j \notin S}\} \), so that \( P^* \) is stable regardless of \( n \).

\[ \text{Proposition 2. For any } n \geq 2, P^* \in C^{pes}. \text{ Moreover, if } n \leq 4, C^{pes} = \{P^*\}. \]

Proof. Note that \( \pi_i(P^*) = \frac{1}{4n}M. \) Suppose that some group \( S \not
subseteq N \) deviates from \( P^* \). Then, by Lemma 1, \( P^* \in C^{pes} \) if and only if

\[ \frac{1}{|S|(n-|S| + 2)^2}M \leq \frac{1}{4n}M \iff 4n \leq |S|(n-|S| + 2)^2 \]

for each \( S \not
subseteq N \). Note that \( \min_{S:1 \leq |S| \leq n-1} |S|(n-|S| + 2)^2 \leq \min\{(n+1)^2, 9(n-1)\} \geq 4n \) because \( n \geq 2 \). Therefore, \( P^* \in C^{pes} \). Uniqueness for \( n \leq 4 \) can be directly checked.

\[ \text{Corollary 2. The pessimistic core is always non-empty.} \]

If \( n \geq 5 \), other partitions can be pessimistic core outcomes, so that \( \{P^*\} \not
subseteq C^{pes} \). Table 1 illustrates the pessimistic core outcomes in the case of \( n = 5 \). For general cases, by Lemma 1 and the above proof of Proposition 2, we provide the following characterization of \( C^{pes} \).

\[ \text{Proposition 3. } P \in C^{pes} \text{ if and only if both of the following two conditions hold:} \]

\[ (i) \left( \frac{n+1}{|P|+1} \right)^2 \geq |T_i| \text{ for all } i \in N. \]
\(\mathcal{P} / \pi_i(\mathcal{P})\) & 1 & 2 & 3 & 4 & 5 & \(C^{pes}\)  \\
12345 & \(\frac{1}{20}\) & \(\frac{1}{20}\) & \(\frac{1}{20}\) & \(\frac{1}{20}\) & \(\frac{1}{20}\) & \(\bigcirc\)  \\
1234|5 & \(\frac{1}{36}\) & \(\frac{1}{36}\) & \(\frac{1}{36}\) & \(\frac{1}{18}\) & \(\frac{1}{18}\) & \(\bigcirc\)  \\
123|45 & \(\frac{1}{27}\) & \(\frac{1}{27}\) & \(\frac{1}{18}\) & \(\frac{1}{18}\) & \(\bigcirc\)  \\
123|4|5 & \(\frac{1}{48}\) & \(\frac{1}{48}\) & \(\frac{1}{16}\) & \(\frac{1}{16}\) & \(\bigcirc\)  \\
12|34|5 & \(\frac{1}{32}\) & \(\frac{1}{32}\) & \(\frac{1}{16}\) & \(\frac{1}{16}\) & \(\bigcirc\)  \\
12|3|4|5 & \(\frac{1}{50}\) & \(\frac{1}{50}\) & \(\frac{1}{25}\) & \(\frac{1}{25}\) & \(\bigcirc\)  \\
1|2|3|4|5 & \(\frac{1}{36}\) & \(\frac{1}{36}\) & \(\frac{1}{36}\) & \(\frac{1}{36}\) & \(\bigcirc\)

Table 1: Case of \(n = 5\). The symbol \(\bigcirc\) indicates \(\mathcal{P} \in C^{pes}\). Since each firm is ex-ante symmetric, we only write the patterns of the partitions.

(ii) \(\left(\frac{2}{|\mathcal{P}| + 1}\right)^2 \geq \frac{|T_i|}{n}\) for some \(i \in N\).

Proof. Observe that

\[
\max_{1 \leq |S| \leq n} \frac{1}{|S|(n - |S| + 2)^2} M = \max\left\{ \frac{1}{(n + 1)^2} M, \frac{1}{4n} M \right\},
\]

that is, the maximum value is attained at either \(|S| = 1\) or \(|S| = n\). Then, by Lemma 1, for any partition \(\mathcal{P} \in \Pi\) and \(S \notin \mathcal{P}\), if \(\mathcal{P}' \succ_S \mathcal{P}\) for any \(\mathcal{P}'\) with \(S \in \mathcal{P}'\), then \(S\) can be taken as either \(S = \{i\}\) for some \(i \in N\) or \(S = N\). Therefore, \(\mathcal{P} \in C^{pes}\) if and only if both \(S = \{i\}\) for all \(i \in N\) and \(S = N\) do not satisfy the condition of definition 3. By Lemma 1, the conditions (i) and (ii) of proposition correspond to the cases \(S = \{i\}\) and \(S = N\), respectively. \(\square\)

In particular, the severe competition in the sense that \(\mathcal{P} = \{\{i\}_{i \in N}\}\) cannot be a pessimistic core outcome because condition (ii) in Proposition 3 is always violated for any \(n \geq 2\); \(0 < (n - 1)^2 \Leftrightarrow \left(\frac{2}{n + 1}\right)^2 < \frac{1}{n}\). That is, \(\mathcal{P} = \{\{i\}_{i \in N}\}\) is blocked by \(N\).

In the following section, we will consider what kind of coalition structures are stable if there is a patent licensing opportunity.

4 Stability and patent licensing

We introduce a possibility of the cost reduction by making a contract with a patent holder of the new technology. In contrast to the previous setup, we assume that there is a non-productive patent holder, which is denoted by 0, and the technology of the patent holder can be used by firms to reduce a cost for the production by making an exclusive contract. Hereafter, with abuse of notation, let \(\Pi\) be the set of all partition on \(N \cup \{0\}\). We call each firm \(i \in T_0 \setminus \{0\}\) a licensee firm and each firm \(i \notin T_0 \setminus \{0\}\) a non-licensee firm. Let
\(c_H, c_L\) with \(\alpha > c_H > c_L > 0\) be the marginal cost of the production for the non-license and licensee firms, respectively. Let \(\lambda = \alpha - c_H\) and \(\tilde{c} = c_H - c_L > 0\). To guarantee the interior solution, we assume that \(\tilde{c} < \lambda\), that is, the cost reduction is non-drastic in the sense that non-licensee firms drop out of the market because of the cost disadvantage.

For \(n\)-firm asymmetric Cournot competition where only one firm has \(c_L\) and other firms have \(c_H\), it is easy to see that the equilibrium payoffs for the licensee \((\pi_L)\) and non-licensee \((\pi_H)\) firms are given as follows:\(^6\)

\[
\pi_L = \frac{1}{(n+1)^2} M_L(n), \quad \pi_H = \frac{1}{(n+1)^2} M_H,
\]

where

\[
M_L(n) = \frac{(\lambda + n\tilde{c})^2}{\beta}, \quad M_H = \frac{(\lambda - \tilde{c})^2}{\beta}.
\]

For each partition \(\mathcal{P} \in \Pi\) with \(\{0\} \notin \mathcal{P}\), we consider \(|\mathcal{P}|\)-firm Cournot competition where only one firm has \(c_L\) and other firms have \(c_H\). Otherwise, we consider \((|\mathcal{P}| - 1)\)-firm Cournot competition where every firm has \(c_H\) because the patent holder does not have any production technology. Therefore, the total payoff of the coalition by licensee \((\pi(T_0, \mathcal{P}))\) and non-licensee \((\pi(T, \mathcal{P}))\) firms are given as follows:

\[
\pi(T_0, \mathcal{P}) = \begin{cases} 
\frac{1}{(|\mathcal{P}|+1)^2} M_L(|\mathcal{P}|) & \text{if } \{0\} \notin \mathcal{P}, \\
0 & \text{if } \{0\} \in \mathcal{P},
\end{cases}
\]

\[
\pi(T, \mathcal{P}) = \begin{cases} 
\frac{1}{(|\mathcal{P}|+1)^2} M_H & \text{if } \{0\} \notin \mathcal{P}, \\
\frac{1}{|\mathcal{P}|^2} M' & \text{if } \{0\} \in \mathcal{P},
\end{cases}
\]

where

\[
M' = \frac{\lambda^2}{\beta}.
\]

Note that \(M_L(n) > M' > M_H\) for all \(n\). As in the previous setup, we assume that the total profit for each coalition is shared equally by its members. Moreover, we also assume that the payoff of the patent holder is fixed amount of licensee firm’s total profit, namely, \(\tau \pi(T_0, \mathcal{P})\) with \(\tau \in (0, 1)\).\(^7\)

\(^6\)See Appendix A for the derivation.

\(^7\)Some varieties of the payment to the patent holder are observed in the real world such as fixed fee, royalty per unit production, ad valorem royalty, and mixed of them. The assumption used here is that payment is based on the ad valorem royalty. In the literature, many studies consider what payment scheme is superior to others in various settings. See Kamien and Tauman (1986), Muto (1993), Wang (1998, 2002), Sen (2005), Sen and Tauman (2007), San Martín and Saracho (2010), Kishimoto and Muto (2012), Colombo and Filippini (2016), Llobet and Padilla (2016) and references therein .
To summarize, we can derive each firm’s and the patent holder’s payoff functions over Π as follows:

\[
\pi_i(\mathcal{P}) = \begin{cases} 
\frac{\tau}{(|\mathcal{P}|+1)^2} M_L(|\mathcal{P}|) & \text{if } \{0\} \notin \mathcal{P} \text{ and } i = 0, \\
0 & \text{if } \{0\} \in \mathcal{P} \text{ and } i = 0, \\
\frac{(1-\tau)}{(|T_0|-1)(|\mathcal{P}|+1)^2} M_L(|\mathcal{P}|) & \text{if } \{0\} \notin \mathcal{P} \text{ and } i \in T_0 \setminus \{0\}, \\
\frac{1}{|\mathcal{P}|(1+|\mathcal{P}|)^2} M_H & \text{if } \{0\} \notin \mathcal{P} \text{ and } i \notin T_0, \\
\frac{1}{|\mathcal{P}|(1+|\mathcal{P}|)^2} M' & \text{if } \{0\} \in \mathcal{P} \text{ and } i \neq 0.
\end{cases}
\]

In the following analyses, we assume that \( n \geq 3 \). Without loss of generality, we can assume that \( \lambda = \beta = 1 \) because this normalization does not affect the stability condition. By this normalization, non-drastic cost reduction, \( \tilde{c} < \lambda \), corresponds to \( \tilde{c} \in (0,1) \).

We first consider the stability of coalition structures in terms of \( C^{pro} \). As we have seen in the previous section, we can show that the partitions with \( \{0\} \in \mathcal{P} \) is not stable. The following lemma shows the possible candidates of \( C^{pro} \).

**Lemma 2.** Suppose that \( n \geq 3 \) and \( \mathcal{P} \in C^{pro} \). Then, \( \mathcal{P} = \{\{0\} \cup S, \{j\}_{j \notin S}\} \) for some \( S \subseteq N \) with \( \max\{n-4,1\} \leq |S| \).

**Proof.** First, we show that \( \mathcal{P} \notin C^{pro} \) for any \( \mathcal{P} \) such that \( \{0\} \in \mathcal{P} \). If \( \mathcal{P} = \{\{0\}, \{j\}_{j \in N}\} \), then

\[
\frac{M'}{4n} > \frac{M'}{(n+1)^2} \iff (n-1)^2 > 0
\]

implies that \( N \) blocks \( \mathcal{P} \). On the other hand, suppose that \( \{0\} \notin \mathcal{P} \) and \( \mathcal{P} = \{\{0\}, \{j\}_{j \in N}\} \), which means that there is \( S \in \mathcal{P} \) such that \( |S| \geq 2 \). \( i \in S \) blocks \( \mathcal{P} \) if and only if

\[
\frac{M'}{|\mathcal{P}|+1} > \frac{M'}{|S||\mathcal{P}|^2} \iff |S| > \left(1 + \frac{1}{|\mathcal{P}|}\right)^2.
\]

Both \( |S| = n, |\mathcal{P}| = 2 \) and \( 2 \leq |S| < n, |\mathcal{P}| \geq 3 \) satisfies this inequality. Therefore \( \mathcal{P} \notin C^{pro} \) for any \( \mathcal{P} \) such that \( \{0\} \in \mathcal{P} \).

Second, we show that \( \mathcal{P} \notin C^{pro} \) for any \( \mathcal{P} \) such that \( \{0\} \notin \mathcal{P} \) and there is \( S \subseteq N \) with \( S \in \mathcal{P} \) and \( |S| \geq 2 \). Since \( |S| \geq 2 > (1 + \frac{1}{2+1})^2 \geq (1 + \frac{1}{|\mathcal{P}|+1})^2 \), we have

\[
\frac{M_H}{(|\mathcal{P}|+2)^2} > \frac{M_H}{|S|(|\mathcal{P}|+1)^2},
\]

which implies that \( i \in S \) blocks \( \mathcal{P} \). Therefore, if \( \mathcal{P} \in C^{pro} \), then it must be that \( \mathcal{P} = \{\{0\} \cup S, \{j\}_{j \notin S}\} \) for some \( S \subseteq N \).

Finally, we show that \( |S| \geq n-4 \). Consider a deviation by \( S' \subseteq N \setminus S \). The payoff for \( i \in S' \) after deviation is given by \( \frac{M_H}{|S'|(n-|S|-|S'|+3)^2} \), which is maximized at \( |S'| = 2 \) or \( |S'| = n-|S| \). If \( |S'| = 2 \), then the inequality

\[
\frac{M_H}{2(n-|S|+1)^2} > \frac{M_H}{(n-|S|+2)^2} \iff (1 + \frac{1}{n-|S|+1})^2 > 2
\]
implies that $S'$ cannot block $\mathcal{P}$. On the other hand, if $|S'| = n - |S|$, then the inequality
\[
\frac{M_H}{9(n - |S|)} > \frac{M_H}{(n - |S| + 2)^2} \iff (n - |S| + 2)^2 > 9(n - |S|)
\]
\[
\iff (n - |S| - 4)(n - |S| - 1) > 0
\]
\[
\iff |S| \leq n - 5 \text{ or } |S| = n
\]
implies that $S' = N \setminus S \neq \emptyset$ blocks $\mathcal{P}$ if and only if $|S| \geq n - 5$, which completes the proof.

In contrast to the case in the previous section, some coalition structures can be stable even if $n \geq 3$. To illustrate the argument, we first consider the case of $n = 3$. Table 2 indicates the possible patterns of partitions and each firm’s corresponding payoff. By Lemma 2, we consider the condition for $\mathcal{P} = \{S \cup \{0\}, \{j\}_{j \in S}\} \in C^{pro}$ for each $|S| = 1, 2, 3$.

In each case, we will show that there is $\tau^{pro}$ and $\overline{\tau}^{pro}$ such that $\mathcal{P} = \{S \cup \{0\}, \{j\}_{j \in S}\} \in C^{pro}$ if and only if $\tau \in [\tau^{pro}, \overline{\tau}^{pro}] \cap (0, 1)$.

<table>
<thead>
<tr>
<th>$\mathcal{P}$/ $\pi_i(\mathcal{P})$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>123</td>
<td>$\frac{\tau}{4}M_L(1)$</td>
<td>$\frac{1-\tau}{12}M_L(1)$</td>
<td>$\frac{1-\tau}{12}M_L(1)$</td>
</tr>
<tr>
<td>0</td>
<td>123</td>
<td>$\frac{\tau}{9}M_L(2)$</td>
<td>$\frac{1-\tau}{16}M_L(2)$</td>
<td>$\frac{1}{18}M_H$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>3</td>
<td>$\frac{\tau}{16}M_L(3)$</td>
<td>$\frac{1-\tau}{16}M_L(3)$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>3</td>
<td>$\frac{\tau}{9}M_L(2)$</td>
<td>$\frac{1-\tau}{18}M_L(2)$</td>
</tr>
<tr>
<td>0</td>
<td>123</td>
<td>0</td>
<td>$\frac{1}{12}M'$</td>
<td>$\frac{1}{12}M'$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>23</td>
<td>0</td>
<td>$\frac{1}{9}M'$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2: The payoff of each firm for $n = 3$. Since each firm is ex-ante symmetric, we only write the patterns of the partitions.

$|S| = 1$: Without loss of generality, we consider $\mathcal{P} = \{\{0,1\}, \{2\}, \{3\}\}$. First, we consider deviations by $S' \subseteq N$, i.e., deviations without the patent holder. In this case, any deviation by $S' \subseteq N$ is not profitable if and only if all the following conditions hold:

$S' = \{1\} : \pi_1(\mathcal{P}) = \frac{1-\tau}{16}M_L(3) \geq \frac{1}{16}M'$,

$S' = \{1, 2\} : \pi_1(\mathcal{P}) = \frac{1-\tau}{16}M_L(3) \geq \frac{1}{18}M'$, or $\pi_2(\mathcal{P}) = \frac{1}{16}M_H \geq \frac{1}{18}M'$,

$S' = \{2, 3\} : \pi_2(\mathcal{P}) = \pi_3(\mathcal{P}) = \frac{1}{16}M_H \geq \frac{1}{18}M_H$,

$S' = N : \pi_1(\mathcal{P}) = \frac{1-\tau}{16}M_L(3) \geq \frac{1}{12}M'$, or $\pi_2(\mathcal{P}) = \pi_3(\mathcal{P}) = \frac{1}{16}M_H \geq \frac{1}{12}M'$. 


By these inequalities, we obtain the upper bound of \( \tau \), that is, \( \tau \leq \pi^{\text{pro}} = 1 - \frac{4}{3} \frac{M'}{M_L(3)} \).

Second, we consider deviations by \( S' \cup \{0\} \) with \( S' \subseteq N \), i.e., deviations with the patent holder. Note that 0 is profitable iff the number of partition elements decrease. Hence, any deviation by \( S' \cup \{0\} \) with \( S' \subseteq N \) is not profitable if and only if all the following conditions hold:

\[
S' = \{0, 1, 2\} : \pi_1(\mathcal{P}) = \frac{1 - \tau}{16} M_L(3) \geq \frac{1 - \tau}{18} M_L(2), \quad \text{or} \quad \pi_2(\mathcal{P}) = \frac{1}{16} M_H \geq \frac{1 - \tau}{18} M_L(2),
\]

\[
S' = \{0, 2, 3\} : \pi_2(\mathcal{P}) = \pi_3(\mathcal{P}) = \frac{1}{16} M_H \geq \frac{1 - \tau}{18} M_L(2),
\]

\[
S' = N \cup \{0\} : \pi_1(\mathcal{P}) = \frac{1 - \tau}{16} M_L(3) \geq \frac{1 - \tau}{12} M_L(1), \quad \text{or} \quad \pi_2(\mathcal{P}) = \pi_3(\mathcal{P}) = \frac{1}{16} M_H \geq \frac{1 - \tau}{12} M_L(1).
\]

Similar to the first case, we obtain the lower bound of \( \tau \), that is

\[
\tau \geq \tau^{\text{pro}} = \begin{cases} 1 - \frac{9}{8} \frac{M_H}{M_L(2)}, & \text{if } \frac{M_L(3)}{M_L(1)} \geq \frac{4}{3}, \\ 1 - \min \left\{ \frac{9}{8} \frac{M_H}{M_L(2)}, \frac{3}{4} \frac{M_H}{M_L(1)} \right\}, & \text{otherwise}. \end{cases}
\]

For such \( \tau^{\text{pro}} \) and \( \pi^{\text{pro}} \), we show that \( \tau \in [\tau^{\text{pro}}, \pi^{\text{pro}}] \cap (0, 1) = \emptyset \) for any \( \tilde{\epsilon} \in (0, 1) \). To see this, it suffices to show that neither \( \frac{4}{3} \frac{M'}{M_L(3)} \leq \frac{9}{8} \frac{M_H}{M_L(2)} \) nor \( \frac{4}{3} \frac{M'}{M_L(3)} \leq \frac{3}{4} \frac{M_H}{M_L(1)} \) can hold for any \( \tilde{\epsilon} \in (0, 1) \). In the former case, \( \frac{4}{3} \frac{M'}{M_L(3)} \leq \frac{9}{8} \frac{M_H}{M_L(2)} \Leftrightarrow 243\tilde{\epsilon}^2 - 324\tilde{\epsilon}^3 - 182\tilde{\epsilon}^2 - 20\tilde{\epsilon} - 5 \leq 0 \), which cannot hold for any \( \tilde{\epsilon} \in (0, 1) \). In the latter case, \( \frac{4}{3} \frac{M'}{M_L(3)} \leq \frac{3}{4} \frac{M_H}{M_L(1)} \Leftrightarrow (9\tilde{\epsilon}^2 - 10\tilde{\epsilon} - 7)(9\tilde{\epsilon}^2 - 2\tilde{\epsilon} + 1) \geq 0 \), which cannot hold for any \( \tilde{\epsilon} \in (0, 1) \). Therefore, if \(|S| = 1\), we can say that \( \mathcal{P} = \{S \cup \{0\}, \{j \mid j \notin S\}\} \notin C^{\text{pro}} \) for any \( \tilde{\epsilon}, \tau \in (0, 1) \).

\(|S| = 2\): Without loss of generality, we consider \( \mathcal{P} = \{\{0, 1, 2\}, \{3\}\} \). By the same logic for \(|S| = 1\), we consider the two cases; deviations without and with the patent holder. First, any deviation by \( S' \subseteq N \) is not profitable if and only if all the following conditions hold:

\[
S' = \{1\} : \pi_1(\mathcal{P}) = \frac{1 - \tau}{18} M_L(2) \geq \frac{1}{16} M_H,
\]

\[
S' = \{1, 2\} : \pi_1(\mathcal{P}) = \pi_2(\mathcal{P}) = \frac{1 - \tau}{18} M_L(2) \geq \frac{1}{18} M',
\]

\[
S' = \{1, 3\} : \pi_1(\mathcal{P}) = \frac{1 - \tau}{18} M_L(2) \geq \frac{1}{18} M_H, \quad \text{or} \quad \pi_3(\mathcal{P}) = \frac{1}{9} M_H \geq \frac{1}{18} M_H,
\]

\[
T = N : \pi_1(\mathcal{P}) = \pi_2(\mathcal{P}) = \frac{1 - \tau}{18} M_L(2) \geq \frac{1}{12} M', \quad \text{or} \quad \pi_3(\mathcal{P}) = \frac{1}{9} M_H \geq \frac{1}{12} M'.
\]

By these inequalities, we obtain

\[
\tau \leq \pi^{\text{pro}} = \begin{cases} 1 - \max \left\{ \frac{9}{8} \frac{M_H}{M_L(2)}, \frac{M'}{M_L(2)} \right\}, & \text{if } \frac{M_H}{M'} \geq \frac{3}{4}, \\ 1 - \frac{3}{2} \frac{M'}{M_L(2)}, & \text{otherwise}. \end{cases}
\]
Second, any deviation by \( S' \cup \{0\} \) with \( S' \subseteq N \) is not profitable if and only if the following condition holds:

\[
S' = N \cup \{0\} : \pi_1(\mathcal{P}) = \pi_2(\mathcal{P}) = \frac{1 - \tau}{18} M_L(2) \geq \frac{1 - \tau}{12} M_L(1) \text{ or } \pi_3(\mathcal{P}) = \frac{1}{9} M_H \geq \frac{1 - \tau}{12} M_L(1).
\]

By these inequalities, we obtain the lower bound of \( \tau \), that is

\[
\tau \geq \tau^{\text{pro}} = \begin{cases} 
0, & \text{if } \frac{M_L(2)}{M_L(1)} \geq \frac{3}{2}, \\
1 - \frac{4}{3} \frac{M_H}{M_L(1)}, & \text{otherwise.}
\end{cases}
\]

We show that there is \( \tilde{c} \in (0, 1) \) such that \([\tau^{\text{pro}}, \tau^{\text{pro}}] \cap (0, 1) \neq \emptyset \). For \( \tau^{\text{pro}} \), since \( \frac{M_H}{M'} \geq \frac{3}{4} \Leftrightarrow (1 - \tilde{c})^2 \geq \frac{3}{4} \) and \( \frac{9}{8} \frac{M_H}{M_L(2)} \geq \frac{M'}{M_L(2)} \Leftrightarrow (1 - \tilde{c})^2 \geq \frac{8}{9} \), we can restate it as

\[
\tau \leq \tau^{\text{pro}} = \begin{cases} 
1 - \frac{9}{8} \left( \frac{3}{2} - 2\tilde{c} \right)^2, & \text{if } \tilde{c} \in (0, 1 - \frac{2\sqrt{3}}{3}], \\
1 - \frac{1}{(2\tilde{c})^2}, & \text{if } \tilde{c} \in (1 - \frac{2\sqrt{3}}{3}, 1 - \frac{\sqrt{3}}{2}], \\
1 - \frac{3}{2} \frac{1}{(1+2\tilde{c})^2}, & \text{if } \tilde{c} \in (1 - \frac{\sqrt{3}}{2}, 1).
\end{cases}
\]

Similarly, for \( \tau^{\text{pro}} \), since \( \frac{M_L(2)}{M_L(1)} \geq \frac{3}{2} \Leftrightarrow 5\tilde{c}^2 + 2\tilde{c} - 1 \geq 0 \), we can restate it as

\[
\tau \geq \tau^{\text{pro}} = \begin{cases} 
0, & \text{if } \tilde{c} \in [\sqrt{\frac{6}{5}} - 1, 1), \\
1 - \frac{4}{3} \left( \frac{1 - \sqrt{3}}{1 - 2\tilde{c}} \right)^2, & \text{if } \tilde{c} \in (0, \sqrt{\frac{6}{5}} - 1).
\end{cases}
\]

As the figure 1 suggests, we can say that \([\tau^{\text{pro}}, \tau^{\text{pro}}] \cap (0, 1) \neq \emptyset \) for any \( \tilde{c} \in (0, 1 - \frac{\sqrt{3}}{2}] \cup [\sqrt{\frac{6}{5}} - 1, 1) \), which implies that \( \mathcal{P} = \{S \cup \{0\}, \{j\}_{j \neq S} \} \subseteq C^{\text{pro}} \) where \( |S| = 2 \) for such \( \tilde{c} \) and \( \tau \).

**|S| = 3**: Since 0 does not have any incentive to deviate from \( \mathcal{P} = \{N \cup \{0\}\} \), it suffices to consider deviations without the patent holder. In this case, the most profitable deviation is \( T = N \), which is not profitable if and only if

\[
\pi_1(\mathcal{P}) = \pi_2(\mathcal{P}) = \pi_3(\mathcal{P}) = \frac{1 - \tau}{12} M_L(1) \geq \frac{1}{12} M'.
\]

Hence, we obtain \( \tau \leq \tau^{\text{pro}} = 1 - \frac{M'}{M_L(1)} = 1 - \left( \frac{1}{1+\tilde{c}} \right)^2 \in (0, 1) \) for any \( \tilde{c} \in (0, 1) \), which implies that we can find \( \tau \) such that \( \mathcal{P} = \{N \cup \{0\}\} \subseteq C^{\text{pro}} \) for any \( \tilde{c} \in (0, 1) \).

The above arguments show that, for \( n = 3 \), \( \{S \cup \{0\}, \{j\}_{j \neq S}\} \subseteq C^{\text{pro}} \) if and only if there is a suitable upper and lower bounds for \( \tau \), the patent holder’s share of licensee firms’ payoff, such that \( \tau \) is chosen in this region. The following our first main result shows that this is true in general.
Proposition 4. For any $n \geq 3$, there are $\tau^{pro}, \tau^{pro}$, such that $\mathcal{P} = \{S \cup \{0\}, \{j\}_{j \notin S}\} \in C^{pro}$ if and only if $\tau \in [\tau^{pro}, \tau^{pro}] \cap (0, 1)$.

To prove the result, we have to check all possible deviations from $\mathcal{P} = \{S \cup \{0\}, \{j\}_{j \notin S}\}$ are not profitable. We explain the key logic for the formal proof, which we give in Appendix B. By the observation from the case of $n = 3$, basically, there are two patterns of deviations by $S' \notin \mathcal{P}$, that is, deviations without and with the patent holder.

For the deviations without the patent holder, there are three sub-cases: (i) $S' \subseteq N \setminus S$, (ii) $S' \subseteq S$, and (iii) $S' \cap S \neq \emptyset, S' \cap (N \setminus S) \neq \emptyset$. The case (i) cannot happen if $|S| \geq \max\{n - 4, 1\}$ by Lemma 2, so that there is no restriction. In case (ii), the most profitable deviation is either by $S' = \{j\}$ for some $j \in S$ or $S' = S$. In case (iii), the most profitable deviation is by $S' = N$. Therefore, to prevent the cases (ii) and (iii), we need $\tau$ should not be so large, which implies that there is an upper bound of $\tau$.

For the deviations with the patent holder, the patent holder has the incentive to deviate from $\mathcal{P}$ with $S' \subseteq N$ if and only if the partition size decreases by that. Hence, we can show that only candidates are to be either $S \subseteq S'$ or $S \cap S' = \emptyset$. In each case, we need $\tau$ should not be so small, which implies that there is a lower bound of $\tau$.

The figure 2 shows a numerical example in the case of $n = 9$. By Lemma 2, only the candidates for $C^{pro}$ is $\mathcal{P} = \{S \cup \{0\}, \{j\}_{j \notin S}\}$ with $|S| \geq 5$. Note that, for any $5 \leq |S| \leq 7$, such partitions are not stable for any $\tilde{c} \in (0, 1)$. The following result shows that $C^{pro}$ can contain $\mathcal{P} = \{S \cup \{0\}, \{j\}_{j \notin S}\}$ with $|S| \geq n - 1$. 

Figure 1: The condition for $\mathcal{P} = \{S \cup \{0\}, \{j\}_{j \notin S}\} \in C^{pro}$ where $|S| = 2$. The horizontal line corresponds to $\tilde{c} \in (0, 1)$. The solid line indicates $\tau$ and the dashed line indicates $\tau$, respectively.
**Proposition 5.** For any \( n \geq 3 \), there are \( \tilde{c} \in (0,1) \) and \( \tau \in (0,1) \) such that \( \mathcal{P} \in C^{pro} \) if and only if \( \mathcal{P} = \{\{0\} \cup N\} \) or \( \mathcal{P} = \{\{0\} \cup N\setminus \{i\}, \{i\}\} \) for some \( i \in N \).

If there is no patent holder, \( C^{opt} = \emptyset \) by Corollary 1. In contrast, if there is a patent holder and the market size is small, even the optimistic can be nonempty.

**Proposition 6.** If \( 3 \leq n \leq 9 \), there are \( \tau^{opt}, \tau^{opt} \), such that \( \mathcal{P} = \{\{0\} \cup S\} \cup \{j\}_{j \notin S} \in C^{opt} \) if and only if \( \tau \in [\tau^{opt}, \tau^{opt}] \cap (0,1) \) where \( \mathcal{P} = \{\{0\} \cup N\} \) or \( \mathcal{P} = \{\{0\} \cup N\setminus \{i\}, \{i\}\} \) for some \( i \in N \). Moreover, if \( n \geq 10 \), \( C^{opt} = \emptyset \).

![Figure 2:](image)

Figure 2: The condition for \( \mathcal{P} = \{\{0\} \cup S\} \cup \{j\}_{j \notin S} \notin C^{pro} \) in the case of \( n = 9 \). The horizontal line corresponds to \( \tilde{c} \in (0,1) \). The solid line indicates \( \tau \) and the dashed line indicates \( \tau^{opt} \), respectively.

Next, we consider the stability of coalition structures in terms of \( C^{pes} \). Since \( \tilde{c} < 1 \), we know that \( \frac{1+\tilde{c}|\mathcal{P}|}{|\mathcal{P}|+1} \) is decreasing with respect to \( |\mathcal{P}| \geq 1 \), which implies that \( \frac{M_L(|\mathcal{P}|)}{(|\mathcal{P}|+1)^2} = \left(\frac{1+\tilde{c}|\mathcal{P}|}{|\mathcal{P}|+1}\right)^2 \) is decreasing with respect to \( |\mathcal{P}| \geq 1 \). This is similar to the benchmark case, namely, there is only a positive externality. Hence, we can obtain the following result.

**Lemma 3.** For each \( \mathcal{P} \in \Pi \) and \( S \notin \mathcal{P} \) with \( S \subseteq N \),

1. For \( i = 0 \),
   \[
   \min_{\mathcal{P}' \in \Pi; S \cup \{0\} \in \mathcal{P}'} \pi_i(\mathcal{P}') = \frac{\tau}{(n - |S| + 2)^2} M_L(n - |S| + 1).
   \]

2. For \( i \neq 0 \),
   \[
   \min_{\mathcal{P}' \in \Pi; S \cup \{0\} \in \mathcal{P}'} \pi_i(\mathcal{P}') = \frac{1 - \tau}{|S|(n - |S| + 2)^2} M_L(n - |S| + 1).
   \]
Thus, the worst partition for $j \in S \cup \{0\}$ is $P' = \{S \cup \{0\}, \{i \in S\}\}$.

For each $P \in \Pi$ with $S \notin P$ with $S \subset N$,

$$\min_{P \in \Pi; S \in P} \pi_i(P') = \frac{1}{|S|(n-|S|+2)^2}M_H.$$  

Thus, the worst partition for $j \in S$ is $P' = \{S, \{i \in S\cup \{0\}\}, \{0 \cup l\}\}$ for any $l \notin S$.

To compare the condition with that of $C^{pro}$, we characterize the condition for the stability of the partitions of the form $P = \{S\cup \{0\}, \{j\}_{j \notin S}\}$ for some $S \subseteq N$. To illustrate the argument, we first consider the case of $n = 3$ as in the case of $C^{pro}$. We will show that there is $\tau_{pes}$ and $\tau_{pes}$ such that $P = \{S\cup \{0\}, \{j\}_{j \notin S}\} \in C^{pes}$ if and only if $\tau \in [\tau_{pes}, \tau_{pes}] \cap (0, 1)$.

$|S| = 2$: Without loss of generality, we consider $P = \{\{0, 1, 2\}, \{3\}\}$. We consider the two cases; deviations without and with the patent holder. First, any deviation by $S' \subseteq N$ is not profitable if and only if all the following conditions hold:

$$S' = \{1\} \colon \pi_1(P) = \frac{1 - \tau}{18}M_L(2) \geq \frac{1}{16}M_H,$$

$$S' = \{1, 2\} \colon \pi_1(P) = \pi_2(P) = \frac{1 - \tau}{18}M_L(2) \geq \frac{1}{18}M_H,$$

$$S' = \{1, 3\} \colon \pi_1(P) = \frac{1 - \tau}{18}M_L(2) \geq \frac{1}{18}M_H,$$

$$S' = \{2\} \colon \pi_2(P) = \frac{1 - \tau}{18}M_L(2) \geq \frac{1}{18}M_H \text{ or } \pi_3(P) = \frac{1}{9}M_H \geq \frac{1}{18}M_H,$$

$T = N \colon \pi_1(P) = \pi_2(P) = \frac{1 - \tau}{18}M_L(2) \geq \frac{1}{12}M', \text{ or } \pi_3(P) = \frac{1}{9}M_H \geq \frac{1}{12}M'$.

In the above calculations, note that incentive for $S = \{1, 2\}$ is different from that of the projective core because, upon deviating, each firm considers the worst case. For the projective core, the partition after the deviation is $P' = \{\{0\}, \{1, 2\}, \{3\}\}$, but, under the pessimistic view, the partition after deviation is $P' = \{\{0, 3\}, \{1, 2\}\}$. Hence, the incentive constraint to prevent the deviation is slightly relaxed; $\pi_1(P) = \pi_2(P) \geq \frac{1}{18}M' > \frac{1}{18}M_H$.

By these inequalities, we obtain

$$\tau \leq \tau_{pes} = \begin{cases} 1 - \frac{9}{8}M_H & \text{if } \frac{M_H}{M'} \geq \frac{3}{4}, \\ 1 - \frac{3}{2} \frac{M'}{M_L(2)}, & \text{otherwise}. \end{cases}$$

Second, any deviation by $S' \cup \{0\}$ with $S' \subseteq N$ is not profitable if and only if the following condition holds:

$$S' = N \cup \{0\} \colon \pi_1(P) = \pi_2(P) = \frac{1 - \tau}{18}M_L(2) \geq \frac{1 - \tau}{12}M_L(1) \text{ or } \pi_3(P) = \frac{1}{9}M_H \geq \frac{1 - \tau}{12}M_L(1).$$

By these inequalities, we obtain the lower bound of $\tau$, that is

$$\tau \geq \tau_{pes} = \begin{cases} 0, & \text{if } \frac{M_L(2)}{M_L(1)} \geq \frac{3}{4}, \\ 1 - \frac{4}{3} \frac{M_H}{M_L(1)}, & \text{otherwise}. \end{cases}$$
Note that the upper bound is different while the lower bound is same in the case of the projective core. As the figure 3 suggests, we can also say that, for any $\tilde{c} \in (0, 1 - \frac{\sqrt{3}}{2}] \cup [\frac{\sqrt{5} - 1}{5}, 1)$, $[\tau_{pes}, \tau_{pes}] \cap (0, 1) \neq \emptyset$, which implies that we can find $\tau$ such that $P = \{S \cup \{0\}, \{j\} \mid j \not\in S]\} \in C^{pes}$ where $|S| = 2$ for such $\tilde{c}$ and $\tau$.

![Figure 3: The condition for $P = \{S \cup \{0\}, \{j\} \mid j \not\in S\} \not\in C^{pes}$ where $|S| = 2$. The horizontal line corresponds to $\tilde{c} \in (0, 1)$. The solid line indicates $\tau$ and the dashed line indicates $\tau_\ast$, respectively.](image)

$|S| = 1, 3$: As we have seen in the case of $|S| = 2$, the incentive constraint to prevent each deviation is different and relaxed relative to the corresponding constraint for the projective core. However, we can obtain that the same condition for the corresponding stability condition of the projective core.

The following is our second main result, which is parallel to Proposition 4.

**Proposition 7.** For any $n \geq 3$, there are $\tau_{pes}, \tau_{pes}$ such that $P = \{S \cup \{0\}, \{j\} \mid j \not\in S\} \in C^{pes}$ if and only if $\tau \in [\tau_{pes}, \tau_{pes}] \cap (0, 1)$.

The proof strategy is same in the case of $C^{pro}$, but incentive constraints are different because each firm is pessimistic. For the deviations without the patent holder, as it is same in the case of $C^{pro}$, there are three sub-cases: (i) $S' \subseteq N \setminus S$, (ii) $S' \subseteq S$, and (iii) $S' \cap S \neq \emptyset, S' \cap (N \setminus S) \neq \emptyset$. In the case (i), the worst partition for each firm $i \in S'$ is the form $P' = \{\{0, j\}, S', \{k\} \mid k \notin S' \cup \{j\}\}$ for some $j \notin S'$. If $|S|$ is relatively small, the payoff of $i \in S'$ can be higher because such a merger mitigates the market competition. Hence, to prevent the case (i), we need that $|S|$ is relatively large. If $\tau$ is so large, the payoff of
$i \in S$ is very small, so that such a firm will have incentive to deviate from $P$. Therefore, to prevent the cases (ii) and (iii), we need $\tau$ should not be so large, which implies that there is an upper bound of $\tau$.

For the deviations with the patent holder, the patent holder has the incentive to deviate from $P$ with $S' \subseteq N$ if and only if $|S'| > |S|$. There are two sub-cases: (iv) $S' \subseteq N \backslash S$ and (v) $S' \cap S \neq \emptyset$. If $|S|$ is relatively large, we can ignore case (iv). Moreover, if the payoff from $P$ is high enough for $i \in S$, any deviation is not attractive, so that there is no restriction. Otherwise, if $\tau$ is so small, both $i \in S$ and $j \in N \backslash S$ will have incentive to deviate from $P$ by making suitable size of $S'$. Hence, to prevent the cases (iv) and (v), we need $\tau$ should not be so small, which implies that there is a lower bound of $\tau$.

The figure 4 shows a numerical example in the case of $n = 9$. As we can see, $P = \{S \cup \{0\}, \{j\}_{j \notin S}\}$ is not stable for any $\tilde{c}, \tau \in (0, 1)$ if $|S| \leq 4$, whereas it can be stable for sufficiently large $\tilde{c}$ if $|S| = 5$. Moreover, it can be also stable for any $|S| \geq 5$, which contrasts with the case of $C^{pro}$. This example suggests a monotonicity about $C^{pes}$ in the following sense, that is, there is a lower bound $|S|$ such that, for any $S \subseteq N$ with $|S| \geq |S|$, $P = \{S \cup \{0\}, \{j\}_{j \notin S}\} \in C^{pes}$ for some $\tilde{c}, \tau \in (0, 1)$. In particular, it will be true for sufficiently large $\tilde{c}$. The following result shows that this monotonicity holds in general.

**Proposition 8.** For any $n \geq 3$, there are $\tilde{c} \in (0, 1)$ and $\tau \in (0, 1)$ such that $P = \{S \cup \{0\}, \{j\}_{j \notin S}\} \in C^{pes}$ if and only if $\frac{n-1}{2} < |S| \leq n$.

This result implies that, even if we assume that each firm is pessimistic, the coalition structure is not stable if the coalition of licensee firms is not large enough. The intuition for the monotonicity is as follows. By Lemma 3, the worst case for a firm in the deviating coalition is a severe market competition where other firms react by making a singleton coalition. If $\tilde{c}$ is sufficiently large and $\tau$ is not large enough, the most profitable deviation for each licensee firm $i \in S$ is the deviation with the patent holder, that is, $0 \in S'$. For the patent holder, a deviation is profitable if and only if $|P'| < |P|$ where $P' = \{S', \{j\}_{j \notin S'}\}$. Since the size of licensee firms are large enough, $|S| > \frac{n-1}{2}$ in our case, it is possible only if both $S' \cap S \neq \emptyset$ and $S' \cap (N \backslash S) \neq \emptyset$. For such a deviation, we can show that $\pi_i(P) > \pi_i(P')$ for $i \in S \cap S'$, so that each licensee firm is not willing to participate in the such a deviation. Hence, by the above arguments, neither licensee firms nor patent holder join any deviation. Moreover, if the size of licensee firms is large enough, any deviation by $S' \subseteq N \backslash S$ is not profitable because $|P'|$ is large enough where $P' = \{\{0, i\}, S', \{j\}_{j \in S \cup \{i\}}\}$. Therefore, summarizing the above arguments, the coalition structure is stable.

By Proposition 5, $C^{pro}$ also has this monotonicity property where $|S| = n - 1$. However, we can show that, even though $C^{opt} \neq \emptyset$, it can be non-monotonic in this sense. More
precisely, when $n = 9$, we can show that $\mathcal{P} = \{ \{0\} \cup \mathbb{N} \} \not\in C^{opt}$ for any $\tilde{c}, \tau \in (0, 1)$ while, for any $i \in \mathbb{N}$, $\mathcal{P} = \{ \{0\} \cup \mathbb{N}\{i\}, \{i\} \} \in C^{pro}$ for some $\tilde{c}, \tau \in (0, 1)$. We provide details in Appendix B.

Figure 4: The condition for $\mathcal{P} = \{ S \cup \{0\}, \{j\}_{j \not\in S} \} \not\in C^{pes}$ in the case of $n = 9$. The horizontal line corresponds to $\tilde{c} \in (0, 1)$. The solid line indicates $\tau$ and the dashed line indicates $\tau$, respectively.

5 Conclusion

Since the seminal paper by Arrow (1962), theoretical studies of the patent licensing games have uncovered that what type of payment scheme is optimal for the patent holder and what payoff allocation is stable among licensee firms. As we already mentioned in Section 1, these studies assume that each firms is prohibited to cooperate with each other and ignore incentive of firms to make another coalition structure.

We consider a coalition formation game in the patent licensing games to analyze whether this assumption is plausible and what kind of coalition structure emerges. We show that the coalition structures can be stable if and only if the size of patentee firms are relatively large. This result is a first theoretical foundation for the basic assumption
in the literature of patent licensing games.

Our study focuses on the Cournot oligopoly market. Hence, whether similar results hold or not in another market structure such as a Bertrand oligopoly market is also a meaningful next question. In this direction, analysis of abstract games, which is the approach by Watanabe and Muto (2008), Kishimoto et al. (2011), Kishimoto (2013), Hirai and Watanabe (2018) and Hirai et al. (2019), will also be a next step to understand what kind of coalition structures are stable without depending upon the specific market structure.

Finally, firms’ forward looking expectation may affect the stability of coalition structures. Hirai et al. (2019) consider the forward looking firms in the patent licensing games and characterize the von Neumann Morgenstern stable set of the payoff distributions. As shown by Abe (2018), a type of forward looking expectation induces the same outcomes in the (myopic) pessimistic core. Pursing such a relationship in our setting is also a remained future work.

Appendix A  Equilibrium derivation

Without loss of generality, let \( i = 1 \) be the firm whose cost is \( c_L \) and other firms \( j \neq 1 \) have \( c_H \). The profit function for each firm is as follows:

\[
f_i(q_1, q_{-1}) = (p_1 - c_L)q_1, \quad f_j(q_j, q_{-j}) = (p_j - c_H)q_j \quad \text{for all } j \neq 1.
\]

Since we assume the interior solution, by the first order conditions, we have

\[
q_1 = \frac{\alpha - c_L - \beta \sum_{j \neq 1} q_j}{2\beta}, \quad q_j = \frac{\alpha - c_H - \beta q_1 - \beta \sum_{k \neq j, 1} q_k}{2\beta} \quad \text{for all } j \neq 1.
\]

These simultaneous equations can be written as

\[
\begin{pmatrix}
\beta & \beta/2 & \ldots & \beta/2 \\
\beta/2 & \ddots & \ddots & \beta/2 \\
\vdots & \ddots & \ddots & \vdots \\
\beta/2 & \ldots & \beta/2 & \beta
\end{pmatrix}
\begin{pmatrix}
q_1 \\
\vdots \\
q_n
\end{pmatrix}
=
\begin{pmatrix}
\alpha - c_L \\
\alpha - c_H \\
\vdots \\
\alpha - c_H
\end{pmatrix}.
\]

Since the coefficient matrix can be invertible in our case, we can obtain the equilibrium output as

\[
\begin{pmatrix}
q_1 \\
\vdots \\
q_n
\end{pmatrix}
=
\left(
\begin{pmatrix}
\beta & \beta/2 & \ldots & \beta/2 \\
\beta/2 & \ddots & \ddots & \beta/2 \\
\vdots & \ddots & \ddots & \vdots \\
\beta/2 & \ldots & \beta/2 & \beta
\end{pmatrix}
\right)^{-1}
\begin{pmatrix}
\alpha - c_L \\
\alpha - c_H \\
\vdots \\
\alpha - c_H
\end{pmatrix}.
\]
which is reduced to
\[ q^*_i = \frac{\lambda + n \tilde{c}}{(n + 1)\beta}, \quad q^*_j = \frac{\lambda - \tilde{c}}{(n + 1)\beta} \] for all \( j \neq 1 \).

Then, substituting these outputs to \( p_i = \alpha - \beta q_i - \beta(\sum_{j \neq i} q_j) \) for any \( i \in \mathbb{N} \), we can obtain \( p_1 - c_L = \beta q^*_1 \) and \( p_j - c_H = \beta q^*_j \) for all \( j \neq 1 \). Finally, we can obtain the equilibrium payoff as
\[ f_1(q^*_1, q^*_{-1}) = \pi_L, \quad f_j(q^*_j, q^*_j) = \pi_H \] for all \( j \neq 1 \).

## Appendix B Proof of results in Section 4

### B.1 Mathematical preliminaries

We first introduce summarize the useful algebraic calculations for the proofs about both \( C^{pro} \) and \( C^{pes} \).

For each natural number \( n, k \in \mathbb{N} \) with \( k \leq n - 2 \), define the equation
\[ P(k, n) \equiv n^2 - (k^2 + 6k)n + (k^3 + 5k^2 + 4). \]

Let \( \underline{n}(k) \) and \( \overline{n}(k) \) where \( \underline{n}(k) < \overline{n}(k) \) be the roots of the equation, that is, \( P(k, n) = (n - \underline{n}(k))(n - \overline{n}(k)) = 0 \).

**Lemma 4.** For any \( k \in \mathbb{N} \), \( \underline{n}(k) \leq k + 1 \).

**Proof.** Fix any \( k \in \mathbb{N} \). Then, by inserting \( n = k + 1 \) into \( P(k, n) \), we have \( P(k, k + 1) = -k^2 - 4k + 5 = -(k + 5)(k - 1) \leq 0 \) for any \( k \in \mathbb{N} \), which implies that \( \underline{n}(k) \leq k + 1 \).

Let us define \( f(x) = \frac{1 + \tilde{c}(n-x+1)}{\sqrt{2(n-x+2)}} \) for each \( x > 0 \).

**Lemma 5.** Suppose that \( n \geq 4 \). Then, \( f \) is decreasing in \([2, S^*]\). Moreover, if \( S^* < n \), then \( f \) is increasing in \([S^*, n]\) where \( S^* = \frac{3 + (2n+1)\tilde{c} - \sqrt{(1-\tilde{c})(9 + (8n-7)\tilde{c})}}{2\tilde{c}} \).

**Proof.** By the direct calculation, the denominator of \( f'(x), x \in (0, n) \), denoted by \( g(x) \), is given by
\[ g(x) = -\tilde{c}x^2 + (3 + (2n + 1)\tilde{c})x - (n + 2)(1 + (n + 1)\tilde{c}). \]

Then, note that discriminant of the quadratic equation \( g(x) = 0 \) is
\[ D = (3 + (2n + 1)\tilde{c})^2 - 4(n + 2)(1 + (n + 1)\tilde{c})\tilde{c} = (1 - \tilde{c})(9 + (8n - 7)\tilde{c}) > 0 \]
because \(1 > \tilde{c}\) and \(9 + (8n - 7)\tilde{c} > 0\). Hence, the quadratic equation \(g(x) = 0\) has two real solutions. The smaller one is given by

\[
\mathcal{S}^* = 3 + (2n + 1)\tilde{c} - \sqrt{(1 - \tilde{c})(9 + (8n - 7)\tilde{c})}
\]

Since \(n \geq 4\),
\[
g(2) = -\left((n^2 - n + 4)\tilde{c} + (n - 4)\right) < 0,
\]
so that \(2 < \mathcal{S}^*\). Therefore, \(f\) is decreasing in \([2, \mathcal{S}^*]\). In addition, if \(\mathcal{S}^* < n\), then \(f\) is increasing in \([\mathcal{S}^*, n]\). \(\square\)

Let us define \(s_1(n) = n - 2(1 - \tilde{c})\sqrt{n} + 2\) and \(s_2(n) = -1 + \sqrt{1 + 2n}\). Note that \(s_2(n) < s_1(n)\).

**Lemma 6.**  
(i) \(s_1(n) > s_2(n)\).

(ii) For any \(n \geq 4\), \(s_1(n) > \frac{n}{2}\).

(iii) For \(n \geq 5\), \(s_1(n) > \mathcal{S}^*\) for any \(\tilde{c} \in (0, 1)\).

**Proof.** (i): Since \(n \geq 3\), we can see that \(n + 3 > 2\sqrt{n}\). Then, note that \(s_1(n) = n - 2(1 - \tilde{c})\sqrt{n} + 2 > n + 2 - \sqrt{n} > -1 + \sqrt{1 + 2n} = s_2(n)\) because \(n + 2 - \sqrt{n} > -1 + \sqrt{1 + 2n} \iff (n + 3) - 2\sqrt{n} + \sqrt{1 + 2n} > 0\).

(ii): \(s_1(n) = n + 2 - 2(1 - \tilde{c})\sqrt{n} > n + 2 - 2\sqrt{n}\) and \(n + 2 - 2\sqrt{n} \geq \frac{n}{2} \iff (\sqrt{n} - 2)^2 \geq 0\). Hence, if \(n \geq 4\), we have \(s_1(n) > \frac{n}{2}\).

(iii): By the direct calculation, we have

\[
s_1(n) > \mathcal{S}^*
\]

\[
\iff 2(n + 2)\tilde{c} - 4\tilde{c}(1 - \tilde{c})\sqrt{n} > 3 + (2n + 1)\tilde{c} - \sqrt{(1 - \tilde{c})(9 + (8n - 7)\tilde{c})}
\]

\[
\iff -(1 - \tilde{c})(4\tilde{c}\sqrt{n} + 3) > -\sqrt{(1 - \tilde{c})(9 + (8n - 7)\tilde{c})}
\]

\[
\iff (1 - \tilde{c})(4\tilde{c}\sqrt{n} + 3)^2 < (9 + (8n - 7)\tilde{c})
\]

\[
\iff (2\tilde{c}^2 - 2\tilde{c} + 1)n - 3(1 - \tilde{c})\sqrt{n} + 2 > 0
\]

\[
\iff (\sqrt{n} - \alpha)(\sqrt{n} - \beta) > 0,
\]

where

\[
\alpha(\tilde{c}) = \frac{3(1 - \tilde{c}) - \sqrt{7\tilde{c}^2 - 2\tilde{c} + 1}}{2(2\tilde{c} - 2\tilde{c} + 1)}, \quad \beta(\tilde{c}) = \frac{3(1 - \tilde{c}) + \sqrt{7\tilde{c}^2 - 2\tilde{c} + 1}}{2(2\tilde{c} - 2\tilde{c} + 1)}.
\]

Since \(\beta(\tilde{c}) \in \mathbb{R}\) if and only if \(\tilde{c} \in \left[\frac{1 - 2\sqrt{2}}{4}, \frac{1 + 2\sqrt{2}}{4}\right]\) and

\[
\max_{\tilde{c} \in \left[\frac{1 - 2\sqrt{2}}{4}, \frac{1 + 2\sqrt{2}}{4}\right]} \beta(\tilde{c}) = \frac{3}{2} + \frac{1}{\sqrt{2}},
\]

we can say that \(s_1(n) > \mathcal{S}^*\) if \(\sqrt{n} > \frac{3}{2} + \frac{1}{\sqrt{2}}\), which is satisfied for \(n \geq 5\). \(\square\)
Finally, the following obvious inequality is useful later.

**Lemma 7.** For any \( a, b, c, d, e \in \mathbb{R} \), \( a > d \Rightarrow \max\{a, b\} > \min\{d, c, e\} \).

### B.2 Proofs: \( C^{pro} \)

First, we consider the condition for which the deviation by \( S' \subseteq N \) is not profitable.

**Lemma 8.** Suppose that \( n \geq 4 \). Then, for any \( S \subseteq N \) with \( \max\{n-4, 1\} \leq |S| \leq n \), there are \( \tau_1^{pro}(S, n) \) and \( \tau_2^{pro}(S, n) \) such that any \( S' \subseteq N \) cannot block \( \mathcal{P} = \{\{0\} \cup S, \{j\}_{j \not\in S'}\} \) if and only if one of the following conditions holds:

(i) \( 1 \leq |S| < s_1(n) \) and \( \tau \in (0, \tau_1^{pro}(S, n)] \).

(ii) \( s_1(n) \leq |S| \leq n \) and \( \tau \in (0, \tau_2^{pro}(S, n)] \).

**Proof.** Note that \( S' \subseteq N \setminus S \) cannot block \( \mathcal{P} \) as in the proof of Lemma 2, so that it is sufficient to consider the deviations by \( S' \subseteq N \) with \( S' \cap S = \emptyset \). Let \( \mathcal{P}' = \{\{0\} \cup (S \setminus S'), S', \{j\}_{j \not\in S \cup S'}\} \) be the partition after the deviation by \( S' \). First, suppose that \( S \setminus S' \neq \emptyset \), that is, \( \{0\} \notin \mathcal{P}' \). We show that \( S' \cap (N \setminus S) = \emptyset \), so that \( S' \subseteq S \). If \( S' \cap (N \setminus S) \neq \emptyset \), then \( 2 \leq |\mathcal{P}'| \leq n - |S| + 1 \). Let

\[
\Pi(\mathcal{P}, S'') = \left\{ \mathcal{P}'' \in \Pi \mid \mathcal{P}'' = \{\{0\} \cup S, S'', \{j\}_{j \not\in S \cup S''}\}, S'' \subseteq N \setminus S, |S''| \geq 2 \right\}
\]

be the set of all possible partitions which can be made after the deviations by \( S' \subseteq N \setminus S \). We can see that \( 2 \leq |\mathcal{P}''| \leq n - |S| \) for any \( \mathcal{P}'' \in \Pi(\mathcal{P}, S'') \). Hence, if \( \pi_i(\mathcal{P}') > \pi_i(\mathcal{P}) \), then there is \( \mathcal{P}'' \in \Pi(\mathcal{P}, S'') \) such that \( \pi_i(\mathcal{P}'') > \pi_i(\mathcal{P}') > \pi_i(\mathcal{P}) \) for each \( i \in S' \cap (N \setminus S) \neq \emptyset \). However, since \( S' \subseteq N \setminus S \) cannot block \( \mathcal{P} \), such \( \mathcal{P}'' \in \Pi(\mathcal{P}, S'') \) does not exist, which implies that any \( i \in N \setminus S \) does not participate in the deviation. Therefore, we only need to consider the deviations by \( S' \subseteq S \). In this case, the most profitable deviations are by \( S' \subseteq S \) with \( |S'| = 1 \), which does not happen if and only if \( |S| = 1 \) or

\[
\frac{(1 - \tau) M_L(n - |S| + 1)}{|S|(n - |S| + 2)^2} \geq \frac{M_H}{(n - |S| + 3)^2} \Leftrightarrow \tau \leq 1 - \frac{|S|(n - |S| + 2)^2}{M_L(n - |S| + 1)} \cdot \frac{M_H}{(n - |S| + 3)^2}.
\]

Next, suppose that \( S \setminus S' = \emptyset \), that is, \( \{0\} \in \mathcal{P}' \). Then, it must be that \( S \subseteq S' \) and the payoff for each \( i \in S' \) after deviation is \( \frac{M'}{|S'|(n - |S'| + 2)^2} \), which is maximized at \( |S'| = |S| \) or \( |S'| = n \). In each case, such a deviation does not happen if and only if

\[
\frac{(1 - \tau) M_L(n - |S| + 1)}{|S|(n - |S| + 2)^2} \geq \frac{M'}{|S'|(n - |S'| + 2)^2} \Leftrightarrow \tau \leq 1 - \frac{|S|(n - |S| + 2)^2}{M_L(n - |S| + 1)} \cdot \frac{M'}{|S'|(n - |S'| + 2)^2},
\]

and

\[
\frac{M_H}{(n - |S'| + 2)^2} \geq \frac{M'}{4n} \Leftrightarrow |S| \geq s_1(n) \text{ or } \frac{(1 - \tau) M_L(n - |S| + 1)}{|S|(n - |S| + 2)^2} \geq \frac{M'}{4n}.
\]
Proof. Let \( S \subseteq N \cup \{0\} \) with \( 0 \in S' \) be the set of all possible partitions which can be made after the deviations by \( T \). Hence, if \( \pi_i(P') > \pi_i(P) \), then there is \( P'' \in \Pi(P,T') \) such that \( \pi_i(P'') > \pi_i(P') > \pi_i(P) \) for each \( i \in T \cap (N\setminus S) \neq \emptyset \), which implies that, for \( j \in S' \cap (N \setminus S) \), the most profitable deviations are by \( S' \cap S = \emptyset \). Therefore, we only need to consider two types of deviations, that is, \( S \subseteq S' \) and \( S \cap S' = \emptyset \).

First, we consider the deviation such that \( S \subseteq S' \). In this case, the payoff for each \( i \in S' \setminus \{0\} \) after the deviation is given by \( (1 - \tau)f^2(|S'|-1) = (1 - \tau)f^2(|S| - 1) \), which is

\[
\tau_1^{pro}(S, n) = 1 - \max \left\{ \frac{M_H}{(n - |S| + 3)^2}, \frac{M'}{(n - |S| + 3)^2}, \frac{M'}{4n} \frac{|S|(n - |S| + 2)^2}{M_L(n - |S| + 1)} \right\},
\]

and

\[
\tau_2^{pro}(S, n) = 1 - \max \left\{ \frac{M_H}{(n - |S| + 3)^2}, \frac{M'}{(n - |S| + 3)^2}, \frac{M'}{4n} \frac{|S|(n - |S| + 2)^2}{M_L(n - |S| + 1)} \right\},
\]

we complete the proof because \( \frac{M_H}{(n+2)^2} < \frac{M'}{4n} \) for any \( n \geq 4 \), which implies that the condition for \( 1 = |S| < s_1(n) \) is equivalent to \( \tau \in (0, \tau_1^{pro}(S, n)) \).
maximized at $|S'| = |S| + 2$ or $|S'| = n + 1$.\(^8\) Then, these coalitions cannot block $\mathcal{P}$ if and only if $|S| = n$, or
\[
\frac{M_H}{(n - |S| + 2)^2} \geq \frac{(1 - \tau)M_L(n - |S|)}{(|S| + 1)(n - |S| + 1)^2} = (1 - \tau)f^2(|S| + 1) \text{ or } f(|S|) \geq f(|S| + 1)
\]
and
\[
\frac{M_H}{(n - |S| + 2)^2} \geq \frac{(1 - \tau)M_L(1)}{4n} = (1 - \tau)f^2(n) \text{ or } f(|S|) \geq f(n).
\]
By Lemma 5, $f(|S|) \geq f(n)$ implies that $f(|S|) \geq f(|S| + 1)$, and $f(n) > f(|S|)$ implies that $f(n) > f(|S| + 1)$. Therefore the above two conditions hold if and only if one of the following conditions holds: $f(|S|) \geq f(n)$, or, $f(n) > f(|S|)$ and $\tau \geq 1 - \frac{4n}{M_L(1)} \frac{M_H}{(n - |S| + 2)^2}$. Second, we consider the deviation such that $S \cap S' = \emptyset$. In this case, 0 participates in the deviation only if $|S| \leq n - 2$. The payoff for each $i \in S \setminus \{0\}$ after the deviation is given by $\frac{(1 - \tau)M_L((n - |S| + 2) - |S'| + 1)}{(|S'| + 1)(n - |S| + 2)^2}$, which is maximized at $|S'| = 3$ or $|S'| = n - |S| + 1$. Then, these coalitions cannot block $\mathcal{P}$ if and only if $|S| \geq n - 1$, or
\[
\frac{M_H}{(n - |S| + 2)^2} \geq \frac{(1 - \tau)M_L(n - |S|)}{2(n - |S| + 1)^2} \text{ and } \frac{M_H}{(n - |S| + 2)^2} \geq \frac{(1 - \tau)M_L(2)}{9(n - |S|)}.
\]
By setting
\[
\tau_3^{pro}(S, n) = 1 - \min\left\{ \frac{2(n - |S| + 1)^2}{M_L(n - |S|)} \cdot \frac{9(n - |S|)}{M_L(2)} \right\} \cdot \frac{M_H}{(n - |S| + 2)^2},
\]
\[
\tau_4^{pro}(S, n) = 1 - \min\left\{ \frac{2(n - |S| + 1)^2}{M_L(n - |S|)} \cdot \frac{9(n - |S|)}{M_L(2)} \cdot \frac{4n}{M_L(1)} \right\} \cdot \frac{M_H}{(n - |S| + 2)^2},
\]
and
\[
\tau_5^{pro}(S, n) = 1 - \frac{4n}{M_L(1)} \cdot \frac{M_H}{9},
\]
we complete the proof. \(\square\)

By combining Lemma 8 and Lemma 9, we characterize the condition for the stability of the partition $\mathcal{P} = \{\{0\} \cup S, \{j\}_{j \in S}\}$ for each $S \subseteq N$ with $\max\{n - 4, 1\} \leq |S| \leq n$.

**Lemma 10.** Suppose that $n \geq 4$. Then, for any $S \subseteq N$ with $\max\{n - 4, 1\} \leq |S| \leq n$, $\mathcal{P} = \{\{0\} \cup S, \{j\}_{j \in S}\} \in C^{pro}$ if and only if one of the following conditions holds:

(A) $f(|S|) \geq f(n)$:

(A-i) $|S| = n, n - 1 < s_1(n)$ and $\tau \in (0, \tau_1^{pro}(S, n)]$.

(A-ii) $|S| = n, n - 1 \geq s_1(n)$ and $\tau \in (0, \tau_2^{pro}(S, n)]$.

(A-iii) $|S| \leq n - 2$, $|S| < s_1(n)$ and $\tau \in [\tau_3^{pro}(S, n), \tau_4^{pro}(S, n)].$

\(^8\)Note that $0 \in S'$. 

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(A-iv) $|S| \leq n - 2$, $|S| \geq s_1(n)$ and $\tau \in [\tau_3(S, n), \tau_2^{pro}(S, n)]$.

(B) $f(|S|) < f(n)$:

- (B-i) $|S| = n - 1 < s_1(n)$ and $\tau \in [\tau_5^{pro}(S, n), \tau_1^{pro}(S, n)]$.
- (B-ii) $|S| = n - 1 \geq s_1(n)$ and $\tau \in [\tau_5^{pro}(S, n), \tau_2^{pro}(S, n)]$.
- (B-iii) $|S| \leq n - 2$, $|S| < s_1(n)$ and $\tau \in [\tau_4^{pro}(S, n), \tau_1^{pro}(S, n)]$.
- (B-iv) $|S| \leq n - 2$, $|S| \geq s_1(n)$ and $\tau \in [\tau_4^{pro}(S, n), \tau_2^{pro}(S, n)]$.

In above conditions, we show that only (A-i), (A-ii), (B-i) and (B-ii) can hold.

**Lemma 11.** Suppose that $n \geq 4$. Then, for any $S \subseteq N$ with $\max\{n - 4, 1\} \leq |S| \leq n - 2$, and $c \in (0, 1)$, conditions (A-iii), (A-iv), (B-ii) and (B-iv) cannot hold.

**Proof.** (A-iv) and (B-iv): By (i) of Lemma 6, $\frac{n}{2} < s_1(n) \leq |S| \leq n - 2 \Rightarrow n > 4$, which implies that (A-iv) and (B-iv) cannot hold when $n = 4$. Moreover, by (ii) of Lemma 6, for any $n \geq 5$, $s_1(n) > S^*$, which implies that $f(|S|) < f(n)$. Hence, (A-iv) cannot hold.

To prove (B-iv) cannot hold, it is sufficient to show that $\tau_4^{pro}(S, n) > \tau_2^{pro}(S, n)$ for each $S \subseteq N$ with $|S| = n - 4, n - 3$, and $n - 2$ and $f(|S|) < f(n)$. By $\frac{n}{2} < s_1(n) \leq |S|$, in each case, it must be that $n \geq 9, n \geq 7, n \geq 5$. For each $n - 4 \leq |S| \leq n - 2$, observe that

$$(n - 1) > (n - |S|) \frac{(n - |S| + 3)^2}{(n - |S| + 2)^2},$$

because, in each case, the inequality is equivalent to $n - 1 > \frac{49}{9}, \frac{108}{25}, \frac{25}{8}$ and it is satisfied for $n \geq 9, 7, 5$, respectively. Then, we have

$$(n - 1) > (n - |S|) \frac{(n - |S| + 3)^2}{(n - |S| + 2)^2} \quad \Rightarrow \quad f^2(n - 1)(n - |S| + 2)^2 > f^2(|S|)(n - |S| + 3)^2 \frac{n - |S|}{n - 1}$$

$$\quad \Leftrightarrow \quad \frac{1}{f^2(|S|)} \frac{M_H}{(n - |S| + 3)^2} > \frac{1}{f^2(n - 1)} \frac{n - |S|}{n - 1} \frac{M_H}{(n - |S| + 2)^2}$$

by Lemma 5. Moreover, by Lemma 7, the last condition in above implies that $\tau_4^{pro}(S, n) > \tau_2^{pro}(S, n)$. Therefore, (B-iv) cannot hold.

(A-iii) and (B-iii): We only need to show that $\tau_3^{pro}(S, n) > \tau_1^{pro}(S, n)$, because $\tau_3^{pro}(S, n) \geq \tau_3^{pro}(S, n)$ by definition. Without loss of generality, suppose that $n \geq 5, n \geq 4$ and $n \geq 3$ for each $|S| = n - 4, n - 3$ and $n - 2$. In each case, $|S|(n - |S| + 2)^2 \geq 4n$.

Hence, we have

$$\frac{\tau_3^{pro}(S, n)}{\tau_1^{pro}(S, n)} \Leftrightarrow \max\{\frac{M_H}{(n - |S| + 3)^2}, \frac{M'}{4n}\} \frac{|S|(n - |S| + 2)^2}{M_L(n - |S| + 1)} > \min\{\frac{2(n - |S| + 1)^2}{M_L(n - |S|)}, \frac{9(n - |S|)}{M_L(2)}\} \frac{M_H}{(n - |S| + 2)^2}.$$
Then, by Lemma 7, it is sufficient to show that
\[
\frac{M' |S|(n - |S| + 2)^2}{4n M_L(n - |S| + 1)} > \frac{2(n - |S| + 1)^2 M_H}{M_L(n - |S|)} \frac{(n - |S| + 2)^2}{(n - |S| + 2)^4}.
\]

For each \(|S| = n - 4, n - 3, \text{ and } n - 2,
\[
\frac{|S|}{n} > \frac{8(n - |S| + 1)^2}{(n - |S| + 2)^4} \iff (1 - \frac{4}{n}) > \frac{25}{16^2}, (1 - \frac{3}{n}) > \frac{128}{625} \text{ and } (1 - \frac{2}{n}) > \frac{9}{32},
\]
all of which are satisfied by \(n \geq 5, 4, 3\). Moreover, by the normalization, \(\frac{M_H}{M_L} > \frac{M_H(n - |S|)}{M_L(n - |S| + 1)} \iff (1 + (n - |S|)\bar{c}) > (1 + (n - |S| + 1)\bar{c})(1 - \bar{c}) \iff (n - |S| + 1)\bar{c}^2 > 0\). Therefore, \(\tau_3^{pro}(S,n) > \tau_1^{pro}(S,n)\), which implies that (A-iii) and (B-iii) cannot hold.

\[\square\]

**Proof of Proposition 4.** By lemma 10 and 11, for each \(S \subseteq N\) with \(\max\{n - 4, 1\} \leq |S| \leq n\), \(\mathcal{P} = \{\{0\} \cup S, \{j\}_{j \in S}\} \subseteq C^{pro}\) if and only if (A-i), (A-ii), (B-i) or (B-ii) holds. If \(|S|\) satisfies the condition (A-i), we can set \(\tau^{pro} = 0\), \(\bar{\tau}^{pro} = \tau_1^{pro}(S,n)\). If \(|S|\) satisfies the condition (A-ii), we can set \(\tau^{pro} = \tau_5^{pro}(S,n)\), \(\bar{\tau}^{pro} = \tau_1^{pro}(S,n)\). Finally, if \(|S|\) satisfies the condition (B-i), we can set \(\tau^{pro} = \tau_5^{pro}(S,n)\), \(\bar{\tau}^{pro} = \tau_2^{pro}(S,n)\).

\[\square\]

**Proof of Proposition 5.** Only if part is obvious from Lemma 2, 10 and 11. We show the if part. First, suppose that \(\mathcal{P} = \{\{0\} \cup N\}\). Then , \(\mathcal{P} \subseteq C^{pro}\) if and only if \(\tau \in (0, \tau_2^{pro}(N,n)] = (0, 1 - \max\{\frac{M_H}{M_L}, \frac{4n}{M_L(1)}\})\) by Lemma 10. Note that \(\tau_2^{pro}(N,n) = \frac{3}{4}\) when \(\bar{c} = 1\). Thus, the continuity of \(\tau_2^{pro}(S,n)\) with respect to \(\bar{c}\) implies that \(\tau_2^{pro}(N,n) > 0\) for large \(\bar{c}\). Therefore, there is \(\tau \in (0, 1)\) and \(\bar{c} \in (0, 1)\) such that \(\mathcal{P} = \{\{0\} \cup N\} \subseteq C^{pro}\). Next, suppose that \(\mathcal{P} = \{\{0\} \cup N\setminus\{i\}, \{i\}\}\) for some \(i \in N\). When \(\bar{c} = 1\), then \(f(n - 1) = \frac{1}{\sqrt{n-1}} > \frac{1}{\sqrt{n}} = f(n)\), \(s_1(n) = n + 2 > n - 1 = |S|, \text{ and } \tau_1^{pro}(N\setminus\{i\}, n) = 1 - \max\{0, \frac{1}{n(n-1)}\} \approx 1 - \frac{1}{4}(1 - \frac{1}{n}) > \frac{3}{4}\). Now, the continuity of \(f, s_1(n)\) and \(\tau_1(S,n)\) with respect to \(\bar{c}\) implies that condition (A-i) holds for sufficiently large \(\bar{c}\) and for any \(\tau \in (0, \frac{3}{4})\).

\[\square\]

**B.3 Proofs: C^{opt}\**

To prove the result, it suffices to show the following lemma.

**Lemma 12.** For any \(n \geq 3\), \(\mathcal{P} \subseteq C^{opt}\) if and only if one of the following two conditions holds:

(A) \(\mathcal{P} \subseteq \{\{0\} \cup N\} \text{ and } \tau \leq 1 - \frac{4n}{M_L(1)} \frac{M'}{9}\).

(B) \(\mathcal{P} = \{\{0\} \cup N\setminus\{i\}, \{i\}\}\) for some \(i \in N\) and:
(B-i) \( f(n-1) \geq f(n) \) and \( \tau \leq 1 - (n-1) \frac{M'}{M_L(2)}. \)

(B-ii) \( f(n-1) < f(n) \) and \( 1 - \frac{4n}{M_L(1)} \frac{M_H}{9} \leq \tau \leq 1 - (n-1) \frac{M'}{M_L(2)}. \)

Moreover, \( C^{\text{opt}} = \emptyset \) when \( n \geq 10. \)

**Proof.** If \( \mathcal{P} \in C^{\text{opt}} \), then it must be that \( \mathcal{P} = \{\{0\} \cup N\} \) or \( \mathcal{P} = \{\{0\} \cup N \setminus \{i\}, \{i\}\} \) for some \( i \in N \) by \( C^{\text{opt}} \subseteq C^{\text{pro}} \) and by Proposition 5.

\( \mathcal{P} = \{\{0\} \cup N\}: \) Note that 0 cannot participate in any deviation because the market size must increase. For any \( S' \subseteq N \), the best partition for each \( i \in S' \) is \( \mathcal{P}' = \{\{0\}, S', N \setminus S'\} \). So, the most profitable deviation is \( S' \subseteq N \) with \( |S'| = 1 \). Therefore, \( \mathcal{P} = \{\{0\} \cup N\} \in C^{\text{opt}} \) if and only if

\[
\frac{(1-\tau)M_L(1)}{4n} \geq \frac{M'}{9} \iff \tau \leq 1 - \frac{4n}{M_L(1)} \frac{M'}{9}.
\]

This corresponds to condition (A). Now, \( 1 - \frac{4n}{M_L(1)} \frac{M'}{9} = 1 - \frac{4n}{9} \frac{1}{(1+\hat{c})^2} < 1 - \frac{n}{9} \) implies that \( \{0\} \cup N \} \notin C^{\text{opt}} \) for \( n \geq 9. \)

\( \mathcal{P} = \{\{0\} \cup N \setminus \{i\}, \{i\}\} \): Note that 0 participates in a deviation only if the deviation is by \( \{0\} \cup N \), which cannot block \( \mathcal{P} \) if and only if \( \frac{(1-\tau)M_L(2)}{9(n-1)} \geq \frac{M'}{9} \iff f(n-1) \geq f(n) \) or

\[
\frac{M_H}{9} \geq \frac{(1-\tau)M_L(1)}{4n} \iff \tau \geq 1 - \frac{4n}{M_L(1)} \frac{M_H}{9}.
\]

In any other deviation, the most profitable deviation is by \( \{j\} (j \neq i) \), which cannot block \( \mathcal{P} \) if and only if

\[
\frac{(1-\tau)M_L(2)}{9(n-1)} \geq \frac{M'}{9} \iff \tau \leq 1 - (n-1) \frac{M'}{M_L(2)}.
\]

Therefore, \( \mathcal{P} = \{\{0\} \cup N \setminus \{i\}, \{i\}\} \in C^{\text{opt}} \) if and only if condition (B-i) or (B-ii) holds. Now, \( 1 - (n-1) \frac{M'}{M_L(2)} = 1 - (n-1) \frac{1}{(1+2\hat{c})^2} < 1 - \frac{n-1}{9} \) implies that \( \{0\} \cup N \setminus \{i\}, \{i\}\} \notin C^{\text{opt}} \) for \( n \geq 10. \)

**Proof of Proposition 6.** By Lemma 12, we consider (A) (B-i) or (B-ii) holds. If the condition (A-i) holds, we can set \( \tau^{\text{opt}} = 0, \tau^{\text{opt}} = 1 - \frac{4n}{M_L(1)} \frac{M'}{9}. \) If the condition (B-i) holds, we can set \( \tau^{\text{opt}} = 0, \tau^{\text{opt}} = 1 - (n-1) \frac{M'}{M_L(2)}. \) Finally, if the condition (B-ii) holds, we can set \( \tau^{\text{opt}} = 1 - \frac{4n}{M_L(1)} \frac{M_H}{9}, \tau^{\text{opt}} = 1 - (n-1) \frac{M'}{M_L(2)}. \)

**Failure of the monotonicity for \( C^{\text{opt}} \).** Suppose that \( n = 9. \) Then, the proof of Lemma 12 shows that \( \mathcal{P} = \{\{0\} \cup N\} \notin C^{\text{opt}} \) for any \( \hat{c}, \tau \in (0, 1). \) In this case, \( f(n-1) > f(n) \iff (1+2\hat{c})^2 > 2(1+\hat{c})^2, \) which is satisfied for \( \hat{c} = 1. \) Hence, by the continuity of \( f \) with respect to \( \hat{c} \), it also holds for sufficiently large \( \hat{c} \). Moreover, \( 1 - (n-1) \frac{M'}{M_L(2)} = 1 - 8(\frac{1}{1+2\hat{c}})^2 \leq \frac{1}{9} \) and equality holds when \( \hat{c} = 1. \) Hence, condition (B-i) is satisfied for sufficiently large \( \hat{c} \in (0, 1) \) and for \( \tau \in (0, \frac{1}{9}) \), which means that \( \{\{0\} \cup N \setminus \{i\}, \{i\}\} \in C^{\text{opt}} \) for each \( i \in N. \)
B.4 Proofs: $C^{pes}$

First, we consider the condition for which the deviation by $S' \subseteq N$ is not profitable.

**Lemma 13.** Suppose that $n \geq 4$. Then, for any $S \subseteq N$ with $1 \leq |S| \leq n - 1$, there are $s_2(n), \bar{n}(|S|), \tau_1^{pes}(S, n),$ and $\tau_2^{pes}(S, n)$ such that any $S' \subseteq N$ cannot block $\mathcal{P} = \{S \cup \{0\}, \{j\}_{j \notin S}\}$ if and only if one of the following conditions holds:

(i) $1 \leq |S| < s_2(n)$, $n \leq \bar{n}(|S|)$, and $\tau \in (0, \tau_1^{pes}(S, n)]$.

(ii) $s_2(n) \leq |S| < s_1(n)$ and $\tau \in (0, \tau_1^{pes}(S, n)]$.

(iii) $s_1(n) \leq |S|$ and $\tau \in (0, \tau_2^{pes}(S, n)]$.

**Proof.** The proofs are divided into the following three claims.

**Claim 1.** Either $|S| \geq s_1(n)$ or $\tau \leq \tau_1^{pes}(S, n)$ must hold.

By Lemma 3, we can see that

$$\max_{1 \leq |S'| \leq n} \min_{\mathcal{P}' \in \Pi; |S'| \leq n} \pi_i(\mathcal{P}') = \frac{1}{4n}M'',$$

which means that most profitable deviation by $S' \subseteq N$ is $S' = N$. To make this deviation unprofitable, one of the following conditions must hold:

$$\frac{1}{(n - |S| + 2)^2}M_H \geq \frac{1}{4n}M' \iff |S| \geq s_1(n),$$

$$\frac{1 - \tau}{|S|(n - |S| + 2)^2}M_L(n - |S| + 1) \geq \frac{1}{4n}M' \iff \tau \leq \tau_1^{pes}(S, n).$$

where

$$\tau_1^{pes}(S, n) \equiv 1 - \frac{|S|(n - |S| + 2)^2}{4n} \frac{M'}{M_L(n - |S| + 1)},$$

Moreover, note that if $|S| \geq s_1(n)$, then $\pi_i(\mathcal{P}) \geq \max_{1 \leq |S'| \leq n} \min_{\mathcal{P}' \in \Pi; |S'| \leq n} \pi_i(\mathcal{P}')$ for all $i \notin S$, which implies that any $S' \subseteq N$ with $S' \cap (N \setminus S) \neq \emptyset$ cannot block $\mathcal{P}$. Similarly, if $\tau \leq \tau_1^{pes}(S, n)$, then $\pi_i(\mathcal{P}) \geq \max_{1 \leq |S'| \leq n} \min_{\mathcal{P}' \in \Pi; |S'| \leq n} \pi_i(\mathcal{P}')$ for all $i \in S$, which implies that any $S' \subseteq N$ with $S' \cap S \neq \emptyset$ cannot block $\mathcal{P}$. Hence, it suffices to show that both $S' \subseteq N \setminus S$ and $S' \subseteq N$ with $S' \subseteq S$ cannot block $\mathcal{P}$ if one of the conditions (i)-(iii) holds.

**Claim 2.** $S' \subseteq N \setminus S$ cannot block $\mathcal{P}$.

If $|S| = n - 1$, it is obvious. So, we assume that $1 \leq |S| \leq n - 2$. By Lemma 3, for any $i \in N$,

$$\max_{2 \leq |S'| \leq (n - |S|)} \min_{\mathcal{P}' \in \Pi; |S'| \leq n} \pi_i(\mathcal{P}') = \max \left\{ \frac{1}{2n^2}M_H, \frac{1}{(n - |S|)(|S| + 2)^2}M_H \right\} = \begin{cases} \frac{1}{(n - |S|)(|S| + 2)^2}M_H & \text{if } 1 \leq |S| < s_2(n), \\ \frac{1}{2n^2}M_H & \text{if } s_2(n) \leq |S| \leq n - 2, \end{cases}$$
First, suppose that $2 \leq |S| < s_2(n)$, so the condition (i) holds. In this case, the most profitable deviation is $S' = N \setminus S$. By this argument, it suffices to show that $S' = N \setminus S$ cannot block $\mathcal{P}$. Since $\pi_j(\mathcal{P}) = \frac{1}{(n-|S|+2)^2} M_H$ for all $j \notin S$, we require that

$$
\frac{1}{(n-|S|+2)^2} M_H \geq \frac{1}{(n-|S|)(|S|+2)^2} M_H \iff (n-|S|)(|S|+2)^2 \geq (n-|S|+2)^2
\iff n^2 - (|S|^2 + 6|S|)n + (|S|^3 + 5|S|^2 + 4) \leq 0
\iff (n - n(|S|))(n - \bar{n}(|S|)) \leq 0
$$

where $n(k) = \frac{1}{2}(k^2 + (6 - \sqrt{k^2 + 4k - 4})k - 2\sqrt{k^2 + 4k - 4})$ and $\bar{n}(k) = \frac{1}{2}(k^2 + (6 + \sqrt{k^2 + 4k - 4})k + 2\sqrt{k^2 + 4k - 4})$. By Lemma 4, note that $n(k) \leq k + 1$ for any $k \in \mathbb{N}$. Since $n(|S|) \leq |S| + 1 \leq n \leq \bar{n}(|S|)$, we can say that $S' = N \setminus S$ cannot block $\mathcal{P}$.

Next, suppose that $s_2(n) \leq |S| < s_1(n)$, so the condition (ii) holds. In this case, the most profitable deviation is $S' = \{j_1, j_2\}$ with $j_1, j_2 \in N \setminus S$ and $j_1 \neq j_2$. By this argument, it suffices to show that such a deviation is not profitable. It requires that

$$
\frac{1}{(n-|S|+2)^2} M_H \geq \frac{1}{2n^2} M_H \iff 2n^2 \geq (n-|S| + 2)^2
\iff n^2 - 2(2 - |S|)n - (2 - |S|)^2 \geq 0
\iff (n - n'(|S|))(n - \bar{n}'(|S|)) \geq 0
$$

where $n'(|S|) = 2 - |S| - \sqrt{2(2-|S|)^2}$ and $\bar{n}'(|S|) = 2 - |S| + \sqrt{2(2-|S|)^2}$. Note that $n'(|S|) \leq 0$ for any $|S|$. Moreover, since $1 \leq s_2(n) \leq |S| \leq n - 1$, we have $\bar{n}'(|S|) \leq |S| + 1 \leq n$. Hence, we can say that $S' = \{j_1, j_2\}$ with $j_1, j_2 \in N \setminus S$ and $j_1 \neq j_2$ cannot block $\mathcal{P}$.

Finally, if the condition (iii) holds, by Claim 1, $S' \subseteq N \setminus S$ cannot block $\mathcal{P}$.

Claim 3. $S' \subseteq N$ with $S' \subset S$ cannot block $\mathcal{P}$.

If $|S| = 1$, it is obvious. So, we assume that $2 \leq |S| \leq n - 1$. If either condition (i) or condition (ii) holds, by Claim 1, $S' \subseteq N$ with $S' \subset S$ cannot block $\mathcal{P}$. So, suppose that the condition (iii) holds.

For any $i \in S$, there is no incentive to deviate from $\mathcal{P}$ by $S' \subset S$ if and only if

$$
\frac{1 - \tau}{|S|(n-|S|+2)^2} M_L(n-|S|+1) \geq \max_{1 \leq |S'| \leq |S|} \frac{1}{|S'|(|S'|+2)^2} \frac{1}{M_H}
\geq \max \left\{ \frac{1}{(n+1)^2}, \frac{1}{|S|(|S|+2)^2} \right\} M_H.
$$

By setting

$$
\tau^\text{pes}_2(S, n) \equiv 1 - \max \left\{ \frac{|S|(n-|S|+2)^2}{(n+1)^2}, 1 \right\} \frac{M_H}{M_L(n-|S|+1)},
$$
this is equivalent to $\tau_2^{pes}(S, n) \geq \tau$. Hence, if the condition (iii) holds, such a deviation is not profitable.

Summarizing Claim 1, 2, and 3, we complete the proof. \hfill \Box

Next, we consider the condition for which the deviation by $S' \subseteq N \cup \{0\}$ with $0 \in S'$ is not profitable.

Lemma 14. Suppose that $n \geq 4$. Then, for any $S \subseteq N$ with $1 \leq |S| \leq n - 1$, there are $\tau_3^{pes}(S, n)$, and $\tau_4^{pes}(S, n)$ such that $S' \subseteq N \cup \{0\}$ with $0 \in S'$ cannot block $\mathcal{P} = \{S \cup \{0\}, \{j\}_{j \notin S}\}$ if and only if one of the following conditions holds:

(i) $f(|S|) \geq f(n), |S| > \frac{n-1}{2}$ and $\tau \in [0,1)$.

(ii) $f(|S|) \geq f(n), |S| \leq \frac{n-1}{2}$ and $\tau \in [\tau_3^{pes}(S, n), 1)$.

(iii) $f(|S|) < f(n)$ and $\tau \in [\tau_4^{pes}(S, n), 1)$.

Proof. For the patent holder, the deviation by $S' \subseteq N \cup \{0\}$ with $0 \in S'$ is profitable if and only if

$$\pi_0(\mathcal{P}) < \min_{\mathcal{P}' \in \Pi, S' \in \mathcal{P}'} \pi_0(\mathcal{P}')$$

$$\Leftrightarrow \frac{\tau}{(n - |S| + 1)^2} M_L(n - |S| + 1) \leq \frac{\tau}{(n - |S'| + 1)^2} M_L(n - |S'| + 1)$$

$$\Leftrightarrow |S| < |S'| - 1.$$ 

Hence, it suffices to consider the deviation by $S' = T \cup \{0\}$ with $|T| \geq |S| + 1$ and $T \subseteq N$. Since $|T| \geq |S| + 1$, there are at least one $j \in N \setminus S$ with $j \in T$.

First, suppose that $f(|S|) \geq f(n)$. Then, by Lemma 5, $f(k) \leq f(|S|)$ for any $k \geq |S| + 1$. This implies that

$$\pi_i(\mathcal{P}) = \frac{(1 - \tau)}{\beta} f^2(|S|)$$

$$\geq \max_{|S| + 1 \leq |T| \leq n} \frac{(1 - \tau)}{\beta} f^2(|T|)$$

$$= \max_{|S| + 1 \leq |T| \leq n} \min_{\mathcal{P}' \in \Pi, T \cup \{0\} \in \mathcal{P}'} \pi_i(\mathcal{P}') .$$

for all $i \in S$. This implies that any deviation by $S' = T \cup \{0\}$ with $|T| \geq |S| + 1$ and $T \cap S \neq \emptyset$ is unprofitable for $i \in S$. Therefore, such kinds of deviations are not profitable if and only if, for $j \in N \setminus S$,

$$\pi_j(\mathcal{P}) \geq \max_{|S| + 1 \leq |T| \leq n - |S|} \min_{\mathcal{P}' \in \Pi, T \cup \{0\} \in \mathcal{P}'} \pi_j(\mathcal{P}')$$

$$\Leftrightarrow \frac{1}{(n - |S| + 2)^2} M_H \geq \max_{|S| + 1 \leq |T| \leq n - |S|} \frac{1 - \tau}{|T| (n - |T| + 2)^2} M_L(n - |T| + 1)$$

$$\Leftrightarrow \frac{1}{(n - |S| + 2)^2} M_H \geq (1 - \tau) \max \left\{ \frac{M_L(n - |S|)}{(|S| + 1)(n - |S| + 1)^2}, \frac{M_L(|S| + 1)}{(n - |S|)(|S| + 2)^2} \right\} .$$

\footnote{Note that $0 \notin S$, but $0 \in S'$.}
If $|S| + 1 > n - |S| \leftrightarrow |S| > \frac{n - 1}{2}$, such a deviation does not occur, that is, condition (i). Otherwise, by setting

$$\tau_3^{pes}(S, n) \equiv 1 - \frac{M_H}{(n - |S| + 2)^2} \max \left\{ \frac{M_L(n - |S|)}{(|S| + 1)(n - |S| + 1)^2}, \frac{M_L(|S| + 1)}{(n - |S|)(|S| + 2)^2} \right\},$$

the condition is equivalent to $\tau \geq \tau_3^{pes}(S, n)$, that is, condition (ii).

Next, suppose that $f(|S|) < f(n)$. Then, by Lemma 5, we can see that

$$\max_{|S| + 1 \leq |T| \leq n} \left\{ \frac{1 - \tau}{|T|(n - |T| + 2)^2} M_L(n - |T| + 1) = \frac{1 - \tau}{4n} M_L(1), \right.$$ 

that is, $|T| = N$. This implies that the deviation by $N \cup \{0\}$ is the most profitable for $i \in S$. In this case, such kinds of deviations are not profitable if and only if, for $j \in N \setminus S$,

$$\frac{1}{(n - |S| + 2)^2} M_H \geq \frac{1 - \tau}{4n} M_L(1).$$

By setting

$$\tau_4^{pes}(S, n) \equiv 1 - \frac{4n}{(n - |S| + 2)^2} \frac{M_H}{M_L(1)},$$

the condition is equivalent to $\tau \geq \tau_4^{pes}(S, n)$, that is, condition (iii).

By combining Lemma 13 and 14, we can characterize the condition for the stability of the partition $\mathcal{P} = \{S \cup \{0\}, \{j\}_{j \not\in S}\}$. Depending on $|S|$, as we have seen before, thresholds range of $\tau$ is different, which can be summarized as follows.

**Lemma 15.** Suppose that $n \geq 4$. Then, for any $S \subseteq N$ with $1 \leq |S| \leq n - 1$, $\mathcal{P} = \{S \cup \{0\}, \{j\}_{j \not\in S}\} \in \mathcal{C}_{pes}$ if and only if one of the following conditions holds:

(A) $f(|S|) \geq f(n)$ and $|S| > \frac{n - 1}{2}$:

1. (A-i) $1 \leq |S| < s_2(n)$, $n \leq \overline{\pi}(|S|)$, and $\tau \in (0, \tau_1^{pes}(S, n)]$.
2. (A-ii) $s_2(n) \leq |S| < s_1(n)$ and $\tau \in (0, \tau_1^{pes}(S, n)]$.
3. (A-iii) $s_1(n) \leq |S|$ and $\tau \in (0, \tau_2^{pes}(S, n)]$.

(B) $f(|S|) \geq f(n)$ and $|S| \leq \frac{n - 1}{2}$:

1. (B-i) $1 \leq |S| < s_2(n)$, $n \leq \overline{\pi}(|S|)$, and $\tau \in (\tau_3^{pes}(S, n), \tau_1^{pes}(S, n)]$.
2. (B-ii) $s_2(n) \leq |S| < s_1(n)$ and $\tau \in (\tau_3^{pes}(S, n), \tau_1^{pes}(S, n)]$.
3. (B-iii) $s_1(n) \leq |S|$ and $\tau \in (\tau_3^{pes}(S, n), \tau_2^{pes}(S, n)]$.

(C) $f(|S|) < f(n)$:

1. (C-i) $1 \leq |S| < s_2(n)$, $n \leq \overline{\pi}(|S|)$, and $\tau \in (\tau_4^{pes}(S, n), \tau_1^{pes}(S, n)]$. 

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(C-II) \( s_2(n) \leq |S| < s_1(n) \) and \( \tau \in (\tau_{4}^{pes}(S,n), \tau_{1}^{pes}(S,n)] \).

(C-III) \( s_1(n) \leq |S| \) and \( \tau \in (\tau_{4}^{pes}(S,n), \tau_{2}^{pes}(S,n)] \).

In the above conditions, we show that either (A-II) or (C-III) can hold.

**Lemma 16.** Suppose that \( n \geq 4 \). Then, for any \( S \subseteq N \) with \( 1 \leq |S| \leq n-1 \), and \( \tilde{c} \in (0,1) \), all the conditions except (A-II) and (C-III) cannot hold.

**Proof.** (A-I): Note that \( s_2(n) = -1 + \sqrt{1+2n} \leq \frac{n-1}{2} \Leftrightarrow 0 \leq n^2 - 6n - 3 \), where this inequality is satisfied for \( n \geq 7 \). Hence, if \( n \geq 7 \), the condition (A-I) cannot hold. When \( n = 4, 5, 6 \), by the direct calculation, there is no integer \( |S| \) satisfying \( \frac{n-1}{2} < |S| < s_2(n) \), which implies that the condition (A-I) cannot hold.

(A-III): First, suppose that \( n = 4 \). In this case, \( |S| > \frac{n-1}{2} = \frac{3}{2} \), so that \( |S| = 2,3 \). Since \( s_1(2) = 2(3-2(1-\tilde{c})) \) and \( \tilde{c} \in (0,1) \), \( s_1(2) > 2 \). For \( |S| = 3 \), \( s_1(2) \leq 3 \Leftrightarrow \tilde{c} \leq \frac{1}{4} \), \( f(3) \geq f(4) \Leftrightarrow \tilde{c} \geq \frac{3\sqrt{3}-4}{8-3\sqrt{3}} > \frac{1}{4} \). Hence, the condition (A-III) cannot hold when \( n = 4 \).

Next, for \( n \geq 5 \), by Lemma 5 and 6, we have \( f(S) < f(n) \), so that the condition (A-III) cannot hold.

(B-I) and (B-II): It suffices to show that \( \tau_{1}^{pes}(S,n) \geq \tau_{3}^{pes}(S,n) \) for any \( S \subseteq N \) with \( 1 \leq |S| \leq \frac{n-1}{2} \), and \( \tilde{c} \in (0,1) \). Note that, by the normalization,

\[
\tau_{1}^{pes}(S,n) = 1 - \frac{|S|(n-|S|+2)^2}{4n} \left( \frac{1}{1+(n-|S|+1)\tilde{c}} \right)^2
\]

and

\[
\tau_{3}^{pes}(S,n) = 1 - \left( \frac{1-\tilde{c}}{n-|S|+2} \right)^2 \min \left\{ \frac{1}{f^2(|S|+1)}, \frac{1}{f^2(n-|S|)} \right\}
\]

Then, by Lemma 7, it suffices to show that

\[
\left( \frac{1-\tilde{c}}{n-|S|+2} \right)^2 \left( \frac{n-|S|+1}{1+(n-|S|)\tilde{c}} \right)^2 (|S|+1) \leq \frac{|S|(n-|S|+2)^2}{4n} \left( \frac{1}{1+(n-|S|+1)\tilde{c}} \right)^2
\]

\[
\Leftrightarrow 4n(n-|S|+1)^2(|S|+1) \left( 1-\tilde{c} \right) (1+(n-|S|+1)\tilde{c})^2 \leq |S|(n-|S|+2)^4 \left( 1+(n-|S|)\tilde{c} \right)^2 \cdots (*)
\]

for any \( S \subseteq N \) with \( 1 \leq |S| \leq \frac{n-1}{2} \), and \( \tilde{c} \in (0,1) \). First, note that

\[
1 + (n-|S|)\tilde{c} - (1-\tilde{c})(1+(n-|S|+1)\tilde{c}) = (n-|S|+1)\tilde{c}^2 \geq 0.
\]

Second, let \( h(x) = x(n-x+2)^4 - 4n(x+1)(n-x+1)^2 \) and we show that \( h(x) \geq 0 \) for any \( x \in [1, \frac{n}{2}] \). Note that \( h(1) = (n+1)^4 - 8n^3 > 0 \) and \( h\left( \frac{n}{2} \right) = \frac{n}{32} (n^4 + 64n + 128) > 0 \) if \( n \geq 4 \). Note also that \( h'(1) = n^4 - 4n^3 + 10n^2 - 8n - 3 > 0 \) and \( h'(1) = -\frac{3}{16} n^4 - n^3 - 2n^2 + 4n + 16 < 0 \) if \( n \geq 4 \). Hence, \( h'(x) = 0 \) has at least one real solution in \( [1, \frac{n}{2}] \). Moreover, we can
see that \( \lim_{x \to -\infty} h'(x) = \lim_{x \to -\infty} h'(x) = \infty \) and, if \( n \geq 4 \), \( h'(n) = 4(4 - 7n + 2n^2) > 0 \) and \( h'(n + 1) = -4n - 3 < 0 \), which implies that \( h'(x) \) has at least one solution in each interval \([\frac{n}{2}, n], [n, n + 1] \) and \([n + 1, \infty) \). Since \( h'(x) \) can have at most four real solutions, \( h'_1(x) = 0 \) can have one real solution in \([1, \frac{n}{2}] \). Therefore, \( h(x) \geq 0 \) for any \( x \in [1, \frac{n}{2}] \). Finally, by above arguments, the condition \( * \) holds.

**(B-iii):** This is done by (ii) of Lemma 6.

**(C-i) and (C-ii):** Note that

\[
\tau_{4}^{pes}(S, n) - \tau_{1}^{pes}(S, n) = \frac{M' |S|(n - |S| + 1)^2}{4n M_L(n - |S| + 1)} \cdot \frac{M_H}{(n - |S| + 1)^2 M_L(1)} - \frac{4n}{M_L(1)} \cdot \frac{M_H}{M_L(1)}
\]

\[
> \frac{4n}{M_L(1)} \left( \frac{M'}{4n} - \frac{M_H}{M_L(1)} \right)
\]

\[
= 0
\]

where the first inequality holds because \( f(|S|) < f(n) \Leftrightarrow \frac{|S|(n - |S| + 1)^2}{M_L(n - |S| + 1)} > \frac{4n}{M_L(1)} \) and the second inequality holds because \( |S| < s_1(n) \Leftrightarrow (n - |S| + 2)^2 > 4n \frac{M_H}{M_L} \).

Finally, by above Lemmas, we can give the proof of Propositions.

**Proof of Proposition 7.** For \( 1 \leq |S| \leq n - 1 \), by Lemma 15 and 16, \( \mathcal{P} = \{S \cup \{0\}, \{j\}_{j \not\in S}\} \in C_{pes}^{pes} \) if and only if either (A-ii) or (C-iii) holds. If \( |S| \) satisfies the condition (A-ii), we can set \( \tau_{pes} = 0 \), \( \tau_{pes} = \tau_1(S, n) \). Similarly, if \( |S| \) satisfies the condition (C-iii), we can set \( \tau_{pes} = \tau_1(S, n), \tau_{pes} = \tau_2(S, n) \).

For \( |S| = n \), by the proof of Lemma 14, any deviation by \( S' \subset N \cup \{0\} \) with \( 0 \in S' \) cannot block \( \mathcal{P} = \{N \cup \{0\}\} \). Hence, we only consider the deviation by \( S' \subset N \) as in the arguments of Lemma 13. Since \( \max_{1 \leq |S| \leq n} \min_{P \in \mathcal{P}} \tau_i(P) = \frac{1}{4n} M' \), we can set \( \tau_{pes} = \tau_1(N, n) \).

**Proof of Proposition 8.** The only if part holds by Lemma 16. To prove the if part, by Lemma 15 and 16, it suffices to show that, for any \( n \geq 4 \) and \( \frac{n-1}{2} < |S| \leq n \), there is \( \tilde{c} \in (0, 1) \) such that the condition (A-ii) holds. First, note that \( s_2(n) = -1 + \sqrt{1 + 2n} < \frac{n+1}{2} + 1 \leq |S| \) for any \( n \geq 4 \). Moreover, there is \( \tilde{c}_1 \in (0, 1) \) such that \( s_1(n) = n + 2 - 2(1 - \tilde{c}) \sqrt{n} > n \) for any \( \tilde{c} \in (\tilde{c}_1, 1) \). Next, we can see that \( f(|S|) \geq f(n) \) for sufficiently large \( \tilde{c} \in (0, 1) \). For \( |S| = n \), \( f(|S|) \geq f(n) \) is obvious. For each \( \frac{n-1}{2} < |S| \leq n - 1 \), notice that, for \( \tilde{c} = 1 \), \( f(|S|) = \frac{1}{\sqrt{|S|}} > \frac{1}{\sqrt{n}} = f(n) \). Hence, by the continuity of \( f \) with respect to \( \tilde{c} \), there is \( \tilde{c}_2 \in (0, 1) \) such that \( f(|S|) \geq f(n) \) for any \( \tilde{c} \in (\tilde{c}_2, 1) \). Finally, for any \( \varepsilon \in (0, \frac{|S|(n - |S| + 3)(n - |S| + 1)}{4n}) \), there is \( \tilde{c}_3(\varepsilon) \in (0, 1) \) such that \( 0 < (1 - \frac{|S|}{4n}) - \tau_1(S, n) < \varepsilon \) for any \( \tilde{c} \in (\tilde{c}_3(\varepsilon), 1) \) and \( 1 - \frac{|S|}{4n} > 0 \) for any \( \frac{n-1}{2} < |S| \leq n \). Therefore, by choosing
\[\varepsilon = 1 - \frac{|S|}{4n}\] and \[\bar{c} = \max\{\bar{c}_1, \bar{c}_2, \bar{c}_3(\varepsilon)\} \in (0, 1),\] we can conclude that the condition (A-ii) holds for any \(\bar{c} \in (\bar{c}, 1).\)

References


