Viral Social Learning

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Abstract

We study social learning with a "viral" feature: on a continuous time line, a group of consumers need to each make a decision of whether to adopt a product, where awareness of the product is transmitted from adopting consumers to new ones. A consumer bases her action on the time she becomes aware at as well as her private signal about product quality. We find a unique equilibrium depicting the product's life cycle: consumers start with herding on adoption given high initial belief and being sensitive to signals given low initial belief, then use mixed strategies which keep beliefs constant, followed by a period of relying on signals, and finally reject the product once and for all when beliefs fall below a threshold. For a strategic producer, viral social learning always emerges as a profit-maximizing choice. As consumers' prior belief differs, a viral product campaign has opposite effects on a producer's incentive for quality improvement.

Keywords: social learning, viral marketing, SIR model, information aggregation

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1 Introduction

A successful product launch nowadays is often associated with a "viral" aspect in marketing – early users of the product make others aware of it via word-of-mouth communication, emails, and social networking websites such as Facebook, Twitter, Instagram and Youtube. According to statistics from the Word of Mouth Marketing Association and TalkTrack[1], around 2.4 billion brand-related conversations take place every day in the US alone, and the average consumer mentions specific brand names 60 times per week in conversations. Given the speed and effectiveness of this peer-to-peer marketing approach, more and more producers begin to fund activities that generate more product exposure through consumers' social networks, for instance inviting star bloggers to write product reviews.

From the perspective of information aggregation among consumers, a product that "goes viral" presents a new problem. Different from the conventional setting that existence of the product is taken as given, a consumer now only perceives the product after she observe others using it, probably through random meeting within her social circle. In addition, and perhaps more realistically, non-adoption of the product does not spread awareness.

These features lead to at least two conceivable differences, in terms of epidemiological dynamics among consumers, from classical social learning models. First, the time that a consumer becomes aware of the product is endogenous, and the consumer's belief on quality hinges on it. Hearing about the product right after its launch and only after several months of the launch, for example, can generate very different even opposing beliefs. Second, due to such belief variation, contrasting behavior – herding on adoption, herding on non-adoption, and being sensitive to private information – may occur over time but may not persist forever. Therefore a product is likely to go through a "life cycle" after launch, whose characteristics are worth investigating. For instance, will a product with high initial belief manage to maintain the reputation? Conversely, will a product with low initial belief always remain unpopular? Furthermore, in a bigger picture where quality is endogenously determined by strategic producers, what can we say about average quality in the market and how is it affected by the market structure?

In this paper, we propose a first theoretical model that captures the essential characteristics of viral social learning, and answer these questions by providing an explicit characterization of equilibrium behavior. Our contribution to the literature is two-fold. On the demand side, we reveal how information aggregation evolves among consumers over time, and how it is determined by initial belief and precision of private

[1]Retrieved from: https://www.crewfire.com/50-peer-to-peer-marketing-statistics/.

information. Our results explain how a product gains initial attention, accumulates sales via peer-to-peer communication, gradually loses consumer trust, and finally dies out. On the supply side, we identify producers' choices of quality under different market structures, and point out the welfare-maximizing market structure. Similar to the traditional Cournot model, a producer's strategic behavior exhibits a regular and continuing pattern when its market power increases.

To model viral social learning, we adopt a framework that originally analyzes biological viruses: the susceptible-infected-recovered model, henceforward referred to as the SIR model. At time 0 on a continuous time line, a new product with unknown quality is launched among a small fraction of consumers, who decide whether to adopt it based on their private information. A consumer's private information consists of a noisy quality-dependent signal whose precision is higher than her initial belief or prior about quality, as in the standard setting of Bayesian social learning. The non-adopting consumers (the "recovered" or "immune") make no further move. The adopting consumers (the "infected") meet new consumers (the "susceptible") at the next time instant and make them aware of the product's existence; the new consumers then take into account this information – the time of their awareness – followed by their private signals, and decide whether to adopt the product. Afterwards, they enter either the infected or the recovered group, and the dynamic process continues. The game stops when every consumer has made their decision.

The consumers' epidemiological dynamics reflect the joint effect of three forces: precision of her private signal, time of awareness, and decisions by her predecessors. The signal structure is exogenously given and invariant, while the other two factors are endogenously determined. Predecessors' different behavior has different impacts on beliefs: neither type of herding provides new information on quality, while being sensitive to signals can be regarded *per se* as favorable to high quality because awareness essentially reveals another good signal. As a result, the effect of time on beliefs is not monotone. Although a good product spreads awareness faster than a bad one, which seems to suggest that a consumer who becomes aware earlier should hold a stronger belief that the product is good, it is also plausible that beliefs rise in time at least for a period when predecessors' actions rely on signals.

Our first main result, Theorem 1, depicts the counterbalance of these forces and characterizes the unique equilibrium among consumers. Right after the product launch, consumers will herd on adoption when their initial belief on quality is high, and be sensitive to signals when the belief is low. In this period of time, consumers' interim beliefs – that is, beliefs upon awareness but before private signals realize – fall in the former case while rise in the latter. When the value of interim beliefs reach $\frac{1}{2}$, consumers begin a period of using mixed strategies, during which beliefs remain unchanged. Afterwards, consumers become sensitive to private signals and their beliefs fall from $\frac{1}{2}$. Finally, as beliefs drop to a certain level, no new consumer will choose adoption and the product dies out. The product's life cycle thus formed can be readily computed numerically using simulation.

This result contributes to the understanding of social learning from two major aspects. Qualitatively, it implies that for products with intrinsically uncertain quality, fads are always transient even if consumers start with high hopes, while products that are initially not so popular may still enjoy a rising reputation and make considerable sales. Such phenomena are widely observed in real-life markets such as medicine and personal care, especially in the current information age where "viral" spread of brand names are prevalent. Quantitatively, the equilibrium dynamics of consumers' actions and beliefs can be readily computed using numerical simulation, in order to generate precise predictions.

Given the methodology for depicting consumers' equilibrium behavior, we then turn our focus to the supply side and analyze the producer's strategic choices. We first allow the producer to select between a viral product campaign and an advertising campaign which makes all susceptible consumers aware of the product instantly. In Proposition 2, we find that the producer will always launch a viral campaign: it can attract some consumers with a bad signal to adopt the product, while an advertising campaign only induces consumers to be sensitive to signals at best. Nevertheless, the producer will usually replace the viral campaign with an advertising one before the product's life cycle ends through viral social learning. Since susceptible consumers decrease at a faster rate for a good product, consumers' interim belief facing an advertising campaign drops over time. Hence, the strategic producer will stop the viral campaign just in time so that the product can still attract the remaining consumers with a good signal.

Our analysis also provides a definite answer to how viral social learning affects the producer's incentive for quality improvement, and hence consumers' welfare. Proposition 3 shows that such incentive is determined by consumers' prior belief. When the prior is high, consumers start with herding in a viral product campaign, which results in a lower difference between profits from a good product and a bad one. Consequently, the longer the viral campaign persists, the less the producer is willing to improve product quality. Conversely, when the prior is low and consumers are initially sensitive to signals, a good product earns an increasingly higher profit than a bad producer over time. The producer then becomes more incentivized for quality improvement as the viral campaign lasts longer.

The rest of the paper is organized as follows. Section 2 reviews related literature. Section 3 presents the model. Section 4 characterizes the unique equilibrium among consumers. Section 5 analyzes a strategic producer's choice of product campaign and quality improvement. Section 6 discusses other extensions of our framework. Section 7 concludes.

2 Literature Review

Early contributions to the literature of social learning by Bayesian agents include seminal papers by Bikhchandani et al. [1992], Banerjee [1992] and Smith and Sorensen [2000]. In these works, before an agent makes their own decision, he/she can observe both a private signal and the entire previous decision history. Herding behavior occurs when the "public" belief from the latter dominates the "private" belief from the former.

Subsequent research on social learning differs by their way of extending the basic model. One branch of literature, for instance Lee [1993], Banerjee [1993] and Celen and Kariv [2004], focuses on more complicated history observation such that agents may not observe the entire but only an independent subset of the history. A more recent paper, Acemoglu et al. [2011], can be regarded as a conclusive generalization of these works. It assumes that each agent observes (some of) their predecessors' actions according to a general stochastic process, and finds that when the private signal structure features unbounded belief, asymptotic learning occurs in each equilibrium if and only if agents always observe close predecessors. Lobel and Sadler [2015] adopt this model to analyze the pattern of learning when agents' observations are correlated. Other recent research in this area include Banerjee and Fudenberg [2004], Gale and Kariv [2003], Callander and Horner [2009] and Smith and Sorensen [2013], which differ from Acemoglu et al. [2011] mainly in relaxing the assumption of known decision order in observation, i.e. agents only observe the number of others taking an action but not their positions in the decision sequence. Guarino et al. [2011], Herrera and Horner [2013] and Monzon and Rapp [2014] consider the case where an agent does not even know their own position in the decision sequence.

Another branch of literature endogenizes the information acquisition process in different ways. On one hand, an agent may have access of direct information about the true state, by paying to acquire an informative signal or to sample an available option and know its value [Hendricks et al., 2012, Mueller-Frank and Pai, 2014, Ali, 2014, Board and Meyer-ter-Vehn, 2018]; on the the other hand, an agent may choose from different sources of indirect information by selecting which part of the previous history to observe [Kultti and Miettinen, 2006, 2007, Celen, 2008, Song, 2016].

Our paper contributes to the literature in two main aspects. First, an agent's awareness of decision, or the timing of him/her facing the choice between adoption and non-adoption, is endogenously determined via the "viral" nature of the model. This stands in contrast to the previous literature which assumes that the timing is exogenously given by a fixed sequence or a stochastic process, and allows us to capture the understudied feature of observational learning that, when more agents adopt a product, awareness of the product grows rapidly in time. We are then able to depict a product's typical life cycle in the unique equilibrium, the exact pattern of which differs by the initial belief and exposure.

We base our analysis on the classical susceptible-infectious-recovered (SIR) model, originated from Ross [1916] and Ross and Hudson [1917a,b] and developed by Kermack and McKendrick [1927, 1932, 1933]. The model was first proposed to simulate spread of contagious disease, and has been widely applied to study various epidemiological problems, such as disease spread on networks [Newman, 2002], antibiotic resistance [Laxminarayan and Brown, 2001, McAdams, 2017, McAdams et al., 2019], and vaccine scares [Bauch and Bhattacharyya, 2012]. Our model provides the first approach of adopting the SIR methodology in the context of observational learning by Bayesian agents.

Second, while the social learning literature mainly focuses on the learning behavior among consumers presented with a single product, our model provides a ready framework to study competition and market structure among producers who seek to maximize the proportion of consumers adopting their own product. Our approach relates to algorithmic diffusion models for viral marketing problems [Kempe et al., 2003, 2005, Mossel and Roch, 2010, Goyal et al., 2011, Borgs et al., 2014], in particular the ones that analyze competitive contagion (for instance Goyal et al. [2014]). These works focus on how the network effect – an agent's choice being directly determined by the choices of their neighbors – together with the network topology, determine product adoption rates in equilibrium. In contrast, the economic force underlying our results is the belief updating dynamics in a Bayesian framework; but similar to these models, our results offer explicit characterization of an easily computable equilibrium.

3 Model and preliminary analysis

Summary. A new product is launched at time $t = 0$. Consumers in a unit-mass population become aware of the product over time and, when first encountering the product, decide whether to adopt it. More widespread adoption speeds the viral diffusion process by which product awareness spreads. The time at which a consumer first encounters the product therefore conveys information about its quality, endogenously determining a time-varying path of consumer adoption, product diffusion, and belief formation that we refer to as "viral social learning." See Figure 1.

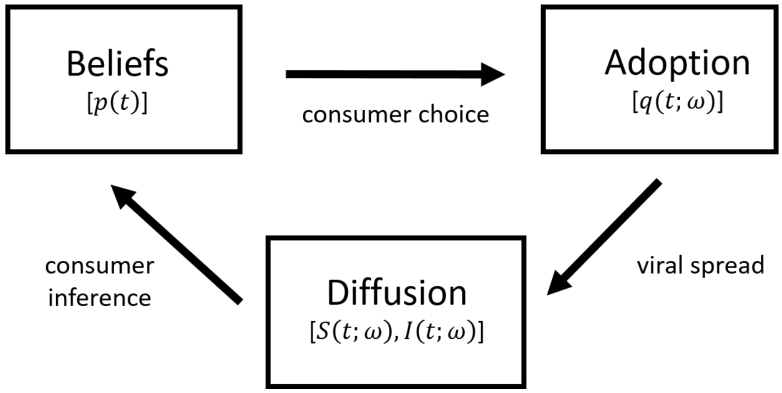


Figure 1: Illustration of viral social learning, whereby the dynamics of consumer beliefs impact the dynamics of product adoption, which in turn impact the epidemiological dynamics of product awareness and new-consumers' beliefs about product quality.

Consumer incentive to adopt. Each consumer i encounters a new product at (random) time t_i, at which point she observes the time, receives a private signal s_i about the product's unobservable quality, and decides whether or not to adopt.[2] The product may be "good" or "bad". Consumers get payoff $u_g > 0$ when adopting a good product, $-u_b < 0$ when adopting a bad product, or zero when not adopting, and seek to maximize their own expected payoff. To simplify equations, suppose that $u_g = u_b$ so that each consumer strictly prefers to adopt if and only if they believe that the product's likelihood of being good exceeds $1/2$.

Consumer belief formation. Let $\alpha \in (0, 1)$ be the ex ante likelihood that the product is good, let $p(t_i)$ be the probability that the product is good conditional on encountering the product at time t_i, and let $p(t_i, s_i)$ denote its likelihood of being good conditional on t_i and private signal s_i. We refer to $\alpha = p(0)$ as consumers' "ex ante belief," $p(t_i)$ as consumer i's "interim belief," and $p(t_i, s_i)$ as their "ex post belief". Interim beliefs are determined according to Bayes' Rule, with each consumer

[2]For the sake of tractability, we assume that consumers must make a once-and-for-all decision whether to adopt when they first encounter the product. In future work, it would be valuable to consider a richer context in which consumers can wait before adopting.

i updating her belief to $p(t_i) = \frac{\alpha f(t_i|\omega=g)}{\alpha f(t_i|\omega=g)+(1-\alpha)f(t_i|\omega=b)}$ or equivalently

$$\frac{p(t_i)}{1-p(t_i)} = \frac{\alpha}{1-\alpha} \times \frac{f(t_i|\omega=g)}{f(t_i|\omega=b)}$$

where $f(t_i|\omega)$ denotes the endogenous[3] p.d.f. of t_i conditional on the true state $\omega \in \{g, b\}$. Consumers' private signals are i.i.d. conditional on the state, with $\Pr(s_i = G|\omega = g) = \Pr(s_i = B|\omega = b) = \rho \in (1/2, 1)$. Given private signal s_i, consumer i updates her belief further to $p(t_i, s_i)$, defined by

$$\frac{p(t_i, s_i)}{1-p(t_i, s_i)} = \frac{p(t_i)}{1-p(t_i)} \times \frac{\Pr(s_i|\omega=g)}{\Pr(s_i|\omega=b)}$$

Among consumers who encounter the product at time t ("time-t consumers"), those who get a good signal ("G consumers") have updated belief $p(t, G) = \frac{p(t)\rho}{p(t_i)\rho+(1-p(t_i))(1-\rho)}$ while those who get a bad signal ("B consumers") have updated belief $p(t, B) = \frac{p(t)(1-\rho)}{p(t_i)(1-\rho)+(1-p(t_i))\rho}$.

Lemma 1 (Adoption patterns)**.** *(i)* Herding on adoption: *If $p(t) > \rho$, then all time-t consumers strictly prefer to adopt.* *(ii)* Herding on non-adoption: *If $p(t) < 1-\rho$, then all time-t consumers strictly prefer not to adopt.* *(iii)* Sensitive to signals: *If $p(t) \in (1-\rho, \rho)$, then all time-t consumers strictly prefer to adopt after a good private signal but not after a bad private signal.*

Proof. Time-t consumers strictly prefer to adopt if and only if their ex post belief $p(t, s_i) > 1/2$. The desired results follow immediately from the fact that $p(t, B) > 1/2$ if and only $p(t) > \rho$ and $p(t, G) > 1/2$ if and only $p(t) > 1-\rho$. □

Viral diffusion. Product awareness spreads through the consumer population much as a virus spreads through a host population, according to a Susceptible-Infected-Recovered (SIR) model. At each point in time, each consumer is in one of three epidemiological states: *susceptible*, if the consumer has not yet been exposed to the product; *infected*, if the consumer previously chose to adopt the product; or *recovered*, if the consumer previously chose not to adopt. We assume that mass $\Delta > 0$ of consumers are exposed to the product at time $t = 0$ regardless of product quality. Those who adopt then become infected and spread product awareness virally, meeting some randomly-selected other consumer at rate $\beta > 0$ and exposing

[3]We will characterize the equilibrium distribution of $t_i|\omega$, showing that $f(t_i|\omega)$ exists and is continuous in t_i at all but finitely-many points.

that other consumer to the product. If susceptible, that other consumer receives a private signal and decides whether or not to adopt, at that point transitioning to either the infected state (if adopting) or the recovered state (if not adopting).

Let $S_\omega(t)$, $I_\omega(t)$, and $R_\omega(t)$ denote the mass of susceptible, infected, and recovered consumers at time t conditional on the unobserved product-quality state $\omega \in \{g, b\}$. Since the population has unit mass, $R_\omega(t) = 1 - S_\omega(t) - I_\omega(t)$ and the overall epidemiological process is described by $(S_\omega(t), I_\omega(t) : t \geq 0, \omega = g, b)$. Let $q_\omega(t)$ denote time-t consumers' likelihood of adopting conditional on ω. Epidemiological dynamics are characterized by the system of differential equations

$$S'_\omega(t) = -\beta I_\omega(t) S_\omega(t) \tag{1}$$

$$I'_\omega(t) = q_\omega(t) \beta I_\omega(t) S_\omega(t) \tag{2}$$

Equation (1) follows from the fact that each infected consumer meets another consumer at rate $\beta > 0$ and fraction $S_\omega(t)$ of others remain suspectible, generating a state-dependent flow $\beta I_\omega(t) S_\omega(t)$ of newly-exposed consumers who are then no longer susceptible. Equation (2) follows immediately from the fact that fraction $q_\omega(t)$ of these newly-exposed consumers choose to adopt. Note that epidemiological dynamics are determined by the adoption process $(q_\omega(t) : t \geq 0, \omega = g, b)$.

Viral social learning. Since the consumer population has unit mass, the flow of newly-exposed consumers can be interpreted as the density of the time-until-exposure t, i.e., $f(t|\omega) = \beta S_\omega(t) I_\omega(t)$. Thus, time-$t$ consumers' interim belief is given by

$$\frac{p(t)}{1 - p(t)} = \frac{\alpha}{1 - \alpha} \times \frac{S_g(t) I_g(t)}{S_b(t) I_b(t)} \tag{3}$$

Equilibrium. Our solution concept is perfect Bayesian equilibrium (PBE or simply "equilibrium"). We will show by construction that a PBE exists and that this equilibrium is essentially unique, in the sense that all PBE generate the same population-wide epidemiological dynamics $(S_\omega(t), I_\omega(t), R_\omega(t) : t \geq 0; \omega \in \{g, b\})$.

Discussion: observability of time since launch. We assume that, when consumers encounter the product, they are able to observe how much time has elapsed since launch. However, our analysis can be easily extended to a setting in which fraction $\gamma \in [0, 1]$ of consumers are unable to observe the time. Since all consumers will eventually be exposed to the product (so long as anyone adopts initially at launch), a consumer who is unable to observe the time will not make any inference about product quality and so will decide whether to adopt *as if* encountering the product at launch. The overall likelihood that a consumer exposed at time $t > 0$ will adopt in product-quality state $\omega \in \{g, b\}$ is therefore $\tilde{q}_\omega(t) = \gamma q_\omega(0) + (1-\gamma)q_\omega(t)$, where $q_\omega(0)$ and $q_\omega(t)$ are the likelihoods that consumers who *can* observe the time will adopt, respectively, at time t and time 0. The rest of our analysis then carries over, with more complex formulae but little additional insight.

4 Equilibrium Product Lifecycle

This section characterizes consumers' equilibrium adoption behavior and the resulting epidemiological dynamics. In so doing, we characterize the endogenous lifecycle of a new product subject to viral social learning. To keep the presentation as simple as possible, we will focus on the case in which the fraction of consumers who encounter the product at launch is small, i.e., we will assume that $\Delta \approx 0$.[4]

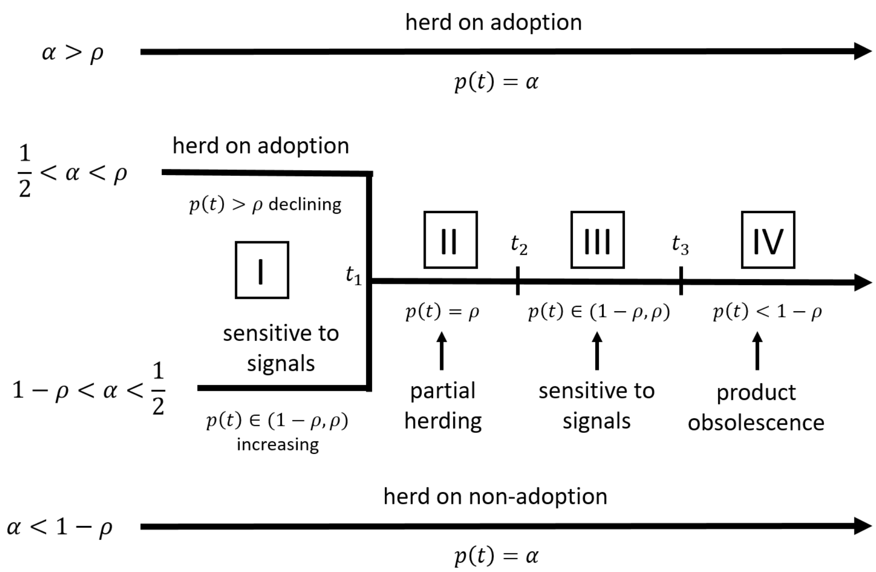


Figure 2: Visual summary of equilibrium adoption behavior and interim beliefs over the product lifecycle, depending on the ex ante likelihood $\alpha = p(0)$ of product quality.

[4]Our analysis and key conclusions (including uniqueness of equilibrium dynamics) extend easily to any $\Delta \in (0, 1]$, but some qualitative features of equilibrium consumer behavior change when Δ is sufficiently large. For instance, in the case when $\alpha \in (1-\rho, 1/2)$, consumers' interim beliefs increase immediately after launch if $\Delta \approx 0$ but decrease immediately after launch if $\Delta > 1/2$.

Figure 2 illustrates the product's equilibrium lifecycle in three main cases: when the product is sufficiently likely to be good that $\alpha > \rho$ (top); when the product is sufficiently likely to be bad that $\alpha < 1 - \rho$ (bottom); and when the product has an intermediate likelihood of being good so that $\alpha \in (1 - \rho, \rho)$ (middle). Proposition 1 characterizes consumers' equilibrium behavior in the first two cases.

Proposition 1. *(i) When $\alpha > \rho$, all consumers adopt the product in equilibrium. (ii) When $\alpha < 1 - \rho$, no consumers adopt the product in equilibrium.*

Proof. When $\alpha > \rho$, consumers herd on adoption at time $t = 0$ and, with an equal mass of "infected" consumers spreading awareness no matter whether the product is good or bad, later-exposed consumers infer nothing about product quality from their time of exposure and so also herd on adoption in the unique equilibrium. Since all consumers eventually encounter the product, all consumers end up adopting. On the other hand, when $\alpha < 1 - \rho$, consumers herd on non-adoption at time $t = 0$ and, with no one "infected" to spread awareness virally, no one else is ever exposed in the unique equilibrium. □

The rest of our analysis focuses on the remaining case in which $\alpha \in (1 - \rho, \rho)$, so that consumers who encounter the product at launch are sensitive to signals.[5] Theorem 1 summarizes our main findings, that equilibrium epidemiological dynamics are uniquely determined and that consumer behavior transitions through (up to) four distinct phases during the product's lifecycle. Behavior in Phase I immediately after launch depends on whether α is greater or less than $1/2$ but, no matter what, there is always a period of partial herding (Phase II), a period in which consumers are sensitive to signals (Phase III), and a final period with zero adoption (Phase IV, referred to as "obsolescence").

Theorem 1. *Suppose that $\alpha \in (1-\rho, \rho)$ and $\Delta \approx 0$. All equilibria generate the same epidemiological dynamics $(S_\omega(t), I_\omega(t), R_\omega(t) : t \geq 0; \omega \in \{g, b\})$ and time-path of interim beliefs $p(t)$. Consumers' post-launch equilibrium behavior transitions through up to four distinct phases.*

[5] We ignore the boundary cases in which $\alpha = \rho$ and $\alpha = 1 - \rho$. These cases are more complex because consumers have multiple best responses at launch, but this extra complexity does not lead to any additional insight. For example, if $\alpha = 1 - \rho$, consumers exposed at launch will adopt with some probability $a_0 \in [0, 1]$ after a good signal but not adopt after a bad signal, resulting in initial infected mass $I_g(0+) = a(0)\rho\Delta$ when the product is good and $I_b(0+) = a(0)(1 - \rho)\Delta$ when it is bad and initial belief $p(0+) = 1/2$. For any a_0, subsequent equilibrium dynamics are then uniquely determined by similar arguments as used here for the case when $\alpha \in (1 - \rho, \rho)$.

Phase I: *(i) If* $\alpha \in (1/2, \rho)$*, then consumers herd on adoption and interim belief* $p(t) > \rho$ *decreases, until time* $t_1 > 0$ *is reached at which* $p(t_1) = \rho$*. (ii) If* $\alpha \in (1-\rho, 1/2)$*, then consumers are sensitive to signals and* $p(t) \in (1/2, \rho)$ *increases until time* $t_1 > 0$ *at which* $p(t_1) = \rho$*. (iii) If* $\alpha = 1/2$*, then* $p(0+) \equiv \lim_{\epsilon \to 0} p(\epsilon) = \rho$ *and Phase I does not occur, i.e.,* $t_1 = 0$*.*

Phase II: *Consumers partially herd on adoption, adopting always after a good signal and with probability* $a_B(t) \in (0, 1)$ *after a bad signal, where* $a_B(t)$ *is decreasing in* t*, until time* $t_2 > t_1$ *is reached at which* $a_B(t_2) = 0$*. Consumers' interim belief* $p(t) = \rho$ *for all* $t \in (t_1, t_2)$*.*

Phase III: *Consumers are sensitive to signals and interim belief* $p(t) \in (1-\rho, \rho)$ *is decreasing in* t*, until time* $t_3 > t_2$ *is reached at which* $p(t_3) = 1 - \rho$*.*

Phase IV: *Consumers forevermore herd on non-adoption, what we refer to as "product obsolescence," and consumers' interim belief* $p(t) < 1 - \rho$ *continues to decrease with* $\lim_{t \to \infty} p(t) = 0$*.*

The rest of this section provides the proof of Theorem 1 through a series of lemmas and propositions, characterizing consumers' equilibrium behavior throughout the product's lifecycle, from launch to (endogenous) obsolescence.

4.1 Dynamics of consumer beliefs

Throughout the subsequent analysis, we make repeated use of the following two lemmas. Lemma 2 provides an easy-to-check condition to determine whether consumers' interim beliefs are increasing ($p'(t) > 0$) or decreasing ($p'(t) < 0$) over time.

Lemma 2. $\frac{p(t)}{1-p(t)}$ *increases exponentially at rate* $\beta X(t)$*, where*

$$X(t) = (q_g(t)S_g(t) - q_b(t)S_b(t)) - (I_g(t) - I_b(t)). \tag{4}$$

In particular: (i) Suppose that consumers herd on adoption at time t*.* $p'(t) < 0$ *if* $I_g(t) > I_b(t)$ *and* $S_g(t) < S_b(t)$*. (ii) Suppose that consumers are sensitive to signal at time* t*.* $p'(t) > 0$ *if and only if the following inequality holds:*

$$\rho S_g(t) - (1-\rho)S_b(t) > I_g(t) - I_b(t). \tag{SS}$$

(We refer to this as "Condition SS," mnemonic for "sensitive to signal") (iii) Suppose that consumers herd on non-adoption at time t*.* $p'(t) < 0$ *if* $I_g(t) > I_b(t)$*.*

Proof. By equation (3), the likelihood ratio $\frac{p(t)}{1-p(t)} = \frac{\alpha}{1-\alpha} \times \frac{S_g(t)I_g(t)}{S_b(t)I_b(t)}$; so,

$$\begin{aligned}\frac{\mathrm{dlog}\left(\frac{p(t)}{1-p(t)}\right)}{\mathrm{d}t} &= \frac{S_g'(t)}{S_g(t)} + \frac{I_g'(t)}{I_g(t)} - \frac{S_b'(t)}{S_b(t)} - \frac{I_b'(t)}{I_b(t)} \\ &= \beta\left(-I_g(t) + q_g(t)S_g(t) + I_b(t) - q_b(t)S_b(t)\right) \\ &= \beta X(t) \qquad (5)\end{aligned}$$

where $\frac{S_\omega'(t)}{S_\omega(t)} = -\beta I_\omega(t)$ and $\frac{I_\omega'(t)}{I_\omega(t)} = \beta q_\omega(t) S_\omega(t)$ follow from equations (1-2). We conclude that $\frac{p(t)}{1-p(t)}$ grows exponentially at rate $\beta X(t)$ and, in particular, that $p'(t) \gtreqless 0$ iff $X(t) \gtreqless 0$.

Implications (i-iii) follow immediately. (i) When consumers find herd on adoption, $q_g(t) = q_b(t) = 1$ and (4) simplifies to $X(t) = (S_g(t) - S_b(t)) - (I_g(t) - I_b(t))$, which is negative whenever $S_g(t) < S_b(t)$ and $I_g(t) > I_b(t)$. (ii) When consumers are sensitive to signal, $q_g(t) = \rho$ and $q_b(t) = 1 - \rho$, and (4) simplifies to $X(t) = (\rho S_g(t) - (1-\rho)S_b(t)) - (I_g(t) - I_b(t))$, which is positive exactly when Condition SS is satisfied. (iii) When consumers herd on non-adoption, $q_g(t) = q_b(t) = 0$ and (4) simplifies to $X(t) = -(I_g(t) - I_b(t))$, which is negative exactly when $I_g(t) > I_b(t)$. □

Lemma 3 provides a useful connection between consumer beliefs and the rates at which consumers encounter and adopt good versus bad products over time.

Lemma 3. *Suppose that $p(t) > \alpha$ for all $t \in (0, \hat{t})$ for some $\hat{t}$. Then $I_g(t) > I_b(t)$, $I_g'(t) > I_b'(t)$, $S_g(t) < S_b(t)$, and $S_g'(t) < S_b'(t)$ for all $t \in (0, \hat{t})$.*

Proof. At launch, mass Δ of consumers are exposed to the product, of whom fraction $q_g(t)$ or $q_b(t)$ adopt when the product is good or bad, respectively; so, $S_g(0) = S_b(0) = 1 - \Delta$, $I_g(0) = q_g(0)\Delta$, and $I_b(0) = q_b(0)\Delta$. By Lemma 1, each exposed consumer is always at least as likely to adopt when the product is good than when it is bad. In particular, it must be that $q_g(0) \geq q_b(0)$, implying that there are at least as many consumers "infected" at launch when the product is good than when it is bad.

At each time $t > 0$ when the product is good, mass $I_g(t)$ of infected consumers each meet another consumer at rate β, and fraction $S_g(t)$ of these meetings result in a *new exposure* to the product. This creates a flow $\beta S_g(t) I_g(t) = -S_g'(t)$ of new exposures to a good product. Similarly, when the product is bad, there is a flow $\beta S_b(t) I_b(t) = -S_b'(t)$ of new exposures.

Suppose that $p(t) > \alpha$. Since $\frac{p(t)}{1-p(t)} = \frac{\alpha}{1-\alpha} \times \frac{S_g(t)I_g(t)}{S_b(t)I_b(t)}$, it must be that $\frac{S_g(t)I_g(t)}{S_b(t)I_b(t)} > 1$ and hence that $S_g(t)I_g(t) > S_b(t)I_b(t)$. Thus, $-S_g'(t) > -S_b'(t)$ whenever $p(t) > \alpha$. Since $I_g'(t) = -q_g(t)S_g'(t)$, $I_b'(t) = -q_b(t)S_b'(t)$, and $q_g(t) \geq q_b(t)$, this implies further that $I_g'(t) > I_b'(t)$ whenever $p(t) > \alpha$. We conclude that, if $p(t) > \alpha$ for all $t \in (0, \hat{t})$, then $S_g(t) < S_b(t)$, $I_g(t) > I_b(t)$, $S_g'(t) < S_b'(t)$, and $I_g'(t) > I_b'(t)$ for all $t \in (0, \hat{t})$. □

4.2 Beginning of product lifecycle (Launch and Phase I)

This section characterizes consumers' equilibrium behavior at launch (Prop 2) and after launch until time t_1 is reached at which consumers' interim belief equals ρ (Props 3-4). We refer to the period of time from launch until t_1 as "Phase I".

Proposition 2. *Suppose that $\alpha \in (1-\rho, \rho)$. In any equilibrium, consumers are sensitive to signals at time $t = 0$ and consumers exposed immediately after launch have interim belief $p(0+) = \frac{\alpha\rho}{\alpha\rho+(1-\alpha)(1-\rho)} > \alpha$.*

Proof. Because $\alpha = p(0) \in (1-\rho, \rho)$, consumers exposed to the product at launch are sensitive to signals (Lemma 1(iii)). Thus, fraction ρ of the $\Delta > 0$ mass of consumers exposed at launch choose to adopt when the product is good, whereas only fraction $1-\rho$ of these consumers adopt when the product is bad. In particular, at any time $t \approx 0$ shortly after launch, there are more "infected" consumers when the product is good than when it is bad: $I_g(t) \approx \rho\Delta$ versus $I_b(t) \approx (1-\rho)\Delta$. On the other hand, there are approximately the same number of still-unexposed "susceptible" consumers: $I_g(t) \approx I_g(t) \approx 1 - \Delta$.

Since more consumers adopt good products at launch, others are more likely to encounter the product shortly after launch when it is good. Consumers exposed to the product shortly after launch therefore interpret their quick awareness of the product as good news about its quality. For any time $t \approx 0$, consumers exposed to the product at time t therefore hold interim belief $p(t) = \frac{\alpha S_g(t) I_g(t)}{\alpha S_g(t) I_g(t)+(1-\alpha) S_b(t) I_b(t)} \approx p(0+) = \frac{\alpha\rho}{\alpha\rho+(1-\alpha)(1-\rho)}$. □

Being quickly exposed to the product is "good news," causing quickly-exposed consumers to update upward their beliefs about product quality. Whether this is enough to prompt such consumers to herd on adoption, however, depends on the ex ante likelihood α that the product is good. If $\alpha > 1/2$, then $p(0+) > \rho$ and consumers will herd on adoption immediately after adoption (Prop 3). On the other hand, if $\alpha < 1/2$, then $p(0+) \in (1/2, \rho)$ and consumers will continue to be sensitive to signals immediately after adoption (Prop 4). Finally, if $\alpha = 1/2$, then $p(0+) = \rho$ and Phase I does not occur, i.e., $t_1 = 0$.

Proposition 3. *Suppose that $\alpha \in (1/2, \rho)$. There exists $t_1 > 0$ such that, in any equilibrium at all times $t \in (0, t_1)$, consumers herd on adoption and consumers' interim belief $p(t) > \rho$ is decreasing over time with $p(t_1) = \rho$.*

Proof. Since $\alpha > 1/2$, a consumer exposed immediately after launch has interim belief $p(0+) > \rho$ (Prop 2) and so has an incentive to adopt no matter what her private signal (Lemma 1(i)). So long as consumers continue to herd on adoption (what we refer to as "Phase I"), the term $X(t)$ in Lemma 2 simplifies to $X(t) = (S_g(t) - S_b(t)) - (I_g(t) - I_b(t))$. Because consumers' interim belief $p(t) \geq \rho$ during Phase I and $\alpha < \rho$, Lemma 3 implies that $I_g(t) > I_b(t)$ and $S_g(t) < S_b(t)$. We conclude that $X(t) < 0$ throughout Phase I. Moreover, because all those who are exposed to the product choose to adopt, the increase in infected consumers during Phase I equals the decrease in susceptibles, i.e., $S_\omega(0+) - S_\omega(t) = I_\omega(t) - I_\omega(0+)$ for $\omega \in \{g, b\}$; so, $X(t)$ is constant and equal to $X(0+)$ throughout Phase I. Finally, because consumers are sensitive to signal at launch, $S_g(0+) = S_b(0+) = 1 - \Delta$, $I_g(0+) = \rho\Delta$, and $I_b(0+) = (1-\rho)\Delta$; so, $X(0+) = -\Delta(2\rho-1)$. Because $\frac{\text{dlog}\left(\frac{p(t)}{1-p(t)}\right)}{\text{d}t} = \beta X(t)$ by Lemma 2, we conclude that $\frac{p(t)}{1-p(t)}$ decreases exponentially over time so long as consumers continue herd on adoption. Let t_1 be the first time at which consumers' interim belief equals ρ, given that consumers are sensitive to signal at launch and herd on adoption at times $t \in (0, t_1)$. □

Proposition 4. *Suppose that $\alpha \in (1 - \rho, 1/2)$ and $\Delta \approx 0$. There exists $t_1 > 0$ such that, in any equilibrium at all times $t \in (0, t_1)$, consumers are sensitive to signals and consumers' interim beliefs $p(t) > \rho$ are increasing over time with $p(t_1) = \rho$.*

Proof. Since $\alpha \in (1 - \rho, 1/2)$, a consumer exposed immediately after launch has interim belief $p(0+) \in (1/2, \rho)$ (Prop 2) and so has an incentive to be sensitive to signal, adopting only after a good private signal. Given that consumers are sensitive to signal, interim belief is increasing while consumers are sensitive to signal so long as Condition SS is satisfied, i.e., so long as

$$X(t) = (\rho S_g(t) - I_g(t)) - ((1 - \rho)S_b(t) - I_b(t)) > 0. \tag{6}$$

Recall that, at times $t \approx 0$, $S_g(t) \approx S_b(t) \approx 1 - \Delta$, $I_g(t) \approx \rho\Delta$, and $I_b(t) \approx (1 - \rho)\Delta$. Thus, $X(t) \approx (2\rho - 1)(1 - 2\Delta) \approx 2\rho - 1 > 0$ and interim beliefs are increasing immediately after launch.

Although interim beliefs are initially increasing, the rate at which $\frac{p(t)}{1-p(t)}$ increases itself decreases over time. To see why, recall that $S_g'(t) = -\beta I_g(t)S_g(t)$, $S_b'(t) = -\beta I_b(t)S_b(t)$, $I_g'(t) = \beta\rho I_g(t)S_g(t)$, and $I_b'(t) = \beta(1 - \rho)I_b(t)S_b(t)$ by equations (1-2); thus,

$$X'(t) = -\beta\left(\rho S_g(t)I_g(t) - (1 - \rho)S_b(t)I_b(t)\right). \tag{7}$$

Note that $X'(t) \geq 0$ iff $\frac{S_g(t)I_g(t)}{S_b(t)I_b(t)} \leq \frac{1-\rho}{\rho}$. By equation (3), that is only possible at times when $\frac{p(t)}{1-p(t)} \leq \frac{\alpha(1-\rho)}{(1-\alpha)\rho}$ which, since $\alpha < 1/2$, implies that $p(t) < 1-\rho$. We conclude that, so long as $p(t) > 1-\rho$ and consumers are sensitive to signal, $X'(t) < 0$.

Because $X'(t) < 0$, consumers' interim beliefs may begin to decline if consumers are sensitive to signal for long enough. However, because Δ is small, this does not happen for a long time. To see why, note that the total mass of consumers exposed by any given time $\tilde{t}$ can be made arbitrarily small by beginning with a sufficiently small initial mass Δ of consumers exposed at launch. In particular, for any time $\tilde{t}$ and any small $\epsilon > 0$, we can find Δ sufficiently small so that (i) $S_g(t), S_b(t) \in (1-\epsilon, 1)$ for all $t < \tilde{t}$ and (ii) $I_g(t), I_b(t) \in (0, \epsilon)$ for all $t < \tilde{t}$, implying that $X(t) > (\rho(1-\epsilon)-\epsilon) - (1-\rho-0) = 2\rho - 1 - \epsilon(1+\rho) > 0$ for all $t \in (0, \tilde{t})$. Recall by Lemma 2 that the likelihood ratio $\frac{p(t)}{1-p(t)}$ rises exponentially at rate $\beta X(t)$; so, for small Δ, $\frac{p(t)}{1-p(t)}$ rises exponentially at approximate rate $\beta(2\rho-1)$ until a time t_1 is reached at which consumers' interim belief equals ρ.

We conclude that, in any equilibrium, consumers must be sensitive to signal at all times $t \in (0, t_1)$ (since interim belief is in $(1/2, \rho)$ at such times) and must not continue to be sensitive to signal immediately after t_1 (since then interim belief would rise above ρ, a contradiction). This completes the proof and, in addition, uniquely characterizes t_1 as the first time at which $p(t_1) = \rho$. □

4.3 Middle of product lifecycle (Phase II)

This section characterizes equilibrium behavior immediately after time t_1. We find that, for a non-empty interval of time, consumers randomize whether to adopt after a bad signal, what we call "partial herding." Over that period of time, consumers' interim belief remains equal to ρ and the likelihood $a_B(t)$ that consumers adopt after a bad signal declines continuously until, at some time t_2, $a_B(t) = 0$ and consumers become sensitive to signal. We refer to the partial-herding period from t_1 until t_2 as "Phase II".

Proposition 5. *Suppose that $\alpha \in (1-\rho, \rho)$ and $\Delta \approx 0$. There exists $t_2 > t_1$ such that, in any equilibrium at all times $t \in (t_1, t_2)$, consumers partially herd with probability $a_B(t) \in (0,1)$ of adoption after a bad signal and consumers' interim belief $p(t) = \rho$ is constant. Moreover, $a_B(t)$ is decreasing over time with $a_B(t_2) = 0$.*

Proof. By construction of the time t_1 at which Phase I ends, $p(t_1) = \rho$ and consumers exposed at time t_1 are indifferent whether to adopt after a bad private signal. We begin by showing that, immediately after time t_1, consumers' interim belief cannot rise above ρ. If it did rise above ρ immediately after time t_1, consumers would herd on adoption. Since $p(t) > \alpha$ for all $t \in (0, t_1)$, $S_g(t_1) < S_b(t_1)$ and $I_g(t_1) > I_b(t_1)$

by Lemma 3; thus, $S_g(t) - I_g(t) < S_b(t) - I_b(t)$ and consumers' interim belief must decline over time by Lemma 2(i), a contradiction.

Consumers' interim belief also cannot fall below ρ immediately after time t_1. As discussed in the proof of Prop 4, the assumption here of a small launch ($\Delta \approx 0$) implies that only a small mass of consumers are exposed to the product prior to Phase II; in particular, $S_g(t_1), S_b(t_1) \in (1-\epsilon, 1)$ and $I_g(t_1), I_b(t_1) \in (0, \epsilon)$ for some small ϵ. Consequently, for all times t shortly after t_1, Condition SS holds: $(\rho S_g(t) - I_g(t)) - ((1-\rho)S_b(t) - I_b(t)) \approx 2\rho - 1 > 0$. Were consumers to be sensitive to signal immediately after time t_1, consumers' interim belief would therefore increase over time by Lemma 2(ii), a contradiction.

We conclude that, in any equilibrium, consumers' interim belief must remain $p(t) = \rho$ for some period of time after t_1. Let $a_B(t)$ denote the likelihood that consumers exposed at time t adopt the product after getting a bad signal. We begin by characterizing $a_B(t_1+)$ and then show that $a_B(t)$ must decline over time after t_1.

By equation (3), interim belief $p(t) = \rho$ requires that $\frac{\rho}{1-\rho} = \frac{\alpha I_g(t) S_g(t)}{(1-\alpha) I_b(t) S_b(t)}$ or, equivalently, $\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)} = \frac{(1-\alpha)\rho}{\alpha(1-\rho)}$. In order for this ratio not to change over time, the ratio of derivatives $\frac{(I_g(t)S_g(t))'}{(I_b(t)S_b(t))'}$ must also equal $\frac{(1-\alpha)\rho}{\alpha(1-\rho)}$. Taking derivatives, using equations (1-2), and re-arranging yields

$$\frac{(1-\alpha)\rho}{\alpha(1-\rho)} = \frac{I_g'(t)S_g(t) + I_g(t)S_g'(t)}{I_b'(t)S_b(t) + I_b(t)S_b'(t)} = \frac{\beta I_g(t)S_g^2(t)q_g(t) - \beta I_g^2(t)S_g(t)}{\beta I_b(t)S_b^2(t)q_b(t) - \beta I_b^2(t)S_b(t)}$$
$$= \frac{I_g(t)S_g(t)(S_g(t)q_g(t) - I_g(t))}{I_b(t)S_b(t)(S_b(t)q_b(t) - I_b(t))}$$

and so it must be that

$$S_g(t)q_g(t) - I_g(t) = S_b(t)q_b(t) - I_b(t). \tag{8}$$

Given that consumers exposed at time $t \in (t_1, t_2)$ always adopt after a good signal and adopt with probability $a_B(t)$ after a bad signal, the overall likelihood that a good product is adopted equals $q_g(t) = \rho + (1-\rho)a_B(t)$; similarly, the likelihood that a bad product is adopted equals $q_b(t) = 1 - \rho + \rho a_B(t)$. Equation (8) can therefore be re-written as

$$(\rho S_g(t) - (1-\rho)S_b(t)) - (I_g(t) - I_b(t)) + a_B(t)\left((1-\rho)S_g(t) - \rho S_b(t)\right) = 0 \tag{9}$$

or, equivalently,

$$a_B(t) = \frac{\rho S_g(t) - (1-\rho)S_b(t) - (I_g(t) - I_b(t))}{\rho S_b(t) - (1-\rho)S_g(t)} \tag{10}$$

The analysis of Section 4.2 uniquely characterizes the time t_1 at which Phase II begins and the initial conditions $(I_g(t_1), I_b(t_1), S_g(t_1), S_b(t_1))$. Now, equation (10) uniquely determines $a_B(t_1+)$, consumers' equilibrium likelihood of adopting after a bad signal immediately after time t_1. Note that, since $I_g(t_1) > I_b(t_1)$ and $S_b(t_1) > S_g(t_1)$ (by Lemma 3), $a_B(t_1+) < 1$. Moreover, because Condition SS holds at time t_1 (discussed earlier), the numerator in (10) is positive; so, $a_B(t_1+) > 0$.

By Lemma 3, $S_b(t) > S_g(t)$ so long as consumers' interim beliefs continue to exceed the initial belief α. Since $p(t) > \alpha$ throughout Phase I and $\rho > \alpha$, we conclude that $S_b(t) > S_g(t)$ so long as consumers continue to partially herd, ensuring that the denominator of (10) remains positive. Consumers therefore continue to partially herd so long as the numerator remains positive, i.e., so long as Condition SS continues to be satisfied. Next, note that

$$a'_B(t) = \frac{\begin{array}{c}(\rho S'_g(t) - (1-\rho)S'_b(t) - (I'_g(t) - I'_b(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ -(\rho S_g(t) - (1-\rho)S_b(t) - (I_g(t) - I_b(t)))(\rho S'_b(t) - (1-\rho)S'_g(t))\end{array}}{(\rho S_b(t) - (1-\rho)S_g(t))^2}.$$

Rearranging and simplifying the numerator, we have

$$\begin{aligned}\text{numerator} =& (\rho^2 - (1-\rho)^2)(S'_g(t)S_b(t) - S'_b(t)S_g(t)) \\ &- (I'_g(t) - I'_b(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ &+ (I_g(t) - I_b(t))(\rho S'_b(t) - (1-\rho)S'_g(t)).\end{aligned}$$

By (1-2), the second term above can be re-written as

$$\begin{aligned}&- (I'_g(t) - I'_b(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ =& - \beta(I_g(t)S_g(t)(\rho + (1-\rho)a_B(t)) - I_b(t)S_b(t)(1-\rho+\rho a_B(t)))(\rho S_b(t) - (1-\rho)S_g(t)) \\ =& - \beta I_b(t)(I_g(t) - I_b(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ &- \beta(I_g(t) - I_b(t))S_g(t)(\rho + (1-\rho)a_B(t))(\rho S_b(t) - (1-\rho)S_g(t)) \qquad (11)\end{aligned}$$

Similarly, the third term above can be re-written is

$$\begin{aligned}&(I_g(t) - I_b(t))(\rho S'_b(t) - (1-\rho)S'_g(t)) \\ =& - \beta(I_g(t) - I_b(t))(\rho I_b(t)S_b(t) - (1-\rho)I_g(t)S_g(t)) \\ =& - \beta I_b(t)(I_g(t) - I_b(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ &+ \beta(I_g(t) - I_b(t))S_g(t)(1-\rho)(I_g(t) - I_b(t)) \qquad (12)\end{aligned}$$

We will show that the entire numerator is negative, by showing that the first term is negative and that the sum of the second term (11) and third term (12) is negative.

To that end, recall by Lemma 3 that $I_g(t) > I_b(t)$, $I'_g(t) > I'_b(t)$, $S_g(t) < S_b(t)$, and $S'_g(t) < S'_b(t)$ so long as consumers continue to partially herd. The fact that the first term is negative now follows immediately from (1-2), since $S'_g(t)S_b(t) - S'_b(t)S_g(t) = -\beta S_g(t)S_b(t)(I_g(t)-I_b(t)) < 0$. Moreover, $\rho S_b(t) > (1-\rho)S_g(t)$ because $S_b(t) > S_g(t)$ and $\rho > 1/2$; so, the first part of (11) and the first part of (12) are negative. To show that the sum of (11) and (12) is negative, it therefore suffices to show that $(\rho + (1-\rho)a_B(t))(\rho S_b(t) - (1-\rho)S_g(t)) > (1-\rho)(I_g(t) - I_b(t))$. But this follows immediately from the fact that $\rho S_b(t) - (1-\rho)S_g(t) > I_g(t) - I_b(t)$ (since Condition SS remains satisfied) and $\rho + (1-\rho)a_B(t) > 1 - \rho$ (since $\rho > 1/2$ and $a_B(t) \geq 0$).

Overall, we conclude that $a_B(t) > 0$ but $a'_B(t) < 0$ so long as the numerator of equation (10) continues to be positive, i.e., so long as Condition SS continues to be satisfied. Moreover, there is a finite time t_2 at which partial herding ceases. To see why, suppose for the sake of contradiction that consumers were to partially herd forever. Because all consumers are eventually exposed to the product, $\lim_{t\to\infty} S_g(t) = \lim_{t\to\infty} S_b(t) = 0$. On the other hand, because $I'_g(t) > I'_b(t)$ so long as $a_B(t) > 0$, $\lim_{t\to\infty}(I_g(t) - I_b(t)) > I_g(t_1) - I_g(t_1) > 0$. All together, then, the numerator of (10) must eventually become negative, a contradiction.

The time t_2 at which Phase II ends is the first time at which $\rho S_g(t) - (1-\rho)S_b(t) = I_g(t) - I_b(t)$; at that time, $p(t_2) = \rho$ but consumers are sensitive to signal because $a_B(t_2) = 0$. □

4.4 End of product lifecycle (Phase III and obsolescence)

This section characterizes equilibrium behavior after time t_2. We have two main findings. First, consumers remain sensitive to signal for a period of time but, even though newly-exposed consumers are only adopting after a good private signal, consumers' interim belief falls until a time t_3 is reached at which $p(t_3) = 1 - \rho$. Second, consumers herd on non-adoption after time t_3, what we refer to as "product obsolescence" and, after obsolescence, interim beliefs continue to decline to zero. We refer to the sensitive-to-signal period from t_2 to t_3 as "Phase III" and the obsolescent period after t_3 as "Phase IV".

Proposition 6. *Suppose that $\alpha \in (1-\rho, \rho)$ and $\Delta \approx 0$. There exists $t_3 > t_2$ such that, in any equilibrium at all times $t \in (t_2, t_3)$, consumers are sensitive to signal and consumers' interim belief $p(t)$ declines over time from $p(t_2) = \rho$ to $p(t_3) = 1 - \rho$. Moreover, at time t_3, more consumers will have adopted the product if it is good than if it is bad, i.e., $I_g(t_3) > I_b(t_3)$.*

Proof. For ease of exposition, we divide the proof into three main steps.

Step 1: After time t_2, $\frac{p(t)}{1-p(t)}$ declines exponentially at an increasing rate until time $\tilde{t}$ is reached at which $p(\tilde{t}) = \max\{1-\rho, \underline{\alpha}\}$, where $\underline{\alpha} \equiv \frac{(1-\rho)\alpha}{(1-\rho)\alpha+\rho(1-\alpha)} \in \left(\frac{(1-\rho)^2}{(1-\rho)^2+\rho^2}, \frac{1}{2}\right)$.

By Lemma 2, $\frac{p(t)}{1-p(t)}$ declines exponentially at rate $\beta X(t)$. So, it suffices to show that $X(t) < 0$ and $X'(t) < 0$ at all times $t \in (t_2, \tilde{t})$, where $\tilde{t}$ is the first time at which $p(\tilde{t}) = \max\{1-\rho, \underline{\alpha}\}$.

By the proof of Prop 5: $p(t_2) = \rho$; consumers are sensitive to signal at time t_2 (because $a_B(t_2) = 0$), and $X(t_2) = (\rho S_g(t_2) - I_g(t_2)) - ((1-\rho)S_b(t_2) - I_b(t_2)) = 0$. So, it suffices to show that $X'(t) < 0$ at all times $t \in [t_2, \tilde{t})$.

So long as consumers are sensitive to signal, $(\rho S_g(t) - I_g(t))' = -2\beta\rho S_g(t) I_g(t)$ and $(\rho S_b(t) - I_b(t))' = -2\beta(1-\rho)S_b(t)I_b(t)$ by equations (1-2) and hence

$$X'(t) = -2\beta\left(\rho S_g(t)I_g(t) - (1-\rho)S_b(t)I_b(t)\right). \tag{13}$$

We conclude that $X'(t) < 0$, causing $\frac{p(t)}{1-p(t)}$ to decline exponentially at an increasing rate, so long as consumers are sensitive to signal and $\frac{S_g(t_2)I_g(t_2)}{S_b(t_2)I_b(t_2)} > \frac{1-\rho}{\rho}$. By equation (3), $\frac{p(t)}{1-p(t)} = \frac{\alpha S_g(t_2)I_g(t_2)}{(1-\alpha)S_b(t_2)I_b(t_2)}$; so, $\frac{S_g(t_2)I_g(t_2)}{S_b(t_2)I_b(t_2)} > \frac{1-\rho}{\rho}$ if and only if $\frac{p(t)}{1-p(t)} > \frac{\alpha(1-\rho)}{(1-\alpha)\rho} = \frac{\underline{\alpha}}{1-\underline{\alpha}}$. Overall, then, $X'(t) < 0$ at times $t > t_2$ so long as (i) $p(t) \in (1-\rho, \rho)$ (so that consumers continue to be sensitive to signal) and (ii) $p(t) > \underline{\alpha}$ (so that the expression in (13) remains negative).

There are two relevant cases. First, suppose that $\alpha \in (1-\rho, 1/2]$. In this case, $\underline{\alpha} \leq 1-\rho$ and so $p(\tilde{t}) = 1-\rho$, i.e., $\tilde{t} = t_3$. Second, suppose that $\alpha \in (1/2, \rho)$ so that $\underline{\alpha} \in (1-\rho, 1/2)$. In this more challenging case, $\underline{\alpha} \in (1-\rho, 1/2)$ and the argument so far only shows that $\frac{p(t)}{1-p(t)}$ declines until consumers' interim belief hits $\underline{\alpha}$. Step 2 provides the additional arguments needed in this case, to establish further that consumers' interim belief continues falling all the way to $1-\rho$.

Step 2: After time $\tilde{t}$ in the case when $\alpha \in (1/2, \rho)$, $\frac{p(t)}{1-p(t)}$ declines exponentially at an decreasing rate from time $\tilde{t}$ (described in Step 1) until a time t_3 is reached at which $p(t_3) = 1-\rho$.

We have shown in Step 1 that $X(t) < 0$ and $X'(t) < 0$ when $t \approx \tilde{t}+$. In other words, $\frac{p(t)}{1-p(t)}$ declines exponentially at an decreasing rate right after $\tilde{t}$. Suppose that it does not exhibit the same decreasing pattern until $p(t)$ reaches $1-\rho$, during which time consumers continue to be sensitive to signals. There are now two possibilities: (1) $\frac{p(t)}{1-p(t)}$ stops decreasing, or increases, before $p(t)$ reaches $1-\rho$. In this case, $X'(t) \geq 0$ at some finite t; (2) $\frac{p(t)}{1-p(t)}$ keeps decreasing but $p(t)$ never reaches $1-\rho$ in finite time. The two possibilities can be unified, in view of Lemma 2, by the existence of some $t' > 0$ ($t' < \infty$ for possibility (1) and $t' = \infty$ for possibility (2)) at which $X(t') = 0$, i.e.

$\rho S_g(t') - I_g(t') = (1-\rho)S_b(t') - I_b(t')$ for the first time after t_2. It means further that there exists $t'' \in (t_2, t')$ such that $((\rho S_g(t'') - I_g(t'')) - ((1-\rho)S_b(t'') - I_b(t'')))' = 0$, i.e. $\frac{I_g(t'')S_g(t'')}{I_b(t'')S_b(t'')} = \frac{1-\rho}{\rho}$. As $X(t) < 0$ for $t \in (t_2, t')$ by assumption, we know that $\frac{I_g(t')S_g(t')}{I_b(t')S_b(t')} < \frac{1-\rho}{\rho}$.

Several equations that follow are quite complex, so we introduce the following shorthand: $a = S_g(t_2)$; $b = S_b(t_2)$; $c = \rho S_g(t_2) - I_g(t_2) = (1-\rho)S_b(t_2) - I_b(t_2)$; and $d = -(\rho S_g(t') - I_g(t')) = -((1-\rho)S_b(t') - I_b(t'))$.

We know that

$$\begin{aligned} c + d &= (\rho S_g(t_2) - I_g(t_2)) - (\rho S_g(t') - I_g(t')) \\ &= \int_{t_2}^{t'} 2\beta\rho I_g(t)S_g(t)dt = 2(I_g(t') - I_g(t_2)) = -2\rho(S_g(t') - S_g(t_2)) \\ &= \int_{t_2}^{t'} 2\beta(1-\rho)I_b(t)S_b(t)dt = 2(I_b(t') - I_b(t_2)) = -2(1-\rho)(S_b(t') - S_b(t_2)), \end{aligned}$$

which implies that

$$\begin{aligned} I_g(t') - I_g(t_2) = I_b(t') - I_b(t_2) &= \frac{c+d}{2} \\ S_g(t') - S_g(t_2) &= -\frac{c+d}{2\rho} \\ S_b(t') - S_b(t_2) &= -\frac{c+d}{2(1-\rho)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{I_g(t')S_g(t')}{I_b(t')S_b(t')} &= \frac{(I_g(t_2) + I_g(t') - I_g(t_2))(S_g(t_2) + S_g(t') - S_g(t_2))}{(I_b(t_2) + I_b(t') - I_b(t_2))(S_b(t_2) + S_b(t') - S_b(t_2))} \\ &= \frac{(a - \frac{c+d}{2\rho})((a\rho - c) + \frac{c+d}{2})}{(b - \frac{c+d}{2(1-\rho)})((b(1-\rho) - c) + \frac{c+d}{2})} \\ &= \frac{a(a\rho - c) + \frac{c^2-d^2}{4\rho}}{b(b(1-\rho) - c) + \frac{c^2-d^2}{4(1-\rho)}} \end{aligned}$$

We already know that $\frac{a(a\rho-c)}{b(b(1-\rho)-c)} = \frac{(1-\alpha)\rho}{\alpha(1-\rho)} > 1$. Hence, no matter whether $c^2 - d^2 \geq 0$ or $c^2 - d^2 < 0$, $\frac{I_g(t')S_g(t')}{I_b(t')S_b(t')} > \frac{1-\rho}{\rho}$, a contradiction. This proves that interim beliefs will keep decreasing until t_3 such that $q(t_3; \emptyset) = 1 - \rho$.

Step 3: At time t_3, more consumers will have adopted the product if it is good than if it is bad, i.e., $I_g(t_3) > I_b(t_3)$. From the previous analysis, we know the following claims are true:

(1) $I_g(t_2) > I_b(t_2)$. It is clear that $I_g(0) > I_b(0)$ initially. Moreover, we know that $\frac{\alpha I_g(t) S_g(t)}{(1-\alpha) I_b(t) S_b(t)} \geq \frac{\rho}{1-\rho}$ for $t \leq t_2$, which means that

$$\frac{I_g'(t)}{I_b'(t)} \geq \frac{S_g(t) I_g(t)}{S_b(t) I_b(t)} \geq \frac{(1-\alpha)\rho}{\alpha(1-\rho)} \geq 1$$

for $t \leq t_2$. Therefore $I_g(t_2) > I_b(t_2)$.

(2) $I_g(t_3) - I_b(t_3) > I_g(t_2) - I_b(t_2)$. Since

$$\begin{aligned} X(t_3) &= (\rho S_g(t_3) - I_g(t_3)) - ((1-\rho) S_b(t_3) - I_b(t_3)) \\ &< X(t_2) = (\rho S_g(t_2) - I_g(t_2)) - ((1-\rho) S_b(t_2) - I_b(t_2)) = 0, \end{aligned}$$

we can apply the same way as characterizing $c + d$ to have

$$\begin{aligned} (\rho S_g(t_3) - I_g(t_3)) - ((1-\rho) S_b(t_3) - I_b(t_3)) &< (\rho S_g(t_2) - I_g(t_2)) - ((1-\rho) S_b(t_2) - I_b(t_2)) \\ (\rho S_g(t_3) - I_g(t_3)) - (\rho S_g(t_2) - I_g(t_2)) &< ((1-\rho) S_b(t_3) - I_b(t_3)) - ((1-\rho) S_b(t_2) - I_b(t_2)) \\ \int_{t_2}^{t_3} -2\beta\rho I_g(t) S_g(t) dt &< \int_{t_2}^{t_3} -2\beta(1-\rho) I_b(t) S_b(t) dt \\ I_g(t_3) - I_g(t_2) &> I_b(t_3) - I_b(t_2). \end{aligned}$$

Therefore $I_g(t_3) - I_b(t_3) > I_g(t_2) - I_b(t_2)$.

Combining (1) and (2), we have $I_g(t_3) > I_b(t_3)$. □

Proposition 7. *After time t_3 in any equilibrium, consumers herd on non-adoption and $p(t)$ continues to decline with $\lim_{t\to\infty} p(t) = 0$.*

Proof. Step 4: After time t_3, $\frac{p(t)}{1-p(t)}$ declines exponentially at a constant rate.

Consider time t_3. In any equilibrium, consumers adopt after good signal with probability $a_G(t_3) \in [0, 1]$ and do not adopt after bad signal. $X(t_3)$ therefore takes the form $X(t_3) = a_G(t_3) \left(\rho S_g(t_3-) - (1-\rho) S_b(t_3-)\right) - \left(I_g(t_3-) - I_b(t_3-)\right)$.

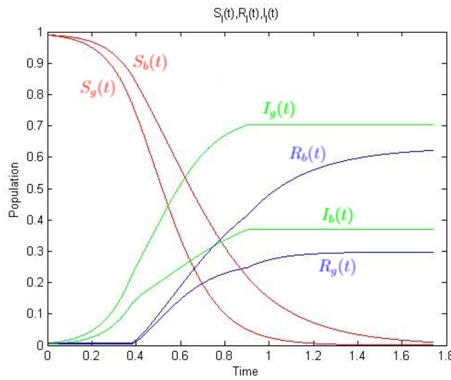
By Step 2, $X(t_3-) = (\rho S_g(t_3) - (1-\rho) S_b(t_3)) - (I_g(t_3) - I_b(t_3)) < 0$.By Step 3, $I_g(t_3) - I_b(t_3) < 0$. Together, these observations imply that $X(t_3) < 0$. Therefore interim belief falls at time t_3.

After time t_3, consumers herd on non-adoption; so $X(t_3+) = -\left(I_g(t_3-) - I_b(t_3-)\right)$, which is less than zero by Step 3. We conclude that $X(t) = X(t_3+)$ for all $t > t_3$, with the implication that $\frac{p(t)}{1-p(t)}$ declines exponentially at a constant rate.

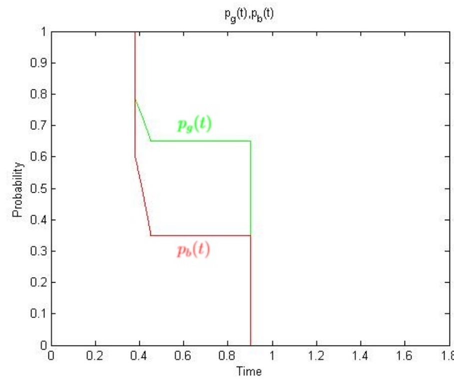
□

After t_2, interim beliefs go down with time even though consumers use the most informative strategy of being sensitive to signals. Hence when $q(t;\emptyset)$ reaches $1-\rho$, which means that G consumers become indifferent, there will not be a second period of mixed strategies, but consumers lose interest in the product once and for all.

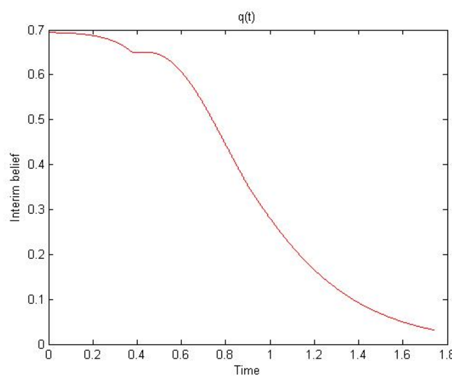
Given the parameter set $(\alpha, \rho, \beta, \Delta)$, the unique equilibrium can be easily simulated by numerical methods. The following figures illustrate how consumer behavior and interim beliefs evolve over time for different initial beliefs[6].



(a) Population of susceptible, infected and recovered consumers



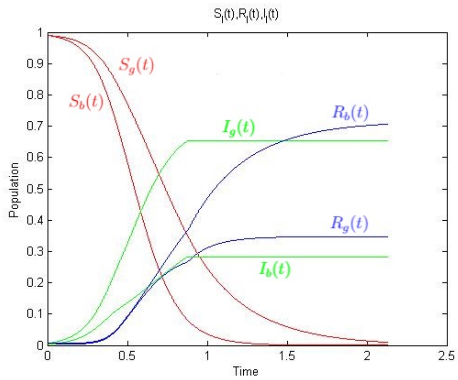
(b) Probability of product adoption



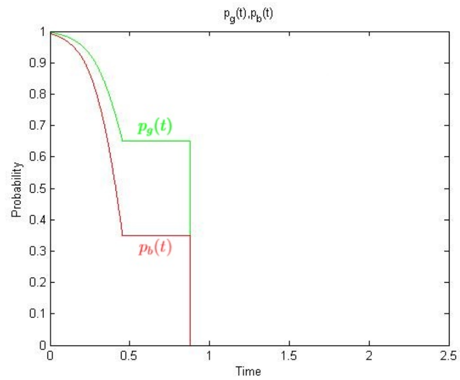
(c) Interim beliefs

Figure 3: Simulation results when $\alpha \in (\frac{1}{2}, \rho)$

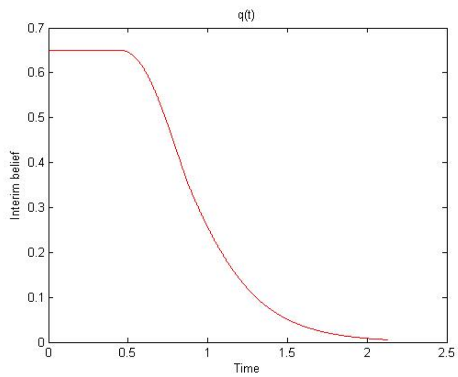
[6]In the simulation, we set $(\rho, \beta, \Delta) = (0.65, 10, 0.001)$. $\alpha = 0.55$ for Figure 1, 0.5 for Figure 2 and 0.45 for Figure 3.



(a) Population of susceptible, infected and recovered consumers

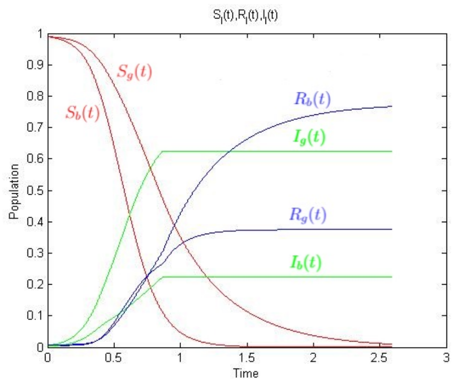


(b) Probability of product adoption

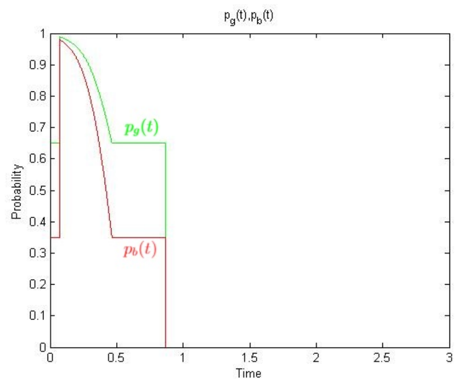


(c) Interim beliefs

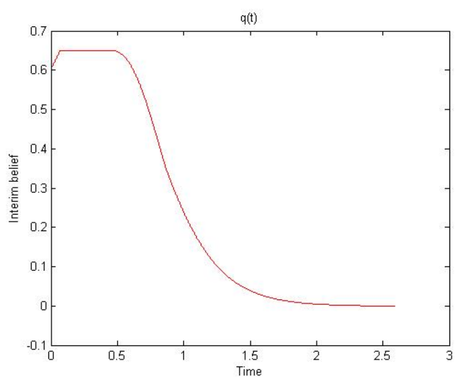
Figure 4: Simulation results when $\alpha = \frac{1}{2}$



(a) Population of susceptible, infected and recovered consumers



(b) Probability of product adoption



(c) Interim beliefs

Figure 5: Simulation results when $\alpha \in (1-\rho, \frac{1}{2})$

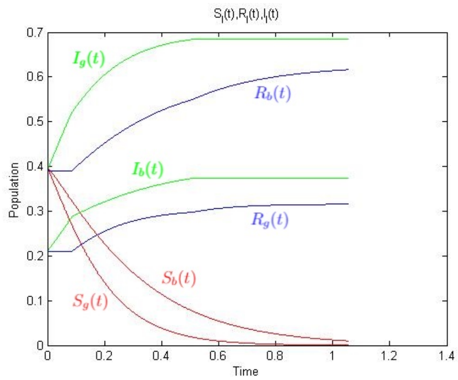
4.5 Special Cases in Equilibrium Characterization

Our analysis so far, which establishes different time cutoffs $t_1 < t_2 < t_3$, is based on two assumptions: (1) a sufficiently small Δ, and (2) $\rho S_g(t_1) - I_g(t_1) > (1-\rho)S_b(t_1) - I_b(t_1)$. In this subsection, we illustrate the unique consumer equilibrium assuming violation of either assumption.

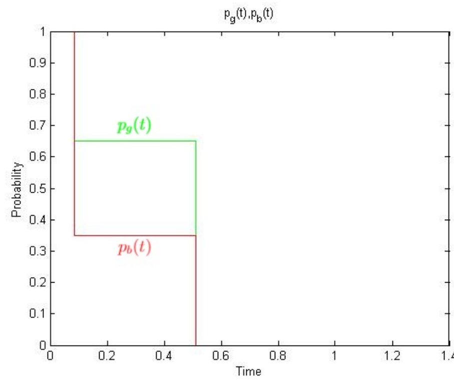
As it turns out, the two conditions are inter-connected: when Δ increases from infinitesimal, it must cause $\rho S_g(t) - I_g(t) < (1-\rho)S_b(t) - I_b(t)$ to occur when $q(t;\emptyset) = \rho$ (if $\alpha \in [\frac{1}{2}, \rho)$) or before $q(t;\emptyset) = \rho$ (if $\alpha \in (1-\rho, \frac{1}{2})$), before Δ reaches its maximum possible value 1. Alternatively, violation of (2) above may also occur if α is very close to ρ, which makes both $S_g(t)$ and $S_b(t)$ relatively small when $q(t;\emptyset) = \rho$.

On one hand, the equilibrium now is characterized by only two cutoffs as mixed strategies on $[t_1, t_2]$ in Theorem 1 have vanished. On the other hand, consumers' behavior around the first cutoff differs by α. When $\alpha \in (\frac{1}{2}, \rho)$, it means that consumers start being sensitive to signals immediately after their interim beliefs reach ρ; when $\alpha = \frac{1}{2}$, it means that consumers start being sensitive to signals immediately after $t = 0$; when $\alpha \in (1-\rho, \frac{1}{2})$, it means that consumers start being sensitive to signals as soon as $\rho S_g(t) - I_g(t) < (1-\rho)S_b(t) - I_b(t)$, which is before interim beliefs reach ρ. The following figures illustrate these three cases[7].

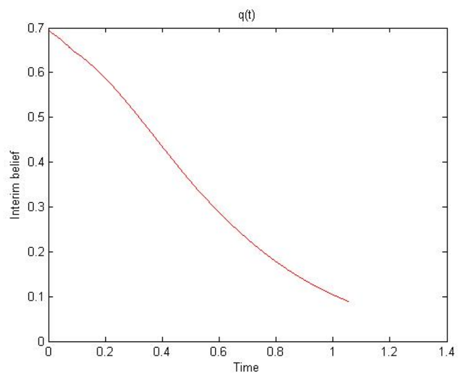
[7] In the simulation, $\alpha = 0.55$ for Figure 4, 0.5 for Figure 5 and 0.45 for Figure 6. We set $(\rho, \beta, \Delta) = (0.65, 10, 0.6)$ for Figures 4 and 5, and $(\rho, \beta, \Delta) = (0.65, 10, 0.3)$ for Figure 6. The difference in Δ is because a larger Δ is needed for consumers to start with being sensitive to signals when $\alpha = 0.5$, but the same Δ would have made Figures 5 and 6 identical. Hence we picked a smaller Δ when $\alpha \in (1-\rho, \frac{1}{2})$ to show that $q(t; \emptyset)$ may still increase initially even though $\rho S_g(t_1) - I_g(t_1) < (1-\rho)S_b(t_1) - I_b(t_1)$.



(a) Population of susceptible, infected and recovered consumers

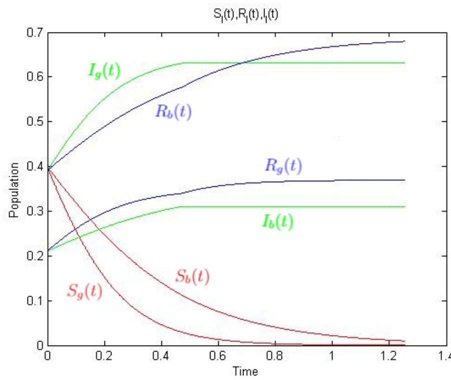


(b) Probability of product adoption

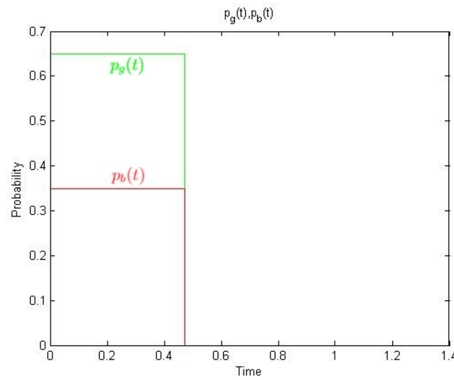


(c) Interim beliefs

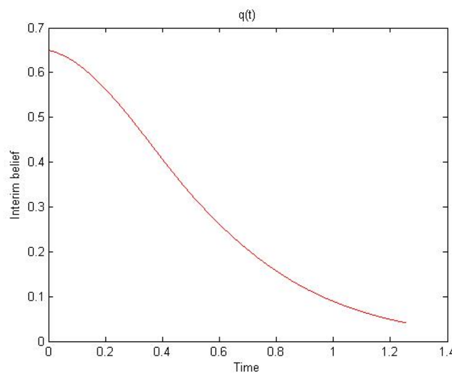
Figure 6: Simulation results when $\alpha \in (\frac{1}{2}, \rho)$



(a) Population of susceptible, infected and recovered consumers

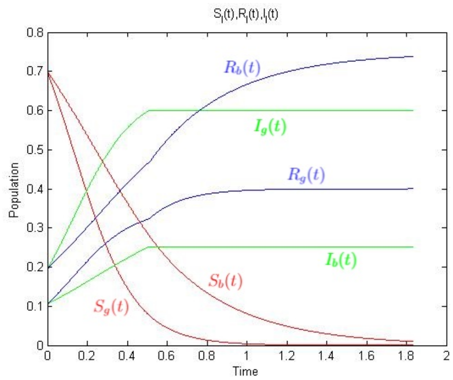


(b) Probability of product adoption

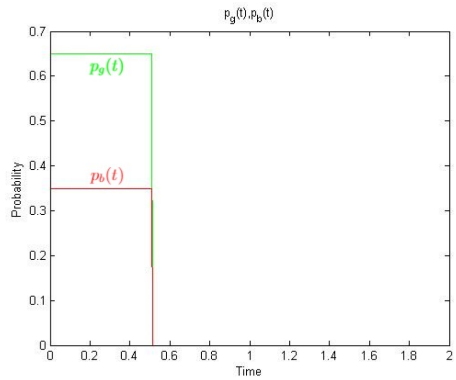


(c) Interim beliefs

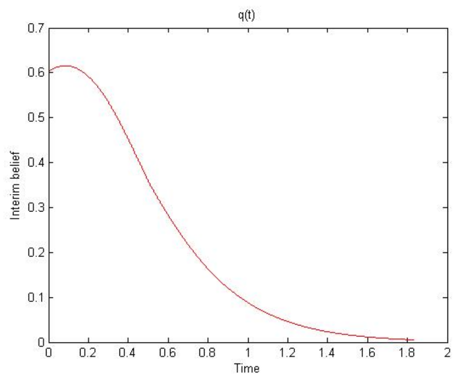
Figure 7: Simulation results when $\alpha = \frac{1}{2}$



(a) Population of susceptible, infected and recovered consumers



(b) Probability of product adoption



(c) Interim beliefs

Figure 8: Simulation results when $\alpha \in (1-\rho, \frac{1}{2})$

4.6 General Cases in Equilibrium Characterization

So far we have assumed that $P = \frac{\bar{u}+\underline{u}}{2}$, but our results can be easily generalized to cases with an arbitrary $P \in (\underline{u}, \bar{u})$. The key to extending the analysis is to determine (1) an upper bound of prior belief $\bar{\alpha}$, above which herding on adoption always occurs; (2) a lower bound of prior belief $\underline{\alpha}$, below which herding on non-adoption always occurs; (3) a cutoff of prior belief $\hat{\alpha}$, at which B consumers always mix when $t \approx 0$.

Analogous to our characterization for the range of α when $P = \frac{\bar{u}+\underline{u}}{2}$, $\bar{\alpha}$, $\underline{\alpha}$ and $\hat{\alpha}$

must satisfy the following conditions:

$$\frac{\bar{\alpha}(1-\rho)}{\bar{\alpha}(1-\rho)+(1-\bar{\alpha})\rho}\bar{u}+\frac{(1-\bar{\alpha})\rho}{\bar{\alpha}(1-\rho)+(1-\bar{\alpha})\rho}\underline{u}=P$$
$$\frac{\underline{\alpha}\rho}{\underline{\alpha}\rho+(1-\underline{\alpha})(1-\rho)}\bar{u}+\frac{(1-\underline{\alpha})(1-\rho)}{\underline{\alpha}\rho+(1-\underline{\alpha})(1-\rho)}\underline{u}=P$$
$$\hat{\alpha}\bar{u}+(1-\hat{\alpha})\underline{u}=P,$$

which implies that

$$\bar{\alpha}=\frac{\rho(P-\underline{u})}{(1-\rho)(\bar{u}-P)+\rho(P-\underline{u})}$$
$$\underline{\alpha}=\frac{(1-\rho)(P-\underline{u})}{\rho(\bar{u}-P)+(1-\rho)(P-\underline{u})}$$
$$\hat{\alpha}=\frac{P-\underline{u}}{\bar{u}-\underline{u}}.$$

Respectively, the conditions characterizing t_1, t_2 and t_3, become

$$q(t_1;\emptyset)=\bar{\alpha}$$
$$\rho S_g(t_2)-I_g(t_2)=(1-\rho)S_b(t_2)-I_b(t_2)$$
$$q(t_3;\emptyset)=\underline{\alpha}.$$

Note that the second condition has not changed because the prior belief does not play a role in determining the mixing probability of B consumers to keep the current belief constant.

Therefore, Theorem 1 can be generalized as follows.

Corollary 1. *Suppose that $P\in(\underline{u},\bar{u})$ and $\alpha\in(\frac{(1-\rho)(P-\underline{u})}{\rho(\bar{u}-P)+(1-\rho)(P-\underline{u})},\frac{\rho(P-\underline{u})}{(1-\rho)(\bar{u}-P)+\rho(P-\underline{u})})$. The equilibrium is generically unique: at $t=0$, consumers are sensitive to signals. Afterwards, there exist time cutoffs t_1, t_2, t_3 such that:*

1. *For all $t\in(0,t_1)$: when $\alpha\in(\frac{P-\underline{u}}{\bar{u}-\underline{u}},\frac{\rho(P-\underline{u})}{(1-\rho)(\bar{u}-P)+\rho(P-\underline{u})})$, consumers herd on adoption; when $\alpha\in(\frac{(1-\rho)(P-\underline{u})}{\rho(\bar{u}-P)+(1-\rho)(P-\underline{u})},\frac{P-\underline{u}}{\bar{u}-\underline{u}})$, consumers are sensitive to signals; t_1 is characterized by $q(t_1;\emptyset)=\frac{\rho(P-\underline{u})}{(1-\rho)(\bar{u}-P)+\rho(P-\underline{u})}$. At $t=t_1>0$, G consumers always adopt while B consumers adopt with an arbitrary probability. When $\alpha=\frac{P-\underline{u}}{\bar{u}-\underline{u}}$, $t_1=0$.*

2. For all $t \in (t_1, t_2)$: G consumers always adopt while B consumers adopt with probability $x(t) = \frac{\rho S_g(t) - (1-\rho)S_b(t) - (I_g(t) - I_b(t))}{\rho S_b(t) - (1-\rho)S_g(t)}$; t_2 is characterized by $\rho S_g(t_2) - I_g(t_2) = (1-\rho)S_b(t_2) - I_b(t_2)$. At $t = t_2$, G consumers always adopt while B consumers adopt with an arbitrary probability.

3. For all $t \in (t_2, t_3)$: consumers are sensitive to signals; t_3 is characterized by $q(t_3; \emptyset) = \frac{(1-\rho)(P-\underline{u})}{\rho(\bar{u}-P)+(1-\rho)(P-\underline{u})}$. At $t = t_3$, G consumers adopt with an arbitrary probability while B consumers never adopt.

4. For all $t > t_3$, consumers herd on non-adoption.

5 Learning Dynamics with Strategic Producer

As we have fully characterized the consumers' equilibrium behavior under viral social learning, a natural and important question is how the availability of a viral campaign affects the supply side. We answer this question from two distinct but relate perspectives. In Section 5.1, we examine whether viral social learning will indeed emerge as the result of a strategic choice. By allowing the producer to select from viral and advertising campaigns, we show that it will always carry out a viral one with positive length at optimum. In Section 5.2, we focus on how viral social learning can impact quality. We show that a viral campaign works in opposite directions, with high and low prior beliefs respectively, in affecting the producer's incentive for quality improvement.

5.1 Viral versus Advertising Campaign

Nature draws the producer's type $\omega \in \{g, b\}$ according to distribution $(\alpha, 1-\alpha)$ where $\alpha \in (1-\rho, \rho)$. The producer knows ω privately, and α is common knowledge. The producer then chooses a time $t^* \in \mathbb{R}^+$. t^* is a time cutoff before which the producer runs a viral campaign and after which it runs an advertising campaign, i.e. make all currently susceptible consumers aware of the product. Upon awareness at t^*, consumers choose once and for all whether to adopt the product. Without loss of generality, the product's price $P \in (\underline{u}, \bar{u})$ is fixed at $\frac{\bar{u}+\underline{u}}{2}$ over time.

Note that the only possible producer equilibrium is pooling, i.e. the two types choose the same t^* (or the same support of t^*, in the case of a mixed-strategy equilibrium). Otherwise, a producer with bad quality always has incentives to mimic one with good quality.

We assume a sufficiently small Δ to focus on generic cases. The following result characterizes the producer's optimal choice of campaign. As there may be a plethora of Bayesian Nash equilibria depending on how off-path beliefs are defined, we focus on equilibria that maximizes the producer's (*ex ante*) expected profit, calculated by $\pi(t^*) = \alpha\pi_g(t^*) + (1-\alpha)\pi_b(t^*)$, where $\pi_\omega(t^*)$ is the profit of producer with quality $\omega \in \{g, b\}$.

Proposition 8. *Suppose that consumers adopt when indifferent. An equilibrium t^* exists and is always positive. When the producer has a lexicographic preference, i.e. it always prefers a higher profit while prefers earning the same profit earlier, t^* is generically unique*[8].

Proof. Given the producer's choice of t^*, consumers start with prior α on $\omega = g$, and update at time t^* facing an advertising campaign using the Bayes' rule, i.e. their ratio of interim beliefs is equal to $\frac{\alpha S_g(t)}{(1-\alpha)S_b(t)}$.

Maintaining our previous notations, let t_1, t_2 and t_3 denote the equilibrium time cutoffs in Theorem 1. We know that

$$(\frac{\alpha S_g(t)}{(1-\alpha)S_b(t)})' = \frac{(-I_g(t) + I_b(t))\alpha S_g(t)}{(1-\alpha)S_b(t)} < 0$$

for all t such that $S_g(t), S_b(t) > 0$. Therefore, there is a unique $t^{**} \in (0, t_3]$ such that $\frac{\alpha S_g(t)}{(1-\alpha)S_b(t)} \geq \frac{1-\rho}{\rho}$ if and only if $t \leq t^{**}$. When $\alpha \in (\frac{1}{2}, \rho)$, $\frac{\alpha S_g(t)}{(1-\alpha)S_b(t)} = \frac{\rho I_b(t)}{(1-\rho)I_g(t)} > 1$ at $t = t_1$; when $\alpha \in (1-\rho, \frac{1}{2})$, in the proof of Theorem 1 we have shown that when Δ is sufficiently small, $\frac{S_g(t_1)}{S_b(t_1)}$ is still close to 1. Hence, we can conclude that $t^{**} > t_1$.

It is clear that $\pi(t)$ is strictly increasing on $[t_1, \min\{t^{**}, t_2\}]$ and a constant on $(\max\{t^{**}, t_2\}, \infty)$. When $t^{**} \geq t_2$, $\pi(t)$ is the same on $[t_2, t^{**}]$, and $\pi(t) < \pi(t^{**})$ for all $t > t^{**}$. Thus $\arg\max_t \pi(t) = [t_2, t^{**}]$. When $t^{**} < t_2$, $\arg\max_t \pi(t) = \{t^{**}\}$ ($[t_3, \infty)$, $\{t^{**}\} \cup [t_3, \infty)$) if $\pi(t^{**}) > (<, =)\pi(t_3)$.

In particular, suppose that $\alpha \in [\frac{1}{2}, \rho)$. At t^{**}, we have $\frac{\alpha S_g(t^{**})}{(1-\alpha)S_b(t^{**})} = \frac{(1-\rho)}{\rho}$, which implies that $S_g(t^{**}) \leq \frac{1-\rho}{\rho} S_b(t^{**})$. Hence $\rho S_g(t^{**}) - (1-\rho)S_b(t^{**}) \leq 0 < I_g(t^{**}) -$

[8] Alternatively, one can think of the producer having a small discount rate. t^* will still be unique for almost every parameter configuration, but multiple equilibria – more than one pure strategy equilibrium, or coexistence of pure and mixed strategy equilibria – might occur in very special cases. For instance, when $t^{**} \geq t_2$, a producer might be indifferent between stopping the viral campaign shortly before t_2 and exactly at t_2. The reason is that $a_B(t)$, the probability that a consumer with a bad signal adopting the product at $t \in [t_1, t_2]$, which marks the marginal benefit of continuing the viral campaign, is continuous in t and equals 0 at $t = t_2$. Of course, when the discount rate goes to zero, the possible multiple equilibria converge to a unique one after all.

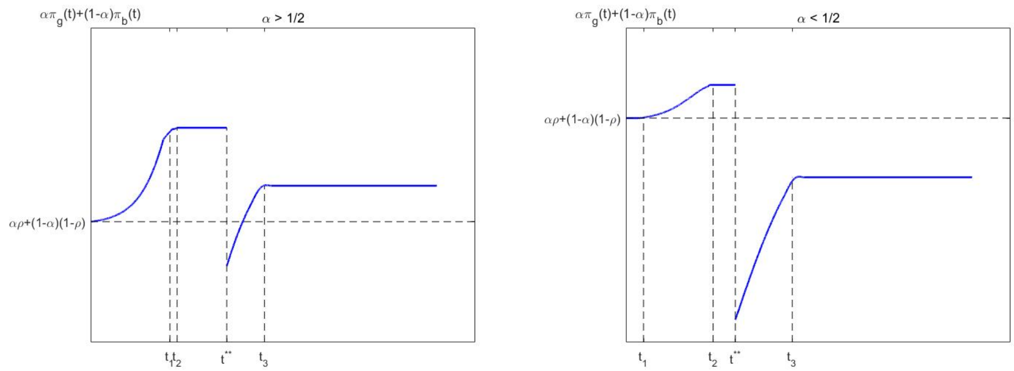
$I_b(t^{**})$, from which we can conclude that $t^{**} > t_2$ by Theorem 1. In other words, the equilibrium t^* takes an arbitrary value on $[t_2, t^{**}]$.

When the producer has a lexicographic preference, the equilibrium t^* is equal to t_2 when $t^{**} \geq t_2$, and is equal to either t^{**} or t_3 when $t^{**} < t_2$. When $\alpha \in [\frac{1}{2}, \rho)$, t^* is equal to t_2. □

While consumers are always sensitive to signals in an instant advertising campaign, a fraction of B consumers in a viral campaign adopt with positive probability. Therefore running a viral campaign for at least some time is beneficial for both types of producers *per se*. However, a strategic producer will not always continue the viral campaign until adoption ceases.

Consider a viral campaign that stops at time t and a consumer who becomes aware of the product via a subsequent advertising campaign. She updates her belief about quality based on the fact that she has not heard about the product previously. Note that this updating is different from that of becoming aware via the viral campaign at approximately t, which is conditional on hearing about the product at the very instant in the viral social learning process. As the measure of susceptible consumers decreases faster for a good product than a bad one, the longer the producer runs the viral campaign, the lower consumers' interim belief is under the advertising campaign. Hence, unless the viral campaign is so profitable that it is worthwhile to give up all other possible revenue, the producer will choose a stopping time at which consumers are still sensitive to signals afterwards.

Figure 9 below illustrates a typical case for both high and low initial beliefs. The producer's profit consists of two parts: profit from the viral campaign, and profit from the advertising campaign. The first part is increasing in t until t_3 regardless of quality. The second part is equal to $\rho S_g(t) + (1-\rho)S_b(t)$ for a good product and $(1-\rho)S_g(t) + \rho S_b(t)$ for a bad product, if $t \leq t^{**}$ and 0 otherwise; this is because the consumers who become exposed after t^{**} from an advertising campaign will form an interim belief of $< 1-\rho$, making the advertising campaign obsolete. Hence the graph drops suddenly at t^{**} because the producer loses the profit from advertising campaign if the viral campaign lasts for longer than $t^*\ast$. As a result, the producer's expected profit is maximized when the viral campaign stops between t_2 and t^{**}.



(a) $\alpha \in (\frac{1}{2}, \rho)$ (b) $\alpha \in (1-\rho, \frac{1}{2})$

Figure 9: Producer's expected profit, as a function of viral campaign's stopping time

5.2 Quality Improvement

Now we further allow the producer to choose its quality at the beginning of the game, on top of selecting the length of a viral campaign. In particular, the producer starts with bad quality and has a private cost $c \in [0, \bar{c}]$, with which it can upgrade the product to be of good quality. The prior distribution of c, denoted $F(c)$, is continuous and strictly increasing in c and is common knowledge.

A *market equilibrium* in this environment is defined as follows: (1) the producer chooses good quality if and only if $c \leq c^*$ for some $c^* \in [0, \bar{c}]$; (2) the consumers' prior belief upon being aware is $\alpha^* = F(c^*)$; (3) the producer, regardless of quality, runs a viral campaign whose length is determined by Proposition 8.

Suppose that no viral campaign takes place, which means that the product is advertised at $t = 0$. We assume that the market equilibrium in this benchmark case induces a prior belief on quality that lies on $(1-\rho, \rho)$. That is, the cost distribution F and the difference in producer's profit by quality when all consumers are sensitive to signals, $2\rho - 1$, are such that $F(2\rho - 1) \in (1-\rho, \rho)$. Let $\hat{\alpha} = F(2\rho - 1)$.

Proposition 9. *Suppose that $\hat{\alpha} \in [\frac{1}{2}, \rho)$. A market equilibrium always exists, and in every market equilibrium, $\alpha^* \leq \hat{\alpha}$. The inequality is strict if $\hat{\alpha} > \frac{1}{2}$.*

Proof. Consider a prior belief $\alpha \in [\frac{1}{2}, \rho)$. We have shown in Proposition 8 that $t^* \leq t^{**}$. The difference between the profits of a good and a bad producer is then

$I_g(t^*) + \rho S_g(t^*) - (I_b(t^*) + (1-\rho)S_b(t^*))$. Consider

$$\begin{aligned} &I_g'(t) + \rho S_g'(t) - (I_b'(t) + (1-\rho)S_b'(t)) \\ =&(\rho + (1-\rho)z(t) - \rho)I_g(t)S_g(t) - (1-\rho+\rho z(t) - (1-\rho))I_b(t)S_b(t) \\ =&z(t)((1-\rho)I_g(t)S_g(t) - \rho I_b(t)S_b(t)), \end{aligned}$$

where $z(t) = \frac{p_g(t)-\rho}{1-\rho} = \frac{p_b(t)-(1-\rho)}{\rho}$ is a consumer's probability of adoption given a bad signal.

We already know that $\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)}$ starts from $\frac{I_g(0)S_g(0)}{I_b(0)S_b(0)} = \frac{\rho}{1-\rho}$ for every α on the interval $(1-\rho, \rho)$. When $\alpha \in (\frac{1}{2}, \rho)$, $\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)}$ decreases on $[0, t_1)$ and stays constant on $[t_1, t_2]$. In addition, $z(t) = 0$ for $t \geq t_2$. Hence, the above difference between profits is strictly decreasing on $[0, \min\{t_2, t^{**}\}]$ and constant on $[t_2, t^{**}]$. Similarly, when $\alpha = \frac{1}{2}$, the difference is constant on $[0, t^{**}]$.

Now consider the difference as a function of α, denoted $G(\alpha)$. In every market equilibrium, α^* must be characterized by $G(\alpha^*) = F^{-1}(\alpha^*)$. We know that $G(\alpha)$ is continuous on $[\frac{1}{2}, \rho)$, $G(\frac{1}{2}) = 2\rho - 1$ and that $G(\alpha) < 2\rho - 1$ on $(\frac{1}{2}, \rho)$. On the other hand, $F^{-1}(\alpha)$ is strictly increasing in α, and $\hat{\alpha} \geq \frac{1}{2}$ implies that $F^{-1}(\frac{1}{2}) \leq 2\rho - 1 = F^{-1}(\hat{\alpha})$ and that $G(\hat{\alpha}) \leq 2\rho - 1$ (both inequalities become strict when $\hat{\alpha} > \frac{1}{2}$). Therefore, $G(\alpha^*) = F^{-1}(\alpha^*)$ is never satisfied for any $\alpha^* > \hat{\alpha}$, is satisfied for $\alpha^* = \frac{1}{2}$ when $\hat{\alpha} = \frac{1}{2}$, and is satisfied for some $\alpha^* \in (\frac{1}{2}, \hat{\alpha})$ when $\hat{\alpha} \in (\frac{1}{2}, \rho)$. This completes the proof. Figure 10 illustrates our argument.

□

Proposition 10. *Suppose that $\hat{\alpha} \in (1-\rho, \frac{1}{2})$. In every market equilibrium where $t^* = t^{**}$, $\alpha^* > \hat{\alpha}$.*

Proof. The proof follows that of Proposition 9. When the prior belief $\alpha \in (1-\rho, \frac{1}{2})$, $\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)}$ increases on $[0, t_1)$ and stays constant on $[t_1, t_2]$. In addition, $z(t) = 0$ for $t \leq t_1$ and $t \geq t_2$. Hence, the difference between profits as a function of the length of viral campaign, is constant on $[0, t_1] \cup [t_2, \max\{t_2, t^{**}\}]$ and strictly increasing on $(t_1, \min\{t_2, t^{**}\})$.

Again letting $G(\alpha)$ denote the difference in profits between a good and a bad product, given that the length of viral campaign is t^{**}. We have $G(\frac{1}{2}) = 2\rho - 1$ and that $G(\alpha) > 2\rho - 1$ on $(1-\rho, \frac{1}{2})$. On the other hand, $F^{-1}(\alpha)$ is strictly increasing in α, and $\hat{\alpha} < \frac{1}{2}$ implies that $F^{-1}(\frac{1}{2}) > 2\rho - 1 = F^{-1}(\hat{\alpha})$ and that $G(\hat{\alpha}) > 2\rho - 1$. As we have shown in Proposition 9 that G and F^{-1} cannot intersect when $\alpha > \frac{1}{2}$, $G(\alpha^*) = F^{-1}(\alpha^*)$ is and can only be satisfied at some $\alpha^* \in (\hat{\alpha}, \frac{1}{2})$. This means that if a market equilibrium is characterized by $t^* = t^{**}$, then we have $\alpha^* > \hat{\alpha}$. Figure 11 illustrates our argument.

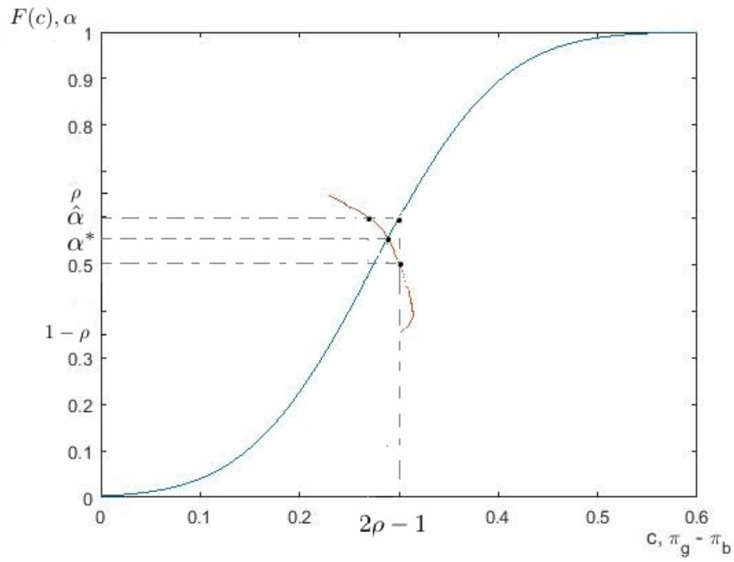


Figure 10: Illustration of equilibrium α^* when $\hat{\alpha} \in (\frac{1}{2}, \rho)$.

□

Figure 12 below illustrates the difference in profits between a good product and a bad one under high and low prior beliefs. When $\alpha \in (\frac{1}{2}, \rho)$, consumers start with herding on adoption in a viral campaign. As private signals no longer matter in this phase, the difference in profits shrinks over time on $[0, t_1]$. It continues to drop on $[t_1, t_2]$ but not as steeply due to mixed strategies of B consumers, and becomes constant after t_2 when consumers become sensitive to signals. As a result, a bad producer is less willing to improve its quality when the viral campaign lasts longer, which in turn leads to a smaller α in the market equilibrium.

When $\alpha \in (1-\rho, \frac{1}{2})$, the difference in profits does not change when consumers are sensitive to signals on $[0, t_1]$ and after t_2. However, since the measure of adopting consumers grows nearly exponentially faster for a good product than a bad one on $[0, t_1]$, a good product has a much larger coverage on $[t_1, t_2]$, which means that more B consumers in this phase adopt the product due to their mixed strategies. Therefore the difference in profits enlarges and a bad producer is more willing to improve its quality. Consequently, a market equilibrium features a larger α.

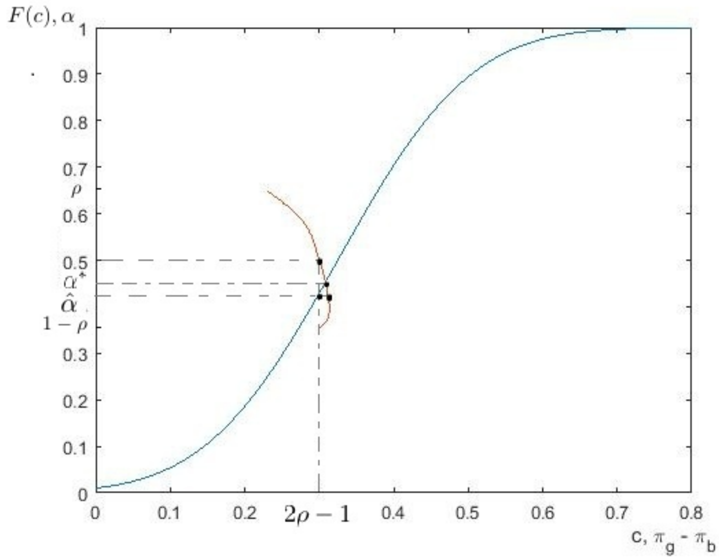
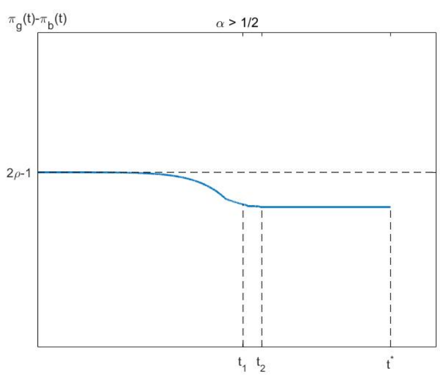
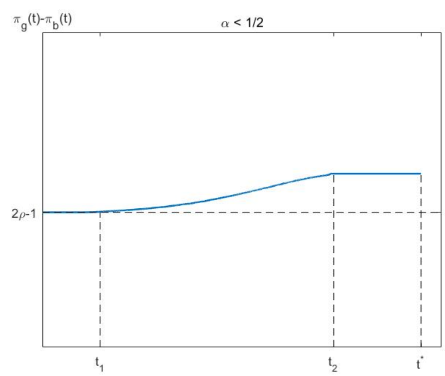


Figure 11: Illustration of equilibrium α^* when $\hat{\alpha} \in (1-\rho, \frac{1}{2})$.



(a) $\alpha \in (\frac{1}{2}, \rho)$



(b) $\alpha \in (1-\rho, \frac{1}{2})$

Figure 12: Profit difference between good and bad product, as a function of viral campaign's stopping time

6 Conclusion

In this paper, we have built a continuous-time social learning model to study the epidemiological dynamics among consumers when information about a product is spread via viral marketing. Our results depict a unique equilibrium life cycle of the product, in which various types of consumer behavior – herding on adoption, being sensitive to signals, and mix strategies – occur and switch from one to another when beliefs evolve over time. We also take strategic producers into account and illustrate how their decisions may affect the length of viral social learning, product quality, and welfare.

We hope that our work can lay a foundation for the study of richer strategic behavior in the social learning process. Possible future directions that extend the current model include active acquisition of information by consumers, different topologies of consumers' social network, time-varying quality improvement by producers, etc.

A Appendix: omitted proofs

A.1 Proof of Prop 1

Suppose first that $\alpha = p(0) > \rho$. Consumers exposed to the product at launch herd on adoption (Lemma 1(i)); so, $I_g(0+) = I_b(0+) = \Delta$ and $S_g(0+) = S_b(0+) = 1 - \Delta$, where we use shorthand $h(x+)$ to refer to a function's right-limit, i.e., $h(x+) = \lim_{\epsilon \to 0} h(x+\epsilon)$. By equation (3), consumers' interim belief right after launch $p(0+)$ is determined by $\frac{p(0+)}{1-p(0+)} = \frac{\alpha}{1-\alpha} \frac{S_g(0+) I_g(0+)}{S_b(0+) I_b(0+)}$; so, $p(0+) = \alpha$ and consumers who encounter the product right after launch also herd on adoption. Note that, so long as consumers have herded on adoption up to time t, $I_g(t) = I_b(t)$ and $S_g(t) = S_b(t)$, implying that interim belief $p(t) = \alpha$. But then consumers must strictly prefer to continue herding on adoption. We conclude that consumers must herd on adoption at all times $t > 0$ and that doing so constitutes an equilibrium. This completes the proof of part (i).

Suppose next that $\alpha = p(0) < 1 - \rho$. Consumers exposed to the product at launch herd on non-adoption (Lemma 1(ii)); so, $I_g(0+) = I_b(0+) = 0$ and $S_g(0+) = S_b(0+) = 1 - \Delta$ in any equilibrium. With no one infected, no one is subsequently exposed to the product; so, $I_g(t) = I_b(t) = 0$ and $S_g(t) = S_b(t) = 1 - \Delta$ for all $t > 0$. Consumers' off-equilibrium beliefs (should they encounter someone who has previously adopted) are indeterminate, and many equilibria exist with different off-equilibrium beliefs. However, all such equilibria are outcome equivalent and exhibit zero adoption. $\square$

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