Dynamic Adverse Selection and Belief Update in Credit Markets*

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Abstract

We develop a dynamic model of debt contracts with adverse selection and belief updates. In the model, entrepreneurs borrow investment goods from lenders to run businesses whose return depends on entrepreneurial productivity and common productivity. The entrepreneurial productivity is the entrepreneur's private information, and the lender constructs beliefs about the entrepreneur's productivity based on the entrepreneur's business operation history, common productivity history, and terms of the contract. The model provides insights on the dynamic and cross-sectional relationship between firm ages and credit risks, cyclical asymmetry of the business cycle, slow recovery after a crisis, and the constructive economic downturn.

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1 Introduction

Financial markets exhibit asymmetric information in that one of the two parties in financial affairs has more information than the other, making a more informed decision and information processing by the less informed party to overcome the informational disadvantages. In debt contracts, for instance, lenders usually know less than borrowers about payoff-relevant borrower attributes. In response to asymmetric information problems, the lender, in practice, estimates the borrower's solvency by looking at not only the borrower's history but also the aggregate states in the past because the borrower's financial state depends on aggregate economic conditions and borrowers' attributes. However, the dynamic construction of lenders' belief about borrowers' credit risks considering borrowers' actions and economic states has received relatively less attention to date.

In this paper, we develop a dynamic equilibrium model of debt contracts with adverse selection and belief updates. We investigate how the information on aggregate economic conditions in the past is used for constructing the lender's belief about the credit risk of borrowers with different histories. We study the dynamic evolution of the borrowing cost as a borrower getting older and the cross-sectional relationship between the borrower's age and the borrowing cost in a given period. We also analyze the effects of positive and negative aggregate shocks on macroeconomic outcomes in the environment with asymmetrically informed borrowers and lenders and with dynamic belief update of lenders.

In the model economy, an entrepreneur can run his/her business using the lender's investment good as inputs in each period. The return from the business operation is a product of common productivity and entrepreneurial productivity. The common productivity is a random variable independently and identically distributed across time, and its realized value is public information. Entrepreneurs are heterogeneous with respect to their entrepreneurial productivity, which is the entrepreneur's private information. To run the business, an entrepreneur must borrow the investment good from the lender, subject to limited commitment. Unsecured credit is feasible in equilibrium due to the threat of punishment toward the defaulters. In particular, if an entrepreneur defaults, then he/she will be excluded from the future credit forever and hence leaves the economy. Bankrupt entrepreneurs are replaced with new entrepreneurs whose productivity is randomly drawn from the given distribution.

The key novel ingredient in our model is that lenders can observe entrepreneurs' business operation histories, i.e., whether an entrepreneur operated his/her business in a specific period in the past. The lender employs the entrepreneur's business operation history in conjunction with the information on the realized common productivity in the past and the terms of the contract to construct beliefs about the entrepreneur's productivity, which is the hidden type. Then, based on the constructed beliefs, the lender decides whether to lend the investment good to the entrepreneur.

In equilibrium, where all entrepreneurs run their business, the only possible contract for a group of entrepreneurs with the same operation history is pooling, and entrepreneurs default only if they have no choice but to default. This implies that given a certain level of the common productivity and a group of entrepreneurs of the same age, there exists a threshold value of entrepreneurial productivity such that only entrepreneurs with productivity lower than the threshold value default and the other entrepreneurs honor the debt contract keeping the access to the credit market in the next period. Therefore, in the next period, lenders can form their beliefs that the productivity of survived entrepreneurs is distributed above the threshold value in contrast to new entrepreneurs who were born in the current period.

Because more productive entrepreneurs tend to stay in the economy for a longer period and less productive entrepreneurs are more likely to leave the economy early, the lender's belief about the entrepreneur's productivity weakly improves over time in terms of first-order-stochastic dominance. As a result, the borrowing cost weakly decreases as the borrower getting older. Furthermore, in the model economy, old entrepreneurs tend to have lower credit risk and borrowing costs than young entrepreneurs on average in a given period, although the reverse is also possible under some conditions.

Our model also provides macroeconomic implications on the effects of aggregate shocks.

First, the negative common productivity shock can change the distribution of entrepreneurial productivity while the positive shock does not. As a result, the arrival of a recession is prompt, and the recovery from a recession appears protracted in the model economy due to the process of replacing low productive entrepreneurs with new ones over time. In particular, a big negative shock on the common productivity makes most of (or all) existing entrepreneurs default, and it can take a long time for the aggregate production to return to the pre-shock level, providing a narrative for the sluggish recovery of production from a crisis (e.g., Ikeda and Kurozumi (2019)). Second, although the negative common productivity shock reduces the current output, the model shows that under some conditions, a mild negative shock on the common productivity can be constructive for the economy by raising the aggregate production in the long term.

We are certainly not the first to study adverse selection problems in credit markets. Jaffee and Russell (1976) and Stiglitz and Weiss (1981) show that credit rationing arises as a means of market response to adverse selection.[1] Bester (1985), Besanko and Thakor (1987b), and Milde and Riley (1988) show that no credit rationing occurs in equilibrium if another instrument, such as collateral and loan size, is used as credit instrument in addition to interest rates to screen borrower's riskiness. Besanko and Thakor (1987a) extend the previous papers and study the effects of credit market structure on the role of collateral and credit allocation.

While these papers analyzed credit markets with asymmetrically informed borrowers and lenders, they studied one time transactions focusing on how adverse selection problems are related to the crediting rationing practices. In contrast, we study the dynamic evolution of lenders' beliefs and terms of debt contracts over time in response to the update of the information on the histories of borrower's actions and aggregate economic conditions in the past.[2] In particular, we focus on providing insights on the relationship between borrowers'

[1]Williamson (1986, 1987) also derives credit rationing as an equilibrium outcome using a costly state verification model. However, in his model, entrepreneurs are ex-ante homogeneous, and hence there is no adverse selection problem.

[2]While models of debt contracts with dynamic adverse selection are limited, there are several papers that

ages and borrowing costs, the economic justification for the cyclical asymmetry of aggregate outputs, and the effects of big and mild productivity shocks on the dynamics of aggregate productions using the model of debt contract with adverse selection.

Boot and Thakor (1994) studied the dynamics of loan interest rates over the course of borrower's life in a repeated game between a lender and a borrower with moral hazard. While the distinction between adverse selection and moral hazard in credit markets is often subtle, the ways of incorporating the two frictions into the model differ profoundly because asymmetric information problem occurs before the transaction occurs in adverse selection and moral hazard arises after the transaction occurs. Furthermore, we introduce aggregate shocks into the model to understand the way of interaction between aggregate shocks and lenders' belief construction, letting our model provide more macroeconomic implications.

Our paper is also related to the literature on unsecured debt contracts with limited commitment and default history. Kehoe and Levine (1993) and Azariadis and Kass (2013) study the condition under which the first-best allocation is obtained in an economy with limited commitment, and Kocherlakota (1996) shows that if individuals are not sufficiently patient, imperfect diversification is optimal. Gu et al. (2013) and Bethune et al. (2018) derive endogenous credit cycles in models of credit with limited commitment, and Sanches and Williamson (2010) study a set of frictions under which money and unsecured credit are both robust as a means of payment. While these previous studies show how unsecured credit is supported by the threat of off-equilibrium punishment and determinants of the credit limit, there is only potential default in those models, which is problematic given regularities in real-world default behaviors. In our model, by contrast, we derive default as an equilibrium outcome by incorporating adverse selection into debt contract models with limited commitment.

The rest of the paper is organized as follows. Section 2 presents the economic environment

study the multi-period adverse selection problems in other areas. For example, Kreps and Wilson (1982) consider a finite-period model to show that high type's precommitment to its action. Noldeke and van Damme (1990) and Swinkels (1999) extend the Spence (1973) signaling model into a multi-period environment. Kaya (2009) and Toxvaerd (2017) consider the infinite-period environment when the sender's type is persistent.

of the model, section 3 describes the bargaining game between borrowers and lenders. Section 4 characterizes the equilibrium, and section 5 presents a number of implications of our model. Section 6 concludes. The omitted proofs are relegated to the Appendix.

2 Model

Time is discrete and goes on forever. Each period t is divided into two subperiods, morning and afternoon. Morning is the planning period, and consumption takes place in the afternoon. There are two risk-neutral agents with a common discount factor $\beta \in (0, 1)$ across periods: A unit measure of entrepreneurs (E) and lenders (L). Specifically, the sets of entrepreneurs and lenders are given as $I_E = [0, 1]$ and $I_L = [-1, 0)$, respectively, in the real space. A lender lives indefinitely, but an entrepreneur may leave the economy being replaced by a new entrepreneur, which will be discussed later.

Each lender receives an indivisible endowment of one unit of an investment good in the morning. The investment good can be lent to an entrepreneur or invested in a saving technology that yields a certain return of r units of the consumption good in the afternoon if one unit is invested in the morning, and yields zero units in the afternoon otherwise. Entrepreneurs do not receive endowments in the morning. Instead, each entrepreneur can operate his/her business that produces a return of w units of consumption goods in the afternoon by investing one unit of investment good in the morning. The outcome of the project (or business) depends on the common productivity, A, and the entrepreneurial productivity θ, as $w = A\theta$.

The common productivity, A, is independently and identically distributed across time according to the uniform distribution with the support of $[0, 1]$. Entrepreneurs can be different types with respect to their productivity θ, and the type (productivity) θ is drawn randomly from the uniform distribution with the support of $\Theta = [\underline{\theta}, \overline{\theta}]$ when an entrepreneur is born for the first time and is fixed until the entrepreneur leaves the economy. We assume that productivity θ is the entrepreneur's private information, so the only entrepreneur can observe the exact realized return of his/her business $w = A\theta$. However, we assume that the cumulative distribution functions (cdf) of A and θ of new entrepreneurs are public information. Throughout, $U_{[a,b]}$ refers to the cdf of uniform distribution with the support $[a, b]$. For notational simplicity, we denote by $U(A_t)$ instead of $U_{[0,1]}(A_t)$ for the cdf of the common productivity A.

An entrepreneur can leave the economy and be replaced by a new entrepreneur whose productivity is drawn from $U_{[\underline{\theta},\overline{\theta}]}(\theta)$. Therefore, the distribution of the productivity of entrepreneurs in period $t > 0$ can be different from $U_{[\underline{\theta},\overline{\theta}]}(\theta)$. Let $\mathcal{M}$ be the set of all feasible cumulative distribution functions on Θ, and let $\Omega_t \in \mathcal{M}$ denote the cdf for the productivity of entrepreneurs who live in period $t \geq 0$. Because an entrepreneur may not run his/her business in a given period, we denote by $\Omega_t^* \subseteq \Omega_t$ the cdf for the productivity of entrepreneurs who run business in period t. Then, the aggregate production in period t, denoted by Y_t, is given as $Y_t = A_t \int_{\underline{\theta}}^{\overline{\theta}} \theta d\Omega_t^* + rL_{h,t}$, where $L_{h,t}$ is the mass of lenders who invests endowments in the saving technology.

Business operation history One of the important assumptions in the model is that the business operation history of an entrepreneur - whether the entrepreneur ran business or not in a given period - is publicly observable. Specifically, consider an entrepreneur $i \in I_E$ who is born in period $s \geq 0$, and define o_t for all $t \in \{-1, 0, 1, 2, \ldots\}$ as follows: 1) $o_{i,t} = \varnothing$ if $t < s$, 2) $o_{i,t} = 1$ if the entrepreneur runs his/her project in period $t \geq s$, and 3) $o_{i,t} = 0$, otherwise. We define $\mathbf{o}_{i,t} \equiv \{o_{-i,1}, o_{i,0}, o_{i,1}, \ldots, o_{i,t}\}$ as sequence of $o_{i,t}$ upto $t \geq 0$. Let $\mathcal{O}_t$ denote the set of all feasible sequences $\mathbf{o}_{i,t}$ in a given period t and $\mathcal{O} = \bigcup_{t \in \mathbb{Z}_+} \mathcal{O}_t$.

Entrepreneurs could have different operation histories because entrepreneurs might be born in different periods and some entrepreneurs may not run their business in some periods. An operation history profile of all entrepreneurs in period t, denoted by $\mathbf{O}_t$, is a measurable function from I_E to $\mathcal{O}_t$, which gives $\mathbf{O}_t(i) = \mathbf{o}_{i,t}$ for all $i \in I_E$. Then, in the morning in period $t \geq 0$, $\mathbf{O}_{t-1}$ is public information. If there is no risk of confusion, we abuse the notation such that $\mathbf{o}_{i,t} \in \mathbf{O}_t$ if $\mathbf{O}_t(i) = \mathbf{o}_{i,t}$ for all $i \in I_E$.

We use $\mathbf{o}_{t-1}$, dropping the index $i \in I_E$, to state a particular operation history, and we call a group of entrepreneurs, who have the operation history $\mathbf{o}_{t-1}$, the "$\mathbf{o}_{t-1}$-group" in the following analysis. Note that entrepreneurs' types are two-dimensional: the productivity θ which is unobservable and 2) the operation history $\mathbf{o}_{t-1}$ which is observable. Thus, $(\theta, \mathbf{o}_{t-1}) \in \Theta \times \mathcal{O}_t$ characterize the entrepreneur's type in period $t \geq 0$.

In reality, most of the firms continuously run their business until they leave the economy instead of running businesses discontinuously. Therefore, when we characterize equilibrium, we focus on the case in which all alive entrepreneurs choose to operate their business until they leave the economy. This implies that what we need to trace the whole operation history of an entrepreneur in equilibrium is the establishment date of the entrepreneur, which can be checked by looking at the issue date of a business license of a firm in reality.

Common productivity history In reality, most countries have an online portal system that provides time-series data of gross domestic production (GDP), although the portal may not provide time-series data of common productivity. Suppose that the aggregate production Y, which represents the GDP of the model economy, is observable. Then, by forming a rational expectation about the distribution Ω_t^* about the productivity of entrepreneurs who ran their business in period $t \geq 0$ and the mass of lenders $L_{h,t}$ who invest endowments in the saving technology, agents can correctly infer the common productivity as $A_t = \frac{Y_\tau - rL_{h,t}}{\int_{\underline{\theta}}^{\overline{\theta}} \theta d\Omega_t^*}$. Furthermore, any entrepreneurs who run projects in period t learn the common productivity A_t from the realized return $w_t = A_t\theta$ because they know their type θ. For these rationals, we assume that the history of the past common productivities is public information to make the analysis straightforward. Specifically, in the morning in period $t \geq 0$, all agents can observe $\mathbf{A}_{t-1} \equiv \{A_{-1}, A_0, A_1, \ldots, A_{t-1}\}$, where $A_{-1} = \varnothing$. Let $\mathbb{A}_t$ denote the set of all feasible sequences of $\mathbf{A}_t$ for all $t \geq 0$ and $\mathbb{A} \equiv \bigcup_{t \in \mathbb{Z}_+} \mathbb{A}_t$.

Parameter assumption Before describing the structure of the model economy further, we impose the following assumption on parameters throughout the paper.

Assumption 1 $\beta > \frac{b(\underline{\theta})-\sqrt{b(\underline{\theta})^2-4b(\underline{\theta})r}}{\underline{\theta}} > 0$ *where* $b(\theta') = \frac{\bar{\theta}-\theta'}{\int_{\theta'}^{\bar{\theta}} \frac{1}{\theta} d\theta}$ *for all* $\theta' \in [\underline{\theta}, \overline{\theta})$ *and* $b(\bar{\theta}) = \lim_{\theta' \to \bar{\theta}} b(\theta') = \bar{\theta}$.

Assumption 1 is a technical condition that is necessary for the existence of an equilibrium in which all entrepreneurs operate their business. This assumption serves to streamline the analysis by restricting attention to relevant cases. Assumption 1 requires that agents are sufficiently patient. Because $\beta < 1$, it must be verified that the set $\{\underline{\theta}, \overline{\theta}, r, \beta\}$ that satisfies the assumption 1 is not empty in advance before making further analysis. The next lemma provides a sufficient condition for the set $\{\underline{\theta}, \overline{\theta}, r, \beta\}$ satisfying the assumption 1 is not empty.

Lemma 1 *If* $\underline{\theta} \geq 4r$, *then there exists* $\beta \in \left(\frac{b(\underline{\theta})-\sqrt{b(\underline{\theta})^2-4b(\underline{\theta})r}}{\underline{\theta}}, 1\right)$.

Bilateral meetings in the morning Entrepreneurs need to borrow investment goods from lenders to run their business, and there is a decentralized market in which there are bilateral meetings between entrepreneurs and lenders in the morning. Entrepreneurs and lenders are randomly matched, and in each meeting, the entrepreneur offers a credit contract that the lender accepts or rejects.

A specific form of a credit contract in a bilateral meeting is as follows. Under a contract, a lender transfers one unit of investment good to an entrepreneur in the morning. Then, after observing the return on the business operation $w \in [0, \overline{\theta}]$ in the afternoon, the entrepreneur emits a signal $w^s \in [0, \overline{\theta}]$ to the lender and pay $R(w^s)$ units of consumption good, where $R(\cdot)$ is a function on $[0, \overline{\theta}]$. Note that the lender cannot observe the realized return on the project because the productivity θ is private information of the entrepreneur, and hence the repayment depends on the entrepreneur's report w^s.

We say that the borrower defaults on loans if he/she does not make payment $R(w^s)$ after reporting the signal w^s to the lender or does not make any payments without reporting the signal, which is feasible because there is no external source of enforcement in the credit market. However, we assume that there is a device that records the default history of entrepreneurs, and an entrepreneur who defaults on loans is excluded from the future credit permanently. For example, an entrepreneur can receive a discharge by filing bankruptcy, but the bankruptcy document is stored in the publicly available court archive, and no lenders would provide loans to that entrepreneur in the future. Because an entrepreneur cannot run projects without borrowing the investment good from lenders, bankrupt entrepreneurs leave the economy and are replaced with new entrepreneurs whose productivities are drawn from $U_{[\underline{\theta},\overline{\theta}]}(\theta)$.

The important assumption in the model economy is that the information about the terms of contracts and the payment amount that each entrepreneur has made in the past are not publicly observable. This implies that if an entrepreneur decides not to default, he/she will always choose w^s so as to minimize the payment to the lender, which implies that the payment is constant, denoted by $x = \min_{w^s \in [0,\overline{\theta}]} R(w^s)$. Thus, the payment x fully describes terms of a contract because the loan size is fixed, and we denote a credit contract by x in the following analysis.

Potentially, the probability of providing loans can be a part of the debt contract. However, we assume that lenders also have limited commitment in terms of the contract. Now, suppose that a lender accepts a debt contract that specifies the repayment x and the probability of loan provision α. The lender accepts this contract because he/she can achieve a trade surplus by receiving the repayment from the entrepreneur. Then, in the case that the lender and entrepreneur should not make the debt contract that occurs with probability $1 - \alpha$, both parties have incentives to clinch the debt contract because it is optimal. Thus, loan provision probability is ineffective and cannot be an instrument of debt contracts, and hence, the repayment x is the only instrument of debt contracts similar to Stiglitz and Weiss (1981).

Although ruling out the loan provision probability from the terms of the contract makes the analysis straightforward without unnecessary distraction, it is not critical for obtaining the main results. Even if we explicitly consider the loan provision probability in terms of contracts, we can still obtain the same results by constructing lenders' out-of-equilibrium beliefs appropriately as standard in signaling literature.

3 Bargaining game

In this section, we describe the bargaining game between the entrepreneur and the lender in a bilateral meeting in the morning. To define the payoffs and strategies in the bargaining game, it is useful to note that the entrepreneur's value at the beginning of the morning is a function of the productivity θ, operation history $\mathbf{o}_{t-1}$, and the history of common productivity $\mathbf{A}_t$. This is because θ affects the realized return on his/her business and the set of public information $\{\mathbf{o}_{t-1}, \mathbf{A}_{t-1}\}$ is used for constructing lender's belief about productivity θ, which in turn affects the set of acceptable credit contracts. Let $V_t(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})$ denote the value function of an type $(\theta, \mathbf{o}_{t-1})$ entrepreneur in period $t \geq 0$, when the history of common productivity is given as $\mathbf{A}_{t-1}$.

The bargaining game between the entrepreneur and the lender in the bilateral meeting has the structure of a signaling game in which the entrepreneur who has a private information about his/her hidden type θ makes an offer to the lender. We let $x = \varnothing$ if the entrepreneur chooses not to offer a contract to the lender. A period $t \geq 0$ strategy for the entrepreneur specifies a contract $x_t \in X \equiv \mathbb{R}_+ \cup \varnothing$ as a function of $(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})$, and a default set $D_t \subset [0, 1]$ as a correspondence of $(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}, x_t)$ such that for all $A_t \in D_t$, the entrepreneur defaults on the loan contract x_t. A period $t \geq 0$ strategy for the lender is an acceptance rule that specifies a set $\mathcal{B}_t \subset \mathbb{R}_+$ of acceptable credit contracts. If there is no risk of confusion, we drop arguments for each decision rule from now on; we use x_t and D_t instead of $x_t(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})$ and $D_t(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}, x_t)$, respectively, for instance.

Note that an entrepreneur decides whether to default or not after observing the realized common productivity A_t, and hence return $w = A_t\theta$ from the business operation. An entrepreneur defaults on the credit contract x_t for two reasons. First, if a type $(\theta, \mathbf{o}_{t-1})$ entrepreneur made a contract x_t in the morning, then he/she has no choice but to default for all $A_t \in \left[0, \frac{x_t}{\theta}\right)$ because he/she does not have sufficient resources to make repayment. Second, when $A_t\theta \geq x$, the entrepreneur strategically defaults on the credit contract x_t if

$$x_t > \beta V_{t+1}(\theta, \mathbf{o}_t, \mathbf{A}_t), \tag{1}$$

where $\mathbf{o}_t = \mathbf{o}_{t-1} \cup \{1\}$ and $\mathbf{A}_t = \mathbf{A}_{t-1} \cup \{A_t\}$, and honors on the contract otherwise. Note, from (1), that the realization of A_t in period t affects $V_{t+1}(\theta, \mathbf{o}_t, \mathbf{A}_t)$ through the history of common productivity $\mathbf{A}_t$.

Payoffs Given the common productivity history $\mathbf{A}_{t-1}$ in period t, the payoff for the type $(\theta, \mathbf{o}_{t-1})$ entrepreneur from the strategy profile $(x_t, D_t, \mathcal{B}_t)$ is

$$\begin{aligned}
&v(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}, x_t, D_t, \mathcal{B}_t) \\
&= \mathbb{I}_{\mathcal{B}_t}(x_t) \left\{ \begin{array}{c} \int_{[0,1]} A_t\theta dU(A_t) + \\ \int_{[0,1]\setminus D} \left[-x_t + \beta V_{t+1}(\theta, \mathbf{o}_{t-1} \cup \{1\}, \mathbf{A}_{t-1} \cup \{A_t\})\right] dU(A_t) \end{array} \right\} \\
&+ (1 - \mathbb{I}_{\mathcal{B}_t}(x_t)) \int_{[0,1]} \beta V_{t+1}(\theta, \mathbf{o}_{t-1} \cup \{0\}, \mathbf{A}_t) dU(A_t),
\end{aligned} \tag{2}$$

where $\mathbb{I}_{\mathcal{B}_t}(x_t)$ is an indicator function that is one if $x_t \in \mathcal{B}_t$. If a contract x_t is accepted, then the entrepreneur receives one unit of investment good from the lender and runs his/her business which produces $\int_{[0,1]} A_t\theta dU_{[0,1]}(A_t)$ units of consumption goods in the afternoon in expectation. Then, the entrepreneur repays x_t units of goods to the lender for all $A_t \in [0, 1] \setminus D$ and proceeds to the next period with updated operation history as $\mathbf{o}_t = \mathbf{o}_{t-1} \cup \{1\}$. If the entrepreneur defaults, then he/she consumes all produced goods from the business operation and leaves the economy. On the other hand, if the lender rejects the contract x_t, the entrepreneur does not run business in period t and enters to the next period with $\mathbf{o}_t = \mathbf{o}_{t-1} \cup \{0\}$.

The lender's payoff function is

$$\left\{ \int_{[0,1] \setminus D_t(\theta, x_t, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})} x_t dU(A_t) \right\} \mathbb{I}_{\mathcal{B}_t}(x_t) + r \left(1 - \mathbb{I}_{\mathcal{B}_t}(x_t)\right), \tag{3}$$

where we explicitely specifies the default sets as a correspondence of $(\theta, x_t, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})$ to clarify that (3) is the lender's payoff when the lender is offered a contract x_t from the type $(\theta, \mathbf{o}_{t-1})$ entrepreneur when the common productivity history is $\mathbf{A}_{t-1}$.

Belief system Because θ is the entrepreneur's private information, the lender needs to form beliefs about the entrepreneur's productivity θ before making an acceptance decision on the proposed contract x_t. To specify the lender's belief system, it is useful to make the following two observations. First, entrepreneurs, in the model economy, can be grouped by their operation history and each group has diferent distribution for θ of the group's entrepreneurs. Second, the cdf for θ of entrepreneurs in each group depends on the realization of the common productivity in the past because the entrepreneur's default decision depends on the realized common productivity. Let $\widehat{\Gamma}_t(\mathbf{o}_{t-1}, \mathbf{A}_{t-1})$ denote the cdf of θ of entrepreneurs in the $\mathbf{o}_{t-1}$-group in the morning in period $t \geq 0$ when the common productivity history is $\mathbf{A}_{t-1}$.

Then, lenders can form a rational expectation about $\widehat{\Gamma}_t(\mathbf{o}_{t-1}, \mathbf{A}_{t-1})$ for all $\mathbf{o}_{t-1} \in \mathbf{O}_{t-1}$, which provides a useful information for belief construction because $\{\mathbf{O}_{t-1}, \mathbf{A}_{t-1}\}$ is public information in the morning of period t. As a result, the lender uses $(x, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})$ to construct the belief. Specifically, we write $\Phi : X \times \mathcal{O} \times \mathbb{A} \rightarrow \mathcal{M}$ for the lender's belief function, assigning a cdf for θ of the matched entrepreneur in a bilateral meeting upon observing $(x, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})$. Thus, $\Phi(\theta | x, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})$ is the lender's conditional belief that the distribution for θ of an entrepreneur, who has the operation history $\mathbf{o}_{t-1}$ and offers x, when the lender observes $(x, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) \in X \times \mathcal{O} \times \mathbb{A}$.

Optimal strategy Given the lender's acceptance rule $\mathcal{B}_t$ and the common productivity history $\mathbf{A}_{t-1}$, the type $(\theta, \mathbf{o}_{t-1})$ entrepreneur optimally chooses the strategy (x_t, D_t). Note that the entrepreneur can always choose not to offer a contract to the lender, i.e., $x_t = \varnothing$. Thus, the entrepreneur's problem in period t is

$$\underset{x_t \in X, D_t \in \mathbb{P}([0,1])}{Max} \left\{ v(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}, x_t, D_t, \mathcal{B}_t) \right\}, \tag{4}$$

where $\mathbb{P}([0,1])$ is a power set of $[0,1]$ and $D_t = \varnothing$ whenever $x_t = \varnothing$. Note that the $(\theta, \mathbf{o}_{t-1})$ entrepreneur, in principle, can offer a contract $x_t \in X \backslash \mathcal{B}_t$, which will be reject by the lender with certainty. However, this is exactly the same as not to make an offer, and we assume that the entrepreneur chooses not to offer a contract instead of offering a contract that will be rejected in the following analysis.

Next, given a belief system Φ, the operation history of the matched entrepreneur $\mathbf{o}_{t-1}$, and common productivity history $\mathbf{A}_{t-1}$, the set of acceptable contracts for a lender is

$$\mathcal{B}_t^*(\Phi, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) = \left\{ x_t \in \mathbb{R}_+ : \int_\theta \int_{[0,1] \backslash D_t} x_t dU(A_t) d\Phi(\theta | x_t, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) \geq r \right\}. \tag{5}$$

For a contract to be acceptable, the expected revenue from the entrepreneur's repayment should not be lower than the payoff from investing the investment good to the saving technology that yields r units of consumption goods in the afternoon with certainty. We assume that a lender accepts a contract that makes the lender indifferent between accepting or rejecting the contract.

4 Equilibrium

We adopt Perfect Bayesian Equilibrium (PBE) as our equilibrium concept to the bargaining game, which is formally stated in the following definition.

Definition 1 *An equilibrium of the bargaining game is a profile of strategies for the en-*

trepreneur and the lender, and belief system, $\langle \{x_t(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}), D_t(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}, x_t)\}, \mathcal{B}_t, \Phi\rangle_{t=0}^{\infty}$, *such that for all* $t \geq 0$, *1)* $\{x_t(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}), D_t(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}, x_t)\}$ *is a solution to (4) for all* $(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) \in \Theta \times \mathcal{O} \times \mathbb{A}$, *2)* $\mathcal{B}_t = \mathcal{B}_t^*(\Phi, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})$ *for all* $(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}) \in \mathcal{O} \times \mathbb{A}$, *and 3)* $\Phi : X \times \mathcal{O} \times \mathbb{A} \to [0, 1]$ *satisfy Bayes' law whenever it is applicable.*

Before characterizing equilibrium, we first show a property of the entrepreneur's optimal strategy for x_t in the next lemma, which provides a useful intermediate step for equilibrium characterization.

Lemma 2 *Take any* $(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}) \in \mathcal{O} \times \mathbb{A}$ *and* $\theta \in$*supp*$(\widehat{\Gamma}_t(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}))$. *If the type* $(\theta, \mathbf{o}_{t-1})$ *entrepreneur offers a contract* $x_t \in \mathcal{B}_t^*(\Phi, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})$ *in equilibrium, then it must be* $x_t = Min\left\{\mathcal{B}_t^*(\Phi, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})\right\}$.

Lemma 2 says that the type $(\theta, \mathbf{o}_{t-1})$ entrepreneur either does not make an offer, i.e., $x_t = \varnothing$, or offers the contract $x_t = Min\left\{\mathcal{B}_t^*(\Phi, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})\right\}$ that does not depend on θ. This implies that the lender cannot use terms of contract meaningfully to update the belief about θ of the matched entrepreneur.

The result of a pooling contract in proposition 2, however, only applies entrepreneurs who offer contracts. In particular, given $(\mathbf{o}_{t-1}, \mathbf{A}_{t-1})$, the entrepreneur can always choose not to make an offer, i.e., $x_t = \varnothing$, if he/she expects that the entrepreneur could have a much better deal in the next period by updating his/her operation history with $\{0\}$ instead of $\{1\}$. Depending on the way of constructing the lender's belief system Φ, multiple equilibria can exist. For example, in one equilibrium, some entrepreneurs do not make offers taking a break from their business in some period to obtain a better deal in the future, and in another equilibrium, all alive entrepreneurs offer contracts in bilateral meetings to raise funds for their business operation.

In reality, most entrepreneurs run their business continuously since the establishment of firms rather than stop running their business occasionally. Thus, in the following analysis, we concentrate on a case in which all alive entrepreneurs run their business every period until they leave the economy, which we call the "*full production equilibrium*". Note that in the full production equilibrium, all entrepreneurs make credits contract with lenders until they leave the economy. Thus, the necessary condition for the existence of full production equilibrium is $\mathcal{B}_t^*(\Phi, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) \neq \varnothing$ for all $(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}) \in \mathcal{O}_{t-1}^* \times \mathbb{A}_{t-1}$, where

$$\mathcal{O}_{t-1}^* = \left\{\mathbf{o}_{t-1} \in \mathcal{O}_{t-1} : o_s \neq 0 \text{ for all } s \leq t-1\right\},$$

for all $t \geq 0$. Here, $\mathcal{O}_{t-1}^*$ is the set of feasible $\mathbf{o}_{t-1}$ in full production equilibrium. Let $\mathcal{O}^* = \underset{t \in \mathbb{Z}_+}{\cup} \mathcal{O}_t^*$. Also, entrepreneurs must have an incentive of offering contracts to lenders, which puts discipline on the lender's belief Φ off the equilibrium path.

Even though focusing on full production equilibrium narrows down equilibria of the original game by disciplining the lender's beliefs in an effective way, it does not guarantee a unique equilibrium outcome in general. Because there is little discipline on the belief Φ for out of equilibrium offer x, the game in bilateral meeting admits a continuum of equilibria. Specifically, we show, in the appendix B, that there exists a subset $\chi \subset \mathbb{R}_+$ such that for any $x' \in \chi$, an equilibrium exists with $\{x_t(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}), D_t(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}, x_t)\} = \left\{x', \left[0, \frac{x'}{\theta}\right)\right\}$ for all $\theta \in \text{supp}\left(\widehat{\Gamma}_t(\mathbf{o}_{t-1}, \mathbf{A}_{t-1})\right)$ and $x' \in \mathcal{B}_t$. To focus on the main issues of the paper, we restrict our attention to the full production equilibrium with the lowest x for each $(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}) \in \mathcal{O} \times \mathbb{A}$, which we denote by the $\boldsymbol{e}^*$ equilibrium.

Proposition 1 *Full production equilibrium exists, and for any* $\mathbf{o}_{t-1} = \{\varnothing, \ldots o_s, \ldots o_{t-1}\} \in \mathcal{O}^*$*, where* $s \geq 0$ *is the birthdate of* $\mathbf{o}_{t-1}$*-group entrepreneurs, and any* $\mathbf{A}_{t-1} \in \mathbb{A}$*, the* $\boldsymbol{e}^*$ *equilibrium has the following properties:*

1. *there exists* $\hat{\theta}_t \in [\underline{\theta}, \overline{\theta}]$ *such that* $\widehat{\Gamma}_t(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}) = U_{[\hat{\theta}_t, \overline{\theta}]}$,
2. *for all* $\theta \in \textit{supp}\left(\widehat{\Gamma}_t(\mathbf{o}_{t-1}, \mathbf{A}_{t-1})\right) = [\hat{\theta}_t, \overline{\theta}]$*, the type* $(\theta, \mathbf{o}_{t-1})$ *entrepreneur offers the contract*

$$x^*(\hat{\theta}_t) \equiv \frac{b(\hat{\theta}_t) - \sqrt{b(\hat{\theta}_t)^2 - 4b(\hat{\theta}_t)r}}{2}, \tag{6}$$

and chooses the default set $D_t = \left[0, \frac{x^*(\hat{\theta}_t)}{\theta}\right)$,

3. *for* $\tau = s, \ldots t$, $\hat{\theta}_\tau = Min\ supp\Big(\widehat{\Gamma}_\tau(\mathbf{o}_{\tau-1}, \mathbf{A}_{\tau-1})\Big)$, *where* $\mathbf{o}_{\tau-1} = \{\varnothing, \ldots o_s, \ldots o_{\tau-1}\}$, *is given as*

$$\widehat{\theta}_s = \underline{\theta} \text{ and } \widehat{\theta}_\tau = Max\left\{\frac{x^*(\widehat{\theta}_{\tau-1})}{A_{\tau-1}}, \widehat{\theta}_{\tau-1}\right\}, \tag{7}$$

for $\tau = s+1, \ldots t$.

Proposition 1 shows the existence of full production equilibrium including the $\boldsymbol{e}^*$ equilibrium and describes the entrepreneur's strategy and the distribution of θ for the $\mathbf{o}_{t-1}$-group in the $\boldsymbol{e}^*$ equilibrium. Proposition 1 provides three implications.

First, in the e^* equilibrium, entrepreneurs do not default strategically and defaults only if they cannot honor the credit contract, i.e., $D_t = \left[0, \frac{x^*(\hat{\theta}_t)}{\theta}\right)$. The intuition for this result is as follows. In the full production equilibrium, the lender's belief system is constructed such that $\mathcal{B}^*_t(\Phi, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) \neq \varnothing$ for all $(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}) \in \mathcal{O}^* \times \mathbb{A}$. Thus, the entrepreneur can always choose to offer an acceptable contract and default on the contract, and the expected payoff from this strategy is $\frac{\theta}{2}$. This implies that $V_{t+1}(\theta, \mathbf{o}_t, \mathbf{A}_t) \geq \frac{\theta}{2}$. Then, by the definition of $x^*(\hat{\theta}_t)$ given in (6), $x^*(\hat{\theta}_t) < \frac{\beta\theta}{2}$ for all $\theta \in [\hat{\theta}_t, \overline{\theta}]$, detailed in the proof, which implies that $x^*(\hat{\theta}_t) < \beta V_{t+1}(\theta, \mathbf{o}_t, \mathbf{A}_t)$. As a result, the entrepreneur defaults only if he/she has no choice but to defeault and the default set is connected as $D_t = \left[0, \frac{x^*(\hat{\theta}_t)}{\theta}\right)$.

Second, the connected default set is a driving force for the first and third parts of proposition 1. To gather intuition, consider entrepreneurs whose θ was randomly drawn from $U_{[\underline{\theta}, \overline{\theta}]}$ when they were born in period $s \geq 0$ as an example. Letting $\hat{\theta}_s = \underline{\theta}$, the second part of proposition 1 says that for all $\theta \in [\hat{\theta}_s, \overline{\theta}]$, the θ entrepreneur offers $x^*(\hat{\theta}_s)$ to the lender and defaults only if $A_s < \frac{x^*(\hat{\theta}_s)}{\theta}$. Therefore, only entrepreneurs with $\theta \geq \frac{x^*(\hat{\theta}_s)}{A_s}$ can survive moving to the next period by making repayment and the set of θ for survived entrepreneurs in period $s+1$ is $\left[\hat{\theta}_{s+1}, \overline{\theta}\right]$, where $\hat{\theta}_{s+1} = Max\left\{\frac{x^*(\hat{\theta}_s)}{A_s}, \underline{\theta}\right\}$ as stated in the third part of proposition 1. Furthermore, because θ is uniformly distributed at period s, θ of survived entrepreneurs in period $s+1$ is also uniformly distributed as $\widehat{\Gamma}_{s+1}(\mathbf{o}_s, \mathbf{A}_s) = U_{\left[\hat{\theta}_{s+1}, \overline{\theta}\right]}$ which is stated in the first part of proposition 1. Note that the above argument holds as long as the initial distribution of θ is the uniform distribution over the connected set of θ, and hence, by induction, it applies for any entrepreneurs with any operation history in any period.

Third, the entrepreneur's strategy $(x_t, D_t) = \left(x^*(\hat{\theta}_t), \left[0, \frac{x^*(\hat{\theta}_t)}{\theta}\right)\right)$ maximizes the entrepreneur's trade surplus subject to the lender's participation constraint. Obviously, the entrepreneur's trade surplus decreases with the repayment x. However, the entrepreneur cannot decrease x unlimitedly because of the lender's participation constraint. Specifically, given that $\widehat{\Gamma}_t(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}) = U_{[\hat{\theta}_t, \overline{\theta}]}$ and all entrepreneurs in the $\mathbf{o}_{t-1}$-group offers the same contract x, the lender's expected payoff from acccepting the contract x is

$$\omega(x, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) = \int_\theta \int_{[0,1] \setminus D_t(\theta, x, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})} x dU(A_t) dU_{[\widehat{\theta}_t, \overline{\theta}]}(\theta). \tag{8}$$

where $D_t(\theta, x_t, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})$ is the optimal default strategy for the entrepreneur $\theta \in [\hat{\theta}_t, \overline{\theta}]$.

As one can see from (8), $\omega(x, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})$ decreases with the measure of $D_t(\theta, x_t, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})$. Thus, the entrepreneur can decrease x without changing the value of $\omega(x, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})$ by reducing the measure of the default set. By imposing the smallest default set, $D_t = \left[0, \frac{x}{\theta}\right)$, into (8) and using the definition of $b(\cdot)$ in assumption 1, we obtain

$$\omega(x, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) = x - \frac{x^2}{b(\hat{\theta}_t)}. \tag{9}$$

Then, the lowest x that satisfies $\omega(x, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) = r$ is $x^*(\hat{\theta}_t)$ defined in (6). Also, the second part of proposition 1 shows that given $x^*(\hat{\theta}_t)$, it is optimal for the entrepreneur to set the default set as $D_t = \left[0, \frac{x^*(\hat{\theta}_t)}{\theta}\right)$. Given that the lender correctly forms the belief about the entrepreneur's productivity, i.e., $\Phi(\theta | x^*(\hat{\theta}_t), \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) = \widehat{\Gamma}_t(\mathbf{o}_{t-1}, A_{t-1})$, in equilibrium, the lender accepts the contract $x^*(\hat{\theta}_t)$. Therefore, $x^*(\hat{\theta}_t)$ is the lowest repayment that the entrepreneur can offer to the lender, maximizing the entrepreneur's trade surplus.

5 Applications

In this section, we consider two applications of our model. In section 5.1, we assess the relationship between entrepreneurs' age and credit risk and in section 5.2, we study the effects of common productivity shocks on the dynamics of aggregate production over time. In the following analysis, whenever we say equilibrium, we mean the e^* equilibrium.

5.1 Entrepreneur age and credit risk

There have been extensive studies on the determinants of firms' default probabilities, and the firm age has been argued as one of the determinants of default probabilities. In this subsection, we use our model to study the relationship between the entrepreneur's ages and the credits risk dynamically and cross-sectionally.

Measuring the credit risk What is the credit risk that lenders face when they lend the investment good to entrepreneurs? In a bilateral meeting, the lender cannot directly observe the entrepreneur's productivity, and the lender must estimate the entrepreneur's credit risk based on the lender's belief Φ.

In the $\boldsymbol{e^*}$ equilibrium, productivity θ of the $\mathbf{o}_{t-1}$-group entrepreneurs is uniformly distributed over $[\widehat{\theta}_t, \overline{\theta}]$ as described in proposition 1. Because the lender's belief system follows the Bayes' rule on the equilibrium path, it must be $\Phi(\cdot|x_t, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) = U_{[\widehat{\theta}_t, \overline{\theta}]}$. Then, given that the entrepreneur does not default opportunistically, the lender perceives that the ax-ante default probability, denoted by λ_t, of the entrepreneur with $\mathbf{o}_{t-1}$ in period t is

$$\lambda(\hat{\theta}_t) = \int_{[\hat{\theta}_t, \overline{\theta}]} \frac{x^*(\hat{\theta}_t)}{\theta} dU_{[\hat{\theta}_t, \overline{\theta}]}. \tag{10}$$

Because $\widehat{\theta}_t$ is an equilibrium outcome that depends on $(\mathbf{o}_{t-1}, \mathbf{A}_{t-1})$ as one can see from proposition 1, λ_t depends on $(\mathbf{o}_{t-1}, \mathbf{A}_{t-1})$.

Lemma 3 *The default probability $\lambda(\hat{\theta}_t)$, defined by (10), decreases with $\widehat{\theta}_t$.*

Lemma 3 says that λ_t decreases with $\widehat{\theta}_t$, which is intuitive given the equation (**??**). As $\widehat{\theta}_t$ rises, the average productivity of entrepreneurs in the $\mathbf{o}_{t-1}$-group increases. Furthermore, $x^*\left(\widehat{\theta}_t\right)$ decreases with $\widehat{\theta}_t$ as one can see from (6). Combined together, the default probability λ_t decreases with $\widehat{\theta}_t$, and $\widehat{\theta}_t$ inversely captures the credit risk of an entrepreneur.

Evolution of credit risk over time We first analyze the dynamic evolution of the entrepreneur's credit risk perceived by lenders over the life of an entrepreneur. Consider an entrepreneur who was born in period $s \geq 0$ and is alive in period $t > s$. The lender's belief about the entrepreneur's productivity θ for the past periods $\tau \in \{s, \ldots t-1\}$ is given as $\Phi(\cdot|x_t, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) = U_{[\widehat{\theta}_\tau, \overline{\theta}]}$, where $\widehat{\theta}_\tau$ is given by (7). As one can see from (7), $\widehat{\theta}_\tau$ weakly increases with τ until the entrepreneur leaves the economy. It means that the lender's belief about the entrepreneur's productivity improves over time in terms of first-order-stochastic dominance, as the entrepreneur becomes older. The improvement of belief, in turn, reduces the entrepreneur's credit risk and the repayment on the credit contract, as stated in the next proposition.

Proposition 2 *In the* $\boldsymbol{e}^*$ *equilibrium, the entrepreneur's credit risk and demanded repayment weakly decrease as the entrepreneur gets older.*

The results of proposition 2 is consistent with the empirical findings in Berger and Udell (1995) and Agarwal and Gort (2002) that document a decline of the firms' default risk and the firm's borrowing cost, respectively, over time. The intuition for the improvement of the lender's belief about the entrepreneur's productivity and the results of proposition 2 is in line with our earlier observations. In equilibrium, an entrepreneur honors the credit contract so far as possible and defaults only if he/she does not have enough income, which is a product of the common productivity and entrepreneur's productivity. Thus, honoring the credit contract in each period indicates that the entrepreneur's productivity is above a certain level, improving the lender's belief. This, in turn, decreases the entrepreneur's perceived credit risks and the demanded repayment.

On a related point, Boot and Thakor (1994) construct a repeated game between a lender and a borrower with a moral hazard problem and demonstrate that loan interest rates decline over time. Although the theoretical prediction is similar to that of ours, the main mechanism is different. In Boot and Thakor (1994), the borrowing cost decreases as a borrower gets older because a decreasing sequence of interest rates incentivizes a borrower to invest more effort into his/her project. On the other hand, we show that borrowing costs can decrease throughout borrower's life as a result of information learning in the credit market where adverse selection problems exist, complementing previous studies.

Cross-sectional differences in credit risk In the model economy, entrepreneurs leave the economy after defaulting on credit contracts and are replaced by new entrepreneurs. Thus, the economy consists of different age groups of entrepreneurs in a given period, and each age group could have different credit risk. We show, in proposition 2, that the credit risk of an individual entrepreneur decreases throughout his/her life. Does it imply that old entrepreneurs have lower credit risk than young entrepreneurs in a given period?

Consider two entrepreneurs: old entrepreneur and young entrepreneur with operation histories $\mathbf{o}^o_{t-1}$ and $\mathbf{o}^y_{t-1}$, respectively in period $t > 0$. Let $s < t$ be the period when the young entrepreneur was born and suppose that the old entrepreneur was born before the period s. As described lemma 3, $\hat{\theta}^i_t = Min\ \text{supp}\Big(\widehat{\Gamma}_t(\mathbf{o}^i_{t-1}, \mathbf{A}_{t-1})\Big)$ for $i = \{o, y\}$ is a sufficient statistic for the lender's belief about entrepreneur's productivity and the entrepreneur's credit risk. Thus, we focus on comparing $\hat{\theta}^o_t$ and $\hat{\theta}^y_t$ in period t in the following analysis.

Note that $\hat{\theta}^y_s = \underline{\theta}$ and $\hat{\theta}^o_s \geq \underline{\theta}$ by the results of proposition 2 when the young entrepreneur was born in period s. Assume that $\hat{\theta}^o_s > \underline{\theta}$ because if $\hat{\theta}^o_s = \underline{\theta}$, then $\hat{\theta}^y_t = \hat{\theta}^o_t$ for all period $t > s$ until one of them leaves the economy after filing bankruptcy. In period s, the old and young entrepreneurs offer $x^*\left(\hat{\theta}^o_s\right)$ and $x^*\left(\underline{\theta}\right)$ to the matched lenders, respectively. Then, assuming that both the old and young entrepreneurs do not default in period s, we obtain $\hat{\theta}^o_{s+1} = Max\left\{\frac{x^*(\hat{\theta}^o_s)}{A_s}, \hat{\theta}^o_s\right\}$ and $\hat{\theta}^y_{s+1} = Max\left\{\frac{x^*(\underline{\theta})}{A_s}, \underline{\theta}\right\}$, respectively, from (7). Because $x^*\left(\hat{\theta}^o_s\right) <$

$x^*(\underline{\theta})$ given the assumption that $\hat{\theta}_s^o > \underline{\theta}$, if $\hat{\theta}_s^o < \frac{x^*(\hat{\theta}_s^y)}{A_s}$, it must be $\hat{\theta}_{s+1}^y > \hat{\theta}_{s+1}^o$, which means that the young entrepreneur has lower credit risk than the old entrepreneur in period $s+1$. Thus, in this economy, the reversal of credit risk between the old and young entrepreneurs can occur depending on the realization of the common productivity. However, the next proposition shows the reversal of credit risk does not occur on average in equilibrium.

Proposition 3 *In the $\boldsymbol{e}^*$ equilibrium, for any t, $\mathbf{A}_{t-1} \in \mathbb{A}_{t-1}$, and $\mathbf{o}_{t-1}^o, \mathbf{o}_{t-1}^y \in \mathcal{O}_{t-1}^*$, if $\min supp\left(\widehat{\Gamma}_t(\mathbf{o}_{t-1}^y, \mathbf{A}_{t-1})\right) < \min supp\left(\widehat{\Gamma}_t(\mathbf{o}_{t-1}^o, \mathbf{A}_{t-1})\right)$, then*

$$\mathbb{E}_{A_t}\left[\hat{\theta}_{t+1}^o - \hat{\theta}_{t+1}^y | supp\left(\widehat{\Gamma}_{t+1}(\mathbf{o}_{t-1}^i \cup \{1\}, \mathbf{A}_{t-1} \cup \{A_t\})\right) \neq \varnothing \text{ for } i = \{o, y\}\right] > 0$$

where $\hat{\theta}_{t+1}^i = \min\ supp\left(\widehat{\Gamma}_{t+1}(\mathbf{o}_{t-1}^i \cup \{1\}, \mathbf{A}_{t-1} \cup \{A_t\})\right)$ for each $i = \{o, y\}$.

Proposition 3 means that an entrepreneur with lower credit risk than the other entrepreneur in the current period maintains lower credit risk on average in the next period. This implies that old entrepreneurs tend to have a lower credit risk than young entrepreneurs on average because when young entrepreneurs were born, it is more likely that old entrepreneurs have lower credit risk than new entrepreneurs.

The negative effects of the firm age on the firm's default probability have been well documented in empirical studies that use cross-sectional data.[3] The supporting argument of those studies is that young firms are more sensitive to external shocks and hence are expected to show higher bankruptcy probabilities than old firms. Through the lens of our model, we can interpret better adaptiveness of old firms as a result that only good firms can deal with a negative external shock and survive for a longer time getting older.

[3]See Altman (1968), Eklund et al. (2001), Benito et al. (2004), Bhimani et al. (2010), and Belaid (2014), for empirical studies.

5.2 Common productivity and aggregate production

In this subsection, we study the effects of common productivity on the dynamics of aggregate production. In the full production equilibrium, the aggregate production in period t is given as $Y_t = A_t \int_{\underline{\theta}}^{\overline{\theta}} \theta d\Omega_t$, where Ω_t is the cdf for θ of entrepreneurs who are alive in period t. The common productivity affects the aggregate production through two channels.

First, A_t has a direct effect on Y_t in period t because entrepreneurs' return on their project is a product of the entrepreneurial productivity and common productivity. Second, the common productivity affects the aggregate production through the cdf Ω_t for θ of alive entrepreneurs. Because A_τ in period $\tau < t$ affects the type of entrepreneurs who defaulted in period τ and defaulted entrepreneurs are replaced with new entrepreneurs, the current cdf, Ω_t, in period t is determined by the history of common productivity $\mathbf{A}_{t-1}$. For instance, all entrepreneurs offer $x^*(\underline{\theta})$ in period 0, and only entrepreneurs with $\theta \geq \frac{x^*(\underline{\theta})}{A_0}$ survive in period 0. If $\frac{x^*(\underline{\theta})}{A_0} > \underline{\theta}$, entrepreneurs with $\theta \in \left[\underline{\theta}, \frac{x^*(\underline{\theta})}{A_0}\right)$ default in period 0 and they are replaced with new entrepreneurs in period 1. Thus, the cdf Ω_1 in period t is an average of two distributions $U_{\left[\frac{x^*(\underline{\theta})}{A_0}, \overline{\theta}\right]}$ and $U_{[\underline{\theta}, \overline{\theta}]}$ weighted by the measure of each distribution, and hence, Ω_1 depends on the realization of A_0. Then, by induction, the cdf Ω_t must depend on $\mathbf{A}_{t-1}$. Then, given a sequence $\mathbf{A} = \{A_\tau\}_{\tau=-1}^{\infty} \in \mathbb{A}_\infty$, we can express the aggregate production in period t as a function of $\mathbf{A}$ such that

$$Y_t = A_t \int_{\underline{\theta}}^{\overline{\theta}} \theta d\Omega_t(\theta|\mathbf{A}_{t-1}) \equiv \widehat{Y}_t(\mathbf{A}), \tag{11}$$

where $\mathbf{A}_{t-1} = \{A_\tau\}_{\tau=-1}^{t-1}$ is a subsequence of $\mathbf{A}$ and $\Omega_t(\theta|\mathbf{A}_{t-1})$ is the associated cdf for θ of alive entrepreneurs in period t given $\mathbf{A}_{t-1}$.

In general, it is hard to trace Ω_t and Y_t over time because the realization of the common productivity in each period is randomly drawn from $U_{[0,1]}$. To gather the intuition about the dynamics of $\Omega_t(\theta|\mathbf{A}_{t-1})$ and Y_t over time, we study a special case in which the realized common productivity is constant such that $A_t = \widetilde{A} \in [0, 1]$ for all $t \geq 0$. For notational convenience, when $A_\tau = \widetilde{A}$ for all $\tau \geq 0$, we denote the sequence of common productivity by $\widetilde{\mathbf{A}}_t = \{A_\tau\}_{\tau=-1}^{t-1}$ and $\widetilde{\mathbf{A}} = \{A_\tau\}_{\tau=-1}^{\infty}$.

Proposition 4 *Suppose that the realized common productivity is constant at $\widetilde{A} \in [0,1]$, i.e., $A_t = \widetilde{A}$, for all $t \geq 0$ in the $\mathbf{e}^*$ equilibrium.*

1. *If $\widetilde{A} \in \left[0, \frac{x^*(\underline{\theta})}{\bar{\theta}}\right] \cup \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$, then $\Omega_t\left(\theta|\widetilde{\mathbf{A}}_{t-1}\right) = U_{[\underline{\theta},\bar{\theta}]}$ and $\widehat{Y}_t\left(\widetilde{\mathbf{A}}\right) = \frac{\widetilde{A}(\underline{\theta}+\bar{\theta})}{2}$ for all $t \geq 0$.*
2. *If $\widetilde{A} \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$, then, letting $\Delta \equiv \left(\frac{\frac{x^*(\underline{\theta})}{\widetilde{A}} - \underline{\theta}}{\bar{\theta}-\underline{\theta}}\right)$,*

$$\Omega_t\left(\theta|\widetilde{\mathbf{A}}_{t-1}\right) = \begin{cases} \Delta^t \frac{\theta-\underline{\theta}}{\bar{\theta}-\underline{\theta}} & \text{for } \theta \leq \frac{x^*(\underline{\theta})}{\widetilde{A}} \\ \Delta^t \frac{\theta-\underline{\theta}}{\bar{\theta}-\underline{\theta}} + (1-\Delta^t)\frac{\theta - \frac{x^*(\underline{\theta})}{\widetilde{A}}}{\bar{\theta} - \frac{x^*(\underline{\theta})}{\widetilde{A}}} & \text{for } \theta > \frac{x^*(\underline{\theta})}{\widetilde{A}} \end{cases} \tag{12}$$

$$\widehat{Y}_t\left(\widetilde{\mathbf{A}}\right) = \triangle^t \frac{\widetilde{A}(\underline{\theta}+\bar{\theta})}{2} + \left[1 - \triangle^t\right] \frac{x^*(\underline{\theta}) + \widetilde{A}\bar{\theta}}{2} \tag{13}$$

for all $t \geq 0$.

Proposition 4 describes the dynamics of $\Omega_t(\theta|\widetilde{\mathbf{A}}_{t-1})$ and $\widehat{Y}_t(\widetilde{\mathbf{A}})$ over time when the realized common productivity is constant at $\widetilde{A} \in [0,1]$ for all $t \geq 0$. The first part of proposition 4 is straightforward: If $\widetilde{A} \in \left[0, \frac{x^*(\underline{\theta})}{\bar{\theta}}\right]$, all entrepreneurs default and are replaced with ones every period, and if $\widetilde{A} \in \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$, all entrepreneurs do not default every period.[4] In either cases, $\Omega_t\left(\theta|\widetilde{\mathbf{A}}_{t-1}\right) = U_{[\underline{\theta},\bar{\theta}]}$ for all $t \geq 0$, and hence, $\widehat{Y}_t(\widetilde{\mathbf{A}}) = \frac{\widetilde{A}(\underline{\theta}+\bar{\theta})}{2}$. When $\widetilde{A} \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$, on the other hand, a certain fraction of new entrepreneurs leaves the economy after default and they are replaced with new entrepreneurs changing the cdf $\Omega_t\left(\theta|\widetilde{\mathbf{A}}_{t-1}\right)$ and, hence $\widehat{Y}_t(\widetilde{\mathbf{A}})$, over time as stated in (12) and (13), resepctively.

Note that $\Omega_t(\theta|\widetilde{\mathbf{A}}_{t-1})$ in (12) improves over time in the sense of first-order-stochastic dominance because $\Delta < 1$ and $\frac{\theta-\underline{\theta}}{\bar{\theta}-\underline{\theta}} < \frac{\theta - \frac{x^*(\underline{\theta})}{\widetilde{A}}}{\bar{\theta} - \frac{x^*(\underline{\theta})}{\widetilde{A}}}$ for all $\theta \in \left[\bar{\theta}, \underline{\theta}\right]$ when $\widetilde{A} \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$. As a consequence, $\widehat{Y}_t(\widetilde{\mathbf{A}})$ increases over time converging to its limit $\frac{x^*(\underline{\theta})+\widetilde{A}\bar{\theta}}{2}$. The intuitive

[4]When $\widetilde{A} = \frac{x^*(\underline{\theta})}{\bar{\theta}}$, $\bar{\theta}$ type entrepreneurs do not default and survive to the next period. However, the measure of survived $\bar{\theta}$ type entrepreneurs is zero every period, so they do not affect the cdf Ω_t.

explanation for these findings is as follows. All new entrepreneurs offer $x^*(\underline{\theta})$ to lenders when they are born. Among them, $1-\Delta$ fraction of entrepreneurs with $\theta \geq \frac{x^*(\underline{\theta})}{\widetilde{A}}$ make repayment $x^*(\underline{\theta})$, and offer $x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)$ to lenders for all succeeding periods staying in the economy.[5] On the other hand, Δ fraction of new entrepreneurs with $\theta < \frac{x^*(\underline{\theta})}{\widetilde{A}}$ leave the economy after default, and they are replaced with new entrepreneurs who go through the same process. In summary, only entrepreneurs with $\theta \geq \frac{x^*(\underline{\theta})}{\widetilde{A}}$ survive in each period and the process of survival of the fittest continues until θ of all entrepreneurs is distributed over $\left[\frac{x^*(\underline{\theta})}{\widetilde{A}}, \overline{\theta}\right]$.

Asymmetric effects of shocks We now study the dynamics of the aggregate production after a temporary shock on the common productivity when the economy stays in the stationary $\boldsymbol{e}^*$ equilibrium. By stationarity, we mean that the cdf Ω_t does not change over time. For example, if $\widetilde{A} \in \left[0, \frac{x^*(\underline{\theta})}{\overline{\theta}}\right] \cup \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$, the economy stays in a stationary equilibrium because $\Omega_t\left(\theta|\widetilde{\mathbf{A}}_{t-1}\right) = U_{[\underline{\theta},\overline{\theta}]}$ for all $t \geq 0$. When $\widetilde{A} \in \left(\frac{x^*(\underline{\theta})}{\overline{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$, $\Omega_t\left(\theta|\widetilde{\mathbf{A}}_{t-1}\right)$ changes over time, but for a sufficiently high $s > 0$, we have $\Omega_t\left(\theta|\widetilde{\mathbf{A}}_{t-1}\right) \approx \Omega_{t+1}\left(\theta|\widetilde{\mathbf{A}}_{t}\right)$ for all $t \geq s$. In this case, we also say that the economy is in a stationary equilibrium, and let $\Omega_t\left(\theta|\widetilde{\mathbf{A}}_{t-1}\right) = \Omega_{t+1}\left(\theta|\widetilde{\mathbf{A}}_{t}\right)$ for all $t \geq s$ without loss of generality.

Consider the sequence $\widetilde{\mathbf{A}}' = \{A_\tau\}_{\tau=-1}^{\infty}$ such that

$$A_\tau = \widetilde{A} \text{ for all } \tau \neq s \text{ and } A_s = A' \tag{14}$$

Suppose that the economy has reached to a stationary equilibrium in period $s' < s$, i.e., $\Omega_t\left(\theta|\widetilde{\mathbf{A}}_{t-1}\right) = \Omega_{t+1}\left(\theta|\widetilde{\mathbf{A}}_{t}\right)$ for $t \in \{s', \ldots s-1\}$. It is obvious that the aggregate output produced by entrepreneurs in period $t = s$ when the shock arrives is given as $\widehat{Y}_s\left(\widetilde{\mathbf{A}}'\right) = \frac{A'}{\widetilde{A}}\widehat{Y}_{s-1}\left(\widetilde{\mathbf{A}}'\right)$. The question is how the dynamics of $\widehat{Y}_t\left(\widetilde{\mathbf{A}}'\right)$ is for $t > s$ after the shock. The results depend on whether a shock is positive, i.e., $A' > \widetilde{A}$ or negative, i.e., $A' < \widetilde{A}$.

If the shock is positive, i.e., $A' > \widetilde{A}$, the return on each entrepreneur's business operation

[5] Note that $x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right) < x^*(\underline{\theta})$ when $A_t = \widetilde{A}$ for all $t \geq 0$, and hence entrepreneurs with $\theta \geq \frac{x^*(\underline{\theta})}{\widetilde{A}}$ can honor the contract $x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)$.

is higher in period $t = s$ than that of previous periods due to an increase in the common productivity. No entrepreneurs default at $t = s$, and hence, $\Omega_{s-1} = \Omega_s$. Given, $A_t = \widetilde{A}$ for $t \geq s+1$, the aggregate output produced by entrepreneurs is reversed to the previous level, $\widehat{Y}_{s-1}\left(\widetilde{\mathbf{A}}'\right)$, in a stationary equilibrium. Thus, the effects of a positive shock $A' > \widetilde{A}$ has temporary effects on the economy. This is formally stated in the next proposition, whose proof is omitted.

Proposition 5 *Take the sequence $\widetilde{\mathbf{A}}'$ given by (14) for some $\widetilde{A} \in (0,1]$, and assume that the economy has reached to the stationary $\mathbf{e}^*$ equilibrium in period $s' < s$. If $A' > \widetilde{A}$, then $\widehat{Y}_t\left(\widetilde{\mathbf{A}}'\right) = \widehat{Y}_{s-1}\left(\widetilde{\mathbf{A}}'\right)$ for all $t \geq s+1$.*

If the shock is negative, i.e., $A' < \widetilde{A}$, on the other hand, the shock could lead a certain type of entrepreneurs to default, which changes the composition of entrepreneurs in the economy. Thus, a negative shock can have persistent effects on $\widehat{Y}_t(\widetilde{\mathbf{A}}')$ for $t \geq s+1$. Specific dynamics of $\widehat{Y}_t(\widetilde{\mathbf{A}}')$ after the shock depends on the level of $\widetilde{A}$ and A' as described in the next proposition.

Proposition 6 *Take the sequence $\widetilde{\mathbf{A}}'$ given by (14) for some $\widetilde{A} \in (0,1]$ with $A' < \widetilde{A}$, and assume that the economy has reached to the stationary $\mathbf{e}^*$ equilibrium in period $s' < s$. Let $\widetilde{\theta} = \frac{x^*(\underline{\theta})}{\widetilde{A}}$, $\Delta = \frac{\frac{x^*(\underline{\theta})}{\widetilde{A}} - \underline{\theta}}{\bar{\theta} - \underline{\theta}}$, $\triangle' = Min\left\{1, \frac{\frac{x^*(\underline{\theta})}{A'} - \underline{\theta}}{\bar{\theta} - \underline{\theta}}\right\}$, and $\widetilde{\triangle}' = Min\left\{1, \frac{\frac{x^*(\widetilde{\theta})}{A'} - \widetilde{\theta}}{\bar{\theta} - \widetilde{\theta}}\right\}$. Then, for $t \geq s+1$, $\widehat{Y}_t(\widetilde{\mathbf{A}}')$ is given as follows:*

1. *Assume that $\widetilde{A} \in \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$.*

   1-a. If $A' \in \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, \widetilde{A}\right)$, then $\widehat{Y}_t\left(\widetilde{\mathbf{A}}'\right) = \frac{\widetilde{A}(\underline{\theta}+\bar{\theta})}{2}$.

   1-b. If $A' \in \left[0, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$, then $\widehat{Y}_t\left(\widetilde{\mathbf{A}}'\right) = \triangle' \frac{\widetilde{A}(\underline{\theta}+\bar{\theta})}{2} + [1 - \triangle'] \frac{\widetilde{A}}{2}\left(\frac{x^*(\underline{\theta})}{A'} + \bar{\theta}\right)$.

2. *Assume that $\widetilde{A} \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$.*

   2-a. If $A' \in \left[\frac{x^*(\widetilde{\theta})}{\widetilde{\theta}}, \widetilde{A}\right)$, then $\widehat{Y}_t\left(\widetilde{\mathbf{A}}'\right) = \frac{\widetilde{A}\left(\widetilde{\theta}+\bar{\theta}\right)}{2}$.

   2-b. If $A' \in \left[0, \frac{x^*(\widetilde{\theta})}{\widetilde{\theta}}\right)$, then $\widehat{Y}_t\left(\widetilde{\mathbf{A}}'\right) = \widetilde{\triangle}'\left\{\triangle^{t-(s+1)} \frac{\widetilde{A}(\underline{\theta}+\bar{\theta})}{2} + \left[1 - \triangle^{t-(s+1)}\right] \frac{x^*(\underline{\theta}) + \widetilde{A}\bar{\theta}}{2}\right\} + \left[1 - \widetilde{\triangle}'\right] \frac{\widetilde{A}}{2}\left(\frac{x^*(\widetilde{\theta})}{A'} + \bar{\theta}\right)$

3. Assume that $\widetilde{A} \in \left(0, \frac{x^*(\underline{\theta})}{\bar{\theta}}\right]$, *then* $\widehat{Y}_t\left(\widetilde{\mathbf{A}}'\right) = \frac{\widetilde{A}(\underline{\theta}+\bar{\theta})}{2}$.

The central implication of proposition 6 is that the dynamics of $\widehat{Y}_t(\widetilde{\mathbf{A}}')$ depends on the measure of defaulted entrepreneurs when the negative shock arrives at $t = s$. First, if A' is not too low as in the cases of 1-a and 2-a in proposition 6, all existing entrepreneurs survive without defaulting in period $t = s$. This implies that $\Omega_t = \Omega_{s-1}$, and hence $Y_t(\widetilde{\mathbf{A}}') = Y_{s-1}(\widetilde{\mathbf{A}}')$, for all $t \geq s+1$. Second, if A' is low enough, a certain fraction ($\triangle'$ and $\widetilde{\triangle}'$ for the cases 1-b and 2-b, respectively) of existing entrepreneurs default in period $t = s$ and are replaced with new entrepreneurs. Thus, $Y_t(\widetilde{\mathbf{A}}')$ for $t \geq s+1$ consists of two parts: 1) goods produced by entrepreneurs who were born after the negative shock and 2) goods produced by the existing entrepreneurs who did not default in the period when the shock arrived. In particular, if A' is sufficiently low, including the case 3 where all entrepreneure default every period, then all existing entrepreneurs leave the economy, and the economy starts with all new entrepreneurs in period $s+1$.

Note, from proposition 6, that when $\widetilde{A} \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$, the time it takes for the aggregate production to recover back to the pre-shock level after a negative shock depends on the size of shock. Specifically, when A' is not too low as $A' \in \left[\frac{x^*(\widetilde{\theta})}{\widetilde{\theta}}, \widetilde{A}\right)$, no entrepreneurs default in period s and total entrepreneurs' production $\widehat{Y}_t\left(\widetilde{\mathbf{A}}'\right)$ moves back to the pre-shock level $\frac{\widetilde{A}\left(\widetilde{\theta}+\bar{\theta}\right)}{2}$ in the next period after the negative shock, i.e., $\widehat{Y}_{s+1}\left(\widetilde{\mathbf{A}}'\right) = \frac{\widetilde{A}\left(\widetilde{\theta}+\bar{\theta}\right)}{2}$. On the other hand, if A' is sufficiently low as $A' < \frac{x^*(\underline{\theta})}{\bar{\theta}}$, then all entrepreneurs default when the shock arrives in period $t = s$ and $\widehat{Y}_t\left(\widetilde{\mathbf{A}}'\right)$ increases for all $t \geq s+1$, converging to $\frac{\widetilde{A}\left(\widetilde{\theta}+\bar{\theta}\right)}{2}$. Finally, suppose that $A' \in \left[\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\widetilde{\theta})}{\widetilde{\theta}}\right)$. Then, from the case 2-b of proposition 6, we obtain

$$\begin{aligned}
&\widehat{Y}_t\left(\widetilde{\mathbf{A}}'\right) - \frac{\widetilde{A}\left(\widetilde{\theta}+\bar{\theta}\right)}{2} \\
&= -\widetilde{\triangle}'\triangle^{t-(s+1)}\left[\frac{x^*\left(\underline{\theta}\right) - \widetilde{A}\underline{\theta}}{2}\right] + \left(1-\widetilde{\triangle}'\right)\left[\frac{x^*\left(\widetilde{\theta}\right)\frac{\widetilde{A}}{A'} - x^*\left(\underline{\theta}\right)}{2}\right]
\end{aligned} \tag{15}$$

for $t \geq s+1$. Substituting $\Delta = \frac{\frac{x^*(\underline{\theta})}{\widetilde{A}} - \underline{\theta}}{\bar{\theta}-\underline{\theta}}$ and $\widetilde{\triangle}' = \frac{\frac{x^*(\widetilde{\theta})}{A'} - \widetilde{\theta}}{\bar{\theta}-\widetilde{\theta}}$ into (15) and using the assumptions

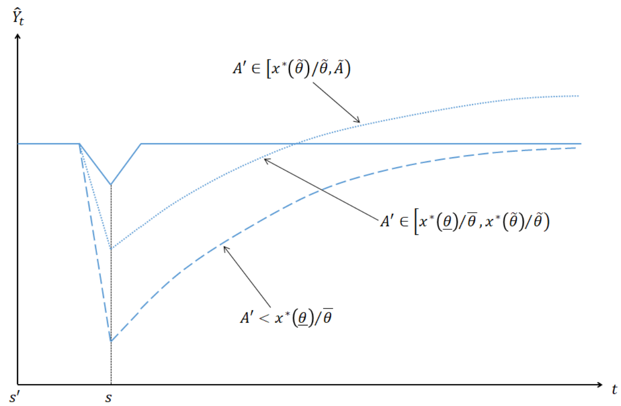

Figure 1: Dynamics of $\widehat{Y}_t\left(\widetilde{\mathbf{A}}'\right)$ when $\widetilde{A} \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$

that $\widetilde{A} > \frac{x^*(\underline{\theta})}{\bar{\theta}}$ and $A' < \frac{x^*(\widetilde{\theta})}{\widetilde{\theta}}$, we obtain that $\widehat{Y}_t\left(\widetilde{\mathbf{A}}'\right) \geq \frac{\widetilde{A}\left(\widetilde{\theta}+\bar{\theta}\right)}{2}$ for all $t \geq \widehat{t}\left(\widetilde{A}, A'\right) + s + 1$, where

$$\widehat{t}\left(\widetilde{A}, A'\right) \equiv \frac{\log\left(\widetilde{\theta} - \underline{\theta}\right) - \log\left(\bar{\theta} - \frac{x^*(\widetilde{\theta})}{A'}\right)}{\log\left(\bar{\theta} - \underline{\theta}\right) - \log\left(\frac{x^*(\underline{\theta})}{\widetilde{A}} - \underline{\theta}\right)}. \tag{16}$$

Note that $\widehat{t}\left(\widetilde{A}, A'\right)$ in (16) decreases with A', and hence it takes more time for the aggregate production to move back to the pre-shock level as A' decreases. The analysis of the above three cases shows that the time for recovery of aggregate production increases as the size of shock, measured by $\frac{\widetilde{A}-A'}{\widetilde{A}}$, increases. Figure 1 summarizes the above analysis.

Although we have focused on the effects of a temporary common productivity shock in a stationary equilibrium, the results that a positive shock does not change the composition of entrepreneurs while a negative shock can change the distribution of entrepreneurial productivity also hold in a non-stationary equilibrium. Thus, given a sequence of $\{A_t\}_{t=0}^{\infty} \in \mathbb{A}_\infty$, where A_t is independently distributed overtime, the pattern of the dynamics of aggregate output is similar to the results in propositions 5 and 6, although the aggregate output fluctuates in response to changes in A_t overtime. Specifically, the model generates a slower pace in increases in the output than declines on average that is the cyclical asymmetry in which the economy behaves differently over the expansion and recession phases of the business cycle, consistent with empirical findings.[6]

A number of studies attempted to provide explanations for the cyclical asymmetry of aggregate time-series data. For example, Acemoglu and Scott (1997) show that intertemporal increasing return can generate persistent output fluctuation over the expansion phases, and Chalkley and Lee (1998) derive similar results using risk-averse agents and noisy information on the aggregate state. In the context of our modeled economy, the cyclical asymmetry of the business cycle and the slow recovery of output back to the pre-crisis level after a big shock is symptomatic of the improvement of entrepreneurial productivity over time through the continuous replacement of low productive entrepreneurs with new ones, complementing previous studies. In particular, our model provides better insights on the recent empirical findings in Reifschneider et al. (2015) and Ikeda and Kurozumi (2019) that protracted drop in productivity is an essential factor of the slow recovery after a crisis once we interpret the total factor productivity as the product of common productivity and the average of entrepreneurial productivity.

Constructive economic downturn One interesting result in proposition 6 is that while the aggregate production drops when the negative shock arives, the aggregate production can exceed the pre-shock level after the shock unless all existing entrepreneurs leave the economy or survive. Specifically, when $\widetilde{A} \in \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$, if $A' \in \left(\frac{x^*(\underline{\theta})}{\underline{\theta}}, \frac{x^*(\underline{\theta})}{\bar{\theta}}\right)$, we obtain

$$\begin{aligned} \widehat{Y}_t\left(\widetilde{\mathbf{A}}'\right) &= \triangle' \frac{\widetilde{A}(\underline{\theta}+\bar{\theta})}{2} + [1-\triangle'] \frac{\widetilde{A}}{2}\left(\frac{x^*(\underline{\theta})}{A'} + \bar{\theta}\right) \\ &> \frac{\widetilde{A}(\underline{\theta}+\bar{\theta})}{2} = \widehat{Y}_{s-1}\left(\widetilde{\mathbf{A}}'\right) \end{aligned}$$

for all $t \geq s+1$. Similarly, when $\widetilde{A} \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$, if $A' \in \left(\frac{x^*(\widetilde{\theta})}{\bar{\theta}}, \frac{x^*(\widetilde{\theta})}{\widetilde{\theta}}\right)$, then $\widehat{Y}_t\left(\widetilde{\mathbf{A}}'\right) \geq \widehat{Y}_{s-1}\left(\widetilde{\mathbf{A}}'\right)$ for all t that satisfies (16). This is because when the negative shock arrives, entrepreneurs with higher productivities than a certain level only survive and they stay in

[6]See Neftçi (1984), Hamilton (1989), and Morley and Piger (2012) for empirical studies.

the economy for all succeeding periods, improving the average entrepreneurial productivity.

Therefore, although a negative shock reduces the total production when the shock arrives, it can raise the aggregate production in the long term. The question is whether a negative shock is beneficial or not. To conduct a cost and benefit analysis of a negative shock on the common productivity, we use the sum of discounted aggregate productions. Specifically, we compare $\sum_{t=0}^{\infty} \beta^t \widehat{Y}_t\left(\widetilde{\mathbf{A}}\right)$ and $\sum_{t=0}^{\infty} \beta^t \widehat{Y}_t\left(\widetilde{\mathbf{A}}'\right)$ for two sequences $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{A}}'$, where $\widetilde{\mathbf{A}}'$ given by (14) for some $\widetilde{A} \in (0,1]$ with $A' < \widetilde{A}$. Note that $\widehat{Y}_t\left(\widetilde{\mathbf{A}}\right) = \widehat{Y}_t\left(\widetilde{\mathbf{A}}'\right)$ for all $t < s$. Given $\widetilde{A}$ and β, define the set of A' as

$$I\left(\widetilde{A}, \beta\right) = \left\{A' < \widetilde{A} : \sum_{t=s}^{\infty} \beta^t [\hat{Y}_t(\mathbf{A}') - \hat{Y}_t(\widetilde{\mathbf{A}})] > 0\right\}.$$

Then, for all $A' \in I\left(\widetilde{A}, \beta\right)$, the negative shock is constructive and the shock is destructive otherwise.

Proposition 7 *Take the sequence $\widetilde{\mathbf{A}}'$ given by (14) for some $\widetilde{A} \in (0,1]$ with $A' < \widetilde{A}$. If β is sufficiently high, there exists $\widetilde{A} \in \left(\frac{x^*(\theta)}{\bar{\theta}}, 1\right]$ such that $I\left(\widetilde{A}, \beta\right)$ is an open interval with the following properties:*

1. *If $\widetilde{A} \in \left[\frac{x^*(\theta)}{\underline{\theta}}, 1\right]$, then $I\left(\widetilde{A}, \beta\right) \subset \left(\frac{x^*(\theta)}{\bar{\theta}}, \frac{x^*(\theta)}{\underline{\theta}}\right)$, and for any $\widetilde{A}_1$, $\widetilde{A}_2 \in \left[\frac{x^*(\theta)}{\bar{\theta}}, 1\right]$ with $\widetilde{A}_1 < \widetilde{A}_2$, $I\left(\widetilde{A}_2, \beta\right) \subset I\left(\widetilde{A}_1, \beta\right)$.*
2. *If $\widetilde{A} \in \left(\frac{x^*(\theta)}{\bar{\theta}}, \frac{x^*(\theta)}{\underline{\theta}}\right)$, then $I\left(\widetilde{A}, \beta\right) \subset \left(\frac{x^*\left(\frac{x^*(\theta)}{\widetilde{A}}\right)}{\bar{\theta}}, \frac{x^*\left(\frac{x^*(\theta)}{\widetilde{A}}\right)}{\frac{x^*(\theta)}{\widetilde{A}}}\right)$, and for any $\widetilde{A}_1$, $\widetilde{A}_2 \in \left(\frac{x^*(\theta)}{\bar{\theta}}, \frac{x^*(\theta)}{\underline{\theta}}\right)$ with $\widetilde{A}_1 < \widetilde{A}_2$, $I\left(\widetilde{A}_1, \beta\right) \subset I\left(\widetilde{A}_2, \beta\right)$.*

Proposition 7 shows that the constructiveness of the negative shock depends on three factors. First, for the negative shock to be constructive, the shock should remove less productive entrepreneurs and improve average entrepreneurial productivity in the long run. Thus, the constructive economic downturn occurs only for A' in the subset of $\left(\frac{x^*(\theta)}{\bar{\theta}}, \frac{x^*(\theta)}{\underline{\theta}}\right)$ or of $\left(\frac{x^*(\widetilde{\theta})}{\bar{\theta}}, \frac{x^*(\widetilde{\theta})}{\widetilde{\theta}}\right)$, depending on $\widetilde{A}$.[7] Second, it takes time for the negative shock to raise

[7]Proposition 6 shows that 1) when $\widetilde{A} \in \left[\frac{x^*(\theta)}{\underline{\theta}}, 1\right]$, the measure of defaulting entrepreneurs $\triangle'$ is in $(0,1)$

the aggregate production in the long run, and hence, it is more likely that the shock is constructive with the higher discount factor β. Third, $\widetilde{A}$ matters because the cdf Ω_t before the shock and the size of shock, $\frac{\widetilde{A}-A'}{\widetilde{A}}$, depend on $\widetilde{A}$. Specifically, when $\widetilde{A} \in \left[\frac{x^*(\underline{\theta})}{\overline{\theta}}, 1\right]$, the value of $\widetilde{A}$ does not affect Ω_t in a steady state and, hence, the type of removed entrepreneurs by the shock. An increase in $\widetilde{A}$ only aggrevates the temporary negative effect of the shock and makes the shock be more destructive. On the other hand, when $\widetilde{A} \in \left(\frac{x^*(\underline{\theta})}{\overline{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$, θ is uniformly distributed over $\left[\frac{x^*(\underline{\theta})}{\widetilde{A}}, \overline{\theta}\right]$ in a stationary equilibrium. If $\widetilde{A}$ is low enough, most of the existing entrepreneurs before the shock is sufficiently productive. Thus, as $\widetilde{A}$ decreases, the positive effects of the negative shock on the aggregate output in the long run decreases, making the shock more destructive.

6 Conclusion

In this paper we have constructed a dynamic equilibrium model of debt contracts with adverse selection and studied the way of constructing lenders' beliefs about borrowers with different business operation histories using the information on aggregate economic conditions in the past. We have shown that credit risk of a borrower perceived by lenders weakly decreases as the borrower gets older because more productive borrowers tend to stay in the economy for longer periods. In equilibrium, the borrowing cost weakly decreases throughout borrower's life, and old borrowers pay lower borrowing costs than young borrowers on average. We have shown that the model was tractable to analyze impulse responses after an aggregate productivity shock analytically. We used the model to provide theoretical explanations for the cyclical asymmetry of aggregate output over the business cycle and a narrative for the sluggish recovery of economic activities after a crisis. The model also shows that a mild negative productivity shock can be constructive, increasing aggregate output in the

for $A' \in \left(\frac{x^*(\underline{\theta})}{\overline{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$, and 2) when $\widetilde{A} \in \left(\frac{x^*(\underline{\theta})}{\overline{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$, the measure of defaulting entrepreneurs $\widetilde{\triangle}'$ is in $(0, 1)$ for $A' \in \left(\frac{x^*(\widetilde{\theta})}{\overline{\theta}}, \frac{x^*(\widetilde{\theta})}{\widetilde{\theta}}\right)$.

long run.

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Appendix A: Proof

Proof of Lemma 1. The proof is done by showing that $0 < \frac{b(\underline{\theta})-\sqrt{b(\underline{\theta})^2-4b(\underline{\theta})r}}{\underline{\theta}} < 1$. First, $\frac{b(\underline{\theta})-\sqrt{b(\underline{\theta})^2-4b(\underline{\theta})r}}{\underline{\theta}} > 0$ is well-defined since $b(\underline{\theta}) = \left[\int_{\underline{\theta}}^{\bar{\theta}} \frac{1}{\theta} \cdot \frac{1}{\bar{\theta}-\underline{\theta}} d\theta\right]^{-1} > \left[\int_{\underline{\theta}}^{\bar{\theta}} \frac{1}{\underline{\theta}} \cdot \frac{1}{\bar{\theta}-\underline{\theta}} d\theta\right]^{-1} = \underline{\theta} \geq 4r$. Since $\frac{b(\underline{\theta})}{\underline{\theta}} - \sqrt{\left(\frac{b(\underline{\theta})}{\underline{\theta}}\right)^2 - \frac{4r}{\underline{\theta}} \frac{b(\underline{\theta})}{\underline{\theta}}}$ strictly decreases in $\frac{b(\underline{\theta})}{\underline{\theta}}$ and $1 < \frac{b(\underline{\theta})}{\underline{\theta}}$, $\frac{b(\underline{\theta})}{\underline{\theta}} - \sqrt{\left(\frac{b(\underline{\theta})}{\underline{\theta}}\right)^2 - \frac{4r}{\underline{\theta}} \cdot \frac{b(\underline{\theta})}{\underline{\theta}}} < 1 - \sqrt{1 - \frac{4r}{\underline{\theta}}} \leq 1$, which finishes the proof. ■

Proof of lemma 2. Consider any entrepreneur θ with $(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}) \in \mathcal{O}^* \times \mathbb{A}$ in period $t \geq 0$. Take any $x_t^i \in \mathcal{B}_t^*(\Phi, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})$ for $i = 1, 2$, where $x_t^1 < x_t^2$. Let D_t^i be the corresponding optimal strategic default set for x_t^i for each $i = 1, 2$. Because $A_t \in D_t^{si}$ if and only if either $x_t^i > \beta V_{t+1}(\theta, \mathbf{o}_{t-1} \cup \{1\}, \mathbf{A}_{t-1} \cup \{A_t\})$ or $x_t^i > A_t\theta$ for each $i = 1, 2$, it must be $D_t^1 \subseteq D_t^2$. Furthermore, because $x_t^2 \in \mathcal{B}_t^*(\Phi, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})$, the lender's expected payoff from accepting x_t^2 is no less than r, which requires that $[0,1] \backslash D_t^2$ has a positive measure. Thus

$$
\begin{aligned}
&v(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}, x_t^1, D_t^1) - v(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}, x_t^2, D_t^2) \\
&= \int_{[0,1]\backslash D_t^1} \left[-x_t^1 + \beta V_{t+1}(\theta, \mathbf{o}_t, \mathbf{A}_{t-1} \cup \{A_t\})\right] dU(A_t) - \int_{[0,1]\backslash D_t^2} \left[-x_t^2 + \beta V_{t+1}(\theta, \mathbf{o}_t, \mathbf{A}_{t-1} \cup \{A_t\})\right] dU(A_t) \\
&\geq \int_{[0,1]\backslash D_t^2} \left[-x_t^1 + \beta V_{t+1}(\theta, \mathbf{o}_t, \mathbf{A}_{t-1} \cup \{A_t\})\right] dU(A_t) - \int_{[0,1]\backslash D_t^{s2}} \left[-x_t^2 + \beta V_{t+1}(\theta, \mathbf{o}_t, \mathbf{A}_{t-1} \cup \{A_t\})\right] dU(A_t) \\
&= \int_{[0,1]\backslash D_t^2} (x_t^2 - x_t^1) dU(A_t) > 0,
\end{aligned}
$$

where $\mathbf{o}_t = \mathbf{o}_{t-1} \cup \{1\}$. Thus, the entrepreneur $(\theta, \mathbf{o}_{t-1})$ strictly prefers x_t^1 to x_t^2. This implies that whenever the entrepreneur makes an acceptable offer from $\mathcal{B}_t^*(\Phi, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})$, he/she chooses $\min\{\mathcal{B}_t^*(\Phi, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})\}$ independent of the productivity θ. ■

Proof of Proposition 1. Fix any $(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}) \in \mathcal{O}^* \times \mathbb{A}$. In a full production equilibrium all entrepreneurs with $(\mathbf{o}_{t-1}, \mathbf{A}_{t-1})$ offer a contract and, by lemma 2, they offer the same contract. Let x be such contract. We show that if $\widehat{\Gamma}(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}) = U_{[\hat{\theta}, \bar{\theta}]}$ then $x = x^*(\hat{\theta})$. We first show $x \geq x^*(\hat{\theta})$. Denoting $\Phi_{\hat{x}} = \Phi(\hat{x}, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})$ for each $\hat{x} \in X$, $\Phi_x = \widehat{\Gamma}(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}) = U_{[\hat{\theta}, \bar{\theta}]}$ and $x = \min \mathcal{B}_t^*(\Phi, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})$. Lender's expected payoff who accepts this contract x is maximized when every entrepreneur $\theta \in$supp $\widehat{\Gamma}(\mathbf{o}_{t-1}, \mathbf{A}_{t-1})$ sets the minimum possible default set $\left[0, \frac{x}{\theta}\right)$. That is,

$$\max_{\{D_\theta\}_{\theta \in [\hat{\theta}, \bar{\theta}]} : [0, \frac{x}{\theta}) \subseteq D_\theta \forall \theta} \int_{[\hat{\theta}, \bar{\theta}]} \int_{[0,1] \setminus D_\theta} x dU(A_t) dU_{[\hat{\theta}, \bar{\theta}]}(\theta) = \int_{[\hat{\theta}, \bar{\theta}]} \int_{[\frac{x}{\theta}, 1]} x dU(A_t) dU_{[\hat{\theta}, \bar{\theta}]}(\theta) = x - \frac{x^2}{b(\hat{\theta})}.$$

Since $x^*(\hat{\theta}) = \min\{x : x - \frac{x^2}{b(\hat{\theta})} \geq r\}$, the lender will never take any offer lower than $x^*(\hat{\theta})$. Now we show that $D_t(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}, x^*(\hat{\theta})) = \left[0, \frac{x^*(\hat{\theta})}{\theta}\right)$ using the following claims, which will indicate that $x^*(\hat{\theta}) \in \mathcal{B}_t^*(\Phi', \mathbf{o}_{t-1}, \mathbf{A}_{t-1})$ for any Φ' such that $\Phi'_{x^*(\hat{\theta})} = U_{[\hat{\theta}, \bar{\theta}]}$.

Claim 1 *In any full production equilibrium, for each $\theta \in supp(\widehat{\Gamma}_t(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}))$,*

(i) $V_{t+1}(\theta, \mathbf{o}_{t-1} \cup \{1\}, \mathbf{A}_{t-1} \cup \{A_t\}) \geq \frac{\theta}{2}$ if $\mathbf{o}_{t-1} \cup \{1\} \in \mathcal{O}_{t+1}^*$.

(ii) If $x < \frac{\beta\theta}{2}$, then $D_t(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}, x) = \left[0, \frac{x}{\theta}\right)$.

(iii) For any $x_1, x_2 \in X$ such that $x_2 > x_1$, if $x_1 \in \mathcal{B}_1$ and $x_2 \in \mathcal{B}_2$ then

$$\max_{D \subset [0,1]} v(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}, x_1, D, \mathcal{B}_1) > \max_{D \subset [0,1]} v(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}, x_2, D, \mathcal{B}_2).$$

Proof of Claim 1. Denote $\mathbf{o}_t = \mathbf{o}_{t-1} \cup \{1\}$. For any $A_t \in [0,1]$ such that $\mathbf{o}_t \in \mathcal{O}_{t+1}^*$, $\mathcal{B}_{t+1} = \mathcal{B}_{t+1}(\mathbf{o}_{t-1} \cup \{1\}, \mathbf{A}_{t-1} \cup \{A_t\}) \neq \varnothing$ in any full production equilibrium so that, for any

$\tilde{x} \in \mathcal{B}_{t+1}$, $V_{t+1}(\theta, \mathbf{o}_t, \mathbf{A}_{t-1} \cup \{A_t\}) = \max_{x,D} v(\theta, \mathbf{o}_t, \mathbf{A}_{t-1} \cup \{A_t\}, x, D, \mathcal{B}_{t+1}) \geq v(\theta, \mathbf{o}_t, \mathbf{A}_{t-1} \cup \{A_t\}, \tilde{x}, [0,1], \mathcal{B}_{t+1}) = \int_{[0,1]} A_t \theta dU(A_t) = \frac{\theta}{2}$.

To show the second statement, notice that each entrepreneur $\theta \in \text{supp}(\widehat{\Gamma}_t(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}))$ with a contract x sets the default set $D_t(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}, x)$ such that $A_t \in D_t(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}, x)$ if and only if either $x > \beta V_{t+1}(\theta, \mathbf{o}_{t-1} \cup \{1\}, \mathbf{A}_{t-1} \cup \{A_t\})$ or $A_t \theta < x$. Since $x < \frac{\beta\theta}{2} \leq \beta V_{t+1}(\theta, \mathbf{o}_{t-1} \cup \{1\}, \mathbf{A}_{t-1} \cup \{A_t\})$ for each $A_t \in [0,1]$ and $\theta \in \text{supp}(\widehat{\Gamma}_t(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}))$, $A_t \in D_t(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}, x)$ if and only if $A_t \theta < x$, that is, $D(\theta, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}, x) = [0, \frac{x}{\theta})$.

The third statement is immediate according to the proof of lemma 2. ■

Claim 2 $\frac{\partial b(\theta)}{\partial \theta} > 0$ *and* $\frac{\partial x^*(\theta)}{\partial \theta} < 0$. *Moreover* $x^*(\theta) < \frac{\beta\theta}{2}$ *for each* $\theta \in \Theta$.

Proof of Claim 2. $\frac{\partial b(\theta)}{\partial \theta} = \frac{\frac{\bar{\theta}}{\theta} - 1 - \ln(\frac{\bar{\theta}}{\theta})}{(\ln \bar{\theta} - \ln \theta)^2} > 0$ for all $\theta < \bar{\theta}$ and $\left.\frac{\partial b(\theta)}{\partial \theta}\right|_{\theta = \bar{\theta}} = \lim_{\theta \to \bar{\theta}} \frac{b(\bar{\theta}) - b(\theta)}{\bar{\theta} - \theta} = \lim_{\theta \to \bar{\theta}} \frac{\partial b(\theta)}{\partial \theta} = \frac{1}{2} > 0$ so that $\frac{\partial x^*(\theta)}{\partial \theta} = \frac{\partial x^*(\theta)}{\partial b(\theta)} \frac{\partial b(\theta)}{\partial \theta} = \left\{1 - \frac{b(\theta) - 2r}{\sqrt{b(\theta)^2 - 4b(\theta)r}}\right\} \frac{\partial b(\theta)}{\partial \theta} < 0$ since $\frac{b(\theta) - 2r}{\sqrt{b(\theta)^2 - 4b(\theta)r}} > 1$. Moreover, since $x^*(\underline{\theta}) < \frac{\beta\underline{\theta}}{2}$ by assumption 1, for each $\theta \in \Theta$, $x^*(\theta) \leq x^*(\underline{\theta}) < \frac{\beta\underline{\theta}}{2} \leq \frac{\beta\theta}{2}$. ■

We complete showing $x = x^*(\bar{\theta})$ in e^* by proving the existence of such e^* at the end of this proof.

We now show that the second statement implies the first statement. The second statement says that if $A_t \hat{\theta} \geq x^*(\hat{\theta})$ then they all survive while if $A_t \hat{\theta} < x^*(\hat{\theta})$ then only the entrepreneurs with the type at least $\frac{x^*(\hat{\theta})}{A_t}$ survive, which results in $\widehat{\Gamma}(\mathbf{o}_{t-1} \cup \{1\}, \mathbf{A}_{t-1} \cup \{A_t\}) = U_{\left[\max\{\hat{\theta}, \frac{x^*(\hat{\theta})}{A_t}\}, \bar{\theta}\right]}$ for each $A_t \in [0,1]$ if $\widehat{\Gamma}(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}) = U_{\left[\hat{\theta}, \bar{\theta}\right]}$. Using this fact, we show the first statement holds by induction on t. First notice that, since every entrepreneur always offers a contract, any operation history at period $t+1$, $\mathbf{o}_t$, satisfies either $\mathbf{o}_t = \mathbf{o}_{t-1} \cup \{1\}$ for some $\mathbf{o}_{t-1} \in \mathcal{O}^*_{t-1}$, or $\mathbf{o}_t = \{\varnothing, \cdots, \varnothing\}$. Moreover, since every newly born entrepreneur's type is randomly assigned according to $U_{[\underline{\theta}, \bar{\theta}]}$, $\widehat{\Gamma}(\{\varnothing, \cdots, \varnothing\}, \mathbf{A}_t) = U_{[\underline{\theta}, \bar{\theta}]}$ for each $\mathbf{A}_t \in \mathbb{A}_t$. So we only need to show that the statement holds for the histories such that $\mathbf{o}_t = \mathbf{o}_{t-1} \cup \{1\}$ for some $\mathbf{o}_{t-1} \in \mathcal{O}^*_{t-1}$ and for all t. If $t = 0$, $\widehat{\Gamma}(\{\varnothing\}, \{\varnothing\}) = U_{[\underline{\theta}, \bar{\theta}]}$ so that $\widehat{\Gamma}(\{\varnothing, 1\}, \{\varnothing, A_0\}) = U_{\left[\max\{\underline{\theta}, \frac{x^*(\hat{\theta})}{A_t}\}, \bar{\theta}\right]}$. Assume that the statement holds for $t = k$, namely, there exists $\hat{\theta}_{\mathbf{o}_{k-1}, \mathbf{A}_{k-1}}$

for all $(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}) \in \mathcal{O}^*_{t-1} \times \mathbb{A}^*_{t-1}$ such that $\widehat{\Gamma}(\mathbf{o}_{k-1}, \mathbf{A}_{k-1}) = U_{\left[\hat{\theta}, \bar{\theta}\right]}$. Then, for each A_t, $\widehat{\Gamma}(\mathbf{o}_{k-1} \cup \{1\}, \mathbf{A}_{k-1} \cup \{A_t\}) = U_{\left[\max\{\hat{\theta}, \frac{x^*(\hat{\theta})}{A_t}\}, \bar{\theta}\right]}(\theta)$.

Now we show the third statement holds given the first and second statement. $\theta_s = \underline{\theta}$ is trivial since the type is randomly assigned according to $U_{[\underline{\theta}, \bar{\theta}]}$ for newly born entrepreneurs. Now consider any $\tau = s+1, \ldots, t$. $\widehat{\Gamma}_\tau(\mathbf{o}_{\tau-1}, \mathbf{A}_{\tau-1} = U_{[\hat{\theta}_{\tau-1}, \bar{\theta}]}$ by the first statement, and therefore each entrepreneur $\theta \in \widehat{\Gamma}_\tau(\mathbf{o}_{\tau-1}, \mathbf{A}_{\tau-1}$ offers $x^*(\hat{\theta}_{\tau-1})$ by the second statement. Finally $\hat{\theta}_\tau = \max\left\{\frac{x^*(\hat{\theta}_{\tau-1})}{A_{\tau-1}}, \hat{\theta}_{\tau-1}\right\}$ by the same logic in the previous paragraph.

To finish the proof, we show the existence of e^* satisfying the second statement in the following claim.

Claim 3 *If the first statement is satisfied, then e^* exists and satisfies the properties of all the statements.*

Proof. According to the first statement, for each t and $(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}) \in \mathcal{O}^*_{t-1} \times \mathbb{A}_{t-1}$ define $\hat{\theta}(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}) \in \Theta$ such that $\widehat{\Gamma}_t(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}) = U_{[\hat{\theta}(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}), \bar{\theta}]}$. Consider a belief system Φ satisfying $\Phi(\theta | x, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) = U_{[\hat{\theta}(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}), \bar{\theta}]}$ if $\mathbf{o}_{t-1} \in \mathcal{O}^*$ and $\Phi(\theta \mid x, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) = U_{[\underline{\theta}, \underline{\theta}]}$ otherwise for every $x \in X$. Also consider the strategy profile in proposition 1.

First of all, by definition of $\mathcal{B}^*_t$, lender's strategy is optimal given Φ. Notice that $\min \mathcal{B}^*_t(\Phi, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) = x^*(\hat{\theta}(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}))$ for each $\mathbf{o}_{t-1} \in \mathcal{O}^*$ since $\Phi(\theta \mid x, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) = U_{[\hat{\theta}(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}), \bar{\theta}]}$ and by the definition of $x^*(\cdot)$ Also, for each $\mathbf{o}_{t-1} \notin \mathcal{O}^*$, if $\mathcal{B}^*_t(\Phi, U_{[\underline{\theta}, \underline{\theta}]}, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) \neq \varnothing$ then $\min \mathcal{B}^*_t(\Phi, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) > \max_{\theta' \in \Theta} x^*(\theta')$ since $U_{[\underline{\theta}, \underline{\theta}]}$ is first order stochastically dominated by $U_{[\theta', \underline{\theta}]}$ for any $\theta' \in \Theta$, and $\mathcal{B}^*_t(\Phi, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) = \varnothing$ otherwise. Using this fact, we show that the entrepreneur's strategy is optimal. By lemma 2, each entrepreneur with any $(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}) \in \mathcal{O}^* \times \mathcal{A}$ offers either $x^*(\hat{\theta}(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}))$ or $\varnothing$. Since newly born entreneur's operation history is in $\mathcal{O}^*$, we are done if we show that any entrepreneur with any $\mathbf{o}_{t-1} \in \mathcal{O}^*$ has no incentive to offer $\varnothing$. If $\mathcal{B}^*_t(\Phi, U_{[\underline{\theta}, \underline{\theta}]}, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) = \varnothing$ then any entrepreneur with $\mathbf{o}_{t-1} \in \mathcal{O}^*$ is has no incentive to offer $\varnothing$ because offering $\varnothing$ results in the operation history belong to $\mathcal{O} \backslash \mathcal{O}^*$ until leaving the market so that the continuation value is zero, while offering any contract to be accepted guarantees a positive continuation value, by claim 1 in the proof of proposition 1. Now suppose $\mathcal{B}_t^*(U_{[\underline{\theta},\bar{\theta}]}, U_{[\underline{\theta},\bar{\theta}]}, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) \neq \varnothing$. Since $\min \mathcal{B}_t^*(\Phi, U_{[\underline{\theta},\bar{\theta}]}, \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) > x^*(\theta)$ for any θ, $\min \mathcal{B}^*(\Phi, \mathbf{o}, \mathbf{A}) < \min \mathcal{B}_t^*(\Phi, U_{[\underline{\theta},\bar{\theta}]}, \mathbf{o}_{t-1}, \mathbf{A}_{t-1})$. Thus, an entrepreneur with $\mathbf{o}_{t-1} \in \mathcal{O}^*$ offering $\varnothing$ has to endure a larger $\min \mathcal{B}_\tau^*$ in the preceeding periods $\tau \geq t+1$, on top of an additional discount factor, so that, by claim 1, it is strictly dominated by offering a contract in $\mathcal{B}_t^*$ in the current period.

We complete the proof by showing that Φ is consistent. According to the strategy profile in which any contract offer is not $\varnothing$, any operation history $\mathbf{o}_{t-1}$ is on the equilibrium path if and only if $\mathbf{o}_{t-1} \in \mathcal{O}^*$ and any contract offer x given $(\mathbf{o}_{t-1}, \mathbf{A}_{t-1})$ is on the equilibrium path if and only if $x = x^*(\hat{\theta}(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}))$. So we only need to show that $\Phi(\theta \mid x^*(\hat{\theta}(\mathbf{o}_{t-1}, \mathbf{A}_{t-1})), \mathbf{o}_{t-1}, \mathbf{A}_{t-1}) = U_{[\hat{\theta}(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}), \bar{\theta}]}$ for all t and $\mathbf{o}_{t-1} \in \mathcal{O}^*$, which is already shown to be true in the proof of proposition 1 given the strategy profile. ■ ■

Proof of lemma 3. Since $\lambda(\theta) = \frac{x^*(\theta)}{b(\theta)} = \frac{b(\theta) - \sqrt{b(\theta)^2 - 4b(\theta)r}}{2b(\theta)} = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{r}{b(\theta)}}$, $\frac{\partial \lambda(\theta)}{\partial b(\theta)} = \frac{\partial}{\partial b(\theta)} \left(1 - \sqrt{1 - \frac{4r}{b(\theta)}}\right) < 0$. Since $\frac{\partial b(\theta)}{\partial \theta} > 0$ by claim 2 in the proof of proposition 1, $\frac{\partial \lambda(\theta)}{\partial \theta} = \frac{\partial \lambda(\theta)}{\partial b(\theta)} \cdot \frac{\partial b(\theta)}{\partial \theta} < 0$. ■

Proof of Proposition 2. Consider any period t and $(\mathbf{o}_{t-1}, \mathbf{A}_{t-1}) \in \mathcal{O}_{t-1} \times \mathbb{A}_{t-1}$. Then, by proposition 1 and letting $s \geq 0$ be the birthdate of $\mathbf{o}_{t-1}$-group entrepreneurs, $\widehat{\Gamma}_\tau(\mathbf{o}_{\tau-1}, \mathbf{A}_{\tau-1}) = U_{[\hat{\theta}_\tau, \bar{\theta}]}$ with $\hat{\theta}_s = \underline{\theta}$ and $\hat{\theta}_\tau = \max\left\{\frac{x^*(\hat{\theta}_{\tau-1})}{A_{\tau-1}}, \hat{\theta}_{\tau-1}\right\}$ for each $\tau = s+1, \ldots t$, and each $\mathbf{o}_{t-1}$-group entrepreneur θ plays $\left(x^*(\hat{\theta}_\tau), \left[0, \frac{x^*(\hat{\theta}_\tau)}{\theta}\right)\right)$ at each period $\tau = s, \ldots, t$. Therefore the credit risk for the lender matched with an entrepreneur in $\widehat{\Gamma}(\mathbf{o}_{\tau-1}, \mathbf{A}_{\tau-1})$ at each period $\tau = s, \ldots t$ is $\lambda(\hat{\theta}_\tau)$. Since $\hat{\theta}_\tau = \max\left\{\frac{x^*(\hat{\theta}_{\tau-1})}{A_{\tau-1}}, \hat{\theta}_{\tau-1}\right\} \geq \hat{\theta}_{\tau-1}$ for each $\tau = s+1, \ldots t$, $\hat{\theta}_\tau$ weakly increases over time. Therefore, by claim 2 in the proof of proposition 1 and lemma 3, both the repayment on the credit contract $x^*(\theta)$ and the credit risk $\lambda(\theta)$ decrease in θ so that they weakly decrease over time. ■

Proof of proposition 3. Let any such $\mathbf{o}_{t-1}^o, \mathbf{o}_{t-1}^y \in \mathcal{O}_{t-1}^*$ in the full production equilibrium. By proposition 1 there exist θ_1, θ_2 such that $\widehat{\Gamma}_t(\mathbf{o}_{t-1}^o, \mathbf{A}_{t-1}) = U_{\left[\theta_1, \bar{\theta}\right]}$ and $\widehat{\Gamma}_t(\mathbf{o}_{t-1}^y, \mathbf{A}_{t-1}) = U_{\left[\theta_2, \bar{\theta}\right]}$. Assume that $\theta_1 > \theta_2$ and let $\theta_1' = \min \operatorname{supp}\left(\widehat{\Gamma}_{t+1}(\mathbf{o}_{t-1}^o \cup \{1\}, \mathbf{A}_{t-1} \cup \{A_t\})\right)$ and $\theta_2' =$

$\min \text{supp}\left(\widehat{\Gamma}_{t+1}(\mathbf{o}^y_{t-1} \cup \{1\}, \mathbf{A}_{t-1} \cup \{A_t\})\right)$. We study the sign of the conditional expectation of $\theta_1' - \theta_2'$ over A_t's such that $\text{supp}\widehat{\Gamma}_t(\mathbf{o}^i_{t-1} \cup \{1\}, \mathbf{A}_{t-1} \cup \{A_t\}) \neq \varnothing$ for each $i \in \{o, y\}$, that is, given that there are survivers who proceed to period $t+1$ in both groups. By proposition 1 each entrepreneur θ with $\mathbf{o}^i_{t-1}$ plays $\left(x^*(\theta_i), \left[0, \frac{x^*(\theta_i)}{\theta}\right)\right)$ and the offer is accepted at period t so that $supp\widehat{\Gamma}_t(\mathbf{o}^i_{t-1} \cup \{1\}, \mathbf{A}_{t-1} \cup \{A_t\}) \neq \varnothing$ for $i \in \{o, y\}$ is equivalent to $\max\left\{\frac{x^*(\theta_1)}{A_t}, \frac{x^*(\theta_2)}{A_t}\right\} \leq \bar{\theta}$, which is in turn equivalent to $A_t \geq \frac{x^*(\theta_2)}{\bar{\theta}}$. We also know from proposition 1 that $\theta_1' = \max\left\{\frac{x^*(\theta_1)}{A_t}, \theta_1\right\}, \theta_2' = \max\left\{\frac{x^*(\theta_2)}{A_t}, \theta_2\right\}$. Notice that $\theta_1' = \frac{x^*(\theta_1)}{A_t}$ when $A_t \in \left[\frac{x^*(\theta_2)}{\bar{\theta}}, \frac{x^*(\theta_1)}{\theta_1}\right]$ and $\theta_1' = \theta_1$ when $A_t \in \left[\frac{x^*(\theta_1)}{\theta_1}, 1\right]$, while $\theta_2' = \frac{x^*(\theta_2)}{A_t}$ when $A_t \in \left[\frac{x^*(\theta_2)}{\bar{\theta}}, \frac{x^*(\theta_2)}{\theta_2}\right]$ and $\theta_2' = \theta_2$ when $A_t \in \left[\frac{x^*(\theta_2)}{\theta_2}, 1\right]$. Moreover it is uncertain whether $\left[\frac{x^*(\theta_2)}{\bar{\theta}}, \frac{x^*(\theta_1)}{\theta_1}\right] = \varnothing$. Let θ^* be such that $\frac{x^*(\theta^*)}{\theta^*}\frac{\bar{\theta}}{x^*(\theta_2)}$. Since $\frac{x^*(\theta)}{\theta}\frac{\bar{\theta}}{x^*(\theta_2)}$ strictly decreases in θ and $\frac{x^*(\theta_2)}{\theta_2}\frac{\bar{\theta}}{x^*(\theta_2)} = \frac{\bar{\theta}}{\theta_2} > 1$, $\theta^* > \theta_2$.

1. Consider that $\frac{x^*(\theta_2)}{\bar{\theta}} \leq \frac{x^*(\theta_1)}{\theta_1}$, that is, $\theta_1 \in (\theta_2, \theta^*]$.

$$
\begin{aligned}
&\left(1 - \frac{x^*(\theta_2)}{\bar{\theta}}\right) E_{A_t}\left[\theta_1' - \theta_2' \mid A_t \geq \frac{x^*(\theta_2)}{\bar{\theta}}\right] \\
&= \int_{\frac{x^*(\theta_2)}{\bar{\theta}}}^{\frac{x^*(\theta_1)}{\theta_1}} \frac{x^*(\theta_1)}{A_t} dA_t + \theta_1\left(1 - \frac{x^*(\theta_1)}{\theta_1}\right) - \int_{\frac{x^*(\theta_2)}{\bar{\theta}}}^{\frac{x^*(\theta_2)}{\theta_2}} \frac{x^*(\theta_2)}{A_t} dA_t - \theta_2\left(1 - \frac{x^*(\theta_2)}{\theta_2}\right) \\
&= (\theta_1 - \theta_2) + (x^*(\theta_2) - x^*(\theta_1)) + x^*(\theta_1)\ln\left(\frac{x^*(\theta_1)}{\theta_1}\frac{\bar{\theta}}{x^*(\theta_2)}\right) - x^*(\theta_2)\ln\frac{\bar{\theta}}{\theta_2}.
\end{aligned}
$$

Denoting $x'(\theta) = \frac{\partial x^*(\theta)}{\partial \theta}$ and $x''(\theta) = \frac{\partial^2 x^*(\theta)}{\partial \theta^2}$, we have $x'(\theta) < 0$ and $\frac{\partial b(\theta)}{\partial \theta} > 0$ by lemma 2. Notice that for each θ, with $u = \frac{\bar{\theta}}{\theta} \geq 1$ and $b'(u) = \frac{u - 1 - \ln u}{(\ln u)^2}$, $\frac{\partial^2 b(\theta)}{\partial \theta^2} = \frac{\partial b'(u)}{\partial u}\frac{\partial u}{\partial \theta} = \left[\left(1 + \frac{1}{u}\right)\ln u - 2\left(1 - \frac{1}{u}\right)\right] \cdot \left(-\frac{\hat{\theta}}{\theta^2(\ln u)}\right) \leq 0$ since $\left(1 + \frac{1}{u}\right)\ln u - 2\left(1 - \frac{1}{u}\right)$ increases in u and $\left(1 + \frac{1}{u}\right)\ln u - 2\left(1 - \frac{1}{u}\right) = 0$ when $u = 1$. Also $x''(\theta) = \frac{1}{2}\left(1 - \frac{b(\theta) - 2r}{\sqrt{b(\theta)^2 - 4b(\theta)r}}\right)\frac{\partial^2 b(\theta)}{\partial \theta^2} + 2r^2\left(b(\theta)^2 - 4b(\theta)\right)^{-\frac{3}{2}}\left(\frac{\partial b(\theta)}{\partial \theta}\right)^2 > 0$ from $1 - \frac{b(\theta) - 2r}{\sqrt{b(\theta)^2 - 4b(\theta)r}} < 0$, $\frac{\partial^2 b(\theta)}{\partial \theta^2} < 0$, and $b(\theta)^2 - 4b(\theta) > 0$.

Define

$$
F(\theta) = (\theta - \theta_2) + (x^*(\theta_2) - x^*(\theta)) + x^*(\theta)\ln\left(\frac{x^*(\theta)}{\theta}\frac{\bar{\theta}}{x^*(\theta_2)}\right) - x^*(\theta_2)\ln\frac{\bar{\theta}}{\theta_2}
$$

for each $\theta \in [\theta_2, \theta^*]$. Notice that $F(\theta_2) = 0$. Also, $F'(\theta) = 1 + x'(\theta) \ln\left(\frac{x^*(\theta)}{\theta} \frac{\bar{\theta}}{x^*(\theta_2)}\right) - \frac{x^*(\theta)}{\theta}$ and $F''(\theta) = x''(\theta) \ln\left(\frac{x^*(\theta)}{\theta} \frac{\bar{\theta}}{x^*(\theta_2)}\right) + (x'(\theta))^2 \cdot \frac{1}{x^*(\theta)} - x'(\theta) \cdot \frac{2}{\theta} + \frac{x^*(\theta)}{\theta^2}$. It is easy to show $F'' > 0$ using claim 2 and $\frac{x^*(\theta)}{\theta} \frac{\bar{\theta}}{x^*(\theta_2)} \geq 1$. That is, $F'(\theta)$ strictly increases in θ. Given $\theta \geq \theta_2$ and $F(\theta_2) = 0$, $F'(\theta_2) \geq 0$ implies $F(\theta) \geq 0$ for all $\theta_1 \in (\theta_2, \theta^*]$. We finish the first part of the proof by showing that $F'(\theta_2) > 0$. $F'(\theta_2) = 1 + x'(\theta_2) \ln\left(\frac{\bar{\theta}}{\theta_2}\right) - \frac{x^*(\theta_2)}{\theta_2}$ increases in θ_2 since $\frac{\partial}{\partial \theta_2}\left[x'(\theta_2) \ln\left(\frac{\bar{\theta}}{\theta_2}\right)\right] = x''(\theta_2) \ln\left(\frac{\bar{\theta}}{\theta_2}\right) - \frac{x'(\theta_2)}{\theta_2} > 0$. Therefore

$$
\begin{aligned}
F'(\theta_2) \geq F'(\underline{\theta}) &= 1 + x'(\underline{\theta}) \ln\left(\frac{\bar{\theta}}{\underline{\theta}}\right) - \frac{x^*(\underline{\theta})}{\underline{\theta}} \\
&= 1 + \frac{1}{2}\left(1 - \frac{b - 2r}{\sqrt{b^2 - 4br}}\right) \cdot \left(\frac{b}{\underline{\theta}} - 1\right) - \frac{b - \sqrt{b^2 - 4br}}{2\underline{\theta}}
\end{aligned}
$$

as we denote $b = b(\underline{\theta}) = \frac{\bar{\theta} - \underline{\theta}}{\ln(\frac{\bar{\theta}}{\underline{\theta}})}$. Notice that $b > 4r$ and $b > \underline{\theta}$. As we define $G(y) = \left(\frac{y - 2r}{\sqrt{y^2 - 4ry}} - 1\right) \cdot (y - \underline{\theta}) + y - \sqrt{y^2 - 4ry}$ for each $y > 4r$, $G'(y) > 0$ for any $y > \max\{4r, \underline{\theta}\}$. Therefore

$$
F'(\theta_2) \geq F'(\underline{\theta}) = 1 - \frac{1}{2\underline{\theta}} G(b) > 1 - \frac{1}{2\underline{\theta}} G(\underline{\theta}) = 1 - \frac{1}{2\underline{\theta}}\left(\underline{\theta} - \sqrt{\underline{\theta}^2 - 4r\underline{\theta}}\right) > 0.
$$

2. If $\frac{x^*(\theta_2)}{\theta} > \frac{x^*(\theta_1)}{\theta_1}$, then

$$
\begin{aligned}
&\left(1 - \frac{x^*(\theta_2)}{\bar{\theta}}\right) E_{A_t}\left[\theta_1' - \theta_2' \mid A_t \geq \frac{x^*(\theta_2)}{\bar{\theta}}\right] \\
&= \theta_1\left(1 - \frac{x^*(\theta_2)}{\bar{\theta}}\right) - \int_{\frac{x^*(\theta_2)}{\bar{\theta}}}^{\frac{x^*(\theta_2)}{\theta_2}} \frac{x^*(\theta_2)}{A_t} dA_t - \theta_2\left(1 - \frac{x^*(\theta_2)}{\theta_2}\right) \\
&= \theta_1 - \theta_2 + x^*(\theta_2)\left[1 - \frac{\theta_1}{\bar{\theta}} - \ln \frac{\bar{\theta}}{\theta_2}\right].
\end{aligned}
$$

Notice that $\theta_1 - \theta_2 + x^*(\theta_2)\left[1 - \frac{\theta_1}{\bar{\theta}} - \ln \frac{\bar{\theta}}{\theta_2}\right]$ strictly increases in θ_1, and θ_1 is restricted to satisfy $\frac{x^*(\theta_2)}{\bar{\theta}} \geq \frac{x^*(\theta_1)}{\theta_1}$. As we plug in the smallest θ_1 in this range, that is, $\frac{x^*(\theta_2)}{\bar{\theta}} = \frac{x^*(\theta_1)}{\theta_1}$,

$$\left(1-\frac{x^*(\theta_2)}{\bar{\theta}}\right) E_{A_t}\left[\theta_1' - \theta_2' \mid A_t \geq \frac{x^*(\theta_2)}{\bar{\theta}} = \frac{x^*(\theta_1)}{\theta_1}\right]$$
$$= \int_{\frac{x^*(\theta_2)}{\bar{\theta}}}^{\frac{x^*(\theta_1)}{\theta_1}} \frac{x^*(\theta_1)}{A_t} dA_t + \theta_1\left(1-\frac{x^*(\theta_1)}{\theta_1}\right) - \int_{\frac{x^*(\theta_2)}{\bar{\theta}}}^{\frac{x^*(\theta_2)}{\theta_2}} \frac{x^*(\theta_2)}{A_t} dA_t - \theta_2\left(1-\frac{x^*(\theta_2)}{\theta_2}\right).$$

Since $E_{A_t}\left[\theta_1' - \theta_2' \mid A_t \geq \frac{x^*(\theta_2)}{\bar{\theta}}\right] > 0$ given $\frac{x^*(\theta_2)}{\bar{\theta}} \leq \frac{x^*(\theta_1)}{\theta_1}$, it is also true when $\frac{x^*(\theta_2)}{\bar{\theta}} > \frac{x^*(\theta_1)}{\theta_1}$.

■

Proof of proposition 4. First consider the case $\widetilde{A} \in \left[0, \frac{x^*(\underline{\theta})}{\bar{\theta}}\right] \cup \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$. Assume that there is a period t such that $\Omega_t = U_{[\underline{\theta},\bar{\theta}]}$. Then, by proposition 1, each θ at period t plays $\left(x^*(\underline{\theta}), \left[0, \frac{x^*(\underline{\theta})}{\theta}\right)\right)$. If $\widetilde{A} \in \left[0, \frac{x^*(\underline{\theta})}{\bar{\theta}}\right)$ then all the entrepreneurs default at the realized $\widetilde{A}$ and if $\widetilde{A} \in \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$ then every entrepreneur survives to the next period given the realized $\widetilde{A}$ so that $\Omega_{t+1} = U_{[\underline{\theta},\bar{\theta}]}$. If $\widetilde{A} = \frac{x^*(\underline{\theta})}{\bar{\theta}}$ then θ survives if and only if $\theta = \bar{\theta}$ so that the mass of the defaulted entrepreneurs at period 0 is 1, that is, $\Omega_{t+1} = U_{[\underline{\theta},\bar{\theta}]}$. Thus, for any $\widetilde{A} \in \left[0, \frac{x^*(\underline{\theta})}{\bar{\theta}}\right] \cup \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$, $\Omega_t = U_{[\underline{\theta},\bar{\theta}]}$ implies $\Omega_{t+1} = \Omega_t$. Since $\Omega_0 = U_{[\underline{\theta},\bar{\theta}]}$, $\Omega_t = U_{[\underline{\theta},\bar{\theta}]}$ for all $t \geq 0$. Therefore the total production by entrepreneurs at each period t is $\int_{[\underline{\theta},\bar{\theta}]} \widetilde{A}\theta dU_{[\underline{\theta},\bar{\theta}]} = \frac{1}{2}\widetilde{A}(\bar{\theta}+\underline{\theta})$.

Now consider that $\widetilde{A} \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$. For any group of entrepreneurs with any mass M and the type distribution $U_{[\underline{\theta},\bar{\theta}]}$, each θ defaults if and only if $\widetilde{A}\theta < x^*(\underline{\theta})$, according to proposition 1. Thus, the survivers from this group are of mass $\frac{\bar{\theta}-\frac{x^*(\underline{\theta})}{\widetilde{A}}}{\bar{\theta}-\underline{\theta}}M$, and their type distribution is $U_{\left[\frac{x^*(\underline{\theta})}{\widetilde{A}},\bar{\theta}\right]}$ in the next period, while there are newly born entrepreneurs of mass $\frac{\frac{x^*(\underline{\theta})}{\widetilde{A}}-\underline{\theta}}{\bar{\theta}-\underline{\theta}}M$, with the type distribution $U_{[\underline{\theta},\bar{\theta}]}$, due to the defaulters. Notice that $\frac{\frac{x^*(\underline{\theta})}{\widetilde{A}}-\underline{\theta}}{\bar{\theta}-\underline{\theta}} \in (0,1)$ since $\frac{x^*(\underline{\theta})}{\widetilde{A}} \in (\underline{\theta},\bar{\theta})$. Each surviver θ in the proceeding period plays $\left(x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right), \left[0, \frac{x^*(\underline{\theta})}{\theta}\right)\right)$ and they all survive again since $x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right) \leq x^*(\underline{\theta}) = \widetilde{A} \cdot \frac{x^*(\underline{\theta})}{\widetilde{A}}$ by claim 2 in the proof of proposition 1 applying $\frac{x^*(\underline{\theta})}{\widetilde{A}} \geq x^*(\underline{\theta})$, which indicates that all the entrepreneurs $\theta \geq \frac{x^*(\underline{\theta})}{\widetilde{A}}$ can afford the repayment $x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)$ under the aggregate production $\widetilde{A}$. Using this fact and $\Omega_0 = U_{[\underline{\theta},\bar{\theta}]}$, Ω_t consists of $U_{[\underline{\theta},\bar{\theta}]}$ with mass $\left(\frac{\frac{x^*(\underline{\theta})}{\widetilde{A}}-\underline{\theta}}{\bar{\theta}-\underline{\theta}}\right)^t$ and $U_{\left[\frac{x^*(\underline{\theta})}{\widetilde{A}},\bar{\theta}\right]}$ with mass $1-\left(\frac{\frac{x^*(\underline{\theta})}{\widetilde{A}}-\underline{\theta}}{\bar{\theta}-\underline{\theta}}\right)^t$, that is,

$$
\begin{aligned}
\Omega_t(\theta) &= \mathbf{1}_{\{\theta\in[\underline{\theta},\bar{\theta}]\}}\left(\frac{\frac{x^*(\underline{\theta})}{\widetilde{A}}-\underline{\theta}}{\bar{\theta}-\underline{\theta}}\right)^t U_{[\underline{\theta},\bar{\theta}]}(\theta)+\mathbf{1}_{\{\theta\in[\frac{x^*(\underline{\theta})}{\widetilde{A}},\bar{\theta}]\}}\left(1-\left(\frac{\frac{x^*(\underline{\theta})}{\widetilde{A}}-\underline{\theta}}{\bar{\theta}-\underline{\theta}}\right)^t\right)U_{[\frac{x^*(\underline{\theta})}{\widetilde{A}},\bar{\theta}]}(\theta) \\
&= \begin{cases} \left(\frac{\frac{x^*(\underline{\theta})}{\widetilde{A}}-\underline{\theta}}{\bar{\theta}-\underline{\theta}}\right)^t\cdot\frac{\theta-\underline{\theta}}{\bar{\theta}-\underline{\theta}} & \text{if } \theta\in\left[\underline{\theta},\frac{x^*(\underline{\theta})}{\widetilde{A}}\right) \\ \left(\frac{\frac{x^*(\underline{\theta})}{\widetilde{A}}-\underline{\theta}}{\bar{\theta}-\underline{\theta}}\right)^t\cdot\frac{\theta-\underline{\theta}}{\bar{\theta}-\underline{\theta}}+(1-\left(\frac{\frac{x^*(\underline{\theta})}{\widetilde{A}}-\underline{\theta}}{\bar{\theta}-\underline{\theta}}\right)^t)\frac{\theta\widetilde{A}-x^*(\underline{\theta})}{\bar{\theta}\widetilde{A}-x^*(\underline{\theta})} & \text{if } \theta\in\left[\frac{x^*(\underline{\theta})}{\widetilde{A}},\bar{\theta}\right]. \end{cases}
\end{aligned}
$$

Thus

$$
\begin{aligned}
Y_t &= \left(\frac{\frac{x^*(\underline{\theta})}{\widetilde{A}}-\underline{\theta}}{\bar{\theta}-\underline{\theta}}\right)^t\int_{[\underline{\theta},\bar{\theta}]}\widetilde{A}\theta dU_{[\underline{\theta},\bar{\theta}]}+\left(1-\left(\frac{\frac{x^*(\underline{\theta})}{\widetilde{A}}-\underline{\theta}}{\bar{\theta}-\underline{\theta}}\right)^t\right)\int_{[\frac{x^*(\underline{\theta})}{\widetilde{A}},\bar{\theta}]}\widetilde{A}\theta dU_{[\frac{x^*(\underline{\theta})}{\widetilde{A}},\bar{\theta}]} \\
&= \left(\frac{\frac{x^*(\underline{\theta})}{\widetilde{A}}-\underline{\theta}}{\bar{\theta}-\underline{\theta}}\right)^t\frac{1}{2}\widetilde{A}(\bar{\theta}+\underline{\theta})+\left(1-\left(\frac{\frac{x^*(\underline{\theta})}{\widetilde{A}}-\underline{\theta}}{\bar{\theta}-\underline{\theta}}\right)^t\right)\frac{1}{2}\left(\widetilde{A}\bar{\theta}+x^*(\underline{\theta})\right).
\end{aligned}
$$

■

Proof of proposition 6. First, if either *i)* $\widetilde{A}\in\left(0,\frac{x^*(\bar{\theta})}{\underline{\theta}}\right]$, which implies $A'\in\left[0,\frac{x^*(\bar{\theta})}{\underline{\theta}}\right)$, or *ii)* $\widetilde{A}\in\left[\frac{x^*(\underline{\theta})}{\underline{\theta}},1\right]$ and $A'\in\left[0,\frac{x^*(\underline{\theta})}{\bar{\theta}}\right)\cup\left[\frac{x^*(\underline{\theta})}{\underline{\theta}},\widetilde{A}\right)$, then by proposition 4.1, $\Omega_{s+1}=U_{[\underline{\theta},\bar{\theta}]}$. Proposition 4.1 in turn indicates that $\Omega_t=U_{[\underline{\theta},\bar{\theta}]}$, and therefore $Y_t(\widetilde{\mathbf{A}}')=\frac{\widetilde{A}(\underline{\theta}+\bar{\theta})}{2}$ for all $t\geq s+1$.

Now consider the case that $\widetilde{A}\in\left[\frac{x^*(\underline{\theta})}{\underline{\theta}},1\right]$ and $A'\in\left(\frac{x^*(\underline{\theta})}{\bar{\theta}},\frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$. According to the proof of proposition 4, Ω_{s+1} consists of $U_{[\underline{\theta},\bar{\theta}]}$ with mass $\frac{\bar{\theta}-\frac{x^*(\underline{\theta})}{A'}}{\bar{\theta}-\underline{\theta}}$ and $U_{[\frac{x^*(\underline{\theta})}{A'},\bar{\theta}]}$ with mass $1-\frac{\bar{\theta}-\frac{x^*(\underline{\theta})}{A'}}{\bar{\theta}-\underline{\theta}}$ so that each entrepreneur θ offers either $x^*(\underline{\theta})$ or $x^*(\frac{x^*(\underline{\theta})}{A'})$. Since $A_{s+1}=\widetilde{A}\geq\frac{x^*(\underline{\theta})}{\underline{\theta}}$ so that $\widetilde{A}\underline{\theta}\geq x^*(\underline{\theta})\geq\frac{x^*(\underline{\theta})}{\widetilde{A}}$, all the entrepreneurs at period $s+1$ survive and therefore $\Omega_{s+1}=\Omega_t$ for all $t\geq s+1$. So $\hat{Y}_t(\mathbf{A}')=\frac{\frac{x^*(\underline{\theta})}{A'}-\underline{\theta}}{\bar{\theta}-\underline{\theta}}\cdot\frac{1}{2}\widetilde{A}(\bar{\theta}+\underline{\theta})+\frac{\bar{\theta}-\frac{x^*(\underline{\theta})}{A'}}{\bar{\theta}-\underline{\theta}}\cdot\frac{1}{2}\widetilde{A}\left(\bar{\theta}+\frac{x^*(\underline{\theta})}{A'}\right)$ for all $t\geq s+1$.

Finally, consider the case that $\widetilde{A}\in\left(\frac{x^*(\underline{\theta})}{\bar{\theta}},\frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$. By proposition 4.2 $\Omega_s=U_{[\frac{x^*(\underline{\theta})}{\widetilde{A}},\bar{\theta}]}$ and every entrepreneur offers $x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)$ at period s. If $A'\cdot\frac{x^*(\underline{\theta})}{\widetilde{A}}\geq x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)$ then all the entrepreneurs survive so that $\Omega_t=U_{[\frac{x^*(\underline{\theta})}{\widetilde{A}},\bar{\theta}]}$ for all $t\geq s+1$, and therefore, by proposition 4, $Y_t(\widetilde{\mathbf{A}}')=\frac{\widetilde{A}\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}+\bar{\theta}\right)}{2}$. To finish the proof, we derive $Y_t(\widetilde{\mathbf{A}}')$ for $t\geq s+1$ when $A'\in\left[0,\frac{x^*(\widetilde{\theta})}{\widetilde{\theta}}\right)$.

If $A'\bar{\theta} < x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right) = x^*(\widetilde{\theta})$ then all the entrepreneurs default so that $\Omega_{s+1} = U_{[\underline{\theta},\bar{\theta}]}$ therefore, by proposition 4.2,

$$\widehat{Y}_t\left(\widetilde{\mathbf{A}}\right) = \triangle^{t-s-1}\frac{\widetilde{A}(\underline{\theta}+\bar{\theta})}{2} + \left[1-\triangle^{t-s-1}\right]\frac{x^*(\underline{\theta})+\widetilde{A}\bar{\theta}}{2} \tag{17}$$

for all $t \geq s+1$. Now consider the case that $A' \in \left[\frac{x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)}{\bar{\theta}}, \frac{\widetilde{A}\cdot x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)}{x^*(\underline{\theta})}\right) = \left[\frac{x^*(\widetilde{\theta})}{\bar{\theta}}, \frac{x^*(\widetilde{\theta})}{\widetilde{\theta}}\right)$. Then each entrepreneur θ defaults if $\theta \in \left[\frac{x^*(\underline{\theta})}{\widetilde{A}}, \frac{x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)}{A'}\right) = \left[\widetilde{\theta}, \frac{x^*(\widetilde{\theta})}{A'}\right)$ and survives if $\theta \in \left[\frac{x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)}{A'}, \bar{\theta}\right] = \left[\frac{x^*(\widetilde{\theta})}{A'}, \bar{\theta}\right]$, and the mass of entrepreneurs that default is $\frac{\frac{x^*(\widetilde{\theta})}{A'}-\widetilde{\theta}}{\bar{\theta}-\widetilde{\theta}} \equiv p(A')$. Notice that $p(A') \in (0,1]$ if $A' \in \left[\frac{x^*(\widetilde{\theta})}{\bar{\theta}}, \frac{x^*(\widetilde{\theta})}{\widetilde{\theta}}\right)$ and $p(A') > 1$ if $A' < \frac{x^*(\widetilde{\theta})}{\bar{\theta}}$. Ω_{s+1} is a combination of $U_{[\underline{\theta},\bar{\theta}]}$ with mass $p(A')$ and $U_{\left[\frac{x^*(\widetilde{\theta})}{A'},\bar{\theta}\right]}$ with mass $1-p(A')$. By proposition 4.2, at period t with $t \geq s+1$, Ω_t is a combination of $U_{[\underline{\theta},\bar{\theta}]}$ with mass $p(A')\cdot\frac{\widetilde{\theta}-\underline{\theta}}{\bar{\theta}-\underline{\theta}} = p(A'^{t-s-1}$, $U_{[\widetilde{\theta},\bar{\theta}]}$ with mass $p(A'^{t-s-1})$, and $U_{\left[\frac{x^*(\widetilde{\theta})}{A'},\bar{\theta}\right]}$ with mass $1-p(A')$ so that

$$\begin{aligned} Y_t(\widetilde{\mathbf{A}}') = {} & p(A')\left[\triangle^{t-s-1}\frac{\widetilde{A}(\underline{\theta}+\bar{\theta})}{2} + \left[1-\triangle^{t-s-1}\right]\frac{x^*(\underline{\theta})+\widetilde{A}\bar{\theta}}{2}\right] \\ & + (1-p(A'))\frac{\widetilde{A}}{2}\left[\frac{x^*(\widetilde{\theta})}{A'}+\bar{\theta}\right]. \end{aligned} \tag{18}$$

From (17) and (18), when $A' \in \left[0, \frac{x^*(\widetilde{\theta})}{\bar{\theta}}\right)$

$$Y_t(\widetilde{\mathbf{A}}') = \widetilde{\triangle}'\left\{\triangle^{t-s-1}\frac{\widetilde{A}(\underline{\theta}+\bar{\theta})}{2} + \left[1-\triangle^{t-s-1}\right]\frac{x^*(\underline{\theta})+\widetilde{A}\bar{\theta}}{2}\right\} + \left[1-\widetilde{\triangle}'\right]\frac{\widetilde{A}}{2}\left(\frac{x^*(\widetilde{\theta})}{A'}+\bar{\theta}\right).$$

■

Proof of proposition 7. First notice that $\sum_{t=0}^{s-1}\beta^t\widehat{Y}_t(\widetilde{\mathbf{A}}') = \sum_{t=0}^{s-1}\beta^t\widehat{Y}_t(\widetilde{\mathbf{A}})$ so that $\sum_{t=0}^{\infty}\beta^t[\widehat{Y}_t(\widetilde{\mathbf{A}}') - \widehat{Y}_t(\widetilde{\mathbf{A}})] = \sum_{t=s}^{\infty}\beta^t[\widehat{Y}_t(\widetilde{\mathbf{A}}') - \widehat{Y}_t(\widetilde{\mathbf{A}})]$. Moreover, since we assume a stationary equilibrium in period $s' < s$ and $A' < \widetilde{A}$, $\widehat{Y}_s(\widetilde{\mathbf{A}}') < \widehat{Y}_s(\widetilde{\mathbf{A}})$. Thus if $\widehat{Y}_t(\widetilde{\mathbf{A}}') \leq \widehat{Y}_t(\widetilde{\mathbf{A}})$ for all $t \geq s+1$, then $\sum_{t=0}^{\infty}\beta^t[\hat{Y}_t(\mathbf{A}') - \hat{Y}_t(\widetilde{\mathbf{A}})] < 0$. So, from proposition 6, it is enough to consider when either *i)*

$\widetilde{A} \in \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$ and $A' \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$, or *ii)* $\widetilde{A} \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$ and $A' \in \left[0, \frac{x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)}{\frac{x^*(\underline{\theta})}{\widetilde{A}}}\right)$. That is, $\widetilde{A} \in \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$ implies $I(\widetilde{A}, \beta) \subset \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$ and $\widetilde{A} \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$ implies $I(\widetilde{A}, \beta) \subset \left[0, \frac{x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)}{\frac{x^*(\underline{\theta})}{\widetilde{A}}}\right)$.

First consider when $\widetilde{A} \in \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$ and $A' \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$. Using proposition 6, $\hat{Y}_t(\mathbf{A}') = \hat{Y}_t(\widetilde{\mathbf{A}}) + \frac{\bar{\theta} - \frac{x^*(\underline{\theta})}{A'}}{\bar{\theta} - \underline{\theta}} \cdot \frac{\widetilde{A}}{2}\left(\frac{x^*(\underline{\theta})}{A'} - \underline{\theta}\right)$ for all $t \geq s+1$ so that

$$\frac{1}{\beta^{s+1}} \sum_{t=0}^{\infty} \beta^t [\hat{Y}_t(\mathbf{A}') - \hat{Y}_t(\widetilde{\mathbf{A}})] = (A' - \widetilde{A}) \frac{\bar{\theta} + \underline{\theta}}{2} + \frac{\beta}{1-\beta} \cdot \frac{\bar{\theta} - \frac{x^*(\underline{\theta})}{A'}}{\bar{\theta} - \underline{\theta}} \cdot \widetilde{A}\left(\frac{x^*(\underline{\theta})}{A'} - \underline{\theta}\right).$$

Notice that $\frac{\beta}{1-\beta} \cdot \frac{\bar{\theta} - \frac{x^*(\underline{\theta})}{A'}}{\bar{\theta} - \underline{\theta}} \cdot \widetilde{A}\left(\frac{x^*(\underline{\theta})}{A'} - \underline{\theta}\right) > 0$ since $A' < \frac{x^*(\underline{\theta})}{\underline{\theta}}$, and $A' - \widetilde{A} < 0$. So, after a rearrangement, when $\widetilde{A} \in \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$ and $A' \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$, $\sum_{t=0}^{\infty} \beta^t [\hat{Y}_t(\mathbf{A}') - \hat{Y}_t(\widetilde{\mathbf{A}})] > 0$ if and only if $\beta > \frac{\bar{\theta}^2 - \underline{\theta}^2}{\bar{\theta}^2 - \underline{\theta}^2 + \frac{\widetilde{A}}{\widetilde{A} - A'} \cdot \left(\bar{\theta} - \frac{x^*(\underline{\theta})}{A'}\right)\left(\frac{x^*(\underline{\theta})}{A'} - \underline{\theta}\right)}$. Notice that, since $\frac{\bar{\theta}^2 - \underline{\theta}^2}{\bar{\theta}^2 - \underline{\theta}^2 + \frac{\widetilde{A}}{\widetilde{A} - A'} \cdot \left(\bar{\theta} - \frac{x^*(\underline{\theta})}{A'}\right)\left(\frac{x^*(\underline{\theta})}{A'} - \underline{\theta}\right)} < 1$, for each $\widetilde{A} \in \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$, there exists such β, which means that $I(\widetilde{A}, \beta)$ is nonempty. We show that if $I(\widetilde{A}, \beta)$ is nonempty then it is an open interval. Notice that $A' \in I(\widetilde{A}, \beta)$ if and only if $F_1(A') = A'^2(A' - \widetilde{A}) \frac{\bar{\theta} + \underline{\theta}}{2} + \frac{\beta}{1-\beta} \cdot \frac{A'\bar{\theta} - x^*(\underline{\theta})}{\bar{\theta} - \underline{\theta}} \cdot \widetilde{A}\left(x^*(\underline{\theta}) - A'\underline{\theta}\right) > 0$, where $F_1(A')$ is a cubic function of A'. Also $F_1\left(\frac{x^*(\underline{\theta})}{\bar{\theta}}\right) < 0$ and $F_1\left(\frac{x^*(\underline{\theta})}{\underline{\theta}}\right) < 0$ so if $F_1(A')$ is monotone in $A' \in \left[\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right]$ then $I(\widetilde{A}, \beta) = \varnothing$. Therefore there exist $A'_1, A'_2 \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$ such that $A'_1 < A'_2$, $F'_1(A'_1) = F'_1(A'_2) = 0$, $F_1(A'_1) > 0$, $F_1(A'_2) < 0$, $F'_1(A') > 0$ for $A' \in \left[\frac{x^*(\underline{\theta})}{\bar{\theta}}, A'_1\right) \cup \left(A'_2, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right]$, and $F'_1(A') < 0$ for $A' \in (A'_1, A'_2)$. Therefore there exist $A''_1 \in \left[\frac{x^*(\underline{\theta})}{\bar{\theta}}, A'_1\right)$ and $A''_2 \in (A'_1, A'_2)$ such that $F_1(A''_1) = F_1(A''_2) = 0$ so that $I(\widetilde{A}, \beta) = (A''_1, A''_2)$, which implies that $I(\widetilde{A}, \beta)$ is an open interval.

To show the property of $I(\widetilde{A}, \beta)$ when $\widetilde{A} \in \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$, let any $\widetilde{A}_1, \widetilde{A}_2 \in \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$ such that $\widetilde{A}_2 > \widetilde{A}_1$. If $A' \in I(\widetilde{A}_2, \beta)$ then, since $\frac{\widetilde{A}}{\widetilde{A} - A'}$ decreases in $\widetilde{A}$,

$$\beta > \frac{\bar{\theta}^2 - \underline{\theta}^2}{\bar{\theta}^2 - \underline{\theta}^2 + \frac{\widetilde{A}_2}{\widetilde{A}_2 - A'} \cdot \left(\bar{\theta} - \frac{x^*(\underline{\theta})}{A'}\right)\left(\frac{x^*(\underline{\theta})}{A'} - \underline{\theta}\right)} > \frac{\bar{\theta}^2 - \underline{\theta}^2}{\bar{\theta}^2 - \underline{\theta}^2 + \frac{\widetilde{A}_1}{\widetilde{A}_1 - A'} \cdot \left(\bar{\theta} - \frac{x^*(\underline{\theta})}{A'}\right)\left(\frac{x^*(\underline{\theta})}{A'} - \underline{\theta}\right)}$$

so that $A' \in I(\widetilde{A}_1, \beta)$, which implies that $I(\widetilde{A}_2, \beta) \subset I(\widetilde{A}_1, \beta)$.

Second, let any $\widetilde{A} \in \left(\frac{x^*(\theta)}{\bar{\theta}}, \frac{x^*(\theta)}{\underline{\theta}}\right]$ and $A' \in \left[0, \frac{x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)}{\frac{x^*(\underline{\theta})}{\widetilde{A}}}\right)$. By the proof of proposition 6, if $A'\bar{\theta} < x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)$ then $\widehat{Y}_t\left(\widetilde{\mathbf{A}}\right) = \triangle^{t-s-1}\frac{\widetilde{A}(\underline{\theta}+\bar{\theta})}{2} + [1-\triangle^{t-s-1}]\frac{x^*(\underline{\theta})+\widetilde{A}\bar{\theta}}{2} \leq \widehat{Y}_t(\widetilde{\mathbf{A}})$ where $\triangle = \frac{\frac{x^*(\underline{\theta})}{\widetilde{A}}-\underline{\theta}}{\bar{\theta}-\underline{\theta}} \in (0,1]$, which is not the case of our interest. That is, $\widetilde{A} \in \left(\frac{x^*(\theta)}{\bar{\theta}}, \frac{x^*(\theta)}{\underline{\theta}}\right)$ implies $I(\widetilde{A},\beta) \subset \left[\frac{x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)}{\bar{\theta}}, \frac{x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)}{\frac{x^*(\underline{\theta})}{\widetilde{A}}}\right)$. Now we consider that $A' \in \left[\frac{x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)}{\bar{\theta}}, \frac{x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)}{\frac{x^*(\underline{\theta})}{\widetilde{A}}}\right)$. By proposition 6, letting $p(A') = \frac{\frac{x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)}{A'} - \frac{x^*(\underline{\theta})}{\widetilde{A}}}{\bar{\theta}-\frac{x^*(\underline{\theta})}{\widetilde{A}}}$, $Y_t(\widetilde{\mathbf{A}}') = p(A')\left\{\triangle^{t-s-1}\frac{\widetilde{A}(\underline{\theta}+\bar{\theta})}{2} + [1-\triangle^{t-s-1}]\frac{x^*(\underline{\theta})+\widetilde{A}\bar{\theta}}{2}\right\} +$ $[1-p(A')]\frac{\widetilde{A}}{2}\left(\frac{x^*(\widetilde{\theta})}{A'}+\bar{\theta}\right)$. After a rearrangement, we can express $\beta^{t-s-1}\hat{Y}_t(\mathbf{A}')$ as follows:

$$\frac{\beta^{t-s-1}}{2}\left[\widetilde{A}\bar{\theta} + x^*(\underline{\theta}) + (1-p(A'))\left(x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)\frac{\widetilde{A}}{A'} - x^*(\underline{\theta})\right)\right] - \frac{(\beta\triangle)^{t-s-1}}{2}p(A'^*(\underline{\theta}) - \widetilde{A}\underline{\theta}].$$

Notice that $x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)\frac{\widetilde{A}}{A'} - x^*(\underline{\theta}) > 0$ and $x^*(\underline{\theta}) - \widetilde{A}\underline{\theta} > 0$. So

$$\begin{aligned}
\sum_{t=s}^{\infty}\beta^{t-s}\hat{Y}_t(\mathbf{A}') &= \hat{Y}_s(\mathbf{A}') + \beta\sum_{t=s+1}^{\infty}\beta^{t-s-1}\hat{Y}_t(\mathbf{A}') \\
&= \frac{1}{2}(A\bar{\theta} + x^*(\underline{\theta})) - \frac{\beta p(A')}{1-\beta\triangle}\cdot\frac{1}{2}[x^*(\underline{\theta}) - \widetilde{A}\underline{\theta}] \\
&+ \frac{\beta}{1-\beta}\cdot\frac{1}{2}\left[\widetilde{A}\bar{\theta} + x^*(\underline{\theta}) + (1-p(A'))\left(x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)\frac{\widetilde{A}}{A'} - x^*(\underline{\theta})\right)\right].
\end{aligned}$$

Using this result, define $F_2(A')$ for each $A' \in \left[\frac{x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)}{\bar{\theta}}, \frac{x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)}{\frac{x^*(\underline{\theta})}{\widetilde{A}}}\right] \equiv [A_m, A_M]$:

$$\begin{aligned}
F_2(A') \equiv \frac{1}{\beta^s}\left[\sum_{t=0}^{\infty}\beta^t\hat{Y}_t(\mathbf{A}') - \sum_{t=0}^{\infty}\beta^t\hat{Y}_t(\widetilde{\mathbf{A}})\right] &= \hat{Y}_s(\mathbf{A}') - \hat{Y}_s(\widetilde{\mathbf{A}}) + \beta\sum_{t=s+1}^{\infty}\beta^{t-s-1}\hat{Y}_t(\mathbf{A}') - \beta\sum_{t=s+1}^{\infty}\beta^{t-s-1}\hat{Y}_t(\widetilde{\mathbf{A}}) \\
&= \frac{\bar{\theta}}{2}(A' - \widetilde{A}) - \frac{\beta}{1-\beta\triangle}\cdot\frac{p(A')}{2}[x^*(\underline{\theta}) - \widetilde{A}\underline{\theta}] \qquad (19) \\
&+ \frac{\beta}{1-\beta}\cdot\frac{1-p(A')}{2}\left(x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)\frac{\widetilde{A}}{A'} - x^*(\underline{\theta})\right) \qquad (20)
\end{aligned}$$

Using $\frac{\partial p(A')}{\partial A'} = \frac{x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)}{\bar{\theta}-\frac{x^*(\underline{\theta})}{\widetilde{A}}}\cdot\left(-\frac{1}{A'^2}\right) = -\frac{x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)\frac{\widetilde{A}}{A'}}{\widetilde{A}\bar{\theta}-x^*(\underline{\theta})}\cdot\frac{1}{A'}$, we have $\frac{\partial(19)}{\partial A} = \frac{\bar{\theta}}{2} + \frac{\beta}{2(1-\beta\triangle)}\cdot\frac{x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)\frac{\widetilde{A}}{A'}}{\widetilde{A}\bar{\theta}-x^*(\underline{\theta})}\cdot$

$\frac{1}{A'} \cdot [x^*(\underline{\theta}) - \widetilde{A}\underline{\theta}] > 0$ and

$$\frac{\partial(20)}{\partial A} = \frac{\beta}{2(1-\beta)} \left[-\frac{\partial p(A')}{\partial A'} \cdot \left(x^* \left(\frac{x^*(\underline{\theta})}{\widetilde{A}} \right) \frac{\widetilde{A}}{A'} - x^*(\underline{\theta}) \right) + (1 - p(A'^* \left(\frac{x^*(\underline{\theta})}{\widetilde{A}} \right) \widetilde{A} \cdot \left(-\frac{1}{A'^2} \right) \right]$$

$$= \frac{\beta}{2(1-\beta)} \cdot \frac{x^* \left(\frac{x^*(\underline{\theta})}{\widetilde{A}} \right) \frac{\widetilde{A}}{A'}}{\widetilde{A}\bar{\theta} - x^*(\underline{\theta})} \cdot \frac{1}{A'} \cdot \left[\left(x^* \left(\frac{x^*(\underline{\theta})}{\widetilde{A}} \right) \frac{\widetilde{A}}{A'} - x^*(\underline{\theta}) \right) + \left(x^* \left(\frac{x^*(\underline{\theta})}{\widetilde{A}} \right) \frac{\widetilde{A}}{A'} - \widetilde{A}\bar{\theta} \right) \right],$$

where $x^* \left(\frac{x^*(\underline{\theta})}{\widetilde{A}} \right) \frac{\widetilde{A}}{A'} - x^*(\underline{\theta}) > 0$ but $x^* \left(\frac{x^*(\underline{\theta})}{\widetilde{A}} \right) \frac{\widetilde{A}}{A'} - \widetilde{A}\bar{\theta} \leq 0$. $F_2'(A_m) > 0$ since $A' = A_m = \frac{x^* \left(\frac{x^*(\underline{\theta})}{\widetilde{A}} \right)}{\bar{\theta}}$ implies $x^* \left(\frac{x^*(\underline{\theta})}{\widetilde{A}} \right) \frac{\widetilde{A}}{A'} - \widetilde{A}\bar{\theta} = 0$. Moreover, it is obvious that $F_2''(A') < 0$. Also notice that $F_2(A_m) = \frac{\bar{\theta}}{2} \left(x^* \left(\frac{x^*(\underline{\theta})}{\widetilde{A}} \right) / \bar{\theta} - \widetilde{A} \right) - \frac{\beta}{1-\beta\triangle} \cdot \frac{1}{2}[x^*(\underline{\theta}) - \widetilde{A}\underline{\theta}] < 0$ and $F_2(A_M) = \frac{\bar{\theta}}{2} \left(\widetilde{A} \cdot \frac{x^* \left(\frac{x^*(\underline{\theta})}{\widetilde{A}} \right)}{x^*(\underline{\theta})} - \widetilde{A} \right) < 0$. It implies that, if $F_2'(A_M) \geq 0$ then $F_2(A') \leq 0$ for all $A' \in [A_m, A_M]$. On the other hand, if $F_2'(A_M) < 0$ then there exists $A^* \in (A_m, A_M)$ such that $F_2'^*) = 0$, and therefore, $F_2(A')$ is maximized at A'^*. Therefore $I(\widetilde{A}, \beta)$ is nonempty as an open subinterval of $[A_m, A_M]$ if and only if $F_2'(A_M) < 0$ and $F_2(A^*) > 0$. We show that there exists a parametric condition that enables $F_2'(A_M) > 0$ and $F_2(A^*) > 0$. First, $F_2'(A_M) = \frac{\bar{\theta}}{2} + \frac{\beta}{1-\beta} \frac{x^*(\underline{\theta})}{\widetilde{A}\bar{\theta} - x^*(\underline{\theta})} \frac{\frac{x^*(\underline{\theta})}{\widetilde{A}}}{x^* \left(\frac{x^*(\underline{\theta})}{\widetilde{A}} \right)} \left[-\frac{\widetilde{A}\bar{\theta} - x^*(\underline{\theta})}{2} + \frac{1-\beta}{1-\beta\triangle} \frac{x^*(\underline{\theta}) - \widetilde{A}\underline{\theta}}{2} \right]$. Notice that $\widetilde{A}\underline{\theta} < x^*(\underline{\theta}) \leq \widetilde{A}\bar{\theta}$ and $\frac{1-\beta}{1-\beta\triangle} \in (0,1)$. So a large enough β and $\widetilde{A}$ enables the second term's absolute value relatively larger than the first term, and $-\frac{\widetilde{A}\bar{\theta} - x^*(\underline{\theta})}{2} + \frac{1-\beta}{1-\beta\triangle} \frac{x^*(\underline{\theta}) - \widetilde{A}\underline{\theta}}{2} < 0$, which guarantees $F_2'(A_M) < 0$. Also,

$$\begin{aligned}
F_2(A^*) &= \frac{\bar{\theta}}{2}(A^* - \widetilde{A}) - \frac{1 - p(A^*)}{4} \frac{A^*\bar{\theta}(\widetilde{A}\bar{\theta} - x^*(\underline{\theta}))}{x^* \left(\frac{x^*(\underline{\theta})}{\widetilde{A}} \right) \frac{\widetilde{A}}{A^*}} \\
&\quad + \beta \left[\frac{1}{1-\beta} \cdot \frac{1 - p(A^*)}{4} \frac{\widetilde{A}\bar{\theta} - x^*(\underline{\theta})}{2} - \frac{1}{1-\beta\triangle} \frac{1 + p(A^*)}{4} \frac{x^*(\underline{\theta}) - \widetilde{A}\underline{\theta}}{2} \right] \\
&= \frac{\bar{\theta}}{2}(A^* - \widetilde{A}) - \frac{A^*\bar{\theta}}{4} \left(\frac{A^*\bar{\theta}}{x^* \left(\frac{x^*(\underline{\theta})}{\widetilde{A}} \right)} - 1 \right) \\
&\quad + \frac{\beta}{1-\beta} \left[\frac{1}{8} \left(\widetilde{A}\bar{\theta} - x^* \left(\frac{x^*(\underline{\theta})}{\widetilde{A}} \right) \frac{\widetilde{A}}{A'} \right) - \frac{1-\beta}{1-\beta\triangle} \frac{1 + p(A^*)}{4} \frac{x^*(\underline{\theta}) - \widetilde{A}\underline{\theta}}{2} \right].
\end{aligned}$$

As $\widetilde{A} \to \frac{x^*(\theta)}{\underline{\theta}}$, $\frac{1}{8}\left(\widetilde{A}\bar{\theta} - x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)\frac{\widetilde{A}}{A'}\right) + \frac{1-\beta}{1-\beta\triangle}\frac{1+p(A^*)}{4}\frac{x^*(\theta)-\widetilde{A}\underline{\theta}}{2} \to \frac{x^*(\theta)}{8\underline{\theta}}\left(\bar{\theta} - \frac{x^*(\theta)}{A'}\right) > 0$ since $A^* > A'$. Therefore, large enough $\widetilde{A}$ and β guarantees $F_2(A^*) > 0$ by the same logic above.

Finally, to show the property of $I(\widetilde{A}, \beta)$ when $\widetilde{A} \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\theta)}{\underline{\theta}}\right)$, we first rearrange the expression of F_2 as follows:

$$2F_2(A'|\widetilde{A}) = \left(x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)\frac{\widetilde{A}}{A'} - x^*(\underline{\theta})\right)\left[\frac{\beta}{1-\beta}\frac{\widetilde{A}\bar{\theta} - x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)\frac{\widetilde{A}}{A'}}{\widetilde{A}\bar{\theta} - x^*(\underline{\theta})} + \frac{\beta}{1-\beta\triangle}\frac{\widetilde{A}\underline{\theta} - x^*(\underline{\theta})}{\widetilde{A}\bar{\theta} - x^*(\underline{\theta})}\right]$$
$$- \bar{\theta}(\widetilde{A} - A') > 0.$$

Denoting $a = \frac{x^*(\underline{\theta})}{\widetilde{A}} \in (\underline{\theta}, \bar{\theta})$,

$$\begin{aligned}
&\frac{\partial}{\partial\widetilde{A}}\left[\frac{1}{\widetilde{A}^2}\left(x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)\frac{\widetilde{A}}{A'} - x^*(\underline{\theta})\right)\cdot\left(\widetilde{A}\bar{\theta} - x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)\frac{\widetilde{A}}{A'}\right)\right] \\
&= \frac{\partial a}{\partial\widetilde{A}}\cdot\frac{\partial}{\partial a}\left[\left(\bar{\theta} - \frac{x^*(a)}{A'}\right)\left(\frac{x^*(a)}{A'} - a\right)\right] \\
&= \frac{\partial a}{\partial\widetilde{A}}\cdot\frac{\partial}{\partial a}\left[\left(\bar{\theta} - \frac{x^*(a)}{A'}\right)\left(\frac{x^{*'}(a)}{A'} - 1\right) + \left(\frac{x^*(a)}{A'} - a\right)\left(-\frac{x^{*'}(a)}{A'}\right)\right] \\
&= \frac{\partial a}{\partial\widetilde{A}}\cdot\left[\left(\bar{\theta} + a - \frac{2x^*(a)}{A'}\right)\frac{x^{*'}(a)}{A'} - \bar{\theta} + \frac{x^*(a)}{A'}\right] < 0
\end{aligned}$$

since $x^{*'}(a) < 0$, $\frac{x^*(a)}{A'} > \bar{\theta}$, $\bar{\theta} + a - \frac{2x^*(a)}{A'} < 2\bar{\theta} - \frac{2x^*(a)}{A'} < 0$, and $\frac{\partial a}{\partial\widetilde{A}} < 0$. Noticing that $\frac{\partial}{\partial\widetilde{A}}\frac{\widetilde{A}}{\widetilde{A}\bar{\theta} - x^*(\underline{\theta})} < 0$, we have that $\left(x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)\frac{\widetilde{A}}{A'} - x^*(\underline{\theta})\right)\frac{\beta}{1-\beta}\frac{\widetilde{A}\bar{\theta} - x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)\frac{\widetilde{A}}{A'}}{\widetilde{A}\bar{\theta} - x^*(\underline{\theta})}$ increases in $\widetilde{A}$. Moreover, $\frac{\partial}{\partial\widetilde{A}}\left(\frac{\widetilde{A}\underline{\theta} - x^*(\underline{\theta})}{\widetilde{A}\bar{\theta} - x^*(\underline{\theta})}\right) > \frac{x^*(\underline{\theta})(\bar{\theta} - \underline{\theta})}{(\widetilde{A}\bar{\theta} - x^*(\underline{\theta}))^2}$ is larger enough than zero, and, since $x^*(\cdot)$ is strictly increasing and strictly concave, by claim 2 in the proof of proposition 3, $\frac{\partial}{\partial\widetilde{A}}\left(x^*\left(\frac{x^*(\underline{\theta})}{\widetilde{A}}\right)\frac{\widetilde{A}}{A'} - x^*(\underline{\theta})\right)$ is larger enough than zero. Therefore, if β is large enough, then $\frac{\partial F_2(A'|\widetilde{A})}{\partial\widetilde{A}} > 0$. Consider such β, any $\widetilde{A}_1, \widetilde{A}_2 \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\theta)}{\underline{\theta}}\right)$ such that $\widetilde{A}_1 < \widetilde{A}_2$, and any $A' \in I(\widetilde{A}_1, \beta)$. Then $F_2(A'|\widetilde{A}_1) > 0$, which implies $F_2(A'|\widetilde{A}_2) > 0$, that is, $A' \in I(\widetilde{A}_2, \beta)$. Therefore $I\left(\widetilde{A}_1, \beta\right) \subset I\left(\widetilde{A}_2, \beta\right)$. ■