Market sentiment and heterogeneous agents in an evolutive financial model

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Abstract

We study a financial market populated by heterogeneous agents, whose decisions are driven by "animal spirits". Each agent may have either optimistic or pessimistic beliefs about the fundamental value, that can be changed from time to time on the basis of an evolutionary mechanism, or he/she may eventually rely on technical analysis. The evolutionary selection of beliefs depends on a weighted evaluation of the general market sentiment perceived by the agents and on a profitability measure of the existent strategies. The market sentiment, being crucial for the decisions of investors, is included within the set of the driving forces for the performance of the financial market. In fact, as the relevance given to the sentiment index increases, a herding phenomenon in agents behavior may take place and the animal spirits can drive the market toward polarized economic regimes, which coexist and are characterized by persistent high or low levels of optimism and pessimism. This conduct is detectable from agents polarized shares and beliefs, which in turn influence the price level. Such polarized economic regimes can consist in stable steady states or can be characterized by endogenous complex dynamics, generating persistent alternating waves of optimism and pessimism, as well as return distributions displaying the typical features of financial time series, such as fat tails, excess of volatility and multifractality. Moreover, we show that if the sentiment has no or low relevance on the selection of beliefs, those stylized facts are abated or are missing at all from the simulated time series.

Keywords: heterogeneous agents; animal spirits; behavioral finance; market sentiment; complex dynamics.

JEL classification: D84, G41, C62, B52

1 Introduction

Representing agents as heterogeneous and boundedly rational actors has become a quite common modeling assumption in several economic contexts. Such an assumption relies on the evidence that the complexity of the economic environment restricts the agents actual capability to have a complete knowledge about it, so that agents take decisions that are unavoidably cursed by uncertainty (see e.g. [56]). In the past years, the analysis of models involving heterogeneous interacting agents (see e.g. [36]) has considerably improved the understanding of financial markets functioning. Moreover, the psychological investigation about humans, and hence about economic agents, shows that most of the decisions are taken on the basis of simple heuristics (see e.g. [27], [32] and [61] among others).

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Individuals, being affected by psychological and emotional factors, rely more on impressions and common feelings than on a precise knowledge and evaluation of the environment they live in. In fact, even before the behavioral paradigm came to the front of the stage in finance and economics, the role of investor sentiment was perceived as a common phenomenon by financial analysts and market participants. The previous statement finds its foundation in the work by De Grauwe (see [18]) who straightly claims that "the notions of *animal spirits* and *rational expectations* do not mix well". In fact, mainstream economic models tend to assume that agents are rational, markets are complete, and information disseminates widely and freely. If, on one hand, these assumptions are convenient for analytical tractability, on the other hand, they leave no room for sentiments or animal spirits, contradicting the observed behavior of investors in the real-world financial markets. And, while conventional wisdom largely supports the idea that bounded rationality and investor sentiment may overwhelm the rationality hypothesis in trading behavior, the sentiment analysis has only started gaining recognition in financial academic research in the past two decades (see e.g. [44], [52] and [58]).

Nowadays, the approach based on boundedly rational agents is widely applied in the modeling of financial markets (see e.g. [6], [16], [33] and [41]), which are intrinsically characterized by a high degree of complexity and in which the behavior of the agents can not be neglected in order to understand and replicate the dynamics exhibited by the real-world economic variables. The literature that stems from these ideas is burgeoning and widespread (see, just to cite a few, [10], [11], [15], [46], [47]). Concerning the research strand that is closer to the present contribution, we mention: the work by Brock and Hommes [11] where when the intensity of choice to switch predictors is high, asset price fluctuations are characterized by an irregular alternation between phases where prices are close to the fundamental, phases of optimism, where traders become excited and extrapolate upward trends, and phases of pessimism, where traders become nervous causing a sharp decline in asset prices; the paper by De Grauwe and Rovira Kaltwasser [19], in which the emergence of waves of optimism/pessimism is explained in terms of an evolutionary selection between optimistic/pessimistic exogenous beliefs about the fundamental value; the work by Naimzada and Pireddu [51], that studies the evolution of agents beliefs with endogenously varying levels of optimism/pessimism; the paper by Cavalli et al. [13], in which endogenously changing optimistic/pessimistic beliefs can not be disentangled from their evolutionary selection in view of understanding the emergence of waves of optimism/pessimism. In all the above mentioned works, the main mechanisms (imitation and evolutionary selection) ground on precise evaluations of economic indicators (like profits or forecasting errors), which are assumed to be correctly estimated by agents and on which agents base their choices.

However, the previous bunch of works deserves two remarks. The first one concerns the modeling side. Although being in line with a general idea of bounded rationality, such a literature systematically disregards the Keynes' insights "that a large proportion of our positive activities depend on spontaneous optimism rather than on a mathematical expectation, whether moral or hedonistic or economic", and that "there is the instability due to the characteristic of human nature", namely that economic agents can act as "animal spirits" (see [39]). In fact, in the aforementioned contributions, the role of animal spirits behaviors is relegated to a qualitative outcome (in terms of waves of optimism/pessimism), because agents divert from a rational expectation assumption and have biased beliefs. However, agents still take decisions "rationally", measuring precise economic quantities. In this sense, animal spirits are not among the endogenous drivers of the economic decisions. The second remark concerns the lack of a relevant result: when the agents decisions are driven by animal spirits and rely on a general perceived opinion, optimism and pessimism should be able to self-sustain themselves, polarizing the majority of agents toward one or the other feeling and leading them to "herd" in persistent groups where almost all members behave as optimists or as pessimists. Thus a natural question arises, which constitutes the motivation of the present research: "What happens when decisions (in the present case, strategies in a financial market) are driven by animal spirits?".

We try to tackle both previous concerns, without losing the undebatably positive qualities of the results arising in the existing literature, in terms of relevant endogenous dynamics and qualitatively good properties in the behavior of the economic variables. Therefore, the present contribution aims at providing a rigorous formal modeling of animal spirits as one of the drivers of the agents and, subsequently, of the market behavior. Our modeling of animal spirits is significantly affected by behavioral features, that are encompassed in the market sentiment, which is undoubtedly a relevant feature to take into account when including elements of market psychology in the design of a financial market.

A further motivation of this approach is the content of a recent survey by Franke and Westerhoff (see [25]), in which a similar viewpoint is expressed. Using their words, the above mentioned literature provides a "weak form" of animal spirits, in the sense that the "model is able to generate waves of, say, an optimistic and pessimistic attitude, or waves of applying a forecast rule 1 as opposed to a forecast rule 2." (p. 3). Conversely a "strong form" of animal spirits modeling approaches "exists if agents also rush toward an attitude, strategy, or so on, simply because it is being applied at the time by the majority of agents." (p. 3). In the present work, agents may rush toward optimism and pessimism depending on what they observe or feel about the behavior of the majority of the other agents, which is what we call the "general sentiment". Hence, differently from the work in [19] that adopts the "weak" idea of animal spirits, the present model provides a "strong" form of animal spirits modeling in order to retrieve the Keynesian seminal idea. This choice is supported by the fact that investing in stock markets is a social activity. Financial operators devote a substantial amount of their activity on reading and discussing about investments, without disregarding others' successes or failures in investing. It is thus reasonable that investors' behavior, as well as stock prices, are influenced by social interaction. Moreover, it is evident that attitudes on investments may change in reaction to some events, and thus it seems plain to include also social facets while analyzing the behavior of stock prices (see [57]). Since boundedly rational investors lack any clear objective evidence regarding prices of speculative assets, the process by which their expectations are formed may be social in nature. Therefore, it is crucial to provide an alternative to the exclusive focus on the expectations about specific stock prices. Since the long-term decisions of the agents are based also on a form of market mood derived from their social activity, what would be needed is actually a setup in which the concept of market sentiment is coupled with the individual evaluation of the market.

Keeping this in mind, some recent approaches that base their arguments on the notion of market sentiments incorporate some kind of informational constraints that limits agents' full access to the market (see e.g. [2], [8] and [45]). These constraints generate sentiment oscillations that give rise to self-fulfilling oscillations, whose source is a mix of psychological and sociological drivers that take the form of sentiment shocks. Hence, although emphasizing the role of interaction and coordination, in the aforementioned contributions fluctuations occur as a consequence of exogenous disturbances. Instead, it is precisely on the endogenous origins of such oscillations that we will focus our attention by dealing with a framework where the sentiment is modeled as the outcome of social contacts, similar to the approach in, e.g., [22] and [29].

It is against this background that we consider a baseline financial market model populated by heterogeneous agents, namely optimists and pessimists who respectively overestimate and underestimate the true fundamental value, as in [13], and can change their belief from time to time on the basis of an evolutionary mechanism. The evolutionary mechanism is based on a combination of the average mood about the status of the market (i.e., the *sentiment index*) and of a measure of profitability of the existent strategies. The sentiment index depends on how many optimists and pessimists populate the market and on how much their beliefs about the fundamental value are optimistically/pessimistically biased. In this way, the psychological and emotional components become a constitutive part of the decision process.

The results emerging from our analysis are interesting under several perspectives. Starting from a static analysis, in addition to the fundamental steady state, we find the existence of sentiment-connected non-fundamental steady states characterized by a either high or low price level. But the analysis proceeds further. Moving from a modeling approach which relies on a weak form of animal spirits to a different one which relies on a strong form, if the final outcomes were the same in both approaches, there would not be much difference between the two forms of animal spirits. But this is not the case. We shall show the emergence of herding phenomena which occurs "because individual agents believe that the majority will probably be better informed and smarter than they themselves" (see [25]), and thus agents can pay more attention to the general sentiment than to the comparison of realizable profits. Herding drives the occurrence of long lasting waves of optimism and pessimism, which are a consequence of the (strong form of) animal spirits behavior of the agents. Furthermore, the emergence of coexisting economic regimes (consisting of stable steady states or of more complex attractors), that did not emerge in the above mentioned literature and that are characterized by persistently polarized levels of optimism and pessimism, mirrors the eventuality of such herding behaviors. When agents "endeavor to conform with the behavior of the majority or the average" [40], it is more likely that animal spirits generate outcomes which can be seen as the result of a herding phenomenon around a polarized situation. The present setting does not rule out the possibility of endogenous dynamics with neat and persistent periods of optimism alternating with periods of pessimism, and the resulting dynamics provide a clear representation of the stylized facts occurring in real-world financial markets.

Whether fundamentalists have beliefs which are affected by the same or by a different bias is also an issue that one may wonder about in order to check if our findings, which highlight the role played by the market sentiment in the emergence of the optimistic/pessimistic steady states, are robust. Considering beliefs which are not symmetrically balanced around the fundamental value confirms the relevance of the role of the sentiment in the selection of the strategies, as two new steady states can still arise. The ex-ante exogenous asymmetry in the agents beliefs reflects on that of the steady states thanks to the joint role of the sentiment weight, of the evolutionary pressure to switch between biases and of the heterogeneity degree of the biases.

The role of the market sentiment in a setting with simple belief types in deviations from fundamentals may also be analyzed in an even larger market setup, in which the investors' attention is posed on charts, price, volume, money flow and other market information. This is the strategy of technical traders, who seek to identify price patterns and market trends in financial markets and who attempt to exploit these patterns. Such class of agents can be considered as a constant presence in the market due to the peculiar feature that characterizes its behavior. They believe that stock prices tend to move in trends which persist for an appreciable length of time and that changes in trend are caused by shifts in demand and supply. Those shifts, no matter why they occur, can be detected sooner or later in the action of the market itself. Accordingly, a thorough investigation of our financial market may benefit by the consideration of such class of agents in order to understand how the price dynamics are influenced by a different type of activity. In agreement with the exiting literature (see e.g. [4], [26] and [28]), the introduction of technical traders is certainly able to spread instabilities within our market setup, but still maintaining the polarization of the sentiment-connected non-fundamental steady states. It is the intrinsic nature of technical traders, that look at turbulent market phases as an opportunity to realize more gains, which may lead the price to divert from the polarized steady states with the emergence of consequent bubbles and crashes in the price dynamics.

As a unifying conclusion, we can certainly state that our outcomes may be ascribed on one hand to the role played by the market sentiment and on the other hand to the agents' heterogeneity: when agents operate in opposite directions, whether they are all fundamentalists or in part technical traders, a considerable heterogeneity degree is embedded into our model. Accordingly, if one aims at explaining the high trading volume and the high volatility observed in many real-world financial markets, it turns out to be necessary the development of a model in which agents are truly heterogeneous.

The remainder of the paper is organized as follows. In Section 2 we outline the baseline model with symmetrically biased optimistic and pessimistic fundamentalists. In Section 3 we deviate from the baseline model considering asymmetric belief biases, while in Section 4 we introduce technical traders to the baseline model. Finally, Section 5 concludes. All the proofs of our analytical results can be found in the Appendix.

2 The baseline model with biased fundamentalists

The baseline evolutive financial model we study takes inspiration from [19] and grounds on a market in which, at each discrete time period t, a normalized population of size 1 is composed by boundedly rational fundamentalists[1].

Agents are divided into optimists and pessimists[2], whose shares are respectively described by $\omega_t \in (0, 1)$ and $1 - \omega_t$. Fundamentalists buy/sell stocks in undervalued/overvalued markets. In particular, pessimistic and optimistic agents, who systematically underestimate and overestimate the true fundamental value F, have biased beliefs respectively equal to $X = F - \Delta/2$ and $Y = F + \Delta/2 = X + \Delta$, where $\Delta > 0$ is the belief bias and assesses the degree of heterogeneity among the agents.[3]

The fraction of each kind of agents varies according to an evolutionary selection of the best performing optimistic/pessimistic heuristic. The stock price P_t is adjusted by a market maker by means of a nonlinear mechanism.

As in [19], the demand functions of pessimistic and optimistic agents are respectively given by $d_{t,pes} = \alpha(X - P_t)$ and $d_{t,opt} = \alpha(Y - P_t)$, where P_t is the asset price at time t and $\alpha > 0$ is a demand reactivity parameter, which we can assume to be the same for the two kinds of agents, being them both fundamentalists. If at time t the share of pessimists (resp. optimists) is $\omega_t \in [0, 1]$ (resp. $1 - \omega_t$), the total excess demand is $D_t = \omega_t\alpha(X - P_t) + (1 - \omega_t)\alpha(Y - P_t)$, which, recalling that $Y = X + \Delta$, can be rewritten as $D_t = \alpha(F - P_t + \Delta(1/2 - \omega_t))$. We assume that the price variation is described by the nonlinear, bounded mechanism

$$P_{t+1} - P_t = f(\gamma D_t) = f(\gamma\alpha(F - P_t + \Delta(1/2 - \omega_t))), \tag{1}$$

where $\gamma > 0$ represents the price adjustment reactivity and $f : \mathbb{R} \to (-a_2, a_1)$, with $-a_2 < 0 < a_1$, is a twice differentiable sigmoidal function, i.e., an increasing function, satisfying $f(0) = 0$, $f'(0) = 1$, $f''(z) > 0$ on $(-\infty, 0)$ and $f''(z) < 0$ on $(0, +\infty)$. Moreover, as it is evident from (1), without loss of generality we can set $\alpha = 1$, encompassing both the demand and the price adjustment reactivities in the parameter γ.

The nonlinear mechanism introduces a cautious price adjustment, as from t to $t + 1$ prices can only increase or decrease by a bounded quantity, respectively given by a_1 or a_2. Namely, the mechanism in (1) encompasses a conservative behavior for the market maker, induced by a central authority

[1] There exist several papers in which agents are endowed with biased beliefs about the fundamental (or target) value of a certain economic variable, such as the price, the inflation, output gap or exchange rate target (see, just to cite a few, Brock and Hommes (1998), Rovira Kaltwasser (2010), Anufriev et al. (2013), Agliari et al. (2017) and Hommes and Lustenhouwer (2017)). Moreover, due to the findings of our work, there is also a deeper reason to consider at first only fundamentalist agents in the model. Namely, in this way, even if a major role is played by the negative feedback of investors on prices (as the behavior of fundamentalists refers to buying and selling decisions that are made with a view to the underlying value of the asset, and tend to push prices back toward fundamentals), we find the emergence of herding phenomena, which are instead usually triggered by a positive feedback, typical of the technical traders behavior. Therefore, including technical traders in our model could conceal the effect of the sentiment index on herding. For a classical work in the herding literature we refer the interested reader to Kirman (1993), which proposes a model of stochastic recruitment to explain both herding and epidemics phenomena observed in financial markets.

[2] A clear explanation of the empirical basis for the assumption that fundamentalists are either optimistic or pessimistic can be found in the introduction of [19]. Indeed, authors recall, e.g., that in the past decade two sets of beliefs emerged about the fundamental value of the US dollar, according to which the large account deficits of the US observed since the second half of the 1990s were unsustainable (see [53] and [54]) or perfectly sustainable (see [31]).

[3] As in [19], we deal with a symmetric framework in order to focus on the most significant form of heterogeneity, i.e. the maximum possible degree of polarization, represented by Δ, between different attitudes of agents toward the reference value F. This also allows us to keep the model analytically tractable and to provide a neater interpretation of the results.

that, trying to limit overreaction phenomena with the consequent occurrence of an excessive stock volatility, imposes limits to price variations (see [23], [30] and [43]). As a consequence, the market maker prudently adjusts prices in the presence of extreme excess demand, while when excess demand is small the price adjustment is nearly proportional to it. In particular, the price variation limiter mechanism can be modeled by a sigmoidal adjustment rule that determines a bounded price variation in every time period, thanks to the presence of two asymptotes that limit the price changes. We recall that, in the literature on behavioral financial markets, nonlinear price adjustment mechanisms have been already considered, among others, by [60, 63]. More precisely, thanks to the previous assumptions on f, from (1) we have that the stock price respectively increases or decreases when the excess demand is positive or negative, and that the variation rate increases as the excess demand vanishes, since the derivative of the right-hand side of (1) with respect to D_t is given by $\gamma f'(\gamma D_t)$, and attains its maximum value γ when $D_t = 0$.

The last part of the model to be described concerns the evolution of the shares of optimists/pessimists, based on an evolutionary competition between the two behavioral rules. Differently from [13, 19], where the fitness measure in the switching mechanism relied only on the comparison between the profits that could be realized by the two groups of agents, in the present contribution the evolutionary selection of beliefs depends also on the general feeling perceived by the agents about the market status. Such a feeling is described by the sentiment index

$$I_t = \omega_t X + (1 - \omega_t)Y - F = X + (1 - \omega_t)\Delta - F, \tag{2}$$

which measures the difference between the average belief about the fundamental value, represented by $\omega_t X + (1 - \omega_t)Y$, and the true fundamental value F. The average belief about fundamental value depends on both the beliefs and the shares, whose effects can not be completely disentangled. The sign of I_t gives information about the general degree of optimism or pessimism in the market, as I_t is positive (negative), portraying the underlying optimistic (pessimistic) perceived market mood, when $\omega_t X + (1 - \omega_t)Y$ is larger (smaller) than F. In particular, since $X = F - \Delta/2$, we find that

$$I_t = \Delta(1/2 - \omega_t). \tag{3}$$

Hence, I_t is positive (negative) when the share of pessimists is below (above) 1/2.
Then, the population fraction composed by pessimists evolves depending on a convex combination of the general market sentiment and of the profits realized by the two kinds of agents, according to the following updating rule

$$\omega_{t+1} = \frac{e^{\beta(\sigma(-I_t)+(1-\sigma)\pi_{X,t+1})}}{e^{\beta(\sigma(-I_t)+(1-\sigma)\pi_{X,t+1})} + e^{\beta(\sigma I_t+(1-\sigma)\pi_{Y,t+1})}} = \frac{1}{1 + e^{\beta(2\sigma I_t+(1-\sigma)(\pi_{Y,t+1}-\pi_{X,t+1}))}}, \tag{4}$$

where β is a positive parameter representing the intensity of choice of the switching mechanism and $\sigma \in [0, 1]$ is the *sentiment weight*.
Equation (4) describes the probability that agents will be pessimistic in their belief. Such a probability depends on two factors: the general market sentiment at time t and a measure of profitability of the existent strategies.[4] The latter is given by the most recent profit

$$\pi_{j,t+1} = (P_{t+1} - P_t)(j - P_t), \quad j \in \{X, Y\}, \tag{5}$$

that would have been realized adopting a pessimistic belief (corresponding to $j = X$) or an optimistic one (corresponding to $j = Y$).[5] When $\sigma = 0$, the evolutionary mechanism in (4) is exactly the same

[4] We stress that the opposite signs preceding I_t in the argument of the exponential functions in (4) are a consequence of the different attitude of optimists and pessimists toward positive or negative values of the sentiment index.

[5] We would like to point out that more refined fitness measures could encompass portfolio balances and wealth, as the literature on market microstructure deals with (see e.g. [21] and [36]). But this is beyond the scope of this paper and

as in [19], while, as σ increases, the relevance assigned by the agents to the general perceived mood increases, raising the impact of the crowd psychology and of the herd instinct. If $\sigma = 1$, the profitability measure is no more influencing the switching mechanism, which only depends on the sentiment index (2), and the share of pessimists will increase (decrease) when I_t is negative (positive), i.e., when the average belief about fundamental value is smaller (larger) than F. We note that in [13] the average belief about the fundamental value (and its generalization over a window of $n > 1$ preceding periods) was introduced and used to study the possible emergence of waves of optimism and pessimism. However, in [13], the index I_t only helped in describing the dynamics without affecting them, being not taken into account by the agents in their decisions. Finally, we stress that β in (4) measures how much relevance agents assign to the forecasting rules. If β is small, agents are quite indifferent to the signals coming from the two heuristics (in terms of general sentiment and/or profitability), and they tend to equally distribute themselves between optimism and pessimism. Conversely, if more relevance is given to the perceived market mood and/or to the eventual profitability, a larger share of agents will switch to the most performing attitude toward the reference value.

Our model is obtained collecting the price adjustment mechanism (1) and the evolutionary mechanism (4), being described by the two-dimensional map $G = (G_1, G_2) : (0, +\infty) \times (0, 1) \to \mathbb{R}^2$, $(P_t, \omega_t) \mapsto (G_1(P_t, \omega_t), G_2(P_t, \omega_t))$, defined as:

$$\begin{cases} P_{t+1} = G_1(P_t, \omega_t) = P_t + f\left(\alpha\gamma(F - P_t + \Delta(1/2 - \omega_t))\right), \\ \omega_{t+1} = G_2(P_t, \omega_t) = \dfrac{1}{1 + e^{\beta(\sigma\Delta(1-2\omega_t) + (1-\sigma)\Delta f(\alpha\gamma(F - P_t + \Delta(1/2 - \omega_t))))}}, \end{cases} \tag{6}$$

where, in the last equation, we replaced I_t with its expression provided in (3) and we employed the identity $\pi_{Y,t+1} - \pi_{X,t+1} = (Y - X)(P_{t+1} - P_t) = \Delta f(\gamma D_t)$.[6]

We stress that for $\sigma = 0$ the model in (6) reduces to that studied in [13] when $\mu = 0$ or in [19] for a linear price adjustment mechanism. Therefore, in what follows we investigate what happens when $\sigma > 0$, focusing in particular on the extreme case $\sigma = 1$ (see Subsection 2.2). It is worth noticing that the present framework significantly diverts from [19] under several aspects. As explained above, the key economic element to be considered here is that the choice between optimism and pessimism is not only driven by a rational computation of the profitability of the existent strategies, but, as σ increases, the psychological and emotional aspects increasingly assume a central role, being the unique impulse when $\sigma = 1$. This reinforces the behavior of agents as "animal spirits", which is partially encompassed in [13, 19] in the optimistically /pessimistically biased beliefs, but which, as $\sigma \to 1$, becomes the main motivational driver of the agents choice about the forecasting rule in the present model. However, the significance of the considered framework is not limited to the economic interpretation of the model, but, as it will become evident from the analytical results in Subsection 2.1 and the numerical simulations in Subsection 2.2, the possible dynamical outcomes arising when the sentiment index drives the stock market significantly differ from those found in the existing literature, so that the proposed approach provides a "strong form" ([25]) of animal spirits modeling, differently from the "weak form" of [13, 19].

2.1 Analytical results on steady states and local stability

In this subsection we determine the possible steady states of the model outlined above and we provide analytical conditions for the local stability of the fundamental steady state. Moreover, we also present some results on how the additional steady states vary when the relevant parameters change.

we leave the investigation of such measures for future research.

[6] In the present work we only consider the case of exogenous biased beliefs, to better focus on the role of the market sentiment on the results. The generalization to the case in which the belief biases depend on the relative ability to guess the actual realized price is considered in [14], from which it is possible to see that the results we shall present in Subsections 2.1 and 2.2 are robust with respect to the endogenization of the beliefs.

We start by investigating the existence of the steady states of (6).

Proposition 1. *System* (6) *has*

a) a unique steady state $S^* = (P^*, \omega^*) = (F, 1/2)$ *if* $\sigma \in [0,1]$ *and*

$$\sigma \leq \frac{2}{\beta\Delta}; \tag{7}$$

b) three steady states $S^*, S^o = (P^o, \omega^o)$ *and* $S^p = (P^p, \omega^p)$ *if* $\frac{2}{\beta\Delta} < \sigma \leq 1$. *In particular,* S^o *and* S^p *are symmetric w.r.t.* S^*, *with* $P^p < P^* < P^o$ *and* $\omega^o < \omega^* < \omega^p$.

Proposition 1 bears relevance for our analysis. In fact, differently from what is found in the existing literature (see e.g. [13, 19, 51]), where the unique steady state is given by the fundamental steady state S^*, when animal spirits affect economic decisions the system can be driven toward a steady state characterized by either greater (P^o) or smaller (P^p) prices than the fundamental value. In these steady states, the population consists of a larger share of optimists ($\omega^o < 1/2$) or pessimists ($\omega^p > 1/2$), respectively. We then have two more steady economic regimes that can be identified as "pessimistic" (S^p) and "optimistic" (S^o), coexisting with S^*. This eventuality occurs if agents give a sufficiently large relevance to the perceived market mood, as the additional steady regimes can emerge only for suitably large sentiment weight values. If agents only rely on a "rational" comparison of the performance of pessimism and optimism in terms of profits (as in [13]), the equilibrium configuration can solely consist in an even distribution of pessimists/optimists, with the stock price corresponding to the true fundamental value. However, as the sentiment weight approaches 1, the switching mechanism is more and more influenced by the sentiment index, whose size is not only determined by the population share of pessimists, but also by the distance Δ between optimistic and pessimistic beliefs. More precisely, if the relevance given by the agents to the perceived mood is small (i.e., β is low), agents will more likely choose indifferently one of the two heuristics, so that deviations from a uniform distribution have a little consequence and shares will settle back to a uniform distribution. Conversely, if the relevance is large (i.e., β is high), even a small excess of pessimistic agents ($\omega_t > 1/2$) triggers a diffusion of pessimism, that leads the majority of agents to become pessimists ($\omega^p > 1/2$). We will come back to this aspect after Proposition 4 with the help of the stability analysis and of some simulated time series. Moreover, as the factors characterizing the agents behaviors become more extreme (i.e., as the intensity of choice and/or the polarization of the beliefs increase), the effect of animal spirits is bolstered and a progressively reduced sentiment weight is enough to trigger the emergence of the polarized steady states. In this respect, we stress that both the relevance given to the evolutionary selection of heuristics ($\beta > 0$) and the heterogeneity degree of beliefs ($\Delta > 0$) are essential, as otherwise only the intermediate steady state is possible (as (7) is fulfilled).

Following what suggested by the previous considerations, we introduce the aggregate index $\mathfrak{s} = \beta\sigma\Delta$, which encompasses the effect of the sentiment weight enhanced by the evolutionary pressure β and by the heterogeneity degree Δ. From now on, we refer to $\mathfrak{s}$ as the sentiment strength. We stress that this non-negative aggregate parameter regulates the threshold value at which two extra steady states emerge in the symmetric biases framework, when S^* loses stability. Namely, (7) can be simply rewritten as $\mathfrak{s} \leq 2$. As we shall see in Propositions 5 and 7, the parameter $\mathfrak{s}$ similarly regulates the threshold value at which the number of steady states increases from one to three when agents' biases are asymmetric and when we deal with technical traders, as well.

However, in the symmetric bias framework, the agents features encompassed in $\mathfrak{s}$ do not only foster the emergence of the steady states S^o and S^p, but also significantly affect their position, as well as the values of the sentiment index I^o and I^p at S^o and S^p, as shown in the next result (see also Proposition 6 for the symmetric bias framework and Proposition 7 in the presence of technical traders).

Proposition 2. *Let* $\mathfrak{s} > 2$, *with* $\sigma \in (0,1]$. *Then, on increasing* σ, β *and* Δ *we have that* ω^o *decreases, while* P^o *and* I^o *increase, and that* ω^p *increases, while* P^p *and* I^p *decrease.*

This result reinforces that in Proposition 1. Firstly, the more the psychological and emotional components are determinant for the choice of optimistic/pessimistic heuristics, the more the polarized steady states divert from the intermediate one. Indeed, increasing the relevance of the perceived market mood leads to final outcomes that are more strongly characterized (both in terms of prices and shares) by optimism and pessimism. Such a feature reflects on the resulting sentiment index at the polarized steady states, which is not "neutral" ($I = 0$) as at S^*, but consistently portrays the pessimistic ($I^p < 0$) and optimistic ($I^o > 0$) mood perceived in the market. Psychological factors can then strengthen the role of the market sentiment in determining the prices and the shares.

The next level of investigations concerns how the stability of S^* is affected by the sentiment weight and, more generally, we analyze the effects of the sentiment index on the resulting dynamics.[7] Before presenting such results, we compare the roles of β, Δ and γ on the stability of S^*, when the evolutionary mechanism is only driven either by the profitability measure[8] ($\sigma = 0$) or by the perceived sentiment ($\sigma = 1$). To this end we recall that, if we consider homogeneous beliefs ($\Delta = 0$) and we consequently neglect the evolutionary switching mechanism ($\beta = 0$), it is easy to see that dynamics only consist in the price adjustment mechanism and that S^* is stable when $\gamma < 2$.

Proposition 3. *When $\sigma = 0$, the steady state S^* is locally asymptotically stable provided that $2(\gamma - 2)/\gamma < \Delta^2\beta < 4/\gamma$. When $\sigma = 1$, the steady state S^* is locally asymptotically stable provided that $\gamma < 2$ and $\beta\Delta < 2$.*

When the price mechanism does not introduce instabilities ($\gamma < 2$), increasing the intensity of choice and the heterogeneity degree has a destabilizing effect for both extremal choices of fitness measures (i.e., $\sigma = 0$ and $\sigma = 1$). Conversely, when the price mechanism introduces instability ($\gamma > 2$), suitable intermediate values of β and Δ can stabilize S^* (as discussed in [19]) when agents choose their strategy on the basis of the profitability measure. Such stabilization does not occur when the fitness measure is given by the sentiment index, as the steady state S^* is unstable regardless of Δ and β.

Proposition 3 shows that the same degree of beliefs heterogeneity has a different effect when the fitness criterion is represented by the profitability measure or by the sentiment index. In the latter case, the role of the parameters affecting the selection mechanism (β) and the agents heterogeneity (Δ) is essentially the same, as the stability of S^* depends on $\Delta\beta$: therefore, increasing either β or Δ by the same amount has an identical effect. Conversely, beliefs heterogeneity has a more intense effect when the strategy choice depends only on profits, as in that case the stability of S^* is affected by $\beta\Delta^2$. Such dissimilarity can be easily understood in terms of the effect of beliefs polarization on the fitness measure: the degree of heterogeneity affects the sentiment index through beliefs only "once" while it affects the profits twice, both directly, through the excess demand, and through prices, resulting in a "squared" influence. This may occur since agents assign more emphasis to the possibility of realizing a profit evaluating their own strategy rather than considering the general mood perceived by the market, which can manifest in a sluggish manner due to some form of slowness in the news diffusion about the market status.

As a consequence, a large (resp. small) belief polarization has a greater (resp. reduced) effect on the stability of the fundamental steady state when the sentiment has no relevance, compared to the case when it drives the choice of strategies. It is then predictable that, ceteris paribus, as σ increases, the effect of Δ on the stability of S^* can increase or decrease. This will help to understand what happens as the relevance of the sentiment index increases, an issue which is studied in the next proposition. In

[7] In what follows, we say that an unconditionally stable/unstable scenario is realized when the steady state is locally asymptotically stable/unstable independently of the parameter values; a stabilizing/destabilizing scenario occurs when the steady state is locally asymptotically stable only above/below a certain threshold and unstable otherwise; a mixed scenario arises when the steady state is locally asymptotically stable only for intermediate parameter values, between two stability thresholds, and unstable otherwise.

[8] Indeed, this case corresponds to that studied in [19, Proposition 1].

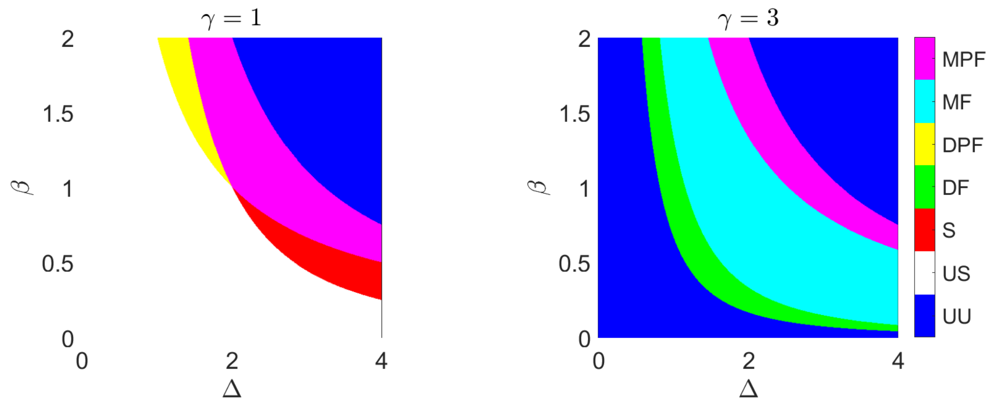


Figure 1: For each couple of β and Δ, different colors correspond to different stability scenarios on increasing σ. Red color denotes values (Δ, β) for which a stabilizing scenario (S) occurs (i.e., according to Proposition 4, S^* is stable for $\sigma \in (\sigma_{ns}, 1]$), green color denotes values for which a destabilizing scenario through a flip bifurcation (DF) occurs (i.e. S^* is stable for $\sigma \in [0, \sigma_{fl})$), yellow color denotes values for which a destabilizing scenario through a pitchfork bifurcation (DPF) occurs (i.e. S^* is stable for $\sigma \in [0, \sigma_{pf})$), cyan color denotes values for which a mixed scenario in which instability occurs through a flip bifurcation (MF) occurs (i.e. S^* is stable for $\sigma \in (\sigma_{ns}, \sigma_{fl})$), magenta color denotes values for which a mixed scenario in which instability occurs through a pitchfork bifurcation (MPF) occurs (i.e. S^* is stable for $\sigma \in (\sigma_{ns}, \sigma_{pf})$). White and blue colors respectively denote values (Δ, β) for which S^* is either stable (US) or unstable (UU) independently of σ.

particular, in order to better focus on the role of σ on varying β and Δ, we make explicit the stability conditions considering two distinct frameworks in relation to the price dynamics. Namely, we study the stability of S^* for System (6) considering two regimes, represented by $\gamma = 1$ and $\gamma = 3$, for which the price mechanism respectively is/is not a source of instability.

Proposition 4. *On varying $\sigma \in [0, 1]$, we have that the sentiment weight can have a destabilizing, stabilizing, mixed or neutral effect on the stability of S^*. In particular, condition (7) is necessary for the stability of S^*. If $\gamma = 1$, then S^* is stable for*

$$\begin{array}{lll}
\sigma \in [0,1] & \textit{if} & \beta < \min\left\{\frac{4}{\Delta^2}, \frac{2}{\Delta}\right\} \\
\sigma \in (\sigma_{ns}, \sigma_{pf}) = \left(\frac{\Delta^2\beta - 4}{\Delta^2\beta}, \frac{2}{\beta\Delta}\right) & \textit{if} & \max\left\{\frac{4}{\Delta^2}, \frac{2}{\Delta}\right\} < \beta < \frac{2\Delta+4}{\Delta^2} \\
\sigma \in [0, \sigma_{pf}) = \left[0, \frac{2}{\beta\Delta}\right) & \textit{if} & \frac{2}{\Delta} < \beta < \frac{4}{\Delta^2} \ \textit{(possible only if } \Delta < 2) \\
\sigma \in (\sigma_{ns}, 1] = \left(\frac{\Delta^2\beta - 4}{\Delta^2\beta}, 1\right] & \textit{if} & \frac{4}{\Delta^2} < \beta < \frac{2}{\Delta} \ \textit{(possible only if } \Delta > 2)
\end{array}$$

and it is unstable for any σ if $\beta > \frac{2\Delta+4}{\Delta^2}$.
Conversely, if $\gamma = 3$, we have that S^* is stable for

$$\begin{array}{lll}
\sigma \in [0, \sigma_{fl}) = \left[0, \frac{3\Delta^2\beta - 2}{\Delta\beta(3\Delta+1)}\right) & \textit{if} & \frac{2}{3\Delta^2} < \beta < \frac{4}{3\Delta^2} \\
\sigma \in (\sigma_{ns}, \sigma_{fl}) = \left(\frac{3\Delta^2\beta - 4}{\Delta\beta(3\Delta+4)}, \frac{3\Delta^2\beta - 2}{\Delta\beta(3\Delta+1)}\right) & \textit{if} & \frac{4}{3\Delta^2} < \beta < \frac{6\Delta+4}{3\Delta^2} \\
\sigma \in (\sigma_{ns}, \sigma_{pf}) = \left(\frac{3\Delta^2\beta - 4}{\Delta\beta(3\Delta+4)}, \frac{2}{\Delta\beta}\right) & \textit{if} & \frac{6\Delta+4}{3\Delta^2} < \beta < \frac{2\Delta+4}{\Delta^2}
\end{array}$$

and it is unstable for any σ if $\beta > \frac{2\Delta+4}{\Delta^2}$ or if $\beta < 2/(3\Delta^2)$.
Finally, crossing the threshold values σ_{fl}, σ_{ns} and σ_{pf} a flip, Neimark-Sacker and pitchfork bifurcation respectively occurs.

We start by analyzing the case $\gamma = 1$, making reference to the left panel in Figure 1. In this case, the occurrence of instability only depends on the switching mechanism, and, according to Proposition

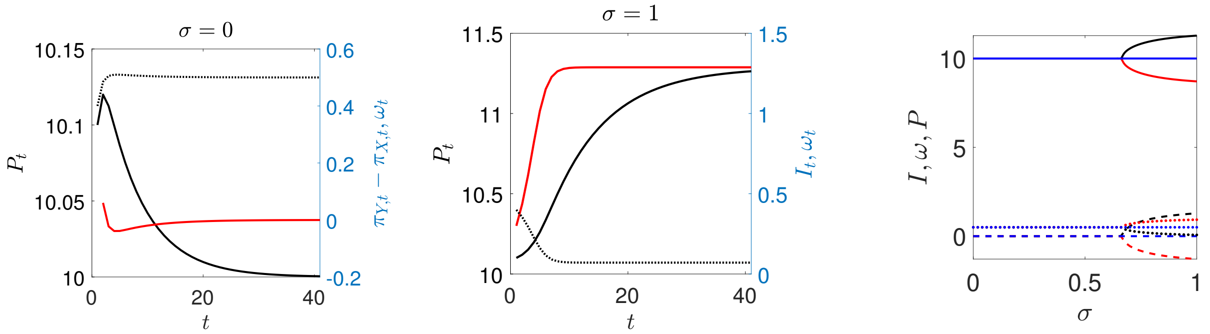


Figure 2: The left and middle panels represent time series when fitness measure coincides with the profitability measure (left panel) with the sentiment index (middle panel). Prices (solid black line, referred to the left axis), shares of pessimists (dotted black line, referred to the right axis) and fitness measure (red line, referred to the right axis) are represented. The right panel represents the evolution of the optimistic (black), pessimistic (red), intermediate (blue) steady states on varying σ. Solid line: P; dotted line: ω; dash-dotted line: I. Parameter values used in all the simulations are $F = 10$, $\beta = 1$, $\gamma = 0.1$ and $\Delta = 3$.

3, in the two extremal frameworks corresponding to $\sigma = 0$ and $\sigma = 1$ we have that S^* is stable for $\Delta^2\beta < 4$ and $\Delta\beta < 2$, respectively.

If β and Δ are suitably small, the increasing relevance given to the perceived sentiment has no effect on stability. Either if the context is affected by a very reduced heterogeneity so that the evolutionary pressure is not enough to induce agents to change strategy or if the beliefs are polarized but the agents do not rely very much on them, we have that S^* is stable regardless of the adopted fitness measure (white region).

If the beliefs polarization is sufficiently large ($\Delta > 2$) and we increase the intensity of choice, since in this case the effect of heterogeneity plays a major role on the profitability measure than on the general sentiment, we have a situation in which S^* is unstable for $\sigma = 0$ and stable for $\sigma = 1$ (see the comments after Proposition 3) and thus the sentiment index has a stabilizing effect on S^* (red region).

Conversely, if the beliefs polarization is small ($\Delta < 2$) and we increase the intensity of choice, since in this case the effect of heterogeneity on the profitability measure is weaker than before, we have a situation in which S^* is stable for $\sigma = 0$ and unstable for $\sigma = 1$. The consequence is that, as the relevance of the sentiment index increases, S^* can lose stability (yellow region). However, such instability does not mean that dynamics become erratic, but that agents rather start herding toward the same strategy, and this drives the evolution of prices toward either the optimistically or the pessimistically biased new steady state. That mechanism can be better understood looking at the left and middle panels in Figure 2, where the parameter setting as well as the initial conditions are essentially the same as in Figure 1.[9] If the fitness measure of the evolutionary process is the profitability measure ($\sigma = 0$, left panel), since initial conditions depict a slightly optimistic state in which $P_0 = 10.1 > F = 10$ and $\omega_0 = 0.4$, there is a positive excess demand ($D_0 = 0.2$) that induces a slight increase in prices at $t = 1$ ($P_1 = 10.12$). However, profits of optimistic agents are only a little larger than those of pessimists, so that the updated share of pessimists is actually larger ($\omega_1 = 0.485$) than the initial one. That is where the typically negative feedback about the market prices of fundamentalists comes in: if prices are too large, pessimists have a more pronounced propensity to sell than optimistic agents have to buy, so that the price increase slows down and a larger share of agents believe that a change is about to occur. This leads prices to decrease at $t = 2$, so that the profits of pessimists are larger than those of optimists and most agents switch to the

[9]We stress that, with respect to the situation reported in Figure 1, in the simulations reported in Figure 2 we have reduced the value of γ, which now is 0.1. This has been done to avoid an uninteresting oscillating convergence toward the steady state and to provide a neater interpretation, which is still valid for any value of $\gamma < 2$.

pessimistic strategy ($\omega_2 > 0.5$). The next trajectories lead prices to monotonically decrease toward the fundamental value, as well as agents choices to evenly distribute between the two strategies. Conversely, if the fitness measure is the sentiment index ($\sigma = 1$, central panel), since at $t = 1$ the price increases, the overall sentiment increases, too. Thus, if the evolutionary tendency to adopt the most fitting strategy is suitably strong and strategies are heterogeneous enough, this leads to a decrease in the share of pessimists. In such a case, the switching mechanism and the heterogeneity strengthen the optimistic mood, which gets a positive feedback by the agents. Indeed, if pessimism is initially quite pronounced, we then observe the opposite effect, and what actually happens on the steady state values as σ increases is reported in the right panel of Figure 2, with the emergence of increasingly polarized prices, shares and general sentiment.

Going back to the role of σ, if we further suitably increase β, for any degree of beliefs heterogeneity $\Delta > 2$ we have that the fundamental steady state is unstable in both the two extremal frameworks identified by $\sigma = 0$ and $\sigma = 1$. However, for intermediate values of σ the negative feedback of the profitability measure is offset by the positive feedback of the sentiment index, and their effects cancel out, giving rise to stable dynamics (magenta region). Finally, for extreme values of β and Δ both mechanisms are very strong and stability can be not recovered (blue region).

Concerning the case of $\gamma = 3$ (right panel in Figure 1), the main difference with respect to the framework with $\gamma = 1$ is that the price mechanism can introduce or bolster instabilities and that, as discussed in [19], an intermediate joint effect of evolutionary pressure and heterogeneity has a stabilizing effect. Hence, when β and Δ are small, the price mechanism introduces instability for $\sigma = 0$ that can not be recovered by increasing the role of the sentiment, due to the reduced relevance and polarization of the beliefs (bottom left blue region). If $\beta\Delta^2$ assumes an intermediate value, the price dynamics are stabilized by the joint effect of evolutionary pressure and heterogeneity when the fitness measure is the profitability measure. However, as the relevance of the sentiment increases, the actual effect of heterogeneity is reduced, and S^* becomes unstable due to the price mechanism when σ becomes suitably large (green region). Since β and Δ are not sufficiently relevant, no herding effect can take place. If β and Δ further increase, the joint effect of evolutionary pressure and heterogeneity now introduces new instability, which can be recovered by appeasing their influence as the sentiment index increases. In this case, we may have again that, if σ is sufficiently large, increasing polarization of the economic variables takes place and dynamics still converge toward a steady state[10] (magenta region).
Finally, when endogenous dynamics induce price fluctuations, agents erratically switch between the optimistic and the pessimistic behaviors.

2.2 Numerical analysis

In the present subsection we complement with numerical simulations the analytical results on the steady states obtained in Subsection 2.1, in order to deepen the understanding of the economic relevance of the arising dynamics in model (6) and to investigate the qualitative properties of the time series when exogenous non-deterministic effects are taken into account.

Deterministic simulations

We consider two sets of simulations, characterized by different values of the heterogeneity degree parameter, in order to provide a portrait of the possible dynamic scenarios that may occur in our model economy and to highlight the influence of the behavioral parameters on the financial market.

[10] We stress that, when the stability of S^* is not recovered, we are in the unconditionally unstable case again, but now dynamics can converge to the polarized steady states at least for intermediate values of σ.

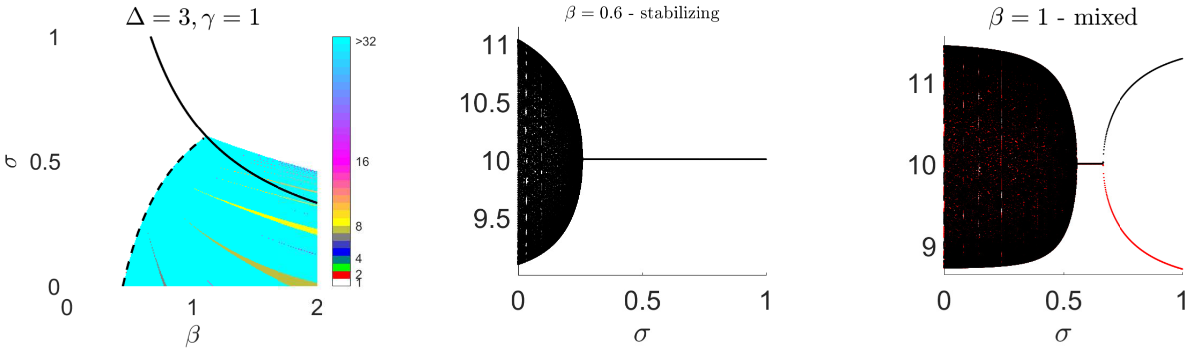


Figure 3: Two-dimensional bifurcation diagram (left plot). The solid and dashed curves represent the bifurcation curves $\sigma = \sigma_{pf}$ and $\sigma = \sigma_{ns}$, respectively, with σ_{pf} and σ_{ns} introduced in Proposition 4. Central and right panels depict two possible scenarios about P on varying σ for different values of β.

To run the simulations, we specify the sigmoid function as

$$f(z) = \begin{cases} a_1 \tanh(z/a_1) & \text{if } z \geq 0, \\ a_2 \tanh(z/a_2) & \text{if } z < 0, \end{cases} \tag{8}$$

setting $F = 10$, $a_1 = 2$, $a_2 = 1$, while we let β, σ, γ and Δ vary from time to time. In the reported two-dimensional bifurcation diagrams we use different colors to identify the degree of complexity of the attractor corresponding to the parameters coupling: white color refers to parameter pairs for which convergence is toward a steady state (either S^*, S^o or S^p), while other colors refer to attractors consisting of more than one point (in particular cyan color identifies cycles of high periodicity, quasi-periodic or complex attractors). The initial datum for the left simulations in Figures 3-4 is $(X_0, P_0, \omega_0) = (X^* + 0.01, P^* + 0.01, \omega^* + 0.01)$. Moreover, the stability threshold curves of the steady state S^* are drawn in black. In particular, the solid line refers to the bifurcation curve $\sigma = \sigma_{pf}$, whose crossing can give rise to a pitchfork bifurcation, while the dashed and dash-dotted lines refer to the bifurcation curves $\sigma = \sigma_{ns}$ and $\sigma = \sigma_{fl}$, whose crossing can give rise to a Neimark-Sacker or to a flip bifurcation, respectively, where the threshold values σ_{pf}, σ_{ns} and σ_{fl} have been introduced in Proposition 4. In the two- and one-dimensional bifurcation diagrams in Figures 3-4 the values of X and ω are related to the price variable, recalling that, when prices are large/small due to the pitchfork bifurcation, shares and beliefs move accordingly. The initial datum in the central plots of Figures 3-4, as well as in the right plots of Figures 3 and 4, is $(X_0, P_0, \omega_0) = (X^* + 0.01, P^* + 0.01, \omega^* + 0.01)$ in black bifurcation diagrams and $(X_0, P_0, \omega_0) = (X^* - 0.01, P^* - 0.01, \omega^* - 0.01)$ in red ones.

The first group of simulations (see Figures 3 and 4) deals with a case in which beliefs are strongly polarized, that is, agents heterogeneity about the fundamental value is high ($\Delta = 3$). We start by considering a moderate ($\gamma = 1$) reactivity to price variation. Looking at the vertical sections of the two-dimensional bifurcation diagram reported in the left plot of Figure 3, we can see examples of unconditionally stable (e.g. for $\beta = 0.1$), stabilizing ($\beta = 0.6$) and mixed ($\beta = 1$) scenarios on varying σ. We stress that in the white region below the black solid line, convergence is toward S^*, while in that above the black solid line, convergence is toward S^o. As shown in the one-dimensional bifurcation diagram in the middle plot of Figure 3, if we keep the value of the intensity of choice fixed at $\beta = 0.6$ and we let σ vary, we observe a stabilizing effect played by the sentiment index on price dynamics. When the sentiment index has no relevance ($\sigma = 0$), the occurring quasi-periodic dynamics are a consequence of the switching mechanism and of a suitably large intensity of choice (in fact, from the left panel in Figure 3 we note that, if we decrease β, we have stable dynamics), and they are transferred to the price mechanism, which would be otherwise stable.[11] In this setting, the evolutionary selection

[11] Indeed, the derivative of the price at the steady states is given by $1 - \gamma$.

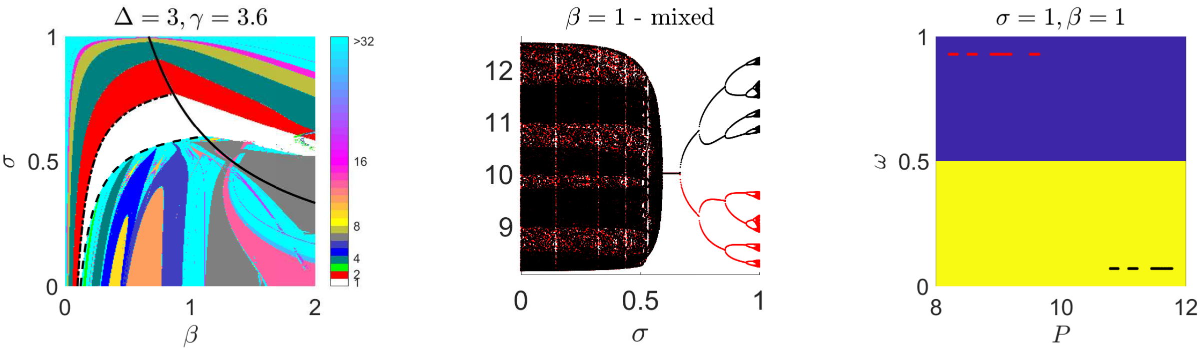


Figure 4: Two-dimensional bifurcation diagram (left panel). The solid, dashed and dash-dotted curves refer to the bifurcation curves $\sigma = \sigma_{pf}$, $\sigma = \sigma_{ns}$ and $\sigma = \sigma_{fl}$, whose crossing can give rise to a pitchfork, to a Neimark-Sacker or to a flip bifurcation, respectively, where the threshold values σ_{pf}, σ_{ns} and σ_{fl} have been introduced in Proposition 4. The one-dimensional bifurcation diagram on the central panel refers to the parameter setting used in the left two-dimensional bifurcation diagram. The right panel depicts the basin of attraction of the optimistic (yellow region) and pessimistic (blue region) attractors, from which the polarization of beliefs is clearly visible.

only depends on the profitability measure, which in turn is affected by excess demand and agents heterogeneity. As σ increases, the strength of the interplay between prices and shares decreases, as the switching mechanism is more affected by the sentiment index and less by the profits, which are in this setting the source of instability. The result is that endogenous oscillations firstly decrease and disappear, so that agents evenly distribute among beliefs and the stock price converges toward the fundamental value. On the other hand, if we slightly increase the intensity of choice up to $\beta = 1$ (see the right plot of Figure 3), we again have the initial stabilizing effect of animal spirits, which then gives rise to a polarization in the shares due to an increased relevance assigned to the utility of being either pessimistic or optimistic, as remarked in the comments after Proposition 1. Depending on an initial deviation in shares toward optimism or pessimism, we have that most of the agents herd around pessimism (red bifurcation diagram) or optimism (black bifurcation diagram). Again, dynamics become convergent toward a stable steady state (either S^* or S^o/S^p) as σ increases.
If we raise γ, we obtain the two-dimensional bifurcation diagram reported in the left plot of Figure 4. In this case, the unconditionally unstable scenario for small values of β is due to the price mechanism. Considering $\beta = 1$ as in the previous simulation, we observe in the central plot of Figure 4 the occurrence of a mixed scenario, similar to the one described in Figure 3, but now the polarized prices undergo a period-doubling cascade of bifurcations that leads to chaotic dynamics. We stress that we still observe a herding phenomenon as σ increases, which now, according to the initial conditions, gives rise to price dynamics that endogenously fluctuate around large or small values. To conclude, in the right plot of Figure 4 we report the basin of attraction obtained when $\sigma = 1$ and $\beta = 1$. We stress that we have checked through simulations that the shape of the basin for $\sigma = 1$ is robust with respect to the considered parameters setting. It is evident that a sufficiently high initial degree of optimism or pessimism uniquely determines the convergence toward an attractor that reflects the same polarized optimism or pessimism. The final state to which the economic variables converge is then affected by the sentiment perceived by the agents in a very neat way, self-sustaining and reinforcing the emergence of more extreme levels of optimism/pessimism. This last aspect, together with the static and local stability analysis of Subsection 2.1, will allow to understand the behavior of economic observables when a non-deterministic effect is introduced.

Stochastic simulations

The analysis performed in the previous subsection showed how animal spirits can drive the market toward economic regimes characterized by either optimism or pessimism. We now consider a stochastically perturbed version of the baseline model in order to see if the model is able to reproduce the qualitative properties of time series of real-world financial markets. For a survey about stylized facts in financial time series we refer to [9, 17, 38, 48]. With respect to those works, we focus on different, increasingly refined, peculiar characteristics of the time series of the economic observables. A first class of stylized facts consists in some qualitative properties of prices and returns time series $R_t = 100((P_{t+1} - P_t)/P_t)$, such as the emergence of bubbles and crashes of stock prices and volatility clustering. A second class includes indicators that aim at estimating the deviation from normality and the persistence of autocorrelation in returns distribution. Such two families of stylized facts are those usually considered in the existing literature. Additionally, we also take into account a third element of investigation, i.e. multifractality, which is observed in stock markets time series and that is identified as a constitutive element of complexity of those markets. More precisely, multifractality is an index to identify the presence of different long-range temporal correlations of observables. A detailed description of what multifractality is can be found in [37]. Understanding the origin of multifractality in financial markets is an issue that has been being addressed, e.g. in [7], and that in many real cases stemmed from the large fluctuations of prices [62].

We recall (see for instance [12]) that a stochastic process $\{X(t)\}$ is named multifractal if it has stationary increments and if

$$E(|X(t)|^q) = c(q)t^{\tau(q)+1}$$

where $t \in [0, T] \subset \mathbb{R}$ and $q \in [-q_0, q_0]$ are constants, while $c : [-q_0, q_0] \to \mathbb{R}$ and $\tau : [-q_0, q_0] \to \mathbb{R}$ are functions of q. The latter, called scaling function, is useful to discriminate between a monofractal process (where τ linearly depends on q) and a multifractal one (for which τ is a concave function of q). To estimate $\tau(q)$, in what follows we perform the Multi-Fractal Detrended Fluctuation Analysis (MFDFA) introduced in [37]. In agreement with the above mentioned literature about multifractality analysis of time series, to detect multifractality we adopt the following strategy. We evaluate the strength of the multifractality process, which is defined by $\Delta\alpha = \alpha_{\max} - \alpha_{\min}$, where $\alpha_{\max}$ and $\alpha_{\min}$ are respectively the maximum and the minimum values of $\alpha(q) = \tau'(q)$. This is an index of the concavity (and, hence, multifractality) degree of $\tau(q)$. Moreover, we study the behavior of $\Delta\alpha$ on increasing the length N of the considered time series, which has to be monotonically increasing for multifractality to be present. Finally, to rule out the possibility that multifractality is due to a broad probability density function of the time series rather than due to long time correlations, we repeat the evaluation of $\Delta\alpha$ considering randomly shuffled time series.

As a reference example, in Figure 5 we report the time series of the returns, the autocorrelogram, the plots of $\Delta\alpha$ depending on N for the actual time series and the shuffled one. The last panel in Figure 5 represents an estimation of the sentiment perceived by the agents, obtained removing the trend from the price series and plotting the resulting zero mean detrended time series (blue line), which actually provides an estimation of the market sentiment. We identify optimistic and pessimistic periods by simply observing its sign and we highlight them through the orange lines. As we can see, we have the alternation of long lasting periods of optimism and pessimism.

We stress that the example reported in Figure 5 is just a qualitative portrait of the prototypical aspects characterizing the time series of economic variables in financial markets. From the quantitative point of view, we may indeed have different values characterizing different stock indexes. However, the common features are the presence of spikes and perceptible volatility clustering in the returns time series (top left panel in Figure 5), slowly decreasing autocorrelation coefficients with significant positive coefficients for large lags (top right panel in Figure 5), multifractality due to a long range correlation of returns (middle row panels in Figure 5), polarization of consecutive periods of optimistic and pessimistic behavior (bottom panel in Figure 5). Consistently with the literature on financial

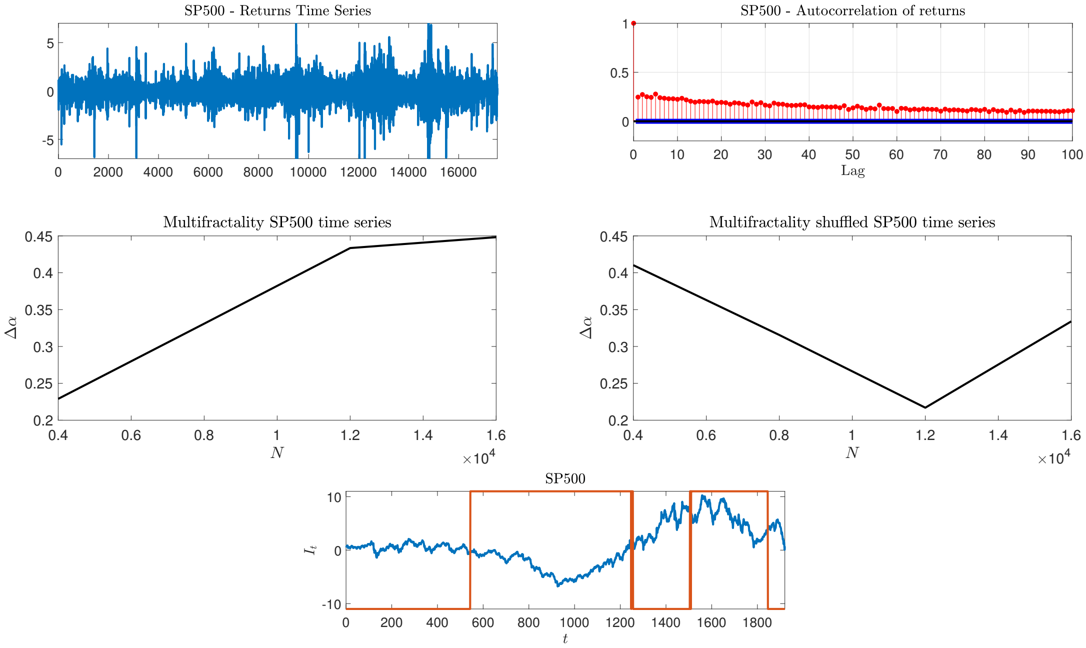


Figure 5: SP500 index stylized facts. The plots in the first row show the time series and the autocorrelation of returns; in the second row we report the multifractality strength $\Delta\alpha$ on increasing the length N of the sample for the original time series (left) and the shuffled one (right); the third row highlights the emergence of optimistic and pessimistic waves.

markets, in what follows we perform a model evaluation in order to check whether the model can qualitatively replicate the above mentioned aspects.

To such aim, we now move to the description of the stochastically perturbed version of model (6). In particular, we assume that the true fundamental value follows the random walk

$$F_{t+1} = F_t + \varepsilon_{F,t} F_t \,, \tag{9}$$

where $\{\varepsilon_{F,t}\}$ are normally distributed random variables with standard deviation $s_1 > 0$ and zero mean. Recalling that in the deterministic version of the model it holds that $X = F - \Delta/2$ and $Y = F + \Delta/2$, we now consider $X_t = F_t - \Delta/2$ and $Y_t = F_t + \Delta/2$, so that the condition $Y_t - X_t = \Delta$ is still fulfilled. We stress that also in [19] the bias about the fundamental remains unchanged, while the fundamental may vary. Moreover, as in [24], we introduce a random perturbation of beliefs, proportional to the price, i.e.,

$$X_{t+1} = F_t - \Delta/2 + \varepsilon_{X,t} P_t, \quad Y_{t+1} = F_t + \Delta/2 + \varepsilon_{Y,t} P_t, \tag{10}$$

where $\{\varepsilon_{X,t}\}$ and $\{\varepsilon_{Y,t}\}$ are normally distributed random variables with standard deviation $s_2 > 0$ and zero mean, which describe a temporary perturbation of the agents heterogeneity level.[12]
The shock on F is an improvement of that considered in [19], as in (9) the random component structurally depends on the size of the fundamental.[13] The resulting stochastic model is then obtained by replacing F, $X = F - \Delta/2$ and $Y = F + \Delta/2$ with F_t, $X_t = F_t - \Delta/2$ and $Y_t = F_t + \Delta/2$, respectively, in the construction of the model in Section 2. This allows to get the stochastic version of System (6),

[12] Being both groups of agents fundamentalists, it is more economically reasonable to consider the same standard deviation for both beliefs perturbations. In any case, we have checked that the results we present are robust with respect to the introduction of a suitable asymmetry in the standard deviations of $\{\varepsilon_{X,t}\}$ and $\{\varepsilon_{Y,t}\}$.

[13] As noted in [24], an approach like that adopted in [19] is able to replicate only a first family of stylized facts, like bubbles, crashes and persistent price deviation from fundamental. However, in order to mimic fat tails, volatility clustering and long memory effects, a structural element has to be introduced in the description of random components.

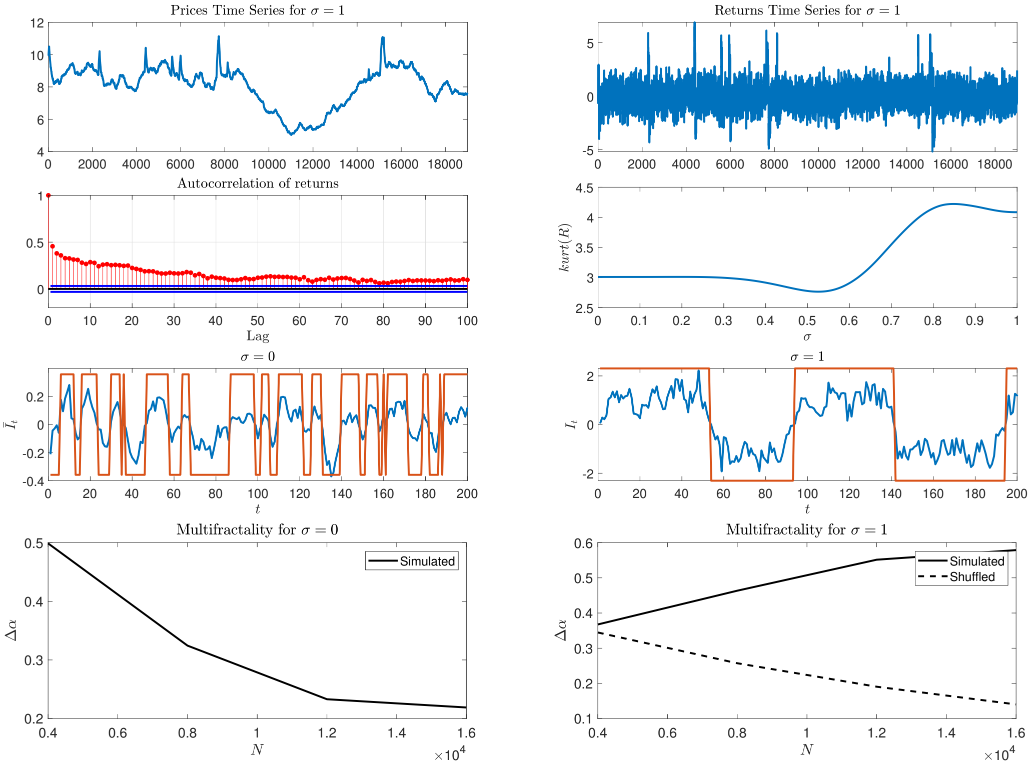


Figure 6: The plots in the first row show the time series of prices and returns; in the second row we show the autocorrelation of returns when $\sigma = 1$ and the kurtosis of returns distribution as σ increases; the third row highlights the emergence of optimistic and pessimistic waves in the market sentiment index for different values of the sentiment weight ($\sigma = 0$ in the left panel and $\sigma = 1$ in the right panel, respectively); the fourth row portrays the behavior of multifractality strength $\Delta\alpha$ as N increases for different values of the sentiment weight ($\sigma = 0$ in the left panel and $\sigma = 1$ in the right panel, respectively).

with F replaced by F_t, to which equations (9) and (10) have to be added. In particular, we note that the shock $\varepsilon_{F,t}$ enters the expression of X_{t+1} and Y_{t+1} through F_t, together with $\varepsilon_{X,t}$ and $\varepsilon_{Y,t}$.

We report some possible outcomes of our stochastic model in Figure 6, where we consider the parameter setting used for the simulation reported in the left panel of Figure 3, with the exception of γ that is set equal[14] to 0.02.

In the presence of exogenous shocks on the fundamental value, periods of high volatility in the price course may alternate with periods in which prices do not depart too much from the fundamental value. Such a behavior can arise when the parameter setting is located near the pitchfork bifurcation boundary and exogenous noise can occasionally spark long-lasting endogenous fluctuations around the new occurring steady states.

More precisely, the top left panel in Figure 6 displays a typical plot for the price time series obtained for $s_1 = 0.003$ and $s_2 = 0.035$, which highlights the very erratic price course with alternating bubbles and crashes. The corresponding time series of returns is reported in the top right panel of Figure 6, which still reflects the alternating periods with high and low volatility, and exhibits volatility clustering, highlighted by the strongly positive, slowly decreasing autocorrelation coefficients of absolute returns (left plot in the second row of Figure 6). Moreover, deviation from normality in the returns distribution only occurs as the herding phenomenon takes place, i.e., as the sentiment index plays an increasing

[14]This is agreement with [24], in which, when structural volatility is considered for the first model taken into account, parameter are changed so that “the price converges monotonically ... though only (very) slowly so”. This also enforces the random walk nature of the asset prices. We stress that results are qualitatively robust with respect to parameters modifications in suitable ranges.

relevant role in determining agents' choices, as shown in the right plot in the second row of Figure 6. The presence of fat tails implies that, when the sentiment index drives the market, large returns often occur, corresponding to strong movements in prices, and thus to more volatility in the financial market, in agreement with the well-known stylized facts empirically observed. The panels in the third row of Figure 6 compare the time series of the sentiment index I_t when $\sigma = 0$ and $\sigma = 1$ (respectively in the left and right plots). When $\sigma = 0$, it is possible to observe the emergence of periods of prevailing optimism or pessimism only if we consider a moving average $\bar{I}_t$ of I_t on a suitable number of periods (in the reported simulation $\bar{I}_t$ is computed considering, at each time period t, the last 5 values assumed by the sentiment index). This means that there is indeed alternation of periods characterized by the prevalence of a certain sentiment, but such phenomenon is quite weak and can be perceived only considering an average behavior over a suitable amount of periods. Conversely, when agents choose strategies on the basis of the sentiment index, there exist waves of optimism and pessimism that are much more long-lasting than when the influence from behavioral aspects is neglected in agents choices. Namely, in such latter case optimism and pessimism quickly alternate due to a continuous and recurrent evaluation of market beliefs based only on the market performance. The rationale for the occurrence of the waves of optimism and pessimism can be explained as follows: suppose that agents have the choice of using biased beliefs about the fundamental value of the asset, and that they seek to opt for the one that provides them a higher profitability. When the price volatility is low, the biases would not diverge too much from the fundamental and agents act more or less independently (some of them being optimists and the others pessimists). Accordingly, the market maker price adjustment will not be too strong and the price volatility remains low. In other words, the negative feedback induced by the traders compounds the one of the market maker, and prices may converge, with alternating periods of optimism and pessimism. On the other hand, when the price dynamics is more turbulent, agents may prefer to observe other agents' choices more closely and possibly imitate them. The resulting herding behavior implies that agents' choices become increasingly aligned (i.e. they behave less independently), eventually on values that differ from the fundamental steady state. This may be the case in which the optimistic and pessimistic steady states emerge. In such an eventuality, agents orders are less balanced around the fundamental value and the market maker is no longer able to mediate among them. Therefore, the market maker's price adjustments over/under react to those misalignments and the volatility remains high.
The emergence of long-lasting alternating periods of optimism and pessimism can be understood in the light of the stability analysis reported in Subsection 2.1. As we have shown, the most economically relevant phenomenon occurring when the market is driven by the general mood is that polarized states emerge, in terms of both possible steady states, attractors and basins of attractions. In the simulation reported in Figure 6, for $\sigma = 1$ the fundamental steady state is unstable and deterministic trajectories can converge toward the polarized steady states. Since the stock price is affected by shocks, thanks to the "polarized" structure of the basins of attractions, the trajectories persist in the basin of the same attractor (e.g. of the optimistic one) for several periods, until a random deviation moves them into the basin of the other attractor (e.g. of the pessimistic one), in which trajectories wander until a similar phenomenon drives them into the basin of the former attractor. In this process, prices are close to a random walk, with optimistic and pessimistic agents frequently switching between the two strategies.

Finally, in the last row of Figure 6 we compare the multifractality of the time series when agents choose their strategy on the basis of the profit evaluation only ($\sigma = 0$) or just on the basis of the sentiment perception ($\sigma = 1$). The results are obtained considering the average values derived from 1000 simulations. As we can see, in the former case $\Delta\alpha$ is decreasing, in contrast with a typical multifractal pattern. For such reason, we do not report the plot of $\Delta\alpha$ obtained with shuffled time series, which is similar to the original one and does not provide further relevant information. Conversely, in the latter case $\Delta\alpha$ is increasing (right panel of Figure 6, solid line), providing evidence for multifractality with a significant strength when $N = 16000$. The result is corroborated by the decreasing behavior of the shuffled time series (right panel of Figure 6, dashed line), which confirms that the source of multifractality is the long time correlation of the observables.

All the previous considerations suggest that the more refined modeling of animal spirits behavior allows for a better agreement between simulated and real time series.

3 Introducing asymmetry in the belief biases

We here extend the baseline setting considered in Section 2 by assuming that the belief biases may differ between optimists and pessimists. In particular, we will suppose that the belief of pessimists is given by $X = F - \Delta_X$ and that the belief of optimists is given by $Y = F + \Delta_Y$, with $\Delta_X, \Delta_Y > 0$ possibly not coinciding, so that we obtain the baseline framework when $\Delta_X = \Delta_Y = \Delta/2$. We still denote the distance between the two biases by $\Delta = \Delta_X + \Delta_Y$, but we remark that when biases are symmetric, as in Section 2, it is enough to specify Δ to characterize Δ_X and Δ_Y, while the present setting can be described providing both Δ and the ratio between the two biases. Moreover, the scenarios arising for $\Delta_X < \Delta_Y$ and $\Delta_X > \Delta_Y$ provide specular results. Hence, in what follows we just consider the former one, i.e., we assume that

$$\Delta_X < \Delta_Y \ \Leftrightarrow \ r = \frac{\Delta_Y}{\Delta_X} > 1, \tag{11}$$

adding some comments to describe how results can be rephrased when we instead consider $\Delta_X > \Delta_Y$. We note that r represents the degree of asymmetry between biases and that the farther r is from 1, the more asymmetric the biases are.

In the new setting, the total excess demand[15] reads as

$$D_t = \omega_t \alpha (F - \Delta_X - P_t) + (1 - \omega_t)\alpha(F + \Delta_Y - P_t),$$

where $\alpha > 0$ is the demand reactivity parameter. Thus the price variation with asymmetric biases is described by

$$P_{t+1} - P_t = f(\gamma D_t) = f(\gamma\alpha(\omega_t(F - \Delta_X - P_t) + (1 - \omega_t)(F + \Delta_Y - P_t))),$$

with $\gamma > 0$ representing the price adjustment reactivity and $f : \mathbb{R} \to (-a_2, a_1)$, with $-a_2 < 0 < a_1$, satisfying the conditions described in Section 2. The share updating rule is still based both on a comparison between the profits of the two groups of agents and on the sentiment index, whose expression reads as

$$I_t = \omega_t(F - \Delta_X) + (1 - \omega_t)(F + \Delta_Y) - F = (1 - \omega_t)\Delta_Y - \omega_t \Delta_X. \tag{12}$$

We observe that $I_t > 0$ when $\omega_t < \Delta_Y/(\Delta_X + \Delta_Y) = r/(1 + r)$, where r is the bias ratio[16]. Hence, as expected, optimism prevails when the share of pessimists is low enough.
The evolutionary mechanism is still represented by (4), i.e.,

$$\omega_{t+1} = \frac{e^{\beta(\sigma(-I_t)+(1-\sigma)\pi_{X,t+1})}}{e^{\beta(\sigma(-I_t)+(1-\sigma)\pi_{X,t+1})} + e^{\beta(\sigma I_t+(1-\sigma)\pi_{Y,t+1})}} = \frac{1}{1 + e^{\beta(2\sigma I_t + (1-\sigma)(\pi_{Y,t+1} - \pi_{X,t+1}))}}, \tag{13}$$

[15] In order not to overburden notation, we will use the same symbols introduced in Section 2 to denote analogous objects.

[16] As we shall see in the proof of Proposition 5, $\omega = \frac{r}{r+1}$ is the inflection point of the map associated to the right-hand side in (A8). We stress that for $r = 1$, i.e., in the absence of asymmetry, we find that the inflection point coincides with $\omega = \frac{1}{2}$, in agreement with the results obtained in Section 2. In this respect, we also remark that when $r \to 1$, the left-hand side in (15) tends to 2, making the similarity between the statements of Propositions 1 and 5 even more apparent.

with $\beta > 0$ representing the intensity of choice, $\sigma \in [0,1]$ describing the sentiment weight and with the profits defined as in (5), so that

$$\pi_{Y,t+1} - \pi_{X,t+1} = (P_{t+1} - P_t)(\Delta_X + \Delta_Y).$$

The model with asymmetric biases is then

$$\begin{cases} P_{t+1} = P_t + f(\gamma\alpha(\omega_t(F - \Delta_X - P_t) + (1-\omega_t)(F + \Delta_Y - P_t))), \\ \omega_{t+1} = \dfrac{1}{1 + e^{\beta(2\sigma((1-\omega_t)\Delta_Y - \omega_t\Delta_X) + (1-\sigma)(\Delta_X+\Delta_Y) f(\gamma\alpha(\omega_t(F-\Delta_X-P_t)+(1-\omega_t)(F+\Delta_Y-P_t))))}} \end{cases} \tag{14}$$

3.1 Steady states analysis

We investigate how the main analytical results on steady states and local stability derived in Subsection 2.1 get modified by the introduction of asymmetric belief biases.

In particular, although most of the conclusions of Proposition 5 still hold true without the biases symmetry assumption, in the present case it is possible to obtain explicit analytical expressions neither for any steady state, nor for the threshold identifying the occurrence of different scenarios. We have the following proposition.

Proposition 5. *Let $\varphi : [2, +\infty) \to \mathbb{R}$ be the strictly increasing function whose graph is reported in Figure 7 and let $\mathfrak{s} = \beta\sigma\Delta$ be the sentiment strength with $\sigma \in [0,1]$. System (14) has*

a) a unique steady state $S^0 = (P^0, \omega^0)$, with $\omega^0 \in (0, 1/2)$ and $P^0 > F$, provided that $\mathfrak{s} \leq 2$ or

$$\varphi(\mathfrak{s}) \leq 1 + r\,; \tag{15}$$

b) three steady states $S^0, S^1 = (P^1, \omega^1)$ and $S^2 = (P^2, \omega^2)$, with $r/(1+r) < \omega^1 < \omega^2$, provided that $\mathfrak{s} > 2$ and $\varphi(\mathfrak{s}) > 1 + r$. At $1 + r = \varphi(\mathfrak{s})$ the two coincident steady states $\omega^1 = \omega^2$ emerge through a fold bifurcation.

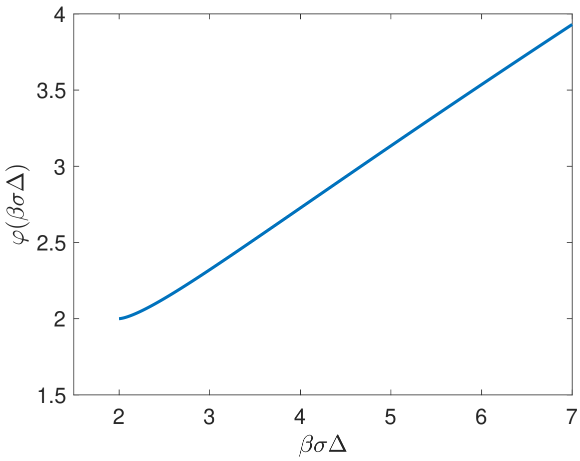


Figure 7: Graph of function φ as defined in Proposition 5.

The main novelty in Proposition 5 concerns the role played by the value of the new parameter r, namely the ratio of the belief biases, which measures their asymmetry, on the emergence of S^1 and S^2. We note that if r decreases but Δ is kept constant (so that the effect of the asymmetry is less relevant than the effect of the degree of heterogeneity), from the inequality $1 + r < \varphi(\mathfrak{s})$ we infer that the emergence of the polarized steady states is facilitated, while it is hindered in when r increases.

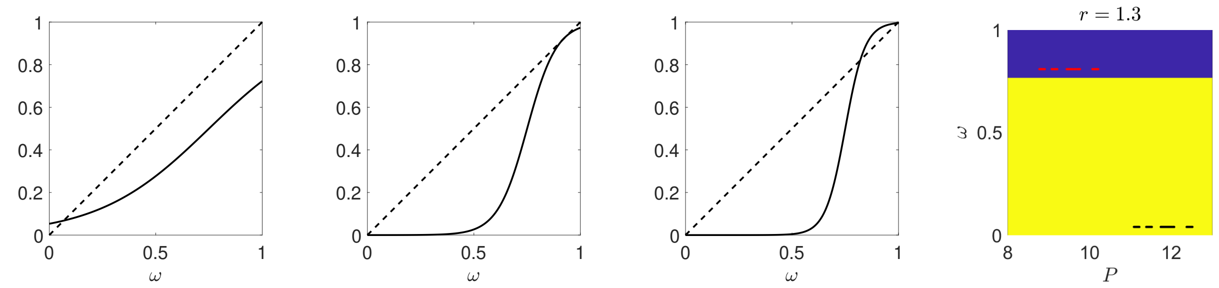
More generally, comparing Propositions 1 and 5, we can highlight several similarities and differences in the results concerning the existence of equilibria in the frameworks with symmetric and asymmetric biases. The key common feature is that in both cases, as the relevance of the role of the sentiment in the selection of strategies increases, two new steady states can arise. If $\sigma = 0$ or if it is too small, both frameworks are characterized by a unique steady state, confirming the role of an animal spirit behavior on the emergence of a multiplicity of steady states. Moreover, after that the three steady states have emerged, their position depend on the aggregate parameter $\mathfrak{s} = \sigma\beta\Delta$, i.e., on the sentiment strength, both for symmetric and asymmetric biases (see Propositions 2 and 6, respectively).

However, going into details, both the scenarios characterized by a unique steady state and that characterized by multiplicity present some own peculiarities. When $\Delta_X = \Delta_Y$, among the non-fundamental steady states we always have the fundamental one S^*, at which the price corresponds to the fundamental value and the population consists in identical shares of optimists and pessimists. Conversely, if $\Delta_X < \Delta_Y$ we have unbalanced biases in favor of optimism. When the sentiment strength is suitably small and only one steady state exists, this does not coincide any more with the fundamental one. Instead, due to a prevalent share of optimistic agents ($\omega^0 < 1/2$), the steady state is characterized by a price above the fundamental and hence by an overall resulting sentiment characterized by optimism ($I^0 > 0$). From the economic viewpoint, the rationale is very evident: the asymmetry in the biases reflects on the steady state features.

Concerning the couple of steady states emerging when the sentiment strength increases, we stress that, as a consequence of the proof of Proposition 1, in the symmetric framework ω^o and ω^p emerge through a pitchfork bifurcation when σ exceeds the threshold value $2/(\beta\Delta)$. Conversely, as it is evident from Proposition 5, with asymmetric belief biases, the condition $\sigma > 2/(\beta\Delta)$ is necessary, but no more sufficient to guarantee the emergence of two further steady states, which now lie on the same side with respect to $r/(1+r)$. In addition, it is indeed necessary that the asymmetry ratio r between the two biases is not too large with respect to their heterogeneity degree Δ. If the optimistic bias strongly dominates the pessimistic one, there is room for only one optimistically biased steady state. Conversely, if the sentiment strength is large enough and the two biases are suitably balanced, in the asymmetric framework the emergence of the two additional steady states occurs through a fold bifurcation, whose threshold value bears a strong resemblance to that of the pitchfork bifurcation in Proposition 1 (see also Footnote 16).

We illustrate the possible scenarios in Figure 8, where we plot the graph of function k fixing $\beta = 3$, and in (A)–(D) we let σ increase, setting also $\Delta_X = 1$ and $\Delta_Y = 3$, so that $\omega^0 \in (0, r/(1+r))$. Indeed $\omega^0 = 0.06 < 0.5$. More precisely, in (A) we consider $\sigma = 0.167 = 2/(\beta\Delta)$ and we observe that, under such condition, the tangent line at the graph of k in correspondence to the inflection point $r/(1+r)$ is parallel to the 45-degree line. Since, differently from the symmetric biases scenario, the inflection point is not a steady state, that value for σ is not large enough to let two further steady states emerge, in addition to ω^0. Indeed, as illustrated in Figure 8 (B), the fold bifurcation through which $\omega^1 = \omega^2$ emerge in $(r/(1+r), 1)$ occurs for $\sigma = 0.6$. Increasing further σ to 0.9 in (C) we witness two distinct values for ω^1 and ω^2 in $(r/(1+r), 1)$. The birth of a pessimistic steady state may be ascribed to the fact that, even if the optimistic bias dominates the pessimistic one, when the role of the market sentiment is sufficiently strong and the initial fraction of pessimistic agents ω is large enough, the role of the asymmetry in the biases (in particular the strength of the optimistic bias), being offset by the role of the market sentiment, is not able to lead the agents towards an optimistic majority. Instead they remain stuck to the pessimistic steady state. However this does not occur if the initial fraction of agents is more balanced and, in this case, we observe a convergence to the optimistic steady state, as the graph of Figure 8 (C) and the basin of attraction in Figure 8 (D) portray.

We stress that if we consider the case of a dominating pessimistic bias, namely opposite to (11), we have $\Delta_X > \Delta_Y$ and the asymmetry degree is better described by the ratio $r = \Delta_X/\Delta_Y$. In this case, similar results to those reported in Proposition 5 still hold true. The threshold discerning between the existence of a unique/multiple steady states is exactly the same (being now r the reciprocal of



(A) (B) (C) (D)

Figure 8: Panels (A), (B) and (C): the graph of map k for $\beta = 3$, $\Delta_X = 1$, $\Delta_Y = 3$, and $\sigma = 0.167$ in (A), $\sigma = 0.6$ in (B) and $\sigma = 0.9$ in (C). Panel (D): basin of attraction of the optimistic (yellow region) and pessimistic (blue region) attractors with asymmetric belief biases.

what defined in (11)), and the always existing steady state is now pessimistically biased, the share of pessimistic agents being larger than 1/2 and the price being smaller than the fundamental.

We now investigate how Proposition 2 is modified by the consideration of possibly not coinciding belief biases. Namely, in the following result we study how the position of the three steady states varies when the value of the main model parameters changes. In Proposition 6 we also analyze the variations in the values of the sentiment index I^0, I^1 and I^2 at S^0, S^1 and S^2, respectively.

Proposition 6. *Let $\sigma \in [0, 1]$ and the parameters configuration be such that $\mathfrak{s} > 2$ and $\varphi(\mathfrak{s}) = \varphi(\beta\sigma\Delta) > 1 + r$ holds true and steady states $S^1 = (P^1, \omega^1)$ and $S^2 = (P^2, \omega^2)$ have already emerged. Then, on increasing σ or β we have that ω^1, P^2 and I^2 decrease, while ω^2, P^1 and I^1 increase for all values of $\Delta_X \neq \Delta_Y$. As concerns ω^0, on increasing σ or β it decreases when $\Delta_X < \Delta_Y$, and it also decreases if the asymmetry in the bias increases, keeping the heterogeneity level fixed. Finally, P^0 and I^0 always have an opposite behavior with respect to ω^0 when raising σ or β.*

We stress that the proof of Proposition 6 allows us to draw some conclusions about the effect produced by an increase in σ or β on the position of ω^0, even when S^1 and S^2 have not emerged yet. Namely, since the map k with respect to σ or β is still decreasing for $\omega \in (0, r/(r + 1))$ and increasing for $\omega \in (r/(r + 1), 1)$, also in this case ω^0 decreases. That same feature of k allows concluding that, since S^1 and S^2 emerge in $(r/(r + 1), 1)$ and k is increasing on $(r/(r + 1), 1)$, then an increase in σ or β facilitates the emergence of S^1 and S^2. We remark that it is possible to show that such finding still holds when $\Delta_X > \Delta_Y$.
In regard to the effects produced by variations in the value of Δ_X and Δ_Y, we focus on the case in which $\Delta = \Delta_X + \Delta_Y$ is constant, while $r > 1$ changes. From a geometric argument similar to that employed in the proof of Proposition 6, we can conclude that, since the inflection point moves towards the 45-degree line when the difference in absolute value between Δ_X and Δ_Y decreases, i.e., when r falls, the stationary value for ω closer to $r/(r + 1)$, that is ω^1 since $\Delta_X < \Delta_Y$, moves in the same direction as $\omega = r/(r + 1)$.

Also concerning the comparative statics of the steady states in the symmetric and asymmetric frameworks, we can highlight some similarities and differences. The main difference is that in the symmetric framework, being the fundamental steady state S^* independent from the considered parameter configuration, when a unique steady state exists, that is not influenced by β, σ and Δ. On the other hand, in the asymmetric framework we have that S^0 becomes "more polarized" as the parameters describing the behavior of the agents become more extreme. We stress that the economic rationale of this last behavior is completely in line with the big picture of the results: the parameters describing the behavioral aspects of the agents tend to enhance any existent deviation from a perfectly symmetric and self-balanced situation. When $\Delta_X = \Delta_Y$, the two population groups are completely symmetric, and there is always a steady state reflecting the lack of any *ex-ante* exogenous asymmetry in the agents. However, as $\mathfrak{s}$ increases, any small deviation from such symmetric steady state can lead, through the previously described herding phenomenon, to the emergence of *ex-post* endogenously polarized steady states. Namely, in the asymmetric framework, there is an *ex-ante* exogenous asymmetry in the agents' beliefs, which, as already noted, reflects on the intrinsic asymmetry embedded in S^0 and which is further polarized by an increase in $\mathfrak{s}$. Moreover, when the two additional steady states emerge, the polarization of the two extreme steady states S^0 and S^2 increases as $\mathfrak{s}$ increases, like it happened to S^p and S^o in the symmetric framework. The difference is that in the symmetric framework the intermediate steady state is the fundamental one, namely the always existing equilibrium S^*, while in the asymmetric framework the intermediate steady state is S^1, that is one of the two steady states arisen through the fold bifurcation. However, despite such dissimilarity, the two frameworks share a common picture, in which an intermediate steady state separates the two increasingly polarized ones as the sentiment strength increases. This result is further confirmed by analyzing the basins of attraction of the two polarized steady states. We have already shown that if the beliefs of optimists and pessimists are respectively $F - \Delta/2$ and $F + \Delta/2$ (see the right panel in Figure 4) the symmetry of the underlying framework reflects on a "symmetric" path dependency with respect to the initial distribution of agents' shares. If we gradually introduce asymmetry in the biases by increasing the ratio r, we have that the basin of attraction of the optimistically polarized attractor increases. Figure 8 (D) is obtained using the same parameter setting used for that reported in Figure 4, but introducing asymmetry in the biases. As a consequence, most of the long run dynamics settle down to the optimistic attractor. We stress that, if we further increase the value of r used for the simulation reported in, as the ratio r between the optimistic and pessimistic biases raises the two attractors become more polarized and most of the long run dynamics settle down to the optimistic attractor, while the pessimistic attractor disappears.

In regard to the comparative statics, we remark that the previous considerations are consistent with the case of $\Delta_X > \Delta_Y$, too.

3.2 Stochastic simulations

We consider the same parameters setting as in the model with symmetric optimistic/pessimistic biases, with the unique difference that we now set $\Delta_X = 1.5$ and $\Delta_Y = 1.2$. We omit the price and return time series, as well as the behavior of kurtosis, as they essentially agree with those reported in Figure 6 for the symmetric biases framework. Moreover, we only focus on the case of $\sigma = 1$, since all the considerations we have made in Subsection 2.2 about the stochastic simulations for the case of $\sigma = 0$ with symmetric biases still hold true for the simulations we collected in the asymmetric biases case. As we can see from a comparison between the plots reported in Figures 6 and 9, the essential features of the stylized facts are still portrayed. However, a significant asymmetry in the biases leads to an asymmetric relevance of the optimistically and pessimistically polarized steady states in the deterministic model, which reflects also on the stochastically perturbed one. In the reported simulations, the stronger bias for the pessimistic agents leads to a pessimistic steady state that attracts more trajectories than the optimistic one, with a generally increased robustness of the pessimistic scenario. Such feature is evident also from the top right panel in Figure 9, from which we can see that even if periods of persistent pessimism and optimism alternate, those characterized by pessimism are more persistent and frequent. This also results in an increase in the autocorrelation effect, as it is clear comparing the top left panel in Figure 9 with the autocorrelation diagram for the symmetric bias case reported in Figure 6. Finally, even if multifractality still emerges from the simulation reported in the second row of Figure 9, the increasing trend of $\Delta\alpha$ is less evident than in Figure 6, as highlighted by the change in monotonicity when N is large enough. We stress that we checked that, if we further increased the asymmetry of bias, the previous phenomena would become much more relevant, while they would attenuate as $\Delta_X \approx \Delta_Y$. All these considerations seem to suggest that a strong asymmetry in the bias

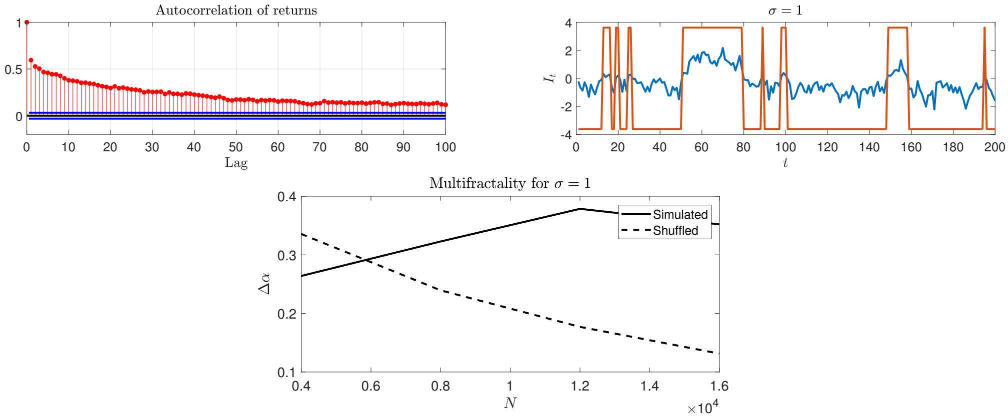


Figure 9: The plots in the first row show the autocorrelation of returns when $\sigma = 1$ and the alternation of optimistic and pessimistic waves in the market sentiment index; the second row shows the behavior of multifractality strength $\Delta\alpha$ for the original (solid line) and shuffled (dashed line) time series.

is not supported by the stochastic simulations and that the model with more balanced biased agents is able to provide more realistic results.

4 A model with technical traders

We shall now study the effects of enriching the set of agents types in order to account also for the presence of technical traders. It is well known (see e.g. [11] and [47]) that agents can choose their strategies not only on the basis of an evaluation of the fundamental value - being called fundamentalists -, but also on the basis of the market trend - being called technical traders -, and that the nonlinear interactions between technical traders and fundamentalists may cause complex price dynamics. In fact, the laboratory evidence (cf. [5] and [34]) and questionnaire studies (see e.g. [49], [50], [59]) indicate that market participants rely on technical or on fundamental trading rules to set their orders. The behavior of technical traders is usually characterized by a positive feedback about price movements, buying and selling when prices respectively increase and decrease, that is, when they observe significant price changes, thus putting greater trust in their technical trading signals. Since technical traders rely on extrapolative rules to forecast future prices and to take their position in the market, they tend to sustain and reinforce current price trends and to possibly amplify the deviations from the fundamental price. Accordingly, forming expectations considering not only fundamental but also technical analysis is consistent with short-run momentum and long-run reversal behavior in financial markets. Hence, because of such a positive feedback, the presence of technical traders can give rise to spontaneous herding phenomena, even when the agents switching choices only depend on profits. This is also a reason for they have not been taken into account in the model analyzed in Sections 2 and 3. We will show below that the results obtained for the model in (6) are robust with respect to the introduction of technical traders. In order to do so, while keeping the model simple and close to (6), we assume that the technical traders share $\omega_c \in [0, 1)$ is exogenously given[17], whereas the share of optimistic/pessimistic agents depends as in (6) on a fitness measure corresponding to a weighted average of a sentiment index and of profits. Namely, we still assume that the beliefs of fundamentalists[18] are optimistically (so that

[17] Such class of agents can be considered as a constant presence in the market due to the peculiar feature that characterizes its behavior. Namely, technical traders believe that stock prices tend to move in trends which persist for an appreciable length of time and that changes in trend are caused by shifts in demand and supply. Those shifts, no matter why they occur, can be detected sooner or later in the action of the market itself.

[18] We stress that the beliefs of fundamentalists regard the fundamental value, not the price. However, in this section, in order to keep the notation more homogeneous to that used for technical traders, we denote the beliefs of fundamentalists by P^e, as well.

they are $P^e_{o,t+1} = Y = F + \Delta/2$) or pessimistically (so that they are $P^e_{p,t+1} = X = F - \Delta/2$) biased, while the beliefs of technical traders are $P^e_{c,t+1} = P_{t-1} + \delta(P_{t-1} - P_{t-2})$, with $\delta > 0$ representing their reactivity. The demand of each kind of agent is $D_{i,t} = \alpha_i(P^e_{i,t+1} - P_t)$, with $i \in \{o, p, c\}$, where $\alpha_i > 0$ are demand parameters and we assume $\alpha_p = \alpha_o = \alpha_F$. The excess demand is then

$$ED_t = \sum_{i\in\{o,p,c\}} \alpha_i\omega_{i,t}(P^e_{i,t+1}-P_t) = \alpha_F\omega_{p,t}(X-P_t)+\alpha_F(1-\omega_{p,t}-\omega_c)(Y-P_t)+\alpha_c\omega_c(P_{t-1}+\delta(P_{t-1}-P_{t-2})-P_t).$$

The sentiment I_t perceived at time t by fundamentalists about the market is given by the difference between the beliefs of each agent and the fundamental, weighted by the share of each group. Now the sentiment index takes into account also the choices of technical traders, i.e.

$$\begin{aligned} I_t = \textstyle\sum_{i\in\{o,p,c\}} \omega_{i,t}(P^e_{i,t+1} - F) &= \omega_{p,t}(X - F) + (1 - \omega_{p,t} - \omega_c)(Y - F) + \omega_c(P_{t-1} + \delta(P_{t-1} - P_{t-2}) - F) \\ &= \omega_{p,t}X + (1 - \omega_{p,t} - \omega_c)Y + \omega_c(P_{t-1} + \delta(P_{t-1} - P_{t-2})) - F. \end{aligned}$$

The model is then given by

$$\begin{cases} P_{t+1} = P_t + f(\gamma ED_t) \\ \omega_{p,t+1} = (1 - \omega_c)\frac{e^{-\sigma\beta I_t+(1-\sigma)\beta\pi_{o,t+1}}}{e^{-\sigma\beta I_t+(1-\sigma)\beta\pi_{p,t+1}}+e^{\sigma\beta I_t+(1-\sigma)\beta\pi_{o,t+1}}} \end{cases} \tag{16}$$

We stress that, due to the presence of technical traders, the joint share of optimists and pessimists satisfies $\omega_{p,t}+\omega_{o,t} = 1-\omega_c$, and thus $\omega_{p,t},\ \omega_{o,t} \in (0, 1-\omega_c)$. Moreover, system (16) can be rewritten as a four-dimensional first order dynamical system by introducing the new variables $C_{t+1} = P_t+\delta(P_t-Z_t)$ and $Z_{t+1} = P_t$, obtaining

$$\begin{cases} P_{t+1} = P_t + f(\gamma ED_t) \\ \omega_{p,t+1} = (1 - \omega_c)\frac{e^{-\sigma\beta I_t+(1-\sigma)\beta\pi_{o,t+1}}}{e^{-\sigma\beta I_t+(1-\sigma)\beta\pi_{p,t+1}}+e^{\sigma\beta I_t+(1-\sigma)\beta\pi_{o,t+1}}} \\ C_{t+1} = P_t + \delta(P_t - Z_t) \\ Z_{t+1} = P_t \end{cases} \tag{17}$$

where

$$ED_t = \alpha_F\omega_{p,t}(X - P_t) + \alpha_F(1 - \omega_{p,t} - \omega_c)(Y - P_t) + \alpha_c\omega_c(C_t - P_t)$$

and

$$I_t = \omega_{p,t}X + (1 - \omega_{p,t} - \omega_c)Y + \omega_cC_t - F.$$

4.1 Analytical Results

We now show that the introduction of technical traders does not affect the results of Section 2.1 about the possible steady states scenarios and their comparative statics. We summarize the generalization of the results encompassed in Propositions 1 and 2 in the following:

Proposition 7. *System* (17) *has*

a) a unique steady state $S^* = (P^*, \omega^*_p, C^*, Z^*) = \left(F, \frac{1-\omega_c}{2}, F, F\right)$ *if* $\sigma \in [0, 1]$ *and*

$$\mathfrak{s} = \beta\sigma\Delta \leq 2;$$

b) three steady states $S^*, S^o = (P^o, \omega^o, P^o, P^o)$ *and* $S^p = (P^p, \omega^p, P^p, P^p)$ *if* $\frac{2}{\beta\Delta} < \sigma \leq 1$. *In particular,* S^o *and* S^p *are symmetric w.r.t.* S^**, with* $P^p < P^* < P^o$ *and* $\omega^o < \omega^* < \omega^p$*. In this case, on increasing* σ, β *and* Δ *we have that* ω^o *decreases, while* P^o *and* I^o *increase, and that* ω^p *increases, while* P^p *and* I^p *decrease.*

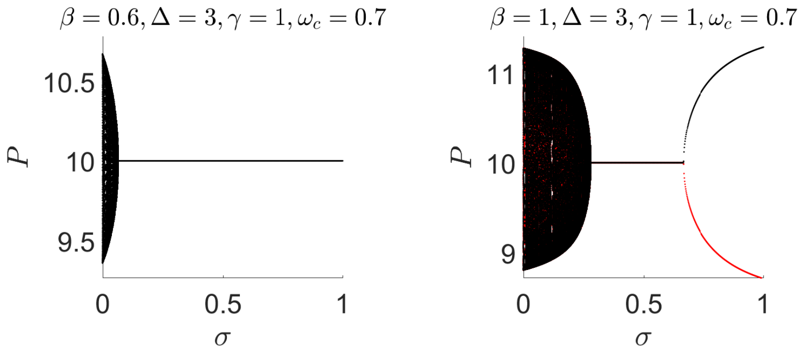


Figure 10: Bifurcation diagrams on varying σ for different values of the intensity of choice β. The parameter values not reported in the plots are: $\alpha_F = 3.5$, $\alpha_c = 0.4$, $\delta = 0.6$. With a stable price mechanism, even a large share of technical traders does not amplify the dynamics, which may settle down to the fundamental steady state S^* (left panel) or to the optimistic/pessimistic steady states S^o, S^p (in black and red, respectively, in the right panel).

The previous proposition shows that the results about the possible steady state scenarios are essentially unaffected by the introduction of technical traders. We again have that a herding phenomenon can take place only when a (whatever small) share of agents takes into account the market sentiment in choosing the strategy to adopt. The economic interpretation for this result is the same of that in Section 2.1, which is valid for the model with technical traders as well.

Concerning the dynamics, due to the high dimensionality of the model, we limit our analysis to present some possible scenarios, to show that also with technical traders all the conclusions drawn in Section 2.1 still hold. If we consider a stable price mechanism (i.e., if $\gamma < 2$ according to Footnote 11), the bifurcation diagrams with respect to γ are basically unaffected by the introduction of an even large share of technical traders, as highlighted by a comparison between Figures 3 and 10.
Conversely, when the price mechanism is unstable, the introduction of technical traders can spread instabilities in the market, as shown in Figure 11, although the polarization of attractors still occurs. This may be due to the fact that a turbulent market is considered as more attractive by the technical traders with higher chances of realizing gains. In turn, if technical traders act aggressively, they may lead the price to divert even from the optimistic or pessimistic steady states, with consequent possible oscillations around them, along trajectories where prices may trace out bubble or crash paths from time to time.

4.2 Stochastic simulations

We calibrate the model considering the same setting used for the simulations of the model with fundamentalist agents and symmetric bias, with the only exceptions of β, which is now set equal to 1, and of the standard deviations of the stochastic terms, which are now $s_1 = 0.075$ and $s_2 = 0.039$. The parameters related to the new components of the model with technical traders agents are $\omega_c = 0.9, \delta = 0.1, \alpha_F = 3.5$ and $\alpha_c = 3.5$. In agreement with the literature on financial markets (see e.g. [59]), we consider a large fraction of technical trader agents. We have thoroughly checked that all the results and considerations we are going to present are still valid when smaller fractions of technical traders are considered, with a possibly different parameters calibration. As we can see from Figure 12, the results obtained therein for $\sigma = 1$ are in very good agreement with those reported in Figure 5. Enriching the original model in (6) with the presence of technical traders, further improves the quality of the results reported in Figure 6, in particular with respect to the autocorrelation and the persistence of consecutive periods of optimistic and pessimistic behaviors, maintaining the same deviation from normality and the multifractal behavior. Nonetheless, although the model in (16) with $\sigma = 0$ performs better than the model in (6) with $\sigma = 0$ and provides a time series description that is qualitatively consistent with

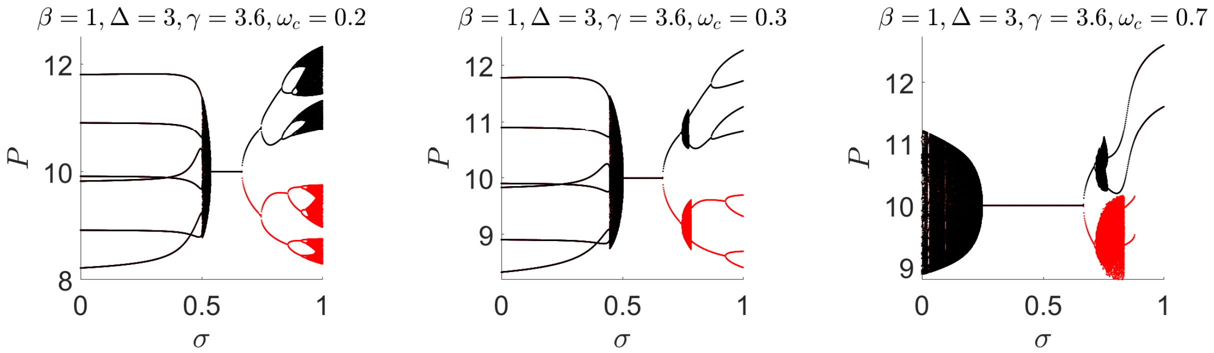


Figure 11: Bifurcation diagrams on varying σ for different values of the share of technical traders ω_c. The parameter values not reported in the plots are: $\alpha_F = 1.25$, $\alpha_c = 0.6$, $\delta = 0.4$. When the price mechanism is unstable, the introduction of technical traders may amplify the resulting dynamics in a way that, after the occurrence of the pitchfork bifurcation, adds further complexity issues, with the occurrence of periodic dynamics due to a period-doubling bifurcation (left panel) or of endogenous oscillations around the polarized regimes as the effect of a secondary Neimark-Sacker bifurcation (central and right panels).

that of SP500, when $\sigma = 0$ we have larger autocorrelation coefficients that decrease faster than those obtained for $\sigma = 1$, as well as a reduced deviation from normality. Finally, the dissimilarity between the time series of the sentiment index in Figures 5-6 and Figure 12 is highly significant: when $\sigma = 0$ in the latter we have very nervous consecutive variations of optimistic and pessimistic behaviors, that differ a lot from those of the real index (even when compared to those obtained from the simulation of the model with fundamentalist agents only reported in the right panel of the third row of Figure 6). The introduction of technical traders seems to amplify the tendency of agents to randomly switch between optimistic and pessimistic behaviors when the choice of the strategy depends on the profit evaluation. Conversely, when the role of the market sentiment dominate the expectation selection mechanism, its polarizing effect is reinforced by technical traders and results in long lasting periods of optimistic/pessimistic perception of the market. We do not report a plot about multifractality when $\sigma = 0$ since in this case the behavior is quite similar to that obtained in Figure 12 for $\sigma = 1$, the unique difference being a less steep increasing/decreasing behavior of $\Delta\alpha$ in the former case.

5 Concluding remarks

In the economic literature there exist several contributions (e.g. [13, 19, 51]) showing that, diverting from the perfect rationality assumption on the agents behavior, the waves of optimism and pessimism observed in financial markets can be explained in terms of endogenous fluctuations originated by the evolutionary selection of simple heterogeneous heuristics and/or by imitation mechanisms in forming beliefs about the fundamental. However, changes in the psychological and emotional perception of the market are not only consequences of the agents' choices, being also part of the process on which decisions are taken. When the mechanism which regulates the evolutionary selection of forecasting rules is based on a combination of the average mood perceived by the agents about the status of the market and a precise evaluation of the profits, new economic regimes arise, different from those occurring when agents decisions are not driven by "animal spirits". Such regimes are characterized by persistently polarized levels of optimism and pessimism, highlighted by high/small beliefs and prices, as well as by a large share of optimists or pessimists. An excess of optimism and pessimism, or an overconfidence placed on technical analysis, may endogenously generate outcomes which can be seen as the result of a self-sustaining herding phenomenon. On the other hand, endogenous waves of optimism and pessimism are not ruled out by animal spirits, especially when decision mechanisms are based on both market sentiment and profits evaluation. Moreover, as the role of the sentiment

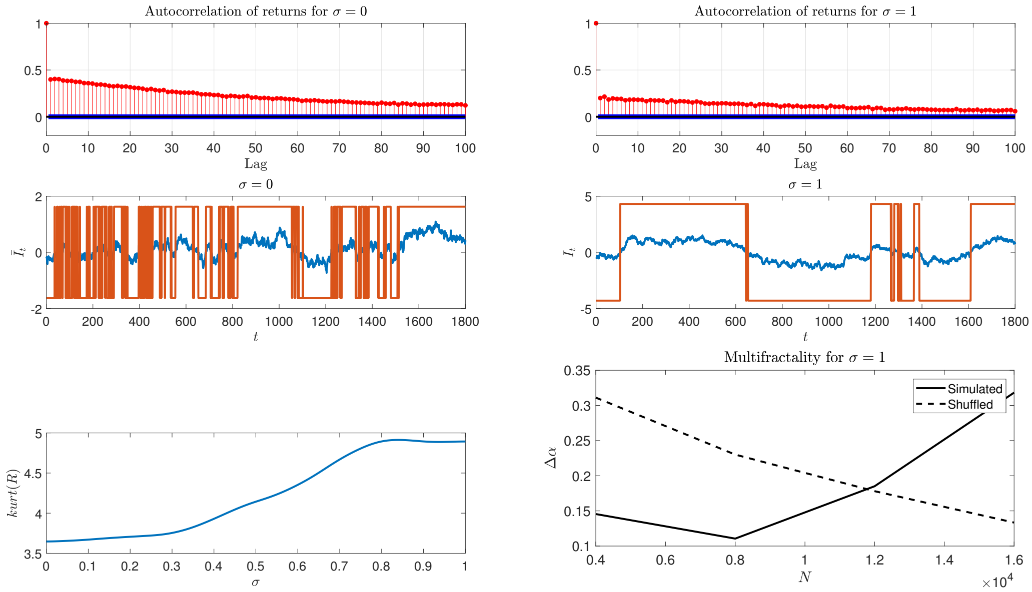


Figure 12: The plots in the first row show the autocorrelation of returns when $\sigma = 0$ and $\sigma = 1$; the plots in the second row show the alternation of optimistic and pessimistic waves in the market sentiment index when $\sigma = 0$ and $\sigma = 1$; in the last row, the left plot shows the kurtosis on varying σ, while the right plot reports the behavior of multifractality strength $\Delta\alpha$ for the original (solid line) and shuffled (dashed line) time series when $\sigma = 1$.

index becomes predominant, those waves are reinforced by possible endogenous dynamics around self-fulfilling economic regimes and, when non-deterministic effects are taken into account, give rise to alternating long-lasting periods of polarized economic regimes. Our future researches will aim at deepening the study of the role of animal spirits as the drivers of economic decisions, extending the pursued approach to other macroeconomic frameworks, also involving the real market side.

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Appendix

A Proof of the analytical results

Proof of Proposition 1. A straightforward check shows that $(P^*, \omega^*) = (F, 1/2)$ is always a steady state. However, in general, it is not the only one. Indeed, setting $P_{t+1} = P_t = P$ and $\omega_{t+1} = \omega_t = \omega$ in (6), we obtain $F - P + \Delta(1/2 - \omega) = 0$, from which, recalling (3), it follows that

$$P = F + \Delta(1/2 - \omega) = F + I \tag{A1}$$

and

$$\omega = \frac{1}{1 + e^{\beta\Delta\sigma(1-2\omega)}}. \tag{A2}$$

Equation (A2) is indeed solved by $\omega^* = 1/2$, so that by (A1) we obtain $P^* = F$.

Let us introduce function $h : (0, 1) \to \mathbb{R}$ whose expression is given by the right-hand side of (A2). We have that $h(1/2) = 1/2$ and, when extending the domain of h to $\mathbb{R}$, $\lim_{\omega\to-\infty} h(\omega) = 0$ and $\lim_{\omega\to+\infty} h(\omega) = 1$. A straightforward check shows that $h'(\omega) > 0$ and that $h'(1/2) = \beta\Delta\sigma/2$. Moreover, h is convex (resp. concave) for $\omega < 1/2$ (resp. $\omega > 1/2$) with $h''(1/2) = 0$.

From the previous considerations, recalling that the left-hand side of (A2) is ω, we can conclude that if $h'(1/2) \leq 1$ there is exactly one solution to (A2), that is, $\omega^* = 1/2$, while if $h'(1/2) > 1$ there are three distinct solutions to (A2). Indeed, it is easy to show that h is symmetric w.r.t. $\omega = 1/2$, i.e., that $h(1/2 + \varepsilon) - h(1/2) = h(1/2) - h(1/2 - \varepsilon)$ for every $\varepsilon > 0$, which, recalling that $h(1/2) = 1/2$, reduces to $h(1/2 + \varepsilon) = 1 - h(1/2 - \varepsilon)$.
Noting that $h'(1/2) \leq 1$ corresponds to (7) allows concluding that, when (7) is violated, (A2) is solved by some $\omega^o < 1/2 < \omega^p$, which satisfy $\omega^o = 1 - \omega^p$, and for which, by (A1), there is a correspondence to $P^p < F < P^o$, respectively. Moreover, replacing in (A1) ω with $1 - \omega$ we obtain $F - I$, which means that P^o and P^p are symmetric with respect to F. □

Proof of Proposition 2. We consider (A2), in which we put in evidence the dependence on the parameter we study of its right-hand side, that we still call h as in the proof of Proposition 1. We start noting that h is greater (resp. smaller) than $1/2$ for $\omega > 1/2$ (resp. $\omega < 1/2$) and we recall that it is continuous on $(0, 1)$. Since the roles of σ and Δ in the expression of h is exactly the same as that of β we can just deal with this last parameter. We have that if $\beta_1 < \beta_2$ then $h_{\beta_2}(\omega) > h_{\beta_1}(\omega)$ (resp. $h_{\beta_2}(\omega) < h_{\beta_1}(\omega)$) for $\omega > 1/2$ (resp. $\omega < 1/2$). A simple geometrical consideration allows concluding that ω^o (resp. ω^p) is strictly decreasing (resp. increasing) with respect to β.

Finally, from the result about ω^o, by (A1) we have that $P^o = F + \Delta(1/2 - \omega^o)$ increases as β, σ or Δ increase and I^o increases, too. □

Proof of Proposition 3. The local asymptotic stability of S^* is guaranteed if

$$\begin{array}{l} 1 + \det(J^*) + \mathrm{tr}(J^*) > 0 \Leftrightarrow -\Delta\beta\sigma\,(\gamma + \Delta\gamma - 2) + \gamma(\Delta^2\beta - 2) + 4 > 0 \\ 1 - \det(J^*) > 0 \Leftrightarrow \Delta\beta\sigma\,(\gamma(2 + \Delta) - 2) - \Delta^2\beta\gamma + 4 > 0 \\ 1 + \det(J^*) - \mathrm{tr}(J^*) > 0 \Leftrightarrow 2 - \Delta\beta\sigma > 0 \end{array} \tag{A3}$$

where

$$J^* = \begin{pmatrix} 1 - \gamma & -\Delta\gamma \\ \dfrac{\Delta\beta\gamma\,(1 - \sigma)}{4} & \dfrac{\beta\Delta\,(2\sigma + \Delta\gamma\,(1 - \sigma))}{4} \end{pmatrix}$$

is the Jacobian matrix of System (6) evaluated at S^*. We recall that when stability is lost due to a violation of the first (resp. second) condition of (A3), steady state S^* incurs a flip (resp. Neimark-Sacker) bifurcation. Conversely, also recalling Proposition 1, the third condition of (A3) is the same as (7), so when it is violated a pitchfork bifurcation occurs.

If $\sigma = 0$, conditions (A3) become

$$\begin{cases} -2\gamma + \Delta^2\beta\gamma + 4 > 0 \\ -\Delta^2\beta\gamma + 4 > 0 \end{cases}$$

which easily provides $2(\gamma - 2) < \Delta^2\beta\gamma < 4$, while if $\sigma = 1$ we have

$$\begin{cases} (\Delta\beta + 2)(\gamma - 2) < 0 \\ \Delta\beta - 2 < 0 \\ \Delta\beta - 2 - \Delta\beta\gamma < 0 \end{cases}$$

which, after noting that the third condition is implied by the second one, leads to the assertion. $\square$

Proof of Proposition 4. Let us set $\gamma = 1$. The conditions in (A3) become

$$\begin{array}{l} \Delta\beta\sigma(1-\Delta) + \Delta^2\beta + 2 > 0 \\ \Delta^2\sigma\beta - \Delta^2\beta + 4 > 0 \\ 2 - \Delta\beta\sigma > 0 \end{array} \tag{A4}$$

As it is easy to check, the first condition in (A4) is satisfied for any $\sigma \in [0, 1]$, both when $\Delta \leq 1$ and when $\Delta > 1$. If fact, if $\Delta \leq 1$ the left-hand side is clearly strictly positive, while if $\Delta > 1$ we have $\Delta\beta\sigma(1-\Delta) + \Delta^2\beta + 2 > -\Delta^2\beta\sigma + \Delta^2\beta + 2 = \Delta^2\beta(1-\sigma) + 2$, which is indeed positive for any $\sigma \in [0, 1]$.
The third condition of (A4) is always satisfied if $\beta < 2/\Delta$ while if $\beta > 2/\Delta$ it holds true provided that

$$\sigma < \frac{2}{\Delta\beta}.$$

The second condition of (A4) is fulfilled for any $\sigma \in [0, 1]$ if $\beta < 4/\Delta^2$, otherwise it is fulfilled for $\sigma \in \left(\frac{\Delta^2\beta-4}{\Delta^2\beta}, 1\right]$ if $\beta > 4/\Delta^2$.

Stability is then unconditional if

$$\beta < \min\left\{\frac{4}{\Delta^2}, \frac{2}{\Delta}\right\}$$

while a pitchfork bifurcation occurs for $\sigma = 2/\Delta\beta$ if

$$\frac{2}{\Delta} < \beta < \frac{4}{\Delta^2}.$$

To have a mixed scenario, we need

$$\begin{cases} \Delta^2\beta > 4 \\ \Delta\beta > 2 \end{cases}$$

and

$$\frac{\Delta^2\beta - 4}{\Delta^2\beta} < \frac{2}{\Delta\beta}$$

which requires $\Delta^2\beta < 2\Delta + 4$, i.e

$$\max\left\{\frac{4}{\Delta^2}, \frac{2}{\Delta}\right\} < \beta < \frac{2\Delta + 4}{\Delta^2},$$

otherwise S^* is unconditionally unstable, i.e. for

$$\beta > \frac{2\Delta + 4}{\Delta^2}.$$

Finally, we have a stabilizing scenario if

$$\frac{4}{\Delta^2} < \beta < \frac{2}{\Delta}.$$

If we set $\gamma = 3$, stability is guaranteed for

$$\begin{aligned} &-\Delta\beta\sigma(3\Delta + 1) + 3\Delta^2\beta - 2 > 0 \\ &\Delta\beta\sigma(3\Delta + 4) - 3\Delta^2\beta + 4 > 0 \\ &2 - \Delta\beta\sigma > 0 \end{aligned} \tag{A5}$$

The first condition of (A5) is satisfied by

$$\sigma < \frac{3\Delta^2\beta - 2}{\Delta\beta(3\Delta + 1)} < 1$$

which requires $\Delta^2\beta > 2/3$ (since the rightmost condition is always fulfilled), otherwise it is never fulfilled for $\sigma \in [0, 1]$.
The second condition of (A5) is always satisfied if $\Delta^2\beta < 4/3$, while if $\Delta^2\beta > 4/3$ it can be rewritten as

$$\sigma > \frac{3\Delta^2\beta - 4}{\Delta\beta(3\Delta + 4)} < 1$$

where the rightmost inequality holds true independently of Δ and β.
The third condition in (A5) is the same as that in (A4).
If $\beta < 2/(3\Delta^2)$ then the steady state is unconditionally unstable.
If $2/(3\Delta^2) < \beta < 4/(3\Delta^2)$, then the steady state is stable on

$$\sigma \in \left[0, \min\left\{\frac{3\Delta^2\beta - 2}{\Delta\beta(3\Delta + 1)}, \frac{2}{\Delta\beta}\right\}\right). \tag{A6}$$

However

$$\frac{3\Delta^2\beta - 2}{\Delta\beta(3\Delta + 1)} < \frac{2}{\Delta\beta}$$

is equivalent to

$$\beta < \frac{6\Delta + 4}{3\Delta^2}$$

so it is granted if $\beta < 4/(3\Delta^2)$. Then (A6) reduces to

$$\sigma \in \left[0, \frac{3\Delta^2\beta - 2}{\Delta\beta(3\Delta + 1)}\right).$$

If $\beta > 4/(3\Delta^2)$, then S^* is stable on

$$\sigma \in \left(\frac{3\Delta^2\beta - 4}{\Delta\beta(3\Delta + 4)}, \min\left\{\frac{3\Delta^2\beta - 2}{\Delta\beta(3\Delta + 1)}, \frac{2}{\Delta\beta}\right\}\right).$$

We start noting that

$$\frac{3\Delta^2\beta - 4}{\Delta\beta(3\Delta + 4)} < \frac{2}{\Delta\beta}$$

if
$$\beta < \frac{2\Delta + 4}{\Delta^2}\,.$$
From the previous considerations we have that if
$$\frac{4}{3\Delta^2} < \beta < \frac{6\Delta + 4}{3\Delta^2} \left(< \frac{2\Delta + 4}{\Delta^2}\right)$$
then S^* is stable on
$$\sigma \in \left(\frac{3\Delta^2\beta - 4}{\Delta\beta(3\Delta + 4)}, \frac{3\Delta^2\beta - 2}{\Delta\beta(3\Delta + 1)}\right)$$
and on decreasing σ we have a Neimark-Sacker bifurcation at $\frac{3\Delta^2\beta-4}{\Delta\beta(3\Delta+4)}$, while on increasing it we have a flip bifurcation at $\frac{3\Delta^2\beta-2}{\Delta\beta(3\Delta+1)}$.

If
$$\frac{6\Delta + 4}{3\Delta^2} < \beta < \frac{2\Delta + 4}{\Delta^2}$$
S^* is stable on
$$\sigma \in \left(\frac{3\Delta^2\beta - 4}{\Delta\beta(3\Delta + 4)}, \frac{2}{\Delta\beta}\right)$$
and on decreasing σ we have a Neimark-Sacker bifurcation at $\frac{3\Delta^2\beta-4}{\Delta\beta(3\Delta+4)}$, while on increasing it we have a pitchfork bifurcation at $2/(\Delta\beta)$.

Finally, if $\beta > \frac{2\Delta+4}{\Delta^2}$ the steady state is unconditionally unstable. □

Proof of Proposition 5. Similarly to the proof of Proposition 1, setting $P_{t+1} = P_t = P$ and $\omega_{t+1} = \omega_t = \omega$ in (14), we obtain $\omega(F - \Delta_X - P) + (1 - \omega)(F + \Delta_Y - P) = 0$, from which, recalling (12), it follows that
$$P = F - \omega\Delta_X + (1 - \omega)\Delta_Y = F + I \tag{A7}$$
and
$$\omega = \frac{1}{1 + e^{2\beta\sigma((1-\omega)\Delta_Y - \omega\Delta_X)}} = \frac{1}{1 + \mathrm{e}^{2\beta\sigma\Delta_X((1-\omega)r-\omega)}}, \tag{A8}$$
where we recall that $r = \Delta_Y/\Delta_X$ is the bias ratio.
Introducing the function $k : (0, 1) \to \mathbb{R}$, $\omega \mapsto \frac{1}{1+\mathrm{e}^{2\beta\sigma\Delta_X((1-\omega)r-\omega)}} = \frac{1}{1+\mathrm{e}^{2\beta\sigma\Delta_X(r-\omega(1+r))}}$, when extending the domain of k to $\mathbb{R}$, we have that $\lim_{\omega\to-\infty} k(\omega) = 0$ and $\lim_{\omega\to+\infty} k(\omega) = 1$, and a straightforward check shows that k is strictly increasing and that it is convex (resp. concave) for $\omega < r/(1 + r)$ (resp. $\omega > r/(1+r)$) with $k(r/(1+r)) = 1/2$, $k'(r/(1+r)) = \beta\sigma\Delta_X((1+r)/2)$ and $k''(r/(1+r)) = 0$. Hence, $\omega = r/(1 + r)$ is a fixed point for k only when $\Delta_X = \Delta_Y$. Nonetheless, extending the domain of k to $[0, 1]$, since $k(0) = 1/(1 + \mathrm{e}^{2\beta\sigma r\Delta_Y}) > 0$ and $k(1) = 1/(1 + \mathrm{e}^{-2\beta\sigma\Delta_X}) < 1$, Bolzano's theorem applied to the map $k(\omega) - \omega$ guarantees the existence of a steady state value $\omega^0 \in (0, 1)$, to which a steady state value P^0 is associated by (A7). In particular, it holds that $r/(1+r) > 1/2$ as $\Delta_X < \Delta_Y$. Hence, $k(r/(1 + r)) = 1/2 < r/(1 + r)$ and, as already observed, $k(0) > 0$ guarantee that $\omega^0 \in (0, r/(1 + r))$ and it is the unique equilibrium therein.
More precisely, since when $r > 1$ we have that $k(1/2) = \frac{1}{1+\mathrm{e}^{\beta\sigma\Delta_X(r-1)}} < 1/2$, we can actually conclude that $\omega^0 \in (0, 1/2)$. A direct computation using (A7) shows that the corresponding value P^0 of the price is larger than the fundamental value, since the sentiment index I^0 is positive.
In the remainder of the proof we show that a new couple of steady states $\omega^1 < \omega^2$ emerges in $(r/(1 + r), 1)$ if and only if condition (15) is violated.
We stress that from the previous considerations, if the maximum slope of k, which is $\beta\sigma\Delta_X((1+r)/2) = \beta\sigma\Delta/2 = \mathfrak{s}/2$ and that is attained at the inflection point $\omega = r/(1+r)$, is smaller than 1, the function

k on $(r/(1+r),1)$ lies strictly below the 45-degree line and no other equilibria emerge.

This means that steady states ω^1 and ω^2 could emerge only if $\mathfrak{s}/2 > 1$ provided that function k becomes tangent to the 45-degree line, i.e. when a fold bifurcation occurs. The necessary and sufficient condition for this is that at some $\omega_f \in (r/(1+r),1)$ we have

$$\begin{cases} k'(\omega_f) = 1 \\ k(\omega_f) = \omega_f \end{cases} \Leftrightarrow \begin{cases} \frac{2\mathfrak{s}x_f}{(1+x_f)^2} = 1 \\ \\ \frac{1}{1+x_f} = \omega_f \end{cases} \tag{A9}$$

where we set

$$x_f = \mathrm{e}^{2\beta\sigma\Delta_X(r-\omega_f(1+r))}.$$

Solving for x_f the latter equation in (A9) and using the resulting expression in the former one, we find

$$2\mathfrak{s}\,\omega_f(\omega_f - 1) + 1 = 0$$

which, solved for ω_f, provides the unique solution larger than 1/2

$$\omega_f = \frac{\mathfrak{s} + \sqrt{\mathfrak{s}(\mathfrak{s}-2)}}{2\mathfrak{s}}.$$

Replacing such expression for ω_f in the latter equation in (A9) we find

$$\frac{1}{1 + e^{-\frac{\mathfrak{s}+\sqrt{\mathfrak{s}(\mathfrak{s}-2)}-r\mathfrak{s}+r\sqrt{\mathfrak{s}(\mathfrak{s}-2)}}{r+1}}} = \frac{\mathfrak{s} + \sqrt{\mathfrak{s}(\mathfrak{s}-2)}}{2\mathfrak{s}}$$

which solved for r provides

$$r + 1 = \frac{2\mathfrak{s}}{\mathfrak{s} - \sqrt{\mathfrak{s}(\mathfrak{s}-2)} - \ln\left(\frac{\mathfrak{s}-\sqrt{\mathfrak{s}(\mathfrak{s}-2)}}{\mathfrak{s}+\sqrt{\mathfrak{s}(\mathfrak{s}-2)}}\right)} = \varphi(\mathfrak{s}) \tag{A10}$$

where in the previous expression we have introduced the function $\varphi : [2,+\infty) \to \mathbb{R}$, $\mathfrak{s} \mapsto \varphi(\mathfrak{s})$. It is easy to check that φ strictly increasing.

Since k is convex on $(0, r/(1+r))$ and $k(r/(1+r)) < r/(1+r)$, the tangency point ω_f between the graph of k and the 45-degree line can lie in $(r/(1+r),1)$ only. Hence, if the ratio between the two biases and the sentiment strength $\mathfrak{s} = \beta\sigma\Delta$ are such that condition (A10) is satisfied, we have two new coincident solutions of (A8) in $(r/(1+r),1)$ corresponding to $\omega^1 = \omega^2 = \omega_f$, and a fold bifurcation occurs.

If, for a given $\mathfrak{s} > 2$ and considering the corresponding r defined by (A10), we take $\tilde{r} < r$ (keeping constant $\mathfrak{s} = \beta\sigma\Delta$, which means that $\beta\sigma\Delta_X$ increases while $\beta\sigma\Delta_Y$ decreases), we have that $k_r(\omega) < k_{\tilde{r}}(\omega)$, which guarantees in particular that $k_{\tilde{r}}(\omega_f) > k_r(\omega_f) = \omega_f$. Namely, k is decreasing in r. Applying Bolzano's Theorem to the map $k_{\tilde{r}}(\omega) - \omega$ we have that there exists $\omega^1 \in (r/(1+r), \omega_f)$ and $\omega^2 \in (\omega_f, 1)$, which guarantee that $\omega^i, i = 1,2$, exist when $1 + r < \varphi(\mathfrak{s})$. The corresponding price values P^1 and P^2 can again be found from (A7).

Let us now consider a given $r > 1$ and the corresponding $\mathfrak{s}$ defined by (A10), which is unique since φ is strictly increasing. If $\hat{\mathfrak{s}} > \mathfrak{s}$ (keeping fixed the ratio corresponding to the new value $\hat{\mathfrak{s}}$), proceeding as before and using that φ is strictly increasing, we again have $1 + r < \varphi(\hat{\mathfrak{s}})$, so that $\omega^i, i = 1,2$, exist. We stress that in the opposite situations we have just one solution, namely when we consider, for a given $\mathfrak{s} > 2$ and the corresponding r defined by (A10), $\hat{r} > r$ (keeping constant $\mathfrak{s}$) or, for a given $r > 1$ and the corresponding $\mathfrak{s}$ defined by (A10), $\tilde{\mathfrak{s}} < \mathfrak{s}$ (keeping fixed the ratio corresponding to the new value $\tilde{\mathfrak{s}}$). $\square$

Proof of Proposition 6. Let us consider (A8), in which we put in evidence the dependence on the parameter that we study of its right-hand side, that we still call k as in the proof of Proposition 5. We start recalling that k is continuous on $(0,1)$. Since the role of σ in the expression of k is exactly the same as that of β, we can just deal with the latter parameter. We have that if $\beta_1 < \beta_2$ then $k_{\beta_2}(\omega) > k_{\beta_1}(\omega)$ (resp. $k_{\beta_2}(\omega) < k_{\beta_1}(\omega)$) for $\omega > r/(r+1)$ (resp. $\omega < r/(r+1)$), where $r = \Delta_Y/\Delta_X$ is the bias ratio. Recalling that $\omega^0 \in (0, r/(r+1))$ and ω^1, $\omega^2 \in (r/(r+1), 1)$, a simple geometrical argument allows concluding that ω^0 and ω^1 decrease, while ω^2 increases when raising β. Since by (12) it holds that

$$I^j = (1-\omega^j)\Delta_Y - \omega^j \Delta_X = \Delta_X(r - \omega^j(1+r)),\ j \in \{1,2,3\},$$

we can immediately infer that the behavior of I^j on increasing β is opposite with respect to that derived for ω^j, $j \in \{1,2,3\}$. If we keep Δ fixed and we raise r, we have that Δ_X decreases and Δ_Y increases. Hence, the exponent of e in the denominator of (A8) increases, namely function k is strictly decreasing with respect to r. This guarantees that ω^0 decreases.

Finally, recalling (A7), the results about P^j, $j \in \{1,2,3\}$, directly follow by those on I^j, $j \in \{1,2,3\}$. This completes the proof. $\square$

Proof of Proposition 7. In order to detect the model steady states we set $\xi_{t+1} = \xi_t = \xi^*$ for $\xi \in \{P, \omega_p, C, Z\}$ in (17), from which we obtain $Z^* = P^* = C^*$. Moreover, we have $ED^* = 0$, where $ED^* = \alpha_F\, \omega_p^*(X - P^*) + \alpha_F(1 - \omega_p^* - \omega_c)(Y - P^*)$, from which, recalling that $X = F - \Delta/2$ and $Y = F + \Delta/2$, we find

$$P^* = \frac{2F - \Delta\omega_c + \Delta - 2\Delta\omega_p^* - 2F\omega_c}{2(1-\omega_c)} \tag{A11}$$

which, used in the equation of ω_p, provides

$$\omega_p^* = \frac{1-\omega_c}{e^{\frac{\Delta\beta\sigma(1-\omega_c-2\omega_p^*)}{1-\omega_c}} + 1} \Leftrightarrow \frac{\omega_p^*}{1-\omega_c} = \frac{1}{e^{\Delta\beta\sigma\left(1-\frac{2\,\omega_p^*}{1-\omega_c}\right)} + 1}.$$

Setting $\omega = \frac{\omega_p^*}{1-\omega_c}$, we find

$$\omega = \frac{1}{e^{\Delta\beta\sigma(1-2\omega)} + 1}$$

which is exactly (A2). The previous implicit equation is studied in the proofs of Propositions 1 and 2. The results we have found for ω in those propositions can be easily rephrased in relation to $\omega_p^* = (1-\omega_c)\,\omega$, from which we have the existence of either one or three steady states depending on $\mathfrak{s} = \sigma\beta\Delta$, as well as their comparative static results. Indeed, when $\omega_p^* = (1-\omega_c)/2$, from (A11) we find $P^* = F$. Moreover, the symmetry of (A2) with respect to $\omega = 1/2$ provides the symmetry of ω_p with respect to $(1-\omega_c)/2 = \omega_p^*$, which reflects also on the other variables. $\square$