Strategy-Proof Allocations with Punishment

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Abstract

A mechanism chooses an allocation of the resource to agents based on their reported type. We discover and describe the set of incentive compatible mechanisms when a monetary punishment to agents who misreport type is possible. This class depends on the punishment function and the probability of punishment. It expands previous characterizations of incentive compatible mechanisms when punishment was no available. Furthermore, when the planner has the ability to select the punishment, the minimal punishment necessary to achieve incentive compatibility and the corresponding class of first-best mechanism is provided. For any punishment, optimal mechanism for the planner are introduced.

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1 Introduction

Consider a planner interested in transmitting a divisible resource to agents (such as money). Although the planner is not directly linked to the agents, it can do so via a group of intermediaries. Intermediaries differ in their ability to transmit the resource. This ability is represented by the total amount of the resource that an intermediary sends to the agents per unit of resource received, as well as by the proportions in which every agent receives a resource relative to another from a given intermediary. Thus, for instance, two intermediaries might be able to reach different agents, and even when they reach the same group of agents, they may transmit different amounts to the agents.

We study the case where the intermediaries' abilities are private information.[1] Therefore, the planner uses a direct mechanism, where intermediaries report their abilities, which are then used to determine the actual transmission rate to the agents, as well as the distribution of the resource among the different intermediaries. In such cases, intermediaries might be able to game the planner by misrepresenting their ability to transfer the resource to agents. Therefore, incentive compatibility of the mechanism, in our case strategy-proofness, is a desirable requirement.

Strategy-proofness is a very robust property that prevents intermediary to misrepresent their ability regardless of the reports of other intermediaries. Restricting to strategy-proof mechanisms might come with a high cost to the planner.[2] In some settings, the planner might be able to alleviate such a cost by enforcing truthful reporting by other means. Indeed, consider the case of auditing, where the planner has the ability to audit the intermediaries in the game (perhaps with some probability) and assign a punishment (expressed in monetary terms) for the intermediaries who are found misreporting their ability. For a given set of abilities, there is always a large enough punishment such that the intermediaries should not feel compelled to misrepresent their preferences. Indeed, any punishment such that the expected punishment is larger than the expected rewards gained by misrepresenting their preferences satisfies that. Thus a natural generalization of strategy-proofness extends the class of mechanisms that are strategy-proof when such a punishment are available for the planner.

The paper introduces a generalization of strategy-proofness when the planner has the ability to monitor and punish the intermediaries for misrepresenting their preferences. Indeed,

[1]In our companion paper, Han and Juarez[15], we study the case where the abilities of the intermediaries are public information. The planner solicits bids from intermediaries to the use of their links and applies this information to select which intermediaries to contract for the transmission of the resource. The main result of Han and Juarez[15] is the necessary and sufficient conditions for the existence of a free intermediation equilibrium, where there is a perfect transmission of the resource to the agents as if there is no intermediation.

[2]This cost is typically measured in efficiency terms, but can also be measured in equity or other terms.

in the domain of quasilinear preferences where money is available, we consider monetary punishments that will depend on an arbitrary function $h(\alpha_i, \beta_i)$, where $h(\alpha_i, \beta_i)$ is the punishment paid by the intermediary represented by the difference between his true ability α_i and reported ability β_i. A mechanism is h-strategy-proof if there is no incentive for any intermediary to misreport under the punishments h. When the planner does not have the ability to monitor the intermediaries, $h = 0$, our property boils down to the traditional strategy-proofness. On the other hand, when $h > 0$ is large, the intermediaries will be punished a large amount $h(\alpha_i, \beta_i)$ and the amount of misreporting will be substantially reduced. This allow us to capture all mechanisms, when $h \to \infty$.[3]

The main contributions of the paper are three-fold. First, it introduces a notion of strategy-proofness with monitoring and punishment. Second, it introduces a new model of resource transmission and intermediation in networks when the abilities of intermediaries are incomplete information and characterizes the entire class of strategy-proof mechanisms when monitoring and punishment are available to the planner. Furthermore, this paper studies the minimal punishment function for a mechanism to be strategy-proof, and the punishment function to achieve first best efficient allocation. On the other hand, the optimal mechanism for the planner is discovered given an arbitrary punishment function.

1.1 Illustrative Example

To illustrate our mechanisms and main results, consider the example of a planner who is connected to three intermediaries, who themselves are connected to two agents (see Figure 1). The planner is interested in transmitting I units of a resource to the agents, but can only do so via the intermediaries. Intermediaries have different quality of intermediation, represented by the proportion in which they transmit their share to the agents for every unit of resource transmitted. In this case, the abilities of intermediaries are $\alpha_1 = (0.7, 0.4)$, $\alpha_2 = (0.6, 0.6)$ and $\alpha_3 = (0.5, 0.8)$, respectively.

In the absence of information, the planner will ask intermediaries to report their abilities to transmit the resource and determine (a) the amount of resource allocated to every intermediary for transmission to agents and (b) the sharing rate charged for every intermediary to transmit at every link based on the information of report.[4] The intermediary's profit is the difference between his true abilities to transmit the resource and his charged sharing rates multiplied by the amount of resource allocated to him.

For instance, consider the traditional first price auction. When intermediaries report

[3]This is true for punishment functions h such that $h(\alpha_i, \alpha_i) = 0$ for all α_i and $h(\alpha_i, \beta_i) = 0$ for $\beta_i \neq \alpha_i$.
[4]For instance, we can imagine the case where the planner might use a second price auction.

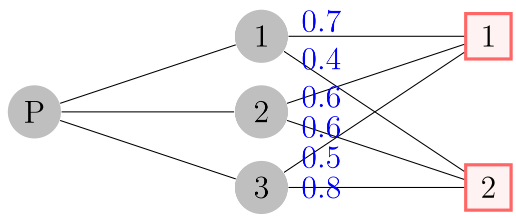

Figure 1: A network with three intermediaries and two agents.

abilities $(\beta_1, \beta_2, \beta_3)$, the planner selects the intermediary with highest reported aggregate ability to transmit all the resource with charged sharing rates $s_i(\beta) = \beta_i$. This mechanism is not strategy-proof. Indeed, when intermediaries report their true abilities $(0.7, 0.4)$, $(0.6, 0.6)$ and $(0.5, 0.8)$, intermediary 3 with aggregate ability 1.3 is selected to transmit all the resource and the charged sharing rates equals his ability $(0.5, 0.8)$. Thus, intermediary 3's profit equals 0. This is not strategy-proof, because he can decrease his report to $(0.5, 0.71)$, where he will get a positive profit equal to $[(0.5, 0.8) - (0.5, 0.71)] * (I, I)^T = 0.09I$.

Now, suppose the planner is able to audit the intermediaries. Suppose that the planner punishes the intermediaries based on the deviation from their true reports with punishment function $h(\alpha_i, \beta_i) = \sum_{m=1}^{M} |\beta_i^m - \alpha_i^m|$. In such a mechanism, intermediaries have no incentive to lie about their reports. Indeed, at the profile α above, when intermediary 3 reports $(0.5, 0.71)$ and planner finds that his true ability is $(0.5, 0.8)$, the expected punishment on intermediary 3 is $(|0.5 - 0.5| + |0.8 - 0.71|) = 0.09$. The expected payoff for intermediary 3 is $(0.09 - 0.09)I = 0$, the same as reporting $\beta_3 = \alpha_3$. Thus, there is no incentive for intermediary 3 to misreport, and first price auction is h-strategy-proof. The set of h-strategy-proof mechanisms expands the set of strategy-proof mechanisms.

1.2 Overview of the Results

We introduce the resource transmission problem of the planner in Section 2 and strategy-proof mechanisms in Section 3. We provide conditions for a mechanism to be h-strategy-proof for arbitrary punishment function h, when the punishment function is differentiable[5] at any truthfully report point (Lemma 1). The class of 0-strategy-proof mechanisms coincide with the class of strategy-proof mechanisms (Theorem 1). Thus, the class of h-strategy-proof mechanisms is largely depending on h, and the comparative static analysis studied in Proposition 1. Furthermore, we study the minimal punishment function for any mechanism in Section 4. Proposition 2 provides the necessary and sufficient condition for minimal

[5]More general punishment functions are discussed in the Appendix.

punishment function, Corollary 1 shows existence of minimal punishment function. We characterize the first-best efficient allocation in Section 4.1. Theorem 2 shows that there exists no symmetric, SP, budget balance and first-best efficient mechanism. The minimal punishment function to achieve first-best efficiency is provided. Finally, Theorem 3 discovers the optimal mechanism of any punishment function for the planner.

1.3 Applications

An application of our game theoretical model is the transmission of advertising money in companies. A company looking to promote their product can use different media (the intermediaries) to reach the advertising target of their product; such intermediaries include TV channels, radio stations, Internet websites, and newspapers. The quality of the connections is relevant because, within the media, there are different channels that target to specific demographics of agents and may influence the planner's objective differently. For instance, two local TV stations based in the same city may be connected to all agents in the city, but the audience may be more biased based on demographics or political preferences —e.g. Fox News and CNN reach the same audience, but they target their programming to attract more conservative or liberal viewers, respectively. Nowadays, the printed version of newspapers are read heavily by older people instead of younger people, and the proportions of older to younger readers are typically available to potential purchasers of advertisements. Therefore, it is in the interest of the planner to choose the media channel that best aligns with his preferences.

Alternatively, consider the case of government contracting. For instance, the allocation of government's money to people in need via charities. The government may decide to send the money via charities that will charge an indirect cost for the use of their services. The connections of the charities, as well as their quality, are exogenous information that the planner cannot control, and they are typically taken into account when making a decision on how to allocate the resources. For instance, charities heavily funded by the government include UNICEF or the Red Cross. While both charities overlap in some of the agents that they serve (e.g. children in need), they also have large difference in their recipients.[6] The quality of the connections of the charities is also important when picking a charity. For instance, inefficiencies happen often in charities and universities, where every dollar spent is often decreased due to indirect cost, which serves to pay for administration.[7] Thus, the

[6]Thus, for instance, the Federal Emergency Management Agency may be more interested in allocating money to the Red Cross, which distribute a large percentage of their resources to helping domestic citizens affected by disasters, as opposed to UNICEF which helps children around the world.

[7]This factor in the quality of the charities is so important that all charities in the US are required by law

planner should care about how their money is distributed to the agents and aligned with its preferences. Our model looks at the case of complete information, which is also the case in this example, as the priorities and activities of the charities are typically reported by them in advance.[8] As such, the planner can make an informed decision on how its money will transmit by the charities chosen.

Finally, the problem has applications to network flow problems. For instance, when there is groundwater that must be distributed to agents via private canals (intermediaries). The planner can decide how to route the water to the canals, but once the water reaches the canal it is distributed to the agents connected to these canals in some fixed proportions that may vary between canals. Conveyance losses are typical in models and may depend on how far the agents are from the source (Jandoc, Juarez, and Roumasset[17] study the optimal allocation of water networks in the presence of these losses). The owners of the canals may charge the planner for the use of their canals, and therefore the planner should consider the trade-offs between allocating goods to cheap canals as opposed to more efficient but expensive canals. The paper studies the case of exogenous quality of the intermediaries.

1.4 Related Literature

The literature on strategy-proofness when money is available has been widely explored. Indeed, the traditional VCG mechanisms in Vickrey[29], Clarke[11], Groves[14] are strategy-proof and efficient. However, one limitation of VCG mechanisms is that they are not budget balance, which does not apply to our model.

The large literature on social choice has been concerned with non-manipulable mechanisms, dating back from Arrow[1] and Gibbard[13], see Barberà[2] for an introduction to strategy-proof social choice functions. Such studies include the case of strategy-proof social choice functions in classical exchange economies (Barberà and Jackson[5]), matching with contracts (Hatfield and Kojima[16]), house allocation with prices (Miyagawa[21]), cost sharing (Moulin and Shenker[25], Moulin[24], Sprumont[27]), preference aggregation (Bossert and Sprumont[8]), social choice (Barberà, Dutta and Sen[4]).

Barberà, Berga and Moreno[3] study group strategy-proof mechanisms, in a general set-

to report the total percentage amount spent in their causes, as opposed to administrative costs. For instance, the current indirect costs for the Red Cross and UNICEF are 9.7% and 4.74%, respectively. Multiple online websites exist that rank charities based on the indirect costs, among other metrics.

[8]The Red Cross publishes at the end of each year 'its activities in the field and at the headquarters during the coming year,' which allow donors to make an informed decision on where the money will go. Earmarking is typically not allowed in such big charities, as 'experience shows that the more restrictive the earmarking policy (whereby donors require that their funds be allocated to a particular region, country, program, project or goods), the more limited the ICRC's operational flexibility, to the detriment of the people that the ICRC is trying to help.'

ting that includes the provision of private good and matchings such as house allocation. Moulin[23], Juarez[18] study group strategy-proof in cost sharing problems.

There is also strategy-proof mechanisms for restricted domain of preferences, such as the class of single-peaked preferences (Moulin[22]). Our focus in the paper is in the entire domain of quasilinear preferences, where the class of strategy-proof mechanisms that satisfy desirable conditions tends to be small. Hence our work expands the class of strategy-proof mechanisms that the planner can use.

There is also a more recent literature dealing with various relaxations and strengthening of strategy-proof notions. There are approximately strategy-proof mechanisms in voting (Birrell and Pass[7]), matching (Pathak and Sönmez[26]), and more generally, Carrol[9] finds that local strategy-proof with single-crossing ordinal preferences implies full strategy-proof. Obviously strategy-proof mechanisms in Li[19] refine the strategy-proof mechanisms by requiring the strategy to be obviously dominant. Pathak and Sönmez[26] develops a rigorous methodology to compare mechanisms based on their vulnerability to manipulation. Unlike this literature on strategy-proofness, our notion of manipulation depends on the punishment function h. This allows for a weaker notion of manipulation that expands the class of strategy-proof mechanisms that a planner can use, hence providing more flexibility when selecting mechanisms. Indeed, our more general version of strategy-proofness can be easily adapted to these settings.

In contrast with the literature on strategy-proofness, our mechanisms are specifically applied to a novel problem of resource transmission in a network. On this line of work, there is only one closely related, our companion paper, Han and Juarez[15], which study the strategic behavior of intermediaries in a more general resource transmission game. Unlike that paper, our model with incomplete information does not restrict the type of mechanisms to a first-price type of mechanism, instead, it characterizes a large class of mechanisms in a more specific resource transmission game in a network.

Townsend[28] first studies costly verification in a principal-agent model with a risk-averse agent. There is a growing interest in mechanism design problem with state verification, Ben-Porath et al.[6] study the principal allocating an indivisible good among agents with an ability to verify agents' type costly, and they don't allow transfer payments. They study the principal's trade-off between allocating the good more efficiently and incurring the cost of verification, and find the optimal mechanism to be a favored agent mechanism, where a pre-determined agent receives the good, unless another agent reports higher than threshold and agent with highest bid will get the good, if his report is verified to be true. Erlanson and Kleiner[12] study similar problem of costly verification in collective choice problem. Li[20] studies costly verification with limited punishment. On the other hand, Carroll and

Egorov[10] studies the mechanism of minimal verification to elicit multidimensional information fully by using a randomized verification strategy and allowing severe punishment. However, we study the mechanism design problem with planner allocating divisible good with report of multidimensional information, allowing exogenous probabilistic verification, and study the minimal punishment as the transfer payments to induce the strategy-proof for a mechanism and also to achieve the first-best allocation.

2 The Model

Each agent $i \in \mathcal{N} = \{1, \ldots, N\}$ is endowed with a **production function** $f_i(\theta_i, x_i) \in \mathbb{R}_+^M$ that generates an outcome $\mathbb{R}_+^M$ based on their type $\theta_i \in \mathbb{R}_+$ and the amount of capital investment $x_i \in \mathbb{R}_+$. We assume that information is asymmetric, agents know their own type but do not know others'.

A planner cares only about the outcome in $\mathbb{R}_+^M$. He is endowed with 1 unit of capital that can be used to invest among the agents to produce the outcome. We assume that the planner does not know agents' types. We study mechanisms where agents reveal their type to the planner, who makes an allocation of the capital among agents and transfers to outcomes based on their report. This is formalized below.

Definition 1 (Mechanism)
A mechanism $\phi = (x(\cdot), t(\cdot))$ is a pair of functions $(x(\cdot), t(\cdot))$ such that

i. $x : \mathbb{R}_+^N \mapsto \mathbb{R}_+^N$ allocates the share of a resource to every agent based on the reported type θ, and $x(\theta) = (x_1(\theta), \ldots, x_N(\theta))$, the amount $x_i(\theta) \in \mathbb{R}_+$ represents the resource allocated to agent i.[9]

ii. $t : \mathbb{R}_+^N \mapsto \mathbb{R}^{N\times M}$ tax of the resource generated by the agents that the planner keeps. Thus, for type θ and $t(\theta) = (t_1(\theta), \ldots, t_N(\theta))$, the vector $t_i(\theta) \in \mathbb{R}_+^M$ is the resource charged by the planner to agent i.

The planner cares about $\sum_{i\in N} t_i(\theta) \in \mathbb{R}^M$. On the other hand, agents care about the total resource generated that was not taken by the planner. For a vector of reports $\tilde{\theta} = (\theta_1, \ldots, \theta_N)$, and true ability of agent i equal to θ_i, his payment equals $(f_i(\theta_i, x_i(\tilde{\theta})) - t_i(\tilde{\theta}))^T \mathbf{1}$, where $\mathbf{1} = (1, \ldots, 1)_{M\times 1}$ is the unitarian vector. Assume the production function $f_i(\theta_i, x_i)$ satisfies $\frac{\partial^2 f_i}{\partial\theta\partial x} \geq 0$.

[9] $x(\theta)$ is fully differentiable except some points with measurement 0.

Example 1 (Examples for f_i, t, x)

1. *Private Good Auctions: Agents have the ability to produce private goods in $\mathbb{R}^M$ with the production function $f_i(\theta_i, x_i)$, where θ_i is the ability of the agent and x_i is the investment of the government. In a mechanism, the government elicits their ability and taxes the agents. A particular case of this, is for one good, where the first-price and second-price auctions have been widely characterized (references).*

2. *Private Equity Investment: A private investor is endowed with a fixed level of capital. Agents have the ability to produce private goods in $\mathbb{R}^M$ with the production function $f_i(\theta_i, x_i)$, where θ_i is the ability of the agent and x_i is the investment of the private investor. Investor receives some shares of the good produced by every agent.*

3. *Government Funding with Intermediation: Government is interested in delivering goods to charities (intermediaries), who can produce goods based on their allocation of resources. Charities may have different abilities.*

4. *Profit-sharing with Punishment: Goods can be produced by agents. Planner is able to allocate time to different agents.*

5.

1. *(Auction) Consider the auction for an indivisible good, there are M aspects of the good. For example, the buyers in an auction of a house care about the address, structures of houses, environment of neighbors, distance to work, etc. They have different valuations about the aspects of house. Buyer i's valuation about the aspect m of the house is θ_i^m. $x(\theta)$ is the vector representing outcome of the auction, and $x_i(\theta) = 1$ means buyer i wins in the auction and gets the good. $f_i(\theta_i, x_i(\theta)) = [\sum_{m=1}^M \theta_i^m] x_i(\theta)$ is the utility of buyer i given the outcome. $t_i(\theta)$ is the payment of buyer to seller.*

2. *(Funding Allocation) Consider the science foundation allocates an amount of funding to support projects of researchers. The decision for allocation of funding is based on the research proposals. The researchers report their potential outputs of research related to different areas, research team i has ability θ_i. The funding allocated to team i is $x_i(\theta)$. Given the funding support of $x_i(\theta)$, the potential output of team i is $f_i(\theta_i, x_i(\theta))$, $t_i(\theta)$ is the requirement of final report by foundation. The elements of production $f_i^j(\theta_i, x_i(\theta))$ measures the contribution to field j, like economics, computer science, mathematics, etc.*

3. *(Matching) Consider the two sided matching problem. Hospital wants to hire a doctor from many candidates, the hospital cares about various ability of the doctor, includ-*

ing watching at night, surgery, medicine knowledge, etc. The information about these abilities of doctors are reported in their resumes received by the hospital, ability of agent i is θ_i. $x_i(\theta) = 1$ represent the doctor i is hired by the hospital, and $f_i(\theta_i, x_i(\theta))$ measures the maximal potential outputs of doctor i for each fields. $t_i(\theta)$ is the requirement of tasks for the doctor by hospital. $t_i(\theta)$ measures the hours of tasks needed to be finished by the doctor.

The generality of a mechanism allows for a variety of properties not covered in previous literature. Our analysis allows for some agents to be charged to only transmit resource to some outcomes.

Example 2
The mechanism $\phi = (x(\cdot), t(\cdot))$ is the

i. ***Equally-Sharing (ES)**: $x_i(\theta) = \frac{1}{N}$, $t_i(\theta) = f_i(\theta_i, \frac{1}{N})$, $\forall i$.*

ii. ***Equally-Sharing rates (ESR)**: $x_i(\theta) = \frac{\bar{f}_i(\theta_i,1)}{\sum_{i=1}^{N} \bar{f}_i(\theta_i,1)}$, $t_i^m(\theta) = \min_{j\in\mathcal{N}} f_{jm}(\theta_j, 1)$, $\forall i$.*

iii. ***Second price mechanism (SPM)**: $x(\theta)$ satisfies: there exists i, s.t. $\bar{f}_i(\theta_i, 1) = \max_{n\in\mathcal{N}} \bar{f}_n(\theta_n, 1)$ and $x_i(\theta) = 1$, $\forall j \neq i$, $x_j(\theta) = 0$. $t(\theta)$ satisfies: $\bar{t}_i(\theta) = \max_{j\neq i} \bar{f}_j(\theta_j, 1)$ and $\bar{t}_j(\beta) = \bar{f}_j(\theta_j, 1)$, $\forall j \neq i$.*

iv. ***First price mechanism (FPM)**: $x(\theta)$ satisfies: there exists i, s.t. $\bar{f}_i(\theta_i, 1) = \max_{n\in\mathcal{N}} \bar{f}_n(\theta_n, 1)$ and $x_i(\theta) = 1$, $\forall j \neq i$, $x_j(\theta) = 0$. $t(\theta)$ satisfies: $\bar{t}_i(\theta) = \bar{f}_i(\theta_i, 1)$.*

ES always allocates the resource equally through each agent. The sharing rates $s_i(\beta)$ equal to the production based on reported type $f_i(\theta_i, \frac{1}{N})$.

The ESR mechanism always allocates resource with the same transfer payment $t_i(\beta)$ through all agents with the share equal to the ratio of agent i's aggregate production $\bar{f}_i(\theta_i, 1)$ over the aggregate production of transmission of all agents $\sum_{i=1}^{N} \bar{f}_i(\theta_i, 1)$.

The second price mechanism always allocates the resource through agent with highest aggregate production and chooses the transfer payment $t_i(\beta)$ equal to second highest aggregate production.

3 h-Strategy-Proof Mechanisms

A punishment function $h_i : \mathbb{R}_+^2 \mapsto \mathbb{R}$, $h_i(a, b)$ can be interpreted as the punishment of agent i to report $b \in R_+$, if when the true ability of transmission is $a \in R_+$. Assume there is no punishment for truthful report, $h_i(a, b) = 0$ if $b = a$.[10] Assume $h : \mathbb{R}_+^{2N} \mapsto \mathbb{R}^N$ is

[10] We do not assume that the punishment is negative, as will be illustrated below.

the vector of punishment function, $h(\theta,\theta')=(h_1(\theta_1,\theta'_1),\ldots,h_i(\theta_i,\theta'_i),\ldots,h_N(\theta_N,\theta'_N))$. The production function $f_i(\theta_i,x_i)\in\mathbb{R}^M_+$, assume $\bar{f}_i(\theta_i,x_i)=f_i(\theta_i,x_i)^T\mathbf{1}$, and $\bar{t}_i(\theta)=t_i(\theta)^T\mathbf{1}$, the profit of agent i is $u_i(\theta)=(f_i(\theta_i,x_i)-t_i(\theta))^T\mathbf{1}=\bar{f}_i(\theta_i,x_i)-\bar{t}_i(\theta)$, which is aggregation over output of production. For the strategy-proofness of mechanism, agents only care about the difference of aggregate production and aggregate charge by planner.

Definition 2 (h-Strategy-Proof)
The mechanism $\phi=(x(\cdot),t(\cdot))$ is h-strategy-proof (h-SP) if for any agent i and for any type θ_i and report θ'_i, there is

$$\bar{f}_i(\theta_i,x_i(\theta))-\bar{t}_i(\theta)\geq\bar{f}_i(\theta_i,x_i(\theta'_i,\theta_{-i}))-\bar{t}_i(\theta'_i,\theta_{-i})-h_i(\theta_i,\theta'_i),\forall\theta_i,\theta'_i,\theta_{-i}.$$

h-strategy-proof mechanisms can be understood in a way that planner has ability to audit the report θ'_i and finds out true value θ_i. Planner imposes a punishment $h_i(a,b)$ when report and true value are different. The agents choose to report the ability of transmission to maximize their expected profit.

Lemma 1 (Conditions for $h-SP$)
Consider a punishment function $h_i(\cdot,\cdot)$ that is differentiable at each point $h_i(\theta,\theta)$ for $\theta>0$.[11]

A mechanism $\phi=(x(\cdot),t(\cdot))$ is h-SP, then there exists a function $\Phi:\mathbb{R}^N_+\mapsto\mathbb{R}^N_+$ such that:

i. The aggregate transfer payment of agent i equals $\bar{t}_i(\theta)=\bar{f}_i(\theta_i,x_i(\theta))-\Phi_i(\theta)$.

ii. For each i and θ_i, $\bar{f}_{i\theta}(\theta_i,x_i(\theta))+h'_{i-}(\theta_i)\leq\frac{\partial\Phi_i(\theta_i,\theta_{-i})}{\partial\theta_i}\leq\bar{f}_{i\theta}(\theta_i,x_i(\theta))+h'_{i+}(\theta_i)$ with $\bar{f}_{i\theta}(\theta_i,x_i(\theta))=\frac{\partial\bar{f}_i(\theta_i,x_i(\theta))}{\partial\theta_i}$, $h'_{i+}(\theta_i)=\lim_{\theta'_i\to\theta_i^+}\frac{h_i(\theta_i,\theta_i)-h_i(\theta_i,\theta'_i)}{\theta_i-\theta'_i}$, $h'_{i-}(\theta_i)=\lim_{\theta'_i\to\theta_i^-}\frac{h_i(\theta_i,\theta_i)-h_i(\theta_i,\theta'_i)}{\theta_i-\theta'_i}$.

From part i, the function

$$\Phi_i(\theta)=\bar{f}_i(\theta_i,x_i(\theta))-\bar{t}_i(\theta)$$

is the profit of agent i when he truthfully reports $\theta'_i=\theta_i$ at the profile θ. From part ii, the profit function Φ_i is monotonic as θ_i increases.

These are local conditions of strategy-proof for deviation of θ to θ'. The conditions are not sufficient for h-SP, and global conditions of strategy-proof are needed. The profit of agent i is affected by aggregate production regardless specific dimension of production.

The h-strategy-proof condition $\bar{f}_i(\theta_i,x_i(\theta))-\bar{t}_i(\theta)\geq\bar{f}_i(\theta_i,x_i(\theta'_i,\theta_{-i}))-\bar{t}_i(\theta'_i,\theta_{-i})-h_i(\theta_i,\theta'_i)$, $\forall\theta_i,\theta'_i,\theta_{-i}$ is equivalent with $\Phi_i(\theta)-\Phi_i(\theta'_i,\theta_{-i})\geq\bar{f}_i(\theta_i,x_i(\theta'_i,\theta_{-i}))-\bar{f}_i(\theta'_i,x_i(\theta'_i,\theta_{-i}))+$

[11] The more general case, when h_i is not differentiable, will be discussed in Appendix.

$h_i(\theta_i, \theta_i) - h_i(\theta_i, \theta_i')$. If h_i is differentiable, take the limit $\theta_i' \to \theta_i$, there is $\frac{\partial \Phi_i(\theta)}{\partial \theta_i} = \frac{\partial \bar{f}_i(\theta_i, x_i(\theta))}{\partial \theta_i} + h_i'(\theta_i)$.

Theorem 1 (Ineffective Punishment Functions)
The following three conditions are equivalent:

i. A mechanism is 0-SP.

ii. For any punishment function $h_i(a, b)$, such that the derivative at the truthful report is zero,[12] $h_i'(\theta_i) = 0$ for any θ_i.

iii. There exists a function $x_i : R_+^N \mapsto \mathbb{R}_+$ non-decreasing in the first coordinate such that for any θ: $\frac{\partial \Phi_i(\theta)}{\partial \theta_i} = \frac{\partial \bar{f}_i(\theta_i, x(\theta))}{\partial \theta_i} + h_i'(\theta_i)$.

There are two important consequences of Theorem 1. On one hand, it provides precise conditions for a mechanisms to be 0-SP, the traditional strategy-proof condition discussed in the literature. First, the allocation of the resource to an agent should depend on his aggregate ability of production instead of specific ability of production. Second, the charged share to an agent depend on the average allocation over all the abilities $\bar{f}_i(\theta_i, x_i(\theta)) - \int_0^{\theta_i'} \bar{f}_{i\theta}(t, x_i(t, \theta_{-i}))dt$.

On the other hand, another important consequence of Theorem 1 is only the punishment functions that at the margin have a positive punishment (i.e., for an arbitrary small deviation) are effective. In other words, punishment function h whose derivative at the truthful report is zero is ineffective, in the sense that h will generate exactly the same class as if there is no punishment, the set of mechanisms in 0-SP.

There is a large number of functions that meet this condition. Moreover, if the punishment function $h(\theta, \theta') = (h_1(\theta_1, \theta_1'), \ldots, h_N(\theta_N, \theta_N'))$ approaches infinite, any mechanism is strategy-proof.[13]

Proposition 1
For any punishment functions h, $\hat{h}$, s.t. $h(\theta, \theta') \leq \hat{h}(\theta, \theta')$, $\forall \theta, \theta' \in \mathbb{R}_+^M$. Then any h-SP mechanism $\phi = (x(\cdot), t(\cdot))$ is $\hat{h}$-SP[14].

This proposition shows the result of comparative static analysis of h-SP mechanisms. As the punishment function h increases, the set of h-SP mechanisms expands. The result is consistent with intuition that punishment would decrease the incentives of agents to misreport.

[12] This happens, for instance, at the large class of polynomial punishment functions $h_i(\theta_i, \theta_i') = \gamma(\theta_i - \theta_i')^k$ for some $k > 1$.

[13] Any mechanism is ∞-SP.

[14] $\hat{h}$-SP here means the mechanism ϕ is h-SP for punishment function $\hat{h}$.

The following remark shows the linear combination of punishment functions, which make mechanisms h-SP, also guarantees the linear combination of the mechanisms to be h-SP.

Remark 1 (Convexity of Punishment Function h)
Suppose the mechanism $\phi = (x(\cdot), t(\cdot))$ and $\hat{\phi} = (\hat{x}(\cdot), \hat{t}(\cdot))$ satisfy $x(\theta) = \hat{x}(\theta)$ for any θ, and ϕ is h-SP for punishment function $h = (h_1, \ldots, h_N)$, $\hat{\phi}$ is h-SP for punishment function $\hat{h} = (\hat{h}_1, \ldots, \hat{h}_N)$, then $\tilde{\phi} = (\tilde{x}(\cdot), \tilde{t}(\cdot)) = \lambda\phi + (1-\lambda)\hat{\phi}$, for which $\tilde{x}(\theta) = x(\theta) = \hat{x}(\theta)$, and $\tilde{t}(\theta) = \lambda t(\theta) + (1-\lambda)\hat{t}(\theta)$. Then $\tilde{\phi}$ is h-SP for punishment function $\tilde{h}$, with $\tilde{h}(\theta, \theta') = \lambda h(\theta, \theta') + (1-\lambda)\hat{h}(\theta, \theta')$ for any θ,θ'.

4 Minimal Punishment Function

It is often the case that a mechanism to allocates goods and services is given whereas the designer of the mechanism has the flexibility to design the punishment function h. In this section, we ask the question: What is the class of punishment functions that makes a given mechanism h-strategy-proof? The answer to this question is related to finding the minimal punishment function h that makes a mechanisms h-SP.

For any mechanism $\phi = (x(\cdot), t(\cdot))$, assume the aggregate transfer payment agent i to the planner is $\bar{t}_i(\theta) = \sum_{j=1}^{M} t_{ij}(\theta)$, and the profit function of agent i is $v_i : \mathbb{R}_+^{N+1} \mapsto \mathbb{R}_+$. If type of agent i is θ_i and reports of all agents are (θ_i', θ_{-i}), the profit of agent i is $v_i(\theta_i, \theta_i', \theta_{-i}) = \bar{f}_i(\theta_i, x_i(\theta_i', \theta_{-i})) - \bar{t}_i(\theta_i', \theta_{-i})$. The profit of agent i for truthful report is $v_i(\theta_i, \theta_i, \theta_{-i}) = \bar{f}_i(\theta_i, x_i(\theta)) - \bar{t}_i(\theta)$. So agent i has incentive to report truthfully if $v_i(\theta_i, \theta) \geq v_i(\theta_i, \theta_i', \theta_{-i}) - h_i(\theta_i, \theta_i')$, for any $\theta_i, \theta_i', \theta_{-i}$.

Definition 3 (Minimal Punishment Function)
For any mechanism $\phi = (x(\cdot), t(\cdot))$, $h_i^{\min} : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ is minimal punishment function for agent i, if for any punishment function $h(\theta_i, \theta_i')$, such that ϕ is h-SP for agent i with punishment h_i, then $h_i(\theta_i, \theta_i') \geq h_i^{\min}(\theta_i, \theta_i')$, $\forall \theta_i, \theta_i'$.

The following result shows that there exists a minimal punishment at every profile and misreport in order to achieve strategy-proofness.

Proposition 2 (Minimal Punishment Function)
Consider the mechanism $\phi = (x(\cdot), t(\cdot))$, the profit of agent i is $v_i(\theta_i, \theta_i', \theta_{-i})$, $v(\theta, \theta') = (v_1(\theta_1, \theta'), \ldots, v_N(\theta_N, \theta'))$ when the true profile is θ and reported profile is θ'. The punishment function h, which guarantees mechanism ϕ to be h-SP, satisfies: $h_i(\theta_i, \theta_i') \geq h_i^{\min}(\theta_i, \theta_i') = \max_{\theta'_{-i}}[v_i(\theta_i, \theta') - v_i(\theta_i, \theta_i, \theta'_{-i})]$, for any θ_i, θ', i.

We can interpret the function $h_i^{\min}(\theta_i, \theta_i') = \max_{\theta'_{-i}}[v_i(\theta_i, \theta') - v_i(\theta_i, \theta_i, \theta'_{-i})]$ as the minimal punishment that agent i needs to incur, when the true profile is θ_i but he actually reports θ_i'.

In particular, we note that the minimal punishment for a strategy proof mechanism satisfies $h_i(\theta_i, \theta_i') = 0$, for any θ_i, θ_i'. Thus any strategy proof mechanism is h-SP. On the other hand, if a mechanism is not strategy proof, the minimal punishment function for the mechanism has to be non-zero.

Corollary 1 (Properties of Minimal Punishment Function $h^{\min}$)

i. For any mechanism $\phi = (x(\cdot), t(\cdot))$, there exists minimal punishment function $h^{\min}$.

ii. If the mechanism is strategy proof, then $h_i^{\min}(\theta_i, \theta_i') = 0$, for any θ_i, θ_i', i.

iii. If the mechanism is not strategy proof, then the minimal punishment function is nonzero. In other words, there exists θ_i, θ_i', i, such that $h_i^{\min}(\theta_i, \theta_i') > 0$.

The following example discusses the minimal punishment function for first price mechanism and second price mechanism.

Example 3

Consider the first price mechanism $\phi_F = (x_F, t_F)$, and second price mechanism $\phi_S = (x_S, t_S)$, $x_S = x_F$ satisfies: for any i, $f_i(\theta_i, 1) < \max_{j \in \mathcal{N}} f_j(\theta_j, 1)$, $x_{Si}(\theta) = 0$. For any i, $f_i(\theta_i, 1) = \max_{j \in \mathcal{N}} f_j(\theta_j, 1)$, $x_{Si}(\theta) = \frac{1}{k(\theta)}$, $k(\theta)$ is the number of agents with largest $f_i(\theta_i, 1)$.

The transfer payment for first price mechanism is $t_F(\theta) = (f_1(\theta_1, 1), \dots, f_N(\theta_N, 1))$, which means the agents are charged at the level of production they report. For second price mechanism, the transfer payment $t_{Si}(\theta) = f_i(\theta_i, 1)$ for i with $f_i(\theta_i, 1) \leq \max_{j \neq i} f_j(\theta_j, 1)$, and $t_{Si}(\beta) = \max_{j \neq i} f_j(\theta_j, 1)$ for i with $f_i(\theta_i, 1) > \max_{j \neq i} f_j(\theta_j, 1)$. The transfer payment t_S and allocation x_S of second price mechanism satisfies $\bar{t}_i(\theta) = \bar{f}_i(\theta_i, 1) - \max_{j \neq i} f_j(\theta_j, 1)$, thus, second price mechanism is strategy proof, so the minimal punishment function for the second price mechanism is $h_i(\theta_i, \theta_i') = 0$.

The minimal punishment function for first price mechanism is $h_i(\theta_i, \theta_i') = (f_i(\theta_i, 1) - f_i(\theta_i', 1))_+$ with $f_i(\theta_i, 1) - f_i(\theta_i', 1)_+ = \max\{f_i(\theta_i, 1) - f_i(\theta_i', 1), 0\}$.

Consider a mechanism ϕ, which is linear combination of ϕ_F and ϕ_S, satisfies $\phi = \epsilon\phi_F + (1 - \epsilon)\phi_S$ with $\epsilon \in [0, 1]$. From Corollary 1, the minimal punishment function for the mechanism ϕ is $h_i^{\min}(\theta_i, \theta_i') = \epsilon(f_i(\theta_i, 1) - f_i(\theta_i', 1))_+$.

Example 4

Consider the equally sharing rule of resource allocation, $x_i(\theta') = \frac{1}{N}$, which means the planner always allocates $\frac{1}{N}$ to each agent. The equally sharing mechanism ϕ_1 in Example 2, *satisfying* $t_i(\theta') = f_i(\theta'_i, 1)$*, is not strategy proof. The agent has higher profit reporting lower than the true ability of transmission. The h-SP condition requires* $h_i(\theta_i, \theta'_i) \geq \max_{\theta'_{-i}}[v_i(\theta_i, \theta') - v_i(\theta_i, \theta_i, \theta'_{-i})]$*, substitute allocation* x_i *and transfer payment* t_i *into the inequality,* $h_i(\theta_i, \theta'_i) \geq \frac{f_i(\theta_i,1)-f_i(\theta'_i,1)}{N} - 0$*, thus,* $h_i(\theta_i, \theta'_i) \geq \frac{f_i(\theta_i,1)-f_i(\theta'_i,1)}{N}$*. The minimal punishment function* $h_i^{\min}(\theta_i, \theta'_i) = (\frac{f_i(\theta_i,1)-f_i(\theta'_i,1)}{N})_+$

The equally sharing strategy proof mechanism ϕ_2 *satisfies* $x_{2i}(\theta') = \frac{1}{N}$ *and* $s_{2i}(\beta) = \mathbf{0}$*.* $\mathbf{0} = (0, \ldots, 0)_{M\times 1}$*. The minimal punishment function for this mechanism is* $h_2^{\min} = 0$*, but the outcome will be* 0.

Finally, notice that Example 4 shows the strategy proof mechanism may result in outcome 0. The goal of planner is to send resource to outcomes via agents, so the class of strategy proof mechanisms may not be good in some circumstances, punishment based on verification is necessary to achieve larger outcomes. The following section will discuss the preferences of planner, how will the punishment help improving the outcome.

4.1 First-Best Efficiency

We need to emphasize that the planner does not care about agents, but only the outcomes. The outcome of resource allocation $y \in \mathbb{R}_+^M$ satisfies $y = \sum_{i=1}^N t_i(\theta')$, θ' is the type reported by agents. Assume planner's preferences $\succeq$ over the outcome of resource allocation y is monotonic, and there exists utility function $u : \mathbb{R}_+^M \mapsto \mathbb{R}$ to represent the preferences.

The following definition states the first best outcome from the perspective of planner.

Definition 4 (First Best Efficient (FBE))
Given the preferences of planner $\succeq$ *and the utility function* $u : \mathbb{R}_+^M \mapsto \mathbb{R}_+$*. A mechanism* ϕ *is first best efficient (FBE), if for any profile of agents* θ*, the outcome of resource allocation* y *maximizes the planner's utility, under the condition of individual rationality. The individual rationality means that agents have nonnegative profit. If the preferences is strictly convex, then there exists a unique resource allocation that maximizes planner's utility given any profile* $\theta = (\theta_1, \ldots, \theta_N)$*.*

Given the type of production θ, the maximal utility $\bar{u}(\theta)$ equals $\max_x u(\sum_{i=1}^N f_i(\theta_i, x_i))$, such that $\sum_{i=1}^N x_i = 1$. Assume $\bar{x} : R_+^N \mapsto \mathbb{R}_+^N$ is the allocation of resource among agents, which maximizes planner's utility when profile of agents are θ, $\bar{x}(\theta) = (\bar{x}_1(\theta), \ldots, \bar{x}_N(\theta)) = \arg\max_x u(\sum_{i=1}^N f_i(\theta_i, x_i))$. Assume $\bar{f}_{i\theta}(\theta, x_i(\theta)) > 0$ for any $x_i(\theta) > 0$, which means if agent i is allocated with positive resource, the marginal effect of type on aggregate production is positive.

Notice that FBE implies the planner allocates resource through agents optimally to achieve maximal utility with the true type of production.

Given mechanism $\phi = (x(\cdot), t(\cdot))$, and the type of production is θ, the utility of planner is $u^*(\theta, \phi) = u(\sum_{i=1}^N t_i(\theta))$, when agents truthfully report their type $\theta' = \theta$.

Theorem 2

Assume the preferences of the planner $\succeq$ is strongly monotonic, continuous, and there exists utility function $u : \mathbb{R}_+^M \mapsto \mathbb{R}$ representing the preferences.

i. *There is no SP, budget balance and first best efficient mechanism.*

ii. *For any FBE mechanism ϕ, the minimal punishment function $h_i^{\min}$ for ϕ to be h-SP satisfies $h_i^{\min}(\theta_i, \theta_i') = \max_{c\in[0,1]} \bar{f}_i(\theta_i, c) - \bar{f}_i(\theta_i', c)$. Any punishment function h that implements a FBE mechanism if and only if $h_i(\theta_i, \theta_i') \geq h_i^{\min}(\theta_i, \theta_i')$ for any $\theta_i \geq \theta_i'$.*

Theorem 2 shows there exists no mechanism satisfying symmetric, budget balance, 0-SP and FBE. It also provides the condition of minimal punishment function for mechanism to achieve the first best efficient. The minimal punishment function is the same with the one in Example 3 to guarantee the first price mechanism to be h-SP, which is not surprising because the first price mechanism is FBE for some special planner's preferences. The following example shows that the first price mechanism is the first best efficient when the outcome of resource is perfect substitute for planner, while the second price mechanism is SP, but not FBE.

Example 5

Consider the preferences of planner is perfect substitute and represented by utility function $u(y) = \sum_{i=1}^M y_i$. Then the first price mechanism ϕ_F is FBE but not SP, and the second price mechanism ϕ_S is SP but not FBE.

When the planner only cares about the sum of outcomes, the first best outcome is to allocate the resource through a agent with largest aggregate production $\bar{f}_i(\theta_i, 1)$ and charge the sharing rates equal to his true ability of transmission. Assume the aggregate production ranking from high to low is $\bar{f}_1(\theta^1, 1) \geq \cdots \geq \bar{f}_N(\theta^N, 1)$, the maximal utility $\bar{u}(\theta) = \max_{i\in\mathcal{N}} \bar{f}_i(\theta^i, 1) = \bar{f}_1(\theta^1, 1)$.

Thus, the first price mechanism, which allocates all resource to agent i with largest aggregate production $\bar{f}_i(\theta_i, 1)$ and transmit with resource equal to reported production. The final allocation $y = \bar{f}_1(\theta^1, 1)$ and utility $u^*(\theta, \phi_F) = \bar{f}_1(\theta^1, 1)$, which is FBE. However, the second price mechanism, which is strategy proof, but it has to pay the agent i with largest aggregate production $\bar{f}_1(\theta^1, 1) - \max_{j\neq 1} \bar{f}_j(\theta^j, 1)$ as information rent, The planner's *utility $u^*(\theta, \phi_S) = \bar{f}_2(\theta^2, 1)$. When $\bar{f}_2(\theta^2, 1) < \bar{f}_1(\theta^1, 1)$, $u^*(\theta, \phi_S) < \bar{u}(\theta)$, the second price mechanism can not achieve FBE.*

This section shows that punishment is necessary to achieve the first best efficient for the planner, and the minimal punishment function for first best efficient mechanism to be h-SP coincides with the minimal punishment function for first price mechanism to be h-SP. Verification and punishment could be used to expand the class of strategy proof mechanisms and achieve higher efficiency for planner.

5 h-optimal Mechanism

Unlike Section 4, that fixes the mechanism and finds (minimal) punishment functions that make it incentive compatible, in this section, we fix the punishment functions and find mechanisms that are 'optimal' for such a punishment function.

Unlike Section 3, that focuses on characterizing the classes of h-SP mechanisms, in this section we find the optimal mechanisms for an arbitrary set of preferences. Our main results in this section shows that the optimal mechanism for the planner given an arbitrary punishment function can be represented as a convex combination of two traditional mechanisms, the first-price mechanism and the second-price mechanism, where the weight between these mechanisms is dependent on the punishment function.

Definition 5
*Consider an arbitrary monotonic preferences $\succeq$ of planner, and let h be an arbitrary punishment function. We say the h-SP mechanism ϕ^* is **h-optimal** if for any h-SP mechanism $\bar{\phi}$ with the same allocation rule $x(\cdot)$, we have that $y^*(\theta) \succeq \bar{y}(\theta)$ for any θ.*

The allocation of resource to the planner is $y(\theta) = \sum_{i=1}^{N} t_i(\theta)$. Consider the mechanisms with allocation rule $x(\theta)$ that allocates resource to agents with highest aggregate production, second price mechanism ϕ_S is 0-SP and first price mechanism ϕ_F is h-SP for punishment function $h_i(\theta_i, \theta'_i) = \max_{c\in[0,1]} |\bar{f}_i(\theta_i, c) - \bar{f}_i(\theta'_i, c)|$. The mechanism $\phi^\lambda = \lambda\phi_F + (1-\lambda)\phi_S$, which is the convex combination of first price mechanism ϕ_F and second price mechanism ϕ_S. ϕ^λ is h-SP with punishment function $h_i^\lambda(\theta_i, \theta'_i) = \lambda \max_{c\in[0,1]} |\bar{f}_i(\theta_i, c) - \bar{f}_i(\theta'_i, c)|$ as a convex combination of punishment functions for ϕ_F and ϕ_S. Assume $\bar{f}_{-i}(\theta_{-i}, 1) = \max_{j\neq i} \bar{f}_j(\theta_j, 1)$.

notes: if there is no feasibility constraint

notes: for any mechanism, $x(\cdot)$, $t(\cdot)$ could be equivalent with certain $\tilde{x}(\cdot)$, $\tilde{t}(\cdot)$ for specific $\tilde{x}$

Definition 6

Consider an arbitrary punishment function h, let function $\lambda_i^h(\theta) \leq \frac{\int_{\underline{\theta}_i(\theta_{-i})}^{\theta_i} -h'_{i-}(q)dq}{\int_{\underline{\theta}_i(\theta_{-i})}^{\theta_i} \bar{f}_{i\theta}(q, x_i(q, \theta_{-i}))dq}$, s.t. $\underline{\theta}_i$. $\underline{\theta}_i(\theta_{-i}) = \sup\{\theta_i | x_i(\theta_i, \theta_{-i}) = 0\}$. Define the mechanism $\phi^h = \lambda^h(\theta)\phi_F + (1 - \lambda^h(\theta))\phi_S$, with $x^h(\theta) = x_S(\theta) = x_F(\theta)$, and $t_i(\theta) = \lambda_i^h(\theta)t_{Fi}(\theta) + (1 - \lambda_i^h(\theta))t_{Si}(\theta)$.

Consider mechanism ϕ^h with allocation rule $x^h(\cdot)$, that planner allocates all resource to agents with highest aggregate production, $x^h(\theta) = x_S(\theta) = x_F(\theta)$. The resource charged by planner is $t_i(\theta) = \lambda_i^h(\theta)t_{Fi}(\theta) + (1 - \lambda_i^h(\theta))t_{Si}(\theta)$. The resource charged in the first price mechanism satisfies $\bar{t}_{Fi}(\theta) = \bar{f}_i(\theta_i, x_i(\theta))$, second price mechanism satisfies $\bar{t}_{Si}(\theta) = \bar{f}_{-i}(\theta_{-i}, 1)x_i(\theta)$.

Then $\underline{\theta}_i(\theta_{-i})$ solves the $\bar{f}_i(\theta_i, 1) = \max_{j \neq i} \bar{f}_j(\theta_j, 1)$. For $\theta_i > \underline{\theta}_i(\theta_{-i})$, $x_i(\theta) = 1$. $\lambda_i^h(\theta)$ satisfies $\lambda_i^h(\theta) \leq \frac{\int_{\underline{\theta}_i(\theta_{-i})}^{\theta_i} -h'_{i-}(q)dq}{\int_{\underline{\theta}_i(\theta_{-i})}^{\theta_i} \bar{f}_{i\theta}(q,1)dq}$. Further, if the production is separable $f_i(\theta_i, x_i(\theta)) = \theta_i x_i(\theta)$, $\bar{f}_{i\theta}(\theta_i, 1) = 1$, $\lambda_i^h(\theta)$ depends on the average marginal punishment of h_i, $\lambda_i^h(\theta) \leq \frac{\int_{\underline{\theta}_i(\theta_{-i})}^{\theta_i} -h'_{i-}(q)dq}{\theta_i - \underline{\theta}_i(\theta_{-i})}$.

This mechanism selects the worst possible type for the planner to monitor, in relation to punishment function h with profile θ. For profile θ, the mechanism $\phi^h = \lambda^h(\theta)\phi_F + (1 - \lambda^h(\theta))\phi_S$ allocates all resource to agent i with highest aggregate production $\bar{f}_i(\theta_i, 1)$, and charge the resource $\lambda_i^h(\theta)\bar{f}_i(\theta_i, x_i(\theta)) + (1 - \lambda_i^h(\theta))\bar{f}_{-i}(\theta_{-i}, 1)x_i(\theta)$. It is worth noting $\lambda_i^h(\theta)$ depends on $\bar{f}_i(\theta_i, 1)$ and $\bar{f}_{-i}(\theta_{-i}, 1)$.

Theorem 3

For an arbitrary punishment function h, the mechanism ϕ^h is h-optimal among mechanisms that allocate to the agent with highest aggregate production.

If the production function satisfies constant return to scale $f_i(\theta_i, x_i(\theta)) = x_i(\theta)f_i(\theta_i, 1)$, the agent with maximal aggregate production is the same as one with maximal marginal productivity $\frac{\partial f_i(\theta_i, x_i)}{\partial x_i}$.

The theorem characterizes the optimal h-SP mechanism for a specific monotonic allocation rule $x(\theta)$. The optimality cannot be guaranteed in the general case. If there is no feasibility constraint, the allocation rule that allocates all resource to agents with highest aggregate production is h-optimal among all mechanisms.

Resource allocating to the agent with maximal aggregate production is necessary for the result. See the following example.

Example 6

Suppose there is no punishment, that is $h_i(\theta_i, \theta'_i) = 0$. Consider the strategy-proof mechanisms. From Theorem 1, the strategy-proof mechanism satisfies $\frac{\partial \Phi_i(\theta_i, \theta_{-i})}{\partial \theta_i} = \bar{f}_{i\theta}(\theta_i, x_i(\theta))$, which is equivalent with $\bar{t}_i(\theta) = \bar{f}_i(\theta_i, x_i(\theta)) - \int_0^{\theta_i} \bar{f}_{i\theta}(t, x_i(t, \theta_{-i}))dt$.

Assume the production function is the same for all agents, $f_i(\theta_i, x_i(\theta)) = f(\theta_i, x_i(\theta)) = \theta_i x_i(\theta) \in \mathbb{R}$, θ_i is the marginal product of agent i. Assume $\theta_1 > \theta_2 \cdots > \theta_N$, for second price mechanism ϕ_S, the planner allocates all resource to agent 1 and charges θ_2.

Consider another strategy-proof mechanism $\hat{\phi} = (\hat{x}(\cdot), \hat{t}(\cdot))$, the allocation rule $\hat{x}(\cdot)$ satisfies $\hat{x}_i(\theta') = \frac{1_{[\theta_2,\theta_1)}(\theta'_i)}{\sum_{i=1}^{N} 1_{[\theta_2,\theta_1)}(\theta'_i)}$[15], for θ' with $\theta_2 \leq \max_{i\in\mathcal{N}} \theta'_i < \theta_1$, and $\hat{x}(\theta') = x_S(\theta')$ for other θ'. In the mechanism $\hat{\phi}$, planner allocates resource equally to agents with report in $[\theta_2, \theta_1)$ if the maximal type is in $[\theta_2, \theta_1)$, and allocate resource to agent with highest production in other situation. Since $\hat{\phi}$ is strategy-proof, if $\theta_2 \leq \max_{i\in\mathcal{N}} \theta'_i < \theta_1$, charge $\hat{t}_i(\theta') = \theta_2 \hat{x}_i(\theta')$ for agents. If $\theta' = \theta$, $\hat{x}_1(\theta') = 1$, and the charged resource $\hat{t}_1(\theta') = \frac{\theta_1+\theta_2}{2} > \theta_2 = t_{S1}(\theta')$. If $\theta'_j = \theta_j$ for $j \geq 3$, and $\theta_2 < \theta'_2 < \theta'_1 < \theta_1$, the charged resource $\hat{t}_1(\theta') = \theta_2 < \theta'_2 = t_{S1}(\theta')$. Thus, neither of the mechanisms are optimal for any SP mechanism.

Corollary 2 (Optimality of ϕ_λ)

i. The mechanism ϕ_λ is the unique h^λ-optimal mechanism among the mechanisms that allocate resource to the agent with maximal aggregate production.

ii. h^λ is the minimal punishment function of the mechanism ϕ_λ.

Corollary 2 discusses the special case of punishment function $h_i^\lambda(\theta_i, \theta'_i) = \lambda_i(\theta)|\bar{f}_i(\theta_i, 1) - \bar{f}_i(\theta'_i, 1)|$, and $\phi_\lambda = \lambda\phi_F + (1-\lambda)\phi_S$ is h^λ-optimal mechanism.

Corollary 3 (Optimality of second-price mechanism)

The second-price mechanism ϕ_S is h-optimal among mechanisms that allocate to the agent with highest aggregate production for a punishment function h that satisfies either of the following conditions:

i. The derivative of h equals to zero, $\frac{\partial h_i(\theta_i,\theta'_i)}{\partial \theta_i}|_{\theta'_i=\theta_i} = 0$, for any θ_i. All polynomial functions satisfy this.

ii. Any punishment function such that for each θ_i, there exists $\theta'_i \in U_\epsilon(\theta_i)$ in neighborhood of θ_i with any $\epsilon > 0$, such that $h_i(\theta_i, \theta'_i) = 0$. This includes, punishment functions that do not punish small deviations.

The first result of Corollary 3 is a natural extension of the results in Theorem 1. The set of strategy-proof mechanisms does not expand if the punishment function h has derivative 0, thus, there is no h-SP mechanism, which allocates resource to agent with maximal aggregate production, can transmit more resource than second price mechanism ϕ_S.

The second result shows if there exists small deviation without any punishment, then the punishment function h does not improve the h-optimal mechanism.

[15] $1_{[\theta_2,\theta_1)}(\cdot)$ is indicator function.

ii is a special case included in i, the condition could be extended to $\frac{h_i'(\theta_i)}{\bar{f}_{i\theta}'(\theta_i,1)} = 0$

Corollary 4 (Optimality of the first-price mechanism)
The first-price mechanism ϕ_F is optimal among mechanisms that allocate resource to the agent with highest aggregate production for any punishment function h that satisfies: $h_i(\theta_i, \theta_i') \geq \max\{\bar{f}_i(\theta_i, 1) - \bar{f}_i(\theta_i', 1), 0\}$ for any θ_i, θ_i'.

6 Extensions

6.1 Arbitrary preferences of the intermediaries

Note that we have been studying the case where agents care about the profit. If more general preferences of the intermediaries are assumed, for instance, when intermediaries can have any potential quasi-concave utilities over the allocation of the goods to agents, then nothing is possible.

Proof: basically any potential slope is possible.

6.2 Specific min-preferences for the intermediaries

Consider the case where intermediaries have specific preferences, for instance, they case about the min of their connections. Then, clearly, we might be able to get different mechanisms.

7 Conclusion

This paper investigates the strategy-proof mechanisms for the problem of resource transmission with intermediation on network. The mechanism requires the intermediaries to report their quality of intermediation, transmits the resource according to the sharing rates based on the report, and imposes punishment for misreporting.

This paper is a start to study the strategy-proof mechanisms with punishments, we discover and describe the sets of strategy-proof mechanisms with various punishment functions. The conditions of strategy-proof mechanism, that require share of resource transmission and sharing rates of intermediaries to satisfy, are provided. We also demonstrate the strategy-proof, symmetric, budget balance mechanisms with three cases of punishment function. With linear punishment function, the strategy-proof, symmetric, budget balance mechanisms require the sharing rule to depend only on the sum of quality of links. The minimal punishment function for a mechanism to be strategy-proof are discussed. On the other hand, given the punishment function, the optimal mechanism for the planner is discovered.

The mechanism design approach in this paper is a complement to game theoretical approach. It studies how to make a plan providing incentives for intermediaries to report the true quality of intermediation, rather than competing in the price charging for using their links.

Appendix: Proofs

In order to prove Theorems and Corollaries, first introduce the follow Lemmas.

Proof of Lemma 1

Proof.

Part i. Given the mechanism $\phi = (x(\cdot), s(\cdot))$, assume $\Phi_i(\theta) = \bar{f}_i(\theta_i, x_i(\theta)) - \bar{t}_i(\theta)$, which represents the profit of agent i.

Part ii. Prove the result for h-SP mechanism, $\frac{\partial \Phi_i(\theta_i,\theta_{-i})}{\partial \theta_i} = \bar{f}_{i\theta}(\theta_i, x_i(\theta)) - h_i'(\theta_i)$.

For mechanism $\phi = (x(\cdot), t(\cdot))$ to be h-SP, $\bar{f}_i(\theta_i, x_i(\theta)) - \bar{t}_i(\theta) \geq \bar{f}_i(\theta_i, x_i(\theta_i', \theta_{-i})) - \bar{t}_i(\theta_i', \theta_{-i}) - h_i(\theta_i, \theta_i')$, $\forall \theta_i, \theta_i', \theta_{-i}$.

$\Phi_i(\theta) = \bar{f}_i(\theta_i, x_i(\theta)) - \bar{t}_i(\theta)$ is the profit of agent i when he truthfully reports his type θ_i, and types of other agents are θ_{-i}. Then the condition for h-strategy-proofness is equivalent with $\Phi_i(\theta) \geq \Phi_i(\theta_i', \theta_{-i}) + \bar{f}_i(\theta_i, x_i(\theta_i', \theta_{-i})) - \bar{f}_i(\theta_i', x_i(\theta_i', \theta_{-i})) - h_i(\theta_i, \theta_i')$, $\forall \theta_i, \theta_i', \theta_{-i}$.

For $\theta_i' > \theta_i$, $\frac{\Phi_i(\theta)-\Phi_i(\theta_i',\theta_{-i})}{\theta_i-\theta_i'} \leq \frac{\bar{f}_i(\theta_i,x_i(\theta_i',\theta_{-i}))-\bar{f}_i(\theta_i',x_i(\theta_i',\theta_{-i}))}{\theta_i-\theta_i'} + \frac{h_i(\theta_i,\theta_i)-h_i(\theta_i,\theta_i')}{\theta_i-\theta_i'}$.

For $\theta_i' < \theta_i$, $\frac{\Phi_i(\theta)-\Phi_i(\theta_i',\theta_{-i})}{\theta_i-\theta_i'} \geq \frac{\bar{f}_i(\theta_i,x_i(\theta_i',\theta_{-i}))-\bar{f}_i(\theta_i',x_i(\theta_i',\theta_{-i}))}{\theta_i-\theta_i'} + \frac{h_i(\theta_i,\theta_i)-h_i(\theta_i,\theta_i')}{\theta_i-\theta_i'}$.

If the profit function Φ_i, production function $\bar{f}_i$ and punishment h_i are differentiable, take the limit $\theta_i' \to \theta_i$, there is $\frac{\partial \Phi_i(\theta_i,\theta_{-i})}{\partial \theta_i} = \bar{f}_{i\theta}(\theta_i, x_i(\theta)) + h_i'(\theta_i)$.

For $h_i(\theta_i, \theta_i') \geq 0$, if h_i is not differentiable, assume $h_{i+}'(\theta_i) = \lim_{\theta_i' \to \theta_i^+} \frac{h_i(\theta_i,\theta_i)-h_i(\theta_i,\theta_i')}{\theta_i-\theta_i'} \geq 0$ with $\theta_i' > \theta_i$, and $h_{i-}'(\theta_i) = \lim_{\theta_i' \to \theta_i^-} \frac{h_i(\theta_i,\theta_i)-h_i(\theta_i,\theta_i')}{\theta_i-\theta_i'} \leq 0$ with $\theta_i' < \theta_i$. Thus, $\bar{f}_{i\theta}(\theta_i, x_i(\theta)) + h_{i-}'(\theta_i) \leq \frac{\partial \Phi_i(\theta_i,\theta_{-i})}{\partial \theta_i} \leq \bar{f}_{i\theta}(\theta_i, x_i(\theta)) + h_{i+}'(\theta_i)$. ■

Proof of Theorem 1

Proof.

Assume $h_i'(\theta_i) = \frac{\partial h_i(\theta_i,\theta_i')}{\partial \theta_i'}$, $\theta = (\theta_i, \theta_{-i})$, $\bar{f}_{i\theta}(\theta_i, x_i(\theta)) = \frac{\partial \bar{f}_i(\theta_i,x_i(\theta))}{\partial \theta_i}$.

From Lemma 1, h-SP mechanism satisfies $\bar{f}_{i\theta}(\theta_i, x_i(\theta)) + h_{i-}'(\theta_i) \leq \frac{\partial \Phi_i(\theta_i,\theta_{-i})}{\partial \theta_i} \leq \bar{f}_{i\theta}(\theta_i, x_i(\theta)) + h_{i+}'(\theta_i)$.

i. ⇒ iii.

For any mechanism $\phi = (x(\cdot), t(\cdot))$ to be 0-SP, $\bar{f}_i(\theta_i, x_i(\theta)) - \bar{t}_i(\theta) \geq \bar{f}_i(\theta_i, x_i(\theta_i', \theta_{-i})) - \bar{t}_i(\theta_i', \theta_{-i})$, equivalent with $\Phi_i(\theta) \geq \Phi_i(\theta_i', \theta_{-i}) + \bar{f}_i(\theta_i, x_i(\theta_i', \theta_{-i})) - \bar{f}_i(\theta_i', x_i(\theta_i', \theta_{-i}))$, $\forall \theta_i, \theta_i', \theta_{-i}$.

For $\theta_i' > \theta_i$, $\frac{\Phi_i(\theta)-\Phi_i(\theta_i',\theta_{-i})}{\theta_i'-\theta_i} \geq \frac{\bar{f}_i(\theta_i,x_i(\theta_i',\theta_{-i}))-\bar{f}_i(\theta_i',x_i(\theta_i',\theta_{-i}))}{\theta_i'-\theta_i}$.

For $\theta_i' < \theta_i$, $\frac{\Phi_i(\theta)-\Phi_i(\theta_i',\theta_{-i})}{\theta_i'-\theta_i} \leq \frac{\bar{f}_i(\theta_i,x_i(\theta_i',\theta_{-i}))-\bar{f}_i(\theta_i',x_i(\theta_i',\theta_{-i}))}{\theta_i'-\theta_i}$.

Take the limit $\theta_i' \to \theta_i$, there is $\frac{\partial \Phi_i(\theta_i,\theta_{-i})}{\partial \theta_i} = \bar{f}_{i\theta}(\theta_i, x_i(\theta))$.

ii. $\Rightarrow$ iii.

For punishment function $h_i(\theta_i, \theta_i')$ with $h_i(\theta_i, \theta_i') = 0$, the h-SP mechanism satisfies $\frac{\partial \Phi_i(\theta_i,\theta_{-i})}{\partial \theta_i} = \bar{f}_{i\theta}(\theta_i, x_i(\theta)) - h_i'(\theta_i) = \bar{f}_{i\theta}(\theta_i, x_i(\theta))$.

iii. $\Rightarrow$ i.

For mechanism ϕ, satisfies $\frac{\partial \Phi_i(\theta_i,\theta_{-i})}{\partial \theta_i} = \bar{f}_{i\theta}(\theta_i, x_i(\theta))$, we have $\Phi_i(\theta_i, \theta_{-i}) = \int_0^{\theta_i} \bar{f}_{i\theta}(t, x_i(t, \theta_{-i}))dt$, $\Phi_i(\theta_i', \theta_{-i}) = \int_0^{\theta_i'} \bar{f}_{i\theta}(t, x_i(t, \theta_{-i}))dt$. The planner charges agent i with $\bar{t}_i(\theta) = \bar{f}_i(\theta_i, x_i(\theta)) - \int_0^{\theta_i} \bar{f}_{i\theta}(t, x_i(t, \theta_{-i}))dt$.

Since $\frac{\partial^2 \bar{f}_i}{\partial \theta \partial x} \geq 0$, $x_i(\theta)$ is non-decreasing in θ_i.

For any $\theta_i \geq t \geq \theta_i'$, $x_i(t, \theta_{-i}) \geq x_i(\theta_i', \theta_{-i})$, $\bar{f}_{i\theta}(t, x_i(t, \theta_{-i})) \geq \bar{f}_{i\theta}(t, x_i(\theta_i', \theta_{-i}))$.

$\Phi_i(\theta_i, \theta_{-i}) - \Phi_i(\theta_i', \theta_{-i}) = \int_{\theta_i'}^{\theta_i} \bar{f}_{i\theta}(t, x_i(t, \theta_{-i}))dt$, and $\bar{f}_i(\theta_i, x_i(\theta_i', \theta_{-i})) - \bar{f}_i(\theta_i', x_i(\theta_i', \theta_{-i})) = \int_{\theta_i'}^{\theta_i} \bar{f}_{i\theta}(t, x_i(\theta_i', \theta_{-i}))dt$. Then we have $\Phi_i(\theta_i, \theta_{-i}) - \Phi_i(\theta_i', \theta_{-i}) \geq \bar{f}_i(\theta_i, x_i(\theta_i', \theta_{-i})) - \bar{f}_i(\theta_i', x_i(\theta_i', \theta_{-i}))$, which is equivalent with the 0-SP condition $\Phi_i(\theta_i, \theta_{-i}) \geq \Phi_i(\theta_i', \theta_{-i}) + \bar{f}_i(\theta_i, x_i(\theta_i', \theta_{-i})) - \bar{f}_i(\theta_i', x_i(\theta_i', \theta_{-i}))$.

For any $\theta_i \leq t \leq \theta_i'$, $x_i(t, \theta_{-i}) \leq x_i(\theta_i', \theta_{-i})$, $\bar{f}_{i\theta}(t, x_i(t, \theta_{-i})) \leq \bar{f}_{i\theta}(t, x_i(\theta_i', \theta_{-i}))$.

Then $\int_{\theta_i}^{\theta_i'} \bar{f}_{i\theta}(t, x_i(t, \theta_{-i}))dt \leq \int_{\theta_i}^{\theta_i'} \bar{f}_{i\theta}(t, x_i(\theta_i', \theta_{-i}))$, which means $-\Phi_i(\theta_i, \theta_{-i}) + \Phi_i(\theta_i', \theta_{-i}) \leq -\bar{f}_i(\theta_i, x_i(\theta_i', \theta_{-i})) + \bar{f}_i(\theta_i', x_i(\theta_i', \theta_{-i}))$, equivalent with $\Phi_i(\theta_i, \theta_{-i}) \geq \Phi_i(\theta_i', \theta_{-i}) + \bar{f}_i(\theta_i, x_i(\theta_i', \theta_{-i})) - \bar{f}_i(\theta_i', x_i(\theta_i', \theta_{-i}))$.

iii. $\Rightarrow$ ii.

From above, any mechanism ϕ satisfying $\frac{\partial \Phi_i(\theta_i,\theta_{-i})}{\partial \theta_i} = \bar{f}_{i\theta}(\theta_i, x_i(\theta))$ is 0-SP, $\Phi_i(\theta) \geq \Phi_i(\theta_i', \theta_{-i}) + \bar{f}_i(\theta_i, x_i(\theta_i', \theta_{-i})) - \bar{f}_i(\theta_i', x_i(\theta_i', \theta_{-i}))$, $\forall \theta_i, \theta_i', \theta_{-i}$. Then $\Phi_i(\theta) \geq \Phi_i(\theta_i', \theta_{-i}) + \bar{f}_i(\theta_i, x_i(\theta_i', \theta_{-i})) - \bar{f}_i(\theta_i', x_i(\theta_i', \theta_{-i})) - h_i(\theta_i, \theta_i')$, with $h_i(\theta_i, \theta_i') \geq 0$. ■

Proof of Proposition 1

Proof.

For any punishment function h and $\hat{h}$. $h(\theta, \theta') \leq \hat{h}(\theta, \theta')$, for any θ and θ'.

Any mechanism ϕ is h-SP, satisfies:

$\bar{f}_i(\theta_i, x_i(\theta)) - \bar{t}_i(\theta) \geq \bar{f}_i(\theta_i, x_i(\theta_i', \theta_{-i})) - \bar{t}_i(\theta_i', \theta_{-i}) - h_i(\theta_i, \theta_i')$, $\forall \theta, \theta_i', i$.

$\bar{f}_i(\theta_i, x_i(\theta_i', \theta_{-i})) - \bar{t}_i(\theta_i', \theta_{-i}) - h_i(\theta_i, \theta_i') \geq \bar{f}_i(\theta_i, x_i(\theta_i', \theta_{-i})) - \bar{t}_i(\theta_i', \theta_{-i}) - \hat{h}_i(\theta_i, \theta_i')$.

Thus, $\bar{f}_i(\theta_i, x_i(\theta)) - \bar{t}_i(\theta) \geq \bar{f}_i(\theta_i, x_i(\theta_i', \theta_{-i})) - \bar{t}_i(\theta_i', \theta_{-i}) - \hat{h}_i(\theta_i, \theta_i')$, $\forall \theta, \theta_i', i$.

The mechanism $\phi = (x(\cdot), t(\cdot))$ is $\hat{h}$-SP. ■

Proof of Remark 1

Proof.

ϕ is h-SP for punishment function h, so $\bar{f}_i(\theta_i, x_i(\theta)) - \bar{t}_i(\theta) \geq \bar{f}_i(\theta_i, x_i(\theta_i', \theta_{-i})) - \bar{t}_i(\theta_i', \theta_{-i}) - h_i(\theta_i, \theta_i')$, $\forall \theta, \theta_i', i$.

$\hat{\phi}$ is h-SP for punishment function $\hat{h}$, so $\bar{f}_i(\theta_i, \hat{x}_i(\theta)) - \bar{\hat{t}}_i(\theta) \geq \bar{f}_i(\theta_i, \hat{x}_i(\theta_i', \theta_{-i})) - \bar{\hat{t}}_i(\theta_i', \theta_{-i}) - \hat{h}_i(\theta_i, \theta_i')$, $\forall \theta, \theta_i', i$.

Then the linear combination of the two inequalities with weighted of λ and $(1-\lambda)$ results $\lambda[\bar{f}_i(\theta_i, x_i(\theta)) - \bar{t}_i(\theta)] + (1-\lambda)[\bar{f}_i(\theta_i, \hat{x}_i(\theta)) - \bar{\hat{t}}_i(\theta)] \geq \lambda[\bar{f}_i(\theta_i, x_i(\theta_i', \theta_{-i})) - \bar{t}_i(\theta_i', \theta_{-i}) - h_i(\theta_i, \theta_i')] + (1-\lambda)[\bar{f}_i(\theta_i, \hat{x}_i(\theta_i', \theta_{-i})) - \bar{\hat{t}}_i(\theta_i', \theta_{-i}) - \hat{h}_i(\theta_i, \theta_i')]$, $\forall \theta, \theta_i', i$.

$x_i(\theta) = \hat{x}_i(\theta)$ and $x_i(\theta_i', \theta_{-i}) = \hat{x}_i(\theta_i', \theta_{-i})$, so rearrange the inequality above, there is $\bar{f}_i(\theta_i, x_i(\theta)) - \lambda \bar{t}_i(\theta) - (1-\lambda)\bar{\hat{t}}_i(\theta) \geq \bar{f}_i(\theta_i, x_i(\theta_i', \theta_{-i})) - \lambda \bar{t}_i(\theta_i', \theta_{-i}) - (1-\lambda)\bar{\hat{t}}_i(\theta_i', \theta_{-i}) - [\lambda h_i(\theta_i, \theta_i') + (1-\lambda)\hat{h}_i(\theta_i, \theta_i')]$, with $x_i(\cdot) = \hat{x}_i(\cdot) = \tilde{x}_i(\cdot)$, $\tilde{h}(\cdot) = \lambda h(\cdot) + (1-\lambda)\hat{h}(\cdot)$.

It is equivalent with $\bar{f}_i(\theta_i, \tilde{x}_i(\theta)) - \bar{\tilde{t}}_i(\theta) \geq \bar{f}_i(\theta_i, \tilde{x}_i(\theta_i', \theta_{-i})) - \bar{\tilde{t}}_i(\theta_i', \theta_{-i}) - \tilde{h}(\theta_i, \theta_i')$, $\forall \theta, \theta_i', i$. Thus, $\tilde{\phi}$ is h-SP for punishment function $\tilde{h}$. ■

Proof of Proposition 2

Proof.

A mechanism is h-SP if and only if $v_i(\theta_i, \theta_i, \theta_{-i}') \geq v_i(\theta_i, \theta') - h_i(\theta_i, \theta_i')$, for any θ', θ_i. It is equivalent with $h_i(\theta_i, \theta_i') \geq v_i(\theta_i, \theta') - v_i(\theta_i, \theta_i, \theta_{-i}')$, for any θ', θ_i.

So if $h_i(\theta_i, \theta_i') \geq \max_{\theta_{-i}'}[v_i(\theta_i, \theta') - v_i(\theta_i, \theta_i, \theta_{-i}')]$ for any θ', θ_i, mechanism ϕ is h-SP.

On the other side, if $h_i(\theta_i, \theta_i') < \max_{\theta_{-i}'}[v_i(\theta_i, \theta') - v_i(\theta_i, \theta_i, \theta_{-i}')]$ for some θ', θ_i. Then there exists θ_{-i}', when intermediary i has ability of transmission θ_i, he will deviate to report θ_i' and achieve higher profit $v_i(\theta_i, \theta') - h_i(\theta_i, \theta_i')$, such that mechanism is not h-SP. ■

Proof of Corollary 1

Proof.

i. By definition, the minimal punishment function $h_i^{\min}(\theta_i, \theta_i') = \max_{\theta_{-i}'}[v_i(\theta_i, \theta') - v_i(\theta_i, \theta_i, \theta_{-i}')]$, for any $\theta_i, \theta_i', \theta_{-i}'$. We have $v_i(\theta_i, \theta') - v_i(\theta_i, \theta_i, \theta_{-i}') = \bar{f}_i(\theta_i, x_i(\theta')) - \bar{t}_i(\theta') - \bar{f}_i(\theta_i, x_i(\theta_i, \theta_{-i}')) + \bar{t}_i(\theta_i, \theta_{-i}')$.

Consider the condition of individual rationality for intermediary to participate, $\bar{f}_i(\theta_i, x_i(\theta_i, \theta_{-i}')) \geq \bar{t}_i(\theta_i, \theta_{-i}')$, so $\bar{f}_i(\theta_i, x_i(\theta')) - \bar{t}_i(\theta') - \bar{f}_i(\theta_i, x_i(\theta_i, \theta_{-i}')) + \bar{t}_i(\theta_i, \theta_{-i}') \leq \bar{f}_i(\theta_i, x_i(\theta')) - \bar{t}_i(\theta') \leq \bar{f}_i(\theta_i, x_i(\theta'))$.

Since $x_i(\theta') \leq 1$, $\bar{f}_i(\theta_i, x_i(\theta')) \leq \bar{f}_i(\theta_i, 1)$ is bounded. $v_i(\theta_i, \theta') - v_i(\theta_i, \theta_i, \theta_{-i}') \leq \bar{f}_i(\theta_i, 1)$, $\forall \theta_i, \theta'$, the upper bound exists. Thus, there exists minimal punishment function $h_i^{\min}(\theta_i, \theta_i') = \max_{\theta_{-i}'}[v_i(\theta_i, \theta') - v_i(\theta_i, \theta_i, \theta_{-i}')]$.

ii. Consider mechanism $\phi = (x(\cdot), s(\cdot))$ is strategy proof. Then $v_i(\theta_i, \theta') \leq v_i(\theta_i, \theta_i, \theta_{-i}')$, for any θ', θ_i. So $\max_{\theta_{-i}'}[v_i(\theta_i, \theta') - v_i(\theta_i, \theta_i, \theta_{-i}')] \leq 0$ for any θ_i', θ_i. Then the minimal punishment function $h_i^{\min}(\theta_i, \theta_i') = 0$, for any θ_i', θ_i.

iii. Consider mechanism $\phi = (x(\cdot), s(\cdot))$ is not strategy proof, there exists θ_i, θ', such that $v_i(\theta_i, \theta_i, \theta'_{-i}) < v_i(\theta_i, \theta')$. Suppose $h_i^{\min}(\theta_i, \theta'_i) = 0$, for any θ'_i, θ_i. Then $h_i^{\min}(\theta_i, \theta'_i) = 0 < v_i(\theta_i, \theta') - v_i(\theta_i, \theta_i, \theta'_{-i})$, contradicts with the h-SP. So $h_i^{\min}$ is nonzero. ■

Proof of Theorem 2

Proof.

i. To prove there does not exist SP mechanism, such that it is FBE.

Assume mechanism $\phi = (x(\cdot), t(\cdot))$ is SP. From Theorem 1, the profit Φ_i and production function f_i satisfies Given the allocation $x(\theta)$, the resource charged $t(\theta)$ of SP mechanism satisfies $\Phi_i(\theta_i, \theta_{-i}) = \int_0^{\theta_i} \bar{f}_{i\theta}(t, x_i(t, \theta_{-i}))dt$, and $\Phi_i(\theta_i, \theta_{-i}) = \bar{f}_i(\theta_i, x_i(\theta)) - \bar{t}_i(\theta)$. ϕ is strategy proof, agents will report truthfully, $\theta' = \theta$. The aggregate resource charged by planner is $\bar{t}_i(\theta) = \bar{f}_i(\theta_i, x_i(\theta)) - \int_0^{\theta_i} \bar{f}_{i\theta}(t, x_i(t, \theta_{-i}))dt$.

Given mechanism ϕ, the resource allocation is $\sum_{i=1}^N t_i(\theta)$.

By definition of FBE, $\bar{u}(\theta) = \max_x u(\sum_{i=1}^N f_i(\theta_i, x_i))$, with $\sum_{i=1}^N x_i(\theta) = 1$. Thus, $\bar{u}(\theta) \geq u(\sum_{i=1}^N f_i(\theta_i, x_i))$ for any mechanism with allocation rule $x(\cdot)$.

To prove there exists θ, such that $\sum_{i=1}^N \bar{f}_i(\theta_i, x_i(\theta)) > \sum_{i=1}^N \bar{t}_i(\theta)$, which is equivalent with $\sum_{i=1}^N \int_0^{\theta_i} \bar{f}_{i\theta}(t, x_i(t, \theta_{-i}))dt > 0$. Without loss of generality, assume for $\tilde{\theta}$, $x_i(\tilde{\theta}) > 0$, and allocation rule $x_i(t, \theta_{-i})$ is monotonic in t. Let $\theta_{-i} = \tilde{\theta}_{-i}$, $\theta_i > \tilde{\theta}_i$, then $x_i(t, \theta_{-i}) > 0$ for any $t > \tilde{\theta}_i$, $\bar{f}_{i\theta}(t, x_i(t, \theta_{-i})) > 0$. Thus, $\int_0^{\theta_i} \bar{f}_{i\theta}(t, x_i(t, \theta_{-i}))dt \geq \int_{\tilde{\theta}_i}^{\theta_i} \bar{f}_{i\theta}(t, x_i(t, \theta_{-i}))dt > 0$.

The aggregate resource allocated to agents under mechanism ϕ is $\bar{t}_i(\theta) = \bar{f}_i(\theta_i, x_i(\theta)) - \int_0^{\theta_i} \bar{f}_{i\theta}(t, x_i(t, \theta_{-i}))dt < \bar{f}_i(\theta_i, x_i(\theta))$, when type of agents is θ.

By definition of maximal utility $\bar{u}(\theta)$, $\bar{u}(\theta) \geq u(\sum_{i=1}^N f_i(\theta_i, x_i(\theta)) > u(\sum_{i=1}^N t_i(\theta))$. The first inequality is from definition of maximal utility $\bar{u}$, and the second inequality comes from strongly monotone of preferences.

So there is no symmetric, SP, budget balance, and FBE mechanism.

ii. Consider ϕ is FBE mechanism. If the transmission ability of agents are θ, and the agents report truthfully about their ability of transmission, $\theta' = \theta$. The first best allocation of resource is $\bar{x}(\theta)$. To achieve the fist best efficient, the charged resource $t_i(\theta)$ should equal to the true quality of production $f_i(\theta_i, x_i(\theta))$ for any agent allocated with positive resource $x_i(\theta) > 0$.

Assume $x(\theta) = (x_1(\theta), \ldots, x_N(\theta))$ solves the utility maximization problem of planner $\max_x u(\sum_{i=1}^N f_i(\theta_i, x_i))$ for any θ, the allocation of resource that maximizes planner's utility is $\bar{y}(\theta) = \sum_{i=1}^N f_i(\theta_i, x_i(\theta))$. Suppose there exists i, the charged resource $t_i(\theta) \leq f_i(\theta_i, x_i(\theta))$, $t_i(\theta) \neq f_i(\theta_i, x_i(\theta))$ and $x_i(\theta) > 0$. Then the resource allocation $y = \sum_{i=1}^N t_i(\theta) \leq \bar{y}(\theta)$ and $y \neq \bar{y}(\theta)$. If the preferences of planner is strongly monotone, then $u(y) < u(\bar{y}(\theta))$, the fist best efficient will not be achieved. For FBE mechanism ϕ, the charged resource

$t_i(\theta) = f_i(\theta_i, x_i(\theta))$, if $x_i(\theta) > 0$, $\theta' = \theta$.

If $x_i(\theta) = 0$, $f_i(\theta_i, x_i(\theta)) = 0$. If $x_i(\theta) > 0$, $t_i(\theta) = f_i(\theta_i, x_i(\theta))$, we have $v_i(\theta_i, \theta) = \bar{f}_i(\theta_i, x_i(\theta)) - \bar{t}_i(\theta) = 0$, and $v_i(\theta_i, \theta') = \bar{f}_i(\theta_i, x_i(\theta')) - \bar{t}_i(\theta') = \bar{f}_i(\theta_i, x_i(\theta')) - \bar{f}_i(\theta'_i, x_i(\theta'))$, with $\theta'_{-i} = \theta_{-i}$. Then $v_i(\theta_i, \theta') - v_i(\theta_i, \theta) = \bar{f}_i(\theta_i, x_i(\theta')) - \bar{f}_i(\theta'_i, x_i(\theta'))$. The minimal punishment function for agent i is $h_i^{\min}(\theta_i, \theta'_i) = \max_{\theta'_{-i}} \bar{f}_i(\theta_i, x_i(\theta')) - \bar{f}_i(\theta'_i, x_i(\theta'))$.

Since $\bar{f}_{i\theta} \geq 0$, $\bar{f}_i(\theta_i, x_i(\theta')) - \bar{f}_i(\theta'_i, x_i(\theta')) \leq 0$ for any $\theta'_i \geq \theta_i$, the minimal punishment $h_i^{\min}(\theta_i, \theta'_i) = 0$ for $\theta'_i \geq \theta_i$. The resource given to agent i is $x_i(\theta') \in [0, 1]$[16], $h_i^{\min}(\theta_i, \theta'_i) = \max_{c\in[0,1]} \bar{f}_i(\theta_i, c) - \bar{f}_i(\theta'_i, c)$, for $\theta'_i < \theta_i$.

From Proposition 2, any punishment function h that implements a FBE mechanism if and only if $h_i(\theta_i, \theta'_i) \geq h_i^{\min}(\theta_i, \theta'_i) = \max_{c\in[0,1]} \bar{f}_i(\theta_i, c) - \bar{f}_i(\theta'_i, c)$ for any $\theta'_i \leq \theta_i$. ■

Proof of Theorem 3

Proof. Consider an arbitrary mechanism ϕ that is h-optimal, and always allocates the full resource to the intermediary with the highest aggregate intermediation quality, so $x = x_F = x_S$. Without loss of generality, assume that $\bar{f}_1(\theta_1, 1) \geq \bar{f}_2(\theta_2, 1) \geq \cdots \geq \bar{f}_N(\theta_N, 1)$, $x_1(\theta) = 1$, charge of planner $\bar{t}_1(\theta) \leq \bar{f}_1(\theta_1, 1)$.

The second price mechanism ϕ_S is strategy-proof, and the planner would charge agent i resource $\bar{t}_{S1}(\theta) = \bar{f}_2(\theta_2, 1)$. So the h-optimal mechanism will transmit no less than $\bar{f}_2(\theta_2, 1)$.

The optimal aggregate charging $\bar{f}_1(\theta_1, 1) \geq \bar{t}_1(\theta) \geq \bar{f}_2(\theta_2, 1)$, there exists $\lambda_1(\theta) \in [0, 1]$, s.t. $\bar{t}_1(\theta) = \lambda_1(\theta)\bar{f}_1(\theta_1, 1) + (1 - \lambda_1(\theta))\bar{f}_2(\theta_2, 1)$. The mechanism ϕ can be written as $\phi(\theta) = \lambda(\theta)\phi_F(\theta) + (1 - \lambda(\theta))\phi_S(\theta)$, for any θ.

By h-SP, $\bar{f}_1(\theta_1, 1) - \bar{t}_1(\theta) \geq \bar{f}_1(\theta_1, x_1(\theta'_1, \theta_{-1})) - \bar{t}_1(\theta'_1, \theta_{-1}) - h_1(\theta_1, \theta'_1)$.

For any θ'_1 satisfying $\bar{f}_{-1}(\theta_{-1}, 1) \leq \bar{f}_1(\theta'_1, 1) \leq \bar{f}_1(\theta_1, 1)$, we have $x_1(\theta'_1, \theta_{-1}) = 1$. The condition of h-SP is equivalent with $h_1(\theta_1, \theta'_1) \geq \bar{t}_1(\theta) - \bar{t}_1(\theta'_1, \theta_{-1})$.

Assume $t_i(\theta) = \lambda_i(\theta)f_i(\theta_i, 1) + (1 - \lambda_i(\theta))f_{-i}(\theta_{-i}, 1)$, there is $t_i(\theta'_i, \theta_{-i}) = \lambda_i(\theta'_i, \theta_{-i})f_i(\theta'_i, 1) + (1 - \lambda_i(\theta'_i, \theta_{-i}))f_{-i}(\theta_{-i}, 1)$. h-SP is equivalent with $h_i(\theta_i, \theta'_i) \geq \lambda_i(\theta)\bar{f}_i(\theta_i, 1) - \lambda_i(\theta'_i, \theta_{-i})\bar{f}_i(\theta'_i, 1) + \bar{f}_{-i}(\theta_{-i}, 1)(\lambda_i(\theta'_i, \theta_{-i}) - \lambda_i(\theta))$.

Take the limit $\theta'_i \to \theta_{i-}$, $h'_i(\theta_i) \geq (\bar{f}_i(\theta_i, 1) - \bar{f}_{-i}(\theta_{-i}, 1))\lambda'_i(\theta) + \lambda_i(\theta)\bar{f}'_{i\theta}(\theta_i, 1)$.

Based on h-SP, $h_i(\theta_i, \theta'_i) \geq \bar{t}_i(\theta) - \bar{t}_i(\theta'_i, \theta_{-i})$, for any θ'_i with $\bar{f}_{-i}(\theta_{-i}, 1) \leq \bar{f}_i(\theta'_i, 1) \leq \bar{f}_i(\theta_i, 1)$. Assume function $r : \mathbb{R}^N \mapsto \mathbb{R}^N$, $r(\theta) = (r_i(\theta_{-i}))_{1\leq i\leq N}$, such that, $\bar{f}_i(r_i(\theta_{-i}), 1) = \bar{f}_{-i}(\theta_{-i}, 1)$. $\bar{t}_i(\theta) \leq \bar{t}_i(r_i(\theta_{-i}), \theta_{-i}) + h_i(\theta_i, r_i(\theta_{-i}))$. For second price mechanism, $\bar{t}_i(r_i(\theta_{-i}), \theta_{-i}) = \bar{f}_{-i}(\theta_{-i}, 1)$, then $\bar{t}_i(\theta) \leq \bar{f}_{-i}(\theta_{-i}, 1) + h_i(\theta_i, r_i(\theta_{-i}))$.

$\bar{t}_i(\theta) = \lambda_i(\theta)\bar{f}_i(\theta_i, 1) + (1 - \lambda_i(\theta))\bar{f}_{-i}(\theta_{-i}, 1)$, then $\lambda_i(\theta)\bar{f}_i(\theta_i, 1) + (1 - \lambda_i(\theta))\bar{f}_{-i}(\theta_{-i}, 1) \leq \bar{f}_{-i}(\theta_{-i}, 1) + h_i(\theta_i, r_i(\theta_{-i}))$, $\lambda_i(\theta)(\bar{f}_i(\theta_i, 1) - \bar{f}_{-i}(\theta_{-i}, 1)) \leq h_i(\theta_i, r_i(\theta_{-i}))$.

[16]The range of $x_i(\theta')$ could be restricted further under certain preferences of planner, in general case, $\min_{\theta'_{-i}} x_i(\theta') \leq x_i(\theta') \leq \max_{\theta'_{-i}} x_i(\theta')$.

Hence, any h-SP mechanism that allocating resource to agent with highest aggregate production would be charged not more than $\bar{t}_i(\theta)$ with $\lambda_i(\theta)$.

To prove the mechanism with $\bar{t}_i(\theta)$ is h-SP. If $h_i(a,b)$ is not increasing in scale of deviation $h_i(a,c) \le h_i(a,b) + h_i(b,c)$, with $a \ge b \ge c$.

Moreover, ϕ^h is h-SP and allocates to agent with highest aggregate production. Then ϕ^h is h-optimal among mechanisms with allocation rule $x_i(\theta)$.

From Theorem1, h-SP mechanism satisfies $\bar{f}_{i\theta}(\theta_i, x_i(\theta)) + h'_{i-}(\theta_i) \le \frac{\partial \Phi_i(\theta_i,\theta_{-i})}{\partial \theta_i} \le \bar{f}_{i\theta}(\theta_i, x_i(\theta)) + h'_{i+}(\theta_i)$.

$\frac{\partial \Phi_i(\theta_i,\theta_{-i})}{\partial \theta_i} = \bar{f}_{i\theta}(\theta_i, x_i(\theta)) + \bar{f}_{ix}(\theta_i, x_i(\theta)) \frac{\partial x_i(\theta)}{\partial \theta_i} - \frac{\partial \bar{t}_i(\theta)}{\partial \theta_i}$, then $\frac{\partial \bar{t}_i(\theta)}{\partial \theta_i} \le \bar{f}_{ix}(\theta_i, x_i(\theta)) \frac{\partial x_i(\theta)}{\partial \theta_i} - h'_{i-}(\theta_i)$.

$h_i(\theta_i, \theta'_i) \ge 0$, and $h'_{i-}(\theta_i) \le 0$.

Then $\bar{t}_i(\theta) \le \int_0^{\theta_i} \bar{f}_{ix}(q, x_i(q, \theta_{-i})) \frac{\partial x_i(q,\theta_{-i})}{\partial \theta_i} - h'_{i-}(q) dq$.

For general allocation rule $x_i(\theta)$, maximal charging is $\bar{t}_i(\theta) = \bar{f}_i(\theta_i, x_i(\theta))$, 0-SP charing is $\bar{f}_i(\theta_i, x_i(\theta)) - \int_0^{\theta_i} \bar{f}_{i\theta}(q, x_i(q, \theta_{-i})) dq$.

$\lambda_i(\theta) \bar{f}_i(\theta_i, x_i(\theta)) + (1-\lambda_i(\theta))(\bar{f}_i(\theta_i, x_i(\theta)) - \int_0^{\theta_i} \bar{f}_{i\theta}(q, x_i(q, \theta_{-i})) dq) \le \int_0^{\theta_i} \bar{f}_{ix}(q, x_i(q, \theta_{-i})) \frac{\partial x_i(q,\theta_{-i})}{\partial \theta_i} - h'_{i-}(q) dq$.

There is $\lambda_i(\theta) \le \frac{\int_0^{\theta_i} \bar{f}_{ix}(q, x_i(q,\theta_{-i})) \frac{\partial x_i(q,\theta_{-i})}{\partial \theta_i} - h'_{i-}(q) dq - \bar{f}_i(\theta_i, x_i(\theta)) + \int_0^{\theta_i} \bar{f}_{i\theta}(q, x_i(q,\theta_{-i})) dq}{\int_0^{\theta_i} \bar{f}_{i\theta}(q, x_i(q,\theta_{-i})) dq}$.

$\int_0^{\theta_i} \bar{f}_{i\theta}(q, x_i(q, \theta_{-i})) dq + \int_0^{\theta_i} \bar{f}_{ix}(q, x_i(q, \theta_{-i})) \frac{\partial x_i(q,\theta_{-i})}{\partial \theta_i} dq = \int_0^{\theta_i} \frac{\partial \bar{f}_i(q, x_i(q,\theta_{-i}))}{\partial \theta_i} dq = \bar{f}_i(\theta_i, x_i(\theta)) - \bar{f}_i(0, x_i(0, \theta_{-i}))$, $\bar{f}_i(0, x_i(0, \theta_{-i})) = 0$.

Thus, the inequality is equivalent with $\lambda_i(\theta) \le \frac{\int_0^{\theta_i} -h'_{i-}(q) dq}{\int_0^{\theta_i} \bar{f}_{i\theta}(q, x_i(q,\theta_{-i})) dq}$.

The inequality need to be modified, since when $\theta_i \le \max \theta_{-i}$, there is no incentive to deviate and punishment is not effective, change the lower bound 0 to lower bound of $\underline{\theta}_i$. $\underline{\theta}_i(\theta_{-i}) = \sup\{\theta_i | x_i(\theta_i, \theta_{-i}) = 0\}$.

$\lambda_i(\theta)$ satisfy $\lambda_i(\theta) \le \frac{\int_{\underline{\theta}_i(\theta_{-i})}^{\theta_i} -h'_{i-}(q) dq}{\int_{\underline{\theta}_i(\theta_{-i})}^{\theta_i} \bar{f}_{i\theta}(q, x_i(q,\theta_{-i})) dq}$.

For the mechanisms that allocate all resource to agent with highest aggregate production, assume the production function $f_i(\theta_i, x_i(\theta)) = \theta_i x_i(\theta)$, $\underline{\theta}_i(\theta_{-i}) = \max_{j \ne i} \theta_j$, $\bar{f}_{i\theta}(q, x_i(q, \theta_{-i})) = x_i(q, \theta_{-i}) = 1$ for $\underline{\theta}_i(\theta_{-i}) \le q \le \theta_i$. $\int_{\underline{\theta}_i(\theta_{-i})}^{\theta_i} \bar{f}_{i\theta}(q, x_i(q, \theta_{-i})) dq = \theta_i - \underline{\theta}_i(\theta_{-i})$. If the punishment function is linear, $h_i(\theta_i, \theta'_i) = c|\theta_i - \theta'_i|$ with $c \in [0, 1]$, then $-h'_{i-}(q) = c$. In this case, $\lambda_i(\theta) \le \frac{c(\theta_i - \underline{\theta}_i(\theta_{-i}))}{\theta_i - \underline{\theta}_i(\theta_{-i})} = c$.

Another way to write, from condition of h-SP, $\bar{f}_{i\theta}(\theta_i, x_i(\theta)) + h'_{i-}(\theta_i) \le \frac{\partial \Phi_i(\theta_i,\theta_{-i})}{\partial \theta_i} \le \bar{f}_{i\theta}(\theta_i, x_i(\theta)) + h'_{i+}(\theta_i)$.

$\frac{\partial \Phi_i(\theta_i,\theta_{-i})}{\partial \theta_i} = \frac{\partial \bar{f}_i(\theta_i, x_i(\theta))}{\partial \theta_i} - \frac{\partial \bar{t}_i(\theta)}{\partial \theta_i} \ge \bar{f}_{i\theta}(\theta_i, x_i(\theta)) + h'_{i-}(\theta_i)$. Then $\frac{\partial \bar{t}_i(\theta)}{\partial \theta_i} \le \frac{\partial \bar{f}_i(\theta_i, x_i(\theta))}{\partial \theta_i} - \bar{f}_{i\theta}(\theta_i, x_i(\theta)) - h'_{i-}(\theta_i)$. For $\theta_i < \underline{\theta}_i(\theta_{-i})$, $\bar{f}_i(\theta_i, x_i(\theta)) = 0$ and $\bar{t}_i(\theta) = 0$. Integral the inequality from $\underline{\theta}_i(\theta_{-i})$ to θ_i, we have the constraint of aggregate charging by planner from agent i. $\bar{t}_i(\theta) \le \bar{f}_i(\theta_i, x_i(\theta)) - \int_{\underline{\theta}_i(\theta_{-i})}^{\theta_i} \bar{f}_{i\theta}(q, x_i(q, \theta_{-i})) dq - \int_{\underline{\theta}_i(\theta_{-i})}^{\theta_i} h'_{i-}(q) dq$.

Since $h'_{i-}(q) \leq 0$, the upper bound of charging by planner increases as the punishment increase, the charging is no less than $\bar{f}_i(\theta_i, x_i(\theta)) - \int_{\underline{\theta}_i(\theta_{-i})}^{\theta_i} \bar{f}_{i\theta}(q, x_i(q, \theta_{-i}))dq$ in 0-SP case.

■

Proof of Corollary 3

Proof.

To verify the punish function h in condition i and ii satisfy $\lambda_i^h(\theta) = 0$.

In condition i, $\frac{\partial h_i(\theta_i,\theta_i')}{\partial \theta_i}|_{\theta_i'=\theta_i} = \lim_{\theta_i'\to\theta_i} \frac{h_i(\theta_i,\theta_i')-h_i(\theta_i,\theta_i)}{\theta_i'-\theta_i} = \lim_{\theta_i'\to\theta_i} \frac{h_i(\theta_i,\theta_i')}{\theta_i'-\theta_i} = 0$.

$\bar{f}_{i\theta}(\theta_i, x_i(\theta)) = \frac{\partial \bar{f}_i(\theta_i,x_i(\theta))}{\partial \theta_i}$, $\lim_{\theta_i'\to\theta_i} \frac{\bar{f}_i(\theta_i',1)-\bar{f}_i(\theta_i,1)}{\theta_i'-\theta_i} = \bar{f}_{i\theta}(\theta_i, 1) > 0$.

$\lim_{\theta_i'\to\theta_i} \frac{h_i(\theta_i,\theta_i')}{\bar{f}_i(\theta_i',1)-\bar{f}_i(\theta_i,1)} = \frac{h_i'(\theta_i)}{\bar{f}_{i\theta}(\theta_i,1)} = 0$. Thus, $\lambda_i^h(\theta) = \min_{\theta_i'} \frac{h_i(\theta_i,\theta_i')}{\bar{f}_i(\theta_i,1)-\bar{f}_i(\theta_i',1)} = 0$, for any θ, and $\phi^h = \phi_S$.

In condition ii, $\lambda_i^h(\theta) = \min_{\theta_i'} \frac{h_i(\theta_i,\theta_i')}{\bar{f}_i(\theta_i,1)-\bar{f}_i(\theta_i',1)} = 0$ also satisfy for any θ, there is $\phi^h = \phi_S$.

From Theorem 3, the h-optimal mechanism is ϕ_S. ■

Proof of Corollary 4

Proof.

From Theorem 2, the minimal punishment function for the first price mechanism is $h_i(\theta_i, \theta_i') = \max_{c\in[0,1]} \bar{f}_i(\theta_i, c) - \bar{f}_i(\theta_i', c)$.

Since $\frac{\partial^2 f_i}{\partial\theta\partial x} \geq 0$, $\max_{c\in[0,1]} \bar{f}_i(\theta_i, c) - \bar{f}_i(\theta_i', c) = \bar{f}_i(\theta_i, 1) - \bar{f}_i(\theta_i', 1)$.

Thus, when $h_i(\theta_i, \theta_i') \geq \bar{f}_i(\theta_i, 1) - \bar{f}_i(\theta_i', 1)$, first price mechanism is h-strategy-proof.

To verify $\lambda_i^h(\theta) = 1$.

$h_i(\theta_i, \theta_i') \geq \bar{f}_i(\theta_i, 1) - \bar{f}_i(\theta_i', 1)$, which means $\frac{h_i(\theta_i,\theta_i')}{\bar{f}_i(\theta_i,1)-\bar{f}_i(\theta_i',1)} \geq 1$. $\lambda_i^h(\theta) = \min_{\theta_i'} \frac{h_i(\theta_i,\theta_i')}{\bar{f}_i(\theta_i,1)-\bar{f}_i(\theta_i',1)}$.

From Theorem 3, the h-optimal mechanism is $\lambda_i^h(\theta)\phi_F + (1 - \lambda_i^h(\theta))\phi_S = \phi_F$. ■

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