The strategy of conflict and cooperation*

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Abstract

The story of conflict and cooperation has started millions of years ago, and now it is everywhere: In biology, computer science, economics, political science, and psychology. Examples include wars, airline alliances, trade, oligopolistic cartels, the evolution of species and genes, and team sports. However, neither cooperative games nor non-cooperative games—in which "each player acts independently without collaboration with any of the others" (Nash, 1951)—fully capture the competition between and across individuals and groups, and the strategic partnerships that give rise to such groups. Thus, one needs to extend the non-cooperative framework to study strategic games like scientific publication, which is a rather competitive game, yet (strategic) collaboration is widespread. In this paper, I propose, to the best of my knowledge, the first solution to the long-standing open problem of strategic cooperation first identified by von Neumann (1928). I introduce the *equilibrium system* solution in *coalitional strategic games* in which players are free to cooperate to coordinate their actions or act independently. Coalitional strategic games unify the study of strategic competition as well as cooperation including logrolling and corruption which have been studied in specific frameworks.

Keywords: strategic cooperation, non-cooperative games, cooperative games, backward induction, extensive form games

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1 Introduction

Cooperation and conflict are not only two of the most studied topics in economics but they are also widely studied in biology, computer science, philosophy, political science, psychology and so on. Wars, transportation, airline alliances, trade, oligopolistic cartels, corruption, the evolution of species and genes, and team sports are examples of strategic situations that involve both conflict and cooperation. Sellers prefer a higher price, whereas buyers prefer a lower price; yet, mutually beneficial trade often takes place. Many elections are games of cooperation as well as competition. In a judicial process, we may have conflicting interests with the opposing side, but we also cooperate with our lawyer and possibly with officials. Global airplane transportation is a giant competitive market, but alliances among airline companies are common. Many popular team sports involve cooperation as well as competition. Conflict and cooperation are widespread across animal species, including humans.[1] However, neither cooperative games nor non-cooperative games in which "each participant acts independently without collaboration with any of the others" (Nash, 1951) fully capture the competition between and across individuals and groups, and the strategic partnerships that give rise to such groups. Thus, one needs to extend the non-cooperative framework to study strategic games like scientific publication, which is a rather competitive game, yet (strategic) collaboration is widespread.

In this paper, I propose, to the best of my knowledge, the first solution to the long-standing open problem of strategic cooperation first identified by von Neumann (1928) and later discussed e.g. by Bernheim, Peleg, and Whinston (1987), who proposed partial solutions.[2] I introduce a novel framework, *coalitional strategic games* in extensive form, and a novel solution concept, *equilibrium system*, which is obtained by applying the *Recursive Backward Induction* (RBI) algorithm, which is a novel generalization of the well-known backward induction algorithm. I show that every coalitional strategic game with possibly imperfect information possesses an equilibrium system. In particular, if the game is of perfect information, then there is an equilibrium system which includes only pure strategies.

Moreover, I apply the equilibrium system to the study of vote-trading à la Casella and Palfrey (2019) and corruption—i.e., buying someone's cooperation to induce them to choose a particular action, which otherwise would not be rational to choose. I introduce

[1]For competition and cooperation among freight carriers, see, e.g., Krajewska et al. (2008); for more examples in multi-agent systems in computer science, Doran et al. (1997); for more applications of game theory, Binmore (2007).

[2]For further discussion, see, subsection 2.2 and subsection 2.3.

a very general logrolling model, which always possesses an equilibrium system. I show that if there is a Condorcet winner in a logrolling game under the majority rule, then the unique equilibrium system outcome is the Condorcet winner, confirming the conjecture of Buchanan and Tullock (1962) that vote-trading would lead to the Condorcet winner when there is one.

Formally, a coalitional strategic game is denoted by $\Gamma = (P, X, I, u, S, H)$, which is an extensive form game with an addition of coalitional utility function for each feasible coalition. P denotes the set of players, X the game tree, I the player function, H the set of all information sets, and S the set of all mixed strategy profiles. For every feasible (possibly singleton) coalition $C \subseteq P$, u_C denotes the von Neumann-Morgenstern utility of player C. Note that coalitional strategic games generalize non-cooperative extensive form games in the sense that if no coalition (with two or more players) is feasible, then a coalitional strategic game would reduce to a standard extensive form game. Unlike in non-cooperative games, the set of players is endogenous—i.e., it may evolve throughout the game according to the following rule: If some players, say i and j, form a coalition $C = \{i, j\}$, then each of them becomes an "agent" of player C.[3,4] Agents then choose their strategy according to player C's utility, u_C. The utility function of an agent does not become irrelevant as it is useful in determining which coalitions are "individually rational." The equilibrium system rationally endogenizes the set of players—a player joins a coalition if and only if the player is better off by joining the coalition.

A *system* is a family of collection of strategy profiles and a family of collection of coalitions. (Note that a solution concept in non-cooperative games consists of only one strategy profile.) A subtree of a game tree (i.e., coalitional strategic game) is like a subgame but any information set can be the root of the subtree. A supertree of a game is a clone of the game such that the player who acts at the root of the game joins a coalition with some other player(s). These notions are formally defined in section 3.

Accordingly, an *equilibrium system* is a system that is both subtree and supertree "perfect." Intuitively, equilibrium system is a system such that at every subtree and supertree, independent players do not have any incentive to deviate unilaterally and the agents that make up the coalitions prefer to be in their respective coalitions than be an independent player or be an agent of another coalition. To illustrate this solution concept, I next provide an example.

[3] A related but somewhat opposite approach called "player splitting" is studied by Perea y Monsuwé et al. (2000) and Mertens (1989), and it was used in the refinement of Nash equilibrium.

[4] If coalition-forming is costly, then this can be incorporated into the coalitional as well as individual utility functions.

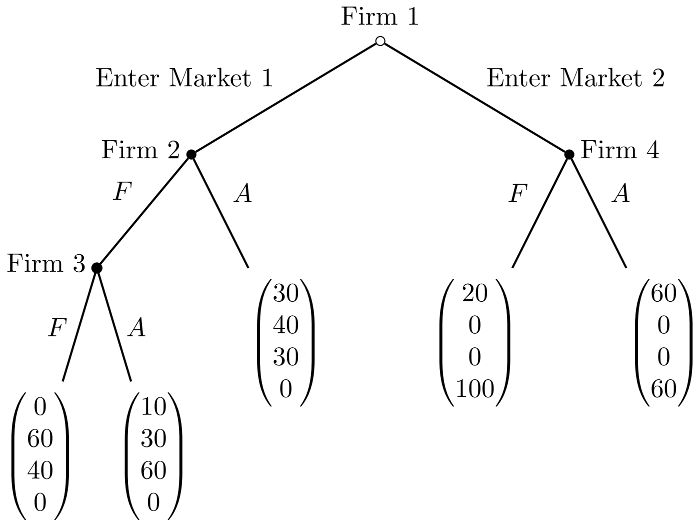


Figure 1: International market entry game

1.1 Illustrative example

Figure 1 presents a stylized international market entry game. Firm 1 from country X chooses to enter the market either in country Y (small) or country Z (large). In country Y there is already a leader (Firm 2) and a follower (Firm 3); in country Z, there is a monopolist (Firm 4). Each firm in Y and Z can choose to fight (F) or accommodate (A). If Firm 1 enters the market in the smaller country, then the total size of the "cake" is 100 units, and if Firm 1 enters the larger economy, then the total size of the cake is 120 units, which are distributed as shown in the figure. Pre-entry profits are normalized to zero. If Firm 1 enters an economy in a country, then by reciprocity the existing firm(s) in that country gain access to the market in country X. Thus, everything else being equal Firm 1's entry is beneficial for local firms, though the distribution of these extra gains depends on the strategic choices of the firms. If the leader (Firm 2) chooses F, then the follower (Firm 3) prefers to choose A. Anticipating this, the leader's best response is to choose A. However, if they collude by both choosing F, then they would both be better off and essentially drive Firm 1 out of the market. The monopolist in country Z prefers to choose A over F.

The equilibrium system solution is based on a recursive procedure called Recursive Backward Induction (RBI), which is formally provided in section 3. To illustrate the steps that lead to the equilibrium system in a simple game, I make the following assumption in the game presented in Figure 1. For each non-empty coalition $C \subseteq \{1, 2, 3, 4\}$, let the

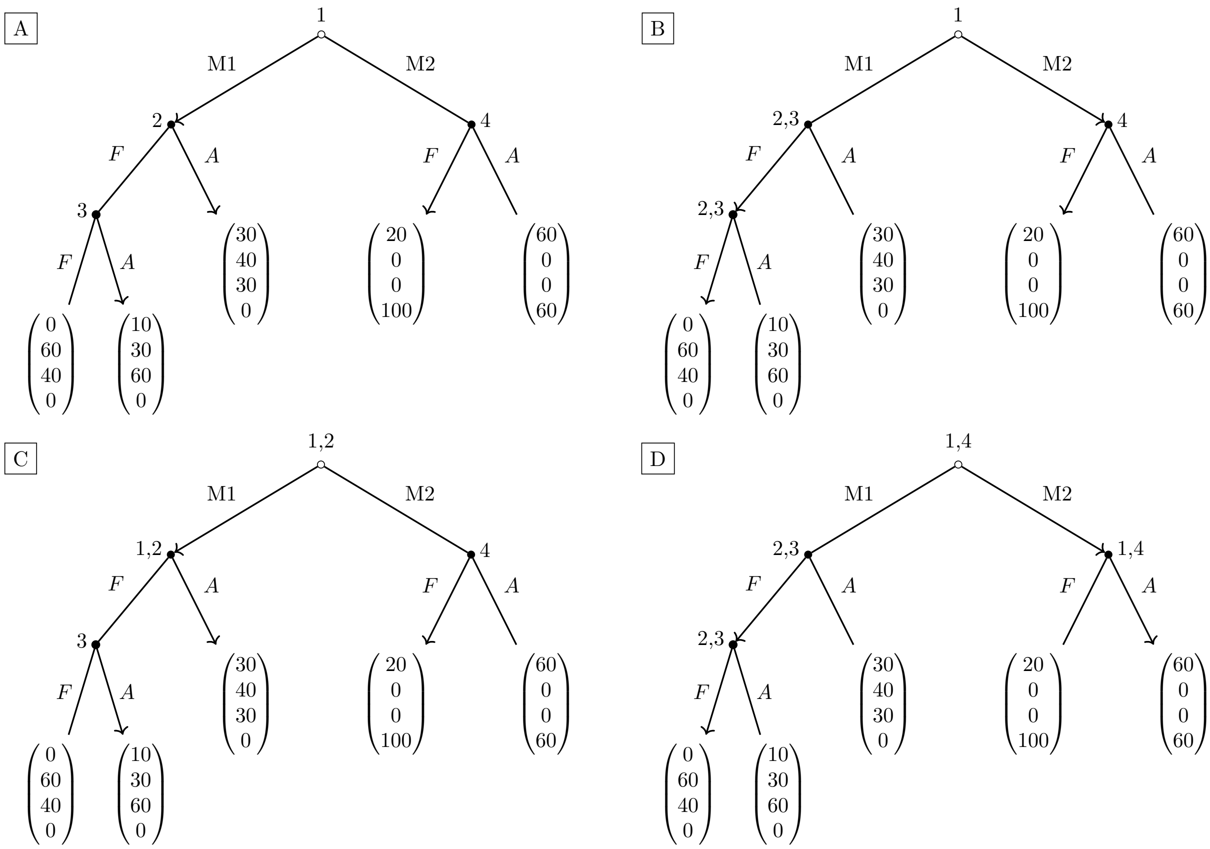


Figure 2: Equilibrium system solution of the international market entry game in four steps A–D. Assume that $u_C(\,\cdot\,) := \min_{i \in C} u_i(\,\cdot\,)$ for every coalition $C \subseteq \{1, 2, 3, 4\}$. Step A: The standard backward induction leads to the outcome (30, 40, 30, 0). Step B: Given the solution in step A, both Firm 2 and Firm 3 benefit from colluding in which both of them choose F, which leads to the outcome (0, 60, 40, 0). Firm 1 anticipates this collusion, so enters Market 2, which leads to the outcome (20, 0, 0, 100). Step C: Both Firm 1 and Firm 2 can do strictly better than the outcome in step B by colluding in which Firm 1 enters market 1, and Firm 2 accommodates, leading to the outcome (30, 40, 30, 0). Step D: Firm 4 anticipates the collusion between Firm 1 and 2, so proposes Firm 1 to cooperate by choosing A provided that Firm 1 enters market 2, which is mutually beneficial compared to (30, 40, 30, 0). Accordingly, the equilibrium system outcome is (60, 0, 0, 60). A step-by-step equilibrium system solution of this game is available at https://youtu.be/I1hIO7CrLnM.

coalitional utility function be defined as $u_C(\,\cdot\,) := \min_{i \in C} u_i(\,\cdot\,)$.[5]

Figure 2 illustrates the equilibrium system of this market entry game in four steps. The first step (A) starts from the standard non-cooperative subgame perfect equilibrium solution, where every player acts independently and non-cooperatively. The outcome of this solution is (30, 40, 30, 0). Given the subgame perfect equilibrium, step B shows that both Firm 2 and Firm 3 benefit from colluding in which both of them choose F, which leads to the outcome (0, 60, 40, 0), where $\min_{2,3}(0, 60, 40, 0) = 40$ which is strictly greater than $\min_{2,3}(30, 40, 30, 0) = 30$. (Notation "$2, 3$" signifies the cooperation between Firm 2 and 3.) Anticipating this collusion, Firm 1 chooses to enter market 2, which leads to the outcome (20, 0, 0, 100). Step C shows that both Firm 1 and Firm 2 collude by best responding to the solution in step B. Firm 1 enters market 1, and Firm 2 accommodates, which leads to the outcome (30, 40, 30, 0), which is strictly better for both than the outcome in step B. Other firms best respond to Firm 1 and Firm 2; Firm 3 chooses A and Firm 4 chooses F. Step D shows that Firm 4 anticipates the collusion between Firm 1 and 2 and offers Firm 1 to collude; Firm 1 and Firm 4 join a coalition $C' = \{1, 4\}$. The best response of player C' to the solution in step C is that agent Firm 1 enters market 2 and agent Firm 4 accommodates, which leads to the mutually beneficial outcome of (60, 0, 0, 60). This is the outcome of the equilibrium system illustrated in Figure 2.

Notice that the reason Firm 4 colludes with Firm 1 in step D is the credible threat of collusion between Firm 1 and Firm 2 in step C, which would lead to a very bad outcome for Firm 4. In turn, the reason Firm 2 colludes with Firm 1 in step C is the credible threat of Firm 1 to choose M2, in which case Firm 2's payoff would be 0.

Everything else being equal, suppose that the outcome (20, 0, 0, 100) is changed to (40, 0, 0, 80). Then, in the unique equilibrium system Firm 1 would enter market 2, and Firm 4 would choose F rather than colluding with Firm 1. This is because Firm 1 now would *not* benefit from cooperation with Firm 2, so such a threat would *not* be credible. This variation shows that off-path credible coalitional threats play a critical role in finding and sustaining the equilibrium system.

1.2 Motivation: Why do we need a framework where strategic cooperation occurs endogenously?

One of the big questions in sciences has been how cooperation has evolved. Somehow, evolution has furnished species with an ability to collaborate and compete to survive

[5]Note that the coalitional strategic games introduced in section 3 allow for any type of von Neumann-Morgenstern coalitional preferences.

and pass their genes onto the next generations. Conflict and cooperation is widespread in animals including humans and other living organisms. Genes, however selfish they may be, engage in cartels. Evolutionary biologist Richard Dawkins coined the term "The Selfish Cooperator," after having noticed that the title of his earlier book, "The Selfish Gene," might have given a wrong impression (Dawkins, 2000, 2006).[6] It has been envisaged that "natural cooperation" could be added as a fundamental principle of evolution beside mutation and natural selection, which is based on fierce competition (Nowak, 2006). Coalitional strategic games do that by adding cooperation as a fundamental principle beside competition.

In another example, First World War was fought between a coalition of Allied Powers and Central Powers, in which members of each coalition cooperated strategically to defeat the other coalition. Payoffs at the end of the war differed among and across coalitions. Russian Empire, from the victorious Allied Powers, collapsed, as well as the three losing empires.

It is commonly believed that cooperative games require an external authority to enforce cooperative behavior, whereas non-cooperative games do not require any external enforcer. However, non-cooperative games do require an authority to induce non-cooperative behavior—the enforcer must guarantee that all players will choose their actions indepedently and, hence, not cooperate and coordinate their actions in any way.[7] If a player knows that others can collude against him or her, then this would potentially change his or her behavior, as I have illustrated in Figure 2. With that in mind, I drop the assumption in non-cooperative games that non-cooperative play can be always enforced throughout the game.[8]

In addition to encouraging competition, modern society is based on rules that in many ways facilitate cooperation, collaboration, and coordination among individuals. For example, citizens are free to make contracts—a simple e-mail can count as a binding agreement—and engage in partnerships such as marriage, employer-employee partnerships, management teams of a company, friendships, and other interpersonal relationships, which are based on formal or informal institutions. To be sure, some types of cooperation such as cartels, drugs and organ trade, and same-sex marriage may be illegal in some countries. However, especially when cooperation is mutually beneficial, it is often more

[6] Since the seminal work of Smith and Price (1973), game theory has been developed and extensively applied to biological sciences as well (Hamilton, 1967; Maynard Smith, 1982; Haigh and Cannings, 1989).

[7] For further discussion between cooperative and non-cooperative games, see, e.g., Serrano (2004).

[8] Note that I do not assume that non-cooperative play can never be enforced; whenever it can be enforced, it is part of the coalitional strategic form.

costly to enforce non-cooperation (if it is possible enforce at all) than cooperation.

In coalitional strategic games, players are free to act independently or form coalitions to coordinate their actions, which could be via formal or informal institutions. But the right to form a coalition may certainly be restricted, and it might be impossible to coordinate actions under certain reasonable situations. If there is an external enforcer that can guarantee non-cooperative behavior among some players, then this will be part of the coalitional strategic game so that all players rationally take this into account. In that sense, coalitional strategic games generalize non-cooperative games: Every non-cooperative extensive form game is a coalitional strategic game, but not vice versa.

In coalitional strategic games, I propose a novel solution concept that is based on a unique procedure that combines backward and some elements of forward induction reasoning in which players rationally cooperate or act non-cooperatively.[9] Just like credible threats play a crucial role in non-cooperative games (Schelling, 1980; Selten, 1965; Brams, 1994), they are indispensable in determining the stability of coalitions in this framework.

2 Related literature

Prior research in game theory—mainly repeated games and the lesser-known farsighted approach to cooperative games—has made many contributions to the study of cooperation from various perspectives. Next, I discuss how the current paper's contribution fits into this existing literature.

2.1 Cooperation in repeated games and the Nash program

The approach in this paper differs from the repeated games and the Nash program approach to cooperation in the following ways. The theory of repeated games mainly asks whether cooperative outcomes in a one-shot game can be sustained as Nash equilibrium or subgame perfect equilibrium when the game is repeated sufficiently many times. Similarly, under the Nash program the main question is whether it is possible to construct a non-cooperative game whose equilibrium outcome coincides with a cooperative solution. Thus, they are not concerned about cooperative behavior of players *per se*, rather they

[9] Backward induction reasoning is based on the assumption that at any point in the game players make rational choices taking into account the future only, so they do not draw any conclusions from past choices. Forward induction reasoning generally assumes that past choices affect future behavior in a rational way. Unlike backward induction, forward induction does not have a unique definition in the literature. For more information, see, e.g., Perea (2012).

study non-cooperative behavior that gives rise to cooperative outcomes. Seminal works in this literature include Nash (1953) and Rubinstein (1982).

To give an example, in repeated games to sustain mutually beneficial outcomes as a subgame perfect equilibrium, players use credible threats, but this is done completely non-cooperatively—i.e., players choose their actions individually and independently at each stage. By contrast, in a coalitional strategic game credible threats are utilized by not only individuals but also group of players who strategically collaborate and coordinate their actions.

2.2 Cooperative approaches in non-cooperative games

A closely related strand of literature is the study of coalition-proofness in non-cooperative games such as strong Nash equilibrium by Aumann (1959), strong perfect equilibrium by Rubinstein (1980), subgame perfect strong Nash equilibrium by Chander and Wooders (2020), and coalition-proof Nash equilibrium by Bernheim et al. (1987).[10] Roughly speaking, strong Nash equilibrium is a Nash equilibrium in which there is no coalitional deviation that can benefit all of its members. Coalition-proof Nash equilibrium is a weaker notion than strong Nash equilibrium, and it additionally requires that coalitional deviations—fixing the strategies of the other players—should be internally consistent in the sense that subcoalitions should not have incentives to further deviate. The most obvious difference between these solutions and the equilibrium system solution is that they refine the set of Nash equilibria (to the extent that they do not always exists), whereas the equilibrium system is neither a refinement nor a coarsening of Nash equilibrium, but it always exists (see Theorem 2).

Of note, as Bernheim et al. (1987, p. 7) themselves point out, coalitional deviations in coalition-proof Nash equilibrium is restrictive in the sense that deviating subcoalitions do not consider forming a coalition with non-deviating players: "This rules out the possibility that some member of the deviating coalition might form a pact to deviate further with someone not included in this coalition. Such arrangements are clearly much more complex than those made entirely by members of the coalition itself... Clearly, further is required to resolve these issues in a fully satisfactory way." Such complexities of strategic formation of coalitions is addressed in coalitional strategic games: Every coalition structure is considered and may potentially emerge as part of the equilibrium system solution in coalitional strategic games.

[10]Note that Rubinstein's (1980) supergame notion refers to a repetition of a game, which differs from the supergame/supertree notion used in this paper (see section 3).

Finally, another major difference is that the concepts in the coalition-proofness literature predict some *strategy profiles* that are coalition-proof according to some notion; whereas, the equilibrium system is formally a family of collection of strategy profiles and coalitions in which the prediction includes a set of coalitions that is "stable."

2.3 Farsighted non-cooperative approaches to cooperative games

The connections between non-cooperative games and cooperative games, which abstracts away from strategic interaction, have been studied since von Neumann (1928), who came up with the maximin solution in a three-person zero-sum game. Von Neumann noticed that any two players might benefit from collaboration in that three-person game, which can destablize the maximin solution. The current paper essentially builds on this observation noted first by von Neumann (1928).

Harsanyi's (1974) seminal work in cooperative games led to a recently a burgeoning body of literature that incorporates elements from non-cooperative games into cooperative games such as farsightedness and backward induction. This integration has greatly improved our understanding of both frameworks and their interrelations. There is a vast literature on the study of coalition-formation in different contexts; see, e.g., Bloch (1996), Brams, Jones, and Kilgour (2005), Herings, Mauleon, and Vannetelbosch (2009), Ray and Vohra (2015), Karos and Kasper (2018), Chander and Wooders (2020), Kimya (2020), and for an informative and extensive review of the literature, see, e.g., Ray (2007) and the references therein.

The current paper differs from the farsighted coalition-formation literature in mainly two respects: the framework and the solution concept. First, various setups used in this literature are directly comparable to neither standard extensive form games nor coalitional strategic games. This is in part due to the 'cyclic' behavior in coalition-formation frameworks and the fact that more than one player or coalition can choose an 'action' at a given decision node, which cannot occur under extensive form games.[11] Second, in cases in which a coalition-formation setup such as Kimya's (2020) is comparable to an extensive form game with perfect information, the solution concept in question generally coincides with standard non-cooperative concepts such as the backward induction. This is not surprising because the main idea is to incorporate non-cooperative notions into cooperative games as first proposed by Harsanyi (1974). Starting from an extensive form game, Chander and Wooders (2020) construct a cooperative game and then introduce

[11]Note that this is not unusual because the frameworks in this literature emerged from cooperative games.

subgame perfection to the notion of core.

While an equilibrium system is neither a refinement nor a generalization of a non-cooperative equilibrium concept, coalitional strategic games generalize extensive form games with possibly imperfect information. I next compare and contrast equilibrium system solution concept with non-cooperative concepts.

2.4 The differences between coalitional strategic games and non-cooperative games

Coalitional strategic games differ from non-cooperative games in three main dimensions: (i) Philosophical/conceptual dimension, (ii) the solution concept, and (iii) the mathematical framework.

First, as Nash (1951, p. 286) points out: "Our theory, in contradistinction, is based on the absence of coalitions in that it is assumed that each participant acts independently, without collaboration or communication with any of the others." By contrast, coalitional strategic games assume that players can form coalitions to act together in an interactive strategic situation unless otherwise specified. If non-cooperation can be enforced among some of the players, then this would be part of the coalitional strategic form. Second, the equilibrium system concept incorporates backward induction reasoning as well as some elements of forward induction reasoning, and it builds on the equilibrium ideas of Cournot (1838), von Neumann (1928), von Neumann and Morgenstern (1944), Nash (1951), and Selten (1965). However, it is neither a refinement nor a coarsening of Nash equilibrium. Third, the framework of coalitional strategic games include extensive form games as a special case.

A natural question arises as to whether we can construct a fully non-cooperative game that is equivalent to a coalitional strategic game, and then apply the standard backward induction solution to the equivalent game to get the same solution as the equilibrium system solution. This does not seem possible without knowing the equilibrium system ex ante, because the solution process is endogenous. Moreover, coalitional strategic games are richer than the standard extensive form games. Reducing the RBI to the backward induction (BI) is akin to reducing backward induction solution to a simple maximization problem on a game tree in which each player chooses only one action. Obviously, once one finds a BI solution, it is possible to cut off all off-path nodes and actions, reducing the solution into a trivial one. However, it is not possible to construct a one-action-per-player game tree that leads to the BI outcome in every extensive form game without actually finding the BI solution ex ante, because the BI solution is endogenous unlike, e.g., the trivial solution concept that assigns everything to be the solution.

3 The setup and the solution concept

3.1 The setup

Let $\Gamma = (P, X, I, u, S, H)$ denote a *coalitional strategic game* with perfect recall, which is a standard extensive form game with an addition of coalitional utility function for each feasible coalition as explained below. Γ will also be referred to as a *coalitional extensive form game.*[12]

Players: Let P be a finite set of players, which may be equal to $N = \{1, 2, ..., n\}$ or a partition of N. Each element of P is called a *coalition* denoted by $C \in P$ or *player* denoted by $i \in P$, and each player $i \in P$ has a finite set of pure actions A_i. With a slight abuse of notation singleton coalition $\{i\}$ is represented as i. The set of players may evolve throughout the game as defined next.

Forming a coalition: If, for some k, players $i_1, i_2, ..., i_k$ form a coalition $C = \{i_1, i_2, ..., i_k\}$, then each individual i_j becomes an "agent" of player C. The agents of player C choose their strategy guided by player C's utility function as defined next.

Utility functions: Let X denote a game tree, $x \in X$ a node in the tree, $|X|$ the cardinality of X, x_0 the root of the game tree where Nature moves (if any), and $z \in Z$ a terminal node, which is a node that is not a predecessor of any other node. Let $u_C : Z \to \mathbb{R}$ denote the payoff function of coalition $C \subseteq N$ where for each terminal node z, $u_C(z)$ denotes the von Neumann-Morgenstern utility of player C, which may be singleton, if z is reached.

Every coalition C for which u_C is given is called *feasible*. Each coalition $C \in P$ is assumed to be feasible. If a coalition is not feasible, then there is no utility function for that coalition.[13] Player i's individual (von Neumann-Morgenstern) utility given a player partition P is denoted by $u_i(\cdot|P)$. Note that $u_i(\cdot|P)$ does not necessarily equal to $u_i(\cdot)$ because of the possibility of synergies (positive or negative) when a player joins a coalition. For example, a player i may personally get a different utility from an outcome when i forms a coalition with j compared to the case in which i joins a coalition with $j' \neq j$.

[12] I refer to standard textbooks such as Fudenberg and Tirole (1991), whose notation I mostly adapt, for details about extensive form games.

[13] Alternatively, the individual utility from that coalition could be defined as minus infinity.

(More generally, a coalition may also affect the utility of the players outside the coalition.) This distinction will be useful in determining which coalitions are "individually rational." When a player joins a coalition C, the strategic decisions are made based on the utility function u_C of the coalition C, whereas player i makes the decision to join a coalition based on the individual utility function u_i given this coalition. Finally, the psychological and monetary costs, if any, incurred due to coalition-forming can be incorporated into the utility functions.

Let u denote the profile of utility functions for each feasible coalition including each singleton player. Clearly, a coalitional strategic game is a generalization of the non-cooperative extensive form game. If no coalition (with two or more person) is feasible, then a coalitional strategic game would reduce to a non-cooperative extensive form game.

Strategies: Let $I : X \to \mathbb{N}$ denote the player function, where $I(x)$ gives the "active" player who moves at node x, and $A(x)$ the set of pure actions at node x. Let $h \in H$ denote an information set, $h(x)$ the information set at node x where there is possibly another node $x' \neq x$ such that $x' \in h(x)$, and the active player at $h(x)$ does not know whether she is at x or at x'. If $h(x)$ is a singleton, then, with a slight abuse of notation, $h(x) = x$. Moreover, if $x \in h(x)$, then $A(x) = A(x')$. Let $A(h)$ denote the set of pure actions at h. Let $A_i = \bigcup_{h_i \in H_i} A(h_i)$ denote player i's set of all pure actions where H_i is player i's set of all information sets. Let $S'_i = \bigtimes_{h_i \in H_i} A(h_i)$ denote the set of all pure strategies of i where a pure strategy $s'_i \in S'_i$ is a function $s'_i : H_i \to A_i$ satisfying $s'_i(h_i) \in A(h_i)$ for all $h_i \in H_i$. Let $s' \in S'$ denote a pure strategy profile and $u_C(s')$ its von Neumann-Morgenstern (expected) utility for player C. Let $\Delta(A(h_i))$ denote the set of probability distributions over $A(h_i)$, $b_i \in \bigtimes_{h_i \in H_i} \Delta(A(h_i))$ a behavior strategy of i, and b a profile of behavior strategies. Let $s \in S$ denote the mixed strategy profile that is equivalent to behavior strategy profile b in the sense of Kuhn (1953). I assume that Γ is common knowledge.

Subgames and subtrees: A *subgame* G of a game Γ is the game Γ restricted to an information set with a singleton node and all of its successors in Γ. The largest subgame at a node x is the subgame that is not a subgame (except itself) of any other subgame starting at x. At the root of a game, the largest subgame is the game itself. Let G be a subgame and H'_i be the set of i's information sets in the subgame. Then, for every $h_i \in H'_i$ function $s_i(\cdot\,|h_i)$ denotes the restriction of strategy s_i to the subgame G. If all information sets are singletons, then Γ is called a game of perfect information.

Let $\bar{x} = |X|$ denote the number of nodes in X, and $succ(x)$ and $Succ(x)$ be the set of immediate successors and all successors of a node x (excluding x), respectively. Let

$root(h)$ denote the node that is the root of the subgame containing information set h such that there is no other subgame starting at an information set between $root(h)$ and h—i.e., $root(h)$ is the closest singleton "ancestor" of h. Note that in perfect information games $x = root(x)$ for all nodes $x \in X$. $Succ(h)$ denotes the set of all successor information sets of information set h. A *subtree* T of a game Γ is the game tree of Γ restricted to an information set (not necessarily singleton) and all of its successors in Γ. So the root of a subtree may be a non-singleton information set.[14] For an information set h, $T(h)$ denotes the largest subtree whose root is h. A subgame is a subtree whoose root is singleton.

Supergames and supertrees: Let $\Gamma(x)$ be the largest subgame starting at x with $|N|$ players. For a coalition C, define $P_C = \{\{j\} \subseteq N | j \in N, j \notin C\} \cup C$—the partition of N in which only the agents in C form a coalition. For a player i, define $F_i = \{C' \in 2^N | i \in C'$ where C' is feasible$\}$—the set of all feasible coalitions in N that include i.

A *supergame* of $\Gamma(h)$, denoted by Γ_{P_C}, is the game Γ in which the set of players N is replaced with the set of players given by the partition P_C where $C \in F_i$ and $i \in I(h)$. The set of all supergames is defined as follows:

$$super(\Gamma(h)) = \{\Gamma_{P_C} | C \in F_i,\, i \in I(h)\}.$$

Note that if the original game is an n-player game, then its supergame Γ_{P_C} is an $(n-|C|+1)$-player game. Note that each $j \in C$ acts as an agent of a single player C in Γ_{P_C}. For example, if Γ is a six-player game where $N = \{1, 2, 3, 4, 5, 6\}$, then $\Gamma_{\{1,2,4\}}$ is the game in which the players are $\{1, 2, 4\}, 3, 5$, and 6. $1, 2$, and 4 are the agents of coalitional player $\{1, 2, 4\}$.

Let $\Gamma(h)$ be the largest subgame starting at $root(h)$ and $T(h)$ be a subtree of game $\Gamma(h)$. A *supertree* of $T(h)$, denoted by T_{P_C}, is the subtree of $\Gamma(h)$ in which the set of players N is replaced with the set of players given by the partition P_C where $C \in F_i$ and $i \in I(h)$. The set of all supertrees is defined as follows:

$$super(T(h)) = \{\Gamma_{P_C} | C \in F_i,\, i \in I(h)\}.$$

Systems: A *system* or a "solution profile" is a pair (σ, π) which is defined as follows. Let $G(h)$ be the set of subgames of the largest subgame starting at $root(h)$ for some information set h. Let $\sigma(h, g)$ denote a strategy profile in the subgame $g \in G(h)$. Note that function $\sigma(h, \cdot)$ gives for each information set a collection of strategy profiles one

[14] Note that the root of an information set h is different than the root of a subtree $T(h)$.

for each successor subgame: $\sigma(h) := (s_g)_{g \in G(h)}$. Moreover, $\sigma := (s_{h,g})_{h \in H, g \in G(h)}$—i.e., σ is a family of collection of strategy profiles. Suppose that g' is a subgame of g. In general $\sigma(h, g|g') \neq \sigma(h, g')$—i.e., $\sigma(h, g')$ is not necessarily equal to the restriction of the strategy profile $\sigma(h, g)$ to the subgame starting at node g'. Similarly, let $\pi(h, g)$ denote a feasible partition of players in subgame $g \in G(h)$. Let $\pi(h) := (\pi_g)_{g \in G(h)}$ and $\pi := (\pi_{h,g})_{h \in H, g \in G(h)}$. The interpretation of a pair $(\sigma(h, g), \pi(h, g))$ is as follows. Let C be some coalition in $\pi(h, g)$. Each agent i of player C chooses their actions based on the strategy profile $\sigma(h, g)$. Their actions are guided by the coalitional utility function $u_C(\,\cdot\,|\pi(h, g))$ in the subgame g.

Non-cooperative solution concepts: Let $\Gamma = (P, X, I, u, S, H)$ be a coalitional strategic game. A mixed strategy profile s is called a Nash equilibrium if for every player $j \in P$, $s_j \in \arg\max_{s'_j \in S'_j} u_j(s'_j, s_{-j})$. A mixed strategy profile s is a called a subgame-perfect Nash equilibrium (SPNE) if for every subgame G of Γ the restriction of s to subgame G is a Nash equilibrium in G.

3.2 The Recursive Backward Induction algorithm: Perfect information games

Let Γ be a coalitional strategic game with perfect information without a chance move. I define *Recursive Backward Induction* (RBI) algorithm by a recursive induction procedure. RBI algorithm outputs a system on each subgame $G(x)$ starting at some node x by inducting on the number of players (n) and number of nodes ($\bar{x}$) in the successor nodes of x including x itself. For example, if x is a terminal node, then $(n, \bar{x}) = (1, 1)$. Let $i = I(x)$ be the active player at x. The solution of the game in which Nature moves at the root of the game is simply the profile of the solutions of the subgames starting at every immediate successor of the root.

1. **Base case**: Let $(n, \bar{x}) = (1, 1)$. Equilibrium system at $G(x)$ is defined by the pair (σ^*, π^*) such that $\sigma^*(x) \in \arg\max_{s \in S} u_1(s)$ and $\pi^* = \{1\}$.

2. **Induction step**: Assume that equilibrium system is defined for all subgames with parameters $(m, \bar{y})$ satisfying $1 \leq m \leq n$, $1 \leq \bar{y} \leq \bar{x}$ such that $(n, \bar{x}) \neq (1, 1)$ and $(m, \bar{y}) \neq (n, \bar{x})$.

   By assumption, $[\sigma^*(x'), \pi^*(x')]$ is defined for all $x' \in Succ(x)$ where x is the root of subgame $G(x)$. The solution is extended to subgame $G(x)$ with parameters $(n, \bar{x})$ as follows. I first define "reference points" to compare the solutions of all supergames with the non-cooperative choice of i at x.

(a) The "autarky" reference point at x, $r_0(x)$: Let $b_i^*(x)$ be a utility-maximizing behavior strategy of i at x given $[\sigma^*(x'), \pi^*(x')]$:

$$b_i^*(x) \in \underset{b_i'(x)\in\Delta A_i(x)}{\arg\max}\ u_i(b_i'(x)|\sigma^*(x'), \pi^*(x')).$$

Let $r_0(x) := [\tau^{\{i\}}(x), \pi^{\{i\}}(x)]$ which is defined as the extension of $[\sigma^*(x'), \pi^*(x')]$ with $x' \in succ(x)$ to the largest subgame starting at node x where i chooses the behavior strategy $b_i^*(x)$, and $\pi^{\{i\}}(x)$ is defined as the partition in which i and other players in the subgame at x act non-cooperatively.[15]

(b) Reference points: For any $(n, \bar{x})$ consider supergame Γ_{P_C} where $C \ni i$ is a feasible non-singleton coalition. Because Γ_{P_C} is an $(n-|C|+1)$-person subgame with $\bar{x}$ nodes, its equilibrium system, $[\tau^C(x), \pi^C(x)]$, is defined by assumption. Let $(r_j(x))_{j=1}^{\bar{j}}$ be a sequence for i where $r_j(x) := [\tau^{C_j}(x), \pi^{C_j}(x)]$ that satisfies (i) for every two indices j and j' with $j < j'$, we have $u_i(\tau^{C_j}(x)|\pi^{C_j}(x)) \leq u_i(\tau^{C_{j'}}(x)|\pi^{C_{j'}}(x))$ (i.e., nondecreasing sequence for i), and (ii) if $u_i(\tau^{C_j}(x)|\pi^{C_j}(x)) = u_i(\tau^{C_{j'}}(x)|\pi^{C_{j'}}(x))$ and $C_j \subset C_{j'}$, then r_j precedes $r_{j'}$ for supergame coalitions C_j and $C_{j'}$ containing i.[16]

(c) Individually rational reference points: Given the autarky reference point $r_0^*(x)$, the following *individually rational* (IR) reference points are defined inductively as follows.

Assume that $r_j(x)$ is IR, denoted as $r_j^*(x)$, for some $j \geq 0$. Let $j' > j$ be the smallest number such that $r_{j'}(x)$ is IR with respect to $r_j^*(x)$—i.e., $u_{i'}(\tau^{C_{j'}}(x)|\pi^{C_{j'}}(x)) > u_{i'}(\tau^{C_j}(x)|\pi^{C_j}(x))$ for every agent $i' \in \bar{C}_{j'}$ where $C_{j'} \subseteq \bar{C}_{j'}$ and $\bar{C}_{j'} \in \pi^{C_{j'}}(x)$.[17]

Let $r_{\bar{l}}^*(x)$ be the greatest IR reference point that maximizes i's utility, and call it *supergame perfect* at x. Note that $r_{\bar{l}}^*(x) = [\tau^{C_{\bar{l}}}(x), \pi^{C_{\bar{l}}}(x)]$. It may be that

[15] Put differently, $\tau^{\{i\}}(x,x)$ is defined as the strategy profile in which i chooses behavior action $b_i^*(x)$, and $\pi^{\{i\}}(x)$ is defined as the partition in which i and other players in the subgame at x act non-cooperatively. And, the rest follows the strategy profile and partitions given in system $[\sigma^*(x'), \pi^*(x')]$ for all successors $x' \in Succ(x)$. Note that $\tau^{\{i\}}(x) = \tau^{\{i\}}(x,x')$ for all $x' \in Succ(x)$—i.e., it gives a strategy profile for every subgame.

[16] Ties are broken arbitrarily.

[17] In other words, $\bar{C}_{j'}$ contains coalition $C_{j'}$ such that $\bar{C}_{j'}$ is part of the solution of the supergame $\Gamma_{P_{C_{j'}}}$.

$r_0^*(x)$ where $\bar{l} = 0$ is the only IR reference point.

Equilibrium system of $\Gamma(x)$ is defined as the system $[\sigma^*, \pi^*]$ where $[\sigma^*(x), \pi^*(x)] = r_{\bar{l}}^*(x)$, which extends $[\sigma^*(x'), \pi^*(x')]$ where $x' \in Succ(x)$. When x is the root of the game tree of Γ, $[\sigma^*, \pi^*]$ is said to be equilibrium system of Γ.

To summarize, an equilibrium system is the one that is both subgame and supergame "perfect" at every subgame and supergame. Notice that the RBI procedure incorporates backward induction reasoning as well as some elements of forward induction reasoning in the sense that the coalitions formed in the 'past' rationally affect the behavior of players in the future. The next theorem shows that an equilibrium system always exists.

Theorem 1. *There exists an equilibrium system in pure strategies in every finite n-person coalitional strategic game with perfect information.*

Proof. The RBI algorithm is well-defined because the game is finite—there are finitely many players and finitely many pure strategies. Base case is the standard utility maximization over finitely many actions, so a pure utility-maximizing action exists. Induction step needs some elaboration. Note that this step is recursive in that it assumes that equilibrium system of "smaller" games has been defined. Then, at every node the procedure compares the solution of each supergame and calculates the greatest individually rational solution of a supergame inductively. This procedure will also end after finitely many steps because there are finitely many players and actions in each supergame. This is because, by definition, a supergame of a subgame has fewer players than the players in the subgame due to forming coalitions. Therefore, the algorithm is guaranteed to terminate. The outcome of this algorithm gives the equilibrium system of the coalitional strategic game. □

Theorem 1 shows that in finite coalitional strategic games with perfect information, there is always an equilibrium system where all the strategies that make up the solution are pure strategies. I call coalitions *stable* if they survive the RBI and players *dynamically rational* if they utilize the RBI in coalitional strategic games.

3.3 Imperfect information games

Let Γ be an extensive form game with imperfect information. I next define Recursive Induction (RBI) algorithm which outputs a system on each subtree $T(h)$ starting at some information set h by inducting on the number of players (n) and number of information sets ($\bar{h}$) in the successor information sets of h including h itself. For example, if h is a terminal information set, then $(n, \bar{h}) = (1, 1)$. Let $i = I(h)$ be the active player at information set h.

1. **Base case**: Let $(n, \bar{h}) = (1, 1)$ and $g_{P'}$ be the largest subgame at $root(h)$ where P' is the set of players in the subgame.[18] Equilibrium system at $T(h)$ is defined by the pair $(\sigma^*(h), \pi^*(h)) := (\sigma^*, \pi^*)$ such that $\sigma^*(h)$ is a subgame-perfect equilibrium in $g_{P'}$ and $\pi^* = P'$ in which no player forms a coalition.[19]

2. **Induction step**:

   Assume that equilibrium system is defined for all subtrees with parameters $(m, \bar{y})$ satisfying $1 \leq m \leq n$, $1 \leq \bar{y} \leq \bar{h}$ such that $(n, \bar{h}) \neq (1, 1)$ and $(m, \bar{y}) \neq (n, \bar{h})$. By assumption, $[\sigma^*(h'), \pi^*(h')]$ is defined for all $h' \in Succ(h)$ where h is the root of subtree $T(h)$. The solution is extended to subtree $T(h)$ with parameters $(n, \bar{h})$ as follows. I first define reference points to iteratively compare the solutions of all supertrees with the non-cooperative choice of i at h.

   (a) The autarky reference point at h, $r_0(h)$: Let $r_0(h) := [\tau^{\{i\}}(h), \pi^{\{i\}}(h)]$ which is defined as the extension of $[\sigma^*(h'), \pi^*(h')]$ with $h' \in succ(h)$ to the largest subgame starting at node $root(h)$ such that $[\tau^{\{i\}}(h), \pi^{\{i\}}(h)]$ is an SPNE in this subgame where all players acting between $root(h)$ and h inclusive choose their strategies non-cooperatively.[20]

   (b) Reference points: For any $(n, \bar{h})$ consider supertree $T_{P_C}(h)$ where $C \ni i$ is a feasible non-singleton coalition. Because $T_{P_C}(h)$ is an $(n - |C| + 1)$-person subtree with $\bar{h}$ information sets, its equilibrium system, $[\tau^C(h), \pi^C(h)]$, is defined by assumption.

   Let $(r_j(h))_{j=1}^{\bar{j}}$ be a sequence for i where $r_j(h) := [\tau^{C_j}(h), \pi^{C_j}(h)]$ that satisfies (i) for every two indices j and j' with $j < j'$, we have $u_i(\tau^{C_j}(h)|\pi^{C_j}(h)) \leq u_i(\tau^{C_{j'}}(h)|\pi^{C_{j'}}(h))$ (i.e., nondecreasing sequence for i), and (ii) if $u_i(\tau^{C_j}(h)|\pi^{C_j}(h)) = u_i(\tau^{C_{j'}}(h)|\pi^{C_{j'}}(h))$ and $C_j \subset C_{j'}$, then r_j precedes $r_{j'}$ for supertree coalitions C_j and $C_{j'}$ containing i.[21]

[18]Note that $g_{P'}$ is the only element in $G(h)$.
[19]Note that $\sigma^*(h)$ is a strategy profile in $g_{P'}$.
[20]Note that SPNE is defined with respect to players, which may be coalitions, which are given by partition $\pi^*(h')$.
[21]Ties are broken arbitrarily.

(c) IR reference points: The autarky reference point $r_0(h)$ is individually rational by definition. The following IR reference points are defined inductively as follows.

Assume that $r_j(h)$ is IR, denoted as $r^*_j(h)$, for some $j \geq 0$. Let $j' > j$ be the smallest number such that $r_{j'}(h)$ is IR with respect to $r^*_j(h)$—i.e., $u_{i'}(\tau^{C_{j'}}(h)|\pi^{C_{j'}}(h)) > u_{i'}(\tau^{C_j}(h)|\pi^{C_j}(h))$ for every $i' \in \bar{C}_{j'}$ where $C_{j'} \subseteq \bar{C}_{j'}$ and $\bar{C}_{j'} \in \pi^{C_{j'}}(h)$.

Let $r^*_{\bar{l}}(h)$ be the greatest IR reference point, which maximizes i's utility. Note that $r^*_{\bar{l}}(h) = [\tau^{C_{\bar{l}}}(h), \pi^{C_{\bar{l}}}(h)]$. It may be that $r^*_0(h)$ where $\bar{l} = 0$ is the only IR reference point.

Equilibrium system of $T(h)$ is defined as $[\sigma^*, \pi^*]$ where $[\sigma^*(h), \pi^*(h)] = r^*_{\bar{l}}(h)$, which extends $[\sigma^*(h'), \pi^*(h')]$ where $h' \in Succ(h)$. When h is the root of the game tree of Γ, $[\sigma^*, \pi^*]$ is said to be equilibrium system of Γ.

In imperfect information games, an equilibrium system is the one that is both subtree and supertree "perfect" for every subtree and supertree. In addition, if the extensive form structure of a normal form game is not given, then the equilibrium system of the game is defined as the combination of the solutions of the extensive form games whose reduced normal form is the given normal form game. Next theorem shows the existence of the equilibrium system in imperfect information games.

Theorem 2. *There exists an equilibrium system in possibly mixed strategies in every finite n-person coalitional strategic game.*

Proof. The RBI procedure for imperfect information games is also well-defined because the game is finite. In the base case, an equilibrium system exists because a subgame-perfect equilibrium exists. The induction step assumes that an equilibrium system of the smaller subtrees has been defined. Then it defines the equilibrium system of the larger subtree at an information set h as the greatest individually rational equilibrium system among the equilibrium systems of the supertrees starting at h. The greatest individually rational referene point exists because (i) by induction hypothesis the equilibrium system of a supertree exists, and (ii) there are finitely many supertrees because there are finitely many coalition-forming possibilities. The whole process ends after finitely many steps because there are finitely many information sets. The outcome of the algorithm gives the equilibrium system of the game. □

3.4 Modifications

Partial, fixed, and mixed cooperation: Suppose that supergame of a game consists not only of the games in which two or more players form a coalition but also the games in which coalitions are subgame-dependent. For example, at an information set h, i cooperates with j but at information set h' i cooperates with j', in which case the active players at h and h' would be defined as $\{i, j\}$ and $\{i, j'\}$, respectively. Since utilities for both $\{i, j\}$ and $\{i, j'\}$ are well defined, the RBI algorithm would be well defined as well—i.e., an equilibrium system would be the one that is both subtree and supertree perfect according to this more general supertree definition. In "fixed cooperation" variation, it is possible to incorporate into coalitional strategic game structure the option that two players, say i and j, act together at an information set h, while acting separately at other nodes. In this case, supertree coalitions at h must include both i and j and the IR reference points at h must be calculated with respect to $u_{\{i,j\}}$. In another variation, it may be that player i at an information set h "mixes" between two or more supertrees, e.g., between coalitions $\{i, j\}$ and $\{i, j'\}$, to achieve a better individually rational reference point at h. Any of these variations can be incorporated into the coalitional strategic games and, mutatis mutandis, the equilibrium system would be well-defined and would exist (i.e., Theorem 2 applies).

Refinements and other solutions: The equilibrium system does not include threats that are non-credible, so it is possible to generalize this solution by allowing non-credible threats in the spirit of Nash equilibrium versus SPNE. Another variation could be as follows. Under the current setting, agents form coalitions provided that they have strict incentives to do so, weakening this assumption is certainly reasonable, though this would also potentially increase the set of solutions. One can also consider concepts such as maximin strategy and maximin equilibrium (optimin criterion) (Ismail, 2014) in the framework of coalitional strategic games. Maximin "coalitional" equilibrium would be a system that simulatenously maximizes the minimum utility for each player under rational deviations at every information set.

4 A fully worked out example

Figure 3 (A) illustrates a three-player coalitional strategic game in which Player 1 (P1) starts by choosing L or R. If cooperation were not possible, then the standard backward induction outcome of this game would be (5, 5, 3), as is illustrated in Figure 3 (B).

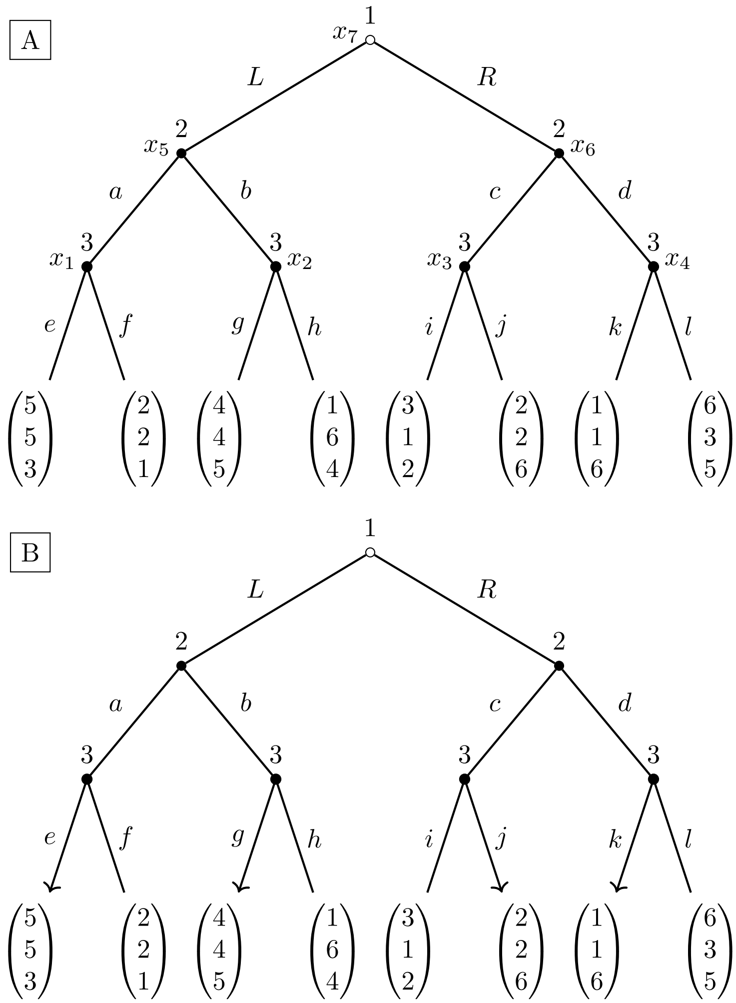


Figure 3: A: A three-player coalitional strategic game. B: The best-response of P3 at the terminal nodes are shown by the lines with arrows.

However, this is only the beginning of the analysis because in this model players may strategically join forces and form coalitions as long as it is mutually beneficial. For simplicity, in this example I assume that a coalition prefers more egalitarian outcomes to less egalitarian outcomes: Coalitional utility function is defined as follows: $u_C(\cdot) := \min_{i \in C} u_i(\cdot)$ where $C \subseteq \{1, 2, 3\}$. That is, one outcome is preferred to the other if the minimum utility a member of the coalition receives from the former outcome is greater than the minimum utility a member receives from the latter. For example, a coalition of P1 and P3, if forms, would prefer $(6, 3, 5)$ to $(1, 1, 6)$ because $min(6, 5) = 5$ and $min(1, 6) = 1$.[22]

[22]Note that the model introduced in section 3 allows for any coalitional preferences as long as represented by a von Neumann-Morgenstern utility function.

I next solve this game's equilibrium system following some of the main steps in the RBI algorithm. (For a more detailed step-by-step solution, see Figure 6.) The best-response of P3 at the terminal nodes are shown in Figure 3 (B) by the lines with arrows. Figure 4 (C) illustrates the left subgame in which it is P2 to make a choice. The backward induction (BI) path in this subgame is a and e, whose outcome is (5, 5, 3), which is the outcome of the autarky reference point r_0 at that node. But if P2 anticipates and identifies a mutually beneficial outcome, then P2 might convince P3 to form a coalition, as they would have the full control of the outcome in the subgame where P2 moves. Note that both P2 and P3 prefer (1, 6, 4) to the autarky reference point, (5, 5, 3). Thus, P2 and P3 will rationally get together and play b and h, because it is mutually beneficial. Because (1, 6, 4) is individually rational for P2 and P3, it overrides the backward induction outcome, and, therefore, the outcome of the solution at the node P2 moves is defined as (1, 6, 4).

But what if P1 anticipates that if P1 plays L, then P2 and P3 will collaborate against P1? Then, the initial backward induction action, L, may not be optimal any more (as we have seen in Example 1), so P1 may need to update its decision based on the new reference point in which P1 receives a payoff of 1. Indeed, if P1 chooses R then the outcome would be (2, 2, 6), which is preferred to the IR reference point, (1, 6, 4), at the left subgame in which P2 moves. Figure 4 (D) illustrates this situation. Therefore, P1's new best response is R, and the new reference point becomes (2, 2, 6), which is the r_0 outcome at the root of the game.

Next, having figured out what would happen if P1 acts non-cooperatively, P1 needs to check using RBI reasoning on supergames whether there are any collaboration opportunities. Notice that P1 would like to form a coalition with P3 to obtain the outcome (6, 3, 5), but P3 would reject such an offer because P3 receives 6 from the current reference point, (2, 2, 6). P1 could threaten to play L, thereby decrease P3's payoff; however, this threat would not be credible because P1 would not rationally carry out this threat: P1 would receive 1 (as opposed to 2) as shown in Figure 4 (C). Actually, P3 would not be interested in forming a coalition with any player at this reference point because P3 receives his highest payoff at (2, 2, 6). I next check whether P1 and P2 can do better in the supergame by forming a coalition, which is illustrated in Figure 5 (E). Notice that if P1 and P2 get together and play L and a, then P3 would best respond to this choice by e. The resulting outcome, (5, 5, 3), is better for P1 and P2 than (2, 2, 6). Therefore, P1 and P2 would form a coalition, which breaks down the alliance between P2 and P3 in this subgame. The outcome of the solution in this supergame is, (5, 5, 3), which has become the outcome again—albeit for a difference reason.

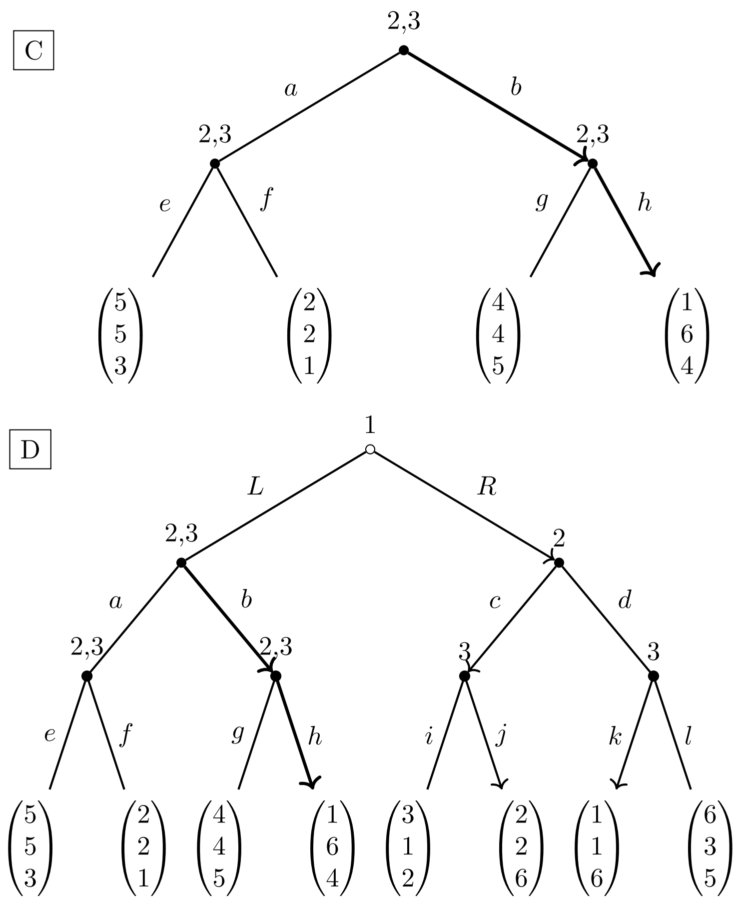
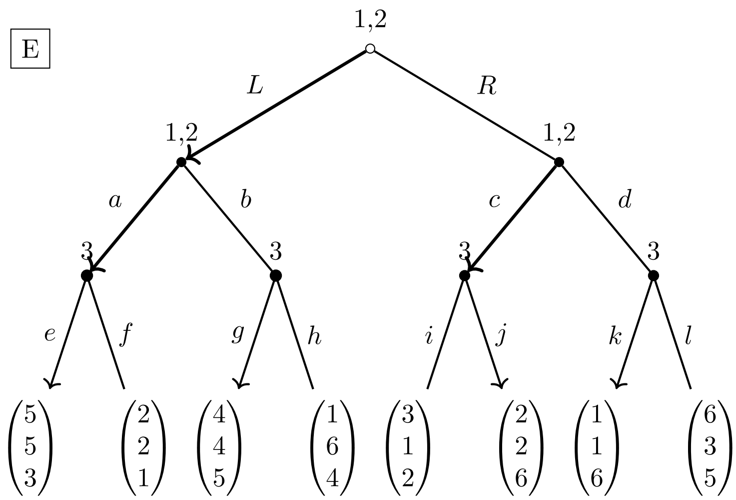


Figure 4: Equilibrium system solution steps (C) and (D). The lines with arrows represent best responses, which could be non-cooperative or coalitional. C: The first coalition forms: Looking forward, P2 forms a coalition with P3 to play b and h. This overrides the backward induction outcome, so the IR reference point is (1, 6, 4). D: P1 anticipates the coalition of P2 and P3 and hence chooses to R, because P1 prefers (2, 2, 6) to (1, 6, 4). The autarky reference point r_0 outcome at the root is (2, 2, 6).



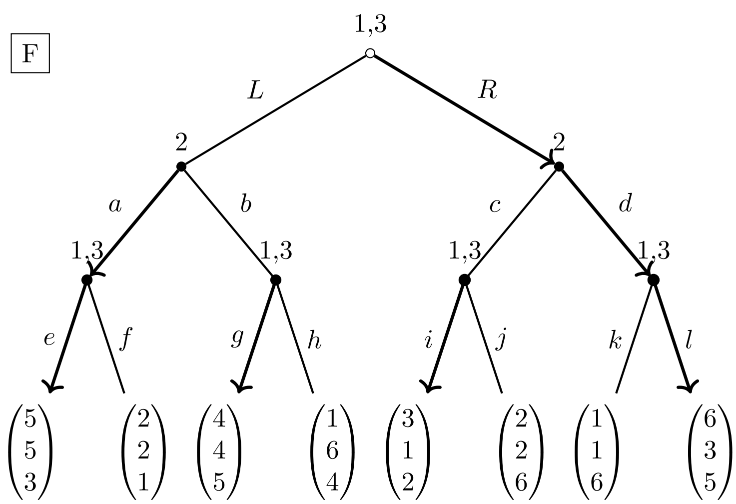


Figure 5: E: The coalition between P2 and P3 breaks down, so a new coalition forms: P1 and P2 cooperate to play L and a to receive 5 each, which is better than 2 each at the previous reference point. The new reference point is, once again, (5, 5, 3). F: A coalition, namely the second one, breaks down once again. P1's forming a coalition with P2 was a credible threat. Anticipating that his payoff will be 3, P3 forms a coalition with P1: P1 will play R and P3 will play l. As a result, (6, 3, 5) is the outcome of this game because no other coalition can do better.

But this is not the end of the analysis because P3 anticipates that in case P3 does not act, the outcome will be (5, 5, 3). Remember that P1 by themself could not credibly threaten P3; however, forming a coalition with P2 sends P3 a credible threat, which brings P3 back to the "bargaining table." I next analyze the supergame in which P3 would seek to form a coalition with P1 to possibly get (6, 3, 5), which is mutually beneficial compared with the current supergame outcome (5, 5, 3). The utility function of the coalition P1 and P3, $u_{\{1,3\}}$, would dictate that that if P1 plays R, then P3 would play l: P2's choice would be d because P2 would receive 3, whereas if P2 plays c, then P2 would get at most 2. Figure 5 (F) illustrates the outcome, (6, 3, 5), when P2 best responds to the coalitional player $\{1, 3\}$. This is the greatest IR reference point at the root of the game because no other coalition can do better.

A complete equilibrium system solution of the game in Figure 3 (A) is presented in Figure 6. But a non-complete ("on-path") solution of this game can be summarized by a list of players, stable coalitions, and their strategies:

$$[\{R\}, \{a, d\}, \{e, g, j, l\}; \{1, 3\}, 2],$$

in which P1 and P3 form a coalition (so each of them becomes an agent of player $\{1, 3\}$), agent P1 chooses R, P2 chooses a and d, and agent P3 chooses e, g, j, and l, from left to right. The outcome of this solution is (6, 3, 5). It is notable that during the solution process we have seen that all coalitions—$\{1, 2\}$, $\{2, 3\}$, and $\{1, 3\}$—except the grand coalition formed, though the only stable coalition turned out to be the one between P1 and P3.[23]

At the outset, it might be tempting to conclude—without running the RBI algorithm—that P1 and P3 will obviously form a coalition to obtain (6, 3, 5). However, this conclusion would be false. To give an example, consider the game in Figure 3 (A) in which, all else being equal, outcome (3, 1, 2) is replaced with outcome (3, 4, 3). This change seems to be irrelevant because P1 and P3 can still form a coalition to obtain (6, 3, 5). However, the outcome of the new game based on the same RBI procedure would be (5, 5, 3), which is significantly different than the previous outcome—why is this? This is because P2 has now a credible threat against the coalition of P1 and P3. Notice that if P1 plays R, then P2 would respond by c, knowing that the coalition would choose i that leads to the more egalitarian outcome, (3, 4, 3).[24] Because the IR reference point in Figure 5 (E) is (5, 5,

[23] Of course, a natural interpretation is that the process first occurs in the minds of the players, and then they form coalitions.

[24] In this example, for simplicity, I assumed that payoff transfers are not possible, but even if they were,

1. Node x_7: The solution at the root of the game:
   (a) Node x_7:
   $$[\{R\}, \{a, d\}, \{e, g, j, l\}; \{1, 3\}, 2].$$
   (b) Node x_6 (the subgame after action R):
      i. Node x_6:
      $$[\{d\}, \{i, l\}; 2, \{1, 3\}].$$
      ii. Nodes x_3 to x_4:
      $$[\{i, l\}; \{1, 3\}].$$
   (c) Node x_5 (the subgame after action L):
      i. Node x_5:
      $$[\{a\}, \{e, g\}; 2, \{1, 3\}].$$
      ii. Nodes x_1 to x_2:
      $$[\{e, g\}; \{1, 3\}].$$
   (d) Nodes x_1 to x_4 (the terminal nodes):
   $$[\{e, g, i, l\}; \{1, 3\}].$$
2. Node x_6: The solution at the subgame after action R:
   (a) Node x_6:
   $$[\{c\}, \{j, k\}; 2, 3].$$
   (b) Nodes x_3 to x_4:
   $$[\{j, k\}; 3].$$
3. Node x_5: The solution at the subgame after action L:
   (a) Node x_5 (the subgame after action L):
   $$[\{b\}, \{h\}; \{2, 3\}].$$
   (b) Nodes x_1 to x_2:
   $$[\{e, g\}; \{3\}].$$
4. Nodes x_1 to x_4: The solution at the terminal nodes:
   (a) Nodes x_1 to x_4:
   $$[\{e, g, j, k\}; 3].$$

Figure 6: An equilibrium system of the game in Figure 3 (A).

3), it would not be individually rational for P1 to collaborate with P3, given P2's credible threat of choosing c instead of d. As a result, a non-complete solution of this game can be summarized as

$$[\{L\}, \{a, c\}, \{e, g, j, k\}; \{1, 2\}, 3],$$

in which P1 and P2 form a coalition, agent P1 chooses L, agent P2 chooses a and c, and P3 chooses e, g, j, and k, from left to right. The outcome of this solution is (5, 5, 3). In this three-person game, a credible threat by P2 prevents P3 destabilizing the coalition of P1 and P2—i.e., the supergame coalition of $\{1, 3\}$ is not individually rational with respect to the supergame coalition of $\{1, 2\}$.

5 Applications

5.1 Logrolling

Early literature on logrolling includes seminal works of Buchanan and Tullock (1962), and Riker and Brams (1973), whose vote-trading paradox shows that individually advantageous vote-trading may lead to Pareto inferior outcomes for everyone including the traders. More recently, Casella and Palfrey (2019) propose a vote-trading model in which voters have separable preferences over proposals and each proposal is decided by majority voting. Their finding is quite general and striking. They show that for any finite number of voters (who are assumed to be myopic), proposals, and any separable preferences and initial vote profile there exists a sequence of strictly payoff-improving trades that leads to a stable vote profile in the sense that no further strictly payoff-improving trade is possible.

Let $L(K, v, r)$ denote a logrolling game, which is defined as a coalitional strategic game $\Gamma = (P, X, I, u, \Sigma, H)$ with the following parameters and interpretation. Each player $i \in P$ is called a voter, $K = \{1, 2, ..., \bar{k}\}$ denotes the number of proposals to vote, and r denotes the rule to determine the acceptance or rejection of a proposal such as the majority rule. Each voter i has $v_i(k)$ votes to vote for or against proposal k. Let $v = (v_1, ..., v_n)$ denote the initial vote profile. Forming a coalition can be interpreted as the agents that make up the coalition trade votes à la Casella and Palfrey (2019).

Next, I give an existence result for this very general logrolling game. The following corollary directly follows from Theorem 2.

Corollary 1. *Every logrolling game possesses an equilibrium system.*

one could construct a similar example.

	1	2	3	4	5
A	4	-7	1	-1	4
B	1	1	-4	4	-1
C	-3	4	2	2	2

Figure 7: A logrolling game by Casella and Palfrey (2019) in which the unique equilibrium system chooses the Condorcet winner. Assume that the initial vote profile is such that for every $i \in \{1, 2, 3, 4, 5\}$ and every $k \in \{A, B, C\}$, $v_i(k) = 1$. The table shows each voter's utility from the accepted proposals. The utility of a rejected proposal is normalized to 0.

In other words, for any finite number of voters and proposals, any initial vote profiles, any voting rule, any type of preferences (seperable or not), there exists an equilibrium system in every logrolling game. Corollary 1 extends Casella and Palfrey's (2019) existence result to logrolling games with (i) dynamically rational voters, (ii) any type of voting rule, and (iii) any type of preferences. In the logrolling game, the dynamic rationality of the voters is arguably an important extension in part because otherwise some voters might engage in myopically payoff-improving trade, which may eventually decrease their utility.

Figure 8 illustrates a vote-trading example by Casella and Palfrey (2019) in which the Condorcet winner, (A, B, C), exists. However, myopic payoff-improving trades may lead to a stable outcome that differs from the Condorcet winner. Consider the following sequence of trades: (i) voters 2 and 3 vote against A and B (i.e., 2 trades with 3 a B vote for an A vote), (ii) 4 and 5 vote for A and B, and then (iii) 1 and 3 vote against B and C. The final outcome is that only A is implemented. As Casella and Palfrey (2019) show, voters at each step myopically benefit from these trades and that the final vote profile is stable. However, note that the utility of voter 2 at outcome A is -7. Thus, if voter 2 were dynamically rational she would not trade votes with voter 3 in the first place.

A logrolling game $L(K, v, r)$ with the majority rule is said to have a Condorcet winner if (i) $u_C(s) > u_C(s')$ whenever $u_i(s) > u_i(s')$ for every $i \in C$, and (ii) there is an outcome that is preferred to any other outcome by a majority of the voters. The following proposition shows that, as a coalitional extensive form game, the unique equilibrium system outcome of the logrolling game is the Condorcet winner.

Proposition 1. *Suppose that a logrolling game with the majority rule has a Condorcet winner. Then, the unique equilibrium system outcome of the logrolling game is the Condorcet winner.*

Proof. To reach a contradiction, suppose that the logrolling game has an equilibrium system outcome s' that differs from the Condorcet winner $\bar{s}$. Then, by definition the

	1	2	3	4	5	6	7
A	2	-1	-1	-1	1	1	1
B	-1	2	-1	-1	1	1	1
C	-1	-1	2	-1	1	1	1
D	-1	-1	-1	2	1	1	1

Figure 8: A 7-voter logrolling example (Casella and Palfrey, 2019) and its equilibrium system

majority of the voters (say, C) strictly prefer $\bar{s}$ to s'. In the relevant supergame, forming C would be individually rational and C can profitably deviate from s', which implies that s' cannot be an outcome of an equilibrium system. □

Note that Proposition 1 shows that the equilibrium system outcome of the logrolling game is Condorcet-consistent. Although the equilibrium system depends on the structure of the coalitional extensive form game, its outcome is always the Condorcet winner, whenever it exists. Casella and Palfrey (2019) showed that payoff-improving myopic vote-trading may lead to a stable outcome that is not the Condorcet winner, which contrasts the conjecture of Buchanan and Tullock (1962) that vote-trading would lead to the Condorcet winner, if there is one. Proposition 1 shows that this conjecture holds if the voters are dynamically rational.

To illustrate an equilibrium system of a logrolling game, consider Figure 8 (Casella and Palfrey, 2019). The coalitional strategic game form of this vote-trading game is too big to illustrate here due to the number of players and strategies. Thus, I next solve this game without explicitly showing the game tree. Suppose that the order in which players choose strategies is 1, 2, ..., 7, and that $u_C(\cdot) := \sum_{i \in C} u_i(\cdot)$ for any coalition $C \subseteq \{1, 2, ..., 7\}$.

It is clear that it is a dominant strategy for voters 5, 6, and 7 to vote for each proposal. First, notice that the following strategy profile forms a subgame perfect equilibrium when considered with the dominant strategies of 5, 6, and 7: Each of the voters in $\{1, 2, 3, 4\}$ votes for their preferred proposal and against the other proposals. As a result, all proposals are accepted. Each player in $\{1, 2, 3, 4\}$ receives a utility of -1, whereas other players each receive a utility of 4. Second, consider the supergame coalition of $\{1, 2, 3, 4\}$ where each agent votes against all proposals. As a result, all proposals are rejected. Thus, every player's payoff would be 0. Clearly, this outcome is preferable to the voters $1-4$. Another supergame coalition is $\{1, 2\}$. Suppose that both 1 and 2 chooses to vote for their preferred proposal and vote against all other proposals with the hope that voters 3 and 4 may reject all proposals, in which case the utility of both 1 and 2 would inrease to 1. Voters 3 and 4 would each receive -2. But then in the supergame $\Gamma_{\{1,2\}}$ and its subgame starting with voter 3, voters 3 and 4 would form a coalition against $\{1,2\}$ and vote for their preferred proposal so that they each strictly benefit. The coalition of $\{3,4\}$ is then a credible threat to the coalition of $\{1,2\}$. Therefore, player 1 would prefer the outcome of the supergame $\Gamma_{\{1,2,3,4\}}$ in which each voter in $\{1,2,3,4\}$ votes against all proposals. In summary, the equilibrium system I have just described includes the coalition $\{1,2,3,4\}$ where every voter votes against all proposals, and players 5, 6, 7 each vote for all proposals. As a result, all proposals are rejected in the equilibrium system of this game.[25]

5.2 Corruption

In this subsection, I show how to study "corruption" by incorporating wealth into coalitonal extensive form games. Let $\Gamma = (P, X, I, u, \Sigma, H, W)$ be a coalitonal extensive form game with wealth in which $W = (W_1, W_2, ..., W_n)$ denotes a wealth profile where $W_i \in \mathbb{R}_+$ denotes the wealth of each player i. Each player starts with an initial wealth and side payments are allowed. Player i can make non-IR supertree coalitions individually rational by transferring wealth to some agents who otherwise would not rationally accept to cooperate at an information set. Obviously, wealth goes into the utility function, and there is a trade-off between transferring wealth and obtaining a more desirable solution. Via this way, one could capture the payments that are made to individuals in order to "buy" their cooperation in strategic situations. As a result, coalitonal extensive form games with wealth would enable social scientists to study the strategic effects of corruption (e.g., buying someone's cooperation to induce them to choose a particular action, which otherwise would not be chosen). Mutatis mutandis, the existence results (Theorem 1 and Theorem 2) would analogously remain valid under this extension.

To given an example, consider Figure 9, which is a slight modified version of the market entry game presented in Figure 1. Note that in the equilibrium system of the original game Firm 1 and Firm 4 colludes, so Firm 1 enters the market 2 and Firm 4 chooses A. The equilibrium system outcome is (60, 0, 0, 60). Suppose that the initial wealth profile is $W = (0, 100, 0, 0)$ and that the utility is linear in money for the sake of simplicity. Notice that Firm 2 would now be willing to transfer 31 units of money to

[25]Note that if individual or coalitional utility functions change in a way that makes it attractive for a voter in $\{1,2,3,4\}$ to join a coalition with the voters 5, 6, or 7, then the equilibrium system might change.

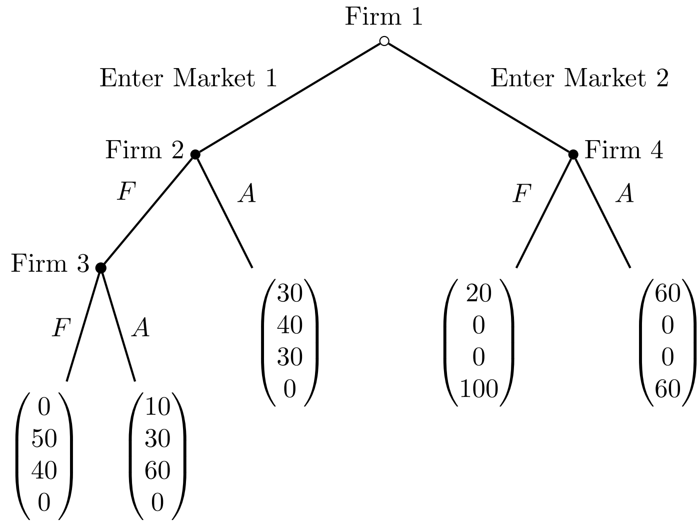


Figure 9: International market entry game with an initial wealth distribution $W = (0, 100, 0, 0)$

Firm 1 to incentivize cooperation.[26] This transfer of money would make Firm 2 better off because otherwise Firm 2 receives a utility of 0 at the equilibrium system outcome, and it clearly makes Firm 1 better off. As a result, Firm 1 would choose to enter market 1 and Firm 2 would choose A under the RBI with the given initial wealth structure. The outcome of the new equilibrium system would be $(30, 40, 30, 0)$ and the final wealth distribution would be $W = (31, 69, 0, 0)$.

Figure 9 presents a simple game that illustrates how money transfers from one player to another can affect the equilibrium system of a game. One can think of more complex coalitonal extensive form games with a different initial wealth distribution. Clearly, the possibility of money transfer between two players may trigger other players to offer "counter-bribes." For example, if Firm 4 had an initial wealth of say $W_4 = 40$, then Firm 4 would be able to "match" Firm 2's offer, hence incentivize Firm 1 to enter market 2.

5.3 Banning policy

Consider the Banning Policy game presented in Figure 10 in which the government (Player 1) chooses between the policy of banning or not banning a market such as drugs, organ trade, abortion etc. A buyer (Player 2), who is considering buying the relevant product/service, chooses between having buying (Y) or not (N); and, and the seller (Player 3) chooses between charging a high (H) or low (L) price. Figure 1 illustrates players'

[26]For simplicity, I assume that only integer unit transfers are possible.

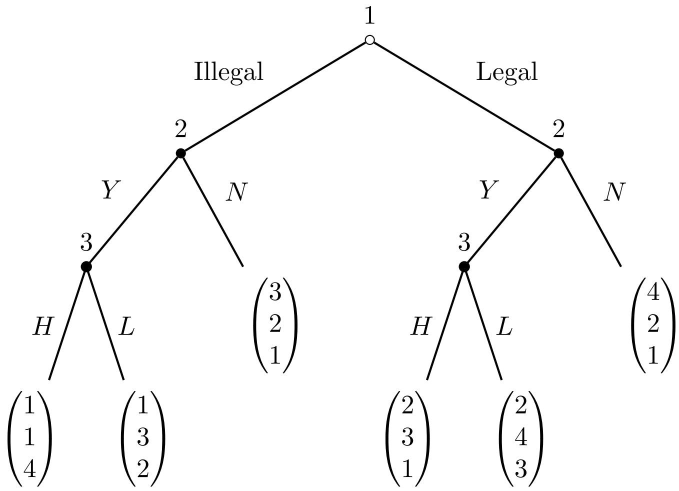


Figure 10: A stylized banning policy game in which government moves first, the buyer who considers buying an illegal good/service moves second, and the seller moves last.

actions and preferences over the outcomes. I assume the following preferences in this stylized example:

1. The government prefers N to Y, and 'Legal' to 'Illegal' in any situation.

2. The buyer's worst outcome is when the choices are 'Illegal, Y, and H', whereas her best outcome is when the choices are 'Legal, Y, and L'. Her second most preferred outcomes are when the choices are 'Illegal, Y, and L' and 'Legal, Y, and H', in which case I assume that she goes to an alternative seller with a lower price as the product is legal. She receives a utility of 2 when she chooses N.

3. The seller's worst outcome is when the buyer does not buy the product from this seller, whereas the seller's best outcome is when the choices are 'Illegal, Y, and H'. When the buyer chooses Y, and the price is L, the seller prefers 'Legal' to 'Illegal', so the seller's utility is 3 and 2, respectively.

First, the subgame perfect equilibrium in this game can be found by following the backward induction procedure: The seller would choose H on the left node (4 vs. 2) and L on the right node (1 vs. 3). Given the seller's choice, the buyer would choose N and Y on the left and right nodes, respectively. Anticipating these choices, the government would choose to ban the product. So, the outcome of this solution would be (3, 2, 1). By banning the product, the government relies on the assumption that the seller and the buyer will act independently and will not cooperate and coordinate their actions.

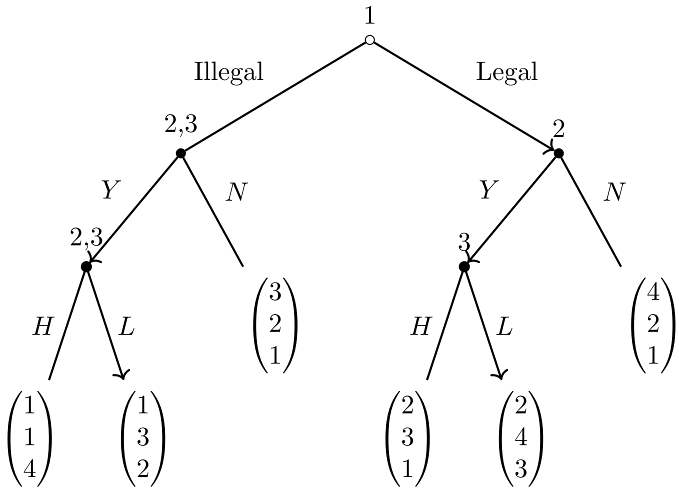


Figure 11: Equilibrium system solution of the banning policy game. Assume that $u_C(\cdot) := \min_{i \in C} u_i(\cdot)$ for any coalition $C \subseteq \{1, 2, 3\}$. The lines with arrows represent best responses, which could be non-cooperative or coalitional. A step-by-step equilibrium system solution of this game is available at https://youtu.be/AzH7WfRu7oY.

However, after the government plays 'Illegal', the buyer and the seller have an incentive to cooperate because they each would prefer outcome (1, 3, 2) to (3, 2, 1), as is illustrated in Figure 11. For simplicity assume that coalitional utility function is defined as follows: $u_C(\cdot) := \min_{i \in C} u_i(\cdot)$ where $C \subseteq \{1, 2, 3\}$. For example, coalitional player $\{2, 3\}$'s payoff from $(1, 3, 2)$ is given $u_{\{2,3\}} = min(3, 2) = 2$ and from $(3, 2, 1)$ is given by $u_{\{2,3\}} = min(2, 1) = 1$. Thus, the best response of coalition $\{2, 3\}$ is to choose Y and L after the choice of Illegal. So, banning the product would not necessarily prevent it from happening because there is a mutually beneficial opportunity to cooperate. If the government is forward-looking and anticipates that the buyer and the seller will cooperate and coordinate their actions (as shown by solid arrows in Figure 11), then it would prefer to legalize the product, in which case the outcome would be (2, 4, 3).[27] Note that the cooperation between Player 2 and Player 3 is a credible threat—if the government chooses 'Illegal', then the threat will be carried out because it is mutually beneficial for the buyer and the seller.

To give more examples, in some countries, it is illegal to cooperate under certain circumstances: oligopolistic cartels, selling/buying drugs, organ trade, forming partnerships—such as dating and same-sex couple marriage or partnership—just to name a few. In these games and other games played outside of restricted lab conditions, it is difficult and costly,

[27] Note that in the right subgame Player 2 and Player 3 choose their actions independently as they would not benefit from cooperation, which is illustrated by the lines with arrows in Figure 11 .

if not impossible, to enforce that players will not exercise their free will to cooperate. Under such strategic interactive decision-making situations, I assume that players rationally take into account strategic partnership opportunities and threats as long as they are "credible."

6 Conclusions

Coalitional strategic games include wars, airline alliances, and scientific publication. I propose a rational solution in such games, which is based on a unique procedure that combines backward and some elements of forward induction reasoning in which players act non-cooperatively or cooperate in a rational way. An equilibrium system is a system which includes a family of strategy profiles and stable coalitions such that independent players do not have any incentive to deviate unilaterally and coalitions are individually rational and stable in the sense that their members prefer to be in the coalition than be out of the coalition.

Traditionally, non-cooperative games were extended first to imperfect-information games (Selten, 1965) and then to incomplete-information games (Harsanyi, 1967). I believe that coalitional strategic games and the associated equilibrium system concept can be extended to more general settings in an analogous way. In essence, any concept defined on standard extensive form games can potentially be extended to coalitional strategic games.

A number of fields may benefit from applications of the framework and the solution concept proposed in this paper. A political scientist may apply the model to conflict and cooperation among countries, a computer scientist to multi-agent systems, a biologist to evolution of species and genes, an ethologist to animal behavior, an operations researcher to freight carriers, and an economist to examples such as airline alliances and oligopolistic cartels.

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